# An overview of the algebraic logic approach for propositional dynamic logic

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# Contents

# Abstract

The general aim of this manuscript is to study the propositional dynamic logic (PDL) from the (contemporary) algebraic logic point of view. We establish which is (are) the class (classes) of algebras associated with PDL. In order to achieve this, we collect the existent algebraic theory about propositional dynamic logic, and develop some new results helping to understand the the algebraic theory of PDL.

# Introduction

First-order dynamic logic (FODL) is a formal system developed for reasoning about computational programs. FODL was presented by Pratt in [15] as related to program verification. The term “dynamic” emphasises the main feature of dynamic logic that distinguishes it from classical logic. In classical logic, the truth is invariant: the truth value of a formula is determined by a valuation of its free variables in some structure. In other words, the truth value given by a valuation on some structure is fixed. In FODL there are explicit syntactic constructions called “programs” whose role is to change the value of the variable, and thus change the truth value of the formulas.

The propositional fragment of FODL was isolated and axiomatized as a multi-modal logic. Unlike FODL, in propositional dynamic logic (PDL) the atomic programs are simply letters from some alphabet. Thus, PDL just studies the pure interaction between programs and propositions. As a multi-modal logic, in PDL the programs are interpreted as binary relations on a set of states $W$, Likewise, the propositions are interpreted as arbitrary subsets of $W$.

# A word from the author

There is already an algebraic logic theory developed for PDL in the literature which is published in a bunch of articles and corresponding to several authors.

This manuscript has the goal to collect in an ordered way (not in a chronological way) those results about the algebraic logic theory for PDL and fill some gaps which I found in the literature.

The most of the concepts and results exposed in the manuscript are in an explicit form in the existing literature. Others are in an implicit form, and others I haven not found in the literature either in an explicit neither implicit form in the literature. For those

La mayoria de los resultados expuestos en estas notas se encuentran, ya se de forma explicita o implicita, en la literatura existente. Incluimos en estas notas otros conceptos y resultados que no aparence o no hemos podidio encontrar en la literatura ni de forma explicita ni implicita. Para estos concpetos y resultados que creemos que son nuevos los marcamos con un asterisco.

En la literatura sobre álgebras dinamicas a hay varios resultados y/o afirmaciones que no entiendo para nada o de las cuales no estoy seguro que significan o tratan de afirmar.

# Chapter 1

# Propositional dynamic logic

In this section, we present the syntactic aspects of propositional dynamic logic (PDL). The main references for this chapter are [12, 10].

## 1.1 Syntax

Propositional dynamic logic (PDL) is a blend of three sort of syntactic objects: propositional formulas, programs, and operators relating the last two. From a logical point of view, PDL can be seeing as a blend of three classical ingredients: propositional logic, modal logic, and the algebra of programs. The language of PDL consist of two sort of expressions: *proposition* or *formulas* $\varphi$, $\psi$, $\chi$, ... and *programs* $\alpha$, $\beta$, $\gamma$, .... In order to build the sets of formulas and programs is needed a countably many atomic symbols for each set. Let $\Phi$ be countable set whose elements are called *atomic propositions* (or *propositional variables* as is used in propositional classical logic) and they are denoted by $p, q, r, \ldots$. Let $\Pi$ be a countable set whose elements are called *atomic programs* and are denoted by $\pi_1, \pi_2, \ldots$. The set of propositions $\mathsf{Fm}(\Phi, \Pi)$ and the set of programs $\mathsf{Prog}(\Phi, \Pi)$ are built inductively from the atomic ones using the following operators:

**Propositional operators:** $\rightarrow$ implication and $\perp$ falsity.

**Program operators:** ; composition, $\sqcup$ choice and $*$ iteration.

**Mixed operators:** $[\,]$ necessity and ? test.

Now, the definition of the sets $\mathsf{Fm}(\Phi, \Pi)$ and $\mathsf{Prog}(\Phi, \Pi)$ is by mutual induction as follows: $\mathsf{Fm}(\Phi, \Pi)$ and $\mathsf{Prog}(\Phi, \Pi)$ are the smallest sets such that:

- $\Phi \subseteq \mathsf{Fm}(\Phi, \Pi)$
- $\Pi \subseteq \mathsf{Prog}(\Phi, \Pi)$
- If $\varphi, \psi \in \mathsf{Fm}(\Phi, \Pi)$, then $\varphi \rightarrow \psi \in \mathsf{Fm}(\Phi, \Pi)$ and $\perp \in \mathsf{Fm}(\Phi, \Pi)$.
- If $\alpha, \beta \in \mathsf{Prog}(\Phi, \Pi)$, then $\alpha; \beta$, $\alpha \sqcup \beta$, and $\alpha^* \in \mathsf{Prog}(\Phi, \Pi)$.

- If $\alpha \in \mathsf{Prog}(\Phi, \Pi)$ and $\varphi \in \mathsf{Fm}(\Phi, \Pi)$, then $[\alpha]\varphi \in \mathsf{Fm}(\Phi, \Pi)$.

- If $\varphi \in \mathsf{Fm}(\Phi, \Pi)$, then $\varphi? \in \mathsf{Prog}(\Phi, \Pi)$.

When there is no danger of confusion, we write $\mathsf{Fm}$ and $\mathsf{Prog}$ instead of $\mathsf{Fm}(\Phi, \Pi)$ and $\mathsf{Prog}(\Phi, \Pi)$, respectively. Notice that the inductive definitions of the sets of propositions $\mathsf{Fm}$ and programs $\mathsf{Prog}$ are intertwined and cannot be separated. Compound propositions and programs have the following intuitive meanings:

| | |
|---|---|
| $[\alpha]\varphi$ | It is necessary that after executing $\alpha$, $\varphi$ is true |
| $\alpha;\beta$ | Execute $\alpha$, then execute $\beta$ |
| $\alpha \sqcup \beta$ | Choose either $\alpha$ or $\beta$ nondeterministically and execute it. |
| $\alpha^*$ | Execute $\alpha$a nondeterministically chosen finte number of times (zero or more) |
| $\varphi?$ | Test $\varphi$; proceed if true, fail if false. |

The possibility operator $\langle\rangle$ is the dual modal operator of the necessity operator $[\,]$. It is defined and has the intuitive meaning as follows:

$\langle\alpha\rangle\varphi := \neg[\alpha]\neg\varphi$ There is a computation of $\alpha$ that terminates in a state satisfying $\varphi$.

## 1.2 Relational Semantics

Let $\Phi$ and $\Pi$ (disjunct) countable sets, and $\mathsf{Fm}(\Phi, \Pi)$ and $\mathsf{Prog}(\Phi, \Pi)$ as in the above section.

Since PDL can be interpreted as multi-modal logic (see [4]), its semantics comes from the semantics for modal logic. Thus, the structures over which the propositions and programs of PDL are interpreted are called *Kripke models*. In general, a multi-modal *Kripke frame* is a structure $\langle S, \{R_i : i \in I\}\rangle$ where $S$ is a non-empty set and for every $i \in I$, $R_i$ is a binary relation on $S$. A multi-modal *Kripke model* is a structure $\langle S, \{R_i : i \in I\}, V\rangle$ such that $\langle S, \{R_i : i \in I\}\rangle$ is a Kripke frame and $V : \Phi \to \mathcal{P}(S)$, where $\mathcal{P}(S)$ denotes the power set of $S$.

Since in PDL there is a model operator $[\alpha]$ for each program $\alpha$, it is necessary a binary relation $R_\alpha$ for each program $\alpha$. Therefore, in the context of this article, we shall say simply that $\mathcal{M} = \langle S, \mathcal{R}, V\rangle$ is a *Kripke model* if $\langle S, \mathcal{R}, V\rangle$ is a Kripke model, where $\mathcal{R} = \{R_\alpha : \alpha \in \mathsf{Prog}\}$.

The definitions of the *satisfiability relation* $\Vdash$ and *validity* are defined as usual in modal logic. Let $\mathcal{M} = \langle S, \mathcal{R}, V\rangle$ be a Kripke model. The relation $\mathcal{M}, s \Vdash \varphi$, which can be read as: *$\varphi$ is satisfied (true) at state $s$ in the model $\mathcal{M}$*, is defined inductively on the formation of $\varphi$ as follows:

- $\mathcal{M}, s \Vdash p$ if and only if $s \in V(p)$;

- $\mathcal{M}, s \Vdash \varphi \to \psi$ if and only if $\mathcal{M}, s \Vdash \varphi$ implies $\mathcal{M}, s \Vdash \psi$;

- $\mathcal{M}, s \nVdash \bot$

- $\mathcal{M}, s \Vdash [\alpha]\varphi$ if and only if for all $t \in S$, $(s, t) \in R_\alpha$ implies $\mathcal{M}, t \Vdash \varphi$.

The definitions of $\Vdash$ for $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$, $\bot$ and $\langle\alpha\rangle\varphi$ follow from the above, and they are also as usual in modal logic. We say that a proposition $\varphi$ is *valid* in a model $\mathcal{M}$, denoted by $\mathcal{M} \Vdash \varphi$, if $\mathcal{M}, s \Vdash \varphi$, for all $s \in S$. Let $\Sigma$ be a set of propositions. We write $\mathcal{M}, s \Vdash \Sigma$ if $\mathcal{M}, s \Vdash \varphi$ for all $\varphi \in \Sigma$, and $\mathcal{M} \Vdash \Sigma$ if $\mathcal{M} \Vdash \varphi$ for all $\varphi \in \Sigma$. We say that an inference rule

$$\frac{\varphi_1, \ldots, \varphi_n}{\varphi}$$

is *sound in* $\mathcal{M}$ if $\mathcal{M} \Vdash \{\varphi_1, \ldots, \varphi_n\}$ implies $\mathcal{M} \Vdash \varphi$.

**Proposition 1.2.1.** *The following are valid formulas in each Kripke model.*

(M1) $[\alpha](\alpha \wedge \psi) \leftrightarrow [\alpha]\varphi \wedge [\alpha]\psi$.

(M2) $[\alpha]\top \leftrightarrow \top$.

(M3) $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$.

(M4) $[\alpha]\varphi \vee [\alpha]\psi \to [\alpha](\varphi \vee \psi)$.

(M5) $[\alpha]\varphi \leftrightarrow \neg\langle\alpha\rangle\neg\varphi$.

(M6) $\langle\alpha\rangle(\alpha \vee \psi) \leftrightarrow \langle\alpha\rangle\varphi \vee \langle\alpha\rangle\psi$.

(M7) $\langle\alpha\rangle\bot \leftrightarrow \bot$.

(M8) $\langle\alpha\rangle(\varphi \wedge \psi) \to (\langle\alpha\rangle\varphi \wedge \langle\alpha\rangle\psi)$.

*Proof.* They are classic results of modal logic. □

**Proposition 1.2.2.** *The following rules of inference are sound in every Kripke model.*

*1.* Modus Ponens (MP)*:*

$$\frac{\varphi, \varphi \to \psi}{\psi}$$

*2.* Necessitation (N) *or* Modal generalization*:*

$$\frac{\varphi}{[\alpha]\varphi}$$

*3.* Monotonicity of $\langle\alpha\rangle$*:*

$$\frac{\varphi \to \psi}{\langle\alpha\rangle\varphi \to \langle\alpha\rangle\psi}$$

*4.* Monotonicity of $[\alpha]$*:*

$$\frac{\varphi \to \psi}{[\alpha]\varphi \to [\alpha]\psi}$$

*Proof.* They are also classic results of modal logic. □

Since the binary relations $R_\alpha$ are used to interpret the modal operators $[\alpha]$ for every program $\alpha$, it is natural to ask that the relations $R_\alpha$ to reflect the intended meaning of programs $\alpha$. Thus, a model $\mathcal{M}$ is said to be *standard* if it satisfies the following conditions:

$$\begin{aligned}
R_{\alpha;\beta} &= R_\alpha \circ R_\beta = \{(s,t) : \exists u((s,u) \in R_\alpha \& (u,t) \in R_\beta\} \\
R_{\alpha \sqcup \beta} &= R_\alpha \cup R_\beta \\
R_{\alpha^*} &= R_\alpha^* = \bigcup_{n \geq 0} R_\alpha^n \\
R_{\varphi?} &= \{(s,s) : \mathcal{M}, s \Vdash \varphi\}.
\end{aligned}$$

Here $R_\alpha^0 = \Delta_S := \{(s,s) : s \in S\}$ and $R_\alpha^n := R_\alpha \circ \cdots \circ R_\alpha$ ($n$ times) for all $n \in \mathbb{N}$. Thus, notice that $R_\alpha^*$ is the reflexive transitive closure of the relation $R_\alpha$, that is, $R_\alpha^*$ is the least reflexive transitive relation containing $R_\alpha$.

Let $\mathsf{PDL}$ be the set of all valid formulas in every standard model. That is,

$$\mathsf{PDL} = \{\varphi \in \mathsf{Fm} : \mathcal{M} \Vdash \varphi \text{ for all standard model } \mathcal{M}\}. \tag{1.2.1}$$

By Propositions 1.2.1 and 1.2.2, it follows that $\mathsf{PDL}$ is a normal multi-modal logic (see for instance [10, pp. 38]). As is usual, we say that $\varphi$ is a *valid formula of* $\mathsf{PDL}$ if $\varphi \in \mathsf{PDL}$, and we write $\vdash_{\mathsf{PDL}} \varphi$.

**Proposition 1.2.3.** *The following are valid formulas of* $\mathsf{PDL}$*:*

(S1) $\langle \alpha \sqcup \beta \rangle \varphi \leftrightarrow \langle \alpha \rangle \varphi \vee \langle \beta \rangle \varphi$.

(S2) $[\alpha \sqcup \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$.

(S3) $\langle \alpha;\beta \rangle \varphi \leftrightarrow \langle \alpha \rangle \langle \beta \rangle \varphi$.

(S4) $[\alpha;\beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$.

(S5) $\langle \varphi? \rangle \psi \leftrightarrow (\varphi \wedge \psi)$.

(S6) $[\varphi?]\psi \leftrightarrow (\varphi \to \psi)$.

*Proof.* The proofs are straightforward. □

**Proposition 1.2.4.** *The following are valid formulas of* $\mathsf{PDL}$*:*

(S7) $[\alpha^*]\varphi \to \varphi$.

(S8) $\varphi \to \langle \alpha^* \rangle \varphi$.

(S9) $[\alpha^*]\varphi \to [\alpha]\varphi$.

(S10) $\langle \alpha \rangle \varphi \to \langle \alpha^* \rangle \varphi$.

(S11) $[\alpha^*]\varphi \leftrightarrow [\alpha^*\alpha^*]\varphi$.

(S12) $\langle \alpha^* \rangle \varphi \leftrightarrow \langle \alpha^*\alpha^* \rangle \varphi$.

(S13) $[\alpha^*]\varphi \leftrightarrow [\alpha^{**}]\varphi$.

(S14) $\langle \alpha^* \rangle \varphi \leftrightarrow \langle \alpha^{**} \rangle \varphi$.

(S15) $[\alpha^*]\varphi \leftrightarrow \varphi \wedge [\alpha][\alpha^*]\varphi$.

(S16) $\langle \alpha^* \rangle \varphi \leftrightarrow \varphi \vee \langle \alpha \rangle \langle \alpha^* \rangle \varphi$.

(S17) $[\alpha^*]\varphi \leftrightarrow \varphi \wedge [\alpha^*](\varphi \to [\alpha]\varphi)$.

(S18) $\langle \alpha^* \rangle \varphi \leftrightarrow \varphi \vee \langle \alpha^* \rangle (\neg\varphi \wedge \langle \alpha \rangle \varphi)$.

*Proof.* The proofs are straightforward. See [12, pp. 122]. □

**Proposition 1.2.5** ([10, pp. 111])**.** *Let* $\alpha \in \mathsf{Prog}$ *and* $\varphi \in \mathsf{Fm}$*. Then:*

(M9) *For all* $n \geq 0$, $[\alpha^n]\varphi \leftrightarrow [\alpha]^n\varphi$ *and* $\langle \alpha^n \rangle \leftrightarrow \langle \alpha \rangle^n$ *are valid formulas of* $\mathsf{PDL}$.

(M10) *For every standard model* $\mathcal{M}$ *and* $s \in S$,

- $\mathcal{M}, s \Vdash [\alpha^*]\varphi$ *if and only if for all* $n \geq 0$, $\mathcal{M}, s \vDash [\alpha^n]\varphi$.
- $\mathcal{M} \Vdash_s \langle \alpha^* \rangle \varphi$ *if and only if there is* $n \geq 0$ *such that* $\mathcal{M} \Vdash_s \langle \alpha^n \rangle \varphi$.

(M11) *For every standard model* $\mathcal{M}$, $\mathcal{M} \vDash [\alpha^*]\varphi$ *if and only if for all* $n \geq 0$, $\mathcal{M} \vDash [\alpha^n]\varphi$.

## 1.3 Hilbert-style Axiomatization of PDL

Let us consider the Hilbert-style system given by the following axioms and rules of inference:

**Axioms**

| | | | |
|---|---|---|---|
| Classical: | All classical tautologies | Mix: | $[\alpha^*]\varphi \to \varphi \wedge [\alpha][\alpha^*]\varphi$ |
| K: | $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$ | Ind: | $\varphi \wedge [\alpha^*](\varphi \to [\alpha]\varphi) \to [\alpha^*]\varphi$. |
| Comp: | $[\alpha;\beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$ | | |
| Alt: | $[\alpha \sqcup \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$ | Test: | $[\varphi?]\psi \leftrightarrow (\varphi \to \psi)$. |

**Rules of inference**

$$\text{(MP)}\ \frac{\varphi,\ \varphi \to \psi}{\varphi} \qquad\qquad \text{(N)}\ \frac{\varphi}{[\alpha]\varphi}$$

The consequence relation $\vdash_{PDL}$ on $\mathsf{Fm}$ is defined through the above Hilbert-style system as usual, and let $\mathcal{L}_{PDL} = \langle \mathsf{Fm}, \vdash_{PDL} \rangle$ be the propositional logic (see [8]) given by $\vdash_{PDL}$. A formula $\varphi$ is said to be a *theorem* of $\mathcal{L}_{PDL}$ if $\vdash_{PDL} \varphi$. Notice that from the modal logic point of view, the set of theorems of the propositional logic $\mathcal{L}_{PDL}$ coincides with the smallest normal logic in $\mathsf{Fm}$ that contains the axioms Comp, Alt, Mix, Ind, and Test (see for instance [10, pp. 111]).

