WEEL03. 수치해석

Kim, Hyun-Min

Department of Mathematics, Pusan National University

Error Analysis

① Bisection method

$f \in C[a,b]$ and $f(a) \cdot f(b) < 0$

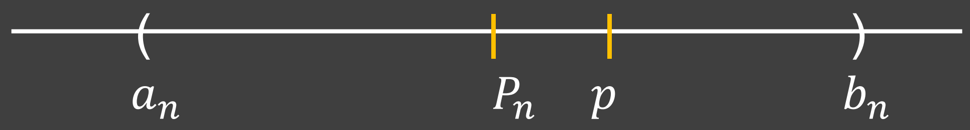
$\{P_n\}_{n=1}^{\infty}$: by the bisection method

$f(p) = 0$, $|P_n - p| \leq \frac{b-a}{2^n}$ when $n \geq 1$

Proof) For each $n \geq 1$,

$$b_n - a_n = \frac{1}{2^{n-1}}(b-a) \text{ and } p \in (a_n, b_n)$$

$$P_n = \frac{1}{2}(a_n - b_n)$$



$$|P_n - p| = \left|\frac{1}{2}(a_n + b_n) - p\right|$$

$$|P_n - p| \leq \frac{1}{2}(b_n - a_n) = \frac{1}{2^n}(b-a)$$

$\{P_n\}_{n=1}^{\infty}$ converges to p

Error Analysis

① Fixed point iteration

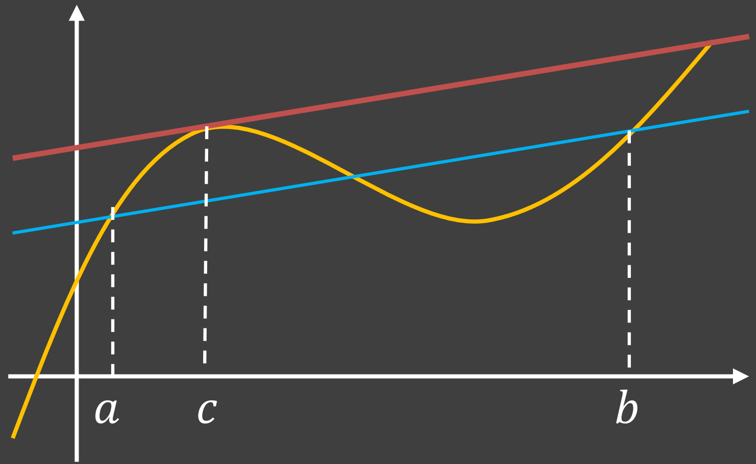
$$|P_n - p| \le k^n \max(P_0 - a, b - P_0)$$

and

$$|P_n - p| \le \frac{k^n}{1-k}|P_1 - P_0|.$$

Proof) $p \in [a, b]$

$$\begin{aligned} |P_n - p| = |g(P_{n-1}) - g(p)| &= |g'(\xi_n)||P_{n-1} - p| \\ &\le k|P_{n-1} - p| \\ &\le k^2|P_{n-2} - p| \\ &\ \ \vdots \\ &\le k^n|P_0 - p| \\ &\le k^n \max(P_0 - a, b - P_0) \end{aligned}$$



Mean value theorem

$\exists c \in (a, b)$ s.t. $\frac{f(b)-f(a)}{b-a} = f'(c)$

Error Analysis

① Fixed point iteration

$$|P_n - p| \le k^n \max(P_0 - a, b - P_0)$$

and

$$|P_n - p| \le \frac{k^n}{1-k}|P_1 - P_0|.$$

Proof) For $n \ge 1$

$$|P_{n+1} - P_n| = |g(P_n) - g(P_{n-1})| \le k|P_n - P_{n-1}| \le \cdots \le k^n|P_1 - P_0|$$

For $1 \le n \le m$

$$\begin{aligned}|P_m - P_n| &= |P_m - P_{m-1} + P_{m-1} - P_{m-2} + \cdots + P_{n+1} - P_n| \\ &\le |P_m - P_{m-1}| + |P_{m-1} - P_{m-2}| + \cdots + |P_{n+1} - P_n| \\ &\le k^{m-1}|P_1 - P_0| + k^{m-2}|P_1 - P_0| + \cdots + k^n|P_1 - P_0| \\ &\le k^n|P_1 - P_0|(k^{m-n-1} + k^{m-n} + \cdots + 1)\end{aligned}$$

$$|p - P_n| = \lim_{m\to\infty}|P_m - P_n| \le \lim_{m\to\infty} k^n|P_1 - P_0| = \frac{k^n}{1-k}|P_1 - P_0|$$

Error Analysis

① Newton's method

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

$$|P_{n+1} - p| = \frac{1}{2}|P_n - p|^2 \frac{f''(\xi_n)}{f'(P_n)}$$