Concurrent Heyting algebras: An algebraic semantics for a intuitionistic modal logic

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ABSTRACT
We semantically define an intuitionistic bimodal logic through certain neighbourhood structures. We call these neighbourhood structures ic-frames. We propose a Hilbert-style axiomatization and prove a completeness theorem with respect to the class of ic-frames. We introduce a class of algebras called concurrent Heyting algebras and prove an algebraic completeness theorem for this class. We show a categorical dual equivalence between a category whose objects are concurrent Heyting algebras and a category of Esakia spaces equipped with an ic-frame structure (see Section 6 for the corresponding morphisms). We present an alternative relational semantics for our logic through specific structures that mix Kripke-semantics and neighbourhood-semantics for the interpretation of the modal operators. We call these structures intuitionistic Kripke Neighbourhood frames (IKN-frames). We show that the category of IKN-frames with certain morphisms is isomorphic to the category of ic-frames.

KEYWORDS
Neighbourhood semantics; Kripke semantics; Intuitionistic modal logic; Heyting algebra; Completeness

1. Introduction

This paper undertakes an algebraic and topological study of an intuitionistic modal logic, denoted by $\Lambda(\mathcal{H})$, with two modal operators: a normal modal operator $\Box$ and a monotone modal operator $\Diamond$. As in many intuitionistic modal logics, the operators are independent but linked through certain axioms or rules. The primary motivation for studying this logic arises from several sources, which we proceed to explain.

The minimal modal logic **K** is an extension of the classical propositional by means of a modal operator $\Box$ together with certain axioms and rules. The logic **K** is usually interpreted using Kripke frames, i.e., pairs $\langle X, S\rangle$ where X is a set and S is a binary relation on X. If $\langle \mathcal{P}(X), \Box_S\rangle$ is the full complex modal algebra of a Kripke frame $\langle X, S\rangle$, where $\Box_S(U) = \{x \in X : S(x) \subseteq U\}$, for each $U \in \mathcal{P}(X)$, then a formula φ in the modal languaje $\{\vee, \wedge, \neg, \Box, \}$ is valid in $\langle X, S\rangle$ if and only if it is valid in $\langle \mathcal{P}(X), \Box_S\rangle$ (Blackburn, De Rijke, & Venema (2002)). In other words, the logic **K** is

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precisely the logic generated by the class of full complex algebras

$$\{\langle \mathcal{P}(X), \Box_S \rangle : \langle X, S \rangle \text{ is a Kripke frame}\}.$$

In the classical case, the modal operator $\Diamond$ is interpreted as $\Diamond_S(U) = \Box_S(U^c)^c$. This interdefinability of the modal operators does not persist in many non-classical logics, in particular in extensions of intuitionistic logic **Int**. Consequently, there exist various intuitionistic modal logics, depending on whether the language admits only one operator or both, and on the possible semantic interpretations of these operators (Božić & Došen (1984), Fischer Servi (1980), Fischer Servi (1984), Orłowska & Rewitzky (2007), Wijesekera (1990), Ono (1977) , Sotirov (1980)).

Neighbourhood frames are a generalisation of Kripke frames and are generally used as a semantic framework for non-normal modal logics (S. A. Celani (2009); Pacuit (2017)). A neighbourhood frame is a pair $\langle X, N \rangle$, where X is a non-empty set and N is a relation between X and $\mathcal{P}(X)$. Given any neighbourhood frame $\langle X, N \rangle$, one can define a monotone modal operator

$$\Diamond_N(U) = \{x : \exists Y \in N(x) \ \ (Y \subseteq U)\}. \tag{1.1}$$

This definition yields a *monotone modal algebra* $\langle \mathcal{P}(X), \Diamond_N \rangle$ called the full complex algebra of the frame. The variety generated by the class of such algebras $\langle \mathcal{P}(X), \Diamond_N \rangle$ is the variety MonBA of monotone Boolean algebras (S. A. Celani (2009); Pacuit (2017)).

Neighbourhood frames can also be used to interpret the modal logic **K**, although this is not the standard approach. The normal modal operator $\Box$ can be interpreted in a neighbourhood frame $\langle X, N \rangle$ as follows: for each $U \in \mathcal{P}(X)$ define

$$\Box_N(U) = \{x \mid \forall Y \in N(x) \ \ (Y \subseteq U)\}. \tag{1.2}$$

It is easy to see that $\langle \mathcal{P}(X), \Box_N \rangle$ is a modal algebra. Hence, the modal logic **K** is also characterized by the class of all neighbourhood frames, when the modal operator $\Box$ is interpreted according to clause (1.2). Thus, neighbourhood frames can be used to interpret both the modal logic **K** and the minimal monotone logic, or a mixture of these two modal logics. In fact, the interpretation of the modal operators $\Box$ and $\Diamond$ by means of the clauses (1.1) and (1.2), respectively, is not new. The literature contains several modal logics in the language with both operators $\Box$ and $\Diamond$, designed to capture different notions: ability and action logics (Brown (1988); Segerberg (1997)), knowledge and belief logics (Askounis, Koutras, & Zikos (2016); Koutras, Moyzes, & Zikos (2017)) and dynamic logic (Goldblatt (1992); Vakarelov (2012)), as well as constructive approaches such as those developed in Wijesekera (1990) and Wijesekera & Nerode (2005). In light of the preceding considerations, a natural question arises: which variety of algebras in the language with two unary operators $\Box$ and $\Diamond$ is generated by the class of full complex algebras

$$\{\langle \mathcal{P}(X), \Box_N, \Diamond_N \rangle : \langle X, N \rangle \text{ is a neighbourhood frame}\}? \tag{1.3}$$

The answer can be found in S. Celani (2011), where the class of concurrent algebras is introduced. A concurrent algebra is a Boolean algebra A endowed with two unary operators $\Box$ and $\Diamond$ such that $\langle A, \Box \rangle$ is a normal modal algebra, $\langle A, \Diamond \rangle$ is a monotonic modal algebra, and certain additional conditions connect the operators $\Box$ and $\Diamond$ (in

S. Celani (2011) the operator $\Diamond$ is denoted by ∇). As shown in S. Celani (2011), the variety of concurrent algebras is generated by the class of full complex algebras (1.3). Its main motivation is that it provides the algebraic semantics for a simplified fragment of Concurrent Propositional Dynamic Logic (**CPDL**) introduced by R. Goldblatt in Goldblatt (1992). This fragment of **CPDL** is the same logic $\mathcal{V}$ studied by M. Brown in Brown (1988), and consequently, the variety of concurrent algebras also serves as its algebraic semantics.

Hence, it is natural to consider some *intuitionistic* version of the logic $\mathcal{V}$, or the simplified fragment of **CPDL**. This issue is the main objective of this paper. We introduce and study an intuitionistic modal logic, denoted by $\Lambda(\mathcal{H})$, with a normal modal operator $\Box$ and a monotone modal operator $\Diamond$. The logic $\Lambda(\mathcal{H})$ is closely related to the modal logics studied by Wijesekera in Wijesekera (1990), but we show that our logic differs from the one presented there. $\Lambda(\mathcal{H})$ is first semantically defined using structures of the form $\langle X, \leq, R\rangle$, where $\langle X, \leq\rangle$ is a poset and $R \subseteq X \times \mathsf{Up}(X)$, with $\mathsf{Up}(X)$ denoting the set of all upsets of X (see in Preliminaries). We call theses structures $\langle X, \leq, R\rangle$ *ic-frames* (see Definition 3.1). These frames generalize the neighbourhood frames used in S. Celani (2011) to study the representation of concurrent algebras. Since we work in an intuitionistic setting, the interpretation of the modal operators in ic-frames must be modified (see Definition 3.1). However, when in an ic-frame the order $\leq$ is the equality we recover the concurrent frames defined in (S. Celani, 2011, Definition 4.1).

We propose a Hilbert-style axiomatization for our logic, and we prove a completeness theorem concerning the class of all ic-frames. We introduce and study a class of algebras (called *concurrent Heyting algebras*) that serve as an algebraic semantics for our logic. We develop a topological representation and a categorical dual equivalence between the category of concurrent Heyting algebras with algebraic homomorphisms and the category whose objects are Esakia spaces (Bezhanishvili (2014)) equipped with neighbourhood-frame structure and whose morphisms are Esakia morphisms satisfying extra conditions. Concerning the algebraic and categorical aspects treated here, we are strongly motivated by the results given in S. Celani (2011). We generalize several concepts and results developed in S. Celani (2011).

The paper is organized as follows. In section 2, we present some concepts and introduce notations necessary for the article's development. Section 3 defines specific neighborhood structures, which we call ic-frames. Then, we use the ic-frames to define semantically an intuitionistic bimodal logic. We give an axiomatization through a Hilbert-style system. We prove soundness and announce completeness, whose proof is postponed to Section 5. In Section 4, we present the concurrent Heyting algebras or CH-algebras, and then we prove that the logic given in Section 3 is complete with respect to this class of algebras. We characterize the congruences of a CH-algebra and give a necessary and sufficient condition for a CH-algebra to be subdirectly irreducible. In Section 5, we present a representation theorem for CH-algebras using ic-frames. According to Theorem 5.4, the variety **CHA** of CH-algebras is generated by the class of complex CH-algebras. That is, if $\mathcal{F} = \langle X, \leq, R\rangle$ is an ic-frame and $A_{\mathcal{F}} = \langle \mathsf{Up}(X), \Box_R, \Diamond_R\rangle$ is the complex CH-algebra associated with $\mathcal{F}$, then $\mathbf{CHA} = \mathcal{V}(\{A_{\mathcal{F}} : \mathcal{F} \text{ is an ic-frame}\})$. We use this representation theorem and the algebraic completeness (Section 4) to prove the relational completeness announce in Section 3. Section 6 introduces a class of Esakia spaces equipped with an ic-frame structure. We called concurrent Esakia spaces. We show how to construct a concurrent Esakia space from a CH-algebra and how to obtain a CH-algebra from a concurrent Esakia space. We define the corresponding morphisms between concurrent Esakia spaces. Then, we formally define the category of CH-algebras with algebraic homomorphisms as morphisms and the category of concurrent Esakia spaces. We prove that these categories are dually equivalent. In Section 7, we present an alternative relational semantics for the logic defined in Section 3 and prove a completeness theorem. For this, we define a new class of frames called IKN-frames, and based on these, we present a new kind of topological space called IKN-spaces. Also, we define morphisms between IKN-spaces and formally define the category of IKN-spaces. Then, we prove that this category is isomorphic to the category of concurrent Esakia spaces. Section 8 presents some conclusions.

2. Preliminaries

In this section, we establish the foundational concepts and notation used in this article.

We begin with category theory. We assume the reader is familiar with the elementary concepts of category theory. For instance, category, (contravariant) functor, natural isomorphism, and equivalence between categories. Our main references for category theory are Mac Lane (1998); Pierce (1991). Let $\mathbb{C}$ be a category. We denote by $\mathcal{O}b(\mathbb{C})$ the collection of all objects of the category $\mathbb{C}$. If $X, Y \in \mathcal{O}b(\mathbb{C})$, we denote by $\mathcal{M}_{\mathbb{C}}(X, Y)$ the set of all morphisms between X and Y.

We assume that the reader is familiar with theoretical notions on distributive lattices and Heyting algebras. Our main references for Order and Lattice Theory and Heyting algebras are Balbes & Dwinger (1974); Davey & Priestley (2002)). Let $\mathcal{P}(X)$ be the set of all subsets of the set X. $X - Y$ denotes the complement of $Y \subseteq X$, and when there is no danger of confusion, we denote it by Y^c. Let $\langle X, \leq \rangle$ be a poset. For each $Y \subseteq X$, let $[Y) = \{x \in X : \exists y \in Y(y \leq x)\}$ and $(Y] = \{x \in X : \exists y \in Y(x \leq y)\}$. We say that Y is an *upset* of X (a *downset* of X) if $Y = [Y)$ $(Y = (Y])$. If $Y = \{y\}$, then we write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively. We denote by $\mathsf{Up}(X)$ the collection of all upsets of X, and we denote by $\mathsf{Do}(X)$ the set of all downsets of X. If $\langle X, \leq \rangle$ is a poset, then $\langle \mathsf{Up}(X), \cup, \cap, \Rightarrow, X, \emptyset \rangle$ is a Heyting algebra (see Balbes & Dwinger (1974)) where

$$U \Rightarrow V = \{x \in X : [x) \cap U \subseteq V\} \tag{$*$}$$

for every $U, V \in \mathsf{Up}(X)$.

Let $\langle A, \vee, \wedge, 0, 1 \rangle$ be a bounded distributive lattice. As usual, the partial order $\leq$ associated with A is defined as follows: for any $x, y \in A$, $x \leq y$ if and only if $x \wedge y = x$. A subset F of A is a *filter* of A if $1 \in F$, F is an upset, and $x \wedge y \in F$ whenever $x, y \in F$. We denote by $\mathrm{Fi}(A)$ the set of all filters of A. Let $X \subseteq A$, the *filter generated* by X (that is, the least filter containing X) is given by $\mathrm{Fg}(X) = \{a \in A : \exists x_1, \cdots, x_n \in X(x_1 \wedge \cdots \wedge x_n \leq a)\}$. In particular, $\mathrm{Fg}(\{x\}) = [x)$. A *proper filter* F of A is a filter of A such that $F \neq A$. A *prime filter* F of A is a proper filter of A such that for all $a, b \in A$, if $a \vee b \in F$, then $a \in F$ or $b \in F$. We denote the set of all prime filters of A by $X(A)$.

We assume that the reader is familiar with the basic notions from topology. Our main reference for General Topology is Engelking (1989). Let $\langle X, \leq, \tau \rangle$ be an ordered topological space, that is, $\langle X, \leq \rangle$ is a poset and $\langle X, \tau \rangle$ is a topological space. We set the following collections:

- $\mathcal{C}\ell(X)$: the collection of all closed subsets of X;

- $\mathcal{CP}(X)$: the collection of all clopen (closed and open) subsets of X;
- $\mathcal{C}\ell\mathsf{Up}(X)$: the collection of all closed upsets of X;
- $\mathcal{CP}\mathsf{Up}(X)$: the collection of all clopen upsets of X;
- $\mathcal{C}\ell\mathsf{Do}(X)$: the collection of all closed downset of X;
- $\mathcal{CP}\mathsf{Do}(X)$: the collection of all clopen downset of X.

A *Priestley space* (see Davey & Priestley (2002); Priestley (1972)) is an ordered topological space $\langle X, \leq, \tau\rangle$ such that $\langle X, \tau\rangle$ is a compact topological space and the *Priestley separation axiom* is satisfied, which means that for every $x, y \in X$ such that $x \nleq y$ there exists a clopen upset U such that $x \in U$ and $y \notin U$. Priestley (Priestley (1972)) shows that there is a dual categorical equivalence between the category of bounded distributive lattices with $\{0,1\}$-lattice homomorphisms and the category of Priestley spaces with order-preserving continuous maps. We briefly sketch this dual equivalence and refer the reader to Davey & Priestley (2002); Gehrke & Van Gool (2024) for the missing details. Let A be a bounded distributive lattice. Consider the set $X(A)$ ordered by the set-theoretical inclusion. Let $\alpha\colon A \to \mathsf{Up}(X(A))$ be the map defined by

$$\alpha(a) = \{P \in X(A) : a \in P\}$$

which is an $\{0,1\}$-embedding of bounded distributive lattices. Let τ_A be the topology on the set $X(A)$ generated by the subbasis $\{\alpha(a) : a \in A\} \cup \{\alpha(a)^c : a \in A\}$. By Priestley's duality theorem, this construction yields a Priestley space $\langle X(A), \subseteq, \tau_A\rangle$, which is called the *dual Priestley space* of A. Let now $\langle X, \leq, \tau\rangle$ be a Priestley space. Then, $\langle \mathcal{CP}\mathsf{Up}(X), \cap, \cup, \emptyset, X\rangle$ is a bounded distributive space. Then, for all bounded distributive lattice A and all Priestley space X, we have that $\alpha\colon A \to \mathcal{CP}\mathsf{Up}(X(A))$ is an isomorphism of bounded distributive lattice, and $\epsilon\colon X \to X(\mathcal{CP}\mathsf{Up}(X))$ is an order-homeomorphism (that is, order-isomorphism and homeomorphism), where ϵ is defined as follows: $\epsilon(x) = \{U \in \mathcal{CP}\mathsf{Up}(X) : x \in U\}$ for all $x \in X$. An important application of Priestley's equivalence is the topological characterisation of the filters of a bounded distributive lattice, which shall be useful for us. If A is a bounded distributive lattice and X is its dual Priestley space, then the lattices $\mathrm{Fi}(A)$ and $\mathcal{C}\ell\mathsf{Up}(X)$ are dually isomorphic. More precisely, if $F \in \mathrm{Fi}(A)$, then $\widehat{F} = \{x \in X : F \subseteq x\} \in \mathcal{C}\ell\mathsf{Up}(X)$. Conversely, if $Y \in \mathcal{C}\ell\mathsf{Up}(X)$, the set $F_Y = \{a \in A : Y \subseteq \alpha(a)\} \in \mathrm{Fi}(A)$. Moreover, for every $F \in \mathrm{Fi}(A)$ and $Y \in \mathcal{C}\ell\mathsf{Up}(X)$ we get $F = F_{\widehat{H}}$ and $Y = \widehat{F_Y}$. Furthermore, a straightforward computation shows that for every $F, G \in \mathrm{Fi}(A)$, $F \subseteq G$ if and only if $\widehat{G} \subseteq \widehat{F}$.

Priestley's equivalence for bounded distributive lattices can be extended to obtain a dual categorical equivalence for Heyting algebras. Esakia independently developed a categorical duality for Heyting algebras through certain ordered topological spaces Esakia (1974), called nowadays *Esakia spaces*. His spaces are special Priestley spaces. We sketch this duality. We refer the reader to Bezhanishvili (2014); Esakia (1974, 2019) for more details about the duality of Heyting algebras.

Esakia space is a Priestley space $\langle X, \leq, \tau\rangle$ such that for every $U \in \mathcal{CP}(X)$, $(A] \in \mathcal{CP}(X)$. Then $\langle \mathcal{CP}\mathsf{Up}(X), \cup, \cap, \Rightarrow, X, \emptyset\rangle$ is a Heyting algebra, where $\Rightarrow$ is defined as in $(*)$. The Esakia space associated with a given Heyting algebra $\langle A, \wedge, \vee, \to, 0, 1\rangle$ is the dual Priestley space $\langle X(A), \subseteq, \tau_A\rangle$ of A. Moreover, the map $\alpha\colon A \to \mathcal{CP}\mathsf{Up}(X(A))$ is an isomorphism of Heyting algebras (that is, it satisfies $\alpha(a \to b) = \alpha(a) \Rightarrow \alpha(b)$). Now we proceed with showing that the dual correspondence holds for morphisms as well. A morphism between Heyting algebras is a Heyting homomorphism, that is, a map preserving the operations $\wedge$, $\vee$, $\rightarrow$, 0 and 1. Let $\langle X_1, \leq_1\rangle$ and $\langle X_2, \leq_2\rangle$ be posets. A map $f\colon X_1 \to X_2$ is called a *p-morphism* if satisfies:

(a) $x \leq_1 y$ then $f(x) \leq_2 f(y)$,
(b) $f(x) \leq_2 u$ then there exist $z \in X_1$ such that $x \leq_1 z$ and $f(z) = u$.

