Bifibrations of Polycategories and Classical Linear Logic

Nicolas Blanco[1]

Computer Science Department
University of Birmingham
Birmingham, UK

Noam Zeilberger[2]

Laboratoire d'Informatique de l'École Polytechnique
Palaiseau, France

Abstract

The main goal of this article is to expose and relate different ways of interpreting the multiplicative fragment of classical linear logic in polycategories. Polycategories are known to give rise to models of classical linear logic in so-called representable $*$-polycategories, which ask for the existence of various polymaps satisfying the different universal properties needed to define tensor, par, and negation. We begin by explaining how these different universal properties can all be seen as instances of a single notion of universality of a polymap parameterised by an input or output object, which also generalises the classical notion of universal multimap in a multicategory. We then proceed to introduce a definition of in-cartesian and out-cartesian polymaps relative to a refinement system (= strict functor) of polycategories, in such a way that universal polymaps can be understood as a special case. In particular, we obtain that a polycategory is a representable $*$-polycategory if and only if it is bifibred over the terminal polycategory $\mathbb{1}$. Finally, we present a Grothendieck correspondence between bifibrations of polycategories and pseudofunctors into **MAdj**, the 2-polycategory of multivariable adjunctions. When restricted to bifibrations over $\mathbb{1}$ we get back the correspondence between $*$-autonomous categories and Frobenius pseudomonoids in **MAdj** that was recently observed by Shulman.

Keywords: Polycategories, linear logic, bifibrations, Grothendieck construction, Frobenius monoids

1 Introduction

In his early studies of the linguistic applications of Gentzen's sequent calculus [Lam61], Lambek observed that the so-called "associative syntactic calculus" of [Lam58] has a natural semantic interpretation, where formulas are interpreted as bimodules of rings and proofs of sequents $A_1, \ldots, A_n \to B$ are interpreted as multilinear maps $A_1 \times \cdots \times A_n \to B$. He mentions that one benefit of the sequent calculus presentation is that it leads to a decision procedure for the existence of canonical mappings, and notes that "it has already been observed by Bourbaki [*Algèbre multilinéaire*, 1948] that linear mappings of the kind we are interested in are best defined with the help of multilinear mappings". These early observations later led Lambek to formally introduce the definition of *multicategories* in [Lam69], which generalise categories by allowing morphisms to have multiple inputs, a paradigmatic example being the multicategory of vector spaces and multilinear maps.

Szabo, a student of Lambek's, introduced *polycategories* in [Sza75], which further generalise multicategories by allowing morphisms to have multiple outputs in addition to multiple inputs. One motivation for studying polycategories from the view of proof theory is that they stand in the same relation to Gentzen's classical sequent calculus LK as multicategories stand in relation to the intuitionistic sequent calculus LJ. For example, the composition operation for morphisms in a polycategory is typed just like the cut rule in classical sequent

[1] Email: n.blanco@pgr.bham.ac.uk

[2] Email: noam.zeilberger@lix.polytechnique.edu

calculus. Lambek and Szabo's work was later revisited from the perspective of linear logic [Gir87] by Cockett and Seely [CS97], see also [BCST96,Hyl02,CS07]. In particular, the notion of a *two-tensor polycategory with duals*, also called a *representable* $*$*-polycategory*, provides a natural source of models for the multiplicative fragment of classical linear logic. Representable $*$-polycategories are equivalent to the $*$-autonomous categories of Barr [Bar91], but have the advantage that all of the logical connectives can be defined by the existence of objects and (poly)morphisms satisfying certain universal properties, rather than as algebraic structures subject to coherence conditions.

This relation between $*$-autonomous categories and representable $*$-polycategories is analogous to the relation between monoidal categories and *representable multicategories* (called *monoidal multicategories* by Lambek [Lam69]), a relation studied carefully by Hermida [Her00]. Hermida noted certain analogies between the theory of representable multicategories and the theory of fibred categories (cf. [Her00, Table 1]), which he later made explicit by introducing a notion of (covariant) *fibration of multicategories* [Her04], in such a way that a representable multicategory is precisely the same thing as a multicategory fibred over the terminal multicategory $\mathbb{1}$. One interest of studying the more general notion of covariant fibration of multicategories $\mathcal{E} \to \mathcal{B}$, where every multimorphism $f : A_1, \dots, A_n \to B$ in $\mathcal{B}$ induces a pushforward functor $\mathbf{push}\langle f \rangle : \mathcal{E}_{A_1} \times \cdots \times \mathcal{E}_{A_n} \to \mathcal{E}_B$, is that it models a much richer class of structures coming from algebra and logic. For example, Hermida notes that an *algebra for an operad* O can be identified with a discrete covariant fibration over O, the latter seen as a one-object multicategory. The appropriate definition of *contravariant* fibration (and of bifibration) of multicategories was not addressed in [Her04]. However, there is a natural definition of contravariant fibration of multicategories, made explicit in the work of Hörmann [HÏ7, A.2] and of Licata, Shulman, and Riley [LSR17], under which each multimorphism of the base multicategory induces a family of pullback operations $\mathbf{pull}[f]^{(i)} : \mathcal{E}^{\mathrm{op}}_{A_1} \times \cdots \times \mathcal{E}^{\mathrm{op}}_{A_{i-1}} \times \mathcal{E}^{\mathrm{op}}_{A_{i+1}} \times \cdots \times \mathcal{E}^{\mathrm{op}}_{A_n} \times \mathcal{E}_B \to \mathcal{E}_{A_i}$, parameterised by the selection of the index $1 \le i \le n$ of a particular input object A_i. One interesting feature of this definition is that *monoidal biclosed categories* in the sense of Lambek [Lam69] are equivalent to multicategories bifibred over $\mathbb{1}$. Moreover, replacing the terminal multicategory by an arbitrary base multicategory leads to a much richer framework for modelling a variety of substructural and modal logics, as discussed by Licata et al. [LSR17], and in a very similar spirit to Melliès and Zeilberger's work on type refinement and monoidal closed bifibrations (cf. [MZ15,MZ16]). In particular, a recurring pattern is that some algebraic gadget in the base (e.g., a monoid object) induces some logical structure (e.g., monoidal closure) on its fibre.

In this paper, we begin to develop a theory of bifibrations of polycategories, guided by the principle that representable $*$-polycategories (and hence $*$-autonomous categories) should be equivalent to polycategories bifibred over the terminal polycategory $\mathbb{1}$. One consequence of this theory is that we recover a nice observation recently made by Shulman [Shu19], that $*$-autonomous categories are equivalent to (pseudo) Frobenius monoids in the (2-)polycategory of multivariable adjunctions. This will follow as a result of a general Grothendieck construction for bifibrations of polycategories, in a similar manner to the pattern mentioned above. Perhaps surprisingly, another one of our motivational examples for developing this theory was the category $\mathbf{FBan_1}$ of finite dimensional Banach spaces and contractive maps. It is a $*$-autonomous category and it comes with a $*$-autonomous forgetful functor into **FVect**, but contrary to the latter it is not compact closed. It provides a model of classical MALL based on finite dimensional vector spaces that is not degenerate, in the sense that the positive and negative fragments do not coincide. The fact that this category and more generally $\mathbf{Ban_1}$ the category of arbitrary Banach spaces and contractive maps provides a model of intuitionistic MALL is well-documented (cf. [BPS94]), however we could not find any mention of the fact that it is $*$-autonomous. Yet the structures needed to interpret the connectives are popular in the study of Banach spaces. For both $\otimes$ and $\parr$ these correspond to the tensor product of vector spaces but equipped with different norms called the projective and the injective (cross)norms. These have the property of being extremal in all the well-behaved norms that can be put on the tensor product. More specifically for any crossnorm $\| - \|$ and any $u \in A \otimes B$ we have $\|u\|_{A \parr B} \le \|u\| \le \|u\|_{A \otimes B}$. We will see that this has a nice explanation from the fact that the projective ($\otimes$) norm and the injective ($\parr$) norm can be defined as pushforwards and pullbacks, respectively, relative to a forgetful functor into vector spaces.

2 Polycategories, linear logic, and universality

2.1 Polycategories

There are several different definitions of "polycategory" in the literature. We will consider the following definition of (non-symmetric) polycategory due to Cockett and Seely [CS97], which differs slightly from Szabo's original definition [Sza75] in imposing a *planarity condition* on composition, although most of the ideas in this paper may transferred in a straightforward way to the setting of symmetric polycategories (cf. [Hyl02,Shu20]) without this condition.

Definition 2.1 A *polycategory* $\mathcal{P}$ consists of:

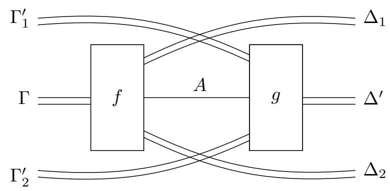
- a collection of *objects* $Ob(\mathcal{P})$
- for any pair of finite lists of objects Γ and Δ, a set $\mathcal{P}(\Gamma;\Delta)$ of *polymaps* from Γ to Δ denoted $f\colon \Gamma \to \Delta$ (we refer to objects in Γ as *inputs* of f, and to objects in Δ as *outputs*)
- for every object A, an *identity* polymap $id_A : A \to A$
- for any pair of polymaps $f\colon \Gamma \to \Delta_1, A, \Delta_2$ and $g\colon \Gamma_1', A, \Gamma_2' \to \Delta'$ satisfying the restriction that [either Δ_1 or Γ_1' is empty] and [either Δ_2 or Γ_2' is empty], a polymap $g \circ_A f\colon \Gamma_1', \Gamma, \Gamma_2' \to \Delta_1, \Delta', \Delta_2$

subject to appropriate *unitality*, *associativity*, and *interchange* laws whenever these make sense:

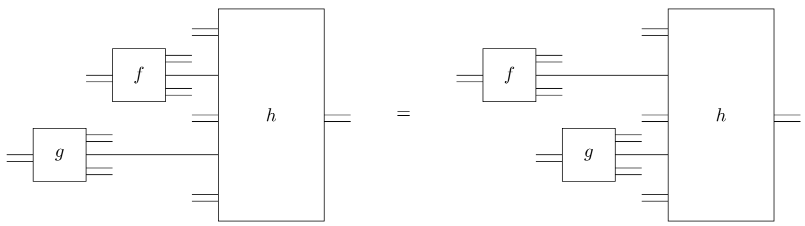
$$id_A \circ_A f = f \tag{1}$$
$$f \circ_A id_A = f \tag{2}$$
$$(h \circ_B g) \circ_A f = h \circ_B (g \circ_A f) \tag{3}$$
$$(h \circ_B g) \circ_A f = (h \circ_A f) \circ_B g \tag{4}$$
$$h \circ_B (g \circ_A f) = g \circ_A (h \circ_B f) \tag{5}$$

Remark 2.2 The notation $\circ_A$ for the composition can be ambiguous when there are multiple copies of the same object. This can be dealt with more carefully by indexing or labelling each input and output of a polymap. However, we will stick with the more relaxed (albeit less precise) notation in this article, since it will never lead to ambiguity in the examples.

Remark 2.3 We will sometimes find it useful to represent polymaps by string diagrams. In this diagrammatic syntax, the composition operation may be depicted schematically as follows:



The restriction on the composition operation that either Δ_1 or Γ_1' is empty and that either Δ_2 or Γ_2' is empty is called a "planarity" condition, since in the picture above it means that there are actually no crossing wires. In general, the string diagram of a polymap corresponds to a planar tree with the edges oriented from left to right, and the polycategory axioms correspond to natural isotopies between diagrams. For example, the interchange law (4) states that when composing along two different inputs, the order should not matter:



This justifies drawing the two polymaps f and g above on the same level, as we will sometimes do in examples.

2.2 Representable polycategories and $*$-polycategories

In this section we briefly recall the notion of representable (or two-tensor) polycategory and of $*$-polycategory, which have been used to model the multiplicative connectives of classical linear logic.

Definition 2.4 Let Γ be a list of objects in a polycategory $\mathcal{P}$. A *tensor product* of Γ is an object $\bigotimes\Gamma$ equipped with a polymap $m_\Gamma : \Gamma \to \bigotimes\Gamma$ such the operation $\mathcal{P}(\Gamma_1, \bigotimes\Gamma, \Gamma_2; \Delta) \to \mathcal{P}(\Gamma_1, \Gamma, \Gamma_2; \Delta)$ of precomposition with m_Γ is invertible. Dually, for any list of objects Δ, a *par product* (or *cotensor product*) of Δ is an object $\parr\Delta$ equipped with a polymap $w_\Delta : \parr\Delta \to \Delta$ such that the operation $\mathcal{P}(\Gamma; \Delta_1, \parr\Delta, \Delta_2) \to \mathcal{P}(\Gamma; \Delta_1, \Delta, \Delta_2)$ of postcomposition with w_Δ is invertible.

Definition 2.5 A polycategory is said to be *representable* (or a *two-tensor polycategory*) if it has tensors and pars of any finite lists of objects.

The definition of representable polycategory may be alternatively stated requiring only the existence of binary and nullary tensors and pars (this being equivalent since the binary and nullary cases are sufficient for building up tensor/pars of arbitrary finite lists of objects). In any case, the definition implies that polymaps $\Gamma \to \Delta$ of a representable polycategory are in one-to-one correspondence with unary maps $\bigotimes \Gamma \to \bigparr \Delta$ of its underlying category. Conversely, Cockett and Seely proved that any *linearly distributive category* $(\mathcal{C}, \otimes, 1, \parr, \bot)$ induces a polycategory where the polymaps $\Gamma \to \Delta$ are defined as maps $\bigotimes \Gamma \to \bigparr \Delta$ in $\mathcal{C}$, and indeed that this extends to an equivalence of 2-categories between representable polycategories and linearly distributive categories [CS97]. One obtains $*$-autonomous categories by moreover asking for the existence of *duals*.

Definition 2.6 A *right dual* of an object A in a polycategory is an object A^* equipped with polymaps $rcup_A : \cdot \to A, A^*$ and $rcap_A : A^*, A \to \cdot$ such that $rcup_A \circ_{A^*} rcap_A = id_A$ and $rcap_A \circ_A rcup_A = id_{A^*}$. A *left dual* of A is an object ${}^*A$ equipped with polymaps $lcup_A : \cdot \to {}^*A, A$ and $lcap_A : A, {}^*A \to \cdot$ such that $lcup_A \circ_{{}^*A} lcap_A = id_A$ and $lcap_A \circ_A lcup_A = id_{{}^*A}$.

