

Modeling of egg production and mating probability for helminth parasites

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Abstract

In the modeling of the transmission dynamics of helminthic parasites, the study of the reproductive characteristics of these parasites is essential.

The reproductive habits of macroparasite are important for the study of the dynamics of their transmission. For populations of parasites distributed by Poisson or negative binomial models, these habits have already been studied. However, other statistical models describe these populations, such as zero-inflated models, but where reproductive characteristics were not analyzed. Using an arbitrary model for the parasite population, we model the distribution of females and males per host, and from these we model the different reproductive variables such as the mean number of fertile females, the mean egg production, the mating probability, and mean fertilized egg production. We show that these variables change due to the effects of negative density-dependence fecundity, a characteristic of helminth parasites. We present the results obtained for particular models.

Keywords: Compound random variable; Macroparasites; Mating probability; Negative binomial distribution; Probability Model; Zero-inflated Model

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61 1 Introduction

62 One of the most important factors in understanding the transmission dynam-
63 ics of soil-transmitted helminths are reproductive behaviors.

64 Most helminths that infect humans are dioecious (separate sexes) and
65 many are assumed to be polygamous (the presence of at least one male
66 guarantee the fertility of all females present), but quantitative data are not
67 available[3].

68 The production of offspring of these parasites is, in general, a function
69 of their population size, the proportion of females, and their reproductive
70 behavior and therefore developing mathematical models that allow under-
71 standing the distribution by sex (female and male) and the reproductive
72 behavior of these parasites is important.

73 In a population where the distribution of parasites per host is described
74 by a Poisson or a negative binomial statistical model, the distribution by
75 sex was studied for the case of a sex ratio 1:1 in [17] and for a variable sex
76 ratio in [16]. Also a dynamic model for the number of fertilized females is
77 presented in [15].

78 In this work we present a generalization of what was developed by these
79 previous works. To model the distribution by sex, we will assume an arbitrary
80 model for the distribution of parasites per host and variable sex ratios. First
81 we consider the case were the distribution of the total population is known,
82 and therefore female and male host burden are not independent random vari-
83 ables. Later we will consider the case were these variables are independent.

84 We then calculated different reproductive variables such as mean number
85 of fertile females, mean egg production, mating probability, and mean fertile
86 egg production.

87 2 Distribution and abundance of parasites

- 88 • discutir la distribución de los parásitos que que presenta sobre-dispersión.
89 binomial negativa como modelo estándar y mencionar los modelos in-
90 flados en cero

91 • discutir y dejar claro el sex ratio que usamos

92 The fractions of female and male parasites in a host are represented by
 93 α and β , respectively, where $\alpha + \beta = 1$. Then the ratio of males to females
 94 is given by $\beta/\alpha : 1$. Also if m is the mean burden of parasites, the mean
 95 number of female and male parasites are given by αm and βm respectively.

96 3 Parasite infection by egg ingestion

97 In this section, we consider that the infection by parasites occurs when a host
 98 ingests fertilized eggs of these parasites. This type of infection can occur when
 99 the host's contaminated hands are placed in the mouth or by consuming fruit
 100 and vegetables that have not been carefully cooked, washed, or peeled. The
 101 parasites responsible for this type of infection are the helminths, such as
 102 *Ascaris lumbricoides*, *Trichuris trichiura*, among others.

103 Since in this type of infection, the host can ingest one or more fertilized
 104 eggs. Then, the host can acquire one or more parasites, which can be male or
 105 female, depending on the sex ratio of the parasite. Therefore, when we ana-
 106 lyze the variables “the number of male parasites per host” and “the number
 107 of female parasites per host”, these variables cannot be independent.

108 In what follows, we develop the study some variables that intervene in the
 109 transmission dynamics of these parasitic infections, such as the mean number
 110 of fertile females per host, the mean number egg production per host, the
 111 mating probability, among others.

112 3.1 Distribution of parasites by sex

113 Let W be a random variable, the number of parasites per host, and F the
 114 number of female parasites per host. We propose that the distribution of
 115 female parasites per host is modeled by a compound random variable ([13])
 116 and its probability generating function (pgf) is the function $G_W \circ G_B$, where
 117 G_B is the pgf of the Bernoulli distribution given by $G_B(s) = \beta + \alpha s$. Therefore
 118 F is given by $F = \sum_{i=1}^W Y_i$ where $Y_i \sim \text{Ber}(\alpha)$, and its pgf is of the form

$$\begin{aligned} G_F(s) &= G_W(\beta + \alpha s) \\ &= \sum_{n \geq 0} \sum_{j=0}^n \Pr(W = n) \binom{n}{j} \alpha^j \beta^{n-j} s^j \end{aligned} \quad (1)$$

119 The first moments of F are

$$\mu_F = \alpha \mu_W \quad \sigma_F^2 = \alpha^2 \sigma_W^2 + \alpha \beta \mu_W \quad (2)$$

120 The coefficient of dispersion, $D = \frac{\sigma_F^2}{\mu_F}$, is given by

$$D = \alpha \frac{\sigma_W^2}{\mu_W} + \beta$$

121 where $\frac{\sigma_W^2}{\mu_W}$ is the coefficient of dispersion of W . Therefore, if W is over-
122 dispersed, so will F .

123 Similarly, if M is the number of male parasites per host, $M = W - F$ and
124 therefore its mean and variance are $\mu_M = \beta\mu_W$ and $\sigma_M^2 = \beta^2\sigma_W^2 + \alpha\beta\mu_W$,
125 respectively. By the definition F and M are dependent random variables.

126 **falta decir algo de las distribuciones por sexo**

127 We denoted by G to the pgf of the distribution of parasites per host G_W .
128 Also, we denoted by $G(s; \theta)$ to $\sum_{n \geq 0} p(n; \theta) s^n$ where p is the probability mass
129 function of the distribution of parasites per host and θ is a parameters vector.
130 For example, θ may include the mean, m , of the distribution of parasites per
131 host.