**Theorem 1.3.1** (Completeness for PDL)**.** *A formula $\varphi$ is valid in* PDL *if and only if $\varphi$ is a theorem of $\mathcal{L}_{PDL}$.*

*Proof.* For a proof, see for instance [10]. □

In other words, PDL is the smallest normal logic[1] that contains the axioms Comp, Alt, Mix, Ind, and Test. In the proof of Theorem 1.3.1 cannot be used the usual technique of canonical model because the canonical model corresponding to the set of theorems of $\mathcal{L}_{PDL}$ is not necessarily standard. An alternative proof is given in [10].

**Author's comment 1.3.2.** En principio estaremos interesados en estudiar, del punto de vista algebraic, a un fragmento de PDL. Este será el fragemento libre de test, es decir, sin el operador ?. Así en el sistema Hilbert antes presentado omitimos el axioma Test. Los modelos relacionales correspondientes a este fragmento son los modelos estandadr anteriores sin la condición par $R_{\varphi?}$. Denotaremos a la lógica (conjunto de fórmulas) sin test por $\mathsf{PDL}^*$ (el conjunto de fórmulas válidas en todos los marcos estandadr sin pedriles $R_{\varphi?}$ )y a la relación consecuencia dado por el sistema de Hilbert sin el axioma Test por $\vdash_{PDL^*}$. La misma prueba para el teorema de completitud para PDL y los teoremas de $\mathcal{L}_{PDL}$ sirve para el fragmento $\mathsf{PDL}^*$ y los teoremas de $\mathcal{L}_{PDL^*}$.

## 1.4 Local and global consequence relation

Let us define the *local and global consequence relation* as usual in modal logic. Let $\varphi$ be a formula and $\Gamma$ a set of formulas.

*Global consequence relation* $\vDash^g$ is defined as follows:

$$\Gamma \vDash^g \varphi \quad \text{iff} \quad \text{for all standard model } \mathcal{M},\ \mathcal{M} \Vdash \Gamma \text{ implies } \mathcal{M} \Vdash \varphi.$$

[1]Here we follow the traditional use of the term *logic* as a set of formulas.

*Local consequence relation* $\models^{\ell}$ is defined as follows:

$$\Gamma \models^{\ell} \varphi \quad \text{iff} \quad \text{for all standard model } \mathcal{M}, \text{ and for all } s \in S, \ \mathcal{M}, s \Vdash \Gamma \text{ implies } \mathcal{M}, s \Vdash \varphi.$$

It is clear that every local consequence is a global consequence, that is, $\Gamma \models^{\ell} \varphi$ implies that $\Gamma \models^{g} \varphi$, but not necessarily vice versa. For instance, taking $\Gamma = \{p\}$ and $\varphi = [\alpha]p$. However, it holds that the global and local consequence relations have the same theorems, that is,

$$\models^{g} \varphi \quad \text{if and only if} \quad \models^{\ell} \varphi.$$

In [12, Theo. 23, pp. 131] is claimed that either $\models^{\ell}$ nor $\models^{g}$ are finitary (consequence relation).

**Question:** Los compañeros finitarios de las relaciones consequencias local y global coinciden??? Alguno de ellos es axiomatozado por los axiomas anteriores???

**Question:** Alguno de los compañeros finitarios de las consequencias global o local es completo con respecto a la aximatization antes dada???

# Chapter 2

# Two-sorted (heterogeneous) algebras

In this chapter we recall the theory of heterogeneous algebras given in [3]. The theory of heterogeneous algebras is an extension of the standard theory of universal algebra for homogeneous algebras [5, 11, 2]. We also extended some other notions and concepts usual in univeral algebra to the context of heterogeneous algebras which is important for our goals. Here we restrict ourselves to a very particular type of heterogeneous algebras instead of the completely general framework given in [3].

For the concepts homogeneous algebras we follow [5]. Roughly speaking, an homogeneous algebra is formed by a non-empty set $A$ (called universe) with a family of operations on $A$ of different arity. A two-sorted algebra consists of two sets say $A$ and $B$, and a set of operations from $X_1 \times \cdots \times X_n \to X_n$ where every $X_i \in \{A, B\}$. As we mentioned, we are interesting only in a particular type of two-sorted algebras. Thus, we restrict ourselves to an specific set of operations.

The definitions and results about heterogeneous algebras are direct extensions from those for homogeneous algebras. Moreover, the proofs about heterogeneous algebras (and in particular for two-sorted algebras) follow relatively straightforwardly from their analogues about homogeneous algebras. Thus, we omit almost all the proofs and leave them to the reader.

## 2.1 Two-sorted algebras and subalgebras

**Definition 2.1.1.** A *two-sorted algebraic language* is a set $\mathrm{L} = \mathrm{L}_1 \cup \mathrm{L}_2 \cup \{\langle\rangle\}$ of functional symbols where $\mathrm{L}_1$ and $\mathrm{L}_2$ are (homogeneous) algebraic language, and $\langle\rangle$ is a *mixed* functional symbol of arity (2,1,1).

**Remark 2.1.2.** The adjective *two-sorted* is used to point out that the functional symbols will be interpreted in two universes. For instance, the mixed functional symbol $\langle\rangle$ of arity (2,1,1) will be interpreted as a two-valued function, where the first coordinated is taking from the second universe, the second coordinate is taking from the first universe and the value is into the first universe.

From now on, let $\mathrm{L} = \mathrm{L}_1 \cup \mathrm{L}_2 \cup \{\langle\rangle\}$ be the two-sorted language as before.

**Definition 2.1.3.** A *two-sorted algebra* of type $\mathrm{L} = \mathrm{L}_1 \cup \mathrm{L}_2 \cup \{\langle\rangle\}$ is a structure $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ where $\mathcal{B}$ is an (homogeneous) algebra of type $\mathrm{L}_1$, $\mathcal{R}$ is an (homogeneous) algebra of type $\mathrm{L}_2$, and $\langle\, , \,\rangle \colon \mathcal{R} \times \mathcal{B} \to \mathcal{B}$.

If $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ is a two-sorted algebra, we denote by $\mathcal{B}$ ($\mathcal{R}$) the algebra of type $L_1$ ($L_2$) and also its universe. We denote also the next notation: for every $a \in \mathcal{B}$ and $p \in \mathcal{R}$,

$$\langle p\rangle a := \langle p, a\rangle. \tag{2.1.1}$$

**Definition 2.1.4.** Let $\mathcal{D}_1 = \langle \mathcal{B}_1, \mathcal{R}_1, \langle\rangle_1\rangle$ and $\mathcal{D}_2 = \langle \mathcal{B}_2, \mathcal{R}_2, \langle\rangle_2\rangle$ be two-sorted algebras of type L. We say that $\mathcal{D}_1$ is a *(two-sorted) subalgebra* of $\mathcal{D}_2$ if:

1. $\mathcal{B}_1$ is a (homogeneous) subalgebra of $\mathcal{B}_2$.

2. $\mathcal{R}_1$ is a (homogeneous) subalgebra of $\mathcal{R}_2$.

3. For all $a \in \mathcal{B}_1$ and $p \in \mathcal{R}_1$, it follows that $\langle p\rangle_1 a = \langle p\rangle_2 a$.

Given a class $\mathbf{K}$ of two-sorted algebras of type L, it defines the two-sorted subalgebra operator as follows:

$$\mathcal{D} \in \mathrm{S}(\mathbf{K}) \text{ iff } \mathcal{D} \text{ is a two-sorted subalgebra of some member of } \mathbf{K}.$$

**Proposition 2.1.5.** *Let $\mathcal{D} = \langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ be a two-sorted algebra of type* L. *If $\{\mathcal{D}_i = \langle \mathcal{B}_i, \mathcal{R}_i, \langle\rangle\rangle : i \in I\}$ is a family of subalgebras of $\mathcal{D}$, then $\bigwedge_{i\in I} \mathcal{D}_i := \left\langle \bigcap_{i\in I} \mathcal{B}_i, \bigcap_{i\in I} \mathcal{R}_i, \langle\rangle \right\rangle$ is a subalgebra of $\mathcal{D}$, and we call it the* intersection *of the family $\{\mathcal{D}_i = \langle \mathcal{B}_i, \mathcal{R}_i, \langle\rangle\rangle : i \in I\}$.*

Notice that in the previous proposition $\bigcap_{i\in I} B_i$ is an (homogeneous) subalgebra of $B$, since every $B_i$ is a (homogeneous) subalgebra of $B$. Similar for $R_i$ and $R$.

Let $\mathcal{D} = \langle B, R, \langle\rangle\rangle$ be a two-sorted algebra of type L. For every pair of subsets $X \subseteq B$ and $Y \subseteq R$, let $\mathcal{D}(X, Y)$ be the intersection of family of all subalgebras $\langle B_i, R_i, \langle\rangle\rangle$ of $\mathcal{D}$ such that $X \subseteq B_i$ and $Y \subseteq R_i$. The two-sorted algebra $\mathcal{D}(X, Y)$ is called *the two-sorted subalgebra of $\mathcal{D}$ generated by $(X, Y)$.*

## 2.2 Morphisms and congruences

**Definition 2.2.1.** Let $\mathcal{D}_1 = \langle B_1, R_1, \langle\rangle_1\rangle$ and $\mathcal{D}_2 = \langle B_2, R_2, \langle\rangle_2\rangle$ be two-sorted algebras of type L. A *two-sorted homomorphism* (or simply a *morphism*) from $\mathcal{D}_1$ into $\mathcal{D}_2$ is a pair $(f, g) \colon \mathcal{D}_1 \to \mathcal{D}_2$ such that:

1. $f \colon B_1 \to B_2$ is a (homogeneous) homomorphism of type $\mathrm{L}_B$.

2. $g \colon R_1 \to R_2$ is a (homogeneous) homomorphism of type $\mathrm{L}_R$.

3. For all $a \in B_1$ and $p \in R_1$, $f(\langle p\rangle_1 a) = \langle g(p)\rangle_2 f(a)$.

We say that $(f,g)\colon \mathcal{D}_1 \to \mathcal{D}_2$ is a *two-sorted embedding* if $f$ and $g$ are (homogeneous) embeddings. We say that the morphism $(f,g)$ is *onto* if $f$ and $g$ are onto homomorphisms. We say that $(f,g)$ is a *two-sorted isomorphism* if both $f$ and $g$ are (homogeneous) isomorphisms.

Given a class $\mathbf{K}$ of two-sorted algebras of type L, it defines the two-sorted isomorphism and homomorphic image operators as follows, respectively:

$$\begin{aligned} \mathcal{D} \in \mathrm{I}(\mathbf{K}) &\quad \text{iff} \quad \mathcal{D} \text{ is two-sorted isomorphic to some member of } \mathbf{K}; \\ \mathcal{D} \in \mathrm{H}(\mathbf{K}) &\quad \text{iff} \quad \mathcal{D} \text{ is a two-sorted homomorphic image of some member of } \mathbf{K}. \end{aligned}$$

**Proposition 2.2.2.** *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two-sorted algebras and $(f,g)\colon \mathcal{D}_1 \to \mathcal{D}_2$ a two-sorted homomorphism. Then $\langle f(\mathcal{B}_1), g(\mathcal{R}_1), \langle\rangle_2\rangle$ is a two-sorted subalgebra of $\mathcal{D}_2$. Moreover, if $(f,g)$ is a two-sorted embedding, then $\mathcal{D}_1$ and $\langle f(\mathcal{B}_1), g(\mathcal{R}_1), \langle\rangle_2\rangle$ are isomorphic.*

**Proposition 2.2.3.** *Let $\mathcal{D}_i = \langle B_i, R_i, \langle\rangle_i\rangle$ be two-sorted algebras of type* L *with $i = 1,2,3$. Let $(f,g)\colon \mathcal{D}_1 \to \mathcal{D}_2$ and $(h,k)\colon \mathcal{D}_2 \to \mathcal{D}_3$ be morphisms. Then $(h,k)\circ(f,g) := (h\circ f, k\circ g)\colon \mathcal{D}_1 \to \mathcal{D}_3$ is a morphism, where $h\circ f$ and $k\circ g$ are the usual set theoretical composition of functions.*

Thus, is straightforward check that any class of two-sorted algebras of type L and its morphisms form a category.

**Definition 2.2.4.** Let $\mathcal{D} = \langle B, R, \langle\rangle\rangle$ be a two-sorted algebra of type L. A *two-sorted congruence* on $\mathcal{D}$ is a pair $(\theta,\eta)$ such that $\theta$ is a (homogeneous) congruence on the (homogeneous) algebra $B$, $\eta$ is a (homogeneous) congruence on the (homogeneous) algebra $R$, and it satisfies: for all $a,b \in B$ and $p,q \in R$,

$$(a,b) \in \theta \;\&\; (p,q) \in \eta \implies (\langle p\rangle a, \langle q\rangle b) \in \theta. \tag{2.2.1}$$

**Proposition 2.2.5.** *Let $(f,g)\colon \mathcal{D}_1 \to \mathcal{D}_2$ be a two-sorted homomorphism. The* kernel *of $(f,g)$ is defined by* $\mathrm{Ker}(f,g) := (\mathrm{Ker}(f), \mathrm{Ker}(g))$*, where* $\mathrm{Ker}(f)$ *and* $\mathrm{Ker}(g)$ *are the (homogeneous) kernels of the (homogeneous) homomorphisms $f\colon \mathcal{B}_1 \to \mathcal{B}_2$ and $g\colon \mathcal{R}_1 \to \mathcal{R}_2$, respectively. Then* $\mathrm{Ker}(f,g)$ *is a two-sorted congruence on the $\mathcal{D}_1$.*

Let $\mathcal{D} = \langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ be a two-sorted algebra of type $L$ and let $\Theta = (\theta,\eta)$ be a two-sorted congruence on $\mathcal{D}$. Then, let $\mathcal{D}/\Theta := \langle \mathcal{B}/\theta, \mathcal{R}/\eta, \langle\rangle_\Theta\rangle$ be the two-sorted algebra where $\mathcal{B}/\theta$ and $\mathcal{R}/\eta$ are the quotient algebras defined by the congruences $\theta$ and $\eta$ on $\mathcal{B}$ and $\mathcal{R}$, respectively, and the operation $\langle\rangle_\Theta\colon \mathcal{R}/\eta \times \mathcal{B}/\theta \to \mathcal{B}/\theta$ defined as follows: for all $a \in \mathcal{B}$ and $p \in \mathcal{R}$,

$$\langle p/\eta\rangle_\Theta a/\theta := \langle p\rangle a/\theta. \tag{2.2.2}$$

The two-sorted algebra $\mathcal{D}/\Theta := \langle \mathcal{B}/\theta, \mathcal{R}/\eta, \langle\rangle_\Theta\rangle$ is called the *two-sorted quotient algebra* of $\mathcal{D}$ by $\Theta$. Moreover, $\pi_\Theta := (\pi_\mathcal{B}, \pi_\mathcal{R})\colon \mathcal{D} \to \mathcal{D}/\Theta$ is an onto two-sorted homomorphism, where $\pi_\mathcal{B}\colon \mathcal{B} \to \mathcal{B}/\theta$ and $\pi_\mathcal{R}\colon \mathcal{R} \to \mathcal{R}/\eta$ are the natural homomorphisms. We call $\pi_\Theta$ the *two-sorted natural homomorphism*.

**Theorem 2.2.6** (Homomorphism Theorem)**.** *Let $(f,g)\colon \mathcal{D}_1 \to \mathcal{D}_2$ be a two-sorted homomorphism. Then, there exists a two-sorted embedding $(\phi_1,\phi_2)\colon \mathcal{D}_1/\mathrm{Ker}(f,g) \to \mathcal{D}_2$ such that $(f,g) = (\phi_1,\phi_2)\circ \pi_{\mathrm{Ker}(f,g)}$, where $\pi_{\mathrm{Ker}(f,g)}\colon \mathcal{D}_1 \to \mathcal{D}_1/\mathrm{Ker}(f,g)$ is the two-sorted natural homomorphism. Moreover, if $(f,g)$ is a two-sorted embedding, then $(\phi_1,\phi_2)$ is a two-sorted isomorphism. Thus $\mathcal{D}_1/\mathrm{Ker}(f,g) \cong \mathcal{D}_2$.*

Notice that if $(f,g)\colon \mathcal{D}_1 \to \mathcal{D}_2$ is a two-sorted homomorphism, then by Proposition 2.2.2, it follows that $\mathcal{D}_1/\mathrm{Ker}(f,g)$ is isomorphic to a two-sorted subalgebra of $\mathcal{D}_2$. That is, $\mathcal{D}_1/\mathrm{Ker}(f,g) \in \mathrm{IS}(\mathcal{D}_2)$.

## 2.3 Direct and subdirect products

**Definition 2.3.1.** Let $\{\mathcal{D}_i = \langle \mathcal{B}_i, \mathcal{R}_i, \langle\rangle_i\rangle : i \in I\}$ be a family of two-sorted algebras of type $\mathrm{L} = \mathrm{L}_1 \cup \mathrm{L}_2 \cup \{\langle\rangle\}$ indexed by the set $I$. It defines the *two-sorted product algebra* $\prod\limits_{i\in I} \mathcal{D}_i := \left\langle \prod\limits_{i\in I} \mathcal{B}_i, \prod\limits_{i\in I} \mathcal{R}_i, \langle\rangle_P \right\rangle$ as follows: $\prod\limits_{i\in I} \mathcal{B}_i$ and $\prod\limits_{i\in I} \mathcal{R}_i$ are the (homogeneous) product algebras of type $L_1$ and $L_2$, respectively, and $\langle\rangle_P\colon \prod\limits_{i\in I} \mathcal{B}_i \times \prod\limits_{i\in I} \mathcal{R}_i \to \prod\limits_{i\in I} \mathcal{B}_i$ is defined by: for every $\overline{a} \in \prod\limits_{i\in I} \mathcal{B}_i$ and $\overline{p} \in \prod\limits_{i\in I} \mathcal{R}_i$,

$$(\langle \overline{p}\rangle_P \overline{a})(i) = \langle p_i\rangle_i a_i \tag{2.3.1}$$

for every $i \in I$.

