2D Nonlinear Systems - Local Analysis & Phase Plane

Example #1

Consider the system given by

$$\dot{x} = f(x, y) \quad \text{where } f(x, y) = 2x + 4xy$$

$$\dot{y} = g(x, y) \quad \text{where } g(x, y) = 6xy - 8y$$

We would like to (1) conduct local analysis, and (2) graph the phase plane to validate our local analysis conclusions.

Local Analysis

In[1]:=
```
f[x_, y_] = 2 x + 4 x y;
g[x_, y_] = 6 x y - 8 y;
```

In[3]:=
```
eqPts = Solve[{f[x, y] == 0, g[x, y] == 0}, {x, y}]
```

Out[3]= $\left\{\left\{x \to \frac{4}{3}, y \to -\frac{1}{2}\right\}, \{x \to 0, y \to 0\}\right\}$

In[4]:=
```
DF[xs_, ys_] =
  {{D[f[x, y], x], D[f[x, y], y]}, {D[g[x, y], x], D[g[x, y], y]}} /. {x → xs, y → ys};
```

In[5]:=
```
For[j = 1, j ≤ Length[eqPts], j++,
 Print[];
 Print[
  "==========================================================================
    =================="];
 Print["At the equilibrium point ", eqPts[[j]],
  " the eigensystem is given by ", MatrixForm[Eigensystem[DF[x, y] /. eqPts[[j]]]]];
 Print[
  "==========================================================================
    =================="];
 Print[];
]
```

```
==========================================================================================
========
```

At the equilibrium point $\left\{x \to \frac{4}{3}, y \to -\frac{1}{2}\right\}$ the eigensystem is given by $\begin{pmatrix} 4\,i & -4\,i \\ \left\{-\frac{4\,i}{3}, 1\right\} & \left\{\frac{4\,i}{3}, 1\right\} \end{pmatrix}$

```
==========================================================================================
========
```

```
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```

At the equilibrium point $\{x \to 0, y \to 0\}$ the eigensystem is given by $\begin{pmatrix} -8 & 2 \\ \{0, 1\} & \{1, 0\} \end{pmatrix}$

```
==========================================================================================
========
```

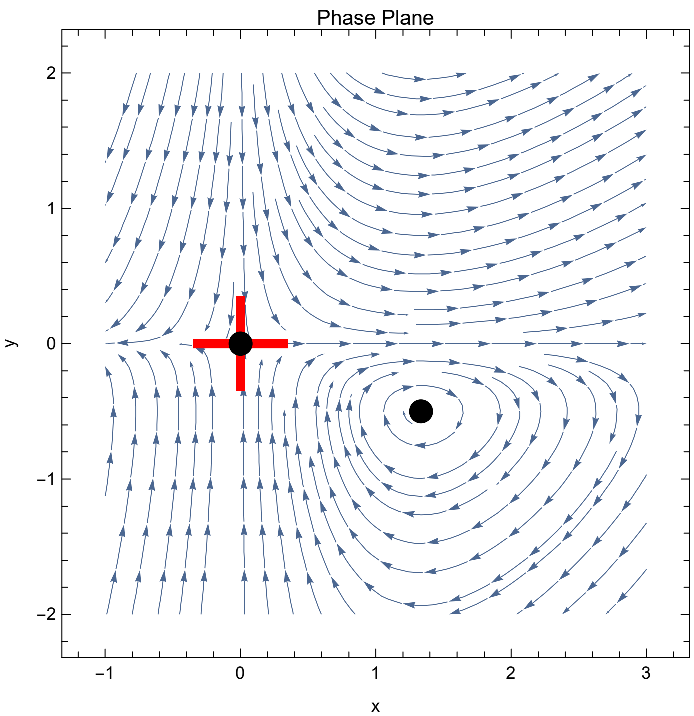
Comparing the Local Analysis with the Phase Plane

First, let’s create a loop that iterates over each of the equilibrium points. We will plot the eigenvectors