Let $\langle X_1, \leq_1, \tau_1\rangle$ and $\langle X_2, \leq_2, \tau_2\rangle$ be Esakia spaces. An *Esakia morphism* is a map $f\colon X_1 \to X_2$ such that it is a p-morphism and continuous. If $f\colon X_1 \to X_2$ is an Esakia morphism, then $f^*\colon \mathcal{CP}\mathsf{Up}(X_2) \to \mathcal{CP}\mathsf{Up}(X_1)$ defined as $f^*(U) = f^{-1}(U)$ is a Heyting homomorphism. Conversely, if A_1 and A_2 are two Heyting algebras and $h\colon A_1 \to A_2$ is a Heyting homomorphism, the map $h_*\colon X(A_2) \to X(A_1)$ defined by $h_*(x) = h^{-1}(x)$ is an Esakia morphism. With these, it is straightforward to show that the category of Heyting algebras with Heyting homomorphisms is dually equivalent to the category of Esakia spaces with Esakia morphisms.

We close this section with the construction of a hyperspace of a Priestley space. We refer the reader to Bezhanishvili, Harding, & Morandi (2023). Let $\langle X, \leq, \tau\rangle$ be a Priestley space. Let us define a topology on $\mathcal{C}\ell\mathsf{Up}(X)$. For every $U \in \mathcal{CP}\mathsf{Up}(X)$ and for every $V \in \mathcal{CP}\mathsf{Do}(X)$, we define the following sets:

$$D_U = \{Y \in \mathcal{C}\ell\mathsf{Up}(X) : Y \subseteq U\} \qquad \text{and} \qquad L_V = \{Y \in \mathcal{C}\ell\mathsf{Up}(X) : Y \cap V \neq \emptyset\}.$$

Let τ_v be the topology on $\mathcal{C}\ell\mathsf{Up}(X)$ generated by the subbasis:

$$\{D_U : U \in \mathcal{CP}\mathsf{Up}(X)\} \cup \{L_V : V \in \mathcal{CP}\mathsf{Do}(X)\}$$

It is straightforward to show that

(1) $D_{U_1} \cup D_{U_2} \subseteq D_{U_1 \cup U_2}$ and
(2) $L_{V_1} \cup L_{V_2} = L_{V_1 \cup V_2}$,

for all $U_1, U_2 \in \mathcal{CP}\mathsf{Up}(X)$ and for all $V_1, V_2 \in \mathcal{CP}\mathsf{Do}(X)$.

In (Bezhanishvili et al., 2023, Props. 3.2 and 3.4), the authors proved that if $\langle X, \leq, \tau\rangle$ is a Priestley space, then the hyperspace $\mathrm{Hy}(X) = \langle \mathcal{C}\ell\mathsf{Up}(X), \subseteq, \tau_v\rangle$ is also a Priestley space.

3. Syntax and neighbourhood semantic

In this section, we define semantically a fragment of intuitionistic concurrent dynamic logic (ICDL) with just one program. Then, we present an axiomatization of this fragment through a Hilbert-style system and prove soundness and completeness.

Our algebraic language is $\mathcal{L} = \{\wedge, \vee, \rightarrow, \Box, \Diamond, \bot\}$ of type (2, 2, 2, 1, 1, 0), and let Fm be the algebra of formulas built over a countably many propositional variables Var as usual. We denote the elements of Var by $p, q, r, \ldots$ and the elements of Fm by $\varphi, \psi, \chi, \ldots$. For us, a *logic* is a subset Λ of Fm. The elements of a logic Λ are called *theorems* of Λ.

We now present one of the central notions of this article: the *intuitionistic concurrent frames*. An intuitionistic concurrent frame consists of a set X, whose elements are called *states*, a partial order $\leq$, and a relation R relating states with sets of states, that is, $R \subseteq X \times \mathcal{P}(X)$. We use the class of all intuitionistic concurrent frames to define a logic, which is a conservative extension of intuitionistic propositional logic.

Definition 3.1. An *intuitionistic concurrent frame* (*ic-frame*) is a structure $\mathcal{F} = \langle X, \leq, R\rangle$ where $\langle X, \leq\rangle$ is a poset, and $R \subseteq X \times \mathsf{Up}(X)$ is a relation satisfying the following condition:

$$(\forall x, y \in X)(\forall Y \in \mathsf{Up}(X)), \text{ if } x \leq y \text{ and } Y \in R(x), \text{ then} \\ \text{there exists } Z \in \mathsf{Up}(X) \text{ such that } Z \in R(y) \text{ and } Z \subseteq Y. \tag{ic}$$

We notice that our frames lie within the *neighbourhood structures*, which can be used to model (non-normal) modal logics. We refer the reader to Pacuit (2017) and Sotirov (1980) for more details on neighbourhood semantics for modal logic.

Let us denote by ICF the class of all ic-fames. Let $\mathcal{F} = \langle X, \leq, R\rangle$ be an ic-frame. A *valuation* on $\mathcal{F}$ is a function $v\colon \mathrm{Var} \to \mathsf{Up}(X)$. A pair $\mathcal{M} = \langle \mathcal{F}, v\rangle$ is called an *ic-model* if $\mathcal{F}$ is an ic-frame and v is a valuation on $\mathcal{F}$. We define recursively the satisfaction relation $\Vdash$ as follows:

- $\mathcal{M}, x \Vdash p \iff x \in v(p)$.
- $\mathcal{M}, x \nVdash \bot$ (i.e. not $\mathcal{M}, x \Vdash \bot$)
- $\mathcal{M}, x \Vdash \varphi \wedge \psi \iff \mathcal{M}, x \Vdash \varphi$ and $\mathcal{M}, x \Vdash \psi$.
- $\mathcal{M}, x \Vdash \varphi \vee \psi \iff \mathcal{M}, x \Vdash \varphi$ or $\mathcal{M}, x \Vdash \psi$.
- $\mathcal{M}, x \Vdash \varphi \to \psi \iff \forall x' \geq x$, if $\mathcal{M}, x' \Vdash \varphi$, then $\mathcal{M}, x' \Vdash \psi$.
- $\mathcal{M}, x \Vdash \Box\varphi \iff \forall x' \geq x, \forall Y \in R(x')$, it follows that $\mathcal{M}, y \Vdash \varphi\ \forall y \in Y$.
- $\mathcal{M}, x \Vdash \Diamond\varphi \iff \exists Y \in R(x)$ such that $\mathcal{M}, y \Vdash \varphi$, $\forall y \in Y$.

As usual, we say that a formula φ is *valid* or *true* in a state x if $\mathcal{M}, x \Vdash \varphi$, and φ is *valid* in an ic-model $\mathcal{M}$, denoted by $\mathcal{M} \Vdash \varphi$, if for all state x, $\mathcal{M}, x \Vdash \varphi$. Also, we say that φ is *valid* in an ic-frame $\mathcal{F}$, if for all valuation v, $\langle \mathcal{F}, v\rangle \Vdash \varphi$; we denoted by $\mathcal{F} \Vdash \varphi$. By induction on the construction of the formulas and by condition (ic), it is straightforward to show that the satisfaction relation $\Vdash$ satisfies the monotonicity property:

$$\text{if } \mathcal{M}, x \Vdash \varphi \text{ and } x' \geq x, \text{ then } \mathcal{M}, x' \Vdash \varphi.$$

We only prove for $\Diamond$ and leave the rest to the reader. Assume that φ satisfies the monotonicity property. Suppose that $\mathcal{M}, x \Vdash \Diamond\varphi$ and $x' \geq x$. Then, there is $Y \in R(x)$ such that $\mathcal{M}, y \Vdash \varphi$ for all $y \in Y$. Since $Y \in R(x)$ and $x \leq x'$, it follows by (ic) that there is $Z \in R(x')$ such that $Z \subseteq Y$. Thus, $Z \in R(x')$ and $\mathcal{M}, z \Vdash \varphi$ for all $z \in Z$. Hence, $\mathcal{M}, x' \Vdash \Diamond\varphi$.

Let us denote by $\Lambda(\mathrm{ICF})$ the logic defined by the class of all ic-frames. That is,

$$\Lambda(\mathrm{ICF}) = \{\varphi \in \mathrm{Fm} : \mathcal{F} \Vdash \varphi, \text{ for all } \mathcal{F} \in \mathrm{ICF}\}.$$

The definition of $\Vdash$ coincides with the one given by Wijesekera in (Wijesekera, 1990, pp 295), with the difference that our frames are different from Wijesekera's frames. Thus, the logic defined by the class of ic-frames is different from the propositional logic given in (Wijesekera, 1990, pp 295).

Example 3.2. The formula $\Diamond\bot \to \bot$ is a theorem of the logic given in (Wijesekera, 1990, pp. 295). However, it is not a theorem of $\Lambda(\mathrm{ICF})$. Let $X = \{x\}$ and $R(x) = \{\emptyset\}$. Then $\mathcal{F} = \langle X, \leq, R\rangle$ is an ic-frame. Let v be any valuation and $\mathcal{M} = \langle \mathcal{F}, v\rangle$. It follows trivially that $\mathcal{M}, x \Vdash \Diamond\bot$ because there is $\emptyset \in R(x)$ such that $\mathcal{M}, y \Vdash \bot$ for all $y \in \emptyset$. Hence, since $\mathcal{M}, x \nVdash \bot$, we have that $x \nVdash \Diamond\bot \to \bot$.

We propose a Hilbert-style system for the logic $\Lambda(\mathrm{ICF})$. Let us consider the following Hilbert-style system $\mathcal{H}$.

Axioms:

(I) Axioms of intuitionistic propositional logic.

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.

($\mathrm{K}_{\Box\Diamond}$) $\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$.

(DB) $(\Diamond\top \to \Box\varphi) \to \Box\varphi$.

Rules of inference:

$$(\mathrm{MP})\ \frac{\varphi \quad \varphi \to \psi}{\psi} \qquad (\mathrm{N})\ \frac{\varphi}{\Box\varphi}.$$

Let $\Lambda(\mathcal{H})$ be the logic defined by the Hilbert-style system $\mathcal{H}$. That is, $\Lambda(\mathcal{H})$ is the least set of formulas including the axioms (I), (K), ($\mathrm{K}_{\Box\Diamond}$) and (DB), closed under substitutions, and closed under the inference rules (MP) and (N). Given a formula φ, we denote $\vdash_{\mathcal{H}} \varphi$ whenever $\varphi \in \Lambda(\mathcal{H})$.

Proposition 3.3. *Let $\varphi, \psi \in \mathrm{Fm}$. Then,*

(1) If $\vdash_{\mathcal{H}} \varphi \to \psi$, then $\vdash_{\mathcal{H}} \Diamond\varphi \to \Diamond\psi$.
(2) $\vdash_{\mathcal{H}} (\Box\varphi \wedge \Diamond\psi) \to \Diamond(\varphi \wedge \psi)$.
(3) $\vdash_{\mathcal{H}} \Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$.

Proof. Recall that IPL means intuitionistic propositional logic.

(1) It is a consequence of (N), Axiom ($\mathrm{K}_{\Box\Diamond}$), and by (MP).

(2)

1.	$\varphi \to (\psi \to \varphi)$	Axiom IPL
2.	$(\psi \to \varphi) \to (\psi \to (\varphi \wedge \psi))$	Theorem IPL
3.	$\Box\varphi \to \Box(\psi \to \varphi)$	1., (N), (K) and (MP)
4.	$\Box(\psi \to \varphi) \to \Box(\psi \to (\varphi \wedge \psi))$	2., (N), (K) and (MP)
5.	$\Box(\psi \to (\varphi \wedge \psi)) \to (\Diamond\psi \to \Diamond(\varphi \wedge \psi))$	($\mathrm{K}_{\Box\Diamond}$)
6.	$\Box\varphi \to (\Diamond\psi \to \Diamond(\varphi \wedge \psi))$	3., 4., 5. and Theorem IPL
7.	$(\Box\varphi \wedge \Diamond\psi) \to \Diamond(\varphi \wedge \psi)$.	

(3)

1.	$(\Box\varphi \wedge \Diamond(\varphi \to \psi)) \to \Diamond(\varphi \wedge (\varphi \to \psi))$	By (2)
2.	$(\varphi \wedge (\varphi \to \psi)) \to \psi$	Theorem IPL
3.	$\Diamond(\varphi \wedge (\varphi \to \psi)) \to \Diamond\psi$	2., (N), $\mathrm{K}_{\Box\Diamond}$ and (MP)
4.	$(\Box\varphi \wedge \Diamond(\varphi \to \psi)) \to \Diamond\psi$	1., 3., and Theorem IPL
5.	$\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$	5. and Theorem IPL. $\Box$

Now, we want to prove that the logic $\Lambda(\mathcal{H})$ is sound and complete with respect to the class of all ic-frames. The proof of soundness, as usual, is routine, and we leave the details to the reader.

Proposition 3.4 (Soundness)**.** $\Lambda(\mathcal{H}) \subseteq \Lambda(\mathrm{ICF})$.

However, the proof of completeness (that is, the inclusion $\Lambda(\mathrm{ICF}) \subseteq \Lambda(\mathcal{H})$) is often more complex. At the end of Section 5, we prove this inclusion (actually the identity) from algebraic techniques. We state it here to continue studying the logic $\Lambda(\mathcal{H})$.

Theorem 3.5 (neighbourhood Completeness)**.** $\Lambda(\mathcal{H}) = \Lambda(\mathrm{ICF})$.

In (Wijesekera, 1990, pp. 294–295), Wijesekera introduces a propositional logic through a Hilbert-style system and a semantics, proving soundness and completeness. Let us denote by $\Lambda(W)$ the logic of Wijesekera defined by the Hilbert-style system:

Axioms:

(i) Axioms of intuitionistic propositional logic.
(ii) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.
(iii) $\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$.
(iv) $(\Box\varphi \wedge \Diamond(\varphi \to \psi)) \to \Diamond\psi$.
(v) $\Diamond\bot \to \bot$
(vi) $\neg(\Diamond\top) \to \Box\bot$
(vii) $(\Diamond\top \to \Box\varphi) \to \Box\varphi$.

Rules of inference:

$$\text{(MP)}\ \frac{\varphi \quad \varphi \to \psi}{\psi} \qquad \text{(N) if } \vdash \varphi, \text{ then } \vdash \Box\varphi.$$

It is clear that $\Lambda(\mathcal{H}) \subseteq \Lambda(W)$. Considering that $\Lambda(\mathcal{H}) = \Lambda(\mathrm{ICF})$, it follows by Example 3.2 that the logics $\Lambda(\mathcal{H})$ and $\Lambda(W)$ are different. Now we show that the logics $\Lambda(\mathcal{H})$ and $\Lambda(W)$ are closely related. Notice that, similar to the proof of (3) in Proposition 3.3, one can prove that Axiom (iv) is derived from the other axioms.

Proposition 3.6. $\Lambda(W) = \Lambda(\mathcal{H} + \Diamond\bot \to \bot)$.

Proof. It is enough to show that Axiom (vi) $\neg(\Diamond\top) \to \Box\bot$ is derivable from the others. First, notice that in IPL $\neg\varphi \equiv \varphi \to \bot$. We have the following proof:

1.	$(\Diamond\top \to \Box\bot) \to \Box\bot$	Axiom (vii)
2.	$\bot \to \Box\bot$	Axiom of IPL
3.	$(\bot \to \Box\bot) \to ((\Diamond\top \to \bot) \to (\Diamond\top \to \Box\bot))$	theorem of IPL
4.	$(\Diamond\top \to \bot) \to (\Diamond\top \to \Box\bot)$	2., 3, and by MP
5.	$(\Diamond\top \to \bot) \to \Box\bot$	1., 4., and theorem of IPL. $\Box$

4. Concurrent Heyting algebras: An algebraic semantics for $\Lambda(\mathcal{H})$

In this section, we introduce the notion of a concurrent Heyting algebra as a generalization of the concurrent algebra defined in S. Celani (2011) (Definition 3.1). We prove that the logic $\Lambda(\mathcal{H})$ is sound and complete with respect to the class of all concurrent Heyting algebras. We characterize the congruences of a concurrent Heyting algebra, and give necessary and sufficient conditions for a concurrent Heyting algebra to be subdirectly irreducible.

Definition 4.1. A *concurrent Heyting algebra*, or *CH-algebra* for short, is an algebra $\langle A, \Box, \Diamond\rangle$, where $A = \langle A, \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra, and $\Box, \Diamond\colon A \to A$ are unary operators on A satisfying the following conditions: for all $a, b \in A$:

(CH1) $\Box(a \wedge b) = \Box a \wedge \Box b$,
(CH2) $\Box 1 = 1$,
(CH3) $\Box(a \to b) \leq \Diamond a \to \Diamond b$,
(CH4) $\Diamond 1 \to \Box a \leq \Box a$.

Heyting algebras with a unary operator $\Box$ that satisfies (CH1) and (CH2) were studied in several papers, for instance Hasimoto (2001); Wolter (1997); Wolter & Zakharyaschev (1997).

From condition (CH3), we get that the operator $\Diamond$ is monotone, i.e., if $a \leq b$, then $\Diamond a \leq \Diamond b$, because if $a \leq b$, then $a \to b = 1$. So, $1 = \Box 1 = \Box(a \to b) \leq \Diamond a \to \Diamond b$.

Let us denote by **CHA** the class of all CH-algebras. Notice that identities can equivalently replace the conditions (CH3) and (CH4). Hence, the class **CHA** is a variety of algebras (Burris & Sankappanavar (1981)).

Remark 4.2. Recall that a *concurrent algebra* is a triple $\langle A, \Box, \Diamond \rangle$ such that A is a Boolean algebra, the modal operators satisfy conditions (CH1), (CH2), and (CH3) of Definition 4.1, and the additional condition (C2): $\Box 0 \vee \Diamond 1 = 1$ (see Definition 3.1 in S. Celani (2011)). We now show that conditions (CH4) and (C2) are equivalent when A is a Boolean algebra. Assume (CH4) holds. In particular, take $a = 0$. Then,

$$\Diamond 1 \to \Box 0 = \neg \Diamond 1 \vee \Box 0 \leq \Box 0.$$

Thus,

$$\Diamond 1 \vee \neg \Diamond 1 \vee \Box 0 \leq \Diamond 1 \vee \Box 0,$$

which simplifies to (C2).Conversely, assume (C2) holds. Then, $\neg \Diamond 1 \leq \Box 0$. Hence,

$$\neg \Diamond 1 \vee \Box a \leq \Box 0 \vee \Box a = \Box a,$$

proving (CH4). We have thus shown that the notion of a *concurrent Heyting algebra* generalizes that of a *concurrent algebra*.

Proposition 4.3. *Let A be a Heyting algebra and let $\Box, \Diamond \colon A \to A$ be maps.*

(I) If $\Box 1 = 1$, then

$$\Box(a \wedge b) = \Box a \wedge \Box b \textit{ if and only if } \Box(a \to b) \leq \Box a \to \Box b$$

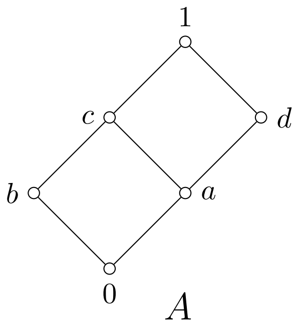
(II) If $\Box$ and $\Diamond$ are monotone, then the following are equivalent:
1. *$\Box(a \to b) \leq \Diamond a \to \Diamond b$.*
2. *$\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$.*
3. *$\Diamond(a \to b) \leq \Box a \to \Diamond b$.*

Proof. (I) It is straightforward by the basic properties of Heyting algebras.

(II) By (Nagy, 2023, Lemma 2.1.2). $\Box$

We present an example a finite CH-algebra.

Example 4.4. Let us consider $\mathbf{A} = \langle A, \vee, \wedge, \to, \Box, \Diamond \rangle$ to be the algebra where the diagram on the left defines the lattice part, and the other operators are defined in the tables shown below.



