WEEK04. Interpolation

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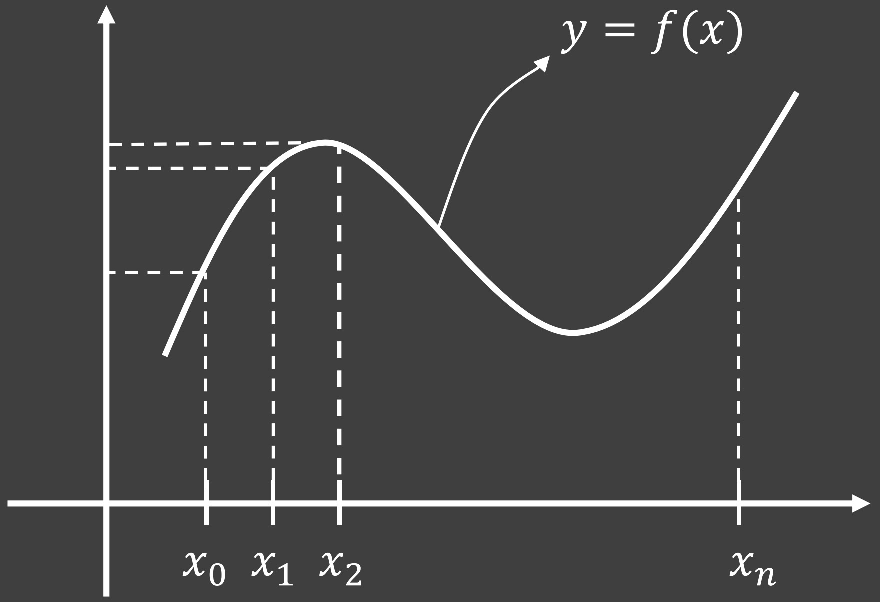
Interpolation



$\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right), \dots, \left(x_n, f(x_n)\right)$

① Lagrange Polynomial

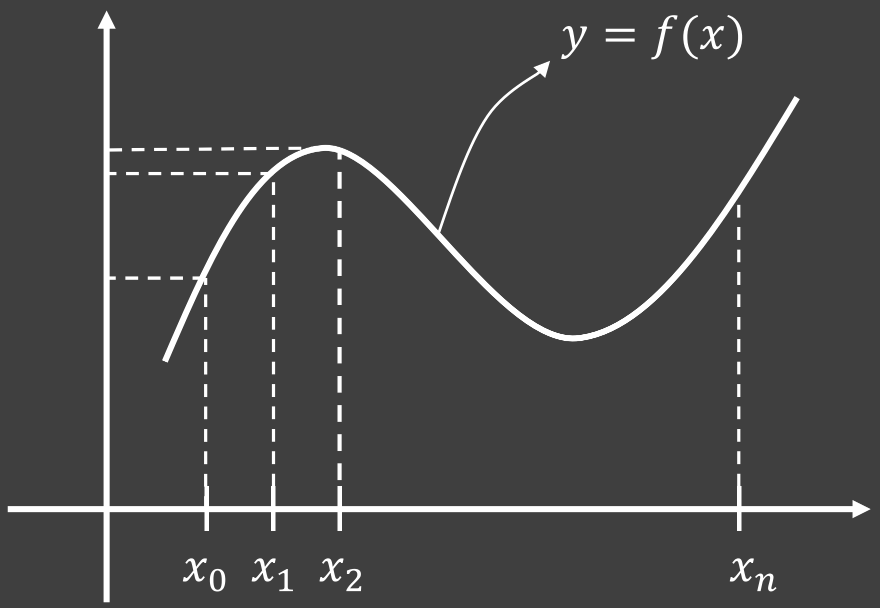
<u>Thm.</u> If $\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right), \dots, \left(x_n, f(x_n)\right)$: Data
f : a function
then a unique polynomial $P(x)$ at degree at most n exists with $f(x_k) = P(x_k)$ for each $k = 0,1,\dots,n$

$$P(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x)$$

where $L_{n,k}$: n th Lagrange interpolating poly.