Definition 2.7 A polycategory is said to be a *$*$-polycategory* (or *have duals*) it every object has both a right and a left dual.

Note that the definition of $*$-polycategory may be simplified in the case of a symmetric polycategory because left and right duals coincide in that case, although following Cockett and Seely we have chosen to consider the more general situation. Cockett and Seely proved that in the symmetric case, the notion of $*$-polycategory coincides with Barr's notion of $*$-autonomous category [Bar91], and that in the non-symmetric case it coincides with a natural notion of "planar" $*$-autonomous category [CS97].

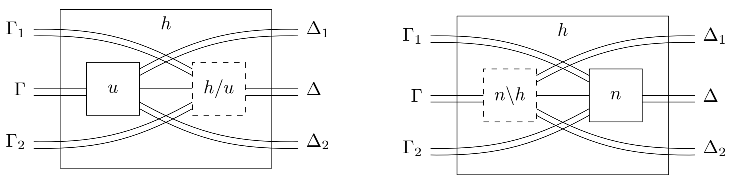
2.3 Representable $*$-polycategories are $*$-representable polycategories

In this section we introduce a notion of "$*$-representability" of a polycategory, and prove that a polycategory is $*$-representable if and only if it is a representable $*$-polycategory.

Definition 2.8 A polymap $u : \Gamma \to \Delta_1, A, \Delta_2$ is said to be *universal in the output* A (or *out-universal* for short, or simply *universal* when there is no ambiguity), written $u : \Gamma \to \Delta_1, \underline{A}, \Delta_2$ if for any polymap $h : \Gamma_1, \Gamma, \Gamma_2 \to \Delta_1, \Delta, \Delta_2$ such that $\Gamma_i = \emptyset$ or $\Delta_i = \emptyset$, there is a unique polymap $h/u : \Gamma_1, A, \Gamma_2 \to \Delta$ such that $h = h/u \circ_A u$.

Dually, a polymap $n : \Gamma_1, A, \Gamma_2 \to \Delta$ is *universal in the input* A (or *in-universal*), written $n : \Gamma_1, \underline{A}, \Gamma_2 \to \Delta$ if for any polymap $h : \Gamma_1, \Gamma, \Gamma_2 \to \Delta_1, \Delta, \Delta_2$ such that $\Gamma_i = \emptyset$ or $\Delta_i = \emptyset$ there is a unique polymap $n \backslash h : \Gamma \to \Delta_1, A, \Delta_2$ such that $h = n \circ_A n \backslash h$.

Graphically, the definitions are summarized in the following diagram:



$(*)$

Remark 2.9 By extension, we say that A is an *out-universal object* (resp. *in-universal object*) with respect to the surrounding context $\Gamma \to \Delta_1, \textvisiblespace, \Delta_2$ (resp. $\Gamma_1, \textvisiblespace, \Gamma_2 \to \Delta$) if there is an out-universal polymap $\Gamma \to \Delta_1, \underline{A}, \Delta_2$ (resp. in-universal polymap $\Gamma_1, \underline{A}, \Gamma_2 \to \Delta$). For a fixed surrounding context, in-universal and out-universal objects are unique up to unique isomorphism.

Definition 2.10 A polycategory is said to be *$*$-representable* if it has all in-universal and out-universal objects, that is, if for any Γ, Δ_1, Δ_2 there is an object A equipped with an out-universal polymap $\Gamma \to \Delta_1, \underline{A}, \Delta_2$, and similarly, for any Γ_1, Γ_2, Δ there is an object A equipped with an in-universal polymap $\Gamma_1, \underline{A}, \Gamma_2 \to \Delta$.

It may be argued that Definition 2.8 is a natural generalisation of the notion of *strong universal* multimap in a multicategory [Her00], and Definition 2.10 the natural generalisation of representability from multicategories to polycategories (*pace* Defn. 2.5). In Section 3, we will see that these concepts are special cases of more general fibrational concepts. Like strong universal multimaps in a multicategory, both in-universal and out-universal polymaps are closed under composition in an appropriate sense.

Proposition 2.11 *In-universal polymaps compose, in the sense that if* $f : \Gamma_1, \underline{A}, \Gamma_2 \to \Delta_1, B, \Delta_2$ *and* $g : \Gamma_1', \underline{B}, \Gamma_2' \to \Delta'$*, then* $g \circ_B f : \Gamma_1', \Gamma_1, \underline{A}, \Gamma_2, \Gamma_2' \to \Delta_1, \Delta', \Delta_2$*. Similarly, out-universal maps compose in the sense that if* $f : \Gamma \to \Delta_1, \underline{B}, \Delta_2$ *and* $g : \Gamma_1', B, \Gamma_2' \to \Delta_1', \underline{C}, \Delta_2'$*, then* $g \circ_B f : \Gamma_1', \Gamma, \Gamma_2' \to \Delta_1, \Delta_1', \underline{C}, \Delta_2', \Delta_2$.

Proof. As we will see later, this is a special case of Proposition 3.4. $\square$

An immediate consequence of these definitions is that tensor products can be considered as out-universal objects, and par products as in-universal objects.

Proposition 2.12 *An object $\bigotimes \Gamma$ equipped with a polymap $m : \Gamma \to \bigotimes \Gamma$ is a tensor product of Γ iff m is out-universal (in its unique output). Dually, an object $\bigparr \Delta$ equipped with a polymap $w : \bigparr \Delta \to \Delta$ is a par product of Δ iff w is in-universal (in its unique input).*

Somewhat more surprisingly, duals can also be characterised as either in-universal or out-universal objects.

Proposition 2.13 *Let A and A^* be objects of a polycategory $\mathcal{P}$. The following are equivalent:*

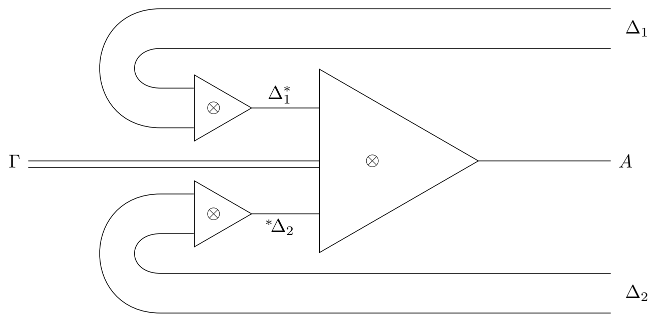
(i) *there is an out-universal map $rcup_A : \cdot \to A, \underline{A^*}$*
(ii) *there is an in-universal map $rcap_A : \underline{A^*}, A \to \cdot$*
(iii) *there is an out-universal map $rcup_A : \cdot \to \underline{A}, A^*$*
(iv) *there is an in-univeresal map $rcap_A : A^*, \underline{A} \to \cdot$*
(v) *A^* is the right dual of A*

Proof. See Appendix A. □

Remark 2.14 There is of course a similar result for left duals with $lcup_A$ and $lcap_A$.

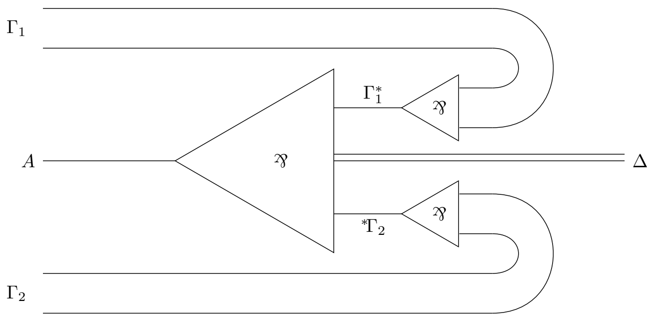
Theorem 2.15 *$\mathcal{P}$ is a representable $*$-polycategory iff it is $*$-representable.*

Proof. The right to left direction follows by propositions 2.12 and 2.13. For the left to right direction we need to prove that we can construct in-universal and out-universal objects for any contexts just using $\otimes$, $\parr$ and $*$. Given contexts $\Gamma, \Delta_1, \Delta_2$ consider the object $A := \Delta_1^* \otimes \bigotimes \Gamma \otimes {}^*\Delta_2$ where $\Delta_1^* := B_{1,n_1}^* \otimes ... \otimes B_{1,1}^*$ for $\Delta_1 = B_{1,1}, ..., B_{1,n_1}$ and similarly for ${}^*\Delta_2$. This object comes with the following polymap. So by proposition 2.11 it is out-universal.



It is a composition of out-universal polymaps on their out-universal object.

So by proposition 2.11 it is out-universal. Similarly we can construct any in-universal object with respect to $\Gamma_1, \Gamma_2, \Delta$ by considering $A := {}^*\Gamma_1 \parr \bigparr \Delta \parr \Gamma_2^*$ and the following polymap.



□

2.4 Examples

Example 2.16 Any linearly distributive category $\mathcal{C}$ gives a polycategory $\mathcal{P}(\mathcal{C})$ called its underlying polycategory. It has the same objects as $\mathcal{C}$ and a polymap $f : A_1, ..., A_m \to B_1, ..., B_n$ in $\mathcal{P}(\mathcal{C})$ is a map $f : A_1 \otimes ... \otimes A_m \to B_1 \parr ... \parr B_n$ in $\mathcal{C}$.

Example 2.17 In particular any monoidal category gives rises to a polycategory with the same objects and with polymaps $f : A_1 \otimes ... \otimes A_m \to B_1 \otimes ... \otimes B_n$.

Example 2.18 The terminal polycategory $\mathbb{1}$ has one object $*$ and a unique arrow $s_{m,n} : *^m \to *^n$ for every arity m and co-arity n.

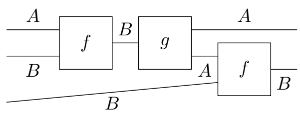
Example 2.19 Any category induces a polycategory with only unary morphisms. Conversely to any polycategory correspond a category by restricting to the unary maps.

Example 2.20 From any multicategory $\mathcal{M}$ we can define two polycategories $\mathcal{M}^+$ and $\mathcal{M}^-$ that have the same objects as $\mathcal{M}$. The polymaps of $\mathcal{M}^+$ have always exactly one output and correspond to multimaps in $\mathcal{M}$ while the polymaps in $\mathcal{M}^-$ have always exactly one input and correspond to multimaps in $\mathcal{M}$ reversed. Conversely from any polycategory we get two multicategories by restricting to polymaps with exactly one output and (reversed) polymaps with exactly one input.

Example 2.21 There are polycategories **Vect** and **FVect** of (finite dimensional) vector spaces and (poly)linear maps. It forms a polycategory since the tensor product of vector spaces defines a monoidal structure on the category of (finite dimensional) vector spaces and linear maps. **FVect** is a representable $*$-polycategory while **Vect** is representable but does not have duals in general. In fact the vector spaces that admit a dual are precisely the finite dimensional ones.

Example 2.22 There are polycategories **Ban**, $\mathbf{Ban_1}$, **FBan** and $\mathbf{FBan_1}$. The first two (resp. last two) have (resp. finite dimensional) Banach spaces as objects. The first and third have continuous/bounded (poly)linear maps as (poly)morphisms. While the secon and last have contractive (poly)maps as morphisms. The definition of these polycategories and their theory are developed in detailed in appendix B. It is well-known that **FBan** is equivalent to **FVect** so **FBan** is a representable $*$-polycategory. However it is degenerate as a model of linear logic since $A \otimes B \simeq A \parr B$ for any A, B. $\mathbf{FBan_1}$ is also a representable $*$-polycategory but where tensor and par are distinguished. The tensor and par both correspond to the tensor product of the vector spaces but with the projective and injective norms respectively. $\mathbf{Ban_1}$ is also representable with the same norms. However in the infinite dimensional case, we need to complete the vector spaces to get Banach spaces. This is not a $*$-polycategory.

Example 2.23 A *poly-signature* Σ consists of a collection of types, together with for any finite lists of types Γ and Δ, a set of operations $\Sigma(\Gamma; \Delta)$. The *free polycategory* generated by Σ, notated $\mathcal{P}(\Sigma)$, has types as objects, and polymaps given by planar oriented trees with a boundary of free edges, whose nodes are labelled by operations and whose edges are labelled by types subject to the constraints specified by the signature. For example, here is a depiction of the composite polymap $f \circ_A (g \circ_B f) : A, B, B \to A, B$ in the free polycategory generated by the signature containing a pair of types A and B and a pair of operations $f : A, B \to B$ and $g : B \to A, A$ (in the diagram, the edges are implicitly oriented from left to right):



In general, composition is performed by grafting two trees along an edge, while the identity on a type A is given by the trivial tree with no nodes and one oriented edge labelled A. Note that free polycategories $\mathcal{P}(\Sigma)$ are *not* representable.

Example 2.24 A one-object multicategory is commonly referred to as an *operad*, while a one-object polycategory is also known as a *dioperad* [Gan03]. In particular, for any polycategory $\mathcal{P}$ and any object $A \in \mathcal{P}$ there is a dioperad called the *endomorphism dioperad of* A, notated $End_{\mathcal{P}}(A)$, defined as the full subpolycategory of $\mathcal{P}$ containing only the object A. In other words it has one object and its polymaps correspond to polymaps $A, ..., A \to A, ..., A$ in $\mathcal{P}$.

3 Bifibrations of polycategories

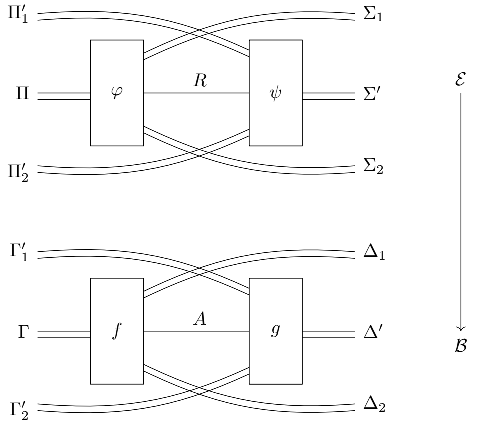
In this section we introduce a notion of bifibration of polycategories, and prove that a polycategory is a representable $*$-polycategory just in case it is bifibred over $\mathbb{1}$. We find it convenient to begin by adapting some terminological and notational conventions from the study of type refinement systems [MZ15,MZ16].