A

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	d	1	1	d	1	1
c	0	d	b	1	d	1
d	b	c	b	c	1	1
1	0	a	b	c	d	1

	$\Box$	$\Diamond$
0	b	a
a	b	a
b	b	d
c	b	d
d	c	a
1	1	d

The reader can easily check that $\mathbf{A}$ is indeed a CH-algebra.

The following are two important examples of CH-algebras. Our first example presents an extension of the *concurrent frame* notion introduced in Definition 4.1 in S. Celani (2011).

Example 4.5. Let $\langle X, \leq, R\rangle$ be an ic-frame. We define two unary operators on $\mathsf{Up}(X)$ as follows. For each $U \in \mathsf{Up}(X)$, let

$$\Box_R(U) = \{x \in X : \forall y \geq x, \forall Y \in R(y)(Y \subseteq U)\}$$

$$\Diamond_R(U) = \{x \in X : \exists Y \in R(x)(Y \subseteq U)\}.$$

Then the algebra $\langle \mathsf{Up}(X), \cup, \cap, \Rightarrow, \Box_R, \Diamond_R, \emptyset, X\rangle$ is a CH-algebra. Indeed, we already know that $\langle \mathsf{Up}(X), \cap, \cup, \Rightarrow, \emptyset, X\rangle$ is a Heyting algebra. It is straightforward show that $\Box_R(U) \in \mathsf{Up}(X)$ for all $U \in \mathsf{Up}(X)$, and that $\Box_R$ satisfies conditions (CH1) and (CH2). Let $U \in \mathsf{Up}(X)$. We prove that $\Diamond_R(U) \in \mathsf{Up}(X)$. Let $x, y \in X$ be such that $x \in \Diamond_R(U)$ and $x \leq y$. Then there exists $Y \in R(x)$ such that $Y \subseteq U$. By condition (ic), there exists $Z \in \mathsf{Up}(X)$ such that $Z \in R(y)$ and $Z \subseteq Y$. Then $Z \subseteq U$. Hence $y \in \Diamond_R(U)$.

(CH3) Let $U, V \in \mathsf{Up}(X)$ and $x \in \Box_R(U \Rightarrow V)$. We prove that $[x) \cap \Diamond_R(U) \subseteq \Diamond_R(V)$. Let $y \in [x) \cap \Diamond_R(U)$. Then there exists $Z \in R(y)$ such that $Z \subseteq U$. Since $x \leq y$, $Z \in R(y)$, and since $x \in \Box_R(U \Rightarrow V)$, it follows that $Z \subseteq U \Rightarrow V$. Then $Z \cap U \subseteq V$, and since $Z \subseteq U$, we get that $Z \subseteq V$. Thus, $y \in \Diamond_R(V)$.

(CH4) Let $U \in \mathsf{Up}(X)$ and $x \in \Diamond_R(X) \Rightarrow \Box_R(U)$. Thus $[x) \cap \Diamond_R(X) \subseteq \Box_R(U)$. We prove that $x \in \Box_R(U)$. Let $y \geq x$ and $Y \in R(y)$. Then $y \in [x) \cap \Diamond_R(X)$, and thus $y \in \Box_R(U)$. Therefore, $x \in \Box_R(U)$.

Let $\mathcal{F} = \langle X, \leq, R\rangle$ be an ic-frame. The algebra $A_{\mathcal{F}} = \langle \mathsf{Up}(X), \Box_R, \Diamond_R\rangle$ is called the *complex CH-algebra* associated with $\mathcal{F}$. In the next section, we show that every CH-algebra can be embedded into a complex CH-algebra.

As mentioned earlier, the interpretation of the operators $\Diamond_R$ and $\Box_R$ in an ic-frame $\langle X, \leq, R\rangle$ extends the modal operators defined in S. Celani (2011). Indeed, when the order $\leq$ is the identity, the operator $\Box_R$ is given by

$$\Box_R(U) = \{x \in X : \forall Y \in R(x),\ Y \subseteq U\},$$

for every $U \in \mathcal{P}(X)$. Likewise, the interpretation of $\Diamond_R(U)$ (denoted as $\nabla_R(U)$ in the notation of S. Celani (2011)) is the same.

Example 4.6. Recall that Fm denotes the algebra of formulas on the algebraic language $\mathcal{L} = \{\wedge, \vee, \rightarrow, \Box, \Diamond, \bot\}$. Let Λ be any extension of the logic $\Lambda(\mathcal{H})$. We define the binary relation $\equiv$ on Fm as usual:

$$\alpha \equiv \beta \quad \text{if and only if} \quad \alpha \leftrightarrow \beta \in \Lambda.$$

Since Λ contains all theorems of intuitionistic propositional logic, it follows that $\equiv$ is an equivalence relation on Fm compatible with the intuitionistic logical connectives $\wedge$, $\vee$ and $\rightarrow$ (see Sec. 1.4 in Galatos, Jipsen, Kowalski, & Ono (2007)). By axioms (K) and ($K_{\Box\Diamond}$), and the inference rules, it follows that $\equiv$ is also compatible with the operations $\Box$ and $\Diamond$. Hence, $\equiv$ is a congruence of the algebra of formula $\langle \mathrm{Fm}, \wedge, \vee, \rightarrow, \Box, \Diamond, \bot, \top\rangle$. Then, it is straightforward to show that $\langle \mathrm{Fm}/\!\equiv, \Box, \Diamond\rangle$ is a CH-algebra. The algebra $\langle \mathrm{Fm}/\!\equiv, \Box, \Diamond\rangle$ is called the *Lindenbaum-Tarski algebra* of the logic Λ.

Now we present the definition of a homomorphism between concurrent Heyting algebras. For the algebraic notions of *trivial algebra, congruence, simple algebra and subdirectly irreducible algebra*, we refer the reader to Burris & Sankappanavar (1981).

Definition 4.7. Let $\langle A, \Box_A, \Diamond_A\rangle$ and $\langle B, \Box_B, \Diamond_B\rangle$ be two concurrent Heyting algebra. We say that a homomorphism of Heyting algebras $h\colon A \rightarrow B$ is a *homomorphism of concurrent Heyting algebras* if $h(\Box_A(x)) = \Box_B h(x)$ and $h(\Diamond_A(x)) = \Diamond_B h(x)$. An *embedding of CH-algebras* is an injective homomorphism of CH-algebras.

Let $\mathbf{A} = \langle A, \Box, \Diamond\rangle$ be a concurrent Heyting algebra. Let us denote by $\Theta(A)$ the set of all congruences relations on $\mathbf{A}$. That is, $\theta \in \Theta(A)$ if θ is compatible with the operations $\wedge, \vee$, $\rightarrow$, $\Box$ and $\Diamond$. Let $\Theta_\Box(A)$ be the set of all congruences of the algebra $\langle A, \Box\rangle$, that is, $\theta \in \Theta_\Box(A)$ if is compatible with $\wedge, \vee$, $\rightarrow$, and $\Box$. It is well-known that the set of congruences of an algebra, ordered with the set-theoretical inclusion, is a complete lattice. An *open filter* of A is a filter of A that is closed under $\Box$, i.e. for all $a \in A$, if $a \in F$ then $\Box a \in F$. Let us call $F_\Box(A)$ the lattice of all open filters of A. Let $a \in A$. We define recursively $\Box^n a$ for all $n \in \mathbb{N} \cup \{0\}$, as follows:

$$\Box^0 a = a \qquad \text{and} \qquad \Box^{n+1} a = \Box(\Box^n a).$$

Since $\Box$ is an operator satisfying (CH1) and (CH2) and A is a Heyting algebra, the lattice $\Theta_\Box(A)$ is isomorphic to $F_\Box(A)$. This result can be found in (Hasimoto, 2001, Proposition 2.1) or in (Wolter, 1997, Proposition 1.6).

Proposition 4.8. *Let $\mathbf{A} = \langle A, \Box, \Diamond\rangle$ be a concurrent Heyting algebra. Then $f : F_\Box(A) \rightarrow \Theta_\Box(A)$ defined as $f(F) = \theta_F = \{(a, b) : (a \rightarrow b) \wedge (b \rightarrow a) \in F\}$ is an isomorphisms between the complete lattices $F_\Box(A)$ and $\Theta_\Box(A)$. The inverse map is given by $g : \Theta(A_\Box) \rightarrow F_\Box(A)$ defined as $g(\theta) = F_\theta = \{a \in A : a\theta 1\}$.*

Proposition 4.9. *Let $\mathbf{A} = \langle A, \Box, \Diamond\rangle$ be a concurrent Heyting algebra. Then $\Theta(A) = \Theta_\Box(A)$, i.e. θ is a congruence of $\langle A, \Box, \Diamond\rangle$ if and only if θ is a congruence of $\langle A, \Box\rangle$.*

Proof. It is obvious that every congruence of the algebra $\langle A, \Box, \Diamond\rangle$ is a congruence of $\langle A, \Box\rangle$. Hence $\Theta(A) \subseteq \Theta_\Box(A)$. Let $\theta \in \Theta_\Box(A)$ and let $(a, b) \in \theta$. Only remains to show that $(\Diamond a, \Diamond b) \in \theta$. By Proposition 4.8, F_θ is an open filter of A, and $\theta_{F_\theta} = \theta$. Thus, $(a \rightarrow b), (b \rightarrow a) \in F_\theta$, which implies $\Box(a \rightarrow b)$, $\Box(b \rightarrow a) \in F_\theta$. Since F_θ is a filter and A satisfies the condition $(CH3)$, it follows that $((\Diamond a \rightarrow \Diamond b) \wedge (\Diamond b \rightarrow \Diamond a)) \in F_\theta$.

Thus, $((\Diamond a \to \Diamond b) \wedge (\Diamond b \to \Diamond a), 1) \in \theta$. Since θ is compatible with $\wedge$, we have that

$$(\Diamond a \wedge (\Diamond a \to \Diamond b) \wedge (\Diamond b \to \Diamond a), \Diamond a) \in \theta, \text{ and } (\Diamond b \wedge (\Diamond a \to \Diamond b) \wedge (\Diamond b \to \Diamond a), \Diamond b) \in \theta$$

Since A is a Heyting algebra, we have $\Diamond a \wedge (\Diamond a \to \Diamond b) \wedge (\Diamond b \to \Diamond a) = \Diamond a \wedge \Diamond b$ and $\Diamond b \wedge (\Diamond a \to \Diamond b) \wedge (\Diamond b \to \Diamond a) = \Diamond a \wedge \Diamond b$. Then, it follows that

$$(\Diamond a \wedge \Diamond b, \Diamond a) \in \theta \text{ and } (\Diamond a \wedge \Diamond b, \Diamond b) \in \theta$$

Hence, $(\Diamond a, \Diamond b) \in \theta$. Therefore, $\theta \in \Theta(A)$. $\square$

The following result is an immediate consequence of the previous proposition.

Proposition 4.10. *Let $\boldsymbol{A} = \langle A, \Box, \Diamond \rangle$ be a concurrent Heyting algebra. Then*

(1) $\boldsymbol{A}$ is simple if and only if $\langle A, \Box \rangle$ is simple.
(2) $\boldsymbol{A}$ is subdirectly irreducible if and only if $\langle A, \Box \rangle$ is subdirectly irreducible.

There is a necessary and sufficient condition for an algebra $\langle A, \Box \rangle$, where A is a Heyting algebra and $\Box$ is an operator on A satisfying (CH1) and (CH2), to be subdirectly irreducible (see Proposition 1.6 in Wolter (1997)).

Proposition 4.11 (Wolter (1997))**.** *Let $\boldsymbol{A} = \langle A, \Box \rangle$ be an algebra such that A is a Heyting algebra and $\Box$ is an unary operation on A satisfying conditions (CH1) and (CH2). If $\boldsymbol{A}$ is non-trivial, then $\boldsymbol{A}$ is subdirectly irreducible if and only if there exists $a \in A$, $a \neq 1$ such that for all $b \in A$, $b \neq 1$ there exists $n \in \mathbb{N} \cup \{0\}$ such that $\Box^0 b \wedge \Box^1 b \wedge \Box^2 b \wedge \cdots \wedge \Box^{n-1} b \wedge \Box^n b \leq a$.*

From this, together with Proposition 4.10, yields a necessary and sufficient condition for a CH-algebra to be subdirectly irreducible.

Proposition 4.12. *Let $\boldsymbol{A} = \langle A, \Box, \Diamond \rangle$ be a CH-algebra non-trivial. Then, $\boldsymbol{A}$ is subdirectly irreducible if and only if there exists $a \in A$, $a \neq 1$ such that for all $b \in A$, $b \neq 1$ there exists $n \in \mathbb{N} \cup \{0\}$ such that $\Box^0 b \wedge \Box^1 b \wedge \Box^2 b \wedge \cdots \wedge \Box^{n-1} b \wedge \Box^n b \leq a$.*

We close this section by proving that the logic $\Lambda(\mathcal{H})$ is sound and complete concerning the class of all CH-algebras. Let A be a CH-algebra. We denote by $\mathrm{Hom}(\mathrm{Fm}, A)$ the set of all algebraic homomorphisms from the algebra of formulas Fm into A. We define the relation $\models$ as follows: for every formula φ and every CH-algebra A,

$$A \models \varphi \iff \forall h \in \mathrm{Hom}(\mathrm{Fm}, A),\ h(\varphi) = 1. \tag{4.1}$$

We say that a formula φ is *valid* in a CH-algebra A if $A \models \varphi$. We write $\mathbf{CHA} \models \varphi$ if $A \models \varphi$ for all $A \in \mathbf{CHA}$. We consider the logic

$$\Lambda(\mathbf{CHA}) = \{\varphi \in \mathrm{Fm} : \mathbf{CHA} \models \varphi\}.$$

Now, we are ready to prove the first completeness theorem for the logic $\Lambda(\mathcal{H})$ defined by the Hilbert-style system $\mathcal{H}$.

Theorem 4.13 (Algebraic completeness)**.** $\Lambda(\mathcal{H}) = \Lambda(\mathbf{CHA})$.

Proof. It is a routine show that, from conditions defining CH-algebras, it follows that $\Lambda(\mathbf{CHA})$ contains the axioms (I), (K), ($K_{\Box\Diamond}$) and (DB), it is closed under substitutions and by the rules (MP) and (N). Hence $\Lambda(\mathcal{H}) \subseteq \Lambda(\mathbf{CHA})$.

Now let us prove that $\Lambda(\mathbf{CHA}) \subseteq \Lambda(\mathcal{H})$. This is a consequence of a standard argument of Algebraic Logic by building the Lindenbaum-Tarski algebra of the logic $\Lambda(\mathcal{H})$. We sketch the proof and leave the details to the reader. Let $\varphi \notin \Lambda(\mathcal{H})$. Let $\langle \mathrm{Fm}/{\equiv}, \Box, \Diamond \rangle$ by the Lindenbaum-Tarski algebra of the logic $\Lambda(\mathcal{H})$. By Example 4.6, we know that $\langle \mathrm{Fm}/{\equiv}, \Box, \Diamond \rangle \in \mathbf{CHA}$. It is easy to show that $\alpha/{\equiv} = \top/{\equiv}$ if and only if $\alpha \in \Lambda(\mathcal{H})$, for all formula α. Then, we have the natural homomorphism $h\colon \mathrm{Fm} \to \mathrm{Fm}/{\equiv}$ such that $h(\varphi) = \varphi/{\equiv} \neq \top/{\equiv}$. Hence $\varphi \notin \Lambda(\mathbf{CHA})$. Therefore, $\Lambda(\mathbf{CHA}) \subseteq \Lambda(\mathcal{H})$. $\Box$

5. Representation and relational completeness

In this section, we develop a representation for the class of CH-algebras through the class of ic-frames. In other words, we show that every CH-algebra is embeddable into a complex CH-algebra associated with an ic-frame. Then, we use this representation to prove the logic $\Lambda(\mathcal{H})$ is complete with respect to the class of all ic-frames.

Let $\langle A, \Box, \Diamond \rangle$ be a CH-algebra. Recall that $X(A)$ denotes the collection of all prime filters of A. Recall that for every $F \in \mathrm{Fi}(A)$, $\widehat{F} = \{x \in X(A) : F \subseteq x\}$. We define the relation

$$R_A \subseteq X(A) \times \mathsf{Up}(X(A))$$

as follows: for every $x \in X(A)$ and $Y \in \mathsf{Up}(X(A))$

$$(x, Y) \in R_A \iff \exists F \in \mathrm{Fi}(A)(Y = \widehat{F} \quad \text{and} \quad \Box^{-1}(x) \subseteq F \subseteq \Diamond^{-1}(x)).$$

Notice that only the sets of the form $\widehat{F}$, with $F \in \mathrm{Fi}(A)$, can be related to an $x \in X(A)$ by the relation R_A.

Remark 5.1. Let $x \in X(A)$. Then, by condition (CH3) and by definition of R_A, it follows that

$$\Diamond 1 \in x \iff \Box^{-1}(x) \subseteq \Diamond^{-1}(x) \iff \widehat{\Box^{-1}(x)} \in R_A(x) \iff R_A(x) \neq \emptyset.$$

The following proposition contains two properties needed to prove the representation theorem (Theorem 5.4).

Proposition 5.2. *Let $\langle A, \Box, \Diamond \rangle$ be a CH-algebra and $x \in X(A)$.*

(1) $\Box a \notin x$ iff $\exists y \in X(A), \exists F \in \mathrm{Fi}(A)$ such that $x \subseteq y$, $(y, \widehat{F}) \in R_A$ and $a \notin F$
(2) $\Diamond a \in x$ iff $\exists F \in \mathrm{Fi}(A), \widehat{F} \in R_A(x) \wedge a \in F$.

Proof. For the first statement, suppose that $\Box a \notin x$. So, we have that $\Diamond 1 \to \Box a \notin x$ and $\Box a \notin \mathrm{Fg}(x \cup \{\Diamond 1\})$. Then, there exists $y \in X(A)$ such that $\mathrm{Fg}(x \cup \{\Diamond 1\}) \subseteq y$ and $\Box a \notin y$, Let us consider the filter $F = \Box^{-1}(y)$. Because $\Diamond 1 \in y$, according to Remark 5.1, we have $\Box^{-1}(y) \subseteq F \subseteq \Diamond^{-1}(y)$. Therefore the pair $(y, \widehat{F}) \in R_A$. The converse direction is straightforward.

Turning to the second statement, suppose that $\Diamond a \in x$. Since $\Diamond$ is a monotone operator, it follows that $\Diamond 1 \in x$. Let us consider $F = \mathrm{Fg}(\Box^{-1}(x) \cup \{a\})$. It is clear that $\widehat{F} \in R_A(x)$ and $a \in F$. The converse direction follows from $F \subseteq \Diamond^{-1}(x)$. $\square$

Proposition 5.3. *Let $\langle A, \Box, \Diamond \rangle$ be a CH-algebra. Then the structure*

$$\mathcal{F}_A = \langle X(A), \subseteq, R_A \rangle$$

is an ic-frame.