$$L_{n,k} = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$$

Interpolation



$(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_n, f(x_n))$

② Divide Difference

Find $P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0) \dots (x - x_{n-1})$

$$a_0 = P_n(x_0) = f(x_0)$$

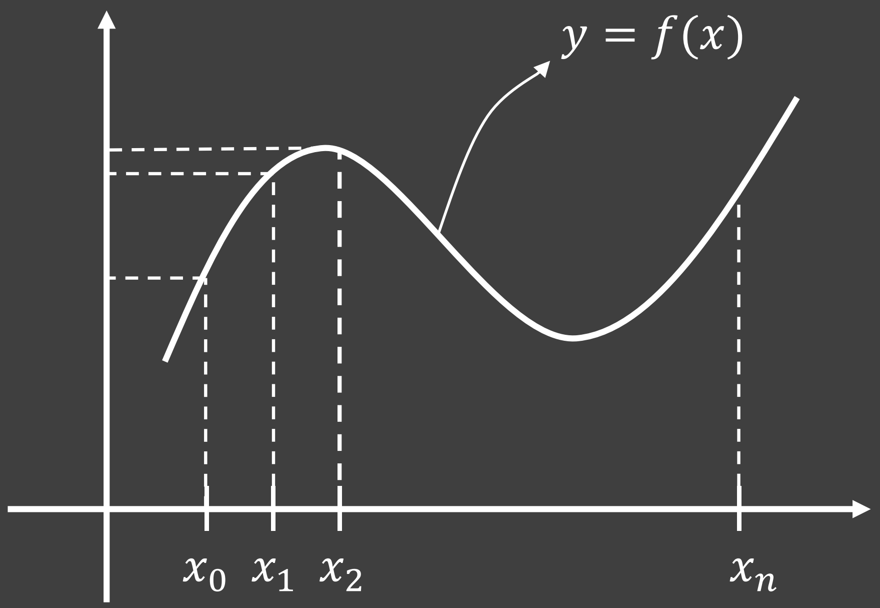
$$P_n(x_1) = f(x_0) + a_1(x_1 - x_0) = f(x_1)$$

$$\therefore a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\vdots$$

$$a_k = ?$$

Interpolation



$(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_n, f(x_n))$

② Divide Difference

Define $f[x_i] = f(x_i)$: zeros divided diff.

$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$: first divided diff.

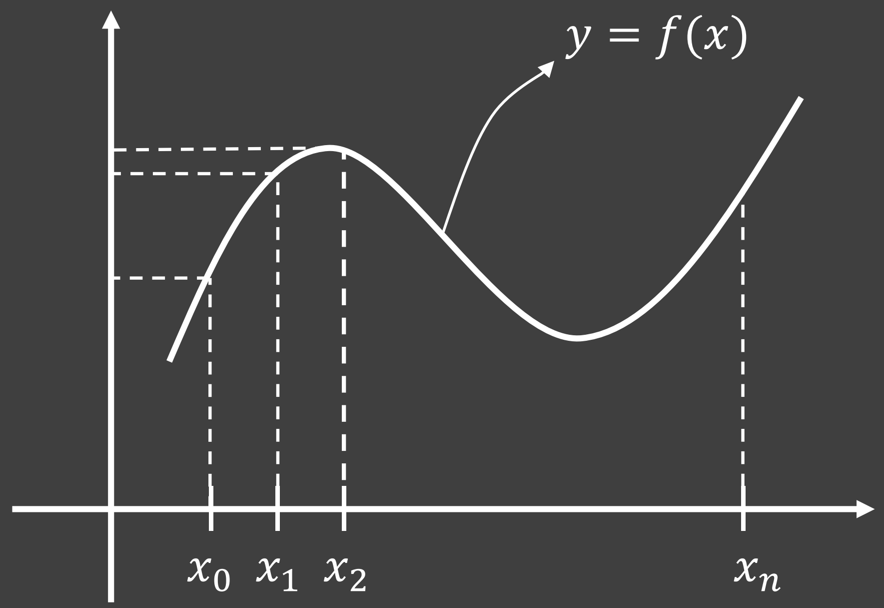
$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$: 2nd divided diff.

$\vdots$

$$f[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{f[x_{i+1}, \dots, x_{i+k}] - f[x_i, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

: kth divided diff.