3.1 Definitions

Definition 3.1 A *poly-refinement system* is defined as a strict functor of polycategories $p : \mathcal{E} \to \mathcal{B}$. Explicitly, p sends objects $R \in \mathcal{E}$ to objects $p(R) \in \mathcal{B}$ and polymaps $\psi : R_1, ..., R_m \to S_1, ..., S_n$ in $\mathcal{E}$ to polymaps

$p(f) : p(R_1), ..., p(R_m) \to p(S_1), ..., p(S_n)$ in $\mathcal{B}$ in such a way that identities and composition are preserved strictly. We write $R \sqsubset A$ (pronounced "R refines A") to indicate that $p(R) = A$, and extend this to lists of objects in the obvious way, writing $\Pi \sqsubset \Gamma$ to indicate that $\Pi = R_1, \ldots, R_n$ and $\Gamma = A_1, \ldots, A_n$ for some $R_1 \sqsubset A_1, \ldots, R_n \sqsubset A_n$. Finally, we write $\psi : \Pi \underset{f}{\Longrightarrow} \Sigma$ to indicate that ψ is a polymap $\Pi \to \Sigma$ in $\mathcal{E}$ such that $p(\psi) = f$, with the implied constraint that $f : \Gamma \to \Delta$ where $\Pi \sqsubset \Gamma$ and $\Sigma \sqsubset \Delta$.

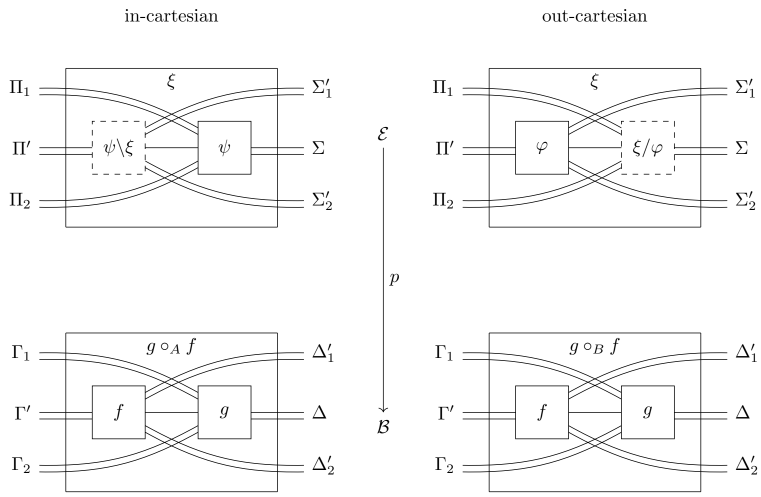
Remark 3.2 We will draw poly-refinement systems vertically, with the top diagram living in $\mathcal{E}$ and the bottom diagram in $\mathcal{B}$ in such a way that an object and polymaps are directly above their image. For example preservation of composition is given by:



Definition 3.3 Fix a poly-refinement system $p : \mathcal{E} \to \mathcal{B}$, and let $\psi : \Pi_1, R, \Pi_2 \underset{g}{\Longrightarrow} \Sigma$ be a polymap in $\mathcal{E}$, with some given input object $R \sqsubset A$. Then ψ is said to be *in-cartesian in R* (relative to p), written $\psi : \Pi_1, \underline{R}, \Pi_2 \underset{g}{\Longrightarrow} \Sigma$, if for any other polymap $\xi : \Pi_1, \Pi', \Pi_2 \underset{g \circ_A f}{\Longrightarrow} \Sigma'_1, \Sigma, \Sigma'_2$ there exists a unique polymap $\psi \backslash \xi : \Pi' \underset{f}{\Longrightarrow} \Sigma'_1, R, \Sigma'_2$ such that $\xi = \psi \circ_R (\psi \backslash \xi)$.

Dually, let $\varphi : \Pi \underset{f}{\Longrightarrow} \Sigma_1, S, \Sigma_2$ be a polymap with some given output object $S \sqsubset B$. Then φ is said to be *out-cartesian in S*, written $\varphi : \Pi \underset{f}{\Longrightarrow} \Sigma_1, \underline{S}, \Sigma_2$, if for any polymap $\xi : \Pi'_1, \Pi, \Pi'_2 \underset{g \circ_B f}{\Longrightarrow} \Sigma_1, \Sigma', \Sigma_2$ there exists a unique polymap $\xi / \varphi : \Pi'_1, S, \Pi'_2 \underset{g}{\Longrightarrow} \Sigma'$ such that $\xi = \xi / \varphi \circ_S \varphi$.

Graphically, the definitions are summarized in the following diagram:



(†)

Proposition 3.4 *In-cartesian polymaps compose, in the sense that if* $\varphi : \Pi_1, \underline{R}, \Pi_2 \underset{g}{\Longrightarrow} \Sigma_1, S, \Sigma_2$ *and* $\psi : \Pi'_1, \underline{S}, \Pi'_2 \underset{f}{\Longrightarrow} \Sigma'$ *then* $\psi \circ_S \varphi : \Pi'_1, \Pi_1, \underline{R}, \Pi_2, \Pi'_2 \underset{g \circ_B f}{\Longrightarrow} \Sigma_1, \Sigma', \Sigma_2$*. Similarly, out-cartesian maps compose in the* *sense that if* $\varphi : \Pi \underset{g}{\Longrightarrow} \Sigma_1, \underline{S}, \Sigma_2$ *and* $\psi : \Pi'_1, S, \Pi'_2 \underset{f}{\Longrightarrow} \Sigma'_1, \underline{T}, \Sigma'_2$ *then* $\psi \circ_S \varphi : \Pi'_1, \Pi, \Pi'_2 \underset{g \circ_B f}{\Longrightarrow} \Sigma_1, \Sigma'_1, \underline{T}, \Sigma'_2, \Sigma_2$.

Definition 3.5 A poly-refinement system $p : \mathcal{E} \to \mathcal{B}$ is said to be a *pull-fibration* if for any $f : \Gamma_1, A, \Gamma_2 \to \Delta$ in $\mathcal{B}$ and any $\Pi_1 \sqsubset \Gamma_1$, $\Pi_2 \sqsubset \Gamma_2$, and $\Sigma \sqsubset \Delta$ there is an object $\mathbf{pull}[f](\Pi_1 \llcorner\!\lrcorner \Pi_2; \Sigma) \sqsubset A$ together with an in-cartesian polymap $\Pi_1, \underline{\mathbf{pull}[f](\Pi_1 \llcorner\!\lrcorner \Pi_2; \Sigma)}, \Pi_2 \underset{f}{\Longrightarrow} \Sigma$. Dually, p is said to be a *push-fibration* if for any $f : \Gamma \to \Delta_1, B, \Delta_2$ in $\mathcal{B}$ and any $\Pi \sqsubset \Gamma$, $\Sigma_1 \sqsubset \Delta_1$, and $\Sigma_2 \sqsubset \Delta_2$ there is an object $\mathbf{push}\langle f\rangle(\Pi; \Sigma_1 \llcorner\!\lrcorner \Sigma_2) \sqsubset B$ together with an out-cartesian polymap $\Pi \underset{f}{\Longrightarrow} \Sigma_1, \underline{\mathbf{push}\langle f\rangle(\Pi; \Sigma_1 \llcorner\!\lrcorner \Sigma_2)}, \Sigma_2$. Finally, p is said to be a *bifibration* if it is both a pull-fibration and a push-fibration.

Remark 3.6 When pulling along a map $f : A \to \Delta$ with only one input, we will write $\mathbf{pull}[f](\Sigma)$ as shorthand for $\mathbf{pull}[f](\llcorner\!\lrcorner; \Sigma)$. Similarly when pushing along a map $f : \Gamma \to A$ with only one output, we will write $\mathbf{push}\langle f\rangle(\Gamma)$ as shorthand for $\mathbf{push}\langle f\rangle(\Gamma; \llcorner\!\lrcorner)$.

3.2 $*$-autonomous categories as bifibrations of polycategories

Comparing diagram (†) with diagram ($*$), the following statements are self-evident.

Proposition 3.7 *Let $\mathcal{P}$ be a polycategory. A polymap $u : \Gamma \to \Delta_1, A, \Delta_2$ (resp. $u : \Gamma_1, A, \Gamma_2 \to \Delta$) is out-universal (resp. in-universal) in A iff it is out-cartesian (resp. in-cartesian) with respect to the unique functor $\mathcal{P} \to \mathbb{1}$ into the terminal polycategory.*

Proposition 3.8 *$\mathcal{P}$ is a $*$-representable polycategory iff $\mathcal{P} \to \mathbb{1}$ is a bifibration of polycategories.*

We then derive the following as a corollary of Theorem 2.15 and Cockett and Seely's connection between $*$-autonomous categories and representable $*$-polycategories.

Theorem 3.9 *There is an equivalence between planar $*$-autonomous categories and bifibrations over the terminal polycategory $\mathbb{1}$.*

This correspondence may be extended in a straightforward way to the case of ordinary (symmetric) $*$-autonomous categories by considering symmetric bibrations, that is, symmetric poly-refinement systems (= functors of symmetric polycategories that strictly preserve identities, composition, and the symmetry actions) which are bifibrations in the above sense. We also expect that this result may be stated more precisely as an equivalence of 2-categories, but we leave this to future work.

One application of Theorem 3.9 is that it provides a way of decomposing a $*$-autonomous structure on a category, using elementary facts about cartesian polymaps.

Proposition 3.10 *Let $p : \mathcal{P} \to \mathcal{E}$ and $q : \mathcal{E} \to \mathcal{B}$ be poly-refinement systems, and let $\psi : \Pi_1, R, \Pi_2 \underset{g}{\Longrightarrow} \Sigma$ be a polymap in $\mathcal{P}$. If ψ is p-in-cartesian in $R \sqsubset A$ and g is q-in-cartesian in $A \sqsubset X$ then ψ is $q \circ p$-in-cartesian in $R \sqsubset X$.*

Remark 3.11 Similarly, a p-out-cartesian polymap over a q-out-cartesian polymap is $(q \circ p)$-out-cartesian.

Proposition 3.12 *Let $p : \mathcal{E} \to \mathcal{B}$ be a poly-refinement system, and suppose that $\mathcal{B}$ is a representable $*$-polycategory. If $\mathcal{E}$ has all in-cartesian liftings of in-universal polymaps and all out-cartesian liftings of out-universal polymaps then $\mathcal{E}$ is a representable $*$-polycategory.*

Proof. By Propositions 3.7 and 3.10. □

Example 3.13 In appendix B we use this idea to deduce that the polycategories $\mathbf{Ban_1}$ and $\mathbf{FBan_1}$ are representable and $*$-representable. We get the definition of the injective and projective norms from the more general one for pullbacks and pushforwards, and moreover, we derive a well-known fact about these norms, namely that they are extremal amongst all the well-behaved norms that can be put on the tensor product.

3.3 Additional examples

Example 3.14 Given two categories considered as polycategories with only unary maps and a functor between those. It is a pull-fibration, push-fibration or bifibration iff it is a fibration, opfibration or bifibration of categories.

Example 3.15 Similarly, we get notion of pull-fibration, push-fibration and bifibration for multicategories. Hermida's notion of covariant fibration in [Her04] correspond to our notion of push-fibration. He proved that multicategories covariantly fibred over the terminal multicategory correspond to representable multicategories, and so to monoidal categories. In the next chapter we will prove a generalisation of this result to polycategories. In particular we will get back Hermida's result and a strenghting of it: that multicategories bifibred over the terminal multicategory correspond to monoidal biclosed categories.

Example 3.16 In appendix B we treat the examples of the poly-refinement systems $\mathbf{Ban_1} \to \mathbf{Vect}$ and $\mathbf{FBan_1} \to \mathbf{FVect}$ in details. We give necessary and sufficient conditions for a (poly)linear map to admits cartesian liftings. In the case of a (unary) linear map f it has in-cartesian lifting if it is injective and out-cartesian lifting if it is surjective. We get bifibrations if we allow for extended pseudonorms instead of norms.

Example 3.17 The forgetful functor $\mathbf{Cat}_* \to \mathbf{Cat}$ from the category of pointed (small) categories to the category of (small) categories is an opfibration of 2-categories. The pushforward of $(\mathcal{A}, A)$ along $F : \mathcal{A} \to \mathcal{B}$ is $(\mathcal{B}, F(A))$. Similarly the forgetful functor $\mathbf{Adj}_* \to \mathbf{Adj}$ of pointed adjunctions is a bifibration of 2-categories. Here a pointed adjunction between pointed categories $(\mathcal{A}, A)$ and $(\mathcal{B}, B)$ consist of an adjunction $F \dashv G : \mathcal{A} \to \mathcal{B}$ and a morphism $f : F(A) \to B$ in $\mathcal{B}$ - or equivalently of a morphism $g : A \to G(B)$ in $\mathcal{A}$. The pushforward is given by the image by F while the pullback is given by the image of G. While working on the polycategorical Grothendieck construction we will define the 2-polycategory of multivariable adjunction $\mathbf{MAdj}$. It also has a pointed variant $\mathbf{MAdj}_*$. The forgetful functor induced is a bifibration of 2-polycategories.

3.4 Frobenius monoids

Definition 3.18 In a polycategory $\mathcal{P}$ a *Frobenius monoid* is an object A equipped with a unique polymap $\overline{(m,n)_A} : A^m \to A^n$ for each $m, n \in (N)$ such that $\overline{(1,1)_A} = id_A$ and such that these polymaps are stable under composition.

Proposition 3.19 *Equivalently a Frobenius monoid in $\mathcal{P}$ is a functor $F : \mathbb{1} \to \mathcal{P}$.*

Proof. The Frobenius monoid corresponds to $F(*)$ and the polymaps $\overline{(m,n)_{F(*)}}$ to $F(\overline{(m,n)})$. The properties needed on the polymaps are exactly functoriality of F. □

Remark 3.20 If $\mathcal{P}$ is representable with $\otimes = ⅋$ we get the usual notion of a Frobenius monoid in a monoidal category.

Definition 3.21 Given a poly-refinement system $p : \mathcal{E} \to \mathcal{B}$ and a Frobenius monoid A in $\mathcal{B}$ the *polyfiber* of p over A, noted $p^{-1}(A)$ is the subcategory of $\mathcal{E}$ with objects the one lying over A and polymaps the one lying over the $\overline{(m,n)_A}$.

