LAB 3 - Part I

- Theorie Part:

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1. To determine de MPE test, as we know that:

$$g_{MPE} = \{ \quad (1)$$

We have to calculate $\Delta(x)$. =

$$\Delta(x) = \frac{\frac{1}{(2\pi)^{N/2}\left(\sqrt{\sigma_W^2+\sigma_S^2}\right)^N} \exp\left(-\frac{1}{2}\frac{\|x\|^2}{\sigma_W^2+\sigma_S^2}\right)}{\frac{1}{(2\pi)^{N/2}\sigma_W^N} \exp\left(-\frac{1}{2}\frac{\|x\|^2}{\sigma_W^2}\right)}$$

As we have in $p_1\ \mathcal{N}(0, \sigma_W^2+\sigma_W^2)$ and $p_0 = \mathcal{N}(0, \sigma_W^2)$

$$\ln(\Delta(x)) = N \ln\left(\frac{\sigma_W}{\sqrt{\sigma_W^2+\sigma_S^2}}\right) - \frac{1}{2}\|x\|^2\left(\frac{1}{\sigma_W^2+\sigma_S^2} - \frac{1}{\sigma_W^2}\right) \quad (2)$$

By the definition given on (1), we decide 1 if

$$\Delta(x) > \frac{\pi_0}{\pi_1} = 1,$$

$$\ln(\Delta(x)) > 0 \quad (3)$$

Therefore, ~~joining~~ with (2) and (3):

$$N \ln\left(\frac{\sigma_W}{\sqrt{\sigma_W^2+\sigma_S^2}}\right) - \frac{1}{2}\|x\|^2\left(\frac{1}{\sigma_W^2+\sigma_S^2} - \frac{1}{\sigma_W^2}\right) > 0$$

Defining ρ as $\frac{\sigma_S^2}{\sigma_W^2}$,

$$N \frac{1}{2} \ln\left(\frac{\sigma_W^2}{\sigma_W^2+\sigma_S^2}\right) > -\frac{1}{2}\|x\|^2 \frac{\sigma_S^2}{(\sigma_S^2+\sigma_W^2)\sigma_W^2}$$

$$-N\ln(\rho+1) > -\frac{\|x\|^2}{\sigma_W^2} \frac{\sigma_S^2}{(\sigma_W^2+\sigma_S^5)}$$

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$$\|X\|^2 > \underbrace{\ln(\rho+1)\,\sigma_w^2\,\frac{(\rho+1)}{\rho}}_{\lambda^2}\, N$$

$$g_{MPE} = \begin{cases} 1, & \text{if} \quad \|X\|^2 > \lambda^2 \\ 0, & \text{otherwise} \end{cases}$$

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PROBABILITY OF ERROR

$$\mathbb{P}(g_{ME} \neq \mathcal{E}) = \overset{1/2}{\widehat{\pi_1}}\, \mathbb{P}(g(\mathcal{X}_1)=0) + \overset{1/2}{\widehat{\pi_0}}\, \mathbb{P}(g(\mathcal{X}_0)=1)$$

- $\mathbb{P}(g(\mathcal{X}_1)=0) = \mathbb{P}(\|x\|^2 > \lambda^2)$

$\cancel{\|x\|^2 \sim \chi^2}$

$$\|x\|^2 = \sum_{i=1}^{N} x_i^2 \cdot \frac{\sigma_x^2}{\sigma_x^2} = \sigma_x^2 \sum_{i=1}^{N} \frac{x_i^2}{\sigma_x^2}$$

$\frac{\|x\|^2}{\sigma_x^2} \sim \chi^2$ N degrees of freedom

$$\mathbb{P}(g(\mathcal{X}_1)=0) = \mathbb{P}(g(\mathcal{X})=0 \mid \mathcal{X}=\mathcal{X}_1) = \mathbb{P}(g(\mathcal{X})=0 \mid \sigma_x^2 = \sigma_w^2+\sigma_s^2)$$

$$= \mathbb{P}(\|x\|^2 < \lambda^2) = 1-\mathbb{P}\left(\frac{\|x\|^2}{\sigma_w^2+\sigma_s^2} > \frac{\lambda^2}{\sigma_w^2+\sigma_s^2}\right)$$

$$= 1-\int_{\frac{\lambda^2}{\sigma_w^2+\sigma_s^2}}^{+\infty} f_{\chi^2_N}(t)\,dt = 1-Q_{\chi^2}\left(\frac{\lambda^2}{\sigma_w^2+\sigma_s^2}\right)$$