132 3.2 Fertilized female parasites

133 The parasites treated in this work present a polygamous mating system, it is
134 conventionally assumed that a male parasite can fertilize all female parasites
135 in the same host [16]. Since the term $\sum_{j=0}^{n-1} p_n \binom{n}{j} \alpha^j \beta^{n-j}$ is the probability of
136 having at least one male parasite in a burden parasite of size n , we obtained
137 the following result

138 **Proposition 3.2.1.** *The mean number of fertilized female parasites is given*
139 *by*

$$\alpha m - \alpha G'(\alpha) \tag{3}$$

Proof. The presence of at least one male parasite in the host ensures the
fertility of all females, so

$$\begin{aligned} \sum_{n \geq 1} \sum_{j=0}^{n-1} j p_n \binom{n}{j} \alpha^j \beta^{n-j} &= \sum_{n \geq 1} p_n \sum_{j=0}^{n-1} j \binom{n}{j} \alpha^j \beta^{n-j} \\ &= \sum_{n \geq 1} p_n (n\alpha - n\alpha^n) \end{aligned}$$

where the last line is obtained from the expression of the mean of $B(n, \alpha)$,

$n\alpha = \sum_{j=0}^n j \binom{n}{j} \alpha^j \beta^{n-j}$. Therefore

$$\begin{aligned} \sum_{n \geq 1} \sum_{j=0}^{n-1} j p_n \binom{n}{j} \alpha^j \beta^{n-j} &= \alpha \sum_{n \geq 1} n p_n (1 - \alpha^{n-1}) \\ &= \alpha \left[\sum_{n \geq 1} n p_n - \sum_{n \geq 1} n \alpha^{n-1} p_n \right] \\ &= \alpha m - \alpha G'(\alpha) \end{aligned}$$

140

□

141 On the other hand, if we consider the quotient between the mean number
142 of fertilized female parasites per host, $\alpha m - \alpha G'(\alpha)$, and the mean number
143 of female parasites per host, αm . We can obtain a simple expression for the
144 mating probability of a female parasite as a function of the mean parasite
145 burden m . Therefore, the mating probability of a female parasite that we
146 denoted by ϕ is given by

$$\phi(m) = 1 - \frac{G'(\alpha; m)}{m} \quad (4)$$

147

decir algo de la denso dependecia

148 3.3 Density-dependent fecundity

149 In population ecology, density-dependent processes occur when population
150 growth rates are regulated by population density.

151 In macroparasites life-cycles, density-dependent processes can influence
152 parasite fecundity, establishment and survival within the host . In the case
153 of soil-transmitted helminths, there is a density-dependent fecundity in which
154 the weight of females and their egg production rates decrease as the parasite
155 burden on the host increases [8, 21].

156 This negative density-dependence can be described mathematically by
157 the negative exponential function

$$\lambda(n) = \lambda_0 \exp[-\gamma(n-1)] \quad (5)$$

158 where $\lambda(n)$ is the per capita female fecundity within a host with a para-
159 site burden of size n , λ_0 is the intrinsic fecundity in absence of density-
160 dependence effects and γ is the density-dependence intensity. A study of
161 density-dependent effects for *Ascaris lumbricoides* is presented in [11].

162 To simplify notation in rest of the text we will express the female fecundity
163 by $\lambda(n) = \lambda_0 z^{n-1}$ where $z = e^{-\gamma}$.

164 3.4 Egg production and mating probability

165 Due to the effects of density-dependent fecundity, the egg production per
 166 female decreases as the parasite burden in host increases. Therefore, if $j\lambda(n)$
 167 is the egg production of j female parasites within a host with n parasites
 168 and $p_n \binom{n}{j} \alpha^j \beta^{n-j}$ is the probability of a host with n parasites having j female
 169 parasites. Then, we obtain the following result for the mean egg production
 170 per host

171 **Proposition 3.4.1.** *The mean egg production per host is given by*

$$\lambda_0 \alpha G'(z) \quad (6)$$

Proof. We consider that all females present in the host can produce eggs according to their per-capita fecundity

$$\begin{aligned} \sum_{n \geq 0} \sum_{j=0}^n j \lambda(n) p_n \binom{n}{j} \alpha^j \beta^{n-j} &= \lambda_0 \sum_{n \geq 0} \sum_{j=0}^n j z^{n-1} p_n \binom{n}{j} \alpha^j \beta^{n-j} \\ &= \lambda_0 \sum_{n \geq 0} z^{n-1} p_n \sum_{j=0}^n j \binom{n}{j} \alpha^j \beta^{n-j} \\ &= \lambda_0 \sum_{n \geq 0} z^{n-1} p_n n \alpha \\ &= \lambda_0 \alpha \sum_{n \geq 0} n z^{n-1} p_n \\ &= \lambda_0 \alpha G'(z) \end{aligned}$$

172

□

173 For the case of the mean fertilized egg production per host, we use the
 174 previous proof, but considering only the egg production by fertilized female
 175 parasites. Therefore, an expression for the mean fertilized egg production is
 176 given by

Proposition 3.4.2. *The mean fertilized egg production per host is given by*

$$\lambda_0 \alpha G'(z) \left[1 - \frac{G'(\alpha z)}{G'(z)} \right]$$

Proof.

$$\begin{aligned}
\sum_{n \geq 0} \sum_{j=1}^{n-1} j \lambda(n) p_n \binom{n}{j} \alpha^j \beta^{n-j} &= \lambda_0 \sum_{n \geq 0} \sum_{j=1}^{n-1} j z^{n-1} p_n \binom{n}{j} \alpha^j \beta^{n-j} \\
&= \lambda_0 \sum_{n \geq 0} z^{n-1} p_n \sum_{j=1}^{n-1} j \binom{n}{j} \alpha^j \beta^{n-j} \\
&= \lambda_0 \sum_{n \geq 0} z^{n-1} p_n (n\alpha - n\alpha^n) \\
&= \lambda_0 \alpha \sum_{n \geq 0} n z^{n-1} p_n (1 - \alpha^{n-1}) \\
&= \lambda_0 \alpha \left[\sum_{n \geq 0} n z^{n-1} p_n - \sum_{n \geq 0} n (\alpha z)^{n-1} p_n \right] \\
&= \lambda_0 \alpha G'(z) \left[1 - \frac{G'(\alpha z)}{G'(z)} \right]
\end{aligned}$$