Proposition 3.22 *$p^{-1}(A)$ is equivalent to the following pullback:*

$$\begin{tikzcd} p^{-1}(A) \arrow[r, hook] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\ \mathbb{1} \arrow[r, "F"'] & \mathcal{B} \end{tikzcd}$$

where F is the functor associated to A

Proposition 3.23 *Given two poly-refinement system $p : \mathcal{E} \to \mathcal{B}$ and functor $s : \mathcal{B}' \to \mathcal{B}$, let $\mathcal{E} \times_{\mathcal{B}} \mathcal{B}'$ be their pullback.*

$$\begin{tikzcd} \mathcal{E} \times_{\mathcal{B}} \mathcal{B}' \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{B}' \arrow[r, "s"'] & \mathcal{B} \end{tikzcd}$$

For a polymap $f : \Gamma_1, A, \Gamma_2 \to \Delta$ in $\mathcal{B}'$ and lists of objects Π_1, Π_2, Σ in $\mathcal{E} \times_{\mathcal{B}} \mathcal{B}'$ lying over Γ_1, Γ_2 and Δ, if there is a pullback $pull^{s(A)}_{s(f)}(\pi_1(\Pi_1)|\pi_1(\Pi_2); \pi_1(\Sigma))$ in $\mathcal{E}$ then there is a pullback $pull^{A}_{f}(\Pi_1|\Pi_2; \Sigma)$ in $\mathcal{E} \times_{\mathcal{B}} \mathcal{B}'$.

Proof. $\mathcal{E} \times_{\mathcal{B}} \mathcal{B}'$ is the polycategory whose objects are pairs of objects (E, B') of $\mathcal{E}$ and $\mathcal{B}$ such that $p(E) = s(B')$ and whose polymaps are pairs of polymaps (f, b') such that $p(f) = s(b')$.

Given a polymap $f : \Gamma_1, A, \Gamma_2 \to \Delta$ in $\mathcal{B}'$ and lists of objects $(\Pi_1, \Gamma_1), (\Pi_2, \Gamma_2), (\Sigma, \Delta)$ in $\mathcal{E} \times_{\mathcal{B}} \mathcal{B}'$ from a pullback $pull^{s(A)}_{s(f)}(\Pi_1|\Pi_2; \Sigma)$ in $\mathcal{E}$ with in-cartesian polymap $\varphi : \Pi_1, pull^{s(A)}_{s(f)}(\Pi_1|\Pi_2; \Sigma), \Pi_2 \to \Sigma$ we get a pullback $pull^A_f((\Pi_1,\Gamma_1)|(\Pi_2,\Gamma_2);(\Sigma,\Delta)) := (pull^{s(A)}_{s(f)}(\Pi_1|\Pi_2;\Sigma),A)$ with in-cartesian polymap (φ,f). □

Remark 3.24 Similarly if the pushforward exists in $\mathcal{E}$ it exists in $\mathcal{E}\times_{\mathcal{B}}\mathcal{B}'$.

Corollary 3.25 *Given a poly-refinement system $p:\mathcal{E}\to\mathcal{B}$ and a Frobenius monoid $(A,\{\overline{(m,n)_A}\})$ in $\mathcal{B}$ if all in-cartesian and out-cartesian polymaps of $\overline{(m,n)_A}$ exist then $p^{-1}(A)$ is a representable $*$-polycategory.*

Remark 3.26 If $\mathcal{B}$ is a representable $*$-polycategory and p has enough in-cartesian and out-cartesian liftings, we will call external the connectives on the global category and internal the connectives in the polyfibers. The internal connectives will be indexed by the Frobenius monoid. For example the external tensor product of two objects R,S lying over A,B is an object $R\otimes S$ lying over $A\otimes B$ while the internal tensor product of two objects R,S lying over a Frobenius monoid A is an object $R\otimes_A S$ lying over A.

4 Grothendieck correspondences

4.1 Categorical Grothendieck correspondences

The usual Grothendieck correspondence has being refined by Bénabou into one that involving distributors. This can be found in [B0́0]. We recall these correspondences below.

- Functor $\mathcal{E}\to\mathcal{B}\longleftrightarrow$ Lax normal functor $\mathcal{B}\to\mathbf{Dist}$
- Fibration $\mathcal{E}\to\mathcal{B}\longleftrightarrow$ pseudofunctor $\mathcal{B}\to\mathbf{Cat}$
- Opfibration $\mathcal{E}\to\mathcal{B}\longleftrightarrow$ pseudofunctor $\mathcal{B}\to\mathbf{Cat}^{\mathrm{op}}$
- Bifibration $\mathcal{E}\to\mathcal{B}\longleftrightarrow$ pseudofunctor $\mathcal{B}\to\mathbf{Adj}$

Where **Dist** is the bicategory of (small) categories and distributors/profunctors and **Adj** is the 2-category of (small) categories and adjunctions. Notice that **Cat**, $\mathbf{Cat}^{\mathrm{op}}$ and **Adj** are all subbicategories of **Dist** corresponding to distributors $A\nrightarrow B$ that are representable in A, in B, and in both, respectively. The explicit constructions are provided in Appendix C.

4.2 Polycategorical Grothendieck correspondences

We want to extend the previous correspondences to polycategories as follow:

- Poly-refinement system $\mathcal{E}\to\mathcal{B}\longleftrightarrow$ Lax normal functor $\mathcal{B}\to\mathbf{Dist}$
- Bifibration $\mathcal{E}\to\mathcal{B}\longleftrightarrow$ Pseudofunctor $\mathcal{B}\to\mathbf{MAdj}$

where **Dist** is the weak 2-polycategory of sets and (multivariable) distributors and **MAdj** is the strict 2-polycategory of sets and multivariable adjunctions. Like in the categorical case, **MAdj** is a sub-2-polycategory of **Dist** consisting of distributors that are representable in each of their variables.

4.2.1 About 2-polycategories

Of course to be able to express these correspondences we would need some theory of weak 2-polycategory, where by 2-polycategory we mean that the 1-cells can have multiple inputs and outputs but we have only unary 2-cells. This has not been carefully worked out to the extent of our knowledge. It would be nice if the weak higher case would mimic the 1-dimensional theory. Especially we would like to have weak 2-polycategories and of $*$-autonomous bicategories such that the results of this paper also holds. In particular, we should get that any compact closed bicategory – as defined by Mike Stay in [Sta13] – is a $*$-autonomous bicategory, and by extension a weak 2-polycategory. This would make **Dist** a weak 2-polycategory. We don't need to assume that much in this paper. We only need a notion of weak 2-polycategory (and lax/pseudo functor) such that **Dist** is an example.

Finally, it is worth noticing that everything considered in the correspondence between bifibrations and pseudofunctors into **MAdj** is strict. Since strict 2-polycategories have been defined before this last part does not require any assumption. This can be found in [Shu20]

4.2.2 Distributors and multivariable adjunctions

In this section we introduce the weak 2-polycategories **Dist** and **MAdj**. We prove that a multivariable adjunction can be understand as a representable distributor.

Definition 4.1 **Dist** is the weak 2-polycategory that has as objects categories, that has as polymaps $f:A_1,...,A_m\to B_1,...,B_n$ distributors $f:A_1\times...\times A_m\nrightarrow B_1,...,B_n$ and that has as 2-cells natural transformations.

Definition 4.2 Given categories $A_1, ..., A_m, B_1, ..., B_n$, a *$(m,n)$-adjunction* or *multivariable adjunction* $(F_l)_{1 \leq l \leq n} \dashv (G_k)_{1 \leq k \leq m} : A_1, ..., A_m \to B_1, ..., B_n$ consists of the following data:

- functors $F_l : \prod_i A_i \times \prod_{j \neq l} B_j^{\mathrm{op}} \to B_l$ for each l
- functors $G_k : \prod_{i \neq k} A_i^{\mathrm{op}} \times \prod_j B_j \to A_k$ for each k
- natural isomorphisms $B_l(F_l(a_1, ..., a_m, b_1, ..., b_n), b_l) \simeq A_k(a_k, G_k(a_1, ..., a_m, b_1, ..., b_n))$ for any k, l

Example 4.3 A (1,1)-adjunction between A, B is a pair of functor $F : A \to B$ and $G : B \to A$ such that $B(F(a), b) = A(a, G(b))$. It is just a usual adjunction.

Example 4.4 Let $(\mathcal{C}, \otimes, I)$ be a biclosed monoidal category. By definition $(A \otimes -)$ as a right adjoint $A \multimap -$ and $(- \otimes B)$ as a right adjoint $- \circ\!\!- B$. We get three functors $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $\multimap: \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ and $\circ\!\!-: \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ such that $\mathcal{C}(A \otimes B, C) \simeq \mathcal{C}(B, A \multimap C) \simeq \mathcal{A}, \mathcal{C} \circ\!\!- \mathcal{B}$, i.e. a $(2,1)$-adjunction $(\otimes) \dashv (\multimap, \circ\!\!-)$.

Proposition 4.5 *A (m,n)-adjunction $(F_l)_{1 \leq l \leq n} \dashv (G_k)_{1 \leq k \leq m} : A_1, ..., A_m \to B_1, ..., B_n$ is the same thing as a distributor $P : A_1 \times ... \times A_m \overset{B}{\nrightarrow}_1 \times ... \times B_n$ that is representable in each of its variables.*

Proof. From any of the F_l we can define a distributor $P_l : A_1 \times ... \times A_m \to B_1 \times ... \times B_n$ representable in B_l by $P_l(-,-) := B_l(F_l(-), -)$. Similarly we can get distributors representable in A_k from the functors G_k by $P^k(-,-) := A_k(-, G_k(-))$. But all of these distributors are naturally isomorphic by definition of a multivariable adjunction.

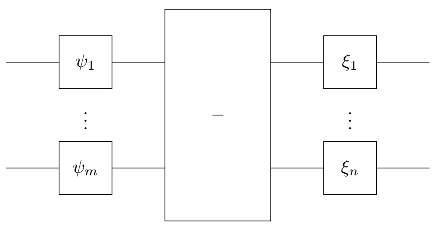
Conversely given a distributor $P : A_1 \times ... \times A_m \to B_1 \times ... \times B_n$, representability in the A_k and B_l produce functors G_k with natural isomorphisms $P(-,-) \simeq A_k(-, G_k(-))$ and functors F_l with natural isomorphisms $P(-,-) \simeq B_l(F_l(-), -)$. □

4.2.3 Fibres of a poly-refinement system and distributors between them

In the following we fix a poly-refinement system $p : \mathcal{E} \to \mathcal{B}$. We will define a lax normal functor into **Dist** by considering the fibres of p like in the categorical case. We will use the following convention for $\Gamma = A_1, ..., A_n$, $p^{-1}(\Gamma) := p^{-1}(A_1) \times ... \times p^{-1}(A_m)$.

$$
\begin{array}{rl}
\partial p : & \mathcal{B} \to \mathbf{Dist} \\
& B \mapsto p^{-1}(B) := \{R \in \mathcal{E} \mid p(R) = B\} \\
& \qquad \partial p(f) : p^{-1}(\Gamma)^{\mathrm{op}} \times p^{-1}(\Delta) \to \mathbf{Set} \\
f : \Gamma \to \Delta \mapsto & \qquad\qquad (\Pi, \Sigma) \mapsto \{\varphi : \Pi \to \Sigma \mid p(\varphi) = f\} \\
& \qquad ((\psi_i)_{1 \leq i \leq m}, (\xi_j)_{1 \leq j \leq n}) \mapsto \xi_1 \circ ... \circ \xi_n \circ - \circ \psi_1 \circ ... \circ \psi_m
\end{array}
$$

Here $\partial p(f)$ define an action of the family of morphisms $\psi_i : R'_i \to R_i$, $\xi_j : S_j \to S'_j$ by precomposition and postcomposition. The order in which we do these composition does not matter due to the interchange and associativity laws. Graphically



The proof that this defines a lax normal functor is similar to the categorical one with some extra bookkeeping because of the presence of contexts of inputs and outputs. In particular given polymaps $f : \Gamma \to \Delta_1, A, \Delta_2$ and $g : \Gamma'_1, A, \Gamma'_2 \to \Delta'$ we define a natural transformation $\partial p(g) \circ_{\partial p(A)} \partial p(f) \Rightarrow \partial p(g \circ_A f)$ by the following family of multimaps:

$$\begin{aligned}(\partial p(g) \circ_{\partial p(A)} \partial p(f))(\Pi_1', \Pi, \Pi_2', \Sigma_1, \Sigma', \Sigma_2) &= \int^R \partial p(g)(\Pi_1', R, \Pi_2', \Sigma') \times \partial p(f)(\Pi, \Sigma_1, R, \Sigma_2) \\ &= \int^R \{\varphi' : \Pi_1', R, \Pi_2' \to \Sigma' \mid p(\varphi') = g\} \times \{\varphi : \Pi \to \Sigma_1, R, \Sigma_2 \mid p(\varphi) = f\} \\ &\to \{\varphi' : \Pi_1', R, \Pi_2' \to \Sigma' \mid p(\varphi') = g\} \times \{\varphi : \Pi \to \Sigma_1, R, \Sigma_2 \mid p(\varphi) = f\} \\ &\to \{\varphi'' : \Pi_1', \Pi, \Pi_2' \to \Sigma_1, \Sigma', \Sigma_2' \mid p(\varphi'') = g \circ_A f\}\end{aligned}$$

Like in the categorical case, for $\partial p(g)$ to be representable in $p^{-1}(A)$ we need to define a functor $p_*^A(g) : p^{-1}(\Gamma_1')^{\mathrm{op}} \times p^{-1}(\Gamma_2')^{\mathrm{op}} \times p^{-1}(\Delta') \to p^{-1}(A)$ establishing a correspondence: $\dfrac{\Pi_1', R, \Pi_2' \overset{\psi}{\underset{g}{\Rightarrow}} \Sigma'}{R \overset{\tilde{\psi}}{\underset{id_A}{\Longrightarrow}} p_*^A(g)(\Pi_1', \Pi_2', \Sigma')}$ and such that $\psi = p_*^A(g)(id_R) \circ \tilde{\psi}$. This is the property of a weak in-cartesian polymap. So if we want $\partial p(g)$ to be representable in all its inputs we get a prefibration. Similarly if we want representability in all variable we get a prebifibration. Finally asking for a pseudofunctor and not just a lax normal functor is the same as asking that weak in-cartesian/opcartesian polymaps compose, so for a bifibration. Having all the representability condition for the distributor is the same as asking for a multivariable adjunction. So we get that any bifibration gives rise to a pseudofunctor in **MAdj**.