- $\mathbb{P}(g(\mathcal{X}_0)=1) = Q_{\chi^2}\left(\frac{\lambda^2}{\sigma_w^2}\right)$

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$$\mathbb{P}(g_{\mathrm{GMIE}}(x) \neq \mathcal{E}) = \frac{1}{2}\left(1 - Q_{\chi^2}\left(\frac{\lambda^2}{\sigma_w^2+\sigma_s^2}\right) + Q_{\chi^2}\left(\frac{\lambda^2}{\sigma_w^2}\right)\right)$$

Q_{χ^2} $\cdot$ $\dfrac{\lambda^2}{\sigma_w^2} = \dfrac{\rho+1}{\rho} \ln(\rho+1) N$

$\cdot$ $\dfrac{\lambda^2}{\sigma_w^2+\sigma_s^2} = \dfrac{\cancel{\rho+1}}{\rho} \ln(\rho+1) \underbrace{\dfrac{\sigma_w^2}{\sigma_w^2+\sigma_s^2}}_{\frac{1}{\cancel{1+\rho}}} N = \dfrac{\ln(\rho+1)}{\rho} N$

$$\boxed{\mathbb{P}(g_{\mathrm{GMIE}}(x) \neq \mathcal{E}) = 1/2\left(1 - Q_{\chi^2}\left(\frac{N\ln(\rho+1)}{\rho}\right) + Q_{\chi^2}\left(N\frac{\rho+1}{\rho}\ln(\rho+1)\right)\right)}$$

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Part II:

(1) Calculating $\Lambda(x)$:

$$P_0(x) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\|x\|^2}$$

$$P_1(x) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\|x - A\xi_0\|^2}$$

$$\ln(\Lambda(x)) = \ln\left(\frac{P_1(x)}{P_0(x)}\right) = \cancel{\ln} -\frac{1}{2\sigma^2}\left(\|x - A\xi_0\|^2 - \|x\|^2\right)$$

$$= \frac{-1}{2\sigma^2}\left(\cancel{\|x\|^2} - 2\cancel{A}x^T A\xi_0 + A^2\overbrace{\|\xi_0\|^2}^{1} - \cancel{\|x\|^2}\right)$$

$$= \frac{-1}{2\sigma^2}\left(A^2 - 2x^T A\xi_0\right) = \frac{-1}{2\sigma^2}\left(A^2 - 2Ax^T\xi_0\right)$$

$\cancel{H_0} \quad \cancel{\ln(\Lambda(x)) \gtrless 0}$

~~$\delta_{MPE}(x)$~~ =

$$\ln(\Lambda(x)) = \frac{-1}{2\sigma^2}\left(A^2 - 2Ax^T\xi_0\right) \cancel{\gtrless 0}$$

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regla de decision

$\cancel{A > 0 : \varphi^{+}_{NP}}$

$\ln(\Lambda(x)) = A^2 - 2 A x^T \xi_0 \cancel{\rightarrow A^2 < 2 A x^T \xi_0 \rightarrow \frac{A}{2} < x^T \xi_0}$

$$\boxed{\ln(\Lambda(x)) = 2 A x^T \xi_0 - A^2}$$

$\cancel{\lambda \doteq \frac{A}{2}}$

$\cancel{\circ \text{ si } A > 0 \rightarrow \frac{A^2}{2A} < x^T \xi_0 \rightarrow x^T \xi_0 > \frac{A}{2}}$

$\cancel{\circ \text{ si } A < 0 \rightarrow \frac{A^2}{2A} > x^T \xi_0 \rightarrow x^T \xi_0 < \frac{A}{2}}$