177

□

178 According to the results previously obtained, if we consider the quotient
 179 between the mean fertilized egg production and the mean egg production, we
 180 can obtain the probability of fecundity of the eggs or mating probability of
 181 female parasites, under the density-dependence effects. Therefore, we obtain
 182 the mating probability of a female parasite as a function of the mean parasite
 183 burden m by

184 **Theorem 3.4.3.** *The mating probabillity of a female parasite is given by*

$$\phi(m) = 1 - \frac{G'(\alpha z; m)}{G'(z; m)} \quad (7)$$

185 From this expression (7) we notice that for the case where there is no
 186 density-dependence ($z \approx 1$) this expression is equivalent to expression (4),
 187 therefore, this is a generalization of the mating probability obtained above.

188 3.5 An application for mean burden-based models for 189 helminth infections

190 In deterministic population models based on the mean parasite burden for
 191 transmission dynamics of helminth infections such as [2, 3, 20], it is necessary
 192 to know the effective transmission contribution of the female population to

the parasite reservoir (in form of eggs or larvae) [7, 8]. The effective transmission contribution is denoted by ψ and we can calculate it as follows [7, 8],

$$\psi = \frac{\sum_{n \geq 0} \sum_{j=1}^n j \lambda(n) p_n^{(n)} \alpha^j \beta^{n-j}}{\sum_{n \geq 0} \sum_{j=0}^n j p_n^{(n)} \alpha^j \beta^{n-j}} \quad (8)$$

where the negative density-dependence function $\lambda(n)$ is redefined as $\lambda(n)/\lambda_0$. This allows the function $\lambda(n)$ to have a maximum value of 1 and separate the density-independent term λ_0 , from the density-dependent processes (n -dependent).

Using the results obtained in this work we can calculate this term ψ as a function of mean parasite burden m by

$$\psi(m) = \frac{G'(z; m)}{m} \quad (9)$$

Therefore, if we know distribution of parasites in hosts, we can calculate the mean egg production per host as

$$\lambda_0 \alpha m \psi(m) = \lambda_0 \alpha G'(z; m) \quad (10)$$

However, only hosts with at least one female parasite and one male parasite will effectively contribute to the parasite reservoir by producing fertilized (or infective) eggs. Then, the mean fertilized eggs production per host is (see, for example, [3])

$$\lambda_0 \alpha m \psi(m) \phi(m) = \lambda_0 \alpha G'(z; m) \left[1 - \frac{G'(\alpha z; m)}{G'(z; m)} \right] \quad (11)$$

where we assume that ψ and ϕ are functions of the mean parasite burden m .

3.6 Some examples

In this section we will consider the most common statistical models used to describe the distribution of parasites among hosts.

3.6.1 Poisson

A simple model for the distribution of parasites per host is the Poisson distribution [14],

$$\Pr(X = x) = \frac{m^x e^{-m}}{x!}, \quad (12)$$

215 where m is the mean parasite burden and its pgf is given by

$$G(s) = e^{m(s-1)} \quad (13)$$

216 For this parasite distribution the effective contribution of female parasites to
217 the transmission cycle is given by (see eq (9))

$$\psi(m) = e^{m(z-1)} \quad (14)$$

218 Another important factor in parasite dynamics is the mating probability ϕ
219 which is given by (see eq 7)

$$\phi(m) = 1 - e^{-mz\beta} \quad (15)$$

220 This expression of ϕ is a generalization for the mating probability obtained
221 in the works [3, 16, 17].

222 3.6.2 Negative binomial

223 In most cases, soil-transmitted helminths, present a distribution of parasites
224 per host that can be well described by a negative binomial distribution [5,
225 12, 18],

$$P(X = x) = \frac{\Gamma(k+x)}{\Gamma(x+1)\Gamma(k)} \left(\frac{k}{k+m}\right)^k \left(\frac{m}{k+m}\right)^x \quad (16)$$

226 where m is the mean parasite burden and k is the inverse dispersion param-
227 eter of the parasites. The corresponding pgf is given by

$$G(s) = \left[1 - \frac{m}{k}(s-1)\right]^{-k} \quad (17)$$

228 Therefore, the expression for ψ , the effective contribution, which is given by
229 (see eq. (9))

$$\psi(m) = \left[1 - \frac{m}{k}(z-1)\right]^{-(k+1)} \quad (18)$$

230 Finally the mating probability, ϕ , is given by (see eq. (7))

$$\phi(m) = 1 - \left[\frac{1 - \frac{m}{k}(\alpha z - 1)}{1 - \frac{m}{k}(z - 1)}\right]^{-(k+1)} \quad (19)$$

231 This expression of ϕ results in a generalization for the mating probability
232 obtained in works [3, 16, 17].

233 3.6.3 Zero-inflated and hurdle Models

Other frequently used models are the zero-inflated and hurdle models (see for example [1, 9, 10, 22]). For a zero-inflated model, its probability mass function is

$$P(Y = y) = \begin{cases} \pi + (1 - \pi)p_0 & y = 0 \\ (1 - \pi)p_y & y \neq 0 \end{cases}$$

where p is the probability mass function of a distribution with no excess of zero counts and G_X the corresponding pgf. Then the pgf of the zero-inflated distribution is

$$G_Y(s) = \pi + (1 - \pi)G_X(s)$$

and the mean burden is

$$m_Y = (1 - \pi)m_X$$

234 For this model the mean number of fertilized female parasites per host is
235 given by

$$\alpha G'_Y(1) \left[1 - \frac{G'_Y(\alpha)}{G'_Y(1)} \right] = \alpha(1 - \pi)G'_X(1) \left[1 - \frac{G'_X(\alpha)}{G'_X(1)} \right]$$