Proof. Let $x, y \in X(A)$ and $F \in \mathrm{Fi}(A)$ such that $x \subseteq y$ and $(x, \widehat{F}) \in R_A$. We aim to show that there exists $H \in \mathrm{Fi}(A)$ such that $(y, \widehat{H}) \in R_A$ and $F \subseteq H$. Since $x \subseteq y$, we have $\Box^{-1}(x) \subseteq \Box^{-1}(y)$. Consider the filter $H = \mathrm{Fg}(F \cup \Box^{-1}(y))$. By its definition, it is clear that $F \subseteq H$ and $\Box^{-1}(y) \subseteq H$. It remains to prove that $H \subseteq \Diamond^{-1}(y)$. Let $a \in H$. By definition of the filter generated by a set, there exists $f \in F$ and $b \in \Box^{-1}(y)$ such that $f \wedge b \leq a$. Since $F \subseteq \Diamond^{-1}(x)$ and $x \subseteq y$, $\Diamond f \in y$. Thus, $\Diamond f \wedge \Box b \in y$. By Proposition 4.3, we have that $\Diamond f \wedge \Box b \leq \Diamond(f \wedge b) \leq \Diamond a$. As y is a filter, $\Diamond a \in y$. Hence $a \in \Diamond^{-1}(y)$,which proves the inclusion $H \subseteq \Diamond^{-1}(y)$. $\square$

Let $\langle A, \Box, \Diamond \rangle$ be a CH-algebra and $\mathcal{F}_A = \langle X(A), \subseteq, R_A \rangle$ its dual ic-frame. Now, by Example 4.5, we have the complex CH-algebra $A_{\mathcal{F}_A} = \langle \mathsf{Up}(X(A)), \Box_{R_A}, \Diamond_{R_A} \rangle$ associated with $\mathcal{F}_A$. Recall that the map $\alpha \colon A \to \mathsf{Up}(X(A))$ is defined as follows: $\alpha(a) = \{x \in X(A) : a \in x\rangle$. for all $a \in A$.

Theorem 5.4 (Representation)**.** *Let $\langle A, \Box, \Diamond \rangle$ be a CH-algebra and $\langle X(A), \subseteq, R_A \rangle$ its dual ic-frame. Then, $\alpha \colon \langle A, \Box, \Diamond \rangle \to \langle \mathsf{Up}(X(A)), \Box_{R_A}, \Diamond_{R_A} \rangle$ is an embedding between CH-algebras.*

Proof. It is known that α is an embedding of Heyting algebras. Therefore, we only need to prove that $\alpha(\Box a) = \Box_{R_A}\alpha(a)$ and $\alpha(\Diamond a) = \Diamond_{R_A}\alpha(a)$.

To prove the first equality, we demonstrate both inclusions. Let $x \in \alpha(\Box a)$, $y \in X(A)$ and $F \in \mathrm{Fi}(A)$ such that $x \subseteq y$ and $\widehat{F} \in R_A(y)$. Given that $\widehat{F} \in R_A(y)$, by the definition of the relation R_A, this implies that $\Box^{-1}(y) \subseteq F \subseteq \Diamond^{-1}(y)$. Our goal is to prove that $\widehat{F} \subseteq \alpha(a)$. Since $x \in \alpha(\Box a)$ and $x \subseteq y$, it follows that $\Box a \in y$, that is $a \in \Box^{-1}(y)$. Since $\Box^{-1}(y) \subseteq F$, we have $a \in F$. Hence, we have shown that $\forall z \in \widehat{F}$, $a \in z$, which proves that $\widehat{F} \subseteq \alpha(a)$. We prove the reverse inclusion by contraposition. Suppose that $x \notin \alpha(\Box a)$, which means that $\Box a \notin x$. By Proposition 5.2(1), there are $y \in X(A)$ and a filter F of A such that $x \subseteq y$, $(y, \widehat{F}) \in R_A$ and $a \notin F$. By the Prime Filter Theorem, there exists $z \in X(A)$ such that $F \subseteq z$ and $a \notin z$. From this, it follows that $z \in \widehat{F}$ and $z \notin \alpha(a)$. Thus, we have shown that $\widehat{F} \nsubseteq \alpha(a)$ which, given that $x \subseteq y$ and $(y, \widehat{F}) \in R_A$, implies that $x \notin \Box_{R_A}\alpha(a)$.

We prove the second equality with a series of equivalences. The first step is to recognise that by definition, $x \in \alpha(\Diamond a)$ if and only if $\Diamond a \in x$. By Proposition 5.2(2) $\Diamond a \in x$ if and only if there exists a filter F of A such that $\widehat{F} \in R_A(x)$ and $a \in F$. Furthermore, $a \in F$ if and only if $a \in z$ for all $z \in \widehat{F}$, which is equivalent to $\widehat{F} \subseteq \alpha(a)$. Combining all the equivalences, we have that $x \in \alpha(\Diamond a)$ if and only if there is $\widehat{F} \in R_A(x)$ such that $\widehat{F} \subseteq \alpha(a)$. By the definition of $\Diamond_{R_A}$, this last condition is equivalent to $x \in \Diamond_{R_A}\alpha(a)$. Thus, we have shown that $x \in \alpha(\Diamond a)$ if and only if $x \in \Diamond_{R_A}\alpha(a)$. $\square$

We are ready to prove the promised completeness theorem (Theorem 3.5) in Section 3. We start with the following result.

Proposition 5.5. *Let* $v\colon \mathrm{Var} \to \mathsf{Up}(X)$ *be a valuation on an ic-frame* $\mathcal{F} = \langle X, \leq, R\rangle$*. The function* $h_v\colon \mathrm{Fm} \to \mathsf{Up}(X)$ *defined by* $h_v(\varphi) = \{x \in X : \mathcal{F}, v, x \Vdash \varphi\}$*, for every* $\varphi \in \mathrm{Fm}$*, is a homomorphism of CH-algebras extending the function* v*.*

Proof. It is straightforward from the definition of the forcing relation $\Vdash$ and the definitions of $\Box_R$ and $\Diamond_R$. $\square$

Considering that every function v from Var into a CH-algebra A can be uniquely extended to a homomorphism from Fm into A, the following result is straightforward by the previous proposition.

Proposition 5.6. *Let* $\mathcal{F} = \langle X, \leq, R\rangle$ *be an ic-frame and* $A_{\mathcal{F}}$ *its complex CH-algebra. For every formula* φ*,*

$$\mathcal{F} \Vdash \varphi \quad \textit{if and only if} \quad A_{\mathcal{F}} \vDash \varphi.$$

Let **CCHA** be the class of all complex CH-algebras $A_{\mathcal{F}} = \langle \mathsf{Up}(X), \Box_R, \Diamond_R\rangle$ given by ic-frames $\mathcal{F} = \langle X, \leq, R\rangle$. Let $\Lambda(\mathbf{CCHA})$ be the logic defined by the class of algebras **CCHA**. Hence, the previous proposition implies the following corollary.

Corollary 5.7. $\Lambda(\mathrm{ICF}) = \Lambda(\mathbf{CCHA})$*.*

Proof. It is immediate from the previous proposition. $\square$

It is on the following proposition where the Representation Theorem (Theorem 5.4) plays a key role.

Proposition 5.8. $\Lambda(\mathbf{CHA}) = \Lambda(\mathbf{CCHA})$*.*

Proof. On the one hand, since $\mathbf{CCHA} \subseteq \mathbf{CHA}$, it follows that $\Lambda(\mathbf{CHA}) \subseteq \Lambda(\mathbf{CCHA})$. On the other hand, by Theorem 5.4, we have that every CH-algebra $\langle A, \Box, \Diamond\rangle$ is isomorphic to a subalgebra of the complex CH-algebra $A_{\mathcal{F}_A} = \langle \mathsf{Up}(\mathbf{X}(A)), \Box_{R_A}, \Diamond_{R_A}\rangle$. Hence, it follows that $\Lambda(\mathbf{CCHA}) \subseteq \Lambda(\mathbf{CHA})$. $\square$

Theorem 5.9 (Neighbourhood completeness: Theorem 3.5)**.** $\Lambda(\mathrm{ICF}) = \Lambda(\mathcal{H})$*.*

Proof. From the two above results and by Theorem 4.13, it follows that

$$\Lambda(\mathrm{ICF}) = \Lambda(\mathbf{CCHA}) = \Lambda(\mathbf{CHA}) = \Lambda(\mathcal{H}). \qquad \square$$

6. Categorical dual equivalence

In the present section, we extend the representation of a CH-algebra through its dual ic-frame (Proposition 5.4) to a categorical dual equivalence by means of specific topological structures. These topological structures are based on the notion of ic-frame and Esakia space. In order to define the dual topological structures of the CH-algebras, we need some auxiliary definitions and notations.

Definition 6.1. Let $\langle X, \leq, R\rangle$ be an ic-frame. We define the auxiliaries relations $S_R \subseteq$

$X \times X$, $\dot{\mathsf{S}}_R \subseteq X \times X$, and $J_R \subseteq X \times \mathsf{Up}(X)$ as follows: for every $x \in X$,

$$S_R(x) = \bigcup\{Y \in \mathsf{Up}(X) : Y \in R(x)\} \tag{6.1}$$

$$\dot{\mathsf{S}}_R(x) = \bigcup\{S_R(y) : y \geq x\} \tag{6.2}$$

$$J_R(x) = \{Y \in \mathsf{Up}(X) : \exists Z \in R(x)(Z \subseteq Y)\}. \tag{6.3}$$

There is a deeper connection between the relation R of an ic-frame $\langle X, \leq, R\rangle$ and the relations $\dot{\mathsf{S}}_R$ and J_R. First, we prove the following result. Then, in Section 7, we show a more specific connection between these relations.

Definition 6.2. Let $\langle X, \leq, R\rangle$ be an ic-frame. For every subset U of X, we define the following sets:

$$\Box_{\dot{\mathsf{S}}_R}(U) = \{x \in X : \dot{\mathsf{S}}_R(x) \subseteq U\} \qquad \text{and} \qquad \Diamond_{J_R}(U) = \{x \in X : \exists Y \in J_R(x)(Y \subseteq U)\}.$$

Proposition 6.3. *Let $\langle X, \leq, R\rangle$ be an ic-frame. Then, for every $U \in \mathsf{Up}(X)$, $\Box_R(U) = \Box_{\dot{\mathsf{S}}_R}(U)$ and $\Diamond_R(U) = \Diamond_{J_R}(U)$.*

Proof. Consider a set $U \in \mathsf{Up}(X)$. We prove the equalities by verifying the double inclusion. Let $x \in \Box_{\dot{\mathsf{S}}_R}(U)$, which means $\dot{\mathsf{S}}_R(x) \subseteq U$. Now, consider any $y \geq x$ and any $Y \in R(y)$. We know that $Y \subseteq S_R(y)$ and since $y \geq x$, it follows that $S_R(y) \subseteq \dot{\mathsf{S}}_R(x)$. Therefore, $Y \subseteq U$. Conversely, let $x \in \Box_R(U)$ and $y \in \dot{\mathsf{S}}_R(x)$. By definition of $\dot{\mathsf{S}}_R(x)$, there exist $z \geq x$ and $Y \in R(z)$ such that $y \in Y$. Since $x \in \Box_R(U)$, we have that $Y \subseteq U$. Thus $y \in U$, thereby showing that $\dot{\mathsf{S}}_R(x) \subseteq U$. Therefore $\Box_R(U) = \Box_{\dot{\mathsf{S}}_R}(U)$.

We now proceed to prove the second equality. First, suppose $x \in \Diamond_{J_R}(U)$, which means that there exists $Y \in J_R(x)$ such that $Y \subseteq U$. The definition of J_R states that there must exist an upset $Z \in R(x)$ such that $Z \subseteq Y$. Consequently, $Z \in R(x)$ and $Z \subseteq U$, we conclude that $x \in \Diamond_R(U)$. For the reverse inclusion, let $x \in \Diamond_R(U)$. This means there exists some $Y \in R(x)$ such that $Y \subseteq U$. Since $R(x)$ is a subset of $J_R(x)$, it follows immediately that $Y \in J_R(x)$. Therefore, we have found an upset $Y \in J_R(x)$ such that $Y \subseteq U$, which means $x \in \Diamond_{J_R}(U)$. Therefore $\Diamond_R(U) = \Diamond_{J_R}(U)$. $\Box$

Now we present the definition of the dual structures to the CH-algebras.

Definition 6.4. We say that $\langle X, \leq, \tau, R\rangle$ is a *concurrent Esakia space* if:

(C1) $\langle X, \leq, \tau\rangle$ is an Esakia space;
(C2) $\langle X, \leq, R\rangle$ is an ic-frame, where $R \subseteq X \times \mathcal{C}\ell\mathsf{Up}(X)$;
(C3) $\Box_R(U), \Diamond_R(U) \in \mathcal{CP}\mathsf{Up}(X)$ for all $U \in \mathcal{CP}\mathsf{Up}(X)$;
(C4) $R(x)$ is closed in $\langle \mathcal{C}\ell\mathsf{Up}(X), \subseteq, \tau_v\rangle$;
(C5) $(x, Y) \in R$ if and only if $Y \subseteq \dot{\mathsf{S}}_R(x)$ and $Y \in J_R(x)$;
(C6) If $R(x) \neq \emptyset$, then $\dot{\mathsf{S}}_R(x)$ is closed in X.

Let $\langle X, \leq, \tau\rangle$ be an Esakia space. Recall that $\mathbf{A}(X) = \langle \mathcal{CP}\mathsf{Up}(X), \cap, \cup, \Rightarrow, \emptyset, X\rangle$ is the dual Heyting algebra of X.

Proposition 6.5. *Let $\langle X, \leq, \tau, R\rangle$ be a concurrent Esakia space. Then, $\langle \mathbf{A}(X), \Box_R, \Diamond_R\rangle$ is a CH-algebra.*

Proof. By condition (C2) and Example 4.5, we have that $\langle \mathsf{Up}(X), \Box_R, \Diamond_R\rangle$ is a

CH-algebra. By condition (C3), it follows that $\langle \mathbf{A}(X), \Box_R, \Diamond_R\rangle$ is a subalgebra of $\langle \mathsf{Up}(X), \Box_R, \Diamond_R\rangle$. Hence $\langle \mathbf{A}(X), \Box_R, \Diamond_R\rangle$ is a CH-algebra. $\Box$

Let $\langle A, \Box, \Diamond\rangle$ be a CH-algebra. Recall that $\langle X(A), \subseteq, \tau_A\rangle$ is the dual Esakia space of A.

Proposition 6.6. *Let $\langle A, \Box, \Diamond\rangle$ be a CH-algebra. Then, the structure $\mathbf{X}(A) = \langle X(A), \subseteq, R_A, \tau_A\rangle$ is a concurrent Esakia space.*

Proof. (C1) By Esakia duality, $\langle X(A), \subseteq, \tau_A\rangle$ is an Esakia space.

(C2) By Proposition 5.3, $\langle X(A), \subseteq, R_A\rangle$ is an ic-frame.

(C3) We must show that $\Box_{R_A}(U)$ and $\Diamond_{R_A}(U)$ belong to $\mathcal{CP}\mathsf{Up}(X(A))$ for all $U \in \mathcal{CP}\mathsf{Up}(X(A))$. Let $U \in \mathcal{CP}\mathsf{Up}(X(A))$. Since $\mathcal{CP}\mathsf{Up}(X(A)) = \{\alpha(a) : a \in A\}$, there exists some $a \in A$ such that $U = \alpha(a)$. It follows that $\Box_{R_A}(U) = \Box_{R_A}(\alpha(a))$. Furthermore, by Theorem 5.4, we have $\Box_{R_A}(\alpha(a)) = \alpha(\Box a)$. Hence $\Box_{R_A}(U) \in \mathcal{CP}\mathsf{Up}(X(A))$. Similarly, for the diamond operator, $\Diamond_{R_A}(U) = \alpha(\Diamond a)$. Therefore $\Diamond_{R_A}(U)$ also belongs to $\mathcal{CP}\mathsf{Up}(X(A))$.

(C4) Our aim is to demonstrate that $R_A(x)$ is closed in $\langle \mathcal{C}\ell\mathsf{Up}(X(A)), \subseteq, \tau_v\rangle$. For this purpose, we consider the families of sets $\mathfrak{B} = \{D_U : U \in \mathcal{CP}\mathsf{Up}(X)\} \cup \{L_V : V \in \mathcal{CP}\mathsf{Do}(X)\}$ and $\mathcal{F} = \{\mathcal{U} \in \mathfrak{B} : R_A(x) \subseteq \mathcal{U}\}$. We prove this result by showing that $R_A(x) = \bigcap \mathcal{F}$. Obviously $R_A(x) \subseteq \bigcap \mathcal{F}$. We now show that $\bigcap \mathcal{F} \subseteq R_A(x)$. Let $\widehat{F} \in \bigcap \mathcal{F}$. This means that $\widehat{F} \in \mathcal{U}$, for all $\mathcal{U} \in \mathcal{F}$. To establish that $\widehat{F} \in R_A(x)$, we must prove that $\Box^{-1}(x) \subseteq F \subseteq \Diamond^{-1}(x)$. We first prove that $\Box^{-1}(x) \subseteq F$. Let $a \in \Box^{-1}(x)$. The condition $a \in F$ is logically equivalent to the set inclusion $\widehat{F} \subseteq \alpha(a)$ (since $a \in F \Leftrightarrow a \in y$ for every $y \in \widehat{F}$). Therefore, we proceed by proving $\widehat{F} \subseteq \alpha(a)$ by contradiction. Suppose it is not the case. This implies $\widehat{F} \notin D_{\alpha(a)}$. Since $D_{\alpha(a)}$ belongs to $\mathfrak{B}$, $D_{\alpha(a)} \notin \mathcal{F}$, meaning that $R_A(x) \nsubseteq D_{\alpha(a)}$. Thus, there exists $\widehat{E} \in R_A(x)$ such that $\widehat{E} \notin D_{\alpha(a)}$. Consequently, we have $\Box^{-1}(x) \subseteq E \subseteq \Diamond^{-1}(x)$ and $a \notin E$. This is a contradiction because $a \in \Box^{-1}(x)$.

Next, to prove the other inclusion, let $a \in F$. It follows that $\widehat{F} \cap \alpha(a)^c = \emptyset$. This implies that $\widehat{F} \notin L_{\alpha(a)^c}$. Consequently, $L_{\alpha(a)^c} \notin \mathcal{F}$, and hence $R_A(x) \nsubseteq L_{\alpha(a)^c}$. Therefore, there exists $\widehat{E} \in R_A(x)$ such that $\widehat{E} \notin L_{\alpha(a)^c}$. This yields $\widehat{E} \subseteq \alpha(a)$, which implies that $a \in E$. Since $\widehat{E} \in R_A(x)$, we have $E \subseteq \Diamond^{-1}(x)$. Hence, $a \in \Diamond^{-1}(x)$.

(C5) Now, we prove that $(x, \widehat{F}) \in R_A$ if and only if $\widehat{F} \subseteq \dot{\mathcal{S}}_{R_A}(x)$ and $\widehat{F} \in J_{R_A}(x)$. First suppose that $(x, \widehat{F}) \in R_A$. Then $\widehat{F} \subseteq S_{R_A}(x) \subseteq \dot{\mathcal{S}}_{R_A}(x)$. Since $\widehat{F} \in R_A(x)$ and $\widehat{F} \subseteq \widehat{F}$, it follows that $\widehat{F} \in J_{R_A}(x)$.

Now suppose that $\widehat{F} \subseteq \dot{\mathcal{S}}_{R_A}(x)$ and $\widehat{F} \in J_{R_A}(x)$. We must prove that $\widehat{F} \in R_A(x)$, namely that $\Box^{-1}(x) \subseteq F \subseteq \Diamond^{-1}(x)$. Suppose that $\Box^{-1}(x) \subseteq F$ is not true . Let $a \in \Box^{-1}(x)$ be such that $a \notin F$. By the Prime Filter Theorem, there exists $z \in X(A)$ such that $F \subseteq z$ and $a \notin z$. Then $z \in \widehat{F}$ y $a \notin z$. Since $\widehat{F} \subseteq \dot{\mathcal{S}}_{R_A}(x)$, there is $y \supseteq x$ such that $z \in S_{R_A}(y)$. So there exists $\widehat{G} \in R_A(y)$ such that $z \in \widehat{G}$. Then, $\Box^{-1}(y) \subseteq G \subseteq \Diamond^{-1}(y)$ and $G \subseteq z$. As $x \subseteq y$, we have $\Box^{-1}(x) \subseteq \Box^{-1}(y) \subseteq G \subseteq z$. Thus $a \in z$, which is a contradiction.