$\cancel{\lambda \doteq \frac{A}{2}}$

$$\begin{cases} A > 0 : \varphi^{+}_{NP}(x) = \begin{cases} 1 & \text{if } x^T \xi_0 > \lambda \\ 0 & \text{otherwise} \end{cases} \\ \\ A < 0 : \varphi^{-}_{NP}(x) = \begin{cases} 1 & \text{if } x^T \xi_0 < \lambda \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

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TEST POWER

- $A > 0$

$$\beta^{+}_{NP} \doteq \mathbb{P}(x^T \xi_0 > \lambda^{+} \mid \mathcal{H}_1) = \mathbb{P}(x_1^T \xi_0 > \lambda^{+}) \quad \text{with } x_1 \sim \mathcal{N}(A\xi_0, \sigma^2 I_N)$$

$$\begin{cases} \mathbb{E}[x_1^T \xi_0] = \mathbb{E}[x_1^T]\xi_0 = A\xi_0^T\xi_0 = A\,\|\xi_0\|^2 = A \\ \mathbb{V}(x_1^T \xi) = \sigma^2 \end{cases} \rightarrow x_1^T \xi_0 \sim \mathcal{N}(A, \sigma^2)$$

$$\beta^{+}_{NP} = 1 - \mathbb{P}[x_1^T \xi_0 < \lambda^{+}] = 1 - \mathbb{P}\left(\frac{x_1^T\xi_0 - A}{\sigma} < \frac{\lambda^{+} - A}{\sigma}\right)$$

$$= Q\left(\frac{\lambda^{+} - A}{\sigma}\right).$$

we calculate λ^{+}

$$\mathbb{P}(x^T \xi_0 > \lambda^{+} \mid \mathcal{H}_0) = \mathbb{P}(x_0^T \xi_0 > \lambda^{+}) = \alpha \quad , x_0 \sim \mathcal{N}(0, \sigma^2 I)$$

$$\mathbb{E}[x_0^T \xi_0] = \mathbb{E}[x_0^T]\xi_0 = 0$$

$$\mathbb{V}(x_0^T \xi_0) = \mathbb{E}[\xi_0^T x_0 x_0^T \xi_0] = \xi_0^T\,\mathbb{E}[x_0 x_0^T]\xi_0 = \xi_0^T \sigma^2 I_N \xi_0$$

$$= \sigma^2 \|\xi_0\|^2 = \sigma^2$$

$$x_0^T \xi_0 \sim \mathcal{N}(0, \sigma^2)$$

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$$\alpha = 1 - \mathbb{P}(x_0^T \xi_0 < \lambda^+) = 1 - \mathbb{P}\left(\frac{x_0^T\xi_0}{\sigma} < \frac{\lambda^+}{\sigma}\right) = 1 - \Phi\left(\frac{\lambda^+}{\sigma}\right) = Q\left(\frac{\lambda^+}{\sigma}\right)$$

$$\boxed{\lambda^+ = Q^{-1}(\alpha)\sigma}$$

$$\boxed{\therefore \beta^+_{NP} = Q\left(\frac{Q^{-1}(\alpha)\sigma - A}{\sigma}\right)}$$

- $A < 0$

$$\beta^-_{NP} \doteq \mathbb{P}(x^T\xi_0 < \lambda | \mathcal{H}_1) \cancel{= 1 - \mathbb{P}(x^T\xi_0 > \lambda | \mathcal{H}_1) = 1 - \beta^+_{NP}}$$

$$\cancel{\beta^-_{NP} = 1 - \beta^+_{NP}}$$

$$\beta^-_{NP} = \mathbb{P}\left(\frac{x^T\xi_0 + A}{\sigma} < \frac{\lambda^- + A}{\sigma}\right) = Q\left(\frac{\lambda^- + A}{\sigma}\right) \quad \left(\text{as } x^T\xi_0 \sim \mathcal{N}(-A, \sigma^2)\right)$$

