UE Computer Vision - Lab Session 3 : Optical flow

Appendix

Let $\boldsymbol{x} = (x, y)$ be the coordinates of a pixel at time t and $\boldsymbol{u}(\boldsymbol{x}) = [u(x, y), v(x, y)]$ its optical flow vector between t and $t + \Delta t$. Let $I(\boldsymbol{x}, t)$ the intensity of $\boldsymbol{x}$ at time t and $I(\boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}), t + \Delta t)$ the intensity of $\boldsymbol{x}$ moved from $\boldsymbol{u}(\boldsymbol{x})$ at instant $t+\Delta t$. The starting point for most optical flow estimation methods is to assume that the intensity of a given point remains constant between two consecutive images, whatever the nature of its motion. This brightness constancy assumption leads to the following optical flow constraint equation:

$$\frac{\partial I(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{u}(\boldsymbol{x}).\nabla I(\boldsymbol{x}, t) = 0 \tag{1}$$

where $\nabla I(\boldsymbol{x}, t) = (\frac{\partial I(\boldsymbol{x},t)}{\partial x}, \frac{\partial I(\boldsymbol{x},t)}{\partial y})^T$ corresponds to the spatial gradient at time t. Assuming small displacements and spatio-temporal differentiability of intensity, we observe that temporal changes are equivalent to the scalar product between spatial changes and apparent motion. Motion estimation using intensity conservation as only constraint is an under-determined problem. The above equation alone does not allow to estimate the two unknown variables $\boldsymbol{u}(\boldsymbol{x}) = [u(x, y), v(x, y)]$. This problem, known as the aperture problem, can be circumvented by adding local or global constraints.

Contrary to *Lucas Kanade*'s optical flow method [1] which exploits local contraints, the so-called global approaches including *Horn Schunck*'s optical flow [2] introduce a hypothesis of continuity (smoothing) of the deformation (velocity) field. The regularization leads to the minimization of a functional with a data term and a smoothing term to model how optical flow varies spatially:

$$min_{(u,v)} \left(\int_\Omega \left[\frac{\partial I(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{u}(\boldsymbol{x}).\nabla I(\boldsymbol{x}, t) \right]^2 + \alpha^2 \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right) d\Omega \right) \tag{2}$$

where Ω corresponds to the regular grid of pixels, $\|\nabla u\|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$ and $\|\nabla v\|^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$

The data attachment term exploits the brightness constancy constraint described above (Eq. 1). Moreover, the spatial regularization term exploits the sum of the squared gradient norm of the two components u and v of the displacement field $\boldsymbol{u}$ and penalizes its strong variations. In other words, the objective is to limit the difference between the displacement at a point and the average motion of its neighbors in order to obtain smooth motion fields. α^2 gives the relative weight between both terms.

We introduce the functional: $\mathcal{L}(x, y, u, v, u_x, u_y, v_x, v_y) = (I_x u + I_y v + I_t)^2 + \alpha^2 \left((u_x)^2 + (u_y)^2 + (v_x)^2 + (v_y)^2 \right)$ and (Eq. 2) can be written more simply:

$$min_{(u,v)} \int_\Omega \mathcal{L}(x, y, u, v, u_x, u_y, v_x, v_y)\, d\Omega \tag{3}$$

This Euler Lagrange equation is satisfied by two functions u and v of the variables (x, y) which are stationary points of the functional. Because $\mathcal{L}$ is a differentiable functional, it is stationary at its local extrema. u and v satisfy the system of Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial u} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial \mathcal{L}}{\partial u_{x_j}} \right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial v} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial \mathcal{L}}{\partial v_{x_j}} \right) = 0$$

Then, the minimization of this functional leads to the following iterative resolution scheme:

$$\begin{cases} u^k = \bar{u}^{k-1} - I_x(I_x \bar{u}^{k-1} + I_y \bar{v}^{k-1} + I_t).(I_x^2 + I_y^2 + \lambda)^{-1} \\ v^k = \bar{v}^{k-1} - I_y(I_x \bar{u}^{k-1} + I_y \bar{v}^{k-1} + I_t).(I_x^2 + I_y^2 + \lambda)^{-1} \end{cases} \tag{4}$$

<u>Question 1</u>. By assuming that the *Laplacian* $\Delta f = \nabla^2 f$ can be approximated by $\kappa(\bar{f} - f)$ where $\bar{f}$ is the average of f in a certain neighborhood of size κ. Retrieve this result analytically.