```
In[6]:= For[j = 1, j ≤ Length[eqPts], j++,
  esys = Eigensystem[DF[x, y] /. eqPts[[j]]];
  evPlots_j =
   ParametricPlot[esys[[2]] * s + Table[{x, y} /. eqPts[[j]] , {k, 1, 2}], {s, -1, 1},
    PlotStyle → {Red, Thickness → .015},
    RegionFunction →
     Function[{u, v, vx, vy, n}, (((u - x)^2 + (v - y)^2) /. eqPts[[j]]) < .1]]];
EVPlot = Show[Table[evPlots_j, {j, 1, Length[eqPts]}]];
eqPtsPlot = ListPlot[{x, y} /. eqPts,
   PlotMarkers → {Automatic, Scaled[.04]},
   PlotStyle → Black];
```

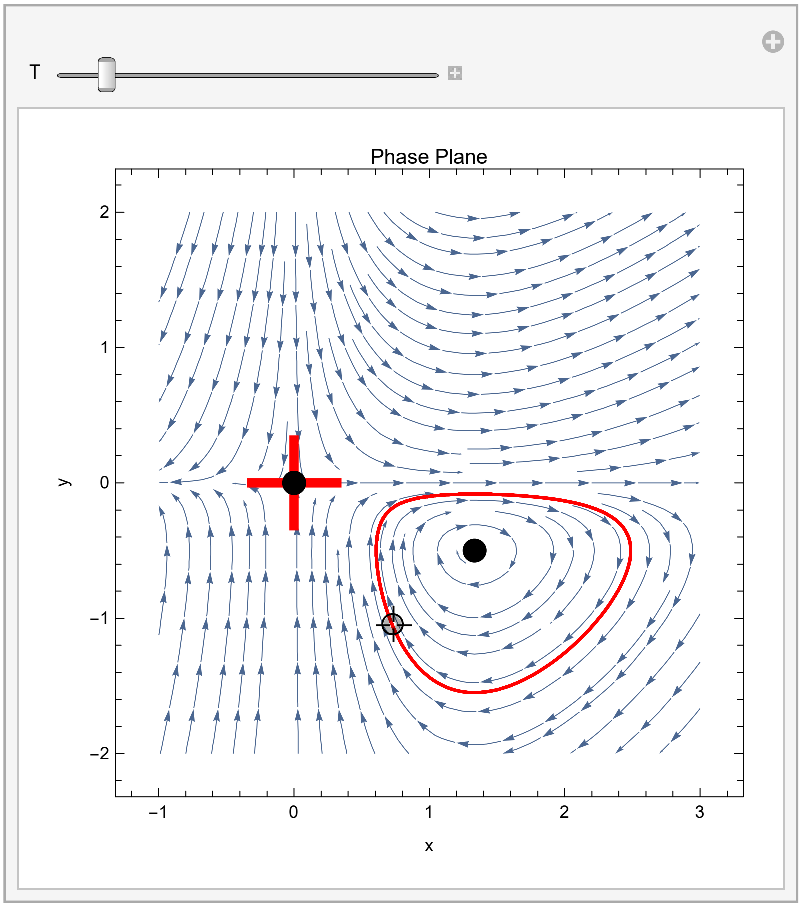
In[9]:=
```
pplanePlot = StreamPlot[{f[x, y], g[x, y]}, {x, -1, 3}, {y, -2, 2},
    FrameLabel → {"x", "y"},
    PlotLabel → "Phase Plane"];