<u>λ^- calcul</u>

$$\mathbb{P}(x^T\xi_0 < \lambda^- | \mathcal{H}_0) = \alpha$$

as $x^T\xi_0 \sim \mathcal{N}(0, \sigma^2)$

$$\alpha = \mathbb{P}(x^T\xi_0 < \lambda^-) = \mathbb{P}\left(\frac{x^T\xi_0}{\sigma} < \frac{\lambda^-}{\sigma}\right) = \Phi\left(\frac{\lambda^-}{\sigma}\right) = 1 - Q\left(\frac{\lambda^-}{\sigma}\right)$$

$$Q\left(\frac{\lambda^-}{\sigma}\right) = 1 - \alpha \to \lambda^- = \sigma Q^{-1}(1-\alpha)$$

$$\boxed{\beta^-_{NP} = Q\left(\frac{\sigma Q^{-1}(1-\alpha) + A}{\sigma}\right)}$$

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②

~~1)~~ $\cancel{\mathcal{H}_1 : \quad x \sim \mathcal{N}(A\xi_0, \sigma^2 I_N)}$

~~we know that~~ $\cancel{MLE(\mu) = \frac{\sum_{i=1}^{N} x_i}{N}}$

$\cancel{\mu = \mathbb{E}[x] = A\xi_0 \implies MLE(A) = \hat{A} =}$

$\cancel{A}$

1) $\mathcal{H}_1 : x \sim \mathcal{N}(A\xi_0, \sigma^2 I_N)$

we know that $MLE(\bar{\mu}) = \frac{\sum_{i=1}^{M} \bar{x}_i}{M}$, $\bar{x}_i \in \mathbb{R}^N$, $\bar{\mu} \in \mathbb{R}^N$

but we have $\bar{\mu} = A\bar{\xi}_0$

$$\ln(f(x_1, \dots, x_M)) = \sum_{i=1}^{M} \ln f(x_i)$$

$$= \sum_{i=1}^{M} \ln\left(\frac{1}{(2\pi\sigma^2)^{N/2}}\right) + \sum_{i=1}^{M} \left(-\frac{1}{2\sigma^2} \|x_i - A\xi_0\|^2\right)$$

$$= M \ln\left(\frac{1}{(2\pi\sigma^2)^{N/2}}\right) - \frac{1}{2\sigma^2} \sum_{i=1}^{M} \|x_i - A\xi_0\|^2$$

$$= M \ln\left(\frac{1}{(2\pi\sigma^2)^{N/2}}\right) - \frac{1}{2\sigma^2} \sum_{i=1}^{M} \left(\|x\|^2 - 2A x^T \xi_0 + A^2 \overbrace{\|\xi_0\|^2}^{1}\right)$$

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$$\mathcal{L} = \ln\left(f(x_i, \dots, x_M)\right) = M \ln\left(\frac{1}{(2\pi\sigma^2)^{N/2}}\right) - \frac{1}{2\sigma^2} \sum_{i=1}^{M} \left(\|x_i\|^2 - 2A x_i^T \xi_0 + A^2\right)$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = -\frac{1}{2\sigma^2} \sum_{i=1}^{M} (2\, x_i - 2\hat{A}\xi_0) = 0$$

$$\sum_{i=1}^{M} x_i = \hat{A}\xi_0 M \rightarrow \hat{A}\xi_0 = \frac{\sum_{i=1}^{M} x_i}{M} = \hat{\mu}$$

$$\xi_0^T \hat{A} \xi_0 = \xi_0^T \hat{\mu}$$

$$\hat{A} \underbrace{\xi_0^T \xi_0}_{1} = \xi_0^T \hat{\mu} \Rightarrow \cancel{\boxed{\hat{A} = \xi_0^T \hat{\mu}}}$$

$$\hat{A} = \xi_0^T \hat{\mu}$$

~~2)~~ we choose $M=1 \rightarrow \hat{\mu} = x \rightarrow \boxed{\hat{A} = \xi_0^T x}$

$\cancel{A}$

2)

$$P_0(x) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\|x\|^2}$$

$$P_1(x) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{\frac{-1}{2\sigma^2}\|x - \hat{A}\xi_0\|^2}$$

$$\ln(\Lambda(x)) = \frac{-1}{2\sigma^2}\left(\hat{A}^2 - 2\hat{A}x\xi_0\right)$$