Now, since $\widehat{F} \in J_{R_A}(x)$, we have that there is $\widehat{G} \in R_A(x)$ such that $\widehat{G} \subseteq \widehat{F}$. Then $\Box^{-1}(x) \subseteq G \subseteq \Diamond^{-1}(x)$ and $F \subseteq G$. Thus, $F \subseteq \Diamond^{-1}(x)$.

(C6) Finally, we must prove that the condition $R_A(x) \neq \emptyset$ implies $\dot{\mathcal{S}}_{R_A}(x)$ is closed.

We prove this by showing that

$$\dot{\mathbb{S}}_{R_A}(x) = \widehat{\Box^{-1}(x)}.$$

Let $z \in \dot{\mathbb{S}}_{R_A}(x)$. By definition, there exists $y \in X(A)$ such that $y \supseteq x$ and $z \in S_{R_A}(y)$. This means that there is $\widehat{F} \in R_A(y)$ such that $z \in \widehat{F}$. Since $\widehat{F} \in R_A(y)$, we have $\Box^{-1}(y) \subseteq F \subseteq \Diamond^{-1}(y)$, and since $z \in \widehat{F}$, we have $F \subseteq z$. It follows that $\Box^{-1}(x) \subseteq \Box^{-1}(y) \subseteq z$. Hence, $z \in \widehat{\Box^{-1}(x)}$. Conversely, using the hypothesis that $R_A(x) \neq \emptyset$ it follows by Remark 5.1 that $\widehat{\Box^{-1}(x)} \in R_A(x)$. Therefore, $\widehat{\Box^{-1}(x)} \subseteq S_{R_A}(x) \subseteq \dot{\mathbb{S}}_{R_A}(x)$.

Since $\widehat{\Box^{-1}(x)}$ is a closed set, the established equality implies that $\dot{\mathbb{S}}_{R_A}(x)$ is closed, completing the proof. $\Box$

Theorem 6.7. *Let $\langle A, \Box, \Diamond\rangle$ be a CH-algebra. Then, $\alpha\colon A \to \mathbf{A}(X(A))$ is an isomorphism between the CH-algebras from $\langle A, \Box, \Diamond\rangle$ onto $\langle \mathbf{A}(\mathbf{X}(A)), \Box_{R_A}, \Diamond_{R_A}\rangle$.*

Proof. The structure $\langle \mathbf{A}(\mathbf{X}(A)), \Box_{R_A}, \Diamond_{R_A}\rangle$ is a CH-algebra by Propositions 6.5 and 6.6. Furthermore, Theorem 5.4 shows that α preserves the modal operators, satisfying the conditions $\alpha(\Box(a)) = \Box_{R_A}\alpha(a)$ and $\alpha(\Diamond a) = \Diamond_{R_A}\alpha(a)$. In addition, the results obtained by Esakia establish that α is an isomorphism of the underlying Heyting algebras. Consequently, since α preserves both the Heyting structure and the modal operators, α is an isomorphism between CH-algebras. $\Box$

Let now $\langle X, \leq, R, \tau\rangle$ be a concurrent Esakia space. Recall that $\langle X(\mathbf{A}(X)), \subseteq, \tau_{\mathbf{A}(X)}\rangle$ is the dual Esakia space of the Heyting algebra $\mathbf{A}(X) = \langle \mathcal{CP}\mathsf{Up}(X), \cap, \cup, \Rightarrow, \emptyset, X\rangle$ and recall that the map $\epsilon\colon X \to X(\mathbf{A}(X))$ defined by $\epsilon(x) = \{U \in \mathcal{CP}\mathsf{Up}(X) : x \in U\}$, for all $x \in X$, is an order-homeomorphism from $\langle X, \leq, \tau\rangle$ onto $\langle X(\mathbf{A}(X)), \subseteq, \tau_{\mathbf{A}(X)}\rangle$.

Theorem 6.8. *Let $\langle X, \leq, R, \tau\rangle$ be a concurrent Esakia space. Then, the order-homeomorphism ϵ satisfies the following: for every $x \in X$ and $Y \in \mathcal{C}\ell\mathsf{Up}(X)$,*

$$(x, Y) \in R \quad \textit{if and only if} \quad (\epsilon(x), \epsilon(Y)) \in R_{\mathbf{A}(X)}.$$

Proof. We first observe that $F_{\varepsilon(Y)} = \{U \in \mathcal{CP}\mathsf{Up}(X) : Y \subseteq U\} = F_Y$ and $\widehat{F_{\varepsilon(Y)}} = \varepsilon(Y)$.

We start by proving that $(x, Y) \in R$ implies $(\varepsilon(x), \varepsilon(Y)) \in R_{A(X)}$. Let $(x, Y) \in R$. We must prove that $\Box_R^{-1}(\varepsilon(x)) \subseteq F_{\varepsilon(Y)} \subseteq \Diamond_R^{-1}(\varepsilon(x))$. For the first part, if $U \in \Box_R^{-1}(\varepsilon(x))$, then $x \in \Box_R(U)$. This condition implies that for every $y \geq x$ and $Z \in R(y)$, $Z \subseteq U$. Since $Y \in R(x)$ by hypothesis, we specifically obtain $Y \subseteq U$. Thus, $U \in F_{\varepsilon(Y)}$, which establishes the first inclusion. To prove the second inclusion, consider $U \in F_{\varepsilon(Y)}$. This means $Y \subseteq U$. Since $Y \in R(x)$ by hypothesis, it immediately follows that $x \in \Diamond_R(U)$; consequently, $\Diamond_R(U) \in \varepsilon(x)$. Therefore, $U \in \Diamond_R^{-1}(\varepsilon(x))$, which concludes the proof of this implication.

Now, we prove the converse implication: $(\varepsilon(x), \varepsilon(Y)) \in R_{A(X)}$ implies $(x, Y) \in R$. We begin by verifying that $R(x)$ is non-empty. By hypothesis, $\varepsilon(Y) \in R_{A(X)}(\varepsilon(x))$,which imples that $\Box_R^{-1}(\varepsilon(x)) \subseteq F_{\varepsilon(Y)} \subseteq \Diamond_R^{-1}(\varepsilon(x))$. Since $F_{\varepsilon(Y)}$ is a filter of $\mathcal{CP}\mathsf{Up}(X)$, X must be an element of $F_{\varepsilon(Y)}$. Thus $X \in \Diamond_R^{-1}(\varepsilon(x))$. Consequently, $x \in \Diamond_R(X)$, meaning there exists $Z \in R(x)$ such that $Z \subseteq X$. This ensures that $R(x)$ is non-empty, which by Definition 6.4 implies that $\dot{\mathbb{S}}_R(x)$ is closed in X.

We now prove that $Y \subseteq \dot{\mathsf{S}}_R(x)$ (and subsequently that $Y \in J_R(x)$). Suppose, for contradiction, that $Y \nsubseteq \dot{\mathsf{S}}_R(x)$. Thus, there exists $z \in Y$ such that $z \notin \dot{\mathsf{S}}_R(x)$. Since $\dot{\mathsf{S}}_R(x)$ is a closed upset, there exists $U \in \mathcal{CP}\mathsf{Up}(X)$ such that $z \notin U$ and $\dot{\mathsf{S}}_R(x) \subseteq U$. Since $z \in Y$ but $z \notin U$, it follows that $Y \nsubseteq U$. This condition implies that $U \notin F_{\varepsilon(Y)}$. Since $\Box_R^{-1}(\varepsilon(x)) \subseteq F_{\varepsilon(Y)}$ holds, we obtain that $U \notin \Box_R^{-1}(\varepsilon(x))$, which in turn gives $x \notin \Box_R(U)$. Conversely, from $\dot{\mathsf{S}}_R(x) \subseteq U$ we have that $x \in \Box_{\dot{\mathsf{S}}_R}(U) = \Box_R(U)$, leading to a contradiction.

We now show that $Y \in J_R(x)$, which requires finding $W \in R(x)$ such that $W \subseteq Y$. We first establish that $R(x) \nsubseteq L_{U_1^c \cup U_2^c \cup \cdots \cup U_n^c}$ for every finite subset $\{U_1, U_2, \cdots, U_n\} \subseteq F_{\varepsilon(Y)}$. Since each $U_i \in \mathcal{CP}\mathsf{Up}(X)$, their complements U_i^c are in $\mathcal{CP}\mathsf{Do}(X)$, justifying the set $L_{U_1^c \cup U_2^c \cup \cdots \cup U_n^c}$. Let $\{U_1, \cdots, U_n\} \subseteq F_{\varepsilon(Y)}$. Since $F_{\varepsilon(Y)}$ is a filter, $U_1 \cap \cdots \cap U_n \in F_{\varepsilon(Y)}$. By hypothesis, we have that $U_1 \cap \cdots \cap U_n \in \Diamond_R^{-1}(\varepsilon(x))$. Consequently, $x \in \Diamond_R(U_1 \cap \cdots \cap U_n)$ which implies there is $Z \in R(x)$ such that $Z \subseteq U_1 \cap \cdots \cap U_n$. From this, we have $Z \in R(x)$ and $Z \notin L_{U_1^c \cup \cdots \cup U_n^c}$. That is, $R(x) \nsubseteq L_{U_1^c \cup \cdots \cup U_n^c}$. Since $R(x)$ is compact (because $\mathcal{C}\ell\mathsf{Up}(X)$ is compact), we deduce that $R(x) \nsubseteq \bigcup\{L_{U^c} : Y \subseteq U\}$. Therefore, there exists $W \in R(x)$ such that $W \notin \bigcup\{L_{U^c} : Y \subseteq U\}$, meaning $W \notin L_{U^c}$ for every $U \in \mathcal{CP}\mathsf{Up}(X)$ such that $Y \subseteq U$. The latter implies $U^c \cap W = \emptyset$ for all $U \in \mathcal{CP}\mathsf{Up}(X)$ with $Y \subseteq U$. This, in turn, implies $W \subseteq \bigcap\{U \in \mathcal{CP}\mathsf{Up}(X) : Y \subseteq U\} = Y$. Hence, we have found $W \in R(x)$ such that $W \subseteq Y$, concluding the proof of the implication. $\Box$

Now we move to define the morphisms between concurrent Esakia spaces and between CH-algebras.

Definition 6.9. Let $\langle X_1, \leq_1, R_1, \tau_1\rangle$ and $\langle X_2, \leq_2, R_2, \tau_2\rangle$ be two concurrent Esakia spaces. A map $f\colon X_1 \to X_2$ is a *concurrent Esakia morphism* if:

(CM 1) f is continuous.
(CM 2) f is a p-morphism.
(CM 3) If $(x, Y) \in R_1$, then $(f(x), f(Y)) \in R_2$, where $f(Y) = \{f(y) : y \in Y\}$.
(CM 4) If $(f(x), Y) \in R_2$, then there exist $Z_1, Z_2 \in R_1(x)$ such that $f(Z_1) \subseteq Y \subseteq f(Z_2)$.

Remark 6.10. Conditions (CM 1) and (CM 2) can be equivalently replaced by: (CM 1') f is an Esakia morphism.

Definition 6.11. Let $\langle X_1, \leq_1, R_1, \tau_1\rangle$ and $\langle X_2, \leq_2, R_2, \tau_2\rangle$ be two concurrent Esakia spaces and let $f\colon X_1 \to X_2$ be a concurrent Esakia morphism. We define $f^*\colon \mathcal{CP}\mathsf{Up}(X_2) \to \mathcal{CP}\mathsf{Up}(X_1)$ as $f^*(U) = f^{-1}(U)$.

Remark 6.12. Since f satisfies conditions (CM 1) and (CM 2) of Definition 6.9, we have by Esakia's duality that f^* is a homomorphism of Heyting algebras.

Proposition 6.13. *Let $\langle X_1, \leq_1, R_1, \tau_1\rangle$ and $\langle X_2, \leq_2, R_2, \tau_2\rangle$ be two concurrent Esakia spaces and let $f\colon X_1 \to X_2$ be a concurrent Esakia morphism. Then $f^*\colon \mathcal{CP}\mathsf{Up}(X_2) \to \mathcal{CP}\mathsf{Up}(X_1)$ is a homomorphism of concurrent Heyting Algebra.*

Proof. We must show that $f^*(\Box_{R_2}(U)) = \Box_{R_1} f^*(U)$ and $f^*(\Diamond_{R_2}(U)) = \Diamond_{R_1} f^*(U)$.

Let $x \in f^*(\Box_{R_2}(U))$. By definition of f^*, $f(x) \in \Box_{R_2}(U)$. This means that for all $u \geq f(x)$ and for all $Z \in R_2(u)$, $Z \subseteq U$. We want to show that $x \in \Box_{R_1} f^*(U)$. Consider $y \geq x$ and $Y \in R_1(y)$. We must prove that $Y \subseteq f^*(U)$. By condition (CM 3) of Definition 6.9, we have $(f(y), f(Y)) \in R_2$.Also, since f is an order-morphism, condition (CM 2) ensures $f(y) \geq f(x)$. Thus, by hypothesis, $f(Y) \subseteq U$. Therefore $Y \subseteq f^*(U)$.

On the other hand, let $x \in \Box_{R_1} f^*(U)$. This means that for all $y \geq x$ and for every $Y \in R_1(y)$, $Y \subseteq f^*(U)$. We aim to show that $f(x) \in \Box_{R_2}(U)$. Consider $u \geq f(x)$ and $Z \in R_2(u)$. We must demonstrate that $Z \subseteq U$. By Definition 6.9, condition (CM2) b), there exists $z \in X_1$ such that $z \geq x$ and $f(z) = u$. Thus, $f(z) \geq f(x)$ and $Z \in R_2(f(z))$. Then by (CM 4), there is $Y \in R_1(z)$ such that $Z \subseteq f(Y)$. By hypothesis, we obtain that $Y \subseteq f^*(U)$, which implies $f(Y) \subseteq U$. Therefore, $Z \subseteq U$. Consequently, the equality $f^*(\Box_{R_2}(U)) = \Box_{R_1} f^*(U)$ is proven.

Suppose that $x \in f^*(\Diamond_{R_2}(U))$. Then $f(x) \in \Diamond_{R_2}(U)$. Thus, there exists $Y \in R_2(f(x))$ such that $Y \subseteq U$. We need to prove that $x \in \Diamond_{R_1} f^*(U)$, which means finding $Z \in R_1(x)$ such that $Z \subseteq f^*(U)$. Since $(f(x), Y) \in R_2$, condition (CM 4) of Definition 6.9 ensures that there exists $Z \in R_1(x)$ such that $f(Z) \subseteq Y$. Since $Y \subseteq U$ by hypothesis, it follows that $f(Z) \subseteq U$. Therefore $Z \subseteq f^*(U)$, as required.

Finally, assume that $x \in \Diamond_{R_1} f^*(U)$. Then there exists $Z \in R_1(x)$ such that $Z \subseteq f^*(U)$. We must prove that $x \in f^*(\Diamond_{R_2}(U))$, or equivalently, that $f(x) \in \Diamond_{R_2}(U)$. This requires us to find $Y \in R_2(f(x))$ such that $Y \subseteq U$. Since $(x, Z) \in R_1$, condition (CM 3) establishes that $(f(x), f(Z)) \in R_2$. Also, since $Z \subseteq f^*(U)$, we immediately have $f(Z) \subseteq U$. Letting $Y = f(Z)$, we find $Y \in R_2(f(x))$ and $Y \subseteq U$. Therefore, $f^*(\Diamond_{R_2}(U)) = \Diamond_{R_1} f^*(U)$ is proven.

Therefore, since f^* preserves the operations of the Heyting algebra and we have shown it preserves $\Box$ and $\Diamond$, f^* is a homomorphism of concurrent Heyting algebras. $\Box$

Definition 6.14. Let $h\colon A \to B$ be a homomorphism between concurrent Heyting algebras. Define $h_*\colon X(B) \to X(A)$ as $h_*(x) = h^{-1}(x)$.

Remark 6.15. It was already proved by Esakia that h_* is well defined and satisfies (CM 1) and (CM 2) of Definition 6.9.

Lemma 6.16. *Let $h\colon A \to B$ be a homomorphism between concurrent Heyting algebras, and let $\widehat{G} \in R_B(x)$ for some $x \in X(B)$. Then $h_*(\widehat{G}) \in \mathcal{C}\ell\mathsf{Up}(X(A))$ and also $h_*(\widehat{G}) = \widehat{h^{-1}(G)}$.*

Proof. Since h_* is continuous and a p-morphism,the image $h_*(\widehat{G})$ is a closed upset in X(A); i.e., $h_*(\widehat{G}) \in \mathcal{C}\ell\mathsf{Up}(X(A))$.

We now show that $h_*(\widehat{G}) = \widehat{h^{-1}(G)}$. Recall that $F_{h_*(\widehat{G})} = \{a \in A : h_*(\widehat{G}) \subseteq \alpha(a)\}$ and $\widehat{F_{h_*(\widehat{G})}} = h_*(\widehat{G})$. Therefore, to prove the set equality, it is sufficient to demonstrate that the corresponding filters are equal: $h^{-1}(G) = F_{h_*(\widehat{G})}$.

$$
\begin{aligned}
a \in F_{h_*(\widehat{G})} &\text{ iff } h_*(\widehat{G}) \subseteq \varphi(a) \\
&\text{ iff } h_*(y) \in \varphi(a) \text{ for every } y \in \widehat{G} \\
&\text{ iff } a \in h_*(y) \text{ for every } y \in \widehat{G} \\
&\text{ iff } h(a) \in y \text{ for every } y \in \widehat{G} \\
&\text{ iff } h(a) \in y \text{ for every } y \in X(B) \text{ such that } G \subseteq y \\
&\text{ iff } h(a) \in G \\
&\text{ iff } a \in h^{-1}(G).
\end{aligned}
$$

$\Box$

Proposition 6.17. *Let $h\colon A \to B$ be a homomorphism between concurrent Heyting* *algebras. Then $h_*\colon X(B) \to X(A)$ is a concurrent Esakia morphism.*

Proof. Based on Remark 6.15, it suffices to prove that h_* satisfies conditions (CM 3) and (CM 4) of Definition 6.9.

(CM 3) Suppose that $\widehat{G} \in R_B(x)$. By definition, this means $\Box_B^{-1}(x) \subseteq G \subseteq \Diamond_B^{-1}(x)$. We must prove that $h_*(\widehat{G}) \in R_A(h_*(x))$. By the previous lemma, we have $h_*(\widehat{G}) \in \mathcal{C}\ell\mathsf{Up}(X(A))$ and $h_*(\widehat{G}) = \widehat{h^{-1}(G)}$. Therefore, we need to show that the dual filters satisfy the condition: $\Box_A^{-1}(h_*(x)) \subseteq h^{-1}(G) \subseteq \Diamond_A^{-1}(h_*(x))$.

Let $a \in \Box_A^{-1}(h_*(x))$. This implies $\Box_A(a) \in h_*(x)$, and thus $h(\Box_A(a)) \in x$. Since h is a homomorphism of concurrent Heyting algebras , $h(\Box_A(a)) = \Box_B(h(a)) \in x$. This means $h(a) \in \Box_B^{-1}(x)$. By hypothesis, we have $h(a) \in G$, consequently $a \in h^{-1}(G)$. Now, suppose $a \in h^{-1}(G)$. Then $h(a) \in G$. By hypothesis, we have $h(a) \in \Diamond_B^{-1}(x)$, which means $\Diamond_B(h(a)) \in x$. Since h is a homomorphism, $\Diamond_B(h(a)) = h(\Diamond_A(a)) \in x$. This yields $a \in \Diamond_A^{-1}(h_*(x))$. Since both inclusions are satisfied, the condition (CM 3) is proven.