Interpolation



$(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_n, f(x_n))$

② Divide Difference

Find $P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\dots(x - x_{n-1})$

$$a_0 = P_n(x_0) = f(x_0)$$

$$P_n(x_1) = f(x_0) + a_1(x_1 - x_0) = f(x_1)$$

$$\therefore a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

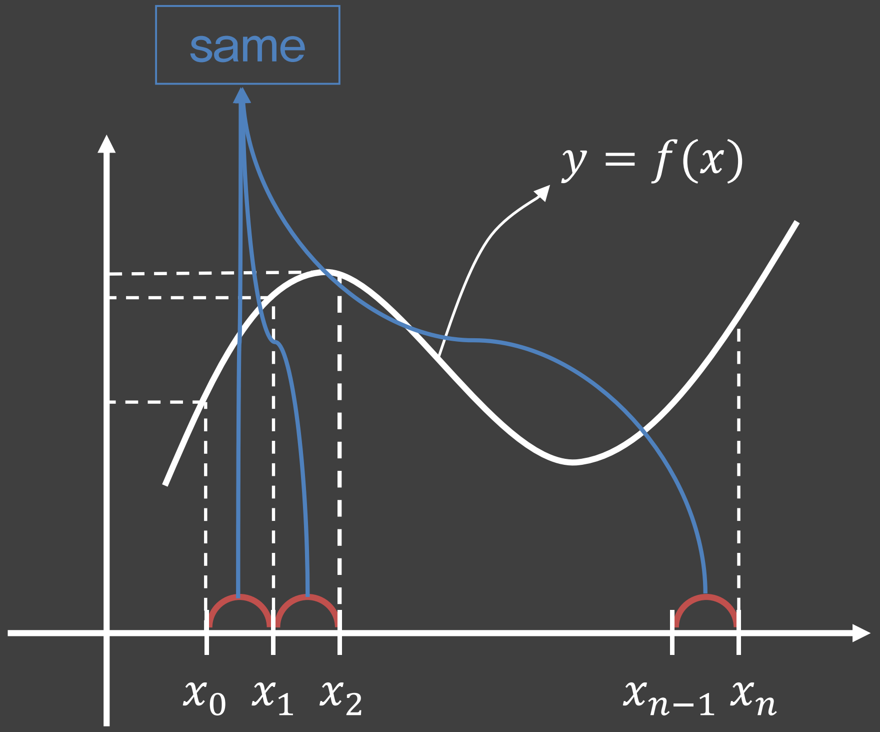
$$\vdots$$

$$a_k = f[x_0, \dots, x_k]$$

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, \dots, x_n](x - x_0)\dots(x - x_{n-1})$$

$$= f[x_0] + \sum_{k=1}^{n} f[x_0, \dots, x_k](x - x_0)\dots(x - x_{k-1})$$

Interpolation



$(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_n, f(x_n))$

③ Binomial Theorem

$h = x_{i+1} - x_i$ for each $i = 0,1, \dots, n-1$

Let $x = x_0 + sh$

Then $x - x_i = (sh - ih) = (s - i)h.$

$$
\begin{aligned}
P_n(x) &= P_n(x_0 + sh) \\
&= f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, \dots, x_n](x - x_0) \dots (x - x_{n-1}) \\
&= f[x_0] + shf[x_0, x_1] + \cdots + s(s-1) \dots 1 \times h^n f[x_0, \dots, x_n]
\end{aligned}
$$

$$= \sum_{k=0}^{n} s(s-1) \dots (s-k+1) h^k f[x_0, \dots, x_k]$$

$$= f[x_0] + \sum_{k=1}^{n} \binom{s}{k} k!\, f[x_0, \dots, x_k]$$

$$= f[x_0] + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} f(x_k)$$

Interpolation

④ Spline

1) Step function spline
2) Linear spline
3) Quadratic spline

   Find $f_i(x) = a_i + b_i x + c_i x^2$ for $i = 1, \dots, n$

   a. $\begin{cases} f_i(x_i) = a_i + b_i x_i + c_1 x_i^2 \\ f_{i+1}(x_i) = a_{i+1} + b_{i+1} x_i + c_{i+1} x_i^2 \end{cases}$ for $i = 1, \dots, n-1$

   b. $\begin{cases} f_1(x_0) = a_1 + b_1 x_0 + c_1 x_0^2 \\ f_n(x_n) = a_n + b_n x_n + c_n x_n^2 \end{cases}$

   c. $f_i'(x) = b_i + 2c_i x$

   $$f_{i+1}'(x) = b_{i+1} + 2c_{i+1} x$$

   $f_i'(x_i) = f_{i+1}'(x_i) \Longrightarrow b_i + 2c_i x_i = b_{i+1} + 2c_{i+1} x_i$ for $i = 1, \dots, n-1$

4) Cubic spline