For every $i \in I$, let $\pi_i := (\pi_{i1}, \pi_{i2})\colon \prod\limits_{i\in I} \mathcal{D}_i \to \mathcal{D}_i$ be the *two-sorted projection map*, where $\pi_{i1}\colon \prod\limits_{i\in I} \mathcal{B}_i \to \mathcal{B}_i$ and $\pi_{i2}\colon \prod\limits_{i\in I} \mathcal{R}_i \to \mathcal{R}_i$ are the (homogeneous) projection maps.

Given a class $\mathbf{K}$ of two-sorted algebras of type L, it defines the two-sorted direct product operator as follows:

$\mathcal{D} \in \mathrm{P}(\mathbf{K})$ iff $\mathcal{D}$ is a two-sorted product of a non-empty family of some members of $\mathbf{K}$.

**Author's comment 2.3.2.** The notion of subdirect product is not considered in [3]. However, it is used in [14] without an explicit definition. Actually, in [14], none of the heterogeneous algebraic concepts are introduced, and no reference is given.

**Definition* 2.3.3.** A two-sorted algebra $\mathcal{D}$ is said to be a *two-sorted subdirect product* of an indexed family $\{\mathcal{D}_i : i \in I\}$ of two-sorted algebras (all of the same type L) if:

1. if there is a two-sorted embedding $(f,g)\colon \mathcal{D} \to \prod\limits_{i\in I} \mathcal{D}_i$
2. $\pi_i \circ (f,g)(\mathcal{D}) = \mathcal{D}_i$, for each $i \in I$.

Given a class $\mathbf{K}$ of two-sorted algebras of type L, it defines the two-sorted subdirect product operator as follows:

$\mathcal{D} \in \mathrm{Ps}(\mathbf{K})$ iff $\mathcal{D}$ is a two-sorted subdirect product of a non-empty family of members of $\mathbf{K}$.

## 2.4 Varieties and class operators

## 2.5 Two-sorted term algebras and two-sorted free algebras

Let $\mathrm{L} = \mathrm{L}_1 \cup \mathrm{L}_2 \cup \{\langle\rangle\}$ be the two-sorted algebraic language as before. Let $X$ and $Y$ be arbitrary sets (disjunct). The pair $T(X,Y) = (T_1(X,Y), T_2(Y))$ is defined by mutual induction as follows: Let $T_1(X,Y)$ and $T_2(Y)$ be the smallest sets such that

1. $X \subseteq T_1(X,Y)$ and $Y \subseteq T_2(Y)$,
2. if $t_1,\dots,t_n \in T_1(X,Y)$ and $f \in \mathrm{L}_1$ is of arity $n$, then $f(t_1,\dots,t_n) \in T_1(X,Y)$,
3. if $s_1,\dots,s_n \in T_2(Y)$ and $g \in \mathrm{L}_2$ is of arity $n$, then $g(s_1,\dots,s_n) \in T_2(Y)$,
4. if $t \in T_1(X,Y)$ and $s \in T_2(Y)$, then $\langle s\rangle t \in T_1(X,Y)$.

**Definition 2.5.1.** Let $\mathrm{L} = \mathrm{L}_1 \cup \mathrm{L}_2 \cup \{\langle\rangle\}$ be the two-sorted algebraic language as before. Let $X$ and $Y$ be arbitrary sets (disjunct). The *two-sorted term algebra* of type L over $(X,Y)$, written $\mathcal{T}(X) = \langle \mathcal{T}_1(X,Y), \mathcal{T}_2(Y), \langle\rangle_{free}\rangle$, has its two-sorted universe $(T_1(X,Y), T_2(Y))$, and the operations:

1. for every $f \in \mathrm{L}_1$ of arity $n$, and all $t_1,\dots,t_n \in T_2(X,Y)$, $f^{\mathcal{T}_1(X,Y)}(t_1,\dots,t_n) := f(t_1,\dots,t_n) \in T_1(X,Y)$;
2. for every $g \in \mathrm{L}_2$ of arity $n$, and all $s_1,\dots,s_n \in T_2(Y)$, $g^{\mathcal{T}_1(Y)}(s_1,\dots,s_n) := g(s_1,\dots,s_n) \in \mathcal{T}_2(Y)$
3. for every $t \in T_1(X,Y)$ and $s \in T_2(Y)$, $\langle s\rangle_{free} t := \langle s\rangle t \in T_1(X,Y)$.

**Remark 2.5.2.** Notice that the construction of the set $T_2(Y)$ depends only from the set $Y$ and the operations in the algebraic language $\mathrm{L}_2$. Thus, the algebra $\mathcal{T}_2(Y)$ is the (homogeneous) term algebra of type $\mathrm{L}_2$ over $Y$. This is a consequence of the particular two-sorted algebraic language we are considering. Unlike the algebra $\mathcal{T}_2(Y)$, the (homogeneous) algebra $\mathcal{T}_1(X,Y)$ depends not only from the set $X$ and the algebraic language $\mathrm{L}_1$ but also of the elements in $T_2(Y)$ and the mixed operation $\langle\rangle_{free}$. Hence the algebra $\mathcal{T}_1(X,Y)$ is not the (homogeneous) term algebra of type $\mathrm{L}_1$ over $X$.

**Proposition 2.5.3.** *Let $\mathcal{D} = \langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ be a two-sorted algebra of type* L, *and $\mathcal{T}(X,Y) = \langle \mathcal{T}_1(X,Y), \mathcal{T}_2(Y)\rangle$ the two-sorted term algebra of type* L *generated by $(X,Y)$. Then, for every pair of functions $(f,g)$ such that $f\colon X \to \mathcal{B}$ and $g\colon Y \to \mathcal{R}$, there exists an unique two-sorted homomorphism $(\widehat{f},\widehat{g})\colon \mathcal{T}(X,Y) \to \mathcal{D}$ extending the maps $(f,g)$. That is, $\widehat{f}(x) = f(x)$ and $\widehat{g}(y) = g(y)$, for all $x \in X$ and $y \in Y$.*

**Definition* 2.5.4.** Let $\mathbf{K}$ be a class of two-sorted algebras of type L. Let $X$ and $Y$ be arbitrary sets. It defines the two-sorted congruence $\Theta_K(X,Y)$ on the two-sorted term algebra $\mathcal{T}(X,Y)$ as follows:

$$\Theta_{\mathbf{K}}(X,Y) := \bigwedge \Phi_{\mathbf{K}}(X,Y) \tag{2.5.1}$$

where

$$\Phi_{\mathbf{K}}(X,Y) := \{\phi = (\phi_1,\phi_2) \in \mathrm{Con}(\mathcal{T}(X,Y)) : \mathcal{T}(X,Y)/\phi \in \mathrm{IS}(\mathbf{K})\}; \tag{2.5.2}$$

and then define $\mathcal{F}_{\mathbf{K}}(\overline{X},\overline{Y})$, the *two-sorted* $\mathbf{K}$*-free algebra over* $(\overline{X},\overline{Y})$, by

$$\mathcal{F}_{\mathbf{K}}(\overline{X},\overline{Y}) := \mathcal{T}(X,Y)/\Theta_{\mathbf{K}}(X,Y). \tag{2.5.3}$$

The definition of $\Theta_{\mathbf{K}}(X,Y)$ in (2.5.1) is as follows. $\Theta_{\mathbf{K}}(X,Y)$ is the two-sorted congruence $\Theta_{\mathbf{K}}(X,Y) = (\Theta_1,\Theta_2)$ given by: (i) $\Theta_1$ is the intersection of all congruences $\phi_1$ on $\mathcal{T}_1(X,Y)$ (of type $\mathrm{L}_1$) such that there is a congruence $\phi_2$ on $\mathcal{T}_2(Y)$ (of type $\mathrm{L}_2$) and $(\phi_1,\phi_2) \in \Phi_{\mathbf{K}}(X,Y)$; and (ii) $\Theta_2$ is the intersection of all congruences $\phi_2$ on $\mathcal{T}_2(Y)$ such that there is a congruence $\phi_1$ on $\mathcal{T}_1(X,Y)$ and $(\phi_1,\phi_2) \in \Phi_{\mathbf{K}}(X,Y)$.

$$\Theta_1 = \bigcap\{\phi_1 \in \mathrm{Con}(\mathcal{T}_1(X,Y)) : \text{there is } \phi_2 \in \mathrm{Con}(\mathcal{T}_2(Y)) \text{ such that } (\phi_1,\phi_2) \in \Phi_{\mathbf{K}}(X,Y)\}. \tag{2.5.4}$$

$$\Theta_2 = \bigcap\{\phi_2 \in \mathrm{Con}(\mathcal{T}_2(Y)) : \text{there is } \phi_1 \in \mathrm{Con}(\mathcal{T}_1(X,Y)) \text{ such that } (\phi_1,\phi_2) \in \Phi_{\mathbf{K}}(X,Y)\}. \tag{2.5.5}$$

Moreover, since $\Theta_{\mathbf{K}}(X,Y) = (\Theta_1,\Theta_2)$, it follows that

$$\mathcal{F}_{\mathbf{K}}(\overline{X},\overline{Y}) := \mathcal{T}(X,Y)/\Theta_{\mathbf{K}}(X,Y) = (\mathcal{T}_1(X,Y)/\Theta_1, \mathcal{T}_2(Y)/\Theta_2, \langle\rangle) \tag{2.5.6}$$

where the mixed operation is defined as in (2.2.2).

For the next proposition we need the following notations. Let $\mathbf{K}$ be a class of two-sorted algebras of type $\mathrm{L} = \mathrm{L}_1 \cup \mathrm{L}_2 \cup \{\langle\rangle\}$. Let us to consider the class of algebras of type $\mathrm{L}_2$ which correspond to the second coordinate of some two-sorted algebra in $\mathbf{K}$. That is, let

$$\mathrm{R}(\mathbf{K}) := \{\mathcal{R} : \mathcal{R} \text{ is an algebra of type } \mathrm{L}_2 \text{ and } \langle\mathcal{B},\mathcal{R},\langle\rangle\rangle \in \mathbf{K} \text{ for some algebra } \mathcal{B} \text{ of type } \mathrm{L}_1\}. \tag{2.5.7}$$

**Proposition* 2.5.5.** *Let* $\mathbf{K}$ *be a class of two-sorted algebras of type* $\mathrm{L} = \mathrm{L}_1 \cup \mathrm{L}_2 \cup \{\langle\rangle\}$*. Let* $X$ *and* $Y$ *sets. Let* $\mathcal{F}_{\mathbf{K}}(\overline{X},\overline{Y}) = (\mathcal{T}_1(X,Y)/\Theta_1, \mathcal{T}_2(Y)/\Theta_2, \langle\rangle)$ *be the two-sorted* $\mathbf{K}$*-free algebra. Then,* $\mathcal{T}_2(Y)/\Theta_2$ *is the (homogeneous)* $\mathrm{R}(\mathbf{K})$*-free algebra (of type* $\mathrm{L}_2$*) over* $\overline{Y}$*.*

*Proof.* First we recall that the term algebra of type $\mathrm{L}_2$ over $Y$ is $\mathcal{T}_2(Y)$. We know also (see [5]) that the $\mathrm{R}(\mathbf{K})$-free algebra of type $\mathrm{L}_2$ over $\overline{Y}$ is

$$\mathcal{F}_{\mathrm{R}(\mathbf{K})}(\overline{Y}) = \mathcal{T}_2(Y)/\Theta_{\mathrm{R}(\mathbf{K})}(Y)$$

where

$$\Theta_{\mathrm{R}(\mathbf{K})}(Y) = \bigcap \Phi_{\mathrm{R}(\mathbf{K})}(Y) \quad \text{with} \quad \Phi_{\mathrm{R}(\mathbf{K})}(Y) = \{\phi \in \mathrm{Con}(\mathcal{T}_2(Y)) : \mathcal{T}_2(Y)/\phi \in \mathrm{IS}(\mathrm{R}(\mathbf{K}))\}.$$

Let us show that $\Theta_{\mathrm{R}(\mathbf{K})}(Y) = \Theta_2$. By the definition of $\Theta_2$ (see (2.5.5)), it is enough to show that

$$\Phi_{\mathrm{R}(\mathbf{K})}(Y) = \{\phi_2 \in \mathrm{Con}(\mathcal{T}_2(Y)) : \text{there is } \phi_1 \in \mathrm{Con}(\mathcal{T}_1(X,Y)) \text{ such that } (\phi_1,\phi_2) \in \Phi_{\mathbf{K}}(X,Y)\}.$$

Let $\Phi_2 := \{\phi_2 \in \mathrm{Con}(\mathcal{T}_2(Y)) : \text{there is } \phi_1 \in \mathrm{Con}(\mathcal{T}_1(X,Y)) \text{ such that } (\phi_1,\phi_2) \in \Phi_{\mathbf{K}}(X,Y)\}$.

($\supseteq$) Let $\phi_2 \in \Phi_2$. Then, there is $\phi_1 \in \mathrm{Con}(\mathcal{T}_1(X,Y))$ such that $\phi = (\phi_1,\phi_2) \in \Phi_{\mathbf{K}}(X,Y)$. Thus $\mathcal{T}(X,Y)/\phi = (\mathcal{T}_1(X,Y)/\phi_1, \mathcal{T}_2(Y)/\phi_2, \langle\rangle) \in \mathrm{IS}(\mathbf{K})$. Then, $\mathcal{T}_2(Y)/\phi_2 \in \mathrm{IS}(\mathrm{R}(\mathbf{K}))$. Hence $\phi_2 \in \Phi_{\mathrm{R}(\mathbf{K})}(Y)$.

($\subseteq$) Let $\phi_2 \in \Phi_{\mathrm{R}(\mathbf{K})}$. So $\mathcal{T}_2(Y)/\phi_2 \in \mathrm{IS}(\mathrm{R}(\mathbf{K}))$. Let $\mathcal{R} \in \mathrm{R}(\mathbf{K})$ and $h\colon \mathcal{T}_2(Y)/\phi \to \mathcal{R}$ an (homogeneous) embedding. Since $\mathcal{R} \in \mathrm{R}(\mathbf{K})$, it follows that $\mathcal{D} := \langle\mathcal{B},\mathcal{R},\langle\rangle\rangle \in \mathbf{K}$ for some algebra $\mathcal{B}$ of type $\mathrm{L}_1$. Let $f\colon X \to \mathcal{B}$ be any map and $g\colon Y \to \mathcal{R}$ given by $g(y) = h\,(y/\phi_2)$. Then, the pair $(f, g)$ can be uniquely extended to a two-sorted homomorphism $(\widehat{f}, \widehat{g})\colon \mathcal{T}(X, Y) \to \mathcal{D}$. Thus, by Theorem 2.2.6, it follows that $\mathcal{T}(X, Y)/\mathrm{Ker}(\widehat{f}, \widehat{g}) \in \mathrm{IS}(\mathbf{K})$ (because $\mathcal{D} \in \mathbf{K}$). Hence $\mathrm{Ker}(\widehat{f}, \widehat{g}) \in \Phi_{\mathbf{K}}(X, Y)$. Now notice that $\widehat{g}(s) = h\,(s/\phi_2)$ for all $s \in \mathcal{T}_2(Y)$. Let $s_1, s_2 \in \mathcal{T}_2(Y)$. Then

$$\begin{aligned}
(s_1, s_2) \in \mathrm{Ker}(\widehat{g}) &\iff \widehat{g}(s_1) = \widehat{g}(s_2) \\
&\iff h\,(s_1/\phi_2) = h\,(s_2/\phi_2) \\
&\iff s_1/\phi_2 = s_2/\phi_2 \\
&\iff (s_1, s_2) \in \phi_2.
\end{aligned}$$

Hence $\mathrm{Ker}(\widehat{g}) = \phi_2$. Thus $(\mathrm{Ker}(\widehat{f}), \phi_2) = (\mathrm{Ker}(\widehat{f}), \mathrm{Ker}(\widehat{g})) = \mathrm{Ker}(\widehat{f}, \widehat{g}) \in \Phi_{\mathbf{K}}(X, Y)$. Therefore, $\phi_2 \in \Phi_2$. This completes the proof. $\square$

definicion de variedad. Identidades. Variedad = clase ecuacional

**Remark 2.5.6.** All the above can be performed for arbitrary two-sorted algebraic languages $\mathrm{L} = \mathrm{L}_1 \cup \mathrm{L}_2 \cup \mathrm{L}_M$ where $\mathrm{L}_1$ and $\mathrm{L}_2$ are (homogeneous) algebraic languages, and $\mathrm{L}_M$ is a set of functional symbols such that a tupla ().

Moreover, other mixed operations can be added than the mix operation $\langle\rangle$, with the caution of adding the corresponding conditions in the definitions. For instance, for a two-sorted language $\mathrm{L} = \{\mathrm{L}_B, \mathrm{L}_R, \langle\rangle, ?\}$ where $\mathrm{L}_B$ and $\mathrm{L}_R$ are (homogeneous) algebraic languages of some types, respectively, $\langle\rangle$ has arity $(2, 1, 1)$ and $(2, 1)$

# Chapter 3

# Dynamic algebras

There is several results connecting PDL with some algebraic structures. In this chapter we try to overview the most relevant algebraic structures associated with PDL and study the algebraization of PDL. We try to give all the details and constructions as it is possible.

In [12, p. 196] it is mentioned that the concept of dynamic algebra relates to PDL as Boolean algebra relates to propositional classical logic. There, it only gives the definition of dynamic logic but neither shows how the Lindembaun-Tarski algebra can be obtained nor proves that it is a dynamic algebra.

In this chapter, we are concerned with the algebraic semantics corresponding to PDL without test. The algebraic structures (two-sorted algebras) studied in the literature associated with PDL are called *dynamic algebras.*

We assume the reader is familiar with order theoretical notions, lattices and Boolean algebras. Our main references for these are [6, 1, 9].

## 3.1 First definitions and properties

Almost of this section corresponds to [14].

Let $\mathrm{L}_D = \mathrm{L}_B \cup \mathrm{L}_R \cup \{\langle\rangle\}$ be the following two-sorted algebraic language:

- $\mathrm{L}_B = \{\wedge, \vee, \neg, 0, 1\}$ is an (homogeneous) algebraic language of type (2,2,1,0,0) (an algebraic language for Boolean algebras).
- $\mathrm{L}_R = \{;, \sqcup, *\}$ is an (homogeneous) algebraic language of type (2,2,1).
- $\langle\rangle$ is a mixed functional symbol of type (2,1,1).