4.2.4 Polycategorical Grothendieck construction

Conversely, given a lax normal functor $F : \mathcal{B} \to \mathbf{Dist}$ we construct a polycategory $\int F$ whose objects are pairs (A, R) with $A \in \mathcal{B}$ and $R \in F(A)$. Then a polymap $(f, \varphi) : (\Gamma, \Pi) \to (\Delta, \Sigma)$ is a polymap $f : \Gamma \to \Delta$ in $\mathcal{B}$ and an element $\varphi \in F(f)(\Pi, \Sigma)$. The identity is given by (id_A, id_R) where $id_R \in Hom_{F(A)}(R, R) = id_{F(A)}(R, R) = F(id_A)(R, R)$. And for (f, φ) and (g, ψ) with $f : \Gamma \to \Delta_1, A, \Delta_2$, $g : \Gamma_1', A, \Gamma_2' \to \Delta'$, $\varphi \in F(f)(\Pi, \Sigma_1, R, \Sigma_2)$ and $\psi \in F(g)(\Pi_1', R, \Pi_2', \Sigma')$ we get a canonical element $\widetilde{(\varphi, \psi)} \in (F(g) \circ_{F(A)} F(f))(\Pi_1', \Pi, \Pi_2', \Sigma_1, \Sigma', \Sigma_2)$ and then one in $F(g \circ_A f)(\Pi_1', \Pi, \Pi_2', \Sigma_1, \Sigma', \Sigma_2)$ using the natural transformation $\mu : F(g) \circ_{F(A)} F(f) \Rightarrow F(g \circ_A f)$. Finally we have $(g, \psi) \circ_{(A,R)} (f, \varphi) = (g \circ_A f, \mu(\widetilde{(\varphi, \psi)}))$. Like in the categorical case the equations of these polymaps follows from the coherence law of a lax functor. Furthermore it is easy to see that these constructions are inverse to each other using the same arguments that for the categorical constructions.

4.3 Frobenius pseudomonoid and Classical Linear Logic

Like in 3.4 there are different ways to define a Frobenius pseudomonoid. The most convenient in our case will be to think of those as (the image of) a pseudofunctor out of $\mathbb{1}$.

Definition 4.6 A *Frobenius pseudomonoid* in a 2-polycategory $\mathcal{C}$ is a pseudofunctor $F : \mathbb{1} \to \mathcal{C}$.

Using the polycategorical Grothendieck construction we recover the result recently advertised by Shulman that Frobenius pseudomonoids in **MAdj** are equivalent to $*$-autonomous categories.

Theorem 4.7 (Shulman [Shu20]) *There is a correspondence between Frobenius pseudomonoid and $*$-autonomous categories.*

Proof. Using the polycategorical Grothendieck construction, pseudofunctors $\mathbb{1} \to$ **MAdj** correspond to bifibrations $p : \mathcal{E} \to \mathbb{1}$. Then using theorem 3.9 these correspond to representable $*$-polycategories. □

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A Equivalence of different notions of duality

Proposition A.1 *Let A and A^* be objects of a polycategory $\mathcal{P}$. The following are equivalent:*

(i) *there is an out-universal map $rcup_A : \cdot \to A, \underline{A^*}$*

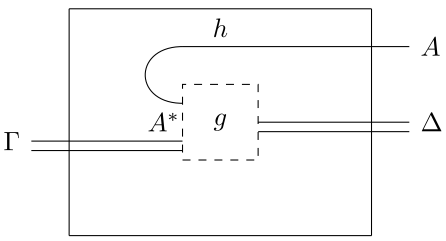
(ii) *there is an in-universal map $rcap_A : \underline{A^*}, A \to \cdot$*

(iii) *there is an out-universal map $rcup_A : \cdot \to \underline{A}, A^*$*

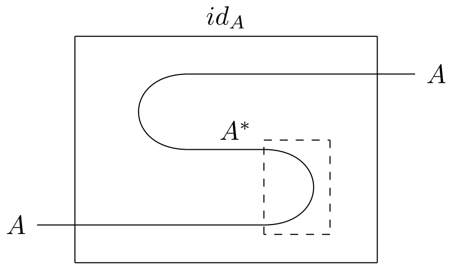
(iv) *there is an in-univeresal map $rcap_A : A^*, \underline{A} \to \cdot$*

(v) *A^* is the right dual of A*

Proof. $1 \Rightarrow 2$ Suppose that there is an out-universal map $rcup_A : \cdot \to A, \underline{A^*}$. Then it verifies the following factorisation property

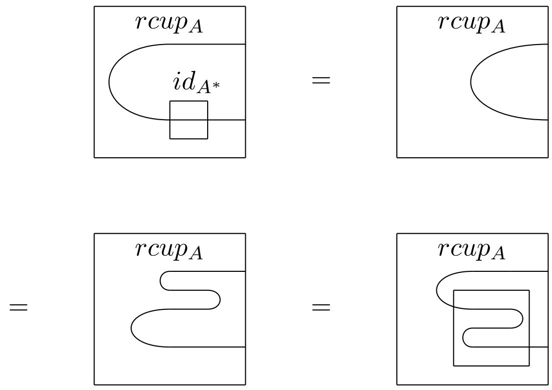


In particular by taking $h = id_A$ we get for $rcap_A := g : A^*, A \to \cdot$:



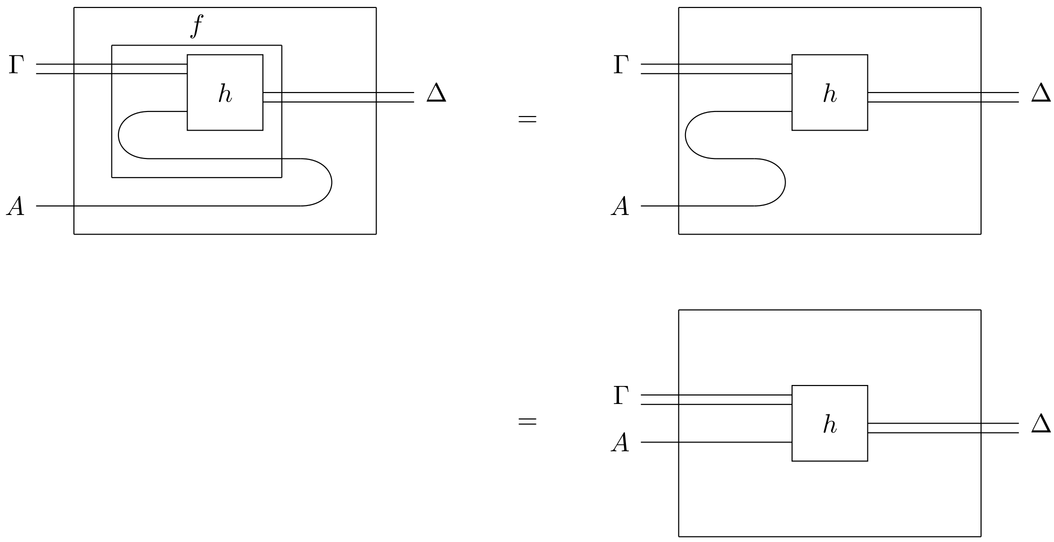
That satisfies the snake identity (where composition take place) in A^* by definition.

Furthermore we have the following equalities where we go from the first line to the second by introducing a snake identity in A^* and we get the last equality by the interchange law for composition:

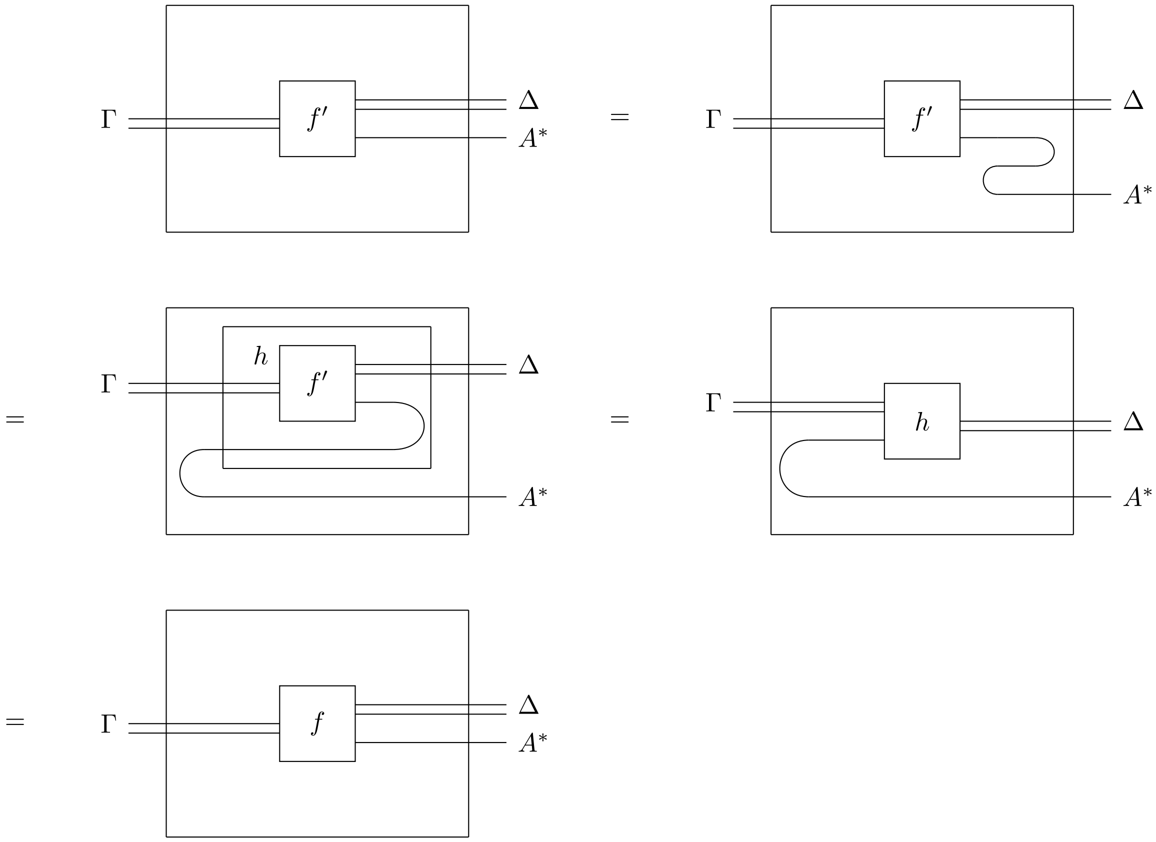


So $rcup_A$ can be factored through itself and id_{A^*} or it can be factored through itself and the snake in A. But universality of $rcup_A$ assure unicity of this factorisation. So we get the second snake identity.

Now given a polymap $h : \Gamma, A \to \Delta$ we can define $f := h \circ_A rcup_A : \Gamma \to \Delta, A^*$ such that $h = rcap_A \circ_{A^*} f$ by the interchange law for composition and the snake identity:



This is the factorisation property needed for $rcap_A$ to be universal in A^*. We still have to prove uniqueness. We get it by the second snake indentity. Indeed, suppose that there is f' such that $h = rcap_A \circ_{A^*} f'$ then:



This concludes the proof that $1 \Rightarrow 2$.

$2 \Rightarrow 1$: By a similar proof we can prove that given $rcap_A$ universal in A^* we can derived $rcup_A$ universal in A^*.

Putting these two results together we get $1 \Leftrightarrow 2$.

$3 \Leftrightarrow 4$: A similar proof will get us an equivalence between the existence of a map $rcup_A$ universal in A and a map $rcap_A$ universal in A.

$1 \Rightarrow 3$: Given $rcup_A$ universal in A^* we proved above that there exists $rcap_A$ such that both snake identities are satisfied. Let $h : \Gamma \to \Delta, A^*$ we can define $g := rcap_A \circ_{A^*} h : \Gamma, A \to \Delta$. We have $h = f \circ_A rcup_A$ by the interchange law for composition and a snake identity. This is the factorisation property needed for $rcup_A$ to be universal in A. We get unicity using the other snake identity.

$3 \Rightarrow 1$: We can easily prove the converse by a similar strategy.

$1 \Leftrightarrow 5$: We have proven that the first four propositions are equivalent. To do so we proved that each proposition induces the existence of a pair $(rcup_A, rcap_A)$ satisfying the snake identities. $\square$

B Application to Banach spaces

In this appendix we will develop enough basic theory of Banach spaces to make the statement above precise. Although the presentation using polycategories is new most of the results are standard and can be found in [Rya02].

In the following we will fix a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. **Vect** is the polycategory of $\mathbb{K}$-vector spaces and $\mathbb{K}$-polylinear maps, where a polylinear map $A_1, ..., A_m \to B_1, ..., B_n$ correspond to a linear map $A_1 \otimes ... \otimes A_m \to B_1 \otimes ... \otimes B_n$. **FVect** is its subpolycategory consisting of finite dimensional vector spaces.

For a polylinear map $f : A_1, ..., A_m \to B_1, ..., B_n$ and elements $a_i \in A_i$ and $\varphi_j \in B_j$ we will write the scalar $(\varphi_1, ..., \varphi_n) f(a_1, ..., a_m) := (\varphi_1 \otimes ... \otimes \varphi_m)(f(a_1 \otimes ... \otimes a_m))$.

Definition B.1 For a vector space V a *pseudonorm* or *seminorm* on V is a function $\| - \| : V \to \mathbb{K}$ such that

- $\|0\| = 0$
- $\|\lambda v\| = |\lambda| \|v\|$ for $\lambda \in \mathbb{K}$ and $v \in V$
- $\|v + w\| \leq \|v\| + \|w\|$ for $v, w \in V$

A *norm* is a pseudonorm such that $\|v\| = 0 \Rightarrow v = 0$.
A vector space V is *complete* for a norm $\|-\|$ if for any sequences $(v_i) \in \mathbb{K}^{\mathbb{N}}$, $\sum\limits_{i=1}^{\infty} \|v_i\| = 0$ implies that $\sum\limits_{i=1}^{\infty} v_i$ converges.
A *Banach space* is a a complete normed vector space.