236 Another important result is the expression for ψ , the mean contribution per
237 female parasite, which is given by

$$\psi = \frac{G'_Y(z)}{m_Y} = \frac{(1 - \pi)G'_X(z; m_X)}{m_Y} = \frac{G'_X\left(z; \frac{m_Y}{1 - \pi}\right)}{\frac{m_Y}{1 - \pi}} \quad (20)$$

238 Finally the mating probability ϕ can be calculated by

$$\phi = 1 - \frac{G'_Y(\alpha z)}{G'_Y(z)} = 1 - \frac{G'_X\left(\alpha z; \frac{m_Y}{1 - \pi}\right)}{G'_X\left(z; \frac{m_Y}{1 - \pi}\right)} \quad (21)$$

A hurdle model is a two-part model, the first part, π , which is the probability of observing the zero value, and the second part which gives the probability of observing non-zero values. The use of hurdle models is often motivated by an excess of zeros in the data, which is not sufficiently accounted for in more standard statistical models [13]. For this model its probability mass function is given by

$$P(Y = y) = \begin{cases} \pi & y = 0 \\ (1 - \pi) \frac{p(y)}{1 - p_0} & y \neq 0 \end{cases}$$

Its pgf G_Y and its mean are of the form

$$G_Y(s) = \pi + (1 - \pi) \frac{G_X(s) - p_0}{1 - p_0}$$

$$m_Y = (1 - \pi) \frac{m_X}{1 - p_0}$$

Therefore

$$\psi = \frac{G'_Y(z)}{m_Y} = \frac{\rho G'_X(z; m_X)}{m_Y} = \frac{G'_X\left(z; \frac{m_Y}{\rho}\right)}{\frac{m_Y}{\rho}}$$

$$\phi = 1 - \frac{G'_Y(\alpha z)}{G'_Y(z)} = 1 - \frac{G'_X\left(\alpha z; \frac{m_Y}{\rho}\right)}{G'_X\left(z; \frac{m_Y}{\rho}\right)} \quad (22)$$

where $\rho = \frac{1-\pi}{1-p_0}$.

3.6.4 Zero-inflated Poisson and zero-inflated negative binomial models

The negative binomial distribution is widely used to describe the distribution of parasites in hosts [9, 18]. However in many cases the negative binomial distribution (or other similar distributions) cannot account for the excess of zeros observed [9]. A solution to this problem are zero-inflated models which have been widely used in the last decade for parasite counting [1, 10, 22].

In Table 1 we present the expressions for the effective contribution and mating probability for the zero-inflated Poisson and zero-inflated negative binomial models.

Table 1: The Effective contribution ψ and the mating probability ϕ for zero-inflated Poisson (ZIPo) and zero-inflated negative binomial (ZINB) models.

Statistical model	effective contribution	mating probability
ZIPo	$\psi = \exp\left(\frac{m}{1-\pi}(z-1)\right)$	$\phi = 1 - \exp\left(-\frac{mz\beta}{1-\pi}\right)$
ZINB	$\psi = \left[1 - \frac{m}{k(1-\pi)}(z-1)\right]^{-(k+1)}$	$\phi = 1 - \left[\frac{1 - \frac{m}{k(1-\pi)}(\alpha z - 1)}{1 - \frac{m}{k(1-\pi)}(z-1)}\right]^{-(k+1)}$

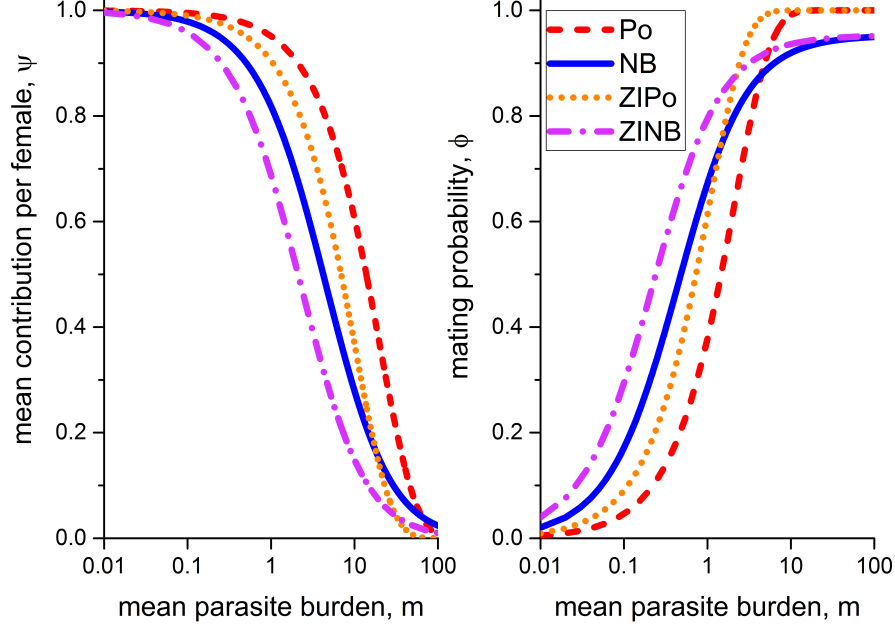


Figure 1: The mean effective contribution per female parasite, ψ (left) and the mating probability, ϕ (right) corresponding to Poisson (dash curve), negative binomial (solid curve), zero-inflated Poisson (dot curve) and zero-inflated negative binomial (dash dot curve) distributions. All as a function of the mean parasite burden m .

In Figure 1 we show plots of the effective mean contribution (ψ) and the mating probability (ϕ) for all the distributions discussed above. We consider the parameters $z = 0.93$, $k = 0.7$, $\pi = 0.3$, $\alpha = 0.574$ ([19]).