**Definition 3.1.1.** A *dynamic algebra* is a two-sorted algebra $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ of type $\mathrm{L}_D$ such such:

(D1) $\mathcal{B} = \langle B, \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra;

(D2) $\mathcal{R} = \langle R, ;, \sqcup, ^*\rangle$ is an (homogeneous) algebra of type $\mathrm{L}_R$.

and for all $a, b \in B$ and $p, q \in R$

(D3) $\langle p\rangle 0 = 0$;

(D4) $\langle p\rangle(a \vee b) = \langle p\rangle a \vee \langle p\rangle b$;

(D5) $\langle p \sqcup q\rangle a = \langle p\rangle a \vee \langle q\rangle a$;

(D6) $\langle p; q\rangle a = \langle p\rangle\langle q\rangle a$;

(D7) $a \vee \langle p\rangle\langle p^*\rangle a \leq \langle p^*\rangle a$;

(D8) $\langle p^*\rangle a \leq a \vee \langle p^*\rangle(\neg a \wedge \langle p\rangle a)$.

**Remark 3.1.2.** Given a dynamic algebra $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$, it follows by identities (D3) and (D4) that for every $p \in \mathcal{R}$, $\langle p\rangle \colon \mathcal{B} \to \mathcal{B}$ is a normal modal operator ([4]) on the Booolean algebra $\mathcal{B}$. Hence, in particular, $\langle p\rangle$ is a monotone operation on $\mathcal{B}$, that is, if $a \leq b$ in $\mathcal{B}$, then $\langle p\rangle a \leq \langle p\rangle b$.

Notice that the inequalities of conditions (D7) and (D8) can be replace by identities. This shows that the class of all dynamic algebras is a (two-sorted) variety. We denote by $\mathbb{DA}$ the variety of all dynamic algebras.

Let $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ by a dynamic algebra. For every $p \in \mathcal{R}$, it defines the dual operator of $\langle\rangle$ as is usual: for every $a \in \mathcal{B}$:

$$[p]a := \neg\langle p\rangle\neg a. \tag{3.1.1}$$

Hence, we have that the dual identities of those defining dynamic algebras hold in every dynamic algebra. Actually, analogous to the context of modal algebras (classical modal logic), a dynamic algebra can be equivalently defined through the operation $[]$. That is,

**Proposition 3.1.3.** *Let $\langle \mathcal{B}, \mathcal{R}, []\rangle$ be a two-sorted algebra over the two-sorted algebraic language $\mathrm{L}_D^{dual} = \mathrm{L}_B \cup \mathrm{L}_R \cup \{[]\}$ where $\mathrm{L}_B$ and $\mathrm{L}_R$ are as on page 3.1 and $[]$ is a mixed functional operation of type (2,1,1). Let $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ be the two-sorted algebra of type $\mathrm{L}_D = \mathrm{L}_B \cup \mathrm{L}_R \cup \{\langle\rangle\}$ where for all $p \in \mathcal{R}$ and $a \in \mathcal{B}$, we define $\langle p\rangle a := \neg[p]\neg a$. Then, $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ is a dynamic algebra if and only if the following conditions hold:*

*(D1)* $\mathcal{B} = \langle B, \wedge, \vee, \neg, 0, 1\rangle$ *is a Boolean algebra;*

*(D2)* $\mathcal{R} = \langle R, ;, \sqcup, ^*\rangle$ *is an (homogeneous) algebra of type* $\mathrm{L}_R$.

*and for all $a, b \in B$ and $p, q \in R$*

*(D3')* $[p]1 = 1$;

*(D4')* $[p](a \wedge b) = [p]a \wedge [p]b$;

*(D5')* $[p \sqcup q]a = [p]a \wedge [q]a$;

*(D6')* $[p; q]a = [p][q]a$;

*(D7')* $[p^*]a \leq a \wedge [p][p^*]a$;

*(D8')* $[p^*](a \to [p]a) \leq a \to [p^*]a$.

*Moreover, for all $p \in \mathcal{R}$ and $a \in \mathcal{B}$, we have $[p]a = \neg\langle p\rangle\neg a$.*

*Proof.* It is straightforward from (3.1.1) and the Boolean properties. $\square$

**Remark 3.1.4.** In the literature about PDL it prefers use the operator $[\,]$ instead of $\langle\,\rangle$. On the contrary, in the literature studying dynamic algebras it prefers to work with the operator $\langle\,\rangle$ instead $[\,]$. The previous proposition tells us it is equivalent to use the operator $[\,]$ or $\langle\,\rangle$.

Let $\langle \mathcal{B}, \mathcal{R}, \langle\,\rangle\rangle$ be a dynamic algebra. For every $p \in \mathcal{R}$, we define recursively $p^n$ for all $n \in \mathbb{N}$ and $\langle p^n\rangle$ for all $n \in \mathbb{N}_0$ as follows:

$$\begin{cases} p^1 = 1 \\ p^{n+1} = p; p^{n-1} \quad \text{for all } n \geq 1 \end{cases} \qquad \begin{cases} \langle p^0\rangle = id_{\mathcal{B}} \\ \langle p^1\rangle = \langle p\rangle \\ \langle p^{n+1}\rangle = \langle p; p^n\rangle = \langle p\rangle\langle p^n\rangle \quad \text{for all } n \geq 1. \end{cases}$$

We present some needed properties for what follows.

**Proposition 3.1.5.** *Let $\langle \mathcal{B}, \mathcal{R}, \langle\,\rangle\rangle$ be a dynamic algebra. Let $[\,]$ be the dual operator of $\langle\,\rangle$. Let $a, b \in \mathcal{B}$ and $p, q \in \mathcal{R}$.*

*(P1)* $\langle p^*\rangle\langle p^*\rangle a = \langle p^*\rangle a$ *and* $[p^*][p^*]a = [p^*]a$.

*(P2)* $\langle p\rangle^n a \leq \langle p^*\rangle a$ *and* $[p^*]a \leq [p]^n a$, *for all* $n \in \mathbb{N}_0$.

*Proof.* (P1) By (D7'), $[p^*]a \leq a$. Then $[p^*][p^*]a \leq [p^*]a$. On the other hand, again by (D7'), $[p^*]a \leq [p][p^*]a$. Then $[p^*]a \to [p][p^*]a = 1$. Thus, by (D3'), $[p^*]([p^*]a \to [p][p^*]a) = 1$. By (D8'), $1 = [p^*]([p^*]a \to [p][p^*]a) \leq [p^*]a \to [p^*][p^*]a$. Hence, $[p^*]a \leq [p^*][p^*]a$. $\square$

**Definition 3.1.6.** Let $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ be a dynamic algebra. It defines the binary relation $\cong$ on $\mathcal{R}$ as follows: for all $p, q \in \mathcal{R}$,

$$p \cong q \iff \langle p\rangle a = \langle q\rangle a \;\forall a \in \mathcal{B}. \tag{3.1.2}$$

A dynamic algebra $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ is said to be *separable* if for all $p, q \in \mathcal{R}$, $p \cong q$ implies that $p = q$.

The reader may ask if the relation $\cong$ is a congruence on the (homogeneous) algebra $\mathcal{R}$. The answer is yes, however to prove this we need a very useful characterization of the operation $^*$. Before, we show the following.

**Proposition* 3.1.7.** *Let $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ be a separable dynamic algebra. Then, the operations on the algebra $\mathcal{R}$ satisfies the following:*

1. *$\sqcup$ is an associative, commutative and idempotent operation;*

2. *; is associative.*

3. *; distributes with respect to $\sqcup$, that is, for all $p, q, r \in \mathcal{R}$, $p; (q \sqcup r) = (p; q) \sqcup (p; r)$ and $(q \sqcup r); p = (q; p) \sqcup (r; p)$.*

*Proof.* 1. It is a consequence of (D5).

2. It is a consequence of (D6).

3. It is a consequence of (D4)–(D6). $\square$

Now we proceed to show a characterization of $^*$. Let $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ be a dynamic algebra. For every $p \in \mathcal{R}$ and $a \in \mathcal{B}$, let

$$p!a := \{b \in \mathcal{B} : a \vee \langle p \rangle b \leq b\}. \tag{3.1.3}$$

Let

(D11) $\min(p!a)$ exists in $\mathcal{B}$ and $\langle p^* \rangle a = \min(p!a)$.

**Theorem 3.1.8.** *Let $\mathcal{D} = \langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ be a two-sorted algebra of type $\mathrm{L}_D$ satisfying the identities (D1)–(D6). Then, $\mathcal{D}$ satisfies identities (D7) and (D8) if and only if $\mathcal{D}$ satisfies the condition (D11).*

*Proof.* This proof is given in [14]. Let $p \in \mathcal{R}$ and $a \in \mathcal{B}$.

($\Rightarrow$) Assume that (D7) and (D8) hold. Condition (D8) asserts that $\langle p^* \rangle a \in p!a$. Let now $b \in p!a$. Thus $a \vee \langle p \rangle b \leq b$. Hence $a \leq b$. Then,

$$\begin{aligned}
a &\leq b & \\
\langle p^* \rangle a &\leq \langle p^* \rangle b & \langle p^* \rangle \text{ is monotone} \\
&\leq b \vee \langle p^* \rangle (\neg b \wedge \langle p \rangle b) & \text{by (D8)} \\
&= b \vee \langle p^* \rangle 0 & a \vee \langle p \rangle b \leq b \\
&= b & \text{by (D3)}
\end{aligned}$$

Hence $\langle p^* \rangle a = \min(p!a)$.

($\Leftarrow$) We assume that $\langle p^* \rangle a = \min(p!a)$. Thus $\langle p^* \rangle a \in p!a$. Hence, by definition of $p!a$, we obtain (D7): $a \vee \langle p \rangle \langle p^* \rangle a \leq \langle p^* \rangle a$. To prove (D8), it is enough to show that $a \vee \langle p^* \rangle (\neg a \wedge \langle p \rangle a) \in p!a$. Thus,

$$\begin{aligned}
a \vee \langle p \rangle [a \vee \langle p^* \rangle (\neg a \wedge \langle p \rangle a)] &= a \vee [\neg a \wedge \langle p \rangle [a \vee \langle p^* \rangle (\neg a \wedge \langle p \rangle a)]] & x \vee y = x \vee (\neg x \wedge y) \\
&= a \vee [\neg a \wedge [\langle p \rangle a \vee \langle p \rangle \langle p^* \rangle (\neg a \wedge \langle p \rangle a)]] & \\
&\leq a \vee [(\neg a \wedge \langle p \rangle a) \vee (\neg a \wedge \langle p \rangle \langle p^* \rangle (\neg a \wedge \langle p \rangle a)) & \\
&\leq a \vee [(\neg a \wedge \langle p \rangle a) \vee \langle p \rangle \langle p^* \rangle (\neg a \wedge \langle p \rangle a)] & \\
&\leq a \vee \langle p^* \rangle (\neg a \wedge \langle p \rangle a) & \text{by (D7)}
\end{aligned}$$

Hence $a \vee \langle p^* \rangle (\neg a \wedge \langle p \rangle a) \in p!a$. Since $\langle p^* \rangle a$ is the minimum of the set $p!a$, it follows (D8): $\langle p^* \rangle a \leq a \vee \langle p^* \rangle (\neg a \wedge \langle p \rangle a)$. □

Now we can show that the binary relation $\cong$ on $\mathcal{R}$ is a congruence on the algebraic language $\mathrm{L}_R = \{;, \sqcup, ^*\}$.

**Proposition 3.1.9.** *Let $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ be a dynamic algebra. Then, the relation $\cong$ as defined in* (3.1.2) *is a congruence on the algebra $\mathcal{R}$.*

*Proof.* It is straightforward to see that $\cong$ is an equivalence relation on $\mathcal{R}$. It follows by (D6) and (D5), respectively, that the relation $\cong$ is compatible with respect to the operation ; and $\sqcup$. Now, let $p, q \in \mathcal{R}$ and assume that $p \cong q$. Thus $\langle p \rangle a = \langle q \rangle a$ for all $a \in \mathcal{B}$. This implies that $p!a = q!a$ for all $a \in \mathcal{B}$. Hence $\langle p^* \rangle a = \min(p!a) = \min(q!a) = \langle q^* \rangle a$ for all $a \in \mathcal{B}$. Therefore, $p^* \cong q^*$. □

**Remark 3.1.10.** Let $\mathcal{D} = \langle \mathcal{B}, \mathcal{R}, \langle\rangle \rangle$ be a dynamic algebra. Then, the two-sorted algebra $\widehat{\mathcal{D}} = \langle \mathcal{B}, \mathcal{R}/\cong, \langle\rangle \rangle$ is a separable dynamic algebra, where $\mathcal{R}/\cong$ is the quotient algebra by the congruence $\cong$, and for every $p \in \mathcal{R}$ and $a \in \mathcal{B}$

$$\langle p/\cong \rangle a := \langle p \rangle a. \tag{3.1.4}$$

It is easy to verify that $\widehat{\mathcal{D}}$ satisfies the identities of dynamic algebras. Moreover, it is clear by definition of $\cong$ that it is separable.

Let us now to present some properties about the operation $^*$. First we introduce some notation. Let $\mathcal{D} = \langle \mathcal{B}, \mathcal{R}, \langle\rangle \rangle$ be a dynamic algebra. We define the binary relation $\sqsubseteq$ on $\mathcal{R}$ as follows: for all $p, q \in \mathcal{R}$,

$$p \sqsubseteq q \iff \langle p \rangle a \leq \langle q \rangle a \ \forall a \in \mathcal{B}. \tag{3.1.5}$$

It is clear that $\sqsubseteq$ is a quasi-order on $\mathcal{R}$ (that is, $\sqsubseteq$ is a reflexive and transitive binary relation on $\mathcal{R}$). Notice that

$$p \cong q \iff p \sqsubseteq q \ \text{ and } \ q \sqsubseteq p.$$

Thus, it is straightforward to see that $\sqsubseteq$ is a partial order on $\mathcal{R}$ (reflexive, transitive and antisymmetric) if and only if the dynamic algebra $\mathcal{D}$ is separable. On the other hand, since the dynamic algebra $\langle \mathcal{B}, \mathcal{R}/\cong, \langle\rangle \rangle$ is separable, it follows that the relation $\sqsubseteq$ defined on $\mathcal{R}/\cong$ is the partial order associated with the semilattice operation $\sqcup$ on $R/\cong$. That is,

$$\begin{aligned} p/\cong \sqsubseteq q/\cong &\iff \langle p/\cong \rangle a \leq \langle q/\cong \rangle a \ \forall a \in \mathcal{B} \\ &\iff \langle p/\cong \rangle a \vee \langle q/\cong \rangle a = \langle p/\cong \rangle a \ \forall a \in \mathcal{B} \\ &\iff \langle p/\cong \vee q/\cong \rangle a = \langle q/\cong \rangle a \ \forall a \in \mathcal{B} \\ &\iff p/\cong \sqcup q/\cong = q/\cong. \end{aligned}$$

Moreover, notice that $p/\cong \sqsubseteq q/\cong \iff \langle p/\cong \rangle a \leq \langle q/\cong \rangle a \ \forall a \in \mathcal{B} \iff \langle p \rangle a \leq \langle q \rangle a \ \forall a \in \mathcal{B} \iff p \sqsubseteq q$.

**Proposition 3.1.11.** *Let $\langle \mathcal{B}, \mathcal{R}, \langle\rangle \rangle$ be a dynamic algebra. Let $p, q \in \mathcal{R}$. Then:*

*(D12)* $p \sqsubseteq p^*$.

*(D13)* $p \sqsubseteq q \implies p^* \sqsubseteq q^*$.

*(D14)* $p^{**} \cong p^*$.

*Proof.* (D12) Let $a \in \mathcal{B}$. Then we obtain that: $a \overset{(D7)}{\leq} \langle p^* \rangle \implies \langle p \rangle a \leq \langle p \rangle \langle p^* \rangle a \overset{(D7)}{\leq} \langle p^* \rangle a$. Hence $p \sqsubseteq p^*$.

(D13) $p \sqsubseteq q \implies \langle p \rangle a \leq \langle q \rangle a \ \forall a \in \mathcal{B} \implies q!a \subseteq p!a \ \forall a \in \mathcal{B} \implies \langle p^* \rangle a = \min(p!a) \leq \min(q!a) = \langle q^* \rangle a \ \forall a \in \mathcal{B} \implies p^* \sqsubseteq q^*$.

(D14) Let $a \in \mathcal{B}$. By Theorem 3.1.8, we have that

$$\langle p^{**} \rangle a = \min(p^*!a) = \min\{b \in \mathcal{B} : a \vee \langle p^* \rangle b \leq b\}.$$

Let us show that $\langle p^* \rangle a$ is the minimum of the set $p^*!a$.

By (D7), we have that $\langle p\rangle\langle p^*\rangle a \le \langle p^*\rangle a$. Thus $\langle p^*\rangle a \vee \langle p\rangle\langle p^*\rangle a \le \langle p^*\rangle a$. Hence, it follows that $\langle p^*\rangle a = \min(\{b \in \mathcal{B} : \langle p^*\rangle a \vee \langle p\rangle b \le b\})$. Then, $\langle p^*\rangle a = \min(p!\langle p^*\rangle a) = \langle p^*\rangle\langle p^*\rangle a$. Thus, in particular, we have $\langle p^*\rangle\langle p^*\rangle a \le \langle p^*\rangle a$, and by (D7) we also have $a \le \langle p^*\rangle a$. Then $a \vee \langle p^*\rangle\langle p^*\rangle a \le \langle p^*\rangle a$. This implies that $\langle p^*\rangle a \in p^*!a$. Now let $b \in p^*!a$. That is, $a \vee \langle p^*\rangle b \le b$. Thus $a \le b$. So $\langle p^*\rangle a \le \langle p^*\rangle b \le b$. Hence, we have proved that $\langle p^*\rangle a = \min(\{b \in \mathcal{B} : a \vee \langle p^*\rangle b \le b\}) = \langle p^{**}\rangle a$. Therefore, $p^* \cong p^{**}$. □

The conditions (D12)–(D14) tell us that if $\mathcal{D} = \langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ is a separable dynamic algebra, then the operation $^*$ is a closure operator on the poset[1] $\langle \mathcal{R}, \sqsubseteq\rangle$ (see [6, 7.1] and [7]). However, in the general case where $\mathcal{D}$ is not necessarily separable, $^*$ is called a *quasiclosure operator* on the quasi-partially ordered set $\langle \mathcal{R}, \sqsubseteq\rangle$, that is, the operation $^*$ on $\mathcal{R}$ satisfies conditions (D12)–(D14).