Any norm on a vector space induces a distance $d(u, v) := \|u - v\|$ that defines a topology on V. So we can talk of continuous linear maps. For a linear map to be continuous it sufficies that it is continuous in 0. Furthermore it is equivalent to ask for the linear map to be bounded, $\|f\| := \sup\limits_{\|x\| \leq 1} \|f(x)\| < \infty$. We could also ask for a scalar $K \in \mathbb{K}$ such that for any $x \in V$ we have $\|f(x)\| \leq K\|x\|$. More generally we will define a notion of bounded polylinear maps.

Definition B.2 A polylinear map $f : A_1, ..., A_m \to B_1, ..., B_n$ between normed vector spaces $(A_i, \|-\|_{A_i})$ and $(B_j, \|-\|_{B_j})$ is *bounded* if there is K such that for any $a_i \in A_i$ and any $\varphi_j \in B_j^*$ we have $|(\varphi_1, ..., \varphi_n)f(a_1, ..., a_m)| \leq K \prod\limits_{i,j} \|a_i\|_{A_i} \|\varphi_j\|_{B_j^*}$.

Proposition B.3 *A unary polymap $f : A \to B$ is bounded if it is bounded as a linear map.*

Proof. Suppose that there is K such that for any $x \in A$ and $\varphi \in B^*$ we have $|\varphi(f(x))| \leq K\|x\|_A\|\varphi\|_{B^*}$. Using the Hahn-Banach theorem given $f(x) \in B$ we can find $\varphi \in B^*$ such that $\varphi(f(x)) = \|f(x)\|_B$ and $\|\varphi\|_{B^*} \leq 1$. So for this choice of φ we get $\|f(x)\|_B \leq K\|x\|_A$ which means that f is bounded.

Conversely suppose that there is K such that for any $x \in A$ $\|f(x)\|_B \leq K\|x\|_A$. Then for any $\varphi \in B^*$ we have $\|\varphi\|_{B^*}\|f(x)\|_B \leq K\|x\|_A\|\varphi\|_{B^*}$. But then by definition, $\|\varphi\|_{B^*} \geq \frac{|\varphi(f(x))|}{\|f(x)\|_B}$ for $x \neq 0$. Finally we get $|\varphi(f(x))| \leq K\|x\|_A\|\varphi\|_{B^*}$ for any $x \neq 0$. For $x = 0$ both sides are null. $\square$

Like in the linear case we can take the smaller such K to be the norm of f. Then a polylinear map is said to be contractive if its norms is smaller than 1.

Definition B.4 A polylinear map $f : A_1, ..., A_m \to B_1, ..., B_n$ between normed vector spaces $(A_i, \|-\|_{A_i})$ and $(B_j, \|-\|_{B_j})$ is *contractive* if for any $a_i \in A_i$ and any $\varphi_j \in B_j^*$ the following inequation holds:

$$|(\varphi_1, ..., \varphi_n)f(a_1 \otimes ... \otimes a_m)| \leq \prod_{i,j} \|a_i\|_{A_i} \|\varphi_j\|_{B_j^*}$$

Bounded polylinear maps compose, so do contractive ones.

Definition B.5 We define the following polycategories:

- **Ban** has Banach spaces for objects and bounded polylinear map as polymaps
- **FBan** is the subpolycategory of **Ban** consisting of finite dimensional Banach spaces
- $\mathbf{Ban_1}$ has Banach spaces for objects and contractive polylinear map as polymaps
- $\mathbf{FBan_1}$ consists of finite dimensional Banach spaces and contractive polymaps

For objects in any of those polycategories to be isomorphic they need to be isomorphic as vector spaces. Two Banach spaces $(A, \|-\|)$ and $(A, \|-\|')$ are isomorphic in **Ban** and $\mathbf{FBan_1}$ if there are scalars K, K' such that for any $a \in A$ we have $K\|a\| \leq \|a\|' \leq K'\|a\|$. Such norms are called equivalent. They are equivalent in $\mathbf{Ban_1}$ and $\mathbf{FBan_1}$ if the norms are equal. In particular **FBan** is not an interesting polycategory.

Proposition B.6 **FBan** *is equivalent to* **FVect**.

Proof. Because all norms on a finite dimensional vector space are equivalent. $\square$

There is are forgetful functors from these polycategories to **Vect** or **FVect** respectively. With the exception of **FBan** $\to$ **FVect** these are not bifibration. In the following we will focus on $\mathbf{Ban_1}$ and $\mathbf{FBan_1}$. We want to understand what are cartesian liftings in these polycategories.

Definition B.7 Given a polylinear map $f : A_1, ..., A_m \to B_1, ..., B_n$, and norms $\|-\|_{A_i}, \|-\|_{B_j}$ on A_i, B_j for

A_i expect A_k and all B_j, we define a function $\|-\|_f^k : A_k \to \mathbb{K}$ as follow:

$$\|x\|_{A_k} := \sup_{a_i,\varphi_j \neq 0} \frac{|(\varphi_1, ..., \varphi_n) f(a_1, ..., x, ..., a_m)|}{\prod\limits_{i \neq k, j} \|a_i\|_{A_i} \|\varphi_j\|_{B_j^*}}$$

Proposition B.8 *$\|-\|_f^k$ is a pseudonorm on A_k.*

Proof. First we want to prove that $\|0\|_f^k = 0$. For that we will use linearity of the maps involved.

$$\begin{aligned}
\|0\|_f^k &= \sup_{a_i,\varphi_j \neq 0} \frac{|(\varphi_1, ..., \varphi_n) f(a_1, ..., 0, ..., a_m)|}{\prod\limits_{i \neq k, j} \|a_i\|_{A_i} \|\varphi_j\|_{B_j^*}} \\
&= \sup_{a_i,\varphi_j \neq 0} \frac{|(\varphi_1 \otimes ... \otimes \varphi_n)(f(a_1 \otimes ... \otimes 0 \otimes ... \otimes a_m))|}{\prod\limits_{i \neq k, j} \|a_i\|_{A_i} \|\varphi_j\|_{B_j^*}} \\
&= \sup_{a_i,\varphi_j \neq 0} \frac{|(\varphi_1 \otimes ... \otimes \varphi_n)(f(0))|}{\prod\limits_{i \neq k, j} \|a_i\|_{A_i} \|\varphi_j\|_{B_j^*}} \\
&= \sup_{a_i,\varphi_j \neq 0} \frac{|(\varphi_1 \otimes ... \otimes \varphi_n)(0)|}{\prod\limits_{i \neq k, j} \|a_i\|_{A_i} \|\varphi_j\|_{B_j^*}} \\
&= \sup_{a_i,\varphi_j \neq 0} \frac{|0|}{\prod\limits_{i \neq k, j} \|a_i\|_{A_i} \|\varphi_j\|_{B_j^*}} \\
&= \sup_{a_i,\varphi_j \neq 0} 0 \\
&= 0
\end{aligned}$$

Similarly using linearity of the maps and the fact that $|-|$ is a norm we get that $\|\lambda x\|_f^k = |\lambda| \|x\|_f^k$ and $\|x + y\|_f^k \leq \|x\|_f^k + \|y\|_f^k$. □

We want to characterise these polymaps for which this pseudonorm is a norm.

Definition B.9 A polylinear map $f : A_1, ..., A_m \to B_1, ..., B_n$ is called *injective in A_k* or *A_k-injective* if the following condition is true $(\forall a_i,\ f(a_1, ..., x, ..., a_m) = 0) \Rightarrow x = 0$

Definition B.10 For a polylinear map $f : A_1, ..., A_m \to B_1, ..., B_n$ its A_k-kernel is the set

$$Ker_{A_k}(f) := \{x \in A_k \mid f(a_1, ..., x, ..., a_m) = 0\ \forall a_i\}$$

.

The A_k-kernel of a polylinear map forms a vector space. The polylinear map is A_k-injective if its A_k-kernel is trivial. Furthermore we have that $Ker_{A_k}(f) = \{x \in A_k \mid (\varphi_1, ..., \varphi_n) f(a_1, ..., x, ..., a_m) = 0\ \forall a_i\}$ by linearity of $\bigotimes\limits_i \varphi_i$.

Remark B.11 For a unary map $f : A \to B$ to be A-injective means that it is injective as a linear map.

Proposition B.12 *For a polylinear map $f : A_1, ..., A_m \to B_1, ..., B_n$ and norms $\|-\|_{A_i}, \|-\|_{B_j}$ the pseudonorm $\|-\|_f^k$ is a norm iff f is A_k-injective.*

Furthermore if A_k is infinite dimensional we note $\widehat{A_k}$ its completion with respect to the norm $\|-\|_f^k$.

Proposition B.13 *Given a A_k-injective polylinear map $f : A_1, ..., A_m \to B_1, ..., B_n$ and norms $\|-\|_{A_i}, \|-\|_{B_j}$, the norm $\|-\|_f^k$ makes f contractive.*

Proof. By definition we have $\|a_k\|^k_f \geq \frac{|(\varphi_1,...,\varphi_n)f(a_1,...,x,...,a_m)|}{\prod_{i\neq k,j}\|a_i\|_{A_i}\|\varphi_j\|_{B^*_j}}$ for any a_i, φ_j. So $|(\varphi_1,...,\varphi_n)f(a_1,...,x,...,a_m)| \leq \prod_{i\neq k,j}\|a_i\|_{A_i}\|a_k\|^k_f\|\varphi\|_{A_j}$. But this is exactly what it means for f to be contractive. □

This norm defines a pullback in $\mathbf{Ban_1}, \mathbf{FBan_1}$.

Proposition B.14 *Given a B-injective polylinear map $g : \Gamma'_1, A, \Gamma'_2 \to \Delta'$ with $\Gamma'_i = A'_{i,1}, ..., A'_{i,m'_i}$ and $\Delta' = B'_1, ..., B'_n$ we fix families of norm $\|-\|_{\Gamma'_i} = (\|-\|_{A'_{i,j}})$ and $\|-\|_\Delta$ makes there respective vector spaces complete. Then* $\mathbf{pull}[g]((\Gamma'_1, \|-\|_{\Gamma'_1}) \smile (\Gamma'_2, \|-\|_{\Gamma'_2}); (\Delta, \|-\|_\Delta)) = (\widehat{A}, \|-\|_g)$.

Proof.

Take a polylinear map $f : \Gamma \to \Delta_1, A, \Delta_2$ and families of norms $\|-\|_\Gamma, \|-\|_{\Delta_1}, \|-\|_{\Delta_2}$ that makes the corresponding vector spaces complete. Furthermore suppose that $g \circ_A f$ is contractive. We want to prove that f is contractive when A is equipped with the norm $\|-\|_g$.

Before starting the proof we should fix some notations. We will note $\overrightarrow{\varphi}_1$ a tuple of elements of Δ^*_1. Similarly we will have $\overrightarrow{\varphi}', \overrightarrow{\varphi}_2$. Finally we will have $\overrightarrow{a}'_1, \overrightarrow{a}, \overrightarrow{a}'_2$ for tuples of the Γs. We will note $\|\overrightarrow{a}\|_\Gamma := \prod\|a_i\|_{A_i}$ and similarly for other tuples. Often we will omit the subscripts on the norms when they can be infered from the context. Also notice that

$$(\overrightarrow{\varphi}_1, \overrightarrow{\varphi}', \overrightarrow{\varphi}_2)(g \circ_A f)(\overrightarrow{a}'_1, \overrightarrow{a}, \overrightarrow{a}'_2) = (\overrightarrow{\varphi}')g(\overrightarrow{a}'_1, (\overrightarrow{\varphi}_1, id_A, \overrightarrow{\varphi}_2)f(\overrightarrow{a}), \overrightarrow{a}'_2) \tag{B.1}$$

where $(\overrightarrow{\varphi}_1, id_A, \overrightarrow{\varphi}_2)f(\overrightarrow{a}) := (\overrightarrow{\varphi}_1 \otimes id_A \otimes \overrightarrow{\varphi}_2) \circ f$ is a multilinear function into A.

To prove that f is contractive we need to prove that for any $\overrightarrow{\varphi}_1, \overrightarrow{\varphi}_2, \overrightarrow{a}$ and any $\alpha \in A^*$

$$|(\overrightarrow{\varphi}_1, \alpha, \overrightarrow{\varphi}_2)f(\overrightarrow{a})| \leq \|\overrightarrow{\varphi}_1\|\|\overrightarrow{\varphi}_2\|\|\overrightarrow{a}\|\|\alpha\|'_g$$

where $\|-\|'_g$ is the norm dual to $\|-\|_g$. Without loss of generality we can take the tuples to not contain 0. By expanding the definition we need to prove

$$\frac{|(\overrightarrow{\varphi}_1, \alpha, \overrightarrow{\varphi}_2)f(\overrightarrow{a})|}{\|\overrightarrow{\varphi}_1\|\|\overrightarrow{\varphi}_2\|\overrightarrow{a}\|} \leq \sup_{x\neq 0}\frac{|\alpha(x)|}{\|x\|_g}$$

We only need to find one x for which this inequality holds. Take $x = (\overrightarrow{\varphi}_1, id_A, \overrightarrow{\varphi}_2)f(\overrightarrow{a})$. We have that $\alpha(x) = (\overrightarrow{\varphi}_1, \alpha, \overrightarrow{\varphi}_2)f(\overrightarrow{a})$. So we just need to prove $\|(\overrightarrow{\varphi}_1, id_A, \overrightarrow{\varphi}_2)f(\overrightarrow{a})\|_g \leq \|\overrightarrow{\varphi}_1\|\|\overrightarrow{\varphi}_2\|\|\overrightarrow{a}\|$. By expanding the definition of the norm we want $\sup\frac{|(\overrightarrow{\varphi}')g(\overrightarrow{a}'_1,(\overrightarrow{\varphi}_1,id_A,\overrightarrow{\varphi}_2)f(\overrightarrow{a}),\overrightarrow{a}'_2)|}{\|\overrightarrow{\varphi}'\|\|\overrightarrow{a}'_1\|\|\overrightarrow{a}'_2\|} \leq \|\overrightarrow{\varphi}_1\|\|\overrightarrow{\varphi}_2\|\|\overrightarrow{a}\|$. Which means that we need to prove that for any $\overrightarrow{\varphi}_1, \overrightarrow{\varphi}', \overrightarrow{\varphi}_2, \overrightarrow{a}'_1, \overrightarrow{a}, \overrightarrow{a}'_2$ we have

$$|(\overrightarrow{\varphi}')g(\overrightarrow{a}'_1, (\overrightarrow{\varphi}_1, id_A, \overrightarrow{\varphi}_2)f(\overrightarrow{a}), \overrightarrow{a}'_2)| \leq \|\overrightarrow{\varphi}_1\|\|\overrightarrow{\varphi}'\|\|\overrightarrow{\varphi}_2\|\|\overrightarrow{a}'_1\|\|\overrightarrow{a}\|\|\overrightarrow{a}'_2\|$$

But then using B.1,

$$|(\overrightarrow{\varphi}_1, \overrightarrow{\varphi}', \overrightarrow{\varphi}_2)(g \circ_A f)(\overrightarrow{a}'_1, \overrightarrow{a}, \overrightarrow{a}'_2)| \leq \|\overrightarrow{\varphi}_1\|\|\overrightarrow{\varphi}'\|\|\overrightarrow{\varphi}_2\|\|\overrightarrow{a}'_1\|\|\overrightarrow{a}\|\|\overrightarrow{a}'_2\|$$

This is true because $g \circ_A f$ is contractive. □

This means that we can take the in-cartesian lifting of any polymap that is injective in the input considered. This injectivity is only needed for $\|-\|_f$ to be a norm.

