WEEK02. 수치해석

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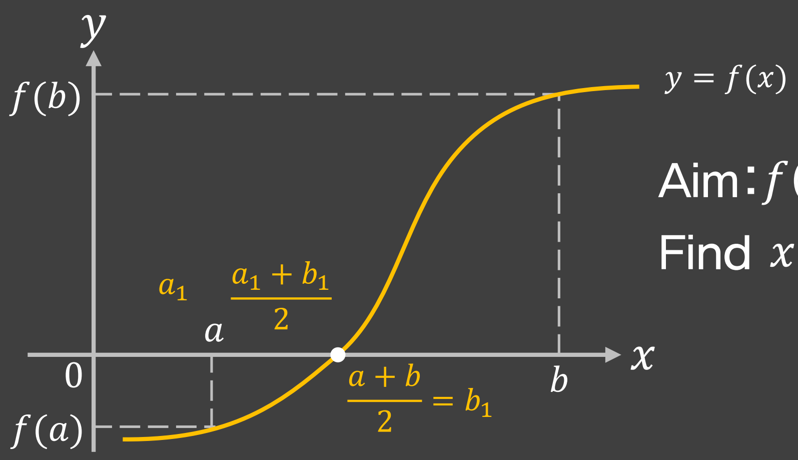
$ax^2 + bx + c = 0$

$anx^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$

$\sin x \quad e^x \quad l_n x$

① Bisection Method for finding solution to equations in 1 variable



Aim: $f(x) = 0$

Find x

Thm $f(x)$: 연속. on $I = [a, b]$

$f(a) \cdot f(b) < 0$

➡ $\exists\, P \in (a, b)$ s.t. $f(p) = 0$

Solution of $f(x) = 0$

$$\left[a, \frac{a+b}{2}\right]$$

$$f\left(\frac{a+b}{2}\right) < 0 [a_1, b_1]$$

$$f\left(\frac{a_1+b_1}{2}\right) > 0 \left[\frac{a_1+b_1}{2}, b_1\right] = [a_2, b_2]$$

Algorithm

Algorithm $f(x) = 0$

Input $a, b,$ tolerance, maximum number of iterations

$f(a) \cdot f(b) < 0$

Set $a_0 = a.\ b_0 = b$

$$p_0 = \frac{a_0 + b_0}{2}$$

solution

If $f(p_0) = 0$ then $p = p_0$

Set $a_1 = a_0 \quad b_1 = p_0$

If $f(p_0) \neq 0$ then if $f(p_0) \cdot f(a_0) < 0 \rightarrow p \in (a_0, p_0)$

if $f(p_0) \cdot f(b_0) < 0 \rightarrow p \in (a_0, b_0)$

$a_1 = p_0 \quad b_1 = b_0$

- ② Fixed Point Iteration
  - What?
  - $g(x)$: a function
  - P : a fixed point for $g(x)$ if $g(P) = P$
  - $f(x) = x - g(x)$
  - $f(x) = 0$, $x = g(x)$

- Theorem
  - (a) $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$
  - Then g has a fixed point in $[a,b]$
  - (b) If $g'(x)$ on (a,b) $\exists\, 0 < k < 1$ with $|g'(x)| \le k$ for all $x \in (a,b)$
  - Then fixed point is unique

- proof. Aim : Find p. s.t $g(p) = P$
  - ① If $g(a) = a$ or $g(b) = b$

    fixed point
  - ② If not

    Let $h(x) = g(x) - x$

    ∵ continuous on $[a, b]$

    $h(a) = g(a) - a > 0$ (∵ $g(a) > a$

    $h(b) = g(b) - b < 0$ $g(x) \in [a, b]$

    (∵ $g(b) < b$)
  - $h(b) < 0 < h(a)$

    $\exists\, p \in (a, b)$ s.t. $h(p) = 0$

    $h(p) = g(p) - p = 0$

    p : a fixed point for $g(x)$

- Theorem
  - (a) $g \in C[a, b]$ and $g(x) \in [a, b]$ for $\underset{x \in [a,b]}{\underline{\text{all}}}$
  - Then g has a fixed point in $[a, b]$
  - (b) If $g'(x)$ on (a, b) $\exists\, 0 < k < 1$

    with $|g'(x)| \le k$ for all $x \in (a, b)$
  - Then fixed point is unique

- proof. Uniqueness
- Let $|g'(x)| \leq k < 1$ and p, q : fixed points
  $g(p) = p$ and $g(q) = q$
- If $p \neq q$
- Mean value Thm
- $g \in C[a, b]$, g : differentiable on (a, b)