Let $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ be a dynamic algebra. A regular element $p \in \mathcal{R}$ is called a *quasi-closed* element[2] of $^*$ if $p \cong p^*$. The quasi-closed elements of $^*$ can be characterize as follows, independently of the operation $^*$. First, we present the following definition and lemma.

**Definition 3.1.12.** Let $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ be a dynamic algebra. An element $p \in \mathcal{R}$ is said to be:

- *reflexive* if $a \le \langle p\rangle a$, for all $a \in \mathcal{B}$;
- *transitive* if $\langle p\rangle\langle p\rangle a \le \langle p\rangle a$, for all $a \in \mathcal{B}$.

**Lemma 3.1.13.** *Let $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ be a dynamic algebra. Let $p \in \mathcal{R}$. The following are equivalent.*

1. *$p$ is reflexive and transitive.*
2. *$a \vee \langle p\rangle\langle p\rangle a \le \langle p\rangle a$, for all $a \in \mathcal{B}$.*
3. *$\langle p\rangle a \in p!a$, for all $a \in \mathcal{B}$.*

*Proof.* (2) ⇔ (3) It follows by the definition of $p!a$.

(1) ⇒ (2) Suppose that $p$ is reflexive and transitive. By reflexivity and the monontonicity, it follows that $\langle p\rangle a \le \langle p\rangle\langle p\rangle a$, for all $a \in \mathcal{B}$. Then, by transitivity, we have that $\langle p\rangle\langle p\rangle a = \langle p\rangle a$, for all $a \in \mathcal{B}$. Hence, by reflexivity, we obtain that $a \vee \langle p\rangle\langle p\rangle a = a \vee \langle p\rangle a = \langle p\rangle a$, for all $a \in \mathcal{B}$.

(2) ⇒ (1). It is straightforward. □

**Proposition 3.1.14.** *Let $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ be a dynamic algebra. An element $p \in \mathcal{R}$ is quasi-closed if and only if $p$ is reflexive and transitive.*

*Proof.* (⇒) Assume that $p$ is a quasi-closed element. Thus $p \cong p^*$. Let $a \in \mathcal{B}$. Then $\langle p\rangle a = \langle p^*\rangle a$. By (D7), we have $a \vee \langle p\rangle\langle p^*\rangle a \le \langle p^*\rangle$. Thus $a \vee \langle p\rangle\langle p\rangle a \le \langle p\rangle a$. By Lemma 3.1.13, $p$ is a reflexive and transitive element of $\mathcal{R}$.

(⇐) Assume that $p$ is a reflexive and transitive element. Let $a \in \mathcal{B}$. By Lemma 3.1.13, $\langle p\rangle a \in p!a$. Then $\langle p^*\rangle a \le \langle p\rangle a$. By (D12), we have that $\langle p\rangle a \le \langle p^*\rangle a$. Hence $\langle p\rangle a = \langle p^*\rangle a$. Therefore, $p \cong p^*$. □

[1] Partially ordered set (poset).

[2] In [14], the quasi-closed elements of $^*$ are called *asterate*.

Now we are in condition to present another characterization of the operation $^*$ in a dynamic algebra.

**Theorem* 3.1.1.** *Let $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ be a dynamic algebra. For every $p \in \mathcal{R}$, we have*

$$p^* = \min_{\sqsubseteq}\{q \in \mathcal{R} : q \textit{ is reflexive and transitive, and } p \sqsubseteq q\}. \tag{3.1.6}$$

*or*

$$p^* \in \min_{\sqsubseteq}\{q \in \mathcal{R} : q \textit{ is reflexive and transitive, and } p \sqsubseteq q\}. \tag{3.1.7}$$

Notice that, since $\sqsubseteq$ is not necessarily antisymmetric on $\mathcal{R}$, the minimum is not necessarily unique. The theorem asserts that a minimum exists and $p^*$ is one of these.

*Proof.* Let us consider the poset $\langle \mathcal{R}/\cong, \sqsubseteq\rangle$ where for all $p, q \in \mathcal{R}$,

$$p/\cong \sqsubseteq q/\cong \iff p \sqsubseteq q.$$

Since $\cong$ is a congruence on the algebra $\mathcal{R}$, we have that $^*$ is an operation on $\mathcal{R}/\cong$, where $(p/\cong)^* = (p^*)/\cong$. From conditions (D12)–(D14), it follows that $^*$ is a closure operation on $\langle \mathcal{R}/\cong, \sqsubseteq\rangle$. Then, it follows that (see for instance [7, Proposition 3.5])

$$(p^*)/\cong = \min_{\sqsubseteq}\{q/\cong \in \mathcal{R}/\cong : (q/\cong)^* = q/\cong \ \text{ and } \ p/\cong \sqsubseteq q/\cong\}.$$

Noting that $(p/\cong)^* = p/\cong \iff p^* \cong p$, it is straightforward to show that

$$p^* = \min_{\sqsubseteq}\{q \in \mathcal{R} : q^* \cong q \ \text{ and } \ p \sqsubseteq q\}.$$

Hence, by Proposition 3.1.14, we obtain that

$$p^* = \min_{\sqsubseteq}\{q \in \mathcal{R} : q \text{ is reflexive and transitive, and } p \sqsubseteq q\}. \qquad \square$$

**Definition 3.1.15.** Let $\mathcal{D} = \langle \mathcal{B}, \mathcal{R}, \langle\,\rangle\rangle$ be a dynamic algebra. We say that:

- $\mathcal{D}$ is *boolean-finite* if $\mathcal{B}$ is finite;
- $\mathcal{D}$ is *regular-finite* if $\mathcal{R}$ is finite.
- $\mathcal{D}$ is *finite* if both $\mathcal{B}$ and $\mathcal{R}$ are finite.

**Proposition* 3.1.16.** *Let $\mathcal{D} = \langle \mathcal{B}, \mathcal{R}, \langle\,\rangle\rangle$ be a separable dynamic algebra. If $\mathcal{D}$ is boolean-finite, then $\mathcal{D}$ is also regular-finite.*

*Proof.* Since $\mathcal{B}$ is finite, the set of all functions from $\mathcal{B}$ into $\mathcal{B}$ is finite. In particular, the set $\{\langle p\rangle : p \in \mathcal{R}\}$ is finite. Now, since $\mathcal{D}$ is separable, we obtain that the set $\{\langle p\rangle : p \in \mathcal{R}\}$ is bijective to $\mathcal{R}$. Hence, $\mathcal{R}$ is finite. $\square$

Example **??** shows a dynamic algebra which is boolean-finite and regular-infinity. The following is a technical result which it is needed in Section **??**.

**Proposition* 3.1.17.** *Let $\mathcal{D} = \langle \mathcal{B}, \mathcal{R}, \langle\,\rangle\rangle$ be a boolean-finite dynamic algebra. Then, for every $p \in \mathcal{R}$, there is $n \in \mathbb{N}$ such that*

$$\langle p^*\rangle a \leq a \vee \langle p\rangle a \vee \langle p^2\rangle a \vee \cdots \vee \langle p^n\rangle a \quad \textit{and} \quad a \wedge [p]a \wedge [p^2]a \wedge \cdots \wedge [p^n]a \leq [p^*]a,$$

*for all $a \in \mathcal{B}$.*

*Proof.* Since $\mathcal{B}$ is finite, there are finitely many functions $f\colon \mathcal{B} \to \mathcal{B}$. Then, by the well-ordered principle in $\mathbb{N}$, we can take the least $k \in \mathbb{N}$ such that $\langle p^k\rangle = \langle p^i\rangle$ for some $i \in \{0, 1, \dots, k-1\}$. Let $n := k - 1$. Notice that $\langle p^{n+1}\rangle = \langle p^k\rangle = \langle p^i\rangle$ with $0 \leq i < k = n + 1$. Let $a \in \mathcal{B}$ and $b := a \vee \langle p\rangle a \vee \cdots \vee \langle p^n\rangle a$. Let us show that $b \in p!a$. That is, we show that $a \vee \langle p\rangle b \leq b$. We have

$$\begin{aligned} a \vee \langle p\rangle b &= a \vee \langle p\rangle(a \vee \langle p\rangle a \vee \cdots \vee \langle p^n\rangle a) \\ &= a \vee \langle p\rangle a \vee \langle p^2\rangle a \cdots \vee \langle p^n\rangle a \vee \langle p^{n+1}\rangle a \\ &= a \vee \langle p\rangle a \vee \cdots \vee \langle p^n\rangle a \\ &= b. \end{aligned}$$

Then $b \in p!a$. Hence, $\langle p^*\rangle a \leq b = a \vee \langle p\rangle a \vee \cdots \vee \langle p^n\rangle a$. □

## 3.2 Some important examples

In this section, we provide some examples of (separable) dynamic algebras relevant to our goals. We start with perhaps one of the most important examples.

**Example 3.2.1.** Let us consider the logic $\mathsf{PDL}$ (see (1.2.1)) and the relational completeness theorem for $\mathsf{PDL}$ (Theorem 1.3.1). Let us build the *Lindendaum-Tarski algebra* of this logic.

Let $\Phi$ and $\Pi$ be two (disjunct) countable sets. Let $\mathsf{Fm}(\Phi, \Pi)$ and $\mathsf{Prog}(\Pi)$ be the two-sorted term algebra of type $\mathrm{L}_D^{\mathrm{dual}} = \mathrm{L}_B \cup \mathrm{L}_R \cup \{[\,]\}$ over $(\Phi, \Pi)$. In other words, $\mathsf{Fm}(\Phi, \Pi)$ and $\mathsf{Prog}(\Pi)$ are respectively the algebras of formulas and programs (without test) over the atomic propositions and programs $(\Phi, \Pi)$, respectively.

For all $\varphi, \psi \in \mathsf{Fm}$, it defines as is usual the equivalence relation $\equiv$ on $\mathsf{Fm}$ as follows:

$$\varphi \equiv \psi \iff \varphi \leftrightarrow \psi \in \mathsf{PDL} \iff \vdash_{PDL} \varphi \leftrightarrow \psi.$$

It is well-known, from the classical tautologies, that $\equiv$ is a congruence of $\mathsf{Fm}$ with respect to the (homogeneous) algebraic language $\mathrm{L}_B$, and the (homogeneous) quotient algebra $\langle \mathsf{Fm}/{\equiv}, \wedge, \vee, \neg, \overline{\bot}, \overline{\top}\rangle$ is a Boolean algebra.

Now let us define a binary relation on $\mathsf{Prog}(\Pi)$. Let $\alpha, \beta \in \mathsf{Prog}$. We define:

$$\alpha \sim \beta \iff \forall \varphi \in \mathsf{Fm}([\alpha]\varphi \leftrightarrow [\beta]\varphi \in \mathsf{PDL}) \iff \forall \varphi \in \mathsf{Fm}([\alpha]\varphi \equiv [\beta]\varphi). \tag{3.2.1}$$

It is clear that $\sim$ is an equivalence relation. Let us show that $\sim$ is a (homogeneous) congruence on $\mathsf{Prog}$ with respect to $\mathrm{L}_R$. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathsf{Prog}$ be such that $\alpha_1 \sim \beta_1$ and $\alpha_2 \sim \beta_2$. Let $\varphi \in \mathsf{Fm}$. Since $\alpha_2 \sim \beta_2$, it follows by (3.2.1) $\vdash_{PDL} [\alpha_2]\varphi \leftrightarrow [\beta_2]\varphi$. From axiom (K) and rule (N), it follows that $\vdash_{PDL} [\alpha_1]\alpha_2]\varphi \leftrightarrow [\alpha_1]\beta_2]\varphi$. Since $\alpha_1 \sim \beta_1$, we have $\vdash_{PDL} [\alpha_1][\beta_2]\varphi \leftrightarrow [\beta_1][\beta_2]\varphi$. Then $\vdash_{PDL} [\alpha_1][\alpha_2]\varphi \leftrightarrow [\beta_1][\beta_2]$. Hence, by axiom (Comp), it follows that $\vdash_{PDL} [\alpha_1;\alpha_2]\varphi \leftrightarrow [\beta_1;\beta_2]\varphi$. Therefore, $\alpha_1;\alpha_2 \sim \beta_1;\beta_2$. Analogously, using axiom (Alt), it can be proved that $\alpha_1 \sqcup \alpha_2 \sim \beta_1 \sqcup \beta_2$. Now let us prove that $\alpha_1^* \sim \beta_1^*$. From what we just proved, it follows that $\alpha_1^n \sim \beta_1^n$ for all $n \in \mathbb{N}$. Then, for every $\varphi \in \mathsf{Fm}$,

$$\begin{aligned}
&\vdash_{PDL} [\alpha_1^n]\varphi \leftrightarrow [\beta_1^n]\varphi, \ \forall n \in \mathbb{N} & \\
&\mathcal{M} \Vdash [\alpha_1^n]\varphi \leftrightarrow [\beta_1^n]\varphi, \text{ for all standard model } \mathcal{M}, \ \forall n \in \mathbb{N} && \text{by Theorem 1.3.1} \\
&\mathcal{M} \Vdash [\alpha_1^*]\varphi \leftrightarrow [\beta_1^*]\varphi, \text{ for all standard model } \mathcal{M}, && \text{by (M10) of Proposition 1.2.5} \\
&\vdash_{PDL} [\alpha_1^*]\varphi \leftrightarrow [\beta_1^*]\varphi && \text{by Theorem 1.3.1}
\end{aligned}$$

Hence $\alpha_1^* \sim \beta_1^*$. Therefore, $\langle \mathsf{Prog}/{\sim}, ;, \sqcup, ^* \rangle$ is an algebra of type $\mathrm{L}_R$.

Now we define the operation $[\,]\colon \mathsf{Prog}/{\sim} \times \mathsf{Fm}/{\equiv} \to \mathsf{Fm}/{\equiv}$ as follows: for all $\alpha \in \mathsf{Prog}$ and $\varphi \in \mathsf{Fm}$,

$$[\alpha/{\sim}]\varphi/{\equiv} := ([\alpha]\varphi)/{\equiv}. \tag{3.2.2}$$

First, we show that every operation is well-defined. Let $\alpha, \beta \in \mathsf{Prog}$ and $\varphi, \psi \in \mathsf{Fm}$. Suppose that $\alpha \sim \beta$ and $\varphi \equiv \psi$. We need to show that $[\alpha/{\sim}]\varphi/{\equiv} = [\beta/{\sim}]\psi/{\equiv}$. On the one hand, since $\alpha \sim \beta$, it follows that

$$\vdash_{PDL} [\alpha]\psi \leftrightarrow [\beta]\psi. \tag{3.2.3}$$

On the other hand, since $\varphi \equiv \psi$, it follows by rule (N) and axiom (K) that

$$\vdash_{PDL} [\alpha]\varphi \leftrightarrow [\alpha]\psi. \tag{3.2.4}$$

Then, by (3.2.3) and (3.2.4), we obtain that $\vdash_{PDL} [\alpha]\varphi \leftrightarrow [\beta]\psi$. Hence $[\alpha/{\sim}]\varphi/{\equiv} = [\beta/{\sim}]\psi/{\equiv}$.

Hence, $\langle \mathsf{Fm}/{\equiv}, \mathsf{Prog}/{\sim}, [\,] \rangle$ is an two-sorted algebra of type $\mathrm{L}_D^{\mathrm{dual}} = \mathrm{L}_B \cup \mathrm{L}_R \cup \{[\,]\}$. Taking into account that for all $\varphi, \psi \in \mathsf{Fm}$,

$$\varphi/{\equiv} \leq \psi/{\equiv} \text{ in } \mathsf{Fm}/{\equiv} \quad \text{if and only if} \quad \vdash_{PDL} \varphi \to \psi,$$

we obtain that the identities (D3')–(D8') are consequence of axioms (K) (Alt), (Comp), (Mix) and (Ind), respectively. Hence, by Proposition 3.1.3, $\langle \mathsf{Fm}/{\equiv}, \mathsf{Prog}/{\sim}, [\,] \rangle$ is a dynamic algebra, and it is called the *Lindenbaum-Tarski algebra of the logic* PDL.

Finally, we show that $\langle \mathsf{Fm}/{\equiv}, \mathsf{Prog}/{\sim}, [\,] \rangle$ is separable. Let $\alpha, \beta \in \mathsf{Prog}$. Then,

$$\begin{aligned}
[\alpha/{\sim}]\varphi/{\equiv} = [\beta/{\sim}]\varphi/{\equiv}, \quad \forall \varphi \in \mathsf{Fm} &\implies ([\alpha]\varphi)/{\equiv} = ([\beta]\varphi)/{\equiv}, \quad \forall \varphi \in \mathsf{Fm} \\
&\implies [\alpha]\varphi \equiv [\beta]\varphi, \quad \forall \varphi \in \mathsf{Fm} \\
&\implies \alpha \sim \beta \\
&\implies \alpha/{\sim} = \beta/{\sim}.
\end{aligned}$$

□

We denote de Lindenbaum-Tarski algebra of the logic PDL by $\mathcal{LT}(\mathsf{PDL}) = \langle \mathsf{Fm}/{\equiv}, \mathsf{Prog}/{\sim}, [\,] \rangle$.

**Example 3.2.2.** Let $W$ be a non-empty set. Let $\mathcal{R}$ be a set of binary relations on $W$ closed under union, composition and reflexive transitive closure. That is:

- if $R, S \in \mathcal{R}$, then $R \cup S \in \mathcal{R}$;
- if $R, S \in \mathcal{R}$, then $R \circ S \in \mathcal{R}$;
- if $R \in \mathcal{R}$, then $R^* := \bigcup_{n \in \mathbb{Z}_{\geq 0}} R^n \in \mathcal{R}$.