Corollary B.15 *There are polycategories* $\mathbf{Ban_1^{ps}}, \mathbf{FBan_1^{ps}}$ *of complete seminormed vector spaces and contractive polylinear maps that come with forgetful functors that are pull-fibred.*

Now we want to determine which polylinear maps have out-cartesian liftings.

Definition B.16 Given a polylinear map $f : \Gamma \to \Delta_1, A, \Delta_2$ and families of norms $\|-\|_\Gamma, \|-\|_{\Delta_1}, \|-\|_{\Delta_2}$, we define a function $\|-\|^f : B_k \to \bar{\mathbb{K}}$ where $\bar{\mathbb{K}}$ is the completion of $\mathbb{K}$, i.e. we add a point at the infinity.

$$\|y\|^f := \inf_{y=\sum_i(\overrightarrow{\varphi}_{1,i}, id_A, \overrightarrow{\varphi}_{2,i})f(\overrightarrow{a}_i)} \sum_i \|\overrightarrow{\varphi}_{1,i}\|\|\overrightarrow{\varphi}_{2,i}\|\|\overrightarrow{a}_i\|$$

where the sum is done on all the possible decompositions of $y = \sum_i (\overrightarrow{\varphi}_{1,i}, id_A, \overrightarrow{\varphi}_{2,i}) f(\overrightarrow{a}_i)$.

Proposition B.17 *$\| - \|^f$ is an extended norm, i.e. a norm with value in $\bar{\mathbb{K}}$.*

Definition B.18 A polylinear map $f : \Gamma \to \Delta_1, A, \Delta_2$ is called *A-surjective* if for any $y \in A$ there exist $\overrightarrow{\varphi}_{1,i}, \overrightarrow{\varphi}_{2,i}, \overrightarrow{a}_i$ such that $y = \sum_i (\overrightarrow{\varphi}_{1,i}, id_A, \overrightarrow{\varphi}_{2,i}) f(\overrightarrow{a}_i)$.
The A-image of f is the set $Im_A(f) := \{\sum_i (\overrightarrow{\varphi}_{1,i}, id_A, \overrightarrow{\varphi}_{2,i}) f(\overrightarrow{a}_i)\}$.

Proposition B.19 *$Im_A(f)$ forms a vector space. f is A-surjective iff $Im_A(f) = A$.*

Remark B.20 A linear map is B-surjective iff it is surjective. Indeed if for $y \in B$ there are x_i such that $y = \sum_i f(x_i)$ then by linearity $y = f(\sum_i x_i)$.

Proposition B.21 *Given a polylinear map $f : \Gamma \to \Delta_1, A, \Delta_2$ and families of norms $\| - \|_\Gamma, \| - \|_{\Delta_1}, \| - \|_{\Delta_2}$, $\| - \|^f$ is a norm iff f is A-surjective.*

Proof. The value of $\|y\|^f$ is non-finite iff we are taking the infimum over the empty set which means that y is not in the A-image of f. $\square$

Proposition B.22 *For $f : \Gamma \to \Delta_1, A, \Delta_2$ a A-surjective polylinear map and the families of norms as usual, $\| - \|^f$ makes f contractive.*

Proof. We want to prove that for any $\overrightarrow{\varphi}_1, \overrightarrow{\varphi}_2, \overrightarrow{a}$ and any $\alpha \in A^*$ $|(\overrightarrow{\varphi}_1, \alpha, \overrightarrow{\varphi}_2) f(\overrightarrow{a})| \leq \|\overrightarrow{\varphi}_1\| \|\overrightarrow{\varphi}_2\| \|\overrightarrow{a}\| \|\alpha\|'$ where $\| - \|'$ is the norm dual to $\| - \|^f$. So we want $|(\overrightarrow{\varphi}_1, \alpha, \overrightarrow{\varphi}_2) f(\overrightarrow{a})| \leq \|\overrightarrow{\varphi}_1\| \|\overrightarrow{\varphi}_2\| \|\overrightarrow{a}\| \sup_y \frac{|\alpha(y)|}{\|y\|^f}$. We just need to find a value of y that satisfy the following inequality: $|(\overrightarrow{\varphi}_1, \alpha, \overrightarrow{\varphi}_2) f(\overrightarrow{a})| \|y\|^f \leq \|\overrightarrow{\varphi}_1\| \|\overrightarrow{\varphi}_2\| \|\overrightarrow{a}\| |\alpha(y)|$

Take $y = (\overrightarrow{\varphi}_1, id_A, \overrightarrow{\varphi}_2) f(\overrightarrow{a})$, by definition $\|(\overrightarrow{\varphi}_1, id_A, \overrightarrow{\varphi}_2) f(\overrightarrow{a})\|^f \leq \|\overrightarrow{\varphi}_1\| \|\overrightarrow{\varphi}_2\| \|\overrightarrow{a}\|$. Furthermore $\alpha(y) = (\overrightarrow{\varphi}_1, \alpha, \overrightarrow{\varphi}_2) f(\overrightarrow{a})$. So by multiplying on both side by $\alpha(y)$ we get the inequality that we wanted. $\square$

This norms define a pushforward on $\mathbf{Ban_1}, \mathbf{FBan_1}$.

Proposition B.23 *For $f : \Gamma \to \Delta_1, A, \Delta_2$ a A-surjective polylinear map and the families of norms as usual, $\mathbf{push}\langle f\rangle(\Gamma; \Delta_1 \smile \Delta_2) = (\widehat{A}, \| - \|^f)$.*

Proof. Suppose that $g : \Gamma'_1, A, \Gamma'_2 \to \Delta'$ is such that $g \circ_A f$ is contractive. Consider any $\overrightarrow{\varphi}'$, $\overrightarrow{a}'_1$, $\overrightarrow{a}'_2$ and $y \in A$. Since f is A-surjective we can find a decomposition of $y = \sum_i (\overrightarrow{\varphi}_{1,i}, id_A, \overrightarrow{\varphi}_{2,i}) f(\overrightarrow{a}_i)$ that reaches the infimum, i.e. such that $\|y\|^f = \sum \|\overrightarrow{\varphi}_{1,i}\| \|\overrightarrow{\varphi}_{2,i}\| \|\overrightarrow{a}_i\|$. Then we have

$$\begin{aligned}
|(\overrightarrow{\varphi}') g(\overrightarrow{a}'_1, y, \overrightarrow{a}'_2)| &= |(\overrightarrow{\varphi}') g(\overrightarrow{a}'_1, \sum_i (\overrightarrow{\varphi}_{1,i}, id_A, \overrightarrow{\varphi}_{2,i}) f(\overrightarrow{a}_i), \overrightarrow{a}'_2)| \\
&= |\sum_i (\overrightarrow{\varphi}') g(\overrightarrow{a}'_1, (\overrightarrow{\varphi}_{1,i}, id_A, \overrightarrow{\varphi}_{2,i}) f(\overrightarrow{a}_i), \overrightarrow{a}'_2)| \\
&= |\sum_i (\overrightarrow{\varphi}_{1,i}, \overrightarrow{\varphi}', \overrightarrow{\varphi}_{2,i}) (g \circ_A f)(\overrightarrow{a}'_1, \overrightarrow{a}_i, \overrightarrow{a}'_2)| \\
&\leq \sum_i |(\overrightarrow{\varphi}_{1,i}, \overrightarrow{\varphi}', \overrightarrow{\varphi}_{2,i}) (g \circ_A f)(\overrightarrow{a}'_1, \overrightarrow{a}_i, \overrightarrow{a}'_2)| \\
&\leq \sum_i \|\overrightarrow{\varphi}_{1,i}\| \|\overrightarrow{\varphi}'\| \|\overrightarrow{\varphi}_{2,i}\| \|\overrightarrow{a}'_1\| \|\overrightarrow{a}_i\| \|\overrightarrow{a}'_2\| \\
&= \|\overrightarrow{\varphi}'\| \|\overrightarrow{a}'_1\| \|\overrightarrow{a}'_2\| \sum_i \|\overrightarrow{\varphi}_{1,i}\| \|\overrightarrow{\varphi}_{2,i}\| \|\overrightarrow{a}_i\| \\
&= \|\overrightarrow{\varphi}'\| \|\overrightarrow{a}'_1\| \|\overrightarrow{a}'_2\| \|y\|^f
\end{aligned}$$

where we used in order, (1) definition of y, (2) linearity, (3) definition of the composition, (4) triangle inequality for the norm $| - |$, (5) contractivity of $g \circ_A f$, (7) that the decomposition consider reach the infimum. This inequality is what is needed to prove that g is contractive. $\square$

This means that we can take the out-cartesian lifting of any polymap that is surjective in the considerd output.

Corollary B.24 *There are polycategories $\mathbf{Ban_1^{ex}}, \mathbf{FBan_1^{ex}}, \mathbf{Ban_1^{ex,ps}}, \mathbf{FBan_1^{ex,ps}}$ of extended normed vector spaces and extended seminormed vector spaces that come with forgetful functors that are push-fibred and bifibred.*

Even when considering $\mathbf{Ban_1}, \mathbf{FBan_1}$ there are enough cartesian polymaps to lift the representability and $*$-representability of $\mathbf{Vect}$ and $\mathbf{FVect}$ to them.

Proposition B.25 *In $\mathbf{Vect}, \mathbf{FVect}$, the universal polylinear maps $A, B \to A \otimes B$ and $A \otimes B \to A, B$ are $A \otimes B$-surjective and $A \otimes B$-injective. In $\mathbf{FVect}$, the universal polylinear maps $A^*, A \to$ is A^*-injective.*

Proof. By definition of the tensor product any $u \in A \otimes B$ is a linear combination of elements from A and B, $u = \sum_i a_i \otimes b_i$. $A \otimes B$-injectivity is trivial. Finally given $\varphi \in A^*$ if for all $a \in A$ $\varphi(a) = 0$ then $\varphi = 0$. □

Corollary B.26 *The category of finite dimensional Banach spaces and contractive linear maps is $*$-autonomous.*

The norm that we get on the dual is the usual dual norms. For the tensor and par we get back well-known norms.

Definition B.27 Let $(A, \| - \|_A)$ and $(B, \| - \|_B)$ be two Banach spaces.
The *projective crossnorm* is the norm defined as follow $\|u\|_{A\otimes B} := \inf_{u = \sum_i a_i \otimes b_i} \|a_i\|_A \|b_i\|_B$.

The *injective crossnorm* is the norm defined by $\|u\|_{A \parr B} := \sup_{\|\varphi\|_{A^*}, \|\psi\|_{B^*} \leq 1} |(\varphi \otimes \psi)(u)|$

These are the norms that we get by taking the pushforward along $A, B \to A \otimes B$ and the pullback along $A \otimes B \to A, B$. So we deduce immediately that they are the norms that make (the completion of) $A \otimes B$ a tensor/par in $\mathbf{FBan_1}$. These norms are known to be extremal into the set of well-behaved norms that we can put on the tensor product.

Definition B.28 Let $(A, \| - \|_A)$ and $(B, \| - \|_B)$ be two Banach spaces. A norm $\| - \|$ on $A \otimes B$ is said to be a *crossnorm* if for any $a, b \in A \times B$ we have $\|a \otimes b\| \leq \|a\|_A \|b\|_B$ and for any $\varphi, \psi \in A^* \otimes B^*$ we have $\|\varphi \otimes \psi\|' \leq \|\varphi\|_{A^*} \|\psi\|_{B^*}$ with $\| - \|'$ the norm dual to $\| - \|$.

Remark B.29 It would have been equivalent to ask for equalities in the definition. A proof can by found in [Rya02].

Proposition B.30 *A norm on $A \otimes B$ is a crossnorm iff it makes the polylinear maps $A, B \to A \otimes B$ and $A \otimes B \to A, B$ contractive.*

Proof. By definition. □

The injective and projective crossnorms are indeed crossnorms. The following property of the injective and projective crossnorm is known. It is this property that first make us consider the injective crossnorm as a potential candidate for interpreting the par. Getting a conceptual comprehension of why this intuition was true is what motivated the theory developed in this paper.