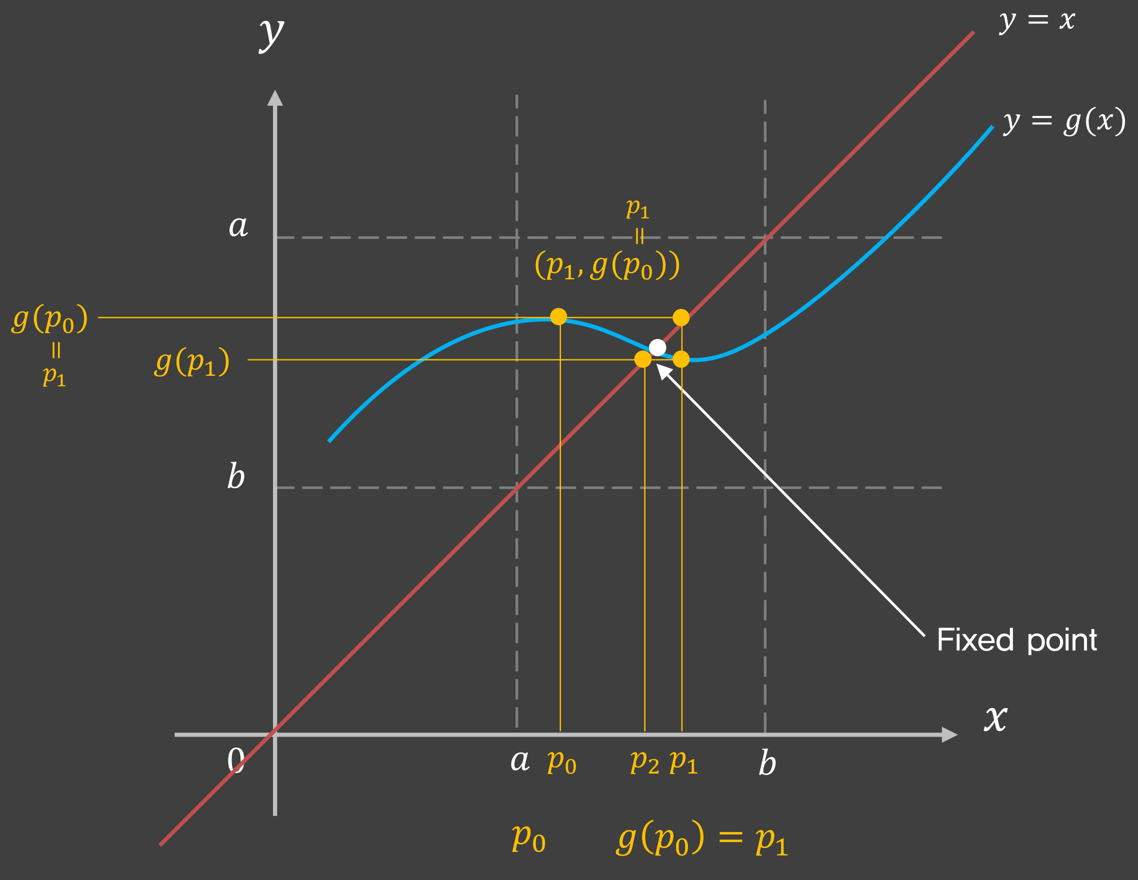
  ➡ $\exists c \in (a, b)$ s.t.

$$g'(c) = \frac{g(b) - g(a)}{b - a}$$

- $\exists c \in (p, q)$ or (q, p)

$$\left|\frac{g(p) - g(q)}{p - q}\right| = |g'(c)|$$

- $\therefore |p - q| = |p - q|\,|g'(c)| = |g(p) - g(q)|$
  $\leq k|p - q| < |p - q|$
- $\therefore p = q$