By $R^n$ we mean that: $R^0 = Id_W$ ($Id_W$ is de identity relation on $W$), and for every $n \geq 0$, $R^{n+1} = R \circ R^n$. Thus, $R^*$ is the least reflexive transitive relation on $W$ containing $R$. It is called the *reflexive transitive closure* of $R$. Now let $\mathcal{D}(W, \mathcal{R}) = \langle \mathcal{B}, \mathcal{R}, \langle\rangle \rangle$ be defined as follows:

- $\mathcal{B} = \langle \mathcal{P}(W), \cup, \cap, {}^*, \emptyset, W \rangle$, the power set Boolean algebra on $W$.
- $\mathcal{R} = \langle \mathcal{R}, \circ, \cup, {}^* \rangle$ as before.
- for every $R \in \mathcal{R}$ and $A \in \mathcal{B}$, let

$$\langle R \rangle A = \{w \in W : R(w) \cap A \neq \emptyset\}. \tag{3.2.5}$$

The definition of $\langle R \rangle A$ corresponds to the definition of the possibility operator in classical modal logic (see [4, 278–279]). Let us prove that $\mathcal{D}(W, \mathcal{R})$ is a separable dynamic algebra.

(D1) and (D2) are obvious.

(D3) and (D4) are consequence of modal logic, see for instance [4, Proposition 5.22].

(D5) and (D6) are direct consequences of the definition of $\langle R \rangle$ and from the corresponding definition of $\cup$ and $\circ$, respectively. We leave the details to the reader.

(D11) Let $R \in \mathcal{R}$. We need to show that $\langle R^* \rangle A = \min(R!A)$. Recall that $R!A = \{B \in \mathcal{B} : A \cup \langle R \rangle B \subseteq B\}$. First we prove that $\langle R^* \rangle A \in R!A$. We need to show that $A \cup \langle R \rangle \langle R^* \rangle A \subseteq \langle R^* \rangle A$. Let $u \in A \cup \langle R \rangle \langle R^* \rangle A$. If $u \in A$, then since $R^*$ is reflexive it follows that $u \in R^*(u) \cap A$. That is, $R^*(u) \cap A \neq \emptyset$. Hence $u \in \langle R^* \rangle A$. Now suppose that $u \in \langle R \rangle \langle R^* \rangle A$. Thus $R(u) \cap \langle R^* \rangle A \neq \emptyset$. Let $w \in R(u)$ and $w \in \langle R^* \rangle A$. Then, $R^*(w) \cap A \neq \emptyset$. Let $v \in A$ be such that $v \in R^*(w)$. Since $R \subseteq R^*$, it follows that $(u, w) \in R^*$. Thus, since $(u, w), (w, v) \in R^*$ and $R^*$ is transitive, we obtain that $v \in R^*(u) \cap A$. Hence $u \in \langle R^* \rangle A$. Therefore, we have proved that $\langle R^* \rangle A \in R!A$. Let now $B \in R!A$. So

$$A \cup \langle R \rangle B \subseteq B. \tag{3.2.6}$$

We need to show that $\langle R^* \rangle A \subseteq B$. Let $u \in \langle R^* \rangle A$. Thus $R^*(u) \cap A \neq \emptyset$. Then, there is $n \in \mathbb{Z}_{\geq 0}$ such that $R^n(u) \cap A \neq \emptyset$. Let $v \in A$ be such that $v \in R^n(u)$. Thus there are $v_0 = u, v_1, \ldots, v_{n-1}, v_n = v$ such that $(v_i, v_{i+1}) \in R$ for all $i = 0, \ldots, n-1$. Notice that $v \in A$, and by (3.2.6) $A \subseteq B$, thus $v \in B$. Then, since $(v_{n-1}, v) \in R$, it follows that $R(v_{n-1}) \cap B \neq \emptyset$. Then, $v_{n-1} \in \langle R \rangle B$. By (3.2.6), it follows that $v_{n-1} \in B$. If we repeat this argument for $v_{n-1}$ and $v_{n-2}$ instead $v_{n-1}$ and $v_n = v$, we obtain that $v_{n-2} \in B$. Continuing in this way, we arrive to $u = v_0 \in B$. Hence $\langle R \rangle A \subseteq B$, for all $B \in R!A$. Therefore, $\langle R^* \rangle A = \min(R!A)$.

Lastly, we show that $\mathcal{D}(W,\mathcal{R})$ is separable. Let $R,S \in \mathcal{R}$ and suppose that $\langle R\rangle A = \langle S\rangle A$, for all $A \in \mathcal{B}$. Let $u \in W$. Then

$$\begin{aligned} u \in \langle R\rangle A &\iff u \in \langle S\rangle A,\ \forall A \in \mathcal{B} \\ R(u)\cap A \neq \emptyset &\iff S(u)\cap A \neq \emptyset,\ \forall A \in \mathcal{B} \\ R(u)\cap \{w\} \neq \emptyset &\iff S(u)\cap \{w\} \neq \emptyset,\ \forall w \in W \\ w \in R(u) &\iff w \in S(u),\ \forall w \in W. \end{aligned}$$

Hence, $R(u) = S(u)$. Therefore, $R = S$. □

A structure $\langle W,\mathcal{R}\rangle$ is said to be a *standard*[3] *Kripke frame* if $W$ is a non-empty set and $\mathcal{R}$ is a set of binary relations on $W$ closed under union, composition and reflexive transitive closure. The (separable) dynamic algebra $\mathcal{D}(W,\mathcal{R})$ associated with a standard Kripke frame $\langle W,\mathcal{R}\rangle$ is called the *complex dynamic algebra* associated with $\langle W,\mathcal{R}\rangle$. When $\mathcal{R}$ is the class of all binary relations on $W$, $\langle W,\mathcal{R}\rangle$ is called a *full Kripke frame*, and in this case the dynamic algebra $\mathcal{D}(W,\mathcal{R})$ is called the *full complex dynamic algebra*[4] associated with $\langle W,\mathcal{R}\rangle$.

**Remark 3.2.3.** Notice that the dual operation $[\,]$ of $\langle\,\rangle$ in a complex dynamic algebra $\mathcal{D}(W,\mathcal{R})$ is given by: for every $R \in \mathcal{R}$ and $A \in \mathcal{B} = \mathcal{P}(W)$,

$$[R]A = (\langle R\rangle A^c)^c = \{w \in W : R(w) \subseteq A\}. \tag{3.2.7}$$

In Chapter 1, we mention that Propositional dynamic logic can be study as a multi-modal logic. The algebraic semantics for classical modal logic is the class of modal algebras ([4]). It is well-know that every modal algebra is embeddable into the complex modal algebra associated with some Kripke frame. Hence, similar to the framework on modal algebras, we can ask ourselves if every (separable) dynamic algebra can be embeddable into the complex dynamic algebra associated with a standard Kripke frame. Pratt [14] shows that it is not the case. That is, he shows that there are separable dynamic algebras which are not embeddable into a complex dynamic algebra for any standard Kripke frame. How is this shown? The complex dynamic algebras enjoy a property P, which is preserved by its two-sorted subalgebras. Pratt [14] gives an example of a separable dynamic algebra which does not satisfies the property P. We proceed to show this.

**Definition 3.2.4.** A dynamic algebra $\mathcal{D} = \langle \mathcal{B},\mathcal{R},\langle\rangle\rangle$ is said to be $^*$-*continuous* ([13]) if for all $p \in \mathcal{R}$ and $a \in \mathcal{B}$ holds

$$\langle p^*\rangle a = \bigvee\{\langle p^n\rangle a : n \in \mathbb{Z}_{\geq 0}\}. \tag{3.2.8}$$

**Proposition 3.2.5.** *A two-sorted subalgebra of a $^*$-continuous dynamic algebra is a $^*$-continuous dynamic algebra.*

*Proof.* It is straightforward. We leave the details to the reader. □

[3]The adjective *standard* come from [10].

[4]In [14], the full complex dynamic algebras are called full Kripke structures.

**Example 3.2.6.** We show that every complex dynamic algebras is $^*$-continuous. Let $\langle W, \mathcal{R}\rangle$ be a standard Kripke frame. Let $\mathcal{D}(W, \mathcal{R}) = \langle \mathcal{P}(W), \mathcal{R}, \langle\rangle\rangle$ be the complex dynamic algebra associated with $\langle W, \mathcal{R}\rangle$. Let $R \in \mathcal{R}$ and $A \in \mathcal{P}(W)$. We have to show that $\langle R^*\rangle A = \bigcup\{\langle R^n\rangle A : n \in \mathbb{Z}_{\geq 0}\}$. Recall that $R^* = \bigcup_{n\in\mathbb{Z}_{\geq 0}} R^n$. Let $u \in W$. Then,

$$\begin{aligned} u \in \langle R^*\rangle A &\iff R^*(u) \cap A \neq \emptyset \\ &\iff R^n(u) \cap A \neq \emptyset, \quad \text{for some } n \in \mathbb{Z}_{\geq 0} \\ &\iff u \in \langle R^n\rangle A, \quad \text{for some } n \in \mathbb{Z}_{\geq 0} \\ &\iff u \in \bigcup\{\langle R^n\rangle A : n \in \mathbb{Z}_{\geq 0}\}. \quad \square \end{aligned}$$

Hence, every dynamic algebra which is embeddable into a complex dynamic algebra is $^*$-continuous. Now we present an example by Pratt [14] of a dynamic algebra which is not $^*$-continuous. Hence, it cannot be embeddable into any complex dynamic algebra. That is,

*(separable) dynamic algebras cannot be representable by standard Kripke frames.*

We start with an auxiliary result and an example.

**Lemma 3.2.7.** *Let $L$ be a complete lattice and let $\Diamond\colon L \to L$ be an order-preserving map. Then, for every $a \in L$, the set $\Diamond!a := \{b \in L : a \vee \Diamond b \leq b\}$ is closed under arbitrary meets. Hence* $\min(\Diamond!a) = \bigwedge(\Diamond!a)$.

*Proof.* Let $a \in L$. Let $S \subseteq \Diamond!a = \{b \in L : a \vee \Diamond b \leq b\}$. We have to prove that $\bigwedge S \in \Diamond!a$. Let $s \in S$. Thus $\bigwedge S \leq s$. Since $\Diamond$ is order-preserving, it follows that $a \vee \Diamond\left(\bigwedge S\right) \leq a \vee \Diamond s \leq s$. Thus, $a \vee \Diamond\left(\bigwedge S\right)$ is a lower bound of $S$. Then $a \vee \Diamond\left(\bigwedge S\right) \leq \bigwedge S$. Hence, $\bigwedge S \in \Diamond!a$.

Now, in particular, $\bigwedge \Diamond!a \in \Diamond!a$. Hence $\bigwedge \Diamond!a = \min(\Diamond!a)$. $\square$

**Example 3.2.8.** Let $\mathcal{B}$ be a complete Boolean algebra. Let $\mathcal{R}$ be the set of all maps $\alpha\colon \mathcal{B} \to \mathcal{B}$ such that $\alpha 0 = 0$ and $\alpha(a \vee b) = \alpha a \vee \alpha b$, for all $a, b \in \mathcal{B}$. We define the operations ;, $\sqcup$ and $^*$ on $\mathcal{R}$ as follows. Let $\alpha, \beta \in \mathcal{R}$. Then $\alpha;\beta,\ \alpha \sqcup \beta,\ \alpha^*\colon \mathcal{R} \to \mathcal{R}$ are defining by:

- $(\alpha;\beta)(a) := \alpha(\beta a)$;
- $(\alpha \sqcup \beta)(a) := \alpha a \vee \beta a$;
- $\alpha^*(a) := \bigwedge\{b \in \mathcal{B} : a \vee \alpha b \leq b\}$.

It is straightforward to show that $\alpha;\beta$ and $\alpha \sqcup \beta$ are well-defined, that is, that $\alpha;\beta,\ \alpha \sqcup \beta \in \mathcal{R}$. We check only that $\alpha^* \in \mathcal{R}$. Notice that, by Lemma 3.2.7 we have that $\alpha^*(a) := \bigwedge\{b \in \mathcal{B} : a \vee \alpha b \leq b\} = \min\{b \in \mathcal{B} : a \vee \alpha b \leq b\}$. Thus for all $a \in \mathcal{B}$, it follows that $a \vee \alpha\alpha^* a \leq \alpha^* a$.

- $\alpha^* 0 = \bigwedge\{b \in \mathcal{B} : 0 \vee \alpha b \leq b\} = \bigwedge\{b \in \mathcal{B} : \alpha b \leq b\} = 0$ because $\alpha 0 = 0$.
- Let $a, b \in \mathcal{B}$. By definition, it is clear that $\alpha^*$ is monotone. Hence $\alpha^* a \vee \alpha^* b \leq \alpha^*(a \vee b)$. For the other inequality, we show that $\alpha^* a \vee \alpha^* b$ is in $\{c \in \mathcal{B} : (a \vee b) \vee \alpha c \leq c\}$. Then, by Lemma 3.2.7, we obtain that

  $$(a \vee b) \vee (\alpha^* a \vee \alpha^* b) = (a \vee \alpha\alpha^* a) \vee (b \vee \alpha\alpha^* b) \leq \alpha^* a \vee \alpha^* b.$$

  Hence, $\alpha^*(a \vee b) \leq \alpha^* a \vee \alpha^* b$. Therefore, $\alpha^* \in \mathcal{R}$.

Now let $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ be the two-sorted algebra where $\mathcal{B}$ and $\mathcal{R}$ are as before, and for every $\alpha \in \mathcal{R}$ and $a \in \mathcal{B}$, $\langle\alpha\rangle a = \alpha a$. Hence, by the definitions, it is straightforward to show that $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ satisfies the identities (D1)–(D6) and condition (D11). Therefore, $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ is a dynamic algebra, which is also clearly separable. □

Now we present an example (by Pratt [14]) of a separable dynamic algebra which is not $^*$-continuous.

**Example 3.2.9.** Let $\mathbb{N} \cup \{\infty\} = \{0, 1, 2, \dots, \infty\}$. Let $\mathcal{B} := \mathcal{P}(\mathbb{N} \cup \{\infty\})$ be the power set Boolean algebra of $\mathbb{N} \cup \{\infty\}$. Let $\mathcal{R}$ be as in Example 3.2.8 on this $\mathcal{B} = \mathcal{P}(\mathbb{N} \cup \{\infty\})$. We show that the separable dynamic algebra $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ (as in Example 3.2.8) is not $^*$-continuous. Let $\alpha \colon \mathcal{B} \to \mathcal{B}$ be defined as follows: for every $A \in \mathcal{B}$,

$$\alpha(A) = \begin{cases} \{n+1 : n \in A\} & \text{if } A \text{ is finite} \\ \{n+1 : n \in A\} \cup \{\infty\} & \text{if } A \text{ is infinite} \end{cases}$$

It follows that $\alpha(\emptyset) = \emptyset$ and for all $A, B \in \mathcal{B}$, $\alpha(A \cup B) = \alpha(A) \cup \alpha(B)$. Thus $\alpha \in \mathcal{R}$. Recall, by the previous example, that $\langle\alpha\rangle = \alpha$. Since $\alpha^n(\{0\}) = \{n\}$ for all $n \in \mathbb{N}$, it follows that $\bigcup\{\alpha^n(\{0\}) : n \in \mathbb{N}\} = \mathbb{N}$. On the other hand, noting that

$$\alpha!\{0\} = \{B \in \mathcal{B} : \{0\} \cup \alpha(B) \subseteq B\} = \{\mathbb{N} \cup \{\infty\}\}^5,$$

we obtain that $\alpha^*(\{0\}) = \min(\alpha!\{0\}) = \mathbb{N} \cup \{\infty\}$. Hence, $\alpha^*(\{0\}) \neq \bigcup\{\alpha^n(\{0\}) : n \in \mathbb{N}\}$. Therefore, this $\langle \mathcal{B}, \mathcal{R}, \langle\rangle\rangle$ is not $^*$-continuous. □

In Section **??**, we will show that every finite separable dynamic algebra is embeddable into a complex dynamic algebra. In other words, every finite separable dynamic algebra is representable through of some standard Kripke frame.

**Example 3.2.10.** We present a dynamic algebra which is boolean-finite and regular-infinite. Thus, by Propostion 3.1.16, it is not separable. Moreover, we show that it is $^*$-continuous.

Let $W$ be a finite set. Let $\mathcal{B} := \mathcal{P}(W)$ be the power set Boolean algebra. For every $n \in \mathbb{N} = \{1, 2, \dots\}$, let $W^n$ be the set of all strings of length $n$. Let $\mathcal{R} := \mathcal{P}\left(\bigcup_{n\in\mathbb{N}} W^n\right)$. That is, $\mathcal{R}$ is the set of all sets of non-empty finite strings over the set $W$. The operations ;, $\sqcup$ and $^*$ are defined as follows: for all $p, q \in \mathcal{R}$,

- $p;q := \{u \dots v \dots w : u \dots v \in p \text{ and } v \dots w \in q\}$;
- $p \sqcup q := p \cup q$;
- $p^* := W \cup p \cup p^2 \cup \dots = \bigcup_{n\geq 0} p^n$, where $p^0 := W$, $p^1 = p$, and $p^{n+1} = p; p^n$ for all $n \geq 1$.

Now let $\mathcal{D} := \langle \mathcal{B}, \mathcal{R}, \langle\,\rangle\rangle$ be the two-sorted algebra defined by: for every $p \in \mathcal{R}$ and $a \in \mathcal{B}$,

- $\langle p\rangle a := \{u \in W : u \dots v \in p \text{ and } v \in a\}$.

[5] Let $B \in \alpha!\{0\}$. Thus $\{0\} \cup \alpha(B) \subseteq B$. Then $0 \in B$. Thus $\mathbb{N} \subseteq \alpha(B) \subseteq B$. Hence $\mathbb{N} \subseteq B$. Now since $B$ is infinite, $\infty \in \alpha(B) \subseteq B$. Hence $\mathbb{N} \cup \{0\} \subseteq B$. That is, $B = \mathbb{N} \cup \{\infty\}$.

It is straightforward to show that conditions (D3)-(D6) hold. For instance, let us shows that $\langle p;q\rangle a = \langle p\rangle\langle q\rangle a$. Let $u \in W$. Then

$$\begin{aligned} u \in \langle p;q\rangle a &\iff \exists v \in a \; s.t. \; u \dots v \in p;q \\ &\iff \exists v \in a \, \exists w \in W \; s.t. \; u \dots w \in p \;\text{ and }\; w \dots v \in q \\ &\iff \exists w \in W \; s.t. \; w \in \langle q\rangle a \;\text{ and }\; u \dots w \in p \\ &\iff u \in \langle p\rangle\langle q\rangle a. \end{aligned}$$

Now let us show that for every $p \in \mathcal{R}$ and $a \in \mathcal{B}$, $\langle p^*\rangle a = \min(p!a) = \min\{b \in \mathcal{B} : a \cup \langle p\rangle b \subseteq b\}$.