Show[pplanePlot, EVPlot, eqPtsPlot]
```

Out[10]=



In[11]:=
```
Manipulate[Show[pplanePlot, EVPlot, eqPtsPlot,
  ParametricPlot[
   Evaluate[First[{x[t], y[t]} /. NDSolve[{x'[t] == f[x[t], y[t]], y'[t] == g[x[t], y[t]],
        Thread[{x[0], y[0]} == point]}, {x, y}, {t, 0, T}]]], {t, 0, T}, PlotStyle → Red]],
 {{T, 10}, 1, 100}, {{point, {1, 0}}, Locator}, SaveDefinitions → True]
```

Out[11]=



NDSolve: At t == 0.34209194796188114`, step size is effectively zero; singularity or stiff system suspected.

NDSolve: At t == 0.27140299148593394`, step size is effectively zero; singularity or stiff system suspected.

Example #2

Consider the system given by

$$\dot{x} = 2\,x\,(x-4) + 4\,x\,y$$
$$\dot{y} = 6\,xy - 8\,y$$

We would like to (1) conduct local analysis, and (2) graph the phase plane to validate our local analysis conclusions as before.

Local Analysis

In[12]:=
```
f2[x_, y_] = 2 x (x - 4) + 4 x y;
g2[x_, y_] = 6 x y - 8 y;
```

In[14]:=
```
eqPts2 = Solve[{f2[x, y] == 0, g2[x, y] == 0}, {x, y}]
```

Out[14]= $\left\{\left\{x \to \frac{4}{3}, y \to \frac{4}{3}\right\}, \{x \to 4, y \to 0\}, \{x \to 0, y \to 0\}\right\}$

In[15]:=
```
DF2[xs_, ys_] =
  {{D[f2[x, y], x], D[f2[x, y], y]}, {D[g2[x, y], x], D[g2[x, y], y]}} /. {x → xs, y → ys};
```

In[16]:=
```
For[j = 1, j ≤ Length[eqPts2], j++,
 Print[];
 Print["======================================================================"];
 Print["At the equilibrium point ", eqPts2[[j]],
  " the eigensystem is given by ", MatrixForm[Eigensystem[DF2[x, y] /. eqPts2[[j]]]]];
 Print["======================================================================"];
 Print[];
]
```

```
======================================================================
```

At the equilibrium point $\left\{x \to \frac{4}{3}, y \to \frac{4}{3}\right\}$ the eigensystem is given by $\begin{pmatrix} 8 & -\frac{16}{3} \\ \{1, 1\} & \left\{-\frac{2}{3}, 1\right\} \end{pmatrix}$

```
======================================================================
```

```
======================================================================
```

At the equilibrium point $\{x \to 4, y \to 0\}$ the eigensystem is given by $\begin{pmatrix} 16 & 8 \\ \{2, 1\} & \{1, 0\} \end{pmatrix}$

```
======================================================================
```

```
======================================================================
```

At the equilibrium point $\{x \to 0, y \to 0\}$ the eigensystem is given by $\begin{pmatrix} -8 & -8 \\ \{0, 1\} & \{1, 0\} \end{pmatrix}$

