Solutions to Exercises

Exercise 1.

(a)

```
>> M = [-3 -2 2 -2; -1 -3 1 -3; 1 -2 1 -2]

M =

    -3    -2     2    -2
    -1    -3     1    -3
     1    -2     1    -2

>> M([1 3],:) = M([3 1],:)

M =

     1    -2     1    -2
    -1    -3     1    -3
    -3    -2     2    -2

>> M(2,:) = M(2,:) + M(1,:)

M =

     1    -2     1    -2
     0    -5     2    -5
    -3    -2     2    -2

>> M(3,:) = M(3,:) + 3*M(1,:)

M =

     1    -2     1    -2
     0    -5     2    -5
     0    -8     5    -8

>> M(2,:) = M(2,:)/(-5)

M =

    1.0000   -2.0000    1.0000   -2.0000
         0    1.0000   -0.4000    1.0000
         0   -8.0000    5.0000   -8.0000

>> M(3,:) = M(3,:) + 8*M(2,:)

M =

    1.0000   -2.0000    1.0000   -2.0000
         0    1.0000   -0.4000    1.0000
         0         0    1.8000         0
```

(b)

```
>> M = [-2 -2 2 1; 1 0 5 -1; 3 2 3 -2]

M =

    -2    -2     2     1
     1     0     5    -1
     3     2     3    -2

>> M([1 2],:) = M([2 1],:)

M =

     1     0     5    -1
    -2    -2     2     1
     3     2     3    -2

>> M(2,:) = M(2,:) + 2*M(1,:)

M =

     1     0     5    -1
     0    -2    12    -1
     3     2     3    -2

>> M(3,:) = M(3,:) - 3*M(1,:)

M =

     1     0     5    -1
     0    -2    12    -1
     0     2   -12     1

>> M(3,:) = M(3,:) + M(2,:)

M =

     1     0     5    -1
     0    -2    12    -1
     0     0     0     0
```

Hence $-2y + 12z = -1$. Let $z = t$, then $y = 6t + 1/2$, and $x = -1 - 5z = -1 - 5t$.

(c)

```
>> M=[-1 0 3 1 2; 2 3 -3 1 2; 2 -2 -2 -1 -2]

M =

    -1     0     3     1     2
     2     3    -3     1     2
     2    -2    -2    -1    -2

>> M([2 3],:) = M([2 3],:) + 2*M([1 1],:)

M =

    -1     0     3     1     2
     0     3     3     3     6
     0    -2     4     1     2

>> M(3,:) = M(3,:) + M(2,:)*2/3

M =

    -1     0     3     1     2
     0     3     3     3     6
     0     0     6     3     6
```

Hence, from row 3, $2x_3 + x_4 = 2$, or $x_3 = 1 - x_4/2$. From row 2, $x_2 + x_3 + x_4 = 2$, or $x_2 = 1 - x_4/2$, and from row 1, $x_1 = -2 + 3x_3 + x_4$, or $x_1 = 1 - x_4/2$.

(d)

```
>> M = [3 -3 1 3 -3; 1 1 -1 -2 3; 4 -2 0 1 0]

M =

     3    -3     1     3    -3
     1     1    -1    -2     3
     4    -2     0     1     0

>> M([1 2],:) = M([2 1],:)

M =

     1     1    -1    -2     3
     3    -3     1     3    -3
     4    -2     0     1     0

>> M([2 3],:) = M([2 3],:) - diag([3 4])*M([1 1],:)

M =

     1     1    -1    -2     3
     0    -6     4     9   -12
     0    -6     4     9   -12

>> M(3,:) = M(3,:) - M(2,:)

M =

     1     1    -1    -2     3
     0    -6     4     9   -12
     0     0     0     0     0
```

Hence, from row 2, $x_2 = 2 + \frac{2}{3}x_3 + \frac{3}{2}x_4$ and from row 1, $x_1 = 3 - x_2 + x_3 + 2x_4 = 1 + \frac{1}{3}x_3 + \frac{1}{2}x_4$.

Exercise 1

Exercise 2.

(a)

```
>> syms a b c
>> M = [1 -2 4 a; 2 1 -1 b; 3 -1 3 c]

M =

[ 1, -2,  4, a]
[ 2,  1, -1, b]
[ 3, -1,  3, c]

>> M(2,:) = M(2,:) - 2*M(1,:)

M =

[ 1, -2,  4,       a]
[ 0,  5, -9, b - 2*a]
[ 3, -1,  3,       c]

>> M(3,:) = M(3,:) - 3*M(1,:)

M =

[ 1, -2,  4,       a]
[ 0,  5, -9, b - 2*a]
[ 0,  5, -9, c - 3*a]

>> M(3,:) = M(3,:) - M(2,:)

M =

[ 1, -2,  4,         a]
[ 0,  5, -9,   b - 2*a]
[ 0,  0,  0, c - b - a]
```

Thus, for the system to be consistent, we need $c - b - a = 0$.