4 Parasite infection by skin-penetrating

Unlike in section 3, we consider that the transmission of parasites occurs through the skin penetration. This type of transmission occurs in parasites such as *Ancylostoma duodenale*, *Necator americanus*, among others [4, 6].

In this type of transmission, the host can acquire a single parasite per infection event. Thus, the host can be infected with only one male or female parasite at a time. Therefore, when analyzing the number of male or female parasites per host, these variables must be independent. Here we will present an analysis of these variables.

263 4.1 Distribution of parasites by sex

264 As in the previous section, let W be the random variable count of the number
 265 of parasites in a host and F , M are the number of female and male parasites,
 266 respectively. In this section we analyze the case in which these variables are
 267 independent and therefore verify the following properties

$$\begin{aligned} W &= F + M \\ G_W(s) &= G_F(s)G_M(s) \end{aligned} \quad (23)$$

268 where G_W , G_F and G_M are probability generating function of the variables
 269 W , F and M , respectively.

270 We present an expression for all the variables developed in section 3,
 271 proofs are in the Appendix A.4

- Mean number of fertilized female parasites

$$\alpha m [1 - p_M(0)] \quad (24)$$

- Mating probability

$$1 - p_M(0) \quad (25)$$

- 272 • Mean egg production per host

$$\lambda_0 G_M(z) G'_F(z) \quad (26)$$

- 273 • Mean fertilized egg production

$$\lambda_0 G_M(z) G'_F(z) \left[1 - \frac{p_M(0)}{G_M(z)} \right] \quad (27)$$

- Mean effective transmission contribution by female parasite

$$\psi = \frac{G_M(z) G'_F(z)}{\alpha m} \quad (28)$$

- 274 • Mating probability and density-dependence effects

$$\phi = 1 - \frac{p_M(0)}{G_M(z)} \quad (29)$$

- 275 • Contribution of mean fertilized egg production for mean-based deter-
 276 ministic model of parasite burden

$$\lambda_0 \alpha m \psi(m) \phi(m) \quad (30)$$

277 4.2 Some examples

278 Distributions for the variables F and M are expected to be the same, but
 279 with different parameter values if the sex ratio it is not 1:1. However the
 280 total parasite burden distribution (M), obtained from the conditions (23),
 281 may have a different distribution.

282 In the examples presented here we show some cases where the variables
 283 W , F and M have all the same statistical model. We work with some of
 284 the most popular distributions used to model parasites and in all cases and
 285 arbitrary sex ratio $\alpha : \beta$, where $\alpha + \beta = 1$, it is assumed.

286 4.2.1 Poisson

For the case where the distribution of parasites per host is Poisson with mean λ , that is, $W \sim \text{Po}(\lambda)$. A solution for the independence of variables F and M are the following distributions

$$F \sim \text{Po}(\alpha\lambda) \quad M \sim \text{Po}(\beta\lambda)$$

$$\begin{aligned} G_F(s)G_M(s) &= e^{\alpha\lambda(s-1)}e^{\beta\lambda(s-1)} \\ &= e^{(\alpha+\beta)\lambda(s-1)} \\ &= e^{\lambda(s-1)} \\ &= G_{F+M}(s) \\ &= G_W(s) \end{aligned}$$

287 Note that the pgf of F and M coincide with what was obtained in section 3.1,
 288 which shows the independence of these variables in that section. We show
 289 some of the expressions obtained in the previous section 3.1 for this case:

- Mean effective transmission contribution by female parasite

$$\psi = \frac{G_M(z)G'_F(z)}{G'_F(1)} = e^{-m(1-z)}$$

- Mating probability and density-dependence effects

$$\phi = 1 - \frac{p_M(0)}{G_M(z)} = 1 - e^{-mz\beta}$$

290 Note that the expression for ψ and ϕ are the same as those obtained in the
 291 section 3.6.

292 4.2.2 Negative binomial

293 If F and M are negative binomial distributed with parameters $m_F = \alpha m$,
 294 $k_F = \alpha k$, $m_M = \beta m$, $k_M = \beta k$,

$$F \sim \text{NB}(\alpha m, \alpha k) \quad M \sim \text{NB}(\beta m, \beta k)$$

295 Then the distribution of $W = F + M$ is the negative binomial distribution
 296 with parameters m and k . In fact, a solution to problem (23) is given by

$$\begin{aligned} G_F(s)G_M(s) &= \left[1 - \frac{\alpha m}{\alpha k}(s-1)\right]^{-\alpha k} \left[1 - \frac{\beta m}{\beta k}(s-1)\right]^{-\beta k} \\ &= \left[1 - \frac{m}{k}(s-1)\right]^{-\alpha k - \beta k} \\ &= \left[1 - \frac{m}{k}(s-1)\right]^{-k} \\ &= G_{F+M}(s) \\ &= G_W(s) \end{aligned}$$

297 For this case, the pgf of F and M are not equal to those obtained in
 298 section 3.1, since it was shown that the variables were not independent. We
 299 show some of the expressions obtained in the previous section 3.1 for case of
 300 independence between variables

- Mean effective transmission contribution by female parasite

$$\psi = \frac{G_M(z)G'_F(z)}{\alpha m} = \left[1 - \frac{m}{k}(z-1)\right]^{-(k+1)} \quad (31)$$

- 301 • Mating probability and density-dependence effects

$$\phi = 1 - \frac{p_M(0)}{G_M(z)} = 1 - \left[\frac{1 + \frac{m}{k}}{1 - \frac{m}{k}(z-1)}\right]^{-\beta k} \quad (32)$$

302 Note that the expression ψ is the same one obtained in the section 3.6.

303 In Figure 2 we show the behavior of the mating probability function for
 304 the cases in which the female and male parasites are distributed together or
 305 independently.