Proposition B.31 *Let $\| - \|$ be a crossnorm then for any $u \in A \otimes B$ we have $\|u\|_{A \parr B} \leq \|u\| \leq \|_{A \otimes B}$*

Proof. We have that $A, B \to A \otimes B$ factors through itself followed by $id_{A\otimes B}$. Since $\| - \|$ is a crossnorm $(A, \| - \|_A), (B, \| - \|_B) \to (A \otimes B, \| - \|)$ is contractive. But then since $(A \otimes B, \| - \|_{A\otimes B})$ is a pullback we can use its universal property to factor $(A, \| - \|_A), (B, \| - \|_B) \to (A \otimes B, \| - \|)$ throught $(A, \| - \|_A), (B, \| - \|_B) \to (A \otimes B, \| - \|_{A\otimes B})$. This means that the identity map lift to a contractive map $(A \otimes B, \| - \|_{A\otimes B}) \to (A \otimes B, \| - \|)$. In other words for any $u \in A \otimes B$, $\|u\| \leq \|u\|_{A\otimes B}$. Similarly we can use the fact that $A \parr B$ is a pushforward to get the other inequality. □

Remark B.32 More than just a model of classical MLL (without negation), $\mathbf{FBan_1}$ ($\mathbf{Ban_1}$) is a model of classical MALL (without negation). The additive connectives are given by the vector space $A \oplus B$ with the norms $\|(a, b)\|_1 := \sum_i \|a\|_A + \|b\|_B$ and $\|(a, b)\|_\infty := \max(\|a\|_A, \|b\|_B)$. These norms are extremal amongst the p-norms.

C Categorical Grothendieck correspondences

We recall a range of categorical Grothendieck correspondences that goes as follow:

- Functor $\mathcal{E} \to \mathcal{B} \longleftrightarrow$ Lax normal functor $\mathcal{B} \to \mathbf{Dist}$
- Fibration $\mathcal{E} \to \mathcal{B} \longleftrightarrow$ pseudofunctor $\mathcal{B} \to \mathbf{Cat}$
- Opfibration $\mathcal{E} \to \mathcal{B} \longleftrightarrow$ pseudofunctor $\mathcal{B} \to \mathbf{Cat}^{\mathrm{op}}$
- Bifibration $\mathcal{E} \to \mathcal{B} \longleftrightarrow$ pseudofunctor $\mathcal{B} \to \mathbf{Adj}$

Where **Dist** is the bicategory of (small) categories and distributors/profunctors and **Adj** is the 2-category of (small) categories and adjunctions. Notice that **Cat**, $\mathbf{Cat}^{\mathrm{op}}$ and **Adj** are all subbicategories of **Dist** corresponding to distributors $A \nrightarrow B$ that are representable in A, in B, and in both, respectively.

Let us look at the details of this correspondence. We will first look at the left-to-right direction. Suppose that you have a functor $p : \mathcal{E} \to \mathcal{B}$ we want to get a lax normal functor $\partial p : \mathcal{B} \to \mathbf{Dist}$. We do that by considering the fibers over $\mathcal{B}$.

$$\begin{aligned}
\partial p :\ & \mathcal{B} \to \mathbf{Dist} \\
& B \mapsto p^{-1}(B) := \{E \in \mathcal{E} \mid p(E) = B\} \\
& f : B \to B' \mapsto \begin{array}{l} \partial p(f) : p^{-1}(B)^{\mathrm{op}} \times p^{-1}(B') \to \mathbf{Set} \\ \qquad (E, E') \mapsto \{\varphi : E \to E' \mid p(\varphi) = f\} \\ \qquad (\psi, \xi) \mapsto \xi \circ - \circ \psi \end{array}
\end{aligned}$$

We can see that this defines a lax normal functor.

$$(\partial p)(id_B)(E, E') = \{\varphi : E \to E' \mid p(\varphi) = id_B\} = Hom_{p^{-1}(B)}(E, E')$$
$$(\partial p)(id_B)(\psi, \xi) = \xi \circ - \circ \varphi = Hom_{p^{-1}(B)}(\varphi, \xi)$$

So $\partial p(id_B) = Hom_{p^{-1}(B)}$ which is the identity profunctor. We also want to find a natural transformation $(\partial p)(f') \circ (\partial p)(f) \Rightarrow (\partial p)(f' \circ f)$. By definition we have

$$\begin{aligned}
((\partial p)(f') \circ (\partial p)(f))(E, E'') &= \int^{E'} ((\partial p)(f') \times (\partial p)(f))(E, E'') \\
&= \int^{E'} \{\varphi' : E' \to E'' \mid p(\varphi') = f'\} \times \{\varphi : E \to E' \mid p(\varphi) = f\}
\end{aligned}$$

Then for any E' we get a function

$$\int^{E'} \{\varphi' : E' \to E'' \mid p(\varphi') = f'\} \times \{\varphi : E \to E' \mid p(\varphi) = f\} \to \{\varphi' : E' \to E'' \mid p(\varphi') = f'\} \times \{\varphi : E \to E' \mid p(\varphi) = f\}$$

And finally a function

$$\{\varphi' : E' \to E'' \mid p(\varphi') = f'\} \times \{\varphi : E \to E' \mid p(\varphi) = f\} \to \{\varphi'' : E \to E'' \mid p(\varphi'') = f' \circ f\}$$

by taking $\varphi'' = \varphi' \circ \varphi$ and using functoriality of p, $p(\varphi' \circ \varphi) = p(\varphi') \circ p(\varphi) = f' \circ f$.

The morphism that we get does not depend on the choice of E' by the universal property of the coend. We still need to prove the naturality condition and the coherence laws for a lax functor. This is left to the reader.

Now suppose that we want for ∂p to factor through $\mathbf{Cat} \hookrightarrow \mathbf{Dist}$. This means that for any $f : B \to B'$ we want a functor $p_*(f) : p^{-1}(B') \to p^{-1}(B)$ such that $\partial p(f) = Hom_{p^{-1}(B)}(-, p_*(-))$. If we unravel the definition we get a correspondence: $\dfrac{E \overset{\varphi}{\underset{f}{\Rightarrow}} E'}{E \xRightarrow[id_B]{\tilde{\varphi}} p_*(f)(E')}$ such that for any polymaps $\psi : E_2 \to E_1$, $\xi : E_1' \to E_2'$ and $\varphi : E_1 \to E_1'$ we have $\xi \circ \varphi \circ \psi = p_*(f)(\xi) \circ \tilde{\varphi} \circ \psi$. In particular for $\psi = id_E$ and $\xi = id_{E'}$ then $\varphi = p_*(f)(id_{E'}) \circ \tilde{\varphi}$. This is the factorisation property of a weak cartesian morphism. A functor such that all the weak cartesian morphisms exists is a prefibration. It is a fibration if weak cartesian morphisms compose. This enforces for ∂p to be a pseudofunctor and not just lax.

Similarly asking for factorisation through $\mathbf{Cat}^{\mathrm{op}} \hookrightarrow \mathbf{Dist}$ corresponds to representability of $\partial p(f)$ on its second variable while factorisation through $\mathbf{Adj} \hookrightarrow \mathbf{Dist}$ is representability in both variables. Combined with pseudofunctoriality this enforced ∂p to be an opfibration and a bifibration respectively.

Conversely, let $F : \mathcal{B} \to \mathbf{Dist}$ be a lax normal functor. We want to get a category $\int F$ and a functor $\int F \to \mathcal{B}$. This is often called the Grothendieck construction, especially when restricted to pseudofunctors $F : \mathcal{B} \to \mathbf{Cat}$. This can be obtained by considering the functor $\mathbf{Dist}_* \to \mathbf{Dist}$ from pointed distributors that forget about the point and consider the pullback $\begin{tikzcd} \int F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Dist}_* \arrow[d] \\ \mathcal{B} \arrow[r] & \mathbf{Dist} \end{tikzcd}$ but for this to be well-defined there are some size considerations. Furthermore this will only give a lax functor $\int F \to \mathcal{B}$. That $\int F$ is a category has to be prove separately. We want to advertise another way to construct $\int F$ that is closer to the philosophy of this paper, using some notion of opcartesian 2-morphisms.

We will work in a slightly more general framework first. Let $\mathcal{B}$ be a category, $\mathcal{C}$ and $\mathcal{D}$ be bicategories, $F : \mathcal{B} \to \mathcal{C}$ be a lax functor and $p : \mathcal{D} \to \mathcal{C}$ a strict functor. Furthermore suppose that for any 2-morphism $\alpha : f \Rightarrow g$ in $\mathcal{C}$ and any morphism φ in $\mathcal{D}$ lying over f, i.e. $p(\varphi) = f$, there is a morphism $push_\alpha(\varphi)$ over g and an opcartesian 2-morphism $\varphi \to push_\alpha(\varphi)$ lying over α. We now define a bicategory $\int_p F$. Its objects are pairs of objects (B, D) from $\mathcal{B}$ and $\mathcal{D}$ such that $F(B) = p(D)$. Similarly morphisms are pair of morphisms with the same image and 2-morphisms correspond to vertical 2-morphisms in $\mathcal{D}$. Given (B, D) an object of $\int_p F$ we want to define $id_{(B,D)} = (f, \varphi)$. We can take $f = id_B$. Then we have $F(f) = F(id_B)$. We would like to take $\varphi = id_D$ but $p(id_D) = id_{p(D)} = id_{F(B)} \neq F(id_B)$. However we have a 2-morphism $\eta_B : F(id_B) \Rightarrow id_{F(B)}$ from the lax functoriality of F. By using the fibrational property of p we can get an object over $id_{F(B)}$ by lifting id_D along η_B. So we have $id_{(B,D)} = (id_B, push_{\eta_B}(id_D))$. Similarly, $(g, \psi) \circ (f, \varphi) = (g \circ f, push_{\mu_{F(f),F(g)}}(\psi \circ \varphi))$ where $\mu_{F(f),F(g)} : F(g) \circ F(f) \Rightarrow F(g \circ f)$. Then we can prove that this are unital and associative up to isomorphism by using the same fact for $\mathcal{B}$ and $\mathcal{D}$ and the lax property of F and fibrational property of p. For example to prove right unitality of identity we consider:

$$\begin{tikzcd} D \arrow[r, "id_D"] \arrow[r, bend right=60, "push_\eta(id_D)"'] \arrow[rr, bend right=80, "push_\mu(\varphi \circ push_\eta(id_D))"'] & D \arrow[r, "\varphi"] & D' \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, "id_D"] \arrow[rr, bend right=80, "\varphi"'] & D \arrow[r, "\varphi"] \arrow[d, phantom, "\Downarrow unit_D"] & D' \\ & {} & \end{tikzcd}$$

$$\begin{tikzcd} F(B) \arrow[r, "id_{F(B)}"] \arrow[r, bend right=60, "F(id_B)"'] \arrow[r, phantom, shift right=3, "\Downarrow\eta"] \arrow[rr, bend right=80, "F(f)"'] & F(B) \arrow[r, "F(f)"] \arrow[d, phantom, "\Downarrow\mu"] & F(B') \\ & {} & \end{tikzcd} = \begin{tikzcd} F(B) \arrow[r, "id_{F(B)}"] \arrow[rr, bend right=80, "F(f)"'] & F(B) \arrow[r, "F(f)"] \arrow[d, phantom, "\Downarrow unit_{F(B)}"] & F(B') \\ & {} & \end{tikzcd}$$

$unit_{F(B)}$ and $unit_D$ are 2-isomorphisms that comes from the bicategorical structure of $\mathcal{C}$ and $\mathcal{D}$. The bottom equality comes from lax functoriality of F. Now using the universal property of cocartesian 2-morphisms we can factors $unit_D$ through $push_\eta(id_D)$ and then through the second pushforward to get a vertical 2-morphism $\chi : push_\mu(\varphi \circ push_\eta(id_D)) \underset{id_{F(f)}}{\Longrightarrow} \varphi$. And we get a vertical 2-morphism χ^{-1} in the other direction by precomposing the diagram on the upper left corner with $unit_D^{-1}$. By definition we have $\chi \circ \chi^{-1} = unit_D \circ unit_D^{-1} = id_\varphi$. We get $\chi^{-1} \circ \chi = id_{push_\eta(\varphi \circ push_\mu(id_D))}$ by using the uniqueness of factorisation through a opcartesian 2-morphism. So left unitality of the identity holds up to vertical 2-isomorphism.

We get the following commutative diagram $\begin{tikzcd} \int_p F \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{B} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$ by considering the projection functors from $\int_p F$.

We get back the Grothendieck construction by taking $p : \mathbf{Dist}_* \to \mathbf{Dist}$ the forgetful functor. Let show that it has opcartesian 2-morphism. Given $P, Q : A \nrightarrow B$ and $\alpha : P \Rightarrow Q$ in $\mathbf{Dist}$, a morphism $(P, \varphi) : (A, a) \to (B, b)$ in $\mathbf{Dist}_*$ over P is given by an element $\varphi \in P(a, b)$. Then we can use $\alpha_{a,b} : P(a, b) \to Q(a, b)$ to get a morphism $(Q, \alpha_{a,b}(\varphi)) : (A, a) \to (B, b)$ over Q with a 2-morphism $\alpha : (P, \varphi) \Rightarrow (Q, \alpha_{a,b}(\varphi))$. This is an opcartesian 2-morphism because for any 2-morphism in $\mathbf{Dist}$ there is at most one 2-morphism over it for some given domain and codomain. So we get $\int F := \int_p F$ for this particular choice of p. If we expand the definition the objects of $\int F$ consists of pairs of objects (B, D) such that $B \in \mathcal{B}$, $D \in \mathbf{Dist}_*$ and $F(B) = p(D)$. Expanding a little more $D = (X, x)$ a pointed set such that $F(B) = X$. So it is just the data of a point in $F(B)$. Similarly, a morphism $(f, \varphi) : (B, x) \to (C, y)$ is a morphism $f : B \to C$ in $\mathcal{B}$ and an element $\varphi \in F(f)(x, y)$. Finally a 2-morphism is a vertical 2-morphism in $\mathbf{Dist}_*$. But the only vertical morphisms are the identities. That makes $\int F$ a category.

These constructions are inverse to each other. First let start with $F : \mathcal{B} \to \mathbf{Dist}$. Let fix an object $B \in \mathcal{B}$. The fiber over B in $\int F$ is the set of objects (B, x) with $x \in F(B)$, which is isomorphic to the set $F(X)$. The profunctor $(\partial \int F)(f)$ is just $F(f)$. Conversely, given $p : \mathcal{E} \to \mathcal{B}$, $\int \partial p$ as for objects pairs (B, x) with $B \in \mathcal{B}$ and $x \in (\partial p)(B)$. But then $(\partial p)(B) = p^{-1}(B)$. So $Ob(\int \partial p) = \{(B, E) \mid p(E) = B\} = ob(\mathcal{E})$. Similarly $Hom_{\int \partial p}((B, E), (B', E')) = \{(f, \varphi) : B \times E \to B' \times E' \mid p(\varphi) = f\} = Hom_{\mathcal{E}}(E, E')$.