(b)

```
>> syms a b c
>> M = [1 -1 2 a; 2 4 -3 b; 4 2 1 c]

M =

[ 1, -1,  2, a]
[ 2,  4, -3, b]
[ 4,  2,  1, c]

>> M(2,:) = M(2,:) - 2*M(1,:)

M =

[ 1, -1,  2,       a]
[ 0,  6, -7, b - 2*a]
[ 4,  2,  1,       c]

>> M(3,:) = M(3,:) - 4*M(1,:)

M =

[ 1, -1,  2,       a]
[ 0,  6, -7, b - 2*a]
[ 0,  6, -7, c - 4*a]

>> M(3,:) = M(3,:) - M(2,:)

M =

[ 1, -1,  2,           a]
[ 0,  6, -7,     b - 2*a]
[ 0,  0,  0, c - b - 2*a]
```

Thus, for the system to be consistent, we need $c - b - 2a = 0$.

Exercise 2

Exercise 4.

$$\left[\begin{array}{ccc|c} k & 1 & 1 & 0 \\ 1 & k & 1 & 0 \\ 1 & 1 & k & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & k & 0 \\ 1 & k & 1 & 0 \\ k & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & k & 0 \\ 0 & k-1 & 1-k & 0 \\ 0 & 1-k & 1-k^2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & k & 0 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & 1-k^2+1-k & 0 \end{array}\right]$$

- For unique solution, need $k^2 + k - 2 \neq 0, \Rightarrow k \neq 1$ and $k \neq -2$
- For one-parameter solution set, need $k^2 + k - 2 = 0$ and $|1 - k| \neq 0, \Rightarrow k = -2$.
- For two-parameter solution set, need $k^2 + k - 2 = 0$ and $|1 - k| = 0, \Rightarrow k = 1$.

Exercise 4

Exercise 5. Form the system and solve for a_0, a_1 and a_2.

$$\begin{aligned} 12 &= a_0 + a_1(1) + a_2(1)^2 \\ 15 &= a_0 + a_1(2) + a_2(2)^2 \\ 16 &= a_0 + a_1(3) + a_2(3)^2 \end{aligned}$$

Solving, we have

```
>> M = [1 1 1 12; 1 2 4 15; 1 3 9 16]

M =

     1     1     1    12
     1     2     4    15
     1     3     9    16

>> M(2,:) = M(2,:) - M(1,:)

M =

     1     1     1    12
     0     1     3     3
     1     3     9    16

>> M(3,:) = M(3,:) - M(1,:)

M =

     1     1     1    12
     0     1     3     3
     0     2     8     4

>> M(3,:) = M(3,:) - 2*M(2,:)

M =

     1     1     1    12
     0     1     3     3
     0     0     2    -2
```

and by back subsitution gives $a_2 = -1$, $a_1 = 6$ and $a_0 = 7$. Exercise 5

Exercise 6.

```
>> syms a b c
>> M = [-2 3 1 a; 1 1 -1 b; 0 5 -1 c]

M =

[ -2, 3,  1, a]
[  1, 1, -1, b]
[  0, 5, -1, c]

>> M([1 2],:) = M([2 1],:)

M =

[  1, 1, -1, b]
[ -2, 3,  1, a]
[  0, 5, -1, c]

>> M(2,:) = M(2,:) + 2*M(1,:)

M =

[ 1, 1, -1,       b]
[ 0, 5, -1, a + 2*b]
[ 0, 5, -1,       c]

>> M(3,:) = M(3,:) - M(2,:)

M =

[ 1, 1, -1,           b]
[ 0, 5, -1,     a + 2*b]
[ 0, 0,  0, c - 2*b - a]
```

(a) it is clear from above that, to be consistent, we need $c - 2b - a = 0$.

(b) thus, if $c - 2b - a \neq 0$, the system is not consistent.

(c) many solutions, since there are two equations with three unknows.

(d) for example, choose $a = b = c = 0$, then if the variables are denoted by x, y and z, then one solution is by setting $z = 1$, then $x = \frac{4}{5}$ and $y = \frac{1}{5}$.

Exercise 6

Exercise 7.

(a)

```
M =

    -2     2    -1     2
     0     3     3    -3
     1    -4     2     2

M =

    -2     2    -1     2
     0     3     3    -3
     1    -4     2     2

M =

   -2.0000    2.0000   -1.0000    2.0000
         0    3.0000    3.0000   -3.0000
         0   -3.0000    1.5000    3.0000

M =

   -2.0000    2.0000   -1.0000    2.0000
         0    3.0000    3.0000   -3.0000
         0         0    4.5000         0
```

```
M =

   -2.0000         0         0    4.0000
         0    3.0000         0   -3.0000
         0         0    4.5000         0

M =

    1.0000         0         0   -2.0000
         0    3.0000         0   -3.0000
         0         0    4.5000         0

M =

    1.0000         0         0   -2.0000
         0    1.0000         0   -1.0000
         0         0    4.5000         0

M =

     1     0     0    -2
     0     1     0    -1
     0     0     1     0
```