• We show first that $a \cup \langle p\rangle\langle p^*\rangle a \subseteq \langle p^*\rangle a$. Let $u \in a \cup \langle p\rangle\langle p^*\rangle a$. Notice that $a \subseteq W \subseteq p^*$. Thus, by definition, $a \subseteq \langle p^*\rangle a$. Now suppose that $u \in \langle p\rangle\langle p^*\rangle a$. Thus $u \dots v \in p$ with $v \in \langle p^*\rangle a$. Then $v \dots w \in p^*$ and $w \in a$. Whence, there is $n \geq 0$ such that $v \dots w \in p^n$. If $n = 0$, then $v \dots w$ is a string of length 1 with $v = w \in a$. Thus $u \dots v \in p \subseteq p^*$ and $v \in a$. Hence $u \in \langle p^*\rangle a$. If $n \geq 1$, then $u \dots v \dots w \in p;p^n = p^{n+1} \subseteq p^*$, and $w \in a$. Hence $u \in \langle p^*\rangle a$. Therefore, $\langle p^*\rangle a \in p!a$.

• Let $b \in p!a$. Thus $a \cup \langle p\rangle b \subseteq b$. Let us prove by induction that for all $n \geq 0$, if $w \dots v \in p^n$ and $v \in b$, then $w \in b$. For $n = 0$, we have $w \in p^0 = W$ and $w \in b$, thus $w \in b$. Suppose that the statement is valid for $n$. Suppose that $w \dots v \in p^{n+1}$ and $v \in b$. Thus, there is $u \in W$ such that $w \dots u \in p$ and $u \dots v \in p^n$ with $v \in b$. Then, by inductive hypothesis, $u \in b$. Thus $w \dots u \in p$ and $u \in b$. Hence $w \in \langle p\rangle b \subseteq b$. Now let $u \in \langle p^*\rangle a$. Thus $u \dots v \in p^*$ and $v \in a$. Then, there is $n \geq 0$ such that $u \dots v \in p^n$ and $v \in a \subseteq b$. Hence, $u \in b$. We have proved that $\langle p^*\rangle a \subseteq b$.

Hence, the two above points implies that $\langle p^*\rangle a = \min(p!a)$. Therefore, $\mathcal{D}$ is a dynamic algebra, which is Boolean-finite and regular-infinite. Now we show that $\mathcal{D} = \langle \mathcal{B}, \mathcal{R}, \langle\,\rangle\rangle$ is $^*$-continuous. Let $p \in \mathcal{R}$ and $a \in \mathcal{B}$. We need to prove that $\langle p^*\rangle a = \bigcup\{\langle p^n\rangle a : n \in \mathbb{Z}_{\geq 0}\}$. Recall that $p^* = \bigcup_{n\geq 0} p^n$. If $u \in \langle p^*\rangle a$, then $u \dots v \in p^*$ and $v \in a$. Thus, there is $n \in \mathbb{N}_0$ such that $u \dots v \in p^n$. Then $u \in \langle p^n\rangle a$. Hence $\langle p^*\rangle a \subseteq \bigcup\{\langle p^n\rangle a : n \in \mathbb{Z}_{\geq 0}\}$. Now let $u \in \bigcup\{\langle p^n\rangle a : n \in \mathbb{Z}_{\geq 0}\}$. Thus $u \in \langle p^n\rangle a$ for some $n \in \mathbb{N}_0$. Then $u \dots v \in p^n \subseteq p^*$ and $v \in a$. Hence $u \in \langle p^*\rangle a$. □

## 3.3 The classes of dynamic algebras

By the results of the previous sections, we have the following classes of two-sorted algebras which are associated (in some sense) to PDL:

- $\mathbb{DA}$ is the variety of dynamic algebras;
- $\mathbb{SDA}$ is the class of separable dynamic algebras;
- $\mathbb{CDA}$ is the class of complex dynamic algebras. That is, $\mathbb{CDA}$ is the class of dynamic algebras $\mathcal{D}(W, \mathcal{R})$ for every standard Kripke frame $\langle W, \mathcal{R}\rangle$, see Example **??** and the paragraph after that.

- $\mathbb{DA}^*$ is the class of $^*$-continuous dynamic algebras (see Definition 3.2.4).

The basic relations between the above classes is:

$$\mathbb{CDA} \subseteq \mathbb{SDA} \subseteq \mathbb{DA} \qquad \mathbb{CDA} \subseteq \mathbb{DA}^* \subseteq \mathbb{DA} \qquad \mathbb{SDA} \not\subseteq \mathbb{DA}^* \qquad \mathbb{DA}^* \not\subseteq \mathbb{SDA}.$$

The following example shows that the class $\mathbb{SDA}$ is not closed under subalgebras neither homomorphic images. Hence, $\mathbb{SDA}$ is not a (two-sorted) variety.

**Example 3.3.1.** Let $\mathcal{D}_4 := \langle \mathcal{B}_4, \mathcal{R}, \langle\,\rangle\rangle$ be the separable dynamic algebra given by:

- $\mathcal{B}_4 := \{0, a, b, 1\}$ the four-element Boolean algebra.
- $\mathcal{R} := \{p, q\}$ with the operations ;, $\sqcup$ and $^*$ (see belowe).
- $\langle\,\rangle\colon \mathcal{R} \times \mathcal{B} \to \mathcal{B}$ defines by: $\langle p\rangle a = 1$, $\langle p\rangle x = x$ for all $x \in \mathcal{B}_4 - \{1\}$; $\langle q\rangle x = x$ for all $x \in \mathcal{B}_4$.

Notice that the operations ;, $\sqcup$ and $^*$ are determined by the fact that the algebra $\mathcal{D}_4 = \langle \mathcal{B}_4, \mathcal{R}, \langle\,\rangle\rangle$ is a dynamic algebra. For instance, it follows that

$$\langle p \sqcup q\rangle x = \langle p\rangle x \vee \langle q\rangle x = \begin{cases} 1 & \text{if } x = a \\ x & \text{if } x \neq a \end{cases}$$

Hence, $p \sqcup q = p$. Also, $q \sqcup p = p$. Analogously, $p; q = q; p = p$. We also have

$$\langle p^*\rangle x = \min\{y \in \mathcal{B}_4 : x \vee \langle y\rangle \leq y\} = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = a \\ b & \text{if } x = b \\ 1 & \text{if } x = 1 \end{cases}$$

Then, $p^* = p$. Also $q^* = q$.

Now let $\mathcal{D}_2 := \langle \mathcal{B}_2, \mathcal{R}, \langle\,\rangle\rangle$ be as follows:

- $\mathcal{B}_2 := \{0, 1\}$ the two-element Boolean algebra;
- $\mathcal{R} := \{p, q\}$ with the operations ;, $\sqcup$ and $^*$ as given before;
- $\langle p\rangle x = x$ and $\langle q\rangle x = x$ for all $x \in \mathcal{B}_2$.

Notice that $\mathcal{D}_2$ is not separable: $\langle p\rangle x = \langle q\rangle x$ for all $x \in \mathcal{B}_2$, but $p \neq q$. It is clear that $\mathcal{D}_2$ is a two-sorted subalgebra of $\mathcal{D}_4$. Let $(f, g)\colon \mathcal{D}_4 \to \mathcal{D}_2$ be defined by $f(a) = f(1) = 1$ and $f(b) = f(0) = 0$, and $g = id_{\mathcal{R}}$. It follows that $(f, g)$ is an onto two-sorted homomorphism. Therefore, the class $\mathbb{SDA}$ is not closed under subalgebras and homomorphic images. $\square$

In Section 3.2, we gave an example of a dynamic algebra which is not embeddable into any complex dynamic algebra. The following theorem shows that every separable finite dynamic algebra is actually isomorphic to a complex dynamic algebra. We need some technical considerations.

Let $\mathcal{D} = \langle \mathcal{B}, \mathcal{R}, \langle\,\rangle \rangle$ be a dynamic algebra. For every $p \in \mathcal{R}$, we define $p^0$ to be an element of $\mathcal{R}$ such that $\langle p^0 \rangle a = a$ for all $a \in \mathcal{B}$. That is, for all $p \in \mathcal{R}$, $\langle p^0 \rangle$ is the identity map on $\mathcal{B}$. From (D5) and (D5'), we have that

$$\langle p^0 \sqcup q \rangle a = \langle p^0 \rangle a \vee \langle q \rangle a = a \vee \langle q \rangle a \quad \text{and} \quad [p^0 \sqcup q]a = [p^0]a \wedge [q]a = a \wedge [q]a$$

for all $p, q \in \mathcal{R}$ and $a \in \mathcal{B}$.

**Theorem 3.3.2.** *Every finite separable dynamic algebra $\mathcal{D}$ is isomorphic to a complex dynamic algebra $\mathcal{D}(W, \mathcal{R})$ for some (finite) standard Kripke frame $\langle W, \mathcal{R} \rangle$.*

**Author's comment 3.3.3.** This theorem corresponds to [14, Theorem 6.1]. In [14], in the previous paragraph to Theorem 6.1, it is mentioned that

*We first prove an easy result about finite Kripke structures.*

Then, Theorem 6.1 is proved by only a sketch of the proof. In this proof, the author uses Stone's representation for finite Boolean algebras and says

*[...], and by separability, $\mathcal{D}$ is isomorphic to a subalgebra of the full [...] dynamic algebra on the power set of the atoms of $\mathcal{B}$ [...].*

However, this is not a straightforward fact.

*Proof.* Let $\mathcal{D} = \langle \mathcal{B}, \mathcal{R}, \langle\,\rangle \rangle$ be a finite separable dynamic algebra. Let us work with the dual operator of $\langle\,\rangle$. Recall that for every $p \in \mathcal{R}$ and $a \in \mathcal{B}$, $[p]a = \neg\langle p \rangle \neg a$, see Proposition 3.1.3.

Let us build a standard Kripke frame $\langle W, \mathfrak{R} \rangle$. Let $W := \mathrm{Ult}(\mathcal{B})$ be the set of ultrafilters of $\mathcal{B}$. For every $p \in \mathcal{R}$, we define the binary relation $R_p \subseteq W \times W$ as follows: let $u, w \in W$,

$$(u, w) \in R_p \iff [p]^{-1}(u) \subseteq w (\iff w \subseteq \langle p \rangle^{-1}(u)). \tag{3.3.1}$$

Let $\mathfrak{R} = \{R_p : p \in \mathcal{R}\}$. Let us prove that $\mathfrak{R}$ is closed under set-theoretical union $\cup$ and composition $\circ$, and reflexive transitive closure $^*$. Let $p, q \in \mathcal{R}$.

(1) $R_p \cup R_q = R_{p \sqcup q}$. Let $u, w \in W$. By (D5'), it follows that $[p \sqcup q]^{-1}(u) = [p]^{-1}(u) \cap [q]^{-1}(u)$. Since $u$ is a filter of $\mathcal{B}$, it follows by (D3')–(D4') that $[p]^{-1}(u)$ and $[q]^{-1}(u)$ are filters of $\mathcal{B}$. Then, using that $w$ is an ultrafilter of $\mathcal{B}$, it follows that

$$\begin{aligned}
(u, w) \in R_{p \sqcup q} &\iff [p \sqcup q]^{-1}(u) \subseteq w \iff [p]^{-1}(u) \cap [q]^{-1}(u) \subseteq w \\
&\iff [p]^{-1}(u) \subseteq w \ \text{ or } \ [q]^{-1}(u) \subseteq w \\
&\iff (u, w) \in R_p \ \text{ or } \ (u, w) \in R_q \\
&\iff (u, w) \in R_p \cup R_q.
\end{aligned}$$

(2) $R_p \circ R_q = R_{p;q}$. Let $u, w \in W$. First notice that

$$(u, w) \in R_{p;q} \iff [p; q]^{-1}(u) \subseteq w \iff ([p][q])^{-1}(u) \subseteq w \iff [q]^{-1}\left([p]^{-1}(u)\right) \subseteq w.$$

- $(\subseteq)$ Assume that $(u, w) \in R_p \circ R_q$. Thus, there is $v \in W$ such that $(u, v) \in R_p$ and $(v, w) \in R_q$. Then $[p]^{-1}(u) \subseteq v$ and $[q]^{-1}(v) \subseteq w$. Hence $[q]^{-1}([p]^{-1}(u)) \subseteq [q]^{-1}(v) \subseteq w$. Therefore, $(u, w) \in R_{p;q}$.

- ($\supseteq$) Suppose that $(u, w) \in R_{p;q}$. Thus $[q]^{-1}\left([p]^{-1}(u)\right) \subseteq w$. From this and (D3')–(D4'), it follows that $[p]^{-1}(u)$ is a proper filter of $\mathcal{B}$. Then, there is at least an ultrafilter $v \in W$ such that $[p]^{-1}(u) \subseteq v$. Now let us suppose that for all $t \in W$ such that $[p]^{-1}(u) \subseteq t$, we have that $[q]^{-1}(t) \nsubseteq w$. Let

$$I := \mathrm{Idg}\left(\bigcup\left\{[q](a) : a \in [q]^{-1}(t) \setminus w, \text{ for every } t \in W \text{ and } [p]^{-1}(u) \subseteq t\right\}\right).$$

Suppose that $I \cap [p]^{-1}(u) = \emptyset$. Since $[p]^{-1}(u)$ is a filter of $\mathcal{B}$, there is an ultrafilter $t \in W$ such that $[p]^{-1}(u) \subseteq t$ and $t \cap I = \emptyset$. Then, there is $a \in [q]^{-1}(t) \setminus w$. Thus $[q](a) \in t$. On the other hand, by definition of $I$, we have that $[q](a) \in I$, which is absurd. Hence, it follows that $I \cap [p]^{-1}(u) \neq \emptyset$. Then, there is $b \in I \cap [p]^{-1}(u)$. Thus, there are $t_1, \ldots, t_n \in W$ and $a_1, \ldots, a_n \in [q]^{-1}(t_i) \setminus w$ with $[p]^{-1}(u) \subseteq t_i$, for all $i = 1, \ldots, n$, such that

$$b \leq [q](a_1) \vee \cdots \vee [q](a_n) \leq [q](a_1 \vee \cdots \vee a_n).$$

Then $[q](a_1 \vee \cdots \vee a_n) \in [p]^{-1}(u)$. Thus $a_1 \vee \cdots \vee a_n \in [q]^{-1}\left([p]^{-1}(u)\right) \subseteq w$. Since $w$ is an ultrafilter, there is $a_i \in w$, which is a contradiction. Hence, we conclude that there is $v \in W$ such that $[p]^{-1}(u) \subseteq v$ and $[q]^{-1}(v) \subseteq w$. That is, $(u, v) \in R_p$ and $(v, w) \in R_q$. Therefore, $(u, w) \in R_p \circ R_q$.

(3) $R_p^* = R_{p^*}$.

- ($\subseteq$) Let us show that $R_{p^*}$ is a reflexive transitive relation on $W$ containing $R_p$.
  - Reflexive: Let $u \in W$. By (D7'), it follows that $[p^*]^{-1}(u) \subseteq u$. Hence $(u, u) \in R_{p^*}$.
  - Transitive: Suppose that $(u, v), (v, w) \in R_{p^*}$. Thus $[p^*]^{-1}(u) \subseteq v$ and $[p^*]^{-1}(v) \subseteq w$. Let $a \in [p^*]^{-1}(u)$. Thus, by (P1), $[p^*][p^*]a = [p^*]a \in u$. Then $[p^*]a \in [p^*]^{-1}(u) \subseteq v$. It follows that $a \in [p^*]^{-1}(v) \subseteq w$. Hence $[p^*]^{-1}(u) \subseteq w$. That is, $(u, w) \in R_{p^*}$.
  - $R_p \subseteq R_{p^*}$: Let $(u, v) \in R_p$. Thus $[p]^{-1}(u) \subseteq v$. Let $a \in [p^*]^{-1}(u)$. By (D7'), it follows that $[p][p^*]a \in u$. Thus $[p^*]a \in [p]^{-1}(u) \subseteq v$. Then, by (D7'), $a \in v$. Hence $[p^*]^{-1}(u) \subseteq v$. That is, $(u, v) \in R_{p^*}$.

  Since $R_p^*$ is the reflexive transitive closure of $R_p$, it follows that $R_p^* \subseteq R_{p^*}$.

- ($\supseteq$)[6]. We know that $R_p^* = R_p^0 \cup R_p \cup R_p^2 \cup \ldots$, where $R_p^0 = \Delta = \{(u, u) : u \in W\}$. Since $\mathcal{B}$ is finite, we have that $W = \mathrm{Ult}(\mathcal{B})$ is finite, and thus there are only finitely many binary relations on $W$. Hence, there is $n \in \mathbb{N}$ such that $R_p^* = R_p^0 \cup R_p \cup \cdots \cup R_p^n$. For what we just proved,

$$R_p^* = R_p^0 \cup R_p \cup \cdots \cup R_p^n = R_{p^0} \cup R_p \cup \cdots \cup R_{p^n} = R_{p^0 \sqcup p \sqcup \cdots \sqcup p^n}. \tag{3.3.2}$$

On the other hand, by Proposition 3.1.17, there is $k \in \mathbb{N}$ such that $a \wedge [p]a \wedge \cdots \wedge [p^k]a \leq [p^*]a$, for all $a \in \mathcal{B}$.

[6]Here is where the hypothesis that the dynamic algebra is finite and separable is fundamental.

– Assume that $n \geq k$. Let $(u, w) \in R_{p^*}$. So $[p^*]^{-1}(u) \subseteq w$. Let $a \in [p^0 \sqcup p \sqcup \cdots \sqcup p^n]^{-1}(u)$. Then, by (D5'), $a \wedge [p]a \wedge \cdots \wedge [p^n]a = [p^0 \sqcup p \sqcup \cdots \sqcup p^n](a) \in u$. Since $n \geq k$, it follows that

$$a \wedge [p]a \wedge \cdots \wedge [p^k]a \wedge \cdots \wedge [p^n]a \leq a \wedge [p]a \wedge \cdots \wedge [p^k]a \leq [p^*]a.$$

Then $[p^*]a \in u$. Thus $a \in [p^*]^{-1}(u) \subseteq w$. Hence, $[p^0 \sqcup p \sqcup \cdots \sqcup p^n]^{-1}(u) \subseteq w$. By (3.3.2), $(u, w) \in R_p^*$.