$x = a$ $\quad g(x) - x = 0$

$g(x) = x$

$f(x) = g(x) - x = 0$

$P_0 \in [a, b]$

$P_n = g(P_{n-1}) \quad n \geq 1$

Converges to $P \in [a, b]$

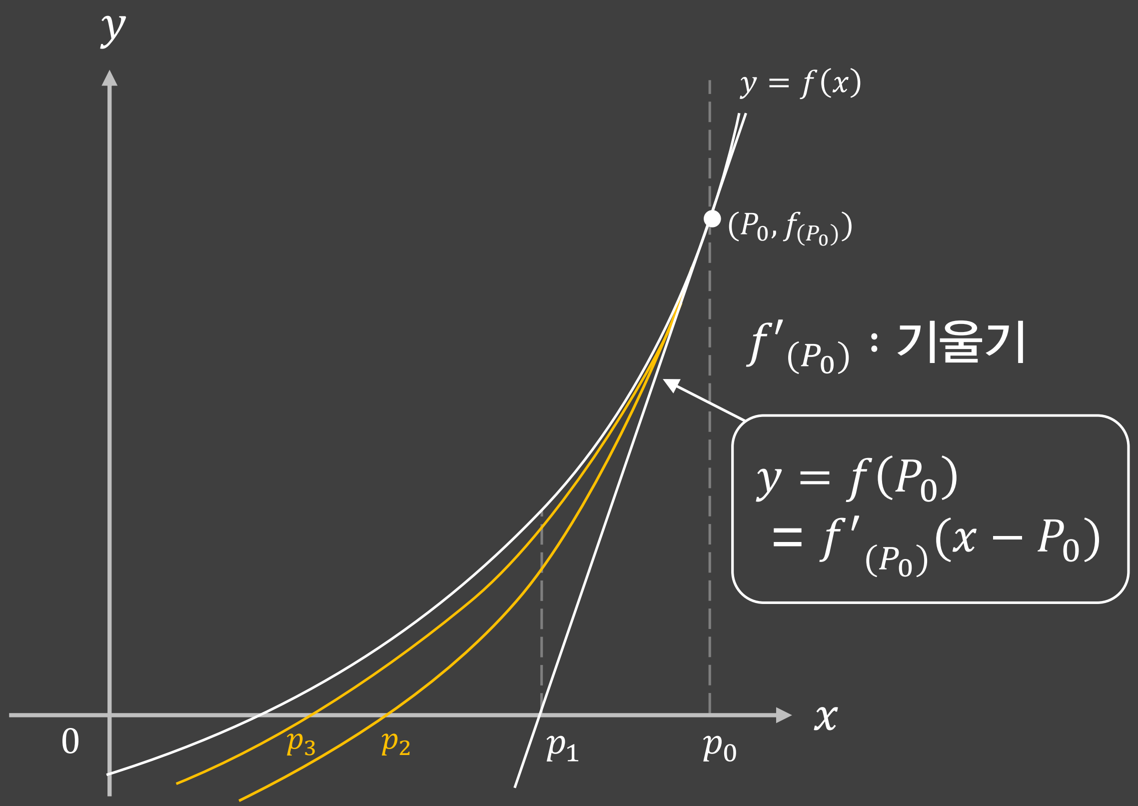
Algorithm

Input P_0, tol, max iter

$P_1 = g(P_0)$ $\quad$ if $|P - P_n| < $ tol $\quad f(P_n) \approx 0$

$P_2 = g(P_1)$ $\quad$ stop

Newton's Method



$$f'(x) = \lim_{h \to \sigma} \frac{f(x+h) - f(x)}{h}$$: derivative of $f(x)$

$$f'(n) = \lim_{h \to \sigma} \frac{f(a+h) - f(a)}{h}$$

$$x = a \quad y = 0$$

$$x = -\frac{f_{(P_0)}}{f'_{(P_0)}} + P_0$$

$$P_n = P_{n-1} - \frac{f_{(P_{n-1})}}{f'_{(P_{n-1})}}$$