(b)

```
M =

     4    -3    -4    -2
    -4     2     1    -4
    -1    -3     1    -4

M =

     4    -3    -4    -2
     0    -1    -3    -6
    -1    -3     1    -4

M =

    4.0000   -3.0000   -4.0000   -2.0000
         0   -1.0000   -3.0000   -6.0000
         0   -3.7500         0   -4.5000

M =

    4.0000   -3.0000   -4.0000   -2.0000
         0   -1.0000   -3.0000   -6.0000
         0         0   11.2500   18.0000

M =

    4.0000   -3.0000   -4.0000   -2.0000
         0   -1.0000         0   -1.2000
         0         0   11.2500   18.0000
```

```
M =

    4.0000   -3.0000         0    4.4000
         0   -1.0000         0   -1.2000
         0         0   11.2500   18.0000

M =

    4.0000         0         0    8.0000
         0   -1.0000         0   -1.2000
         0         0   11.2500   18.0000

M =

    1.0000         0         0    2.0000
         0   -1.0000         0   -1.2000
         0         0   11.2500   18.0000

M =

    1.0000         0         0    2.0000
         0    1.0000         0    1.2000
         0         0   11.2500   18.0000

M =

    1.0000         0         0    2.0000
         0    1.0000         0    1.2000
         0         0    1.0000    1.6000
```

(c) The above were produced by the MATLAB script below:

```
% RRE via Gaussian Elimination
% Straight forward, No error checking
% No pivoting

% For EE2007
% 13 Aug 2016
% Ling KV


M = [-2 2 -1 2; 0 3 3 -3; 1 -4 2 2];

M = [4 -3 -4 -2; -4 2 1 -4; -1 -3 1 -4]

[N, zob] = size(M);


for j = 1:N-1
    for i=j+1:N, M(i,:) = M(i,:) - M(j,:)*M(i,j)/M(j,j), end
end

for j = N:-1:2
    for i=j-1:-1:1, M(i,:) = M(i,:) - M(j,:)*M(i,j)/M(j,j), end
end

for i=1:N
    M(i,:) = M(i,:)/M(i,i)
end
```

Exercise 7

Exercise 8. Straightforward.

```
>> A = [2 0 -1; 1 0 -2]

A =

     2     0    -1
     1     0    -2

>> B = [-3 1 1; -3 1 1]

B =

    -3     1     1
    -3     1     1

>> C = [3 -1; -1 -3]

C =

     3    -1
    -1    -3

>> 2*A'-B'

ans =

     7     5
    -1    -1
    -3    -5

>> B'-2*A
Error using  - 
Matrix dimensions must agree.

>> A*B'

ans =

    -7    -7
    -5    -5
```

```
>> B*A'

ans =

    -7    -5
    -7    -5

>> (A'+B')*C

ans =

    -1     7
     2    -4
     1     3

>> C*(A'+B')
Error using  * 
Inner matrix dimensions must agree.

>> (A'*C)*B

ans =

     0     0     0
     0     0     0
   -18     6     6

>> (A'*B')*C
Error using  * 
Inner matrix dimensions must agree.
```

Exercise 8

Exercise 10.

Short Answer Straightforward. See answer in the tutorial sheet.

Longer Answer If A is a $n \times n$ matrix, then we have, $\det(\alpha A) = \alpha^n \det(A)$. Also $\det(A^{-1}) = 1/\det(A)$. Thus

$$\begin{aligned}
\det(3A) &= 3^3 \det(A) = 3^3 \times 10 = 270. \\
\det(2A^{-1}) &= 2^3 \det(A^{-1}) = 2^3/\det(A) = 2^3/10 = 4/5. \\
\det((2A)^{-1}) &= 1/\det(2A) = 1/(2^3 \det(A)) = 1/80.
\end{aligned}$$

Exercise 10

Exercise 11.

$$\begin{aligned}
\det\begin{bmatrix} -6 & 4 & 5 \\ 2 & 8 & 2 \\ -1 & -4 & 2 \end{bmatrix} &= -\det\begin{bmatrix} -1 & -4 & 2 \\ 2 & 8 & 2 \\ -6 & 4 & 5 \end{bmatrix} \\
&= \det\begin{bmatrix} 1 & 4 & -2 \\ 2 & 8 & 2 \\ -6 & 4 & 5 \end{bmatrix} \\
&= \det\begin{bmatrix} 1 & 4 & -2 \\ 0 & 0 & 6 \\ 0 & 28 & -7 \end{bmatrix} \\
&= -\det\begin{bmatrix} 1 & 4 & -2 \\ 0 & 28 & -7 \\ 0 & 0 & 6 \end{bmatrix} \\
&= -168
\end{aligned}$$

Exercise 11

Exercise 12.

$$E : R_2 \leftarrow R_2 + R_1; \quad F : R_1 \leftarrow R_1 + R_2$$

EF means F, then E, giving

$$F : R_1 \leftarrow R_1 + R_2$$

$$E : R_2 \leftarrow R_2 + R_1 = R_2 + R_1 + R_2$$

FE means E, then F, giving

$$E : R_2 \leftarrow R_2 + R_1$$

$$F : R_1 \leftarrow R_1 + R_2 = R_1 + R_2 + R_1$$

Hence $EF \neq FE$. Exercise 12

Exercise 14.

