Sea la matriz A definida por

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Determine el polinomio característico de la matriz A.

Solución.

Se define el polinomio característico de un matriz como $p(\lambda) = \det(A - \lambda I) = 0$, se tiene que

$$\det\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = 0 \quad => \det\left(\begin{pmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{pmatrix}\right) = 0$$

Desarrollamos el determinante por la primera columna. Se tiene

$$\underbrace{(a_{11} - \lambda)\det\begin{pmatrix} a_{22} - \lambda & a_{23} \\ a_{32} & a_{33} - \lambda \end{pmatrix}}_{P_1} - \underbrace{a_{21}\det\begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} - \lambda \end{pmatrix}}_{P_2} + \underbrace{a_{31}\det\begin{pmatrix} a_{12} & a_{13} \\ a_{22} - \lambda & a_{23} \end{pmatrix}}_{P_3} = 0$$

Desarrollamos cada uno de los términos

$P_1 = (a_{11} - \lambda)\big((a_{22} - \lambda)(a_{33} - \lambda) - a_{23}a_{32}\big) \;=>$

$P_1 = (a_{11} - \lambda)(a_{22}a_{33} - (a_{22} + a_{33})\lambda - a_{23}a_{32} + \lambda^2)$

$= (a_{11}a_{22}a_{33} - a_{11}(a_{22} + a_{33})\lambda - a_{11}a_{23}a_{32} - a_{22}a_{33}\lambda + (a_{22} + a_{33})\lambda^2 + a_{23}a_{32}\lambda + a_{11}\lambda^2 - \lambda^3)$

Reorganizando

$P_1 = -\lambda^3 + (a_{22} + a_{33} + a_{11})\lambda^2 + (a_{23}a_{32} - a_{11}a_{22} - a_{11}a_{33} - a_{22}a_{33})\lambda + (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32})$

Vamos con P2

$P_2 = a_{21}(a_{12}(a_{33} - \lambda) - a_{13}a_{32}) = a_{21}(a_{12}a_{33} - a_{12}\lambda - a_{12}a_{32}) = a_{21}a_{12}a_{33} - a_{21}a_{12}\lambda - a_{21}a_{12}a_{32}$

Terminamos con P3

$P_3 = a_{31}\big(a_{12}a_{23} - a_{13}(a_{22} - \lambda)\big) = a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22} + a_{31}a_{13}\lambda$

Agrupamos los términos $P_1 - P_2 + P_3 = 0$

$$\begin{aligned} -\lambda^3 + \overbrace{(a_{22} + a_{33} + a_{11})}^{A}\lambda^2 + \overbrace{(-a_{11}a_{22} - a_{11}a_{33} - a_{22}a_{33} + a_{23}a_{32} - a_{22}a_{33} + a_{21}a_{12} + a_{31}a_{13})}^{B}\lambda \\ + \underbrace{(a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + a_{21}a_{12}a_{32} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22})}_{C} = 0 \end{aligned}$$

Se define

$$A = th(A) = a_{11} + a_{22} + a_{33}$$

Observamos el termino B

$$B = -\left(\overbrace{a_{22}a_{33} - a_{23}a_{32}}^{\det(M_{11})} + \overbrace{a_{11}a_{33} - a_{31}a_{13}}^{\det(M_{22})} + \overbrace{a_{11}a_{22} - a_{21}a_{12}}^{\det(M_{33})}\right)$$

Por lo que

$$B = -\sum_{i=1}^{3} \det(M_{ii})$$

El termino C es determinante de A, si desarrollamos el determinante de A por la primera columna

$$\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{13}a_{23} - a_{13}a_{22})$$

Por lo que se obtiene

$$\det(A) = a_{11}a_{22}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{13}a_{23} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} - a_{31}a_{13}a_{22}$$

Por lo que podemos concluir que

$$p(\lambda) = -\lambda^3 + A\lambda^2 - B\lambda + C = 0$$

Multiplicando por negativo

$$\boldsymbol{p(\lambda) = \lambda^3 - A\lambda^2 + B\lambda - C = 0}$$

Donde

$$\boldsymbol{A = th(A)} \quad ; \quad \boldsymbol{B = \sum_{i=1}^{3} \det(M_{ii})} \quad ; \quad \boldsymbol{C = \det(A)}$$