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$$\frac{-1}{2\sigma^2}\left((\xi_0^T x)^2 - 2(\xi_0^T x)^2\right) = \frac{1}{2\sigma^2}(\xi_0^T x)^2$$

$$\ln(\Lambda(x)) = \frac{(\xi_0^T x)^2}{2\sigma^2}$$

$\cancel{A} \gtrless 0$

$$\varphi_{GLRT}(x) = \begin{cases} 1 & \text{if } |\xi_0^T x| > \lambda \\ 0 & \text{otherwise} \end{cases}$$

λ calcul

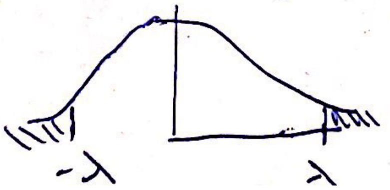
$$\cancel{\mathbb{P}(x^T \xi_0 > \lambda^* \mid \mathcal{H}_0) = \mathbb{P}(x_0^T \xi_0 > \lambda^*) = \alpha,}$$

$$\cancel{x_0^T \xi_0 \sim \mathcal{N}(0,\sigma^2)}$$

$$\cancel{\alpha = 1 - \mathbb{P}\left(\frac{x_0^T \xi_0}{\sigma} < \frac{\lambda}{\sigma}\right) = 1 - \Phi\ \lambda}$$

$$\mathbb{P}(|\xi_0^T x| > \lambda \mid \mathcal{H}_0) = \mathbb{P}(|x_0^T \xi_0| > \lambda) = \alpha \quad , \quad x_0^T \xi \sim \mathcal{N}(0,\sigma^2)$$

$$\alpha = 1 - \mathbb{P}\left(\frac{-\lambda}{\sigma} < \frac{x_0^T \xi}{\sigma} < \frac{\lambda}{\sigma}\right) = Q\left(\frac{\lambda}{\sigma}\right) + \Phi\left(\frac{-\lambda}{\sigma}\right) = 2Q\left(\frac{\lambda}{\sigma}\right)$$



$$\boxed{\lambda = \sigma Q^{-1}(\alpha/2)}$$

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3)
$$\beta_{GLRT} \doteq \mathbb{P}(|x^T \xi_0| > \lambda \mid \mathcal{H}_1) = \mathbb{P}_1(|x_1^T \xi_0| > \lambda)$$

$$\cancel{x_1^T \xi_0 \sim \mathcal{N}(A \xi_0, \sigma^2 I_N)}$$

with $x_1^T \xi_0 \sim \mathcal{N}(\hat{A}, \sigma^2)$

$$\cancel{\beta_{GLRT} = \mathbb{P}\left(\left|\frac{x^T \xi_0 - A}{\sigma}\right|\right)}$$

$$\cancel{\beta_{GLRT} = \mathbb{P}\left(\frac{|x^T \xi_0| - A}{\sigma} > \frac{\lambda - A}{\sigma}\right) = Q\left(\frac{\lambda - A}{\sigma}\right) \cdot 2}$$

$$\cancel{\beta_{GLRT} = 2Q\left(\frac{\sigma Q^{-1}(\alpha/2) - A}{\sigma}\right)}$$

$$\beta_{GLRT} \doteq \mathbb{P}(x_1^T \xi_0 > \lambda) + \mathbb{P}(x_1^T \xi_0 < -\lambda)$$
$$= \mathbb{P}\left(\frac{x_1^T \xi_0 - A}{\sigma} > \frac{\lambda - A}{\sigma}\right) + \mathbb{P}\left(\frac{x_1^T \xi_0 - A}{\sigma} < \frac{-\lambda - A}{\sigma}\right)$$
$$= Q\left(\frac{\lambda - A}{\sigma}\right) + \Phi\left(-\frac{\lambda + A}{\sigma}\right) = Q\left(\frac{\lambda - A}{\sigma}\right) + Q\left(\frac{\lambda + A}{\sigma}\right)$$

as $\lambda = \sigma Q^{-1}(\alpha/2)$

$$\boxed{\beta_{GLRT} \doteq Q\left(\frac{\sigma Q^{-1}(\alpha/2) - A}{\sigma}\right) + Q\left(\frac{\sigma Q^{-1}(\alpha/2) + A}{\sigma}\right)}$$

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