(a) $E = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad |E| = -1, \quad E^{-1} = E$

(b) $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad |E| = 5, \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $E = \begin{bmatrix} 1 & 8 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad |E| = 1, \quad E^{-1} = \begin{bmatrix} 1 & -8 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Exercise 14

Exercise 15.

$$\begin{bmatrix} -6 & 4 & 5 \\ 2 & 8 & 2 \\ -1 & -4 & 2 \end{bmatrix} \underset{\sim}{E_1:R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 8 & 2 \\ -6 & 4 & 5 \\ -1 & -4 & 2 \end{bmatrix}$$
$$\underset{\sim}{E_2:R_1 \leftarrow 0.5*R_1} \begin{bmatrix} 1 & 4 & 1 \\ -6 & 4 & 5 \\ -1 & -4 & 2 \end{bmatrix}$$
$$\underset{\sim}{E_3:R_2 \leftarrow R_2+6R_1} \begin{bmatrix} 1 & 4 & 1 \\ 0 & 28 & 11 \\ -1 & -4 & 2 \end{bmatrix}$$
$$\underset{\sim}{E_4:R_3 \leftarrow R_3+R_1} \begin{bmatrix} 1 & 4 & 1 \\ 0 & 28 & 11 \\ 0 & 0 & 3 \end{bmatrix}$$

Hence

$$\begin{bmatrix} 1 & 4 & 1 \\ 0 & 28 & 11 \\ 0 & 0 & 3 \end{bmatrix} = E_4 E_3 E_2 E_1 \begin{bmatrix} -6 & 4 & 5 \\ 2 & 8 & 2 \\ -1 & -4 & 2 \end{bmatrix}$$

$$\Rightarrow \det\left(\begin{bmatrix} 1 & 4 & 1 \\ 0 & 28 & 11 \\ 0 & 0 & 3 \end{bmatrix}\right) = \det(E_4)\det(E_3)\det(E_2)\det(E_1)\det\left(\begin{bmatrix} -6 & 4 & 5 \\ 2 & 8 & 2 \\ -1 & -4 & 2 \end{bmatrix}\right)$$

$$\Rightarrow \det\left(\begin{bmatrix} -6 & 4 & 5 \\ 2 & 8 & 2 \\ -1 & -4 & 2 \end{bmatrix}\right) = \frac{\det\left(\begin{bmatrix} 1 & 4 & 1 \\ 0 & 28 & 11 \\ 0 & 0 & 3 \end{bmatrix}\right)}{\det(E_4)\det(E_3)\det(E_2)\det(E_1)} = \frac{28*3}{(1)(1)(0.5)(-1)} = -168$$

Exercise 15

Exercise 16.

(a) Matrix A is pre- and post-multiplied by elementary matrices. With $n = 1$, the post-mulitplication matrix interchanges columns 1 and 3, while the pre-multiplication matrix interchanges rows 1 and 3. If we first perform the post-multiplication, then we get

$$\begin{bmatrix} a_{13} & a_{12} & a_{11} \\ a_{23} & a_{22} & a_{21} \\ a_{33} & a_{32} & a_{31} \end{bmatrix},$$

pre-multiplication gives the end result as

$$B = \begin{bmatrix} a_{33} & a_{32} & a_{31} \\ a_{23} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{bmatrix}$$

Alternatively, pre-multiplication first gives

$$\begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix},$$

followed by post-multiplication giving the same end result

$$B = \begin{bmatrix} a_{33} & a_{32} & a_{31} \\ a_{23} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{bmatrix}$$

(b) With n odd, e.g., when $n = 3$, rows/columns 1 and 3 will be interchanged three times, and this is the same as interchanging rows/columns 1 and 3 once. Hence, the answer is same as Part (i).

$$B = \begin{bmatrix} a_{33} & a_{32} & a_{31} \\ a_{23} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{bmatrix}$$

Pre-multiplicaton change the sign of determinant, and post-multiplication again change the sign of the determinant. Hence the determinant of $B = |A| = 5$.

(c) With n even, e.g., when $n = 2$, rows/columns 1 and 3 will be interchanged twice and no change will be made to matrix A. Hence $B = A$, and $|B| = |A| = 5$.

Exercise 16

Exercise 17.

Short Answer As explained in the answer in the tutorial sheet.

Longer Answer By expanding along the first row

$$\begin{aligned} \det\begin{vmatrix} I_n & 0 \\ 0 & B \end{vmatrix} &= 1\det\begin{vmatrix} I_{n-1} & 0 \\ 0 & B \end{vmatrix} \\ &\vdots \\ &= \underbrace{1\times 1\times\ldots 1}_{n\,\text{times}}\det(B) = \det(B) \end{aligned}$$

Similarly, by expanding along the last row

$$\det\begin{vmatrix} A & C \\ 0 & I_m \end{vmatrix} = 1\det\begin{vmatrix} A & C_{m-1} \\ 0 & I_{m-1} \end{vmatrix} = \cdots = \det(A)$$

Since

$$P = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} A & C \\ 0 & I_m \end{bmatrix}$$

and we have $\det(XY) = \det(X)\det(Y)$. Thus

$$\det(P) = \det(\begin{bmatrix} I_n & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} A & C \\ 0 & I_m \end{bmatrix}) = \det(B)\det(A) = \det(A)\det(B)$$

Exercise 17

Exercise 18.

(a) $U = E_2E_1A \Rightarrow A = (E_2E_1)^{-1}U = LU$ where $L = E_1^{-1}E_2^{-1}$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ -1/2 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

By inspection Since inverse is an UNDO operation

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ -1/2 & 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Hence

$$L = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & & & \\ 1/2 & 1 & & \\ 1/2 & 0 & 1 & \\ -1/2 & -1 & 0 & 1 \end{bmatrix}$$

(b) $U = E_2E_1A \Rightarrow A = (E_2E_1)^{-1}U = LU$ where $L = E_1^{-1}E_2^{-1}$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

By inspection

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ -1/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Hence

$$L = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & & & \\ 1/2 & 1 & & \\ -1/2 & -1 & 1 & \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Exercise 18

Exercise 19.