(CM 4) Let $\widehat{F} \in R_A(h_*(x))$, which means $\Box_A^{-1}(h_*(x)) \subseteq F \subseteq \Diamond_A^{-1}(h_*(x))$. We must find $\widehat{G_1}, \widehat{G_2} \in R_B(x)$ such that $h_*(\widehat{G_1}) \subseteq \widehat{F} \subseteq h_*(\widehat{G_2})$. Let G_1 be the filter defined by $G_1 := \mathrm{Fg}(h(F) \cup \Box_B^{-1}(x))$. The inclusion $\Box_B^{-1}(x) \subseteq G_1$ is clear by construction. To prove the second inclusion, assume $g \in G_1$. Then there exist $a \in F$ and $b \in \Box_B^{-1}(x)$ such that $h(a) \wedge b \leq g$. It follows that $b \leq h(a) \to g$, and thus $h(a) \to g \in \Box_B^{-1}(x)$. Therefore $\Box_B(h(a) \to g) \in x$. Since $\Box_B(h(a) \to g) \leq \Diamond_B(h(a)) \to \Diamond_B(g)$, we have $\Diamond_B(h(a)) \to \Diamond_B(g) \in x$. On the other hand, from $a \in F$, the hypothesis gives $a \in \Diamond_A^{-1}(h_*(x))$. This means $h(\Diamond_A(a)) \in x$, which, by homomorphism, is $\Diamond_B(h(a)) \in x$. Since both $\Diamond_B(h(a)) \to \Diamond_B(g) \in x$ and $\Diamond_B(h(a)) \in x$, we deduce $\Diamond_B(g) \in x$ or equivalently, $g \in \Diamond_B^{-1}(x)$. Thus, $\widehat{G}_1 \in R_B(x)$. Now we show the set inclusion $h_*(\widehat{G}_1) \subseteq \widehat{F}$. By construction of G_1, $h(F) \subseteq G_1$. It follows that $F \subseteq h^{-1}(G_1)$. Thus, $\widehat{h^{-1}(G_1)} \subseteq \widehat{F}$. This proves $h_*(\widehat{G}_1) \subseteq \widehat{F}$ (by Lemma 6.16). We claim that the required set is $\widehat{G_2} = \widehat{\Box_B^{-1}(x)}$. Since $\widehat{G}_1 \in R_B(x)$, it is immediate that $\widehat{\Box_B^{-1}(x)} \in R_B(x)$. We now prove the required inclusion $\widehat{F} \subseteq h_*(\widehat{G}_2)$, i.e., $\widehat{F} \subseteq h_*(\widehat{\Box_B^{-1}(x)})$. By the previous lemma, this holds if and only if the dual filters satisfy the reverse inclusion: $h^{-1}(\Box_B^{-1}(x)) \subseteq F$. Let $a \in h^{-1}(\Box_B^{-1}(x))$. Then $\Box_B(h(a)) \in x$, which implies that $h(\Box_A(a)) \in x$. Since h is a homomorphism, $\Box_B(h(a)) = h(\Box_A(a)) \in x$. Hence $a \in \Box_A^{-1}(h_*(x))$. By hypothesis, we conclude that $a \in F$. Therefore, the inclusion $h^{-1}(\Box_B^{-1}(x)) \subseteq F$ is established, which proves $\widehat{F} \subseteq h_*(\widehat{G}_2)$. This completes the proof. $\Box$

Our objective now is to prove, in categorical terms, that the categories of concurrent Esakia spaces and concurrent Heyting algebras, together with their morphisms, are dually equivalent. For this purpose, it is necessary to define formally the categories. We denote by $\mathbb{CHA}$ the category whose objects are concurrent Heyting algebras and whose morphisms are homomorphisms of concurrent Heyting algebras. We denote by $\mathbb{CES}$ the category with concurrent Esakia spaces as objects and concurrent Esakia morphisms as arrows.

Theorem 6.18. *The category $\mathbb{CHA}$ is dually equivalent to the category $\mathbb{CES}$.*

Proof. We first define the contravariant functors that establish the desired equivalence. The functor $(-)_*\colon \mathbb{CHA} \to \mathbb{CES}$ is defined as follows: for each object $A \in \mathcal{O}b(\mathbb{CHA})$, $A_* = \langle X(A), \subseteq, R_A, \tau_A \rangle$ is the dual concurrent Esakia space of A, and for each morphism $h \in \mathcal{M}_{\mathbb{CHA}}(A, B)$, $h_* \colon B_* \to A_*$ is defined by $h_*(x) = h^{-1}(x)$.

The functor $(-)^* \colon \mathbb{CES} \to \mathbb{CHA}$ is defined as follows: for each object $X \in \mathcal{O}b(\mathbb{CES})$, $X^* = \langle \mathcal{CP}\mathsf{Up}(X), \Box_R, \Diamond_R \rangle$ is the dual concurrent Heyting algebra, and for each morphism $f \in \mathcal{M}_{\mathbb{CES}}(X, Y)$, $f^* \colon Y^* \to X^*$ is defined by $f^*(U) = f^{-1}(U)$.

To prove that the functors establish a dual equivalence, we must construct a natural isomorphism α between the composite functor $((-)_*)^*$ and the identity functor in $\mathbb{CHA}$, $id_{\mathbb{CHA}}$ and another ε between $((-)^*)_*$ and the identity functor in $\mathbb{CES}$, $id_{\mathbb{CES}}$.

We first construct the natural isomorphism $\alpha \colon ((-)_*)^* \Rightarrow id_{\mathbb{CHA}}$. For each concurrent Heyting algebra A, we define the concurrent Heyting algebra isomorphism $\alpha_A \colon A \to (A_*)^*$ by $\alpha_A(a) = \{x \in A_* : a \in x\}$. Let $\alpha = \{\alpha_A : A \in \mathcal{O}b(\mathbb{CHA})\}$. We verify naturality. Suppose $A, B \in \mathcal{O}b(\mathbb{CHA})$ and $h \in \mathcal{M}_{\mathbb{CHA}}(A, B)$. We must prove that $(h_*)^* \circ \alpha_A = \alpha_B \circ h$. Let $a \in A$, then

$$y \in (h_*)^*(\alpha_A(a)) \iff h_*(y) \in \alpha_A(a) \iff a \in h_*(y) \iff h(a) \in y \iff y \in \alpha_B(h(a)).$$

We next construct the natural isomorphism $\varepsilon \colon id_{\mathbb{CES}} \Rightarrow ((-)^*)_*$. For each concurrent Esakia space $X \in \mathcal{O}b(\mathbb{CES})$ we define the homeomorphism of concurrent Esakia spaces $\varepsilon_X \colon X \to (X^*)_*$ by $\varepsilon_X(x) = \{U \in X^* : x \in U\}$. Let $\varepsilon = \{\varepsilon_X : X \in \mathcal{O}b(\mathbb{CES})\}$. We verify the naturality. Suppose $X, Y \in \mathcal{O}b(\mathbb{CES})$ and $f \in \mathcal{M}_{\mathbb{CES}}(X, Y)$. We must prove that $(f^*)_* \circ \varepsilon_X = \varepsilon_Y \circ f$. Let $x \in X$, then

$$U \in (f^*)_* \varepsilon_X(x) \iff f^*(U) \in \varepsilon_X(x) \iff x \in f^*(U) \iff f(x) \in U \iff U \in \varepsilon_Y(f(x)).$$

In consequence, the category $\mathbb{CHA}$ is dually equivalent to the category $\mathbb{CES}$. $\Box$

7. IKN-spaces and a new categorical dual equivalence

In Section 6, we show that the two relations $\dot{\mathsf{S}}_R \subseteq X \times X$ and $J_R \subseteq X \times \mathsf{Up}(X)$ (Definition 6.1) can be defined from an ic-frame $\langle X, \leq, R \rangle$. Then, we define the concurrent Esakia spaces in such a way that the relations R, $\dot{\mathsf{S}}_R$, and J_R are related (see condition (C5) in Definition 6.4). Thus, we may consider the structures $\langle X, \leq, \dot{\mathsf{S}}, J \rangle$, where $\dot{\mathsf{S}} \subseteq X \times X$ and $J \subseteq X \times \mathsf{Up}(X)$, to obtain a different relational semantics for the logic $\Lambda(\mathcal{H})$. Notice that these structures $\langle X, \leq, \dot{\mathsf{S}}, J \rangle$ are a blend of an intuitionistic Kripke frame and a neighbourhood frame.

This section considers the structures $\langle X, \leq, \dot{\mathsf{S}}, J \rangle$ satisfying some conditions. We show that these structures are, under certain conditions, inter-definable concerning the ic-frames. We provide a new dual equivalence for the category of CH-algebras and a completeness result for the logic $\Lambda(\mathcal{H})$.

Definition 7.1. An *intuitionistic Kripke neighbourhood frame* or *IKN-frame* is a structure $\langle X, \leq, \dot{\mathsf{S}}, J \rangle$, where $\langle X, \leq \rangle$ is a poset, $\dot{\mathsf{S}} \subseteq X \times X$ and $J \subseteq X \times \mathsf{Up}(X)$, such that the following conditions are satisfied:

(K1) $y \geq x$ implies $\dot{\mathsf{S}}(y) \subseteq \dot{\mathsf{S}}(x)$.

(K2) If $Y \in J(x)$ and $y \geq x$, then there exists $Y' \in \mathsf{Up}(X)$ such that $Y' \subseteq Y$, $Y' \subseteq \dot{\mathsf{S}}(y)$ and $Y' \in J(y)$.

(K3) If $y \in \dot{\mathsf{S}}(x)$, there exist $z \geq x$ and $Y \in \mathsf{Up}(X)$ such that $Y \subseteq \dot{\mathsf{S}}(z)$, $Y \in J(z)$ and $y \in Y$.

(K4) If $Y \in J(x)$ and $Z \in \mathsf{Up}(X)$ such that $Y \subseteq Z$, then $Z \in J(x)$.

Let us denote by IKNF the class of IKN-frames. In the following definition and proposition, let us show how to define an ic-frame from an IKN-frame (cf. condition (C5) Definition 6.4).

Definition 7.2. Let $\mathcal{N} = \langle X, \leq, \dot{\mathsf{S}}, J\rangle$ be an IKN-frame. We define $R_{\mathcal{N}} \subseteq X \times \mathsf{Up}(X)$ as

$$Y \in R_{\mathcal{N}}(x) \text{ iff } Y \subseteq \dot{\mathsf{S}}(x) \text{ and } Y \in J(x)$$

for all $Y \in \mathsf{Up}(X)$ and every $x \in X$.

Proposition 7.3. *Let $\mathcal{N} = \langle X, \leq, \dot{\mathsf{S}}, J\rangle$ be an IKN-frame. Then $\mathcal{F}_{\mathcal{N}} = \langle X, \leq, R_{\mathcal{N}}\rangle$ is an ic-frame.*

Proof. Suppose $x, y \in X$ and $Y \in \mathsf{Up}(X)$ such that $y \geq x$ and $Y \in R_{\mathcal{N}}(x)$. By definition of $R_{\mathcal{N}}$, the condition $Y \in R_{\mathcal{N}}(x)$ implies that $Y \subseteq \dot{\mathsf{S}}(x)$ and $Y \in J(x)$. Applying condition (K2) of Definition 7.1, since $y \geq x$ and $Y \in J(x)$, there exists $Y' \in \mathsf{Up}(X)$ such that $Y' \subseteq Y$, $Y' \subseteq \dot{\mathsf{S}}(y)$ and $Y' \in J(y)$. Thus, since $Y' \subseteq \dot{\mathsf{S}}(y)$ and $Y' \in J(y)$, it follows by definition that $Y' \in R_{\mathcal{N}}(y)$. As $Y' \subseteq Y$ is satisfied, the required condition for $\mathcal{F}_{\mathcal{N}}$ to be an ic-frame is met. $\square$

Conversely, let us show how to define an IKN-frame from an ic-frame.

Proposition 7.4. *Let $\mathcal{F} = \langle X, \leq, R\rangle$ be an ic-frame. Then $\mathcal{N}_{\mathcal{F}} = \langle X, \leq, \dot{\mathsf{S}}_R, J_R\rangle$ is an IKN-frame, where $\dot{\mathsf{S}}_R$ and J_R are defined as in Definition 6.1.*

Proof. Let $\langle X, \leq, R\rangle$ be an ic-frame. We verify the IKN-frame conditions (K1)-(K4).

(K1) If $y \geq x$, the inclusion $\dot{\mathsf{S}}_R(y) \subseteq \dot{\mathsf{S}}_R(x)$ follows immediately by the definition of $\dot{\mathsf{S}}_R$.

(K2) Suppose $y \geq x$ and $Y \in J_R(x)$. By definition of J_R there exists $Z \in \mathsf{Up}(X)$ such that $Z \in R(x)$ and $Z \subseteq Y$. Since $y \geq x$ and $Z \in R(x)$ the ic-frame condition (CI) ensures the existence of $Y' \in R(y)$ such that $Y' \subseteq Z$.This upset Y' satisfies the required properties: first, since $Y' \in R(y)$, it follows immediately by the definition of J_R that $Y' \in J_R(y)$; second, because $Y' \in R(y)$ by definition of S_R and $\dot{\mathsf{S}}_R$ we have $Y' \subseteq S_R(y) \subseteq \dot{\mathsf{S}}_R(y)$; finally, as $Y' \subseteq Z$ and $Z \subseteq Y$, the inclusion $Y' \subseteq Y$ holds. Thus, Y' is the required upset.

(K3) Now, let $y \in \dot{\mathsf{S}}_R(x)$. By definition of $\dot{\mathsf{S}}_R$, there exists $z \geq x$ such that $y \in S_R(z)$.The definition of S_R then guarantees the existence of an upset $Y \in R(z)$ such that $y \in Y$. Since $Y \in R(z)$ it follows that $Y \subseteq \dot{\mathsf{S}}_R(z)$ and $Y \in J_R(z)$, which satisfies the required property.

(K4) Finally, suppose $Y \in J_R(x)$ and $Z \in \mathsf{Up}(X)$ such that $Y \subseteq Z$. Since $Y \in J_R(x)$, the definition of J_R requires the existence of $Y' \in R(x)$ such that $Y' \subseteq Y$. Given that $Y' \subseteq Y$ and $Y \subseteq Z$, we have $Y' \subseteq Z$. As we found an upset $Y' \in R(x)$ such that $Y' \subseteq Z$, it follows immediately that$Z \in J_R(x)$. $\square$

We now show that if we start from an IKN-frame $\mathcal{N}$, we obtain that $\mathcal{N} = \mathcal{N}_{\mathcal{F}_{\mathcal{N}}}$ by the previous constructions. However, if we start from an ic-frame $\mathcal{F}$ and we consider the ic-frame $\mathcal{F}_{\mathcal{N}_{\mathcal{F}}}$ given by the previous constructions, then it is not necessarily the case that $\mathcal{F} = \mathcal{F}_{\mathcal{N}_{\mathcal{F}}}$. We need an extra condition to achieve this.

Proposition 7.5. *Let $\mathcal{N} = \langle X, \leq, \dot{\mathbb{S}}, J\rangle$ be an IKN-frame. Let us consider the IKN-frame $\mathcal{N}_{\mathcal{F}_{\mathcal{N}}} = \langle X, \leq, \dot{\mathbb{S}}_{R_{\mathcal{N}}}, J_{R_{\mathcal{N}}}\rangle$. Then $\dot{\mathbb{S}} = \dot{\mathbb{S}}_{R_{\mathcal{N}}}$ y $J = J_{R_{\mathcal{N}}}$.*

Proof. We first prove $\dot{\mathbb{S}}(x) = \dot{\mathbb{S}}_{R_{\mathcal{N}}}(x)$. Let $x \in X$. Suppose $y \in \dot{\mathbb{S}}(x)$. By Definition 7.1 (K3), there exist $z \geq x$ and $Y \in \mathsf{Up}(X)$ such that $y \in Y$, $Y \subseteq \dot{\mathbb{S}}(z)$, and $Y \in J(z)$. By definition of $R_{\mathcal{N}}$, we have $Y \in R_{\mathcal{N}}(z)$. Since $y \in Y$ and $Y \in R_{\mathcal{N}}(z)$ where $z \geq x$, it follows by the definition of $\dot{\mathbb{S}}_{R_{\mathcal{N}}}$ that $y \in \dot{\mathbb{S}}_{R_{\mathcal{N}}}(x)$. Conversely, let $y \in \dot{\mathbb{S}}_{R_{\mathcal{N}}}(x)$. Thus, there exist $z \geq x$ and $Y \in R_{\mathcal{N}}(z)$ such that $y \in Y$. By the definition of $R_{\mathcal{N}}$, the condition $Y \in R_{\mathcal{N}}(z)$ implies $Y \subseteq \dot{\mathbb{S}}(z)$ and $Y \in J(z)$. Given $z \geq x$, condition (K1) ensures that $\dot{\mathbb{S}}(z) \subseteq \dot{\mathbb{S}}(x)$. Since $y \in Y$ and $Y \subseteq \dot{\mathbb{S}}(z) \subseteq \dot{\mathbb{S}}(x)$, we conclude that $y \in \dot{\mathbb{S}}(x)$. Therefore $\dot{\mathbb{S}} = \dot{\mathbb{S}}_{R_{\mathcal{N}}}$.

We next prove $J = J_{R_{\mathcal{N}}}$. Let $x \in X$. Suppose $Y \in J(x)$. By condition (K2) of Definition 7.1, there exists $Y' \in \mathsf{Up}(X)$ such that $Y' \subseteq Y$, $Y' \subseteq \dot{\mathbb{S}}(x)$ and $Y' \in J(x)$. Since $Y' \subseteq \dot{\mathbb{S}}(x)$ and $Y' \in J(x)$, it follows by definition of $R_{\mathcal{N}}$ that $Y' \in R_{\mathcal{N}}(x)$. As we found $Y' \in R_{\mathcal{N}}(x)$ such that $Y' \subseteq Y$, the definition of $J_{R_{\mathcal{N}}}$ implies $Y \in J_{R_{\mathcal{N}}}(x)$. Conversely, let $Y \in J_{R_{\mathcal{N}}}(x)$. By definition of $J_{R_{\mathcal{N}}}$, there exists $Y' \subseteq Y$ such that $Y' \in R_{\mathcal{N}}(x)$. The definition of $R_{\mathcal{N}}$ implies that $Y' \subseteq \dot{\mathbb{S}}(x)$ and $Y' \in J(x)$. Consequently, since $Y' \in J(x)$ and $Y' \subseteq Y$, condition (K4) of Definition 7.1 ensures that $Y \in J(x)$. Therefore, $J = J_{R_{\mathcal{N}}}$. □

Proposition 7.6. *Let $\mathcal{F} = \langle X, \leq, R\rangle$ be an ic-frame and let τ be a topology on X such that $\langle X, \leq, R, \tau\rangle$ is a concurrent Esakia space. Let us consider the ic-frame $\mathcal{F}_{\mathcal{N}_{\mathcal{F}}} = \langle X, \leq, R_{\mathcal{N}_{\mathcal{F}}}\rangle$. Then $R = R_{\mathcal{N}_{\mathcal{F}}}$*

Proof. Let $x \in X$ and $Y \in \mathsf{Up}(X)$. By condition (C5) of Definition 6.4, $Y \in R(x)$ if and only if $Y \subseteq \dot{\mathbb{S}}_R(x)$ and $Y \in J_R(x)$. By definition, $R_{\mathcal{N}_{\mathcal{F}}}$ is characterized by the exact same condition: $Y \in R_{\mathcal{N}_{\mathcal{F}}}(x)$ if and only if $Y \subseteq \dot{\mathbb{S}}_R(x)$ and $Y \in J_R(x)$. Thus, $R = R_{\mathcal{N}_{\mathcal{F}}}$. □

We use the two previous propositions to show that there is a one-to-one correspondence between concurrent Esakia spaces and specific structures $\langle X, \leq, \dot{\mathbb{S}}, J, \tau\rangle$ where $\langle X, \leq, \dot{\mathbb{S}}, J\rangle$ is an IKN-frame and $\langle X, \leq, \tau\rangle$ is an Esakia space satisfying additional conditions.