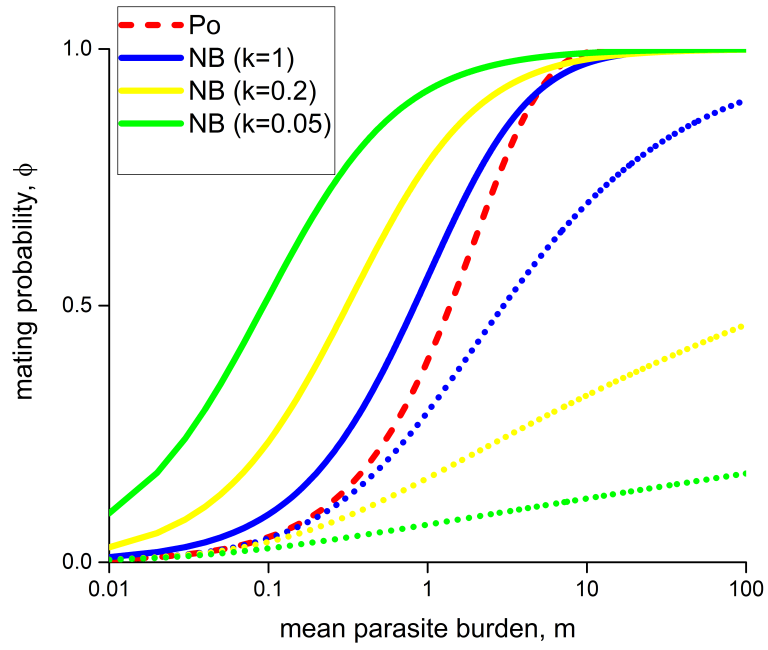


Figure 2: Mating probability as a function of mean parasite load. The dashed curve (red) corresponds to a Poisson distribution ($k \rightarrow \infty$). The solid and dotted curves correspond to a negative binomial distribution with joint or independent distribution by sex, respectively, where $k = 1$ (blue), $k = 0.2$ (yellow) and $k = 0.05$ (green).

306 4.2.3 Zero-inflated negative binomial

- 307 • Mean effective transmission contribution by female parasite

$$\begin{aligned}
 \psi &= \frac{G_M(z)G'_F(z)}{\alpha m} \\
 &= \pi \left[1 - \frac{m}{(1-\pi)k}(z-1) \right]^{-(\alpha k+1)} \\
 &\quad + (1-\pi) \left[1 - \frac{m}{(1-\pi)k}(z-1) \right]^{-(k+1)}
 \end{aligned} \tag{33}$$

- 308 • Mating probability and density-dependence effects

$$\begin{aligned}
 \phi &= 1 - \frac{p_M(0)}{G_M(z)} \\
 &= 1 - \frac{\pi + (1-\pi) \left[1 + \frac{m}{(1-\pi)k} \right]^{-\beta k}}{\pi + (1-\pi) \left[1 - \frac{m}{(1-\pi)k}(z-1) \right]^{-\beta k}}
 \end{aligned} \tag{34}$$

309 5 Monte Carlo simulation

310 5.1 Parasite infection by egg ingestion

311 For this case we consider that the transmission of parasites is produced by the
 312 ingestion of fertilized eggs of these parasites. Here we can consider infection
 313 by *Ascaris lumbricoides* and *Trichuris trichiura*.

314 5.1.1 Model assumptions

315 Simulation algorithms presented in this section are based on the following
 316 assumptions and rules:

- 317 • We considered a host population of size N .
- 318 • The parasite burden of each host is a random variable W .
- 319 • In the ingestion of infectious eggs, a host may acquire one or more
 320 parasites per transmission event. Thus, a host may acquire female or
 321 male parasites in the same transmission event.

- 322 • The female and male parasite burden are dependent random variables
 323 F and M , respectively. The female parasite burden F is obtained by
 324 doing W Bernoulli trials with the parameter α , where α is the sex
 325 ratio of the female parasites. The male parasite burden is given by
 326 $M = W - F$.
- 327 • The egg production per host is given by Fz^{W-1} , where $z = \exp(\sigma)$ with
 328 σ (Falta)
- 329 • The infective egg production per host is given by $I_{M>0}(M)Fz^{W-1}$,
 330 where $I_{M>0}$ is the indicator function of the set $M > 0$.
- 331 • The mating probability is obtained by quotient between the mean in-
 332 infective egg production and the mean egg production.
- 333 • The mean effective contribution per female parasite is obtained by quo-
 334 tient between the mean egg production per host and the mean of female
 335 parasites per host.

336 All the simulations were carried out in RStudio (Version 1.0.136) .

337 5.2 Parasite infection by skin-penetrating

338 5.3 Some examples

339 5.3.1 Negative Binomial

$$\psi = \left[1 - \frac{m}{k}(z - 1)\right]^{-(k+1)} \quad (35)$$

340 Finally the mating probability, ϕ , is given by (see eq. (7))

$$\phi = 1 - \left[\frac{1 - \frac{m}{k}(\alpha z - 1)}{1 - \frac{m}{k}(z - 1)}\right]^{-(k+1)} \quad (36)$$

341 Mating probability and density-dependence effects

$$\phi = 1 - \frac{p_M(0)}{G_M(z)} = 1 - \left[\frac{1 + \frac{m}{k}}{1 - \frac{m}{k}(z - 1)}\right]^{-\beta k} \quad (37)$$