(a)

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \overset{R_1 \leftarrow a^{-1}R_1}{\sim} \left[\begin{array}{cc|cc} 1 & a^{-1}b & a^{-1} & 0 \\ c & d & 0 & 1 \end{array}\right]$$
$$\overset{R_2 \leftarrow R_2 - cR_1}{\sim} \left[\begin{array}{cc|cc} 1 & a^{-1}b & a^{-1} & 0 \\ 0 & \underbrace{d - ca^{-1}b}_{\Delta} & -ca^{-1} & 1 \end{array}\right] \overset{R_2 \leftarrow \Delta^{-1}R_2}{\sim} \left[\begin{array}{cc|cc} 1 & a^{-1}b & a^{-1} & 0 \\ 0 & 1 & -\Delta^{-1}ca^{-1} & \Delta^{-1} \end{array}\right]$$
$$\overset{R_1 \leftarrow R_1 - a^{-1}bR_2}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & a^{-1} + a^{-1}b\Delta^{-1}ca^{-1} & -a^{-1}b\Delta^{-1} \\ 0 & 1 & -\Delta^{-1}ca^{-1} & \Delta^{-1} \end{array}\right]$$

Hence, if $a \neq 0$ and $\Delta \neq 0$, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} + a^{-1}b\Delta^{-1}ca^{-1} & -a^{-1}b\Delta^{-1} \\ -\Delta^{-1}ca^{-1} & \Delta^{-1} \end{bmatrix} = a^{-1}\Delta^{-1} \begin{bmatrix} \Delta + bca^{-1} & -b \\ -c & a \end{bmatrix}$$
$$= \quad \ldots = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and the formula works even if $a = 0$ so long as $ad - bc \neq 0$.

(b) For block partition matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, the steps are essentially the same, except that one has to be careful with multiplication of matrices, i.e. $AB \neq BA$ in general.

$$\left[\begin{array}{cc|cc} A & B & I & 0 \\ C & D & 0 & I \end{array}\right] \overset{R_1 \leftarrow A^{-1}R_1}{\sim} \left[\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ C & D & 0 & I \end{array}\right]$$
$$\overset{R_2 \leftarrow R_2 - CR_1}{\sim} \left[\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ 0 & \underbrace{D - CA^{-1}B}_{\Delta} & -CA^{-1} & I \end{array}\right] \overset{R_2 \leftarrow \Delta^{-1}R_2}{\sim} \left[\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ 0 & I & -\Delta^{-1}CA^{-1} & \Delta^{-1} \end{array}\right]$$
$$\overset{R_1 \leftarrow R_1 - A^{-1}bR_2}{\sim} \left[\begin{array}{cc|cc} I & 0 & A^{-1} + A^{-1}B\Delta^{-1}CA^{-1} & -A^{-1}B\Delta^{-1} \\ 0 & I & -\Delta^{-1}CA^{-1} & \Delta^{-1} \end{array}\right]$$

Hence, if A^{-1} and Δ^{-1} exist, we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B\Delta^{-1}CA^{-1} & -A^{-1}B\Delta^{-1} \\ -\Delta^{-1}CA^{-1} & \Delta^{-1} \end{bmatrix}$$

Exercise 19

Exercise 20. Please go through with the class the relevant pages in the lecture notes.

Exercise 20

Exercise 21.

(a) $(-1)\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} + (3)\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$

(b) Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$.

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix}$$

$$A\mathbf{b}_1 = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3\mathbf{a}_1 + 2\mathbf{a}_2$$

$$A\mathbf{b}_2 = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2\mathbf{a}_1 + 5\mathbf{a}_2$$

(c) All linear combinations of the three vectors can be represented as

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 7 \\ -2 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

In order that the above has a solution (i.e., one can solve for c_1, c_2 and c_3), we need the condition: $3a - b + c = 0$. In other words, the linear combinations of the three vectors can only form vector $\begin{bmatrix} a & b & c \end{bmatrix}^T$ where a, b, c satisfy $3a - b + c$.

Exercise 21

Exercise 22.
Short Answer

(a) Yes, since

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$$

(b) Yes, since

$$\begin{bmatrix} 4 & 5 \end{bmatrix} = 4 \begin{bmatrix} 1 & -1 \end{bmatrix} + 9 \begin{bmatrix} 0 & 1 \end{bmatrix} + 0 \begin{bmatrix} 3 & -3 \end{bmatrix}$$

(c) As explain in the answer in the tutorial sheet.

Longer Answer
Row space of A is the set of vectors generated by taking linear combinations of the rows of A:

$$\begin{bmatrix} x & y \end{bmatrix} = c_1 \begin{bmatrix} 1 & -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 3 & -3 \end{bmatrix}$$

Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & x \\ -1 & 1 & -3 & y \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & x \\ 0 & 1 & 0 & x+y \end{array}\right]$$

always has solution for any x, y. Therefore $row(A) = \mathcal{R}^2$.