Definition 7.7. Let $\langle X, \leq, \dot{\mathbb{S}}, J\rangle$ be an IKN-frame. We define the sets $\Box_{\dot{\mathbb{S}}}$ and $\Diamond_J$ as

$$\Box_{\dot{\mathbb{S}}}(U) = \{x \in X : \dot{\mathbb{S}}(x) \subseteq U\} \qquad \text{and} \qquad \Diamond_J(U) = \{x \in X : \exists Y \in J(x)(Y \subseteq U)\}$$

for all $U \in \mathsf{Up}(X)$.

Proposition 7.8. *Let $\mathcal{N} = \langle X, \leq, \dot{\mathbb{S}}, J\rangle$ be an IKN-frame. Let us consider the ic-frame $\mathcal{F}_{\mathcal{N}} = \langle X, \leq, R_{\mathcal{N}}\rangle$. Then for all $U \in \mathsf{Up}(X)$*

$$\Box_{\dot{\mathbb{S}}}(U) = \Box_{R_{\mathcal{N}}}(U) \qquad \textit{and} \qquad \Diamond_J(U) = \Diamond_{R_{\mathcal{N}}}(U).$$

Proof. We first prove $\Box_{\dot{\mathbb{S}}}(U) = \Box_{R_{\mathcal{N}}}$. Let $x \in \Box_{\dot{\mathbb{S}}}(U)$, so $\dot{\mathbb{S}}(x) \subseteq U$. Suppose $y \geq x$ and $Y \in R_{\mathcal{N}}(y)$. By definition of $R_{\mathcal{N}}$, $Y \subseteq \dot{\mathbb{S}}(y)$. By (K1), $\dot{\mathbb{S}}(y) \subseteq \dot{\mathbb{S}}(x)$. Thus $Y \subseteq \dot{\mathbb{S}}(x) \subseteq U$, which implies $Y \subseteq U$. Therefore, $x \in \Box_{R_{\mathcal{N}}}(U)$. Conversely, let $x \in \Box_{R_{\mathcal{N}}}(U)$ and let $y \in \dot{\mathbb{S}}(x)$. By (K3), there exist $z \geq x$ and $Y \in \mathsf{Up}(X)$ such that $y \in Y$, $Y \subseteq \dot{\mathbb{S}}(z)$, and $Y \in J(z)$. It follows that $Y \in R_{\mathcal{N}}(z)$ by definition. Since $x \in \Box_{R_{\mathcal{N}}}(U)$, the condition

$z \geq x$ and $Y \in R_{\mathcal{N}}(z)$ implies $Y \subseteq U$. As $y \in Y$, we conclude that $y \in U$. It follows that$\dot{\mathsf{S}}(x) \subseteq U$, so $x \in \Box_{\dot{\mathsf{S}}}(U)$.

We now establish the second required equality. Let $x \in \Diamond_J(U)$. By definition, there is $Y \in J(x)$ such that $Y \subseteq U$. By (K2) (since $x \geq x$), there exists $Y' \in \mathsf{Up}(X)$ such that $Y' \subseteq Y$, $Y' \subseteq \dot{\mathsf{S}}(x)$, and $Y' \in J(x)$. Since $Y' \subseteq \dot{\mathsf{S}}(x)$ and $Y' \in J(x)$, it follows that $Y' \in R_{\mathcal{N}}(x)$. Given that $Y' \subseteq Y \subseteq U$, we have $Y' \subseteq U$ with $Y' \in R_{\mathcal{N}}(x)$. Therefore $x \in \Diamond_{R_{\mathcal{N}}}(U)$. For the reverse inclusion, let $x \in \Diamond_{R_{\mathcal{N}}}(U)$. By definition, there exists $Y \in R_{\mathcal{N}}(x)$ such that $Y \subseteq U$. The definition of $R_{\mathcal{N}}$ implies that $Y \in J(x)$. Since we found $Y \in J(x)$ such that $Y \subseteq U$, we conclude $x \in \Diamond_J(U)$. $\square$

Proposition 7.9. *Let $\mathcal{N} = \langle X, \leq, \dot{\mathsf{S}}, J\rangle$ be an IKN-frame and let $\mathcal{F} = \langle X, \leq, R\rangle$ be an ic-frame. Let us consider the structures $A(\mathcal{N}) = \langle \mathsf{Up}(X), \Box_{\dot{\mathsf{S}}}, \Diamond_J\rangle$, $A(\mathcal{F}_{\mathcal{N}}) = \langle \mathsf{Up}(X), \Box_{R_{\mathcal{N}}}, \Diamond_{R_{\mathcal{N}}}\rangle$, $A(\mathcal{F}) = \langle \mathsf{Up}(X), \Box_R, \Diamond_R\rangle$ and $A(\mathcal{N}_{\mathcal{F}}) = \langle \mathsf{Up}(X), \Box_{\dot{\mathsf{S}}_R}, \Diamond_{J_R}\rangle$. Then*

(1) $A(\mathcal{N}) = A(\mathcal{F}_{\mathcal{N}})$,
(2) $A(\mathcal{F}) = A(\mathcal{N}_{\mathcal{F}})$.

In consequence $A(\mathcal{N})$ and $A(\mathcal{N}_{\mathcal{F}})$ are concurrent Heyting algebras.

Proof. The two required equalities are established by direct reference to the equivalence of the modal operators. By Proposition 7.8, $\Box_{\dot{\mathsf{S}}}(U) = \Box_{R_N}(U)$ and $\Diamond_J(U) = \Diamond_{R_N}(U)$ for all $U \in \mathsf{Up}(X)$. It follows that $A(\mathcal{N}) = A(\mathcal{F}_{\mathcal{N}})$. Similarly, the equality $A(\mathcal{F}) = A(\mathcal{N}_{\mathcal{F}})$ follows directly from the definitions of the operators and Proposition 6.3, which establishes $\Box_R(U) = \Box_{\dot{\mathsf{S}}_R}(U)$ and $\Diamond_R(U) = \Diamond_{J_R}(U)$ for all $U \in \mathsf{Up}(X)$. In consequence, since $\mathcal{F}_{\mathcal{N}} = \langle X, \leq, R_N\rangle$ and $\mathcal{F}$ are ic-frames, it is known by Example 4.5 that their associated algebras, $A(\mathcal{F}_{\mathcal{N}})$ and $A(\mathcal{F})$, are concurrent Heyting algebras. Given the proven equalities, it is immediate that $A(\mathcal{N})$ and $A(\mathcal{N}_{\mathcal{F}})$ are also concurrent Heyting algebras. $\square$

Let **KCHA** be the class of all Concurrent Heyting Algebras (CH-algebras) of the form $\langle \mathsf{Up}(X), \Box_{\dot{\mathsf{S}}}, \Diamond_J\rangle$ given by IKN-frames $\langle X, \leq, \dot{\mathsf{S}}, J\rangle$. Recall that **CCHA** is the class of all CH-algebras of the form $\langle \mathsf{Up}(X), \Box_R, \Diamond_R\rangle$ given by ic-frames $\langle X, \leq, R\rangle$. Let $\mathbf{V}(\mathbf{CCHA})$ be the variety generated by the class of algebras **CCHA**, and let $\mathbf{V}(\mathbf{KCHA})$ be the variety generated by the class of algebras **KCHA**.

Corollary 7.10. $\mathbf{KCHA} = \mathbf{CCHA}$ *and* $\mathbf{V}(\mathbf{KCHA}) = \mathbf{V}(\mathbf{CCHA})$.

Proof. The equality $\mathbf{KCHA} = \mathbf{CCHA}$ follows from Proposition 7.9. Since the classes of algebras are identical, the varieties generated by them must coincide. Hence, $\mathbf{V}(\mathbf{KCHA}) = \mathbf{V}(\mathbf{CCHA})$. $\square$

Definition 7.11. An *IKN-space* is a structure $\langle X, \leq, \dot{\mathsf{S}}, J, \tau\rangle$ that satisfies:

(EK 1) $\langle X, \leq, \tau\rangle$ is an Esakia space.
(EK 2) $\mathcal{N} = \langle X, \leq, \dot{\mathsf{S}}, J\rangle$ is an IKN-frame, where $\dot{\mathsf{S}} \subseteq X \times X$, $J \subseteq X \times \mathcal{C}\ell\mathsf{Up}(X)$.
(EK 3) $R_{\mathcal{N}}(x) \neq \emptyset$ implies $\dot{\mathsf{S}}(x) \in \mathcal{C}\ell\mathsf{Up}(X)$.
(EK 4) $\Box_{\dot{\mathsf{S}}}(U), \Diamond_J(U) \in \mathcal{CP}\mathsf{Up}(X)$ for all $U \in \mathcal{CP}\mathsf{Up}(X)$.
(EK 5) $Y \in J(x)$ if and only if there exists $C \in \mathcal{C}\ell\mathsf{Up}(X)$ such that $C \subseteq Y$ and $C \in J(x)$.
(EK 6) $C \in J(x)$ if and only if for every $U \in \mathcal{CP}\mathsf{Up}(X)$, $C \subseteq U$ implies $U \in J(x)$.

Proposition 7.12. *Let $\langle X, \leq, R, \tau\rangle$ be a concurrent Esakia space. Then $\langle X, \leq, \dot{\mathsf{S}}_R, J_R, \tau\rangle$ is an IKN-space.*

Proof. Let $\langle X, \leq, R, \tau\rangle$ be a concurrent Esakia space.

(EK 1) $\langle X, \leq, \tau\rangle$ is an Esakia space. This follows from (C1) of Definition 6.4.

(EK 2) $\mathcal{N} = \langle X, \leq, \dot{\mathrm{S}}_R, J_R\rangle$ is an IKN-frame. This follows from Proposition 7.4.

(EK 3) Let $x \in X$. The equality $R_{\mathcal{N}}(x) = R(x)$ holds by Lemma 7.6. Thus, if $R_{\mathcal{N}}(x) \neq \emptyset$, it follows by (C6) that $\dot{\mathrm{S}}_R(x)$ is closed. Since $\dot{\mathrm{S}}_R(x)$ is always an upset, $\dot{\mathrm{S}}_R(x) \in \mathcal{C}\ell\mathsf{Up}(X)$.

(EK 4) The condition that $\Box_{\dot{\mathrm{S}}_R}(U), \Diamond_{J_R}(U) \in \mathcal{CP}\mathsf{Up}(X)$ for all $U \in \mathcal{CP}\mathsf{Up}(X)$ follows from (C3) of Definition 6.4 and by Proposition 6.3.

(EK 5) Let $x \in X$. If $Y \in J_R(x)$, then there exists $C \in R(x)$ such that $C \subseteq Y$. Since $C \subseteq C$, C satisfies the required condition, so $C \in J_R(x)$. Conversely, assume $C \subseteq Y$ and $C \in J_R(x)$. By definition of J_R, there exists $Z \in R(x)$ such that $Z \subseteq C$. Then $Z \subseteq Y$, and consequently $Y \in J_R(x)$.

(EK 6) Let $x \in X$. Assume $C \in J_R(x)$ and let $U \in \mathcal{CP}\mathsf{Up}(X)$ such that $C \subseteq U$. By definition of J_R the condition $C \in J_R(x)$ implies there exists $Z \in R(x)$ such that $Z \subseteq C$. Since $C \subseteq U$, we have $Z \subseteq U$. By definition, the existence of $Z \in R(x)$ such that $Z \subseteq U$ implies that $U \in J_R(x)$. To prove the converse, we will establish the contra reverse: if $C \notin J_R(x)$ then there exists some $U \in \mathcal{CP}\mathsf{Up}(X)$ such that $C \subseteq U$ and $U \notin J_R(x)$. Let $C \notin J_R(x)$. Thus, for every $Y \in R(x)$, $Y \nsubseteq C$. Since $C \in \mathcal{C}\ell\mathsf{Up}(X)$, for each $Y \in R(x)$ there is $U_Y \in \mathcal{CP}\mathsf{Up}(X)$ such that $Y \nsubseteq U_Y$ and $C \subseteq U_Y$. The condition $Y \nsubseteq U_Y$ is equivalent to $Y \cap U_Y^c \neq \emptyset$, which means $Y \in L_{U_Y^c}$. We obtain the covering:

$$R(x) \subseteq \bigcup \{L_{U_Y^c} : C \subseteq U_Y\}.$$

Since $R(x)$ is closed in $\mathcal{C}\ell\mathsf{Up}(X)$ by (C4) and thus compact, this cover has a finite subcover. Hence, there are $U_{Y_1}, \dots, U_{Y_n}$ such that

$$R(x) \subseteq L_{U_{Y_1}^c \cup \cdots \cup U_{Y_n}^c}.$$

Let $U^c = U_{Y_1}^c \cup \cdots \cup U_{Y_n}^c$. We define U as the complement of U^c, i.e., $U = (U^c)^c$. U is a clopen upset containing C. Since $R(x) \subseteq L_{U^c}$ it follows that for all $Y \in R(x)$, $Y \nsubseteq U$. This means $x \notin \Diamond_R(U)$. Since $\Diamond_R = \Diamond_{J_R}$ (Proposition 6.3), $x \notin \Diamond_{J_R}(U)$. Thus, for all $Z \in J_R(x)$, $Z \nsubseteq U$. Therefore, $U \notin J_R(x)$, and we have found $U \in \mathcal{CP}\mathsf{Up}(X)$ such that $C \subseteq U$ and $U \notin J_R(x)$, completing the proof of the contra reverse. $\Box$

Proposition 7.13. *Let $\mathcal{N} = \langle X, \leq, \dot{\mathrm{S}}, J, \tau\rangle$ be an IKN-space. Then $\langle X, \leq, R_{\mathcal{N}}, \tau\rangle$ is a concurrent Esakia space.*

Proof. Let $\langle X, \leq, \dot{\mathrm{S}}, J, \tau\rangle$ be an IKN-space.

(C1) The structure $\langle X, \leq, \tau\rangle$ is an Esakia space. This follows directly from (EK 2) of Definition 7.11,

(C2) $\langle X, \leq, R_{\mathcal{N}}\rangle$ is an ic-frame. This follows from Proposition 7.3.

(C3) The condition $\Box_{R_{\mathcal{N}}}(U), \Diamond_{R_{\mathcal{N}}}(U) \in \mathcal{CP}\mathsf{Up}(X)$ for all $U \in \mathcal{CP}\mathsf{Up}(X)$. follows from (EK 4) and Proposition 7.8.

(C4) We show that $R_{\mathcal{N}}(x)$ is closed in $\mathcal{C}\ell\mathsf{Up}(X)$ by proving $R_{\mathcal{N}}(x) = \overline{R_{\mathcal{N}}(x)}$ for an arbitrary $x \in X$. If $R_{\mathcal{N}}(x) = \emptyset$, the set is trivially closed. Assume $R_{\mathcal{N}}(x) \neq \emptyset$. Let $Y \notin R_{\mathcal{N}}(x)$. By definition of $R_{\mathcal{N}}$, this means that either $Y \nsubseteq \dot{\mathrm{S}}(x)$ or $Y \notin J(x)$. If $Y \nsubseteq \dot{\mathrm{S}}(x)$, then there exists $y \in Y$ such that $y \notin \dot{\mathrm{S}}(x)$. Since $\dot{\mathrm{S}}(x) \in \mathcal{C}\ell\mathsf{Up}(X)$ by (EK 3), there exists $U \in \mathcal{CP}\mathsf{Up}(X)$ such that $y \notin U$ and $\dot{\mathrm{S}}(x) \subseteq U$. The condition

$\dot{\mathrm{S}}(x) \subseteq U$ implies $x \in \Box_{\dot{\mathrm{S}}}(U)$, which by Proposition 7.8 is equivalent to $x \in \Box_{R_{\mathcal{N}}}(U)$. In consequence, for all $Z \in R_{\mathcal{N}}(x)$, $Z \subseteq U$. This implies $R_{\mathcal{N}}(x) \cap L_{U^c} = \emptyset$. Since $y \in Y$ and $y \notin U$, we have $Y \in L_{U^c}$. Thus Y is contained in the open set L_{U^c} disjoint from $R_{\mathcal{N}}(x)$, so $Y \notin \overline{R_{\mathcal{N}}(x)}$. On the other hand, if $Y \notin J(x)$, it follows by (EK 6) that there is $U \in \mathcal{CP}\mathsf{Up}(X)$ such that $Y \subseteq U$ and $U \notin J(x)$. By (EK 5), the condition $U \notin J(x)$ is equivalent to $x \notin \Diamond_J(U)$. By Proposition 7.8, this means $x \notin \Diamond_{R_{\mathcal{N}}}(U)$. In consequence, for all $Z \in R_{\mathcal{N}}(x)$, $Z \nsubseteq U$. This implies $R_{\mathcal{N}}(x) \cap D_U = \emptyset$. Since $Y \subseteq U$, we have $Y \in D_U$. Thus, $Y \notin \overline{R_{\mathcal{N}}(x)}$. Therefore $R_{\mathcal{N}}(x) = \overline{R_{\mathcal{N}}(x)}$.

(C5) The equivalence $(x, Y) \in R_{\mathcal{N}} \iff Y \subseteq \dot{\mathrm{S}}_{R_{\mathcal{N}}}(x)$ and $Y \in J_{R_{\mathcal{N}}}(x)$ is established by Proposition 7.5.

(C6) By Proposition 7.5, if $R_{\mathcal{N}}(x) \neq \emptyset$, it follows directly from (EK 3) that $\dot{\mathrm{S}}_{R_{\mathcal{N}}}(x)$ is closed in X. $\Box$

Remark 7.14. Let $\langle X, \leq, R, \tau\rangle$ be a concurrent Esakia space. Notice that if $R(x) \neq \emptyset$, then $\dot{\mathrm{S}}_R(x) \in J_R(x)$. Indeed, if $R(x) \neq \emptyset$, there exists $Y \in R(x)$. By (C6) of Definition 6.4, $\dot{\mathrm{S}}_R(x) \in \mathcal{C\ell}\mathsf{Up}(X)$, and by (C5), $Y \subseteq \dot{\mathrm{S}}_R(x)$. Hence $\dot{\mathrm{S}}_R(x) \in J_R(x)$ by definition of J_R. Notice that if $\langle X, \leq, \dot{\mathrm{S}}, J, \tau\rangle$ is an IKN-space such that $R_{\mathcal{N}}(x) \neq \emptyset$, then it follows by Proposition 7.5 that $\dot{\mathrm{S}}(x) \in J(x)$.

Propositions 7.12 and 7.13 show a one-to-one correspondence between concurrent Esakia spaces and IKN-spaces. In what follows, we define the morphisms between IKN-spaces corresponding to the concurrent Esakia morphisms. Then, we define a new category for IKN-spaces and show that it is isomorphic to the category of concurrent Esakia spaces.