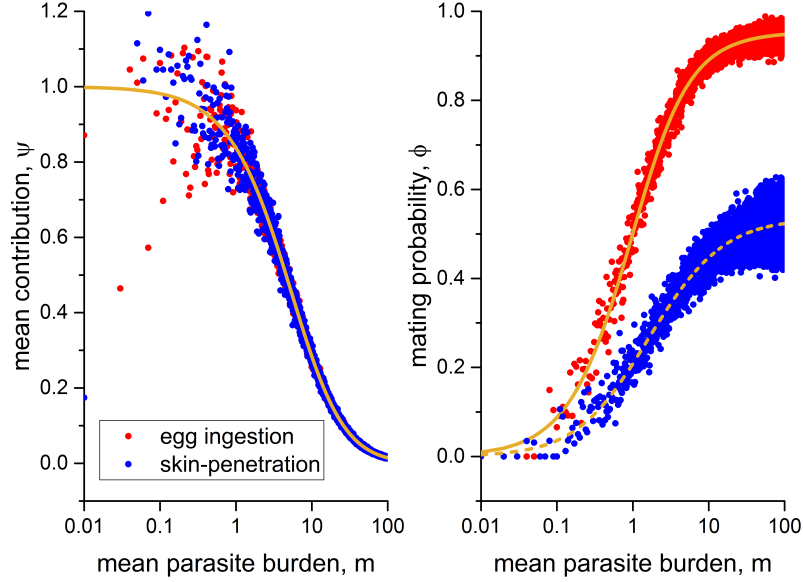


Figure 3: The mean contribution, ψ , and the mating probability, ϕ , as a function of the mean parasite burden, m , for a parasite population with a negative binomial distribution. The dots (red and blue) are the empirical values of ψ and ϕ , obtained for the host population simulated. of the simulations. The curves (continue and dash) correspond to the theoretical models ψ and ϕ obtained in REF. The red dots and continue curves correspond to the infection by egg ingestion. The blue dots and dash curves correspond to the infection by skin-penetrating.

342 5.3.2 Zero-inflated negative binomial

$$\psi = \left\{ \pi + (1 - \pi) \left[1 - \frac{m}{(1 - \pi)k} (z - 1) \right]^{-\beta k} \right\} \left[1 - \frac{m}{(1 - \pi)k} (z - 1) \right]^{-(1 + \alpha k)} \quad (38)$$

343 6 Discussion and Conclusions

344 In most cases total macro-parasites distribution is determined by the infec-
 345 tion process and therefore the variables F and M (number of female and
 346 male parasites within the host) are not independent variables. We presented
 347 a general form to obtain the parasite female burden distribution in hosts
 348 from the observed total parasite distribution.

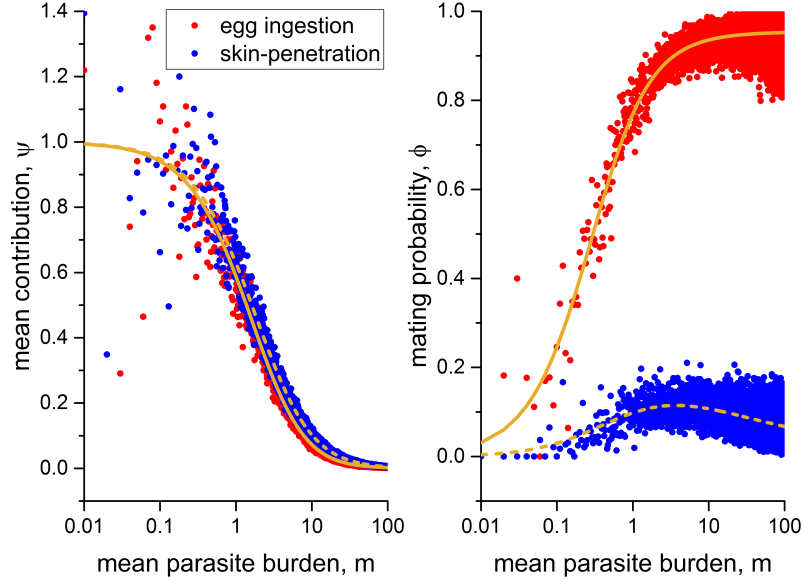


Figure 4: The mean egg contribution, ψ , and the mating probability, ϕ , as functions of the mean parasite burden, m , for a parasite population with a zero-inflated negative binomial distribution. The dots (red and blue) are the simulation values of ψ and ϕ . The curves (continue and dash) correspond to the theoretical models ψ and ϕ obtained in REF. The red dots and continued curves correspond to the infection by egg ingestion (dependiented distributions by sex). The blue dots and dashed curves correspond to the infection by skin-penetrating.

349 Different reproductive variables of parasites of importance for population
 350 dynamics, such as the mean number of fertilized female parasites, mean egg
 351 production, mating probability, mean fertilized egg production and mating
 352 probability, were obtained.

353 The expressions obtained for these reproductive variables in the different
 354 examples are generalizations (for the case of density-dependent fertility on
 355 reproductive behavior of parasites) of those obtained in [15, 16, 17].

356 When parasites are acquired individually we expect the random variables
 357 F and M to be independent. We also expect that these variables have the
 358 same type of distribution.

359 But the total host parasite burden $W = F + M$ not necessarily will
 360 inherit the same distribution os F amd M . There are some obvious cases
 361 where it is known that the distribution of the sum of random variables have

the same distribution of the the variables like in the case of independent Poisson distributed variables. However for the important case of negative binomial distributed variables this is not generally true. In this work we show that only if $F \sim \text{NB}(\alpha m, \alpha k)$ and $M \sim \text{NB}(\beta m, \beta k)$ then the total burden is negative binomial distributed with parameters m and k .

One of the main limitations of this work is that it only considers parasites with a polygamous mating system and we do not consider monogamous and hermaphroditic parasites.

In conclusion, in this work we obtained a general expression for egg production and the mating probability of the parasites. We show how these expressions depend on the sex distribution of the parasites and whether these distributions are considered joint or independent. We also show that these expressions vary due to the effects of the density-dependence of the parasites.

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Conflict of Interest

The authors have declared no conflict of interest.

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A Appendix

We will assume that p is the probability mass function of the distribution of parasites per host and G its probability generating function.