Similarly, for column space of A, repeat the above steps, now looking at columns of A instead.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 1 & y \\ 3 & -3 & z \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 1 & y \\ 0 & 0 & z-3x \end{array}\right]$$

Thus, need $z - 3x = 0$ so that the system has solution. Exercise 22

Exercise 23.

(a)

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0$$

$\Rightarrow$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0$$

$\Rightarrow$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0.$
Hence linearly independent.

(b)

$$\alpha_1 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = 0$$

$\Rightarrow$

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0$$

$\Rightarrow$

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -2 \\ 0 & 3 & -3 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0.$
Hence linearly independent.

(c)

$$\alpha_1 p(x) + \alpha_2 q(x) + \alpha_3 h(x) = 0$$

$$\alpha_1(1 + x) + \alpha_2(1 - x) + \alpha_3(1 - x^2) = 0 + 0x + 0x^2$$

$\Rightarrow$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0$$

$\Rightarrow$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0.$
Hence linearly independent.

Exercise 23

Exercise 24.

(a)

$$\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

$\Rightarrow$

$$\begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & 4 & 5 \\ 3 & 0 & -3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 6 & 6 \\ 0 & -6 & -6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 6 & 6 \\ 0 & 0 & 0 \end{array}\right]$$

Unique solution, hence spanned.

(b)

$$2 - 3x + x^2 = \alpha_1(1 + x) + \alpha_2(1 + x^2)$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$\Rightarrow$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -1 & -5 \\ 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -1 & -5 \\ 0 & 0 & -4 \end{array}\right]$$

There is no solution, hence not spanned.

Exercise 24

Exercise 25. By definition, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are LI iff the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$$

has $c_1 = c_2 = c_3 = 0$ as the only solution.

Re-writing the above equation in terms of $\mathbf{w}$, we have

$$\begin{aligned} & c_1(\mathbf{w}_2 + \mathbf{w}_3) + c_2(\mathbf{w}_1 + \mathbf{w}_3) + c_3(\mathbf{w}_1 + \mathbf{w}_2) = 0 \\ \Rightarrow \quad & (c_2 + c_3)\mathbf{w}_1 + (c_1 + c_3)\mathbf{w_2} + (c_1 + c_2)\mathbf{w}_3 = 0 \end{aligned}$$

Since $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are LI, the above implies that

$$c_2 + c_3 = 0, c_1 + c_3 = 0, c_1 + c_2 = 0$$

or equivalently in matrix notation

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

Solving, we get $c_1 = c_2 = c_3 = 0$, hence proved.

Exercise 25

Exercise 26. If A is a $m \times n$ matrix, then $\text{rank}(A) + \text{nullity}(A) = n$. Since A is 3×5, the rank(A) can take values of 1,2 or 3, and hence the possible values for $\text{nullity}(A) = 4, 3$ or 2 respectively. The $rank(A)$ can at most be 3, and hence there can be at most 3 independent columns. Thus the columns of A must be dependent. Exercise 26

Exercise 27. See proof in lecture notes: $AP = PD$ where D is a diagonal matrix containing the eigenvalues of matrix A, and P is a matrix containing the corresponding eigenvectors of A. Hence A is diagonalisable, i.e. $D = P^{-1}AP$ if P is invertible, which implies that the eigenvectors of A need to be linearly independent. Exercise 27

Exercise 28.

(a)

$$|\lambda I - A| = \begin{vmatrix} \lambda & 0 & -2 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda + 1 \end{vmatrix} = 0 \quad \Rightarrow \lambda = 0, 2, -1$$

(b) No, because we don't know yet whether the eigenvectors are linearly independent. Yes, if the students know the fact that since the eigenvalues are distinct, the eigenvectors will be linearly independent; but this fact is not covered in the class.

(c) Straightforward to compute the eigenvectors. For example, set $\lambda_1 = 0$, and solve for $\mathbf{v}_1$:

$$(\lambda_1 I - A)\mathbf{v}_1 = 0, \Rightarrow \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0, \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

Thus,

$$x_2 = x_3 = 0, \text{ and } x_1 \text{ arbitrary. Hence } v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly, set $\lambda_2 = 2$, and solve for $\mathbf{v}_2$:

$$(\lambda_2 I - A)\mathbf{v}_2 = 0, \Rightarrow \begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0, \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

Thus,

$$x_1 = x_3 = 0, \text{ and } x_2 \text{ arbitrary. Hence } v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Finally, set $\lambda_3 = -1$, and solve for $\mathbf{v}_3$:

$$(\lambda_3 I - A)\mathbf{v}_3 = 0, \Rightarrow \begin{bmatrix} -1 & 0 & -2 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0, \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

Thus,

$$x_2 = 0, \text{ and } x_1 = -2x_3. \text{ Hence } v_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

(d) To check whether the eigenvectors are linearly independent, we can form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$$

and solve for c_1, c_2 and c_3. If $c_1 = c_2 = c_3 = 0$ is the only solution, then the eigenvectors are linearly independent; otherwise, not.

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0, \Rightarrow c_1 = c_2 = c_3 = 0, \text{ hence the eigenvectors are LI.}$$

(e) Yes, because the eigenvectors are LI, and hence the matrix $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ is invertible, so one can write $D = P^{-1}AP$, where D is a diagonal matrix containing the eigenvalues of A.