Definition 7.15. Let $\mathcal{N}_1 = \langle X_1, \leq_1, \dot{\mathrm{S}}_1, J_1, \tau_1\rangle$ and $\mathcal{N}_2 = \langle X_2, \leq_2, \dot{\mathrm{S}}_2, J_2, \tau_2\rangle$ be two IKN-spaces. A map $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ is an *IKN-morphism* if satisfies:

(KM 1) f is continuous.
(KM 2) f is a p-morphism.
(KM 3) If $y \in \dot{\mathrm{S}}_1(x)$, then $f(y) \in \dot{\mathrm{S}}_2(f(x))$.
(KM 4) If $Y \in J_1(x)$, then $f(Y) \in J_2(f(x))$.
(KM 5) If $Y \in J_2(f(x))$, then there exists $Z \in J_1(x)$ such that $f(Z) \subseteq Y$.
(KM 6) If $y \in \dot{\mathrm{S}}_2(f(x))$, then there exists $z \in \dot{\mathrm{S}}_1(x)$ such that $f(z) = y$.

Let us denote by $\mathbb{IKN}$ the category whose objects are IKN-spaces and whose morphisms are IKN-morphisms.

Proposition 7.16. *Let $\mathcal{N}_1 = \langle X_1, \leq_1, \dot{\mathrm{S}}_1, J_1, \tau_1\rangle$ and $\mathcal{N}_2 = \langle X_2, \leq_2, \dot{\mathrm{S}}_2, J_2, \tau_2\rangle$ be two IKN-spaces. Let $\mathcal{F}_{\mathcal{N}_1} = \langle X_1, \leq_1, R_{\mathcal{N}_1}, \tau_1\rangle$ and $\mathcal{F}_{\mathcal{N}_2} = \langle X_2, \leq_2, R_{\mathcal{N}_2}, \tau_2\rangle$ be the concurrent Esakia spaces associates to $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively. If $f\colon X_1 \to X_2$ is an IKN-morphism between the IKN-spaces $\mathcal{N}_1$ and $\mathcal{N}_2$, then f is a concurrent Esakia morphism between the concurrent Esakia spaces $\mathcal{F}_{\mathcal{N}_1}$ and $\mathcal{F}_{\mathcal{N}_2}$.*

Proof. Let $f\colon X_1 \to X_2$ be an IKN-morphism.

(CM 1) f is continuous by (KM 1) of Definition 7.15.

(CM 2) f is a p-morphism by (KM 2) of Definition 7.15 .

(CM 3) Suppose that $Y \in R_{\mathcal{N}_1}(x)$. By definition, this means $Y \subseteq \dot{\mathrm{S}}_1(x)$ and $Y \in J_1(x)$. From $Y \subseteq \dot{\mathrm{S}}_1(x)$ and (KM 3), it follows that $f(Y) \subseteq \dot{\mathrm{S}}_2(f(x))$. From $Y \in J_1(x)$ and (KM 4),it follows directly that $f(Y) \in J_2(f(x))$. Since both conditions are satisfied, we conclude that $f(Y) \in R_{\mathcal{N}_2}(f(x))$.

(CM 4) Suppose $Y \in R_{\mathcal{N}_2}(f(x))$. By definition, $Y \subseteq \dot{\mathrm{S}}_2(f(x))$ and $Y \in J_2(f(x))$.We must show there exist $Z_1, Z_2 \in R_{\mathcal{N}_1}(x)$ such that $f(Z_1) \subseteq Y \subseteq f(Z_2)$. Since $Y \in J_2(f(x))$, by (KM 5), there exists $Z \in J_1(x)$ such that $f(Z) \subseteq Y$. We then use (EK 2) of Definition 7.1, which guarantees the existence of a set $Z_1 \subseteq Z$ such that $Z_1 \subseteq \dot{\mathrm{S}}_1(x)$ and $Z_1 \in J_1(x)$. By the definition of $R_{\mathcal{N}_1}$, $Z_1 \in R_{\mathcal{N}_1}(x)$. Because $Z_1 \subseteq Z$ and $f(Z) \subseteq Y$, it follows that $f(Z_1) \subseteq Y$, which establishes the first required inclusion.

The IKN-morphism conditions, (KM 3) and (KM 6), together ensure the equality $\dot{\mathrm{S}}_2(f(x)) = f(\dot{\mathrm{S}}_1(x))$. Since our initial assumption is $Y \subseteq \dot{\mathrm{S}}_2(f(x))$, we immediately have $Y \subseteq f(\dot{\mathrm{S}}_1(x))$. We choose $Z_2 := \dot{\mathrm{S}}_1(x)$. The fact that $Z_1 \in R_{\mathcal{N}_1}(x)$ implies $R_{\mathcal{N}_1}(x) \neq \emptyset$. By Remark 7.14, $\dot{\mathrm{S}}_1(x) \in J_1(x)$. Since $\dot{\mathrm{S}}_1(x)$ is a subset of itself, we confirm that $Z_2 \in R_{\mathcal{N}_1}(x)$, and the inclusion $Y \subseteq f(Z_2)$ holds. □

Proposition 7.17. *Let $\mathcal{F}_1 = \langle X_1, \leq_1, R_1, \tau_1\rangle$ and $\mathcal{F}_2 = \langle X_2, \leq_2, R_2, \tau_2\rangle$ be two concurrent Esakia spaces. Let $\mathcal{N}_{\mathcal{F}_1} = \langle X_1, \leq_1, \dot{\mathrm{S}}_{R_1}, J_{R_1}, \tau_1\rangle$ and $\mathcal{N}_{\mathcal{F}_2} = \langle X_2, \leq_2, \dot{\mathrm{S}}_{R_2}, J_{R_2}, \tau_2\rangle$ be the IKN-spaces associates to $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. If $g\colon X_1 \to X_2$ is a concurrent morphism between the concurrent Esakia spaces $\mathcal{F}_1$ and $\mathcal{F}_2$, then g is an IKN-morphism between the IKN-spaces $\mathcal{N}_{\mathcal{F}_1}$ and $\mathcal{N}_{\mathcal{F}_2}$.*

Proof. Let $g\colon X_1 \to X_2$ be a concurrent morphism.

(KM 1) By (CM 1) of Definition 6.9, g is continuous.

(KM 2) By (CM 2) of Definition 6.9, g is a p-morphism.

(KM 3) Suppose $y \in \dot{\mathrm{S}}_{R_1}(x)$. By definition of $\dot{\mathrm{S}}_{R_1}$, there exists $Y \in R_1(z)$ for some $z \geq x$ such that $y \in Y$. By (CM 3), $g(Y) \in R_2(g(z))$. Since g is a p-morphism, we have $g(z) \geq g(x)$. As $g(y) \in g(Y)$, it follows that $g(y) \in \dot{\mathrm{S}}_{R_2}(g(x))$.

(KM 4) Now, if $Y \in J_{R_1}(x)$, then there exists $Z \in R_1(x)$ such that $Z \subseteq Y$. By (CM 3), $g(Z) \in R_2(g(x))$, and moreover $g(Z) \subseteq g(Y)$. Thus, $g(Y) \in J_{R_2}(g(x))$.

(KM 5) Let $Y \in J_{R_2}(g(x))$. By definition of J_{R_2}, there exists $Z \in R_2(g(x))$ such that $Z \subseteq Y$. Then, by (CM 4), there exists $W \in R_1(x)$ such that $g(W) \subseteq Z$. By the definition of R_1, $W \in J_{R_1}(x)$. Since $g(W) \subseteq Z$ and $Z \subseteq Y$, it follows that $g(W) \subseteq Y$.

(KM 6) Finally, suppose $y \in \dot{\mathrm{S}}_{R_2}(g(x))$. So, there exist $t \geq g(x)$ and $W \in R_2(t)$ such that $y \in W$. Since g is a p-morphism, there is $s \in X_1$ with $s \geq x$ such that $g(s) = t$. Thus $s \geq x$, $W \in R_2(g(s))$ and $y \in W$. We must prove that $y \in g(\dot{\mathrm{S}}_{R_1}(x))$, which is equivalent to proving there exists $a \in \dot{\mathrm{S}}_{R_1}(x)$ such that $g(a) = y$. Since $a \in \dot{\mathrm{S}}_{R_1}(x)$ means there are $z \geq x$ and $Z \in R_1(z)$ such that $a \in Z$, we must show that there are $z \geq x$ and $Z \in R_1(z)$ such that $y \in g(Z)$. Since we already established $s \geq x$, it suffices to show there exists $Z \in R_1(s)$ such that $y \in g(Z)$. Suppose not. Suppose that for all $Z_i \in R_1(s)$, $y \notin g(Z_i)$. Since each $g(Z_i) \in \mathcal{C}\ell\mathsf{Up}(X)$ (as g is continuous and p-morphism) and $y \notin g(Z_i)$, there exists $U_i \in \mathcal{CP}\mathsf{Up}(X_2)$ such that $g(Z_i) \subseteq U_i$ and $y \notin U_i$. From the construction of U_i, we have $Z_i \subseteq g^{-1}(U_i)$, which means $Z_i \in D_{g^{-1}(U_i)}$ for each $Z_i \in R_1(s)$. Then

$$R_1(s) \subseteq \bigcup \{D_{g^{-1}(U_i)} : Z_i \subseteq U_i\}.$$

As $R_1(s)$ is closed in $\mathcal{C}\ell\mathsf{Up}(X)$, it is compact. Thus there are $U_1, \cdots, U_n$ such that

$$R_1(s) \subseteq D_{g^{-1}(U_1)\cup\cdots\cup g^{-1}(U_n)}.$$

Let us call $U := U_1 \cup \cdots \cup U_n$. Since g^{-1} is an homomorphism of concurrent Heyting algebras (6.13), we obtain that $g^{-1}(U_1) \cup \cdots \cup g^{-1}(U_n) = g^{-1}(U)$. Then $R_1(s) \subseteq$

$D_{g^{-1}(U)}$. Thus, for all $Z_i \in R_1(s)$ we have $Z_i \subseteq g^{-1}(U)$, or equivalently, we have $g(Z_i) \subseteq U$. Now, we have that $W \in R_2(g(s))$. By (CM 4), there exist $T_1, T_2 \in R_1(s)$ such that $g(T_1) \subseteq W \subseteq g(T_2)$. Since $T_2 \in R_1(s)$, we must have $g(T_2) \subseteq U$. This implies $W \subseteq U$. However, we established by construction that $y \in W$ and $y \notin U$ (since $y \notin U_i$ for every i). This means $W \nsubseteq U$, which is the desired absurdity, concluding the proof of (KM 6). □

Theorem 7.18. *The categories* $\mathbb{IKN}$ *and* $\mathbb{CES}$ *are isomorphic.*

Proof. We establish the isomorphism by defining two covariant functors, F and G, and showing that FG and GF are the identity functors on their respective categories.

Let $F\colon \mathbb{IKN} \to \mathbb{CES}$ be the functor that, for each IKN-space $\mathbf{X} = \langle X, \leq, \dot{\mathbb{S}}, J, \tau\rangle$ assigns the concurrent Esakia space $F(\mathbf{X}) := \langle X, \leq, R_N, \tau\rangle$. For each IKN-morphism $f \in \mathcal{M}_{\mathbb{IKN}}(X, Y)$, $F(f) \in \mathcal{M}_{\mathbb{CES}}(F(X), F(Y))$ is the concurrent Esakia morphism defined by $F(f)(x) = f(x)$.

Let $G\colon \mathbb{CES} \to \mathbb{IKN}$ be the functor that, for each concurrent Esakia space $\mathbf{X} := \langle X, \leq, R, \tau\rangle$, assigns the IKN-space $G(\mathbf{X}) := \langle X, \leq, \dot{\mathbb{S}}_R, J_R, \tau\rangle$. For each concurrent Esakia morphism $g\colon X \to Y$, $G(g)\colon G(X) \to G(Y)$ is the IKN-morphism defined by $G(g)(x) = g(x)$.

The fact that F and G are well-defined on objects and morphisms follows from the definitions and the equivalence of morphisms established in the preceding propositions. By definition, both F and G are covariant functors. We verify that $FG = id_{\mathbb{CES}}$ and $GF = id_{\mathbb{IKN}}$ where $id_{\mathbb{CES}}$ and $id_{\mathbb{IKN}}$ are the identity functors in the categories $\mathbb{CES}$ and $\mathbb{IKN}$ respectively.

Let $\langle X, \leq, \dot{\mathbb{S}}, J, \tau\rangle$ be an IKN-space.

$$GF(\langle X, \leq, \dot{\mathbb{S}}, J, \tau\rangle) = G(\langle X, \leq, R_{\mathcal{N}}, \tau\rangle) = \langle X, \leq, \dot{\mathbb{S}}_{R_{\mathcal{N}}}, J_{R_{\mathcal{N}}}, \tau\rangle$$

By Proposition 7.5, $\dot{\mathbb{S}} = \dot{\mathbb{S}}_{R_{\mathcal{N}}}$ and $J = J_{R_{\mathcal{N}}}$, hence $GF(\langle X, \leq, \dot{\mathbb{S}}, J, \tau\rangle) = \langle X, \leq, \dot{\mathbb{S}}, J, \tau\rangle$.

Let $\langle X, \leq, R, \tau\rangle$ be an Esakia concurrent space.

$$FG(\langle X, \leq, R, \tau\rangle) = F(\langle X, \leq, \dot{\mathbb{S}}_R, J_R, \tau\rangle) = \langle X, \leq, R_{\mathcal{N}_{\mathcal{F}}}, \tau\rangle$$

By Proposition 7.6, $R = R_{\mathcal{N}_{\mathcal{F}}}$, hence $FG(\langle X, \leq, R, \tau\rangle) = \langle X, \leq, R, \tau\rangle$.

It is obvious, by the definitions of F and G, that $FG(g) = g$ and $GF(f) = f$ for all $g \in \mathcal{M}_{\mathbb{CES}}(X, Y)$ and $f \in \mathcal{M}_{\mathbb{IKN}}(Z, T)$. Thus, F and G are inverse functors, and the categories $\mathbb{CES}$ and $\mathbb{IKN}$ are isomorphic. □

The above theorem and Theorem 6.18 allow us to obtain a new dual equivalence for the category of CH-algebras.

Corollary 7.19. *The category* $\mathbb{IKN}$ *is dually equivalent to the category* $\mathbb{CHA}$.

Proof. By Theorems 6.18 and 7.18. □

We close this section by proposing an alternative semantics for the logic $\Lambda(\mathcal{H})$ and proving a completeness theorem.

Let $\mathcal{N} = \langle X, \leq, \dot{\mathbb{S}}, J\rangle$ be an IKN-frame. We say that $M = \langle \mathcal{N}, v\rangle$ is an *IKN-model* if $\mathcal{N}$ is an IKN-frame and $v\colon Var \to \mathsf{Up}(X)$ is a valuation. We define recursively the satisfaction relation $\Vdash$ as follows: for propositional variables and for the connectives $\{\bot, \wedge, \vee, \to\}$ are the same definitions as in page 7, and

- $\mathcal{M}, x \Vdash \Box\varphi \iff \forall y \in \dot{\mathbb{S}}(x)$, $\mathcal{M}, y \Vdash \varphi$.
- $\mathcal{M}, x \Vdash \Diamond\varphi \iff \exists Y \in J(x)$ such that $\mathcal{M}, y \Vdash \varphi$, $\forall y \in Y$.

Let us denote by $\Lambda(\mathrm{IKNF})$ the logic defined by the class of all IKN-frames. That is,

$$\Lambda(\mathrm{IKNF}) = \{\varphi \in \mathrm{Fm} : \mathcal{N} \Vdash \varphi \text{ , for all } \mathcal{N} \in \mathrm{IKNF}\}$$

Let $\Lambda(\mathbf{KCHA})$ be the logic defined by the class of algebras **KCHA** (recall the definition of the satisfaction relation $\vDash$ given in (4.1)). In order to prove that $\Lambda(\mathrm{IKNF}) = \Lambda(\mathbf{KCHA})$, we prove the following.

Proposition 7.20. *Let $v\colon \mathrm{Var} \to \mathsf{Up}(X)$ be a valuation on an IKN-frame $\mathcal{N} = \langle X, \leq, \dot{\mathbb{S}}, J\rangle$. The function $h_v\colon \mathrm{Fm} \to \mathsf{Up}(X)$ defined by $h_v(\varphi) = \{x \in X : \mathcal{N}, v, x \Vdash \varphi\}$, for every $\varphi \in \mathrm{Fm}$, is a homomorphism of CH-algebras extending the function v.*

Proof. It is straightforward by definition. $\Box$

Proposition 7.21. *Let $\mathcal{N} = \langle X, \leq, \dot{\mathbb{S}}, J\rangle$ be an IKN-frame. For every formula φ,*

$$\mathcal{N} \Vdash \varphi \quad \textit{if and only if} \quad \mathsf{Up}(X)\varphi.$$

Proof. It follows straightforwardly from the previous proposition. $\Box$

Corollary 7.22. $\Lambda(\mathrm{IKNF}) = \Lambda(\mathbf{KCHA})$.

Proof. It is immediate from the previous proposition. $\Box$

Theorem 7.23 (Algebraic completeness)**.** $\Lambda(\mathcal{H}) = \Lambda(\mathbf{KCHA})$.

Proof. By Corollary 7.10, Proposition 5.8 and Theorem 4.13, it follows that

$$\Lambda(\mathbf{KCHA}) = \Lambda(\mathbf{CCHA}) = \Lambda(\mathbf{CHA}) = \Lambda(\mathcal{H}). \qquad \Box$$

Theorem 7.24 (Completeness with respect to IKN-frames)**.** $\Lambda(\mathrm{IKNF}) = \Lambda(\mathcal{H})$.

Proof. By Corollary 7.22 and Theorem 7.23, we have

$$\Lambda(\mathrm{IKNF}) = \Lambda(\mathbf{KCHA}) = \Lambda(\mathcal{H}). \qquad \Box$$

8. Conclusions

We have presented the class ICF of ic-frames (Definition 3.1), which is used to define in a neighbourhood-semantics way a logic $\Lambda(\mathrm{ICF})$. By definition, the logic $\Lambda(\mathrm{ICF})$ has an intuitionistic base and two modal operators. Since the ic-frames fall within the class of neighbourhood structures, and taking into account the axioms axiomatising this logic, we might consider the logic $\Lambda(\mathrm{ICF})$ as a simplified version of intuitionistic dynamic logic (see Wijesekera (1990) for similar considerations). We have proposed a Hilbert-style axiomatisation for the logic $\Lambda(\mathrm{ICF})$. We denote by $\Lambda(\mathcal{H})$ the logic defined by the axioms proposed by the axiomatisation for $\Lambda(\mathrm{ICF})$. Then, we have introduced a class of algebras, called concurrent Heyting algebras (CH-algebras), as an algebraic semantics for $\Lambda(\mathcal{H})$. We prove that the logic $\Lambda(\mathcal{H})$ is complete with respect to the class of CH-algebras. After that, we presented a representation theorem for the class of CH-algebras through the class of ic-frames. This representation allowed us to prove that the logic $\Lambda(\mathcal{H})$ is complete with respect to the class of ic-frames ICF, that is, $\Lambda(\mathrm{ICF}) = \Lambda(\mathcal{H})$. In Section 7, we proposed a new relational semantics for the logic $\Lambda(\mathcal{H})$. Unlike the ic-frames, the new structures, called intuitionistic Kripke neighbourhood frames (IKN-frames), consist of two relations. One relation is used to model the normal operator $\Box$, and the other is used to model the non-normal operator $\Diamond$. This approach provides an alternative perspective for studying the logic $\Lambda(\mathcal{H})$.

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