A.1 Mean number of fertilized female parasites

Proposition A.1.1. *The mean number of fertilized female parasites is given by*

$$\alpha m - \alpha G'(\alpha)$$

Proof. The presence of at least one male parasite in the host ensures the fertility of all females, so

$$\begin{aligned} \sum_{n \geq 0} \sum_{j=1}^{n-1} j p_n \binom{n}{j} \alpha^j \beta^{n-j} &= \sum_{n \geq 0} p_n \sum_{j=1}^{n-1} j \binom{n}{j} \alpha^j \beta^{n-j} \\ &= \sum_{n \geq 0} p_n (n\alpha - n\alpha^n) \end{aligned}$$

where the last line is obtained from the expression of the mean of $B(n, \alpha)$, $n\alpha = \sum_{j=0}^n j \binom{n}{j} \alpha^j \beta^{n-j}$. Therefore

$$\begin{aligned} \sum_{n \geq 0} \sum_{j=1}^{n-1} j p_n \binom{n}{j} \alpha^j \beta^{n-j} &= \alpha \sum_{n \geq 0} n p_n (1 - \alpha^{n-1}) \\ &= \alpha \left[\sum_{n \geq 0} n p_n - \sum_{n \geq 0} n \alpha^{n-1} p_n \right] \\ &= \alpha m - \alpha G'(\alpha) \end{aligned}$$

454

□

455 A.2 Mean egg production per host

Proposition A.2.1. *The mean egg production per host is given by*

$$\lambda_0 \alpha G'(z)$$

Proof. We consider that all females present in the host can produce eggs according to their per-capita fecundity

$$\begin{aligned} \sum_{n \geq 0} \sum_{j=0}^n j \lambda(n) p_n \binom{n}{j} \alpha^j \beta^{n-j} &= \lambda_0 \sum_{n \geq 0} \sum_{j=0}^n j z^{n-1} p_n \binom{n}{j} \alpha^j \beta^{n-j} \\ &= \lambda_0 \sum_{n \geq 0} z^{n-1} p_n \sum_{j=0}^n j \binom{n}{j} \alpha^j \beta^{n-j} \\ &= \lambda_0 \sum_{n \geq 0} z^{n-1} p_n n \alpha \\ &= \lambda_0 \alpha \sum_{n \geq 0} n z^{n-1} p_n \\ &= \lambda_0 \alpha G'(z) \end{aligned}$$

456

□

A.3 Mean fertilized egg production per host

Proposition A.3.1. *The mean fertilized egg production per host is given by*

$$\lambda_0 \alpha G'(z) \left[1 - \frac{G'(\alpha z)}{G'(z)} \right]$$

Proof. Identical to the previous demonstration but considering only fertilized females

$$\begin{aligned} \sum_{n \geq 0} \sum_{j=1}^{n-1} j \lambda(n) p_n \binom{n}{j} \alpha^j \beta^{n-j} &= \lambda_0 \sum_{n \geq 0} \sum_{j=1}^{n-1} j z^{n-1} p_n \binom{n}{j} \alpha^j \beta^{n-j} \\ &= \lambda_0 \sum_{n \geq 0} z^{n-1} p_n \sum_{j=1}^{n-1} j \binom{n}{j} \alpha^j \beta^{n-j} \\ &= \lambda_0 \sum_{n \geq 0} z^{n-1} p_n (n\alpha - n\alpha^n) \\ &= \lambda_0 \alpha \sum_{n \geq 0} n z^{n-1} p_n (1 - \alpha^{n-1}) \\ &= \lambda_0 \alpha \left[\sum_{n \geq 0} n z^{n-1} p_n - \sum_{n \geq 0} n (\alpha z)^{n-1} p_n \right] \\ &= \lambda_0 \alpha G'(z) \left[1 - \frac{G'(\alpha z)}{G'(z)} \right] \end{aligned}$$

□

A.4 Independence in the variables F and M

- Mean number of fertilized female parasites

$$\begin{aligned} \sum_{i \geq 1} \sum_{j \geq 0} j p_F(j) p_M(i) &= \sum_{i \geq 1} p_M(i) \sum_{j \geq 0} j p_F(j) \\ &= [1 - p_M(0)] \alpha m \end{aligned}$$

- Mating probability

$$\begin{aligned} \frac{\sum_{i \geq 1} \sum_{j \geq 0} j p_F(j) p_M(i)}{\sum_{j \geq 0} j p_F(j)} &= \frac{[1 - p_M(0)] \alpha m}{\alpha m} \\ &= 1 - p_M(0) \end{aligned}$$

- Mean egg production per host

$$\begin{aligned}
\sum_{i \geq 0} \sum_{j \geq 1} j \lambda(i+j) p_F(j) p_M(i) &= \sum_{i \geq 0} \sum_{j \geq 1} j \lambda_0 z^{i+j-1} p_F(j) p_M(i) \\
&= \lambda_0 \sum_{i \geq 0} z^i p_M(i) \sum_{j \geq 1} j z^{j-1} p_F(j) \\
&= \lambda_0 G_M(z) G'_F(z)
\end{aligned}$$

- Mean fertilized egg production per host

$$\begin{aligned}
\sum_{i \geq 1} \sum_{j \geq 1} j \lambda(i+j) p_F(j) p_M(i) &= \sum_{i \geq 1} \sum_{j \geq 1} j \lambda_0 z^{i+j-1} p_F(j) p_M(i) \\
&= \lambda_0 \sum_{i \geq 1} z^i p_M(i) \sum_{j \geq 1} j z^{j-1} p_F(j) \\
&= \lambda_0 [G_M(z) - p_M(0)] G'_F(z) \\
&= \lambda_0 G_M(z) G'_F(z) \left[1 - \frac{p_M(0)}{G_M(z)} \right]
\end{aligned}$$

- Mean effective transmission contribution by female parasite

$$\psi = \frac{\sum_{i \geq 0} \sum_{j \geq 1} j \lambda(i+j) p_F(j) p_M(i)}{\sum_{j \geq 1} j p_F(j)} = \frac{G_M(z) G'_F(z)}{\alpha m}$$

- Mating probability and density-dependence effects

$$\phi = 1 - \frac{p_M(0)}{G_M(z)}$$