```
======================================================================
```

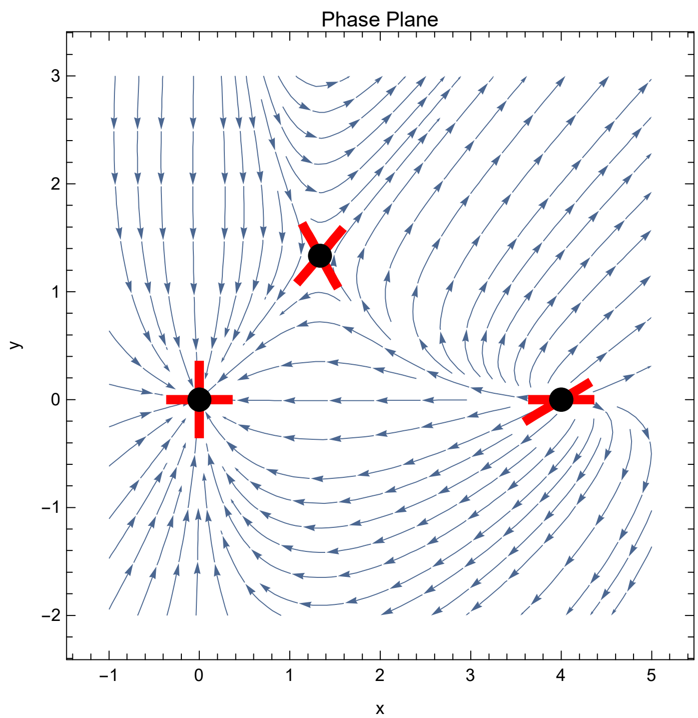
Comparing the Local Analysis with the Phase Plane

First, let’s create a loop that iterates over each of the equilibrium points. We will plot the eigenvectors

In[17]:=
```
For[j = 1, j ≤ Length[eqPts2], j++,
  esys = Eigensystem[DF2[x, y] /. eqPts2[[j]]];
  evPlots2j =
   ParametricPlot[esys[[2]] * s + Table[{x, y} /. eqPts2[[j]] , {k, 1, 2}], {s, -1, 1},
    PlotStyle → {Red, Thickness → .015},
    RegionFunction →
     Function[{u, v, vx, vy, n}, (((u - x)^2 + (v - y)^2) /. eqPts2[[j]]) < .1]]];
EVPlot2 = Show[Table[evPlots2j, {j, 1, Length[eqPts]}]];
eqPtsPlot2 = ListPlot[{x, y} /. eqPts2,
   PlotMarkers → {Automatic, Scaled[.04]},
   PlotStyle → Black];
```

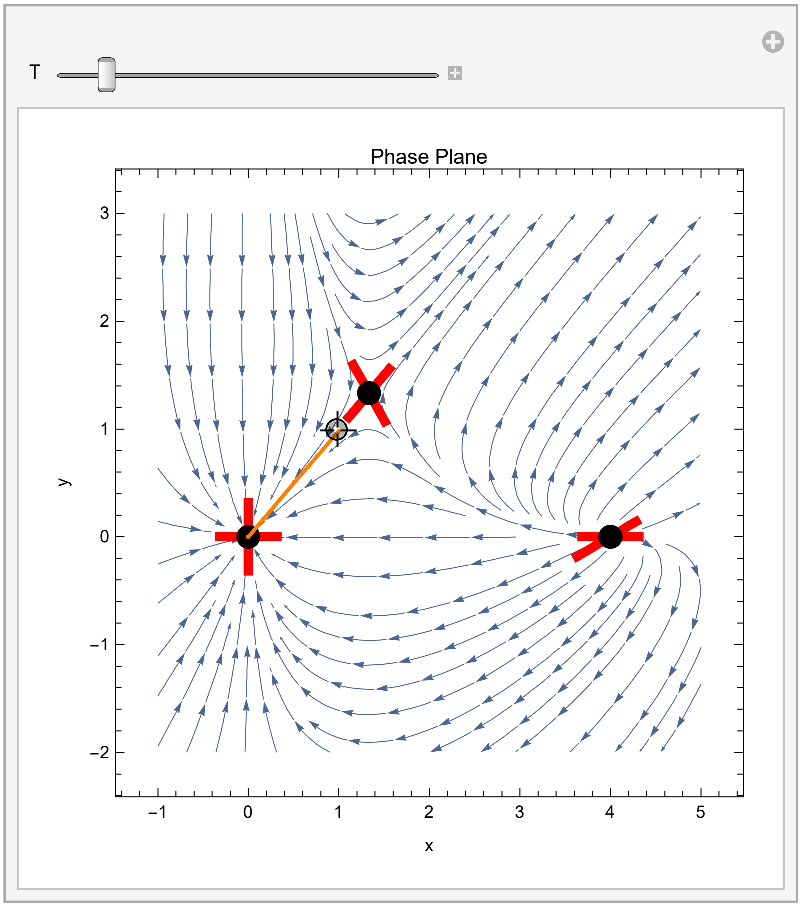
In[20]:=
```
pplanePlot2 = StreamPlot[{f2[x, y], g2[x, y]}, {x, -1, 5}, {y, -2, 3},
   FrameLabel → {"x", "y"},
   PlotLabel → "Phase Plane",
   StreamPoints → 400];

Show[pplanePlot2, EVPlot2, eqPtsPlot2]
```

Out[21]=



In[22]:=
```
Manipulate[Show[pplanePlot2, EVPlot2, eqPtsPlot2,
  ParametricPlot[Evaluate[First[{x[t], y[t]} /. NDSolve[
       {x'[t] == f2[x[t], y[t]], y'[t] == g2[x[t], y[t]], Thread[{x[0], y[0]} == point]},
       {x, y}, {t, 0, T}]]], {t, 0, T}, PlotStyle → {Orange, Thick}]],
 {{T, 10}, 1, 100}, {{point, {1, 0}}, Locator}, SaveDefinitions → True]
```

Out[22]=



NDSolve: At t == 0.07380327641083968`, step size is effectively zero; singularity or stiff system suspected.

NDSolve: At t == 0.36393125690619826`, step size is effectively zero; singularity or stiff system suspected.

NDSolve: At t == 0.38581031137231614`, step size is effectively zero; singularity or stiff system suspected.