– Assume that $n < k$. Thus

$$R_p^* = R_p^0 \cup R_p \cup \cdots \cup R_p^n = R_p^0 \cup R_p \cup \cdots \cup R_p^n \cup \cdots \cup R_p^k = R_{p^0 \sqcup p \sqcup \cdots \sqcup p^n \sqcup \cdots \sqcup p^k}.$$

Let $(u, w) \in R_{p^*}$. So $[p^*]^{-1}(u) \subseteq w$. Let $a \in [p^0 \sqcup p \sqcup \cdots \sqcup p^k]^{-1}(u)$. Then $[p^*]a \in u$, and thus $a \in w$. Then $[p^0 \sqcup p \sqcup \cdots \sqcup p^k]^{-1}(u) \subseteq w$. Hence $(u, w) \in R_p^*$.

Therefore, $R_{p^*} \subseteq R_p^*$.

Hence, we have proved that $\langle W, \mathfrak{R} \rangle$ is a standard Kripke frame. Let $\mathcal{D}(W, \mathfrak{R}) = \langle \mathcal{P}(W), \mathfrak{R}, \langle\,\rangle \rangle$ be the complex dynamic algebra associated with $\langle W, \mathfrak{R} \rangle$. Now let us define $(f, g) \colon \langle \mathcal{B}, \mathcal{R}, \langle\,\rangle \rangle \to \langle \mathcal{P}(W), \mathfrak{R}, \langle\,\rangle \rangle$ as follows:

- let $f \colon \mathcal{B} \to \mathcal{P}(W)$ be defined by: for every $a \in \mathcal{B}$, $f(a) = \{u \in W : a \in u\}$;
- let $g \colon \mathcal{R} \to \mathfrak{R}$ be defined by: for every $p \in \mathcal{R}$, $g(p) = R_p$.

Since $\mathcal{B}$ is finite, we know that $f$ is a Boolean isomorphism. From what we just proved $g$ is an onto (homogeneous) homomorphism from $\langle \mathcal{R}, ;, \sqcup, ^* \rangle$ onto $\langle \mathfrak{R}, \circ, \cup, ^* \rangle$. Let us show that $g$ is injective. Let $p, q \in \mathcal{R}$ and assume that $p \neq q$. Since $\mathcal{D}$ is separable, there is $a \in \mathcal{B}$ such that $[p]a \neq [q]a$. Suppose that $[p]a \nleq [q]a$. Then, there exists an ultrafilter $u \in W$ such that $[p]a \in u$ and $[q]a \notin u$. That is, $a \in [p]^{-1}(u)$ and $a \notin [q]^{-1}(u)$. Since $[q]^{-1}(u)$ is a filter of $\mathcal{B}$, there exists an ultrafilter $v \in W$ such that $[q]^{-1}(u) \subseteq v$ and $a \notin v$. Then, $(u, v) \in R_q$ and $(u, v) \notin R_p$. Hence $g(p) \neq g(q)$.

Finally, we show that $f([p]a) = [g(p)]f(a)$ for all $p \in \mathcal{R}$ and $a \in \mathcal{B}$. Let $p \in \mathcal{R}$ and $a \in \mathcal{B}$. Notice that

$$f([p]a) = \{u \in W : [p]a \in u\} = \{u \in W : a \in [p]^{-1}(u)\}$$

and

$$[g(p)]f(a) = [R_p]f(a) = \{u \in W : R_p(u) \subseteq f(a)\}.$$

Then

$$\begin{aligned} u \in f([p]a) &\iff a \in [p]^{-1}(u) \\ &\iff \forall v \in \mathrm{Ult}(\mathcal{B})([p]^{-1}(u) \subseteq v \implies a \in v) \\ &\iff \forall v \in R_p(u),\ v \in f(a) \\ &\iff R_p(u) \subseteq f(a) \\ &\iff u \in [g(p)]f(a). \end{aligned}$$

Therefore, $(f, g) \colon \langle \mathcal{B}, \mathcal{R}, \langle\,\rangle \rangle \to \langle \mathcal{P}(W), \mathfrak{R}, \langle\,\rangle \rangle$ is a (two-sorted) isomorphism. □

**Author's comment 3.3.4.** Si bien muchos de los argumentos dados en la prueba anterior pueden omitirse por considerarse *faciles de deducir o de rutina* y achicar la prueba, el argumento para mostrar que $R_{p^*} \subseteq R_p^*$ no es tan trivial.

**Author's comment 3.3.5.** Podemos notar que toda la prueba anterior sirve para cualquier algebra dinamica separable $\mathcal{D}$ (no necesariamente finita) salvo el punto donde hay que porbar que $R_{p^*} \subseteq R_p^*$. Y la función $f$ en lugar de ser un isomorfismo es un embedding. Por lo tanto, se puede obtener un teorema de representacion para las algebras dinamicas sin la operacion $^*$. Es decir, para el fragmente $\{;, \sqcup\}$.

**Author's comment 3.3.6.** The following theorem is stronger than [14, Theorem 5.4] (in the sense of our theorem implies of that in [14]) and our proof seems to be simpler of that in [14].

We need an auxiliary result.

**Lemma 3.3.7.** *Let $\mathcal{D} = \langle \mathcal{B}, \mathcal{R}, \langle\,\rangle \rangle$ be a dynamic algebra. Let $\theta$ be a (homogeneous) congruence on $\mathcal{B}$. Then, $\mathcal{D}/\theta := \langle \mathcal{B}/\theta, \mathcal{R}, \langle\,\rangle_\theta \rangle$ is a dynamic algebra, where for all $a \in \mathcal{B}$ and $p \in \mathcal{R}$, $\langle p \rangle_\theta (a/\theta) := (\langle p \rangle a)/\theta$.*

*Proof.* It is straightforward to show the conditions (D1)–(D8). □

**Theorem* 3.3.1.** *Every dynamic algebra is subdirect product of two-boolean dynamic algebras.*

*Proof.* Let $\mathcal{D} = \langle \mathcal{B}, \mathcal{R}, \langle\,\rangle \rangle$ be a dynamic algebra. Let $J := \{\{a, b\} : a, b \in \mathcal{B}$ such that $a \neq b\}$. For every $\{a, b\} \in J$, let $U_{ab} \in \mathrm{Ult}(\mathcal{B})$ be such that $U_{ab}$ contains one and only one of $a$ and $b$. Let $\theta_{ab}$ be the (homogeneous) congruence on $\mathcal{B}$ defined by $U_{ab}$ as is usual:

$$(x, y) \in \theta_{ab} \iff x \to y,\ y \to x \in U_{ab} \iff x, y \in U_{ab} \text{ or } x, y \notin U_{ab}.$$

Notice that $(a, b) \notin U_{ab}$. For every $\{a, b\} \in J$, we have that $\mathcal{B}/\theta_{ab} \cong \mathbf{2}$. Since $\Delta = \bigcap\{\theta_{ab} : \{a, b\} \in J\}$, it follows that the Boolean algebra $\mathcal{B}$ is (homogeneous) subdirect product of the family $\{\mathcal{B}/\theta_{ab} : \{a, b\} \in J\}$. Let $f \colon \mathcal{B} \to \prod_{j \in J} \mathcal{B}/\theta_j$ be the subdirect embedding. That is, for every $a \in \mathcal{B}$ and $j \in J$, $f(a)(j) = a/\theta_j$.

Now for every $j \in J$, let $\mathcal{D}_j = \langle \mathcal{B}/\theta_j, \mathcal{R}, \langle\,\rangle_{\theta_j} \rangle$ be the dynamic algebra given by Lemma 3.3.7. Now let $g \colon \mathcal{R} \to \mathcal{R}^J$ be defined as follows: $g(p) = (p)_{j \in J}$. It is clear that $g$ is a (homogeneous) embedding. Now we show that $(f, g) \colon \langle \mathcal{B}, \mathcal{R}, \langle\,\rangle \rangle \to \left\langle \prod_{j \in J} \mathcal{B}/\theta_j, \mathcal{R}^J, \langle\,\rangle_P \right\rangle$ is a two-sorted homomorphism. Let $x \in \mathcal{B}$ and $p \in \mathcal{R}$. Then, for every $j \in J$,

$$f(\langle p \rangle x)(j) = (\langle p \rangle x)/\theta_j = \langle p \rangle_{\theta_j} x/\theta_j = \langle g(p)(j) \rangle_{\theta_j} x/\theta_j = (\langle g(p) \rangle_P f(x))(j).$$

Hence $f(\langle p \rangle x) = \langle g(p) \rangle_P f(x)$. Moreover, for every $j \in J$

$$(\pi_{j1} \circ f)(\mathcal{B}) = \mathcal{B}/\theta_j \qquad \text{and} \qquad (\pi_{j2} \circ g)(\mathcal{R}) = \mathcal{R}.$$

All this shows that $(f, g) \colon \mathcal{D} \to \prod_{j \in J} \mathcal{D}_j$ is a two-sorted subdirect embedding. □

**Corollary 3.3.8** ([14, Theorem 5.4]). *Every free dynamic algebra $\mathcal{D}$ is subdirect product of Boolean-finite dynamic algebras.*

**Corollary 3.3.9.** *The variety of dynamic algebras is generated by the class of all Boolean-finite dynamic algebras.*

**Author's comment 3.3.10.** The following theorem is also stronger than [14, Theorem 5.5] (in the sense of our theorem implies of that in [14]).

**Theorem* 3.3.2.** *Every separable dynamic algebra is subdirect product of finite separable dynamic algebras.*

*Proof.* Let $\mathcal{D} = \langle \mathcal{B}, \mathcal{R}, \langle\,\rangle\rangle$ be a separable dynamic algebra. Let $J$, $\theta_j$ for $j \in J$, $f\colon \mathcal{B} \to \prod_{j\in J} \mathcal{B}/\theta_j$, and $\mathcal{D}_j = \langle \mathcal{B}/\theta_j, \mathcal{R}, \langle\,\rangle_{\theta_j}\rangle$ as in Theorem 3.3.1. Now for every $j \in J$, let $\widehat{D_j} = \langle \mathcal{B}/\theta_j, \mathcal{R}/{\cong_j}, \langle\,\rangle_{\theta_j}\rangle$ be the separable dynamic algebra given as in Remark 3.1.10. Notice that for every $p \in \mathcal{R}$ and $x \in \mathcal{B}$,

$$\langle p/{\cong_j}\rangle_{\theta_j} x/\theta_j = \langle p\rangle_{\theta_j} x/\theta_j = (\langle p\rangle a)/\theta_j.$$

Now let us show that $\Delta_{\mathcal{R}} = \bigcap\{\cong_j \colon j \in J\}$. The inclusion $\subseteq$ is trivial. Let $p, q \in \mathcal{R}$ and assume that $p \neq q$. Since $\mathcal{D}$ is separable, there is $c \in \mathcal{B}$ such that $\langle p\rangle c \neq \langle q\rangle c$. Since $\Delta_{\mathcal{B}} = \bigcap\{\theta_j : j \in J\}$, there is $j \in J$ such that $(\langle p\rangle c)/\theta_j \neq (\langle q\rangle c)/\theta_j$. Thus $\langle p\rangle_{\theta_j} c/\theta_j \neq \langle q\rangle_{\theta_j} c/\theta_j$. Then, $p \ncong_j q$. Hence, $\mathcal{R}$ is (homogeneous) subdirect product of the family $\{\mathcal{R}/{\cong_j} : j \in J\}$. Let $g\colon \mathcal{R} \to \prod_{j\in J} \mathcal{R}/{\cong_j}$ be the subdirect embedding, that is, $g(p) = p/{\cong_j}$ for all $p \in \mathcal{R}$. It only remains to show that $f(\langle p\rangle a) = \langle g(p)\rangle_P f(a)$ for all $p \in \mathcal{R}$ and $a \in \mathcal{B}$. Let $p \in \mathcal{R}$ and $a \in \mathcal{B}$. Then, for every $j \in J$,

$$\begin{aligned} f(\langle p\rangle a)(j) &= (\langle p\rangle a)/\theta_j \\ &= \langle p/{\cong_j}\rangle_{\theta_j} a/\theta_j \\ &= \langle g(p)\rangle_{\theta_j} f(a)(j) \\ &= (\langle g(p)\rangle f(a))(j). \end{aligned}$$

Hence, $(f, g)\colon \langle \mathcal{B}, \mathcal{R}, \langle\,\rangle\rangle \to \left\langle \prod_{j\in J} \mathcal{B}/\theta_j, \prod_{j\in J} \mathcal{R}/{\cong_j}, \langle\,\rangle_P \right\rangle$ is a two-sorted subdirect produc, where every $\widehat{\mathcal{D}_j} = \langle \mathcal{B}/\theta_j, \mathcal{R}/{\cong_j}, \langle\,\rangle_{\theta_j}\rangle$ is a finite separable dynamic algebra. $\square$

**Corollary 3.3.11** ([14, Theorem 5.5]). *Every free separable dynamic algebra is subdirect product of finite separable dynamic algebras.*

**Theorem* 3.3.3.** *We have*

$$\mathrm{HSP}(\mathbb{SDA}) = \mathrm{HSP}(\mathbb{SDA}_f) = \mathrm{HSP}(\mathbb{CDA}_f) = \mathrm{HSP}(\mathbb{CDA}).$$

*Proof.* Since $\mathbb{SDA}_f \subseteq \mathbb{SDA}$, it follows that $\mathrm{HSP}(\mathbb{SDA}_f) \subseteq \mathrm{HSP}(\mathbb{SDA})$. By Theorem 3.3.2, we have

$$\mathbb{SDA} \subseteq \mathrm{Ps}(\mathbb{SDA}_f) \subseteq \mathrm{SP}(\mathbb{SDA}_f) \subseteq \mathrm{HSP}(\mathbb{SDA}_f).$$

Then, $\mathrm{HSP}(\mathbb{SDA}) \subseteq \mathrm{HSP}(\mathbb{SDA}_f)$. Hence $\mathrm{HSP}(\mathbb{SDA}) = \mathrm{HSP}(\mathbb{SDA}_f)$.

By Theorem 3.3.2 and Example 3.2.2, it follows that $\mathbb{SDA}_f = \mathrm{I}(\mathbb{CDA}_f)$. Then $\mathrm{HSP}(\mathbb{SDA}_f) = \mathrm{HSP}(\mathbb{CDA}_f)$.

Now, since $\mathbb{CDA}_f \subseteq \mathbb{CDA}$, it follows that $\mathrm{HSP}(\mathbb{SDA}) = \mathrm{HSP}(\mathbb{CDA}_f) \subseteq \mathrm{HSP}(\mathbb{CDA})$. On the other hand, by Example 3.2.2 we obtain that $\mathbb{CDA} \subseteq \mathbb{SDA}$. Then $\mathrm{HSP}(\mathbb{CDA}) \subseteq \mathrm{HSP}(\mathbb{SDA})$. Hence $\mathrm{HSP}(\mathbb{SDA}) = \mathrm{HSP}(\mathbb{CDA})$. □

**Remark 3.3.12.** The class $\mathbb{DA}^*$ of $^*$-continuous dynamic algebras is not a variety. Suppose it does. That is, suppose that $\mathbb{DA}^* = \mathrm{HSP}(\mathbb{DA}^*)$. By the previous theorem we have that $\mathbb{SDA} \subseteq \mathrm{HSP}(\mathbb{CDA})$, and by Example 3.2.6, we have $\mathbb{CDA} \subseteq \mathbb{DA}^*$. Thus $\mathbb{SDA} \subseteq \mathrm{HSP}(\mathbb{CDA}) \subseteq \mathrm{HSP}(\mathbb{DA}^*) = \mathbb{DA}^*$. Then $\mathbb{SDA} \subseteq \mathbb{DA}^*$, which is a contradiction by Example 3.2.9.

In [14, Example 2], the author defines a class $\mathbb{KRI}$ of dynamic algebras, which are called *Kripke structures*. The class $\mathbb{KRI}$ is defined as follows:

$$\mathbb{KRI} := \mathrm{S}(\mathbb{FCDA}).$$

That is, $\mathbb{KRI}$ is the closure under subalgebras of the full complex dynamic algebras. Notice that, $\mathrm{S}(\mathbb{FCDA}) = \mathrm{S}(\mathbb{CDA})$[7]. Hence, from the above result, we can derive the following result given in [14].

**Corollary 3.3.13** ([14, Theorem 6.4])**.** $\mathrm{HSP}(\mathbb{SDA}) = \mathrm{HSP}(\mathbb{KRI})$.

**Author's comment 3.3.14.** In [14, pp. 589], the author add a footnote claimed that:

*In fact every free dynamic algebra is separable, as Némit [**?**] subsequently pointed out. This is immediate consequence of the initiality of free algebras [...].*

What means "of the initiality of free algebras"?

In [**?**] it is proved that every free $\mathrm{HSP}(\mathbb{SDA})$-algebra is separable. However, in [**?**], it is not mention at all that every free dynamic algebra is separable.

So, we ask:

Is every free dynamic algebra separable?

[7]On the one hand, it is clear that $\mathbb{FCDA} \subseteq \mathbb{CDA}$. Hence $\mathrm{S}(\mathbb{FCDA}) \subseteq \mathrm{S}(\mathbb{CDA})$. On the other hand, by definition of $\mathbb{CDA}$, it follows that $\mathbb{CDA} \subseteq \mathrm{S}(\mathbb{FCDA})$. Hence $\mathrm{S}(\mathbb{CDA}) \subseteq \mathrm{S}(\mathbb{FCDA})$.

# Chapter 4

# Cosas que ir agregando

- La definición de $\alpha^n$ para $n \in \mathbb{N}$.
- consistencia entre $\mathcal{M} \Vdash \varphi$ y $\mathcal{M} \vDash \varphi$
- decidir si usar $\langle \mathcal{B}, \mathcal{R}, \langle\rangle \rangle$ o $\langle \mathcal{B}, \mathcal{R}, \langle\ \rangle \rangle$
- Agregar un comentarios sobre los conceptos basicos como Boolean algebras, etc y referencias correspondientes.

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