(f) $P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & & \\ & 2 & \\ & & -1 \end{bmatrix}$

Exercise 28

Exercise 29.

(a)

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 & -1 & 0 \\ -1 & \lambda-1 & -1 & 0 \\ 0 & 0 & \lambda & 0 \\ -1 & 0 & -1 & \lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda^2(\lambda-1)^2 = 0, \quad \Rightarrow \lambda = 0, 0, 1, 1$$

(b) No, because we don't know yet whether the eigenvectors are linearly independent.

(c) Straightforward to compute the eigenvectors. For example, set $\lambda_1 = 0$, and solve for $\mathbf{v}_1$:

$$(\lambda_1 I - A)\mathbf{v}_1 = 0, \Rightarrow \begin{bmatrix} -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & - & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0, \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.$$

Thus,

$$x_1 = -x_3, \text{ let } x_3 = \alpha, x_2 = -x_1 - x_3 = 0, \text{ and } x_4 arbitrary, let x_4 = \beta.$$

Hence, $v_1 = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, i.e., there are two LI eigenvectors for $\lambda_1 = 0$.

Similarly, set $\lambda_2 = 1$, and solve for $\mathbf{v}_2$:

$$(\lambda_2 I - A)\mathbf{v}_2 = 0, \Rightarrow \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0, \Rightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.$$

Thus, $x_1 = x_3 = x_4 = 0$, and x_2 arbitrary. Hence $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

(d) There is a repeated eigenvectors for $\lambda = 1$. So the eigenvectors are not LI.

(e) Yes, A is not diagonalisable because its eigenvectors are not LI.

Exercise 29

Exercise 30.

(a) Since

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \end{bmatrix} = (a+b) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

hence, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with corresponding eigenvalue $a+b$.

(b) $\begin{bmatrix} a & b \\ b & a \end{bmatrix} 1 - 1 = (a-b) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, hence $(a-b)$ is another eigenvalue of A and the corresponding eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(c) A can be diagonalised by $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, i.e. $P^{-1}AP = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}$.

Exercise 30

Exercise 31. From the information given, we can write

$$B\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -2\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \text{ and } B\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 5\begin{bmatrix} 3 \\ 4 \end{bmatrix},$$

or, equivalently

$$B\begin{bmatrix} -1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$$

Hence

$$B = \begin{bmatrix} -1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

In general, we can write

$$BP = PD$$

where D is the diagonal matrix containing the eigenvalues of B, and the columns of P contains the corresponding eigenvectors.

If P is invertible, and $P = \alpha P^{-T}$ where α is a non-zero scalar, then we have

$$\begin{aligned} B &= PDP^{-1} \\ &= (\alpha P^{-T})D(\alpha P^{-T})^{-1} \\ &= \alpha P^{-T}D\alpha^{-1}P^{T} \\ &= P^{-T}DP^{T} \\ &= (PDP^{-1})^{T} \\ &= B^{T} \end{aligned}$$

Thus, if P is invertible and $P = \alpha P^{-T}$ where α is a non-zero scalar, then we can conclude that $B = B^T$, i.e., B is symmetric, without needing to compute B.

Exercise 31

Exercise 32. First, let's investigate how the eigenvalues of P are related to the eigenvalues of A and B.

$$\begin{vmatrix} (\lambda I - A) & -C \\ 0 & (\lambda I - B) \end{vmatrix} = 0 \quad \Rightarrow |\lambda I - A| \cdot |\lambda I - B| = 0$$

Thus, we conclude that eigenvalues of P are the eigenvalues of A and B, i.e.,

$$eig(P) = eig(A) \cup eig(B).$$

Next, let's investigate how the eigenvectors are related.

Let $\mathbf{v_A}$ and $\mathbf{v_B}$ be eigenvectors of A and B respectively, i.e., $A\mathbf{v_A} = \lambda_A \mathbf{v_A}$ and $B\mathbf{v_B} = \lambda_B \mathbf{v_B}$, or equivalently, $(\lambda_A I - A)\mathbf{v_A} = 0$ and $(\lambda_B I - B)\mathbf{v_B} = 0$. Let also $\mathbf{v_P}$ be the eigenvectors of P and we partition $\mathbf{v_P} = \begin{bmatrix} \mathbf{v_1} \\ \mathbf{v_2} \end{bmatrix}$ so that the following matrix-vector multiplications make sense.

For simplicity, assume that A and B do not share the same eigenvalues.

(a) Let's find the eigenvectors of P correspond to eigenvalues λ_A, i.e., for what $\mathbf{v_p}$ so that $P\mathbf{v_P} = \lambda_A \mathbf{v_P}$?

$$P\mathbf{v_P} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} \mathbf{v_1} \\ \mathbf{v_2} \end{bmatrix} = \begin{bmatrix} A\mathbf{v_1} + C\mathbf{v_2} \\ B\mathbf{v_2} \end{bmatrix} = \begin{bmatrix} \lambda_A \mathbf{v_1} \\ \lambda_A \mathbf{v_2} \end{bmatrix}$$

From the above, $B\mathbf{v_2} = \lambda_A \mathbf{v_2} \Rightarrow \mathbf{v_2} = 0$. Substituting $\mathbf{v_2} = 0$ into $A\mathbf{v_1} + C\mathbf{v_2} = \lambda_A \mathbf{v_1}$, we have $A\mathbf{v_1} = \lambda_A \mathbf{v_1} \Rightarrow \mathbf{v_1} = \mathbf{v_A}$.

Thus we conclude that for eigenvalues λ_A, the corresponding eigenvector of P are

$$\mathbf{v_P} = \begin{bmatrix} \mathbf{v_A} \\ 0 \end{bmatrix}.$$

(b) Similarly, let's now find the eigenvectors of P correspond to eigenvalues of λ_B, i.e, for what $\mathbf{v_P}$ so that $P\mathbf{v_P} = \lambda_B \mathbf{v_P}$?

$$P\mathbf{v_P} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} \mathbf{v_1} \\ \mathbf{v_2} \end{bmatrix} = \begin{bmatrix} A\mathbf{v_1} + C\mathbf{v_2} \\ B\mathbf{v_2} \end{bmatrix} = \begin{bmatrix} \lambda_B \mathbf{v_1} \\ \lambda_B \mathbf{v_2} \end{bmatrix}$$

From the above, $B\mathbf{v_2} = \lambda_B \mathbf{v_2} \Rightarrow \mathbf{v_2} = \mathbf{v_B}$. Substituting $\mathbf{v_2} = \mathbf{v_B}$ into $A\mathbf{v_1} + C\mathbf{v_2} = \lambda_B \mathbf{v_1}$, we have

$$A\mathbf{v_1} + C\mathbf{v_B} = \lambda_B \mathbf{v_1} \Rightarrow \mathbf{v_1} = (\lambda_B I - A)^{-1} C\mathbf{v_B}, \quad \text{assuming the inverse exists.}$$

Thus we conclude that for eigenvalues λ_B, the corresponding eigenvector of P are

$$\mathbf{v_P} = \begin{bmatrix} (\lambda_B I - A)^{-1} C\mathbf{v_B} \\ \mathbf{v_B} \end{bmatrix}.$$

The following MATLAB code demonstrate the above claims

```
% eigenvalues and eigenvectors of
% block diagonal matrices

% Ling KV, 16 Aug 2013

clear all
m=2; n=3; %choose the dimensions of A and B matrices

A=rand(m,m);
B=rand(n,n);
C=rand(m,n);
P=[A C; zeros(n,m) B];

% this line shows that
% eig(P) = eig(A) union eig(B)
[sort(eig(P))' ;
 sort([eig(A); eig(B)])']

[Va,Da]=eig(A);
[Vb,Db]=eig(B);
[Vp,Dp]=eig(P);

% this line shows that P has eigenvector
% [Va; zeros(n,1)]
[Vp(:,1:m) [Va; zeros(n,m)]]

for i=1:n
  x(:,i) = [inv(Db(i,i)*eye(m,m)-A)*C*Vb(:,i); Vb(:,i)];
  x(:,i) = x(:,i)/x(1,i);
  Vp_normalised(:,i) = Vp(:,m+i)/Vp(1,m+i);
end

% this line show that P has eigenvector
% [ inv(Db*eye(m,m)-A)*C*Vb ; Vb]
[Vp_normalised x]
```

Exercise 32

Exercise 33. Let $A = \begin{bmatrix} -\frac{5}{50} & \frac{5}{100} \\ \frac{5}{50} & -\frac{5}{100} \end{bmatrix}$ giving eigenvalues and corresponding eigenvectors

$$\lambda_1 = -\frac{3}{20}, \mathbf{v_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \lambda_2 = 0, \mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Then original system can be de-coupled and solved as follows:

$$\begin{aligned} \dot{\mathbf{y}} &= A\mathbf{y} = PDP^{-1}\mathbf{y} \\ \Rightarrow \quad P^{-1}\dot{\mathbf{y}} &= DP^{-1}\mathbf{y} \\ \Rightarrow \quad \dot{\mathbf{w}} &= D\mathbf{w} \text{ where } \mathbf{w} = P^{-1}\mathbf{y} \\ \Rightarrow \quad \mathbf{w} &= e^{Dt}\mathbf{w}(0) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix} \end{aligned}$$

where we have let $\mathbf{w}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Thus

$$\mathbf{y}(t) = P\mathbf{w}(t) = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

Given the initial conditions $y_1(0) = 8$ and $y_2(0) = 0$, we can solve for c_1 and c_2 as follows:

$$\mathbf{y}(0) = \begin{bmatrix} 8 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

giving $c_1 = -16/3$ and $c_2 = 8/3$.

In conclusion

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{-16}{3} e^{-\frac{3}{20}t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{8}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Exercise 33

Exercise 34. The matrix T has eigenvalues and eigenvectors

$$\lambda_1 = 1, \mathbf{v_1} = \begin{bmatrix} q/p \\ 1 \end{bmatrix}, \quad \lambda_2 = 1 - p - q, \mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Hence $T = PDP^{-1}$, and $T^n = PD^nP^{-1}$ where

$$P = \begin{bmatrix} q/p & -1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Working out the algebra gives

$$T^n = \frac{p}{p+q} \begin{bmatrix} \frac{q}{p} + \lambda_2^n & \frac{q}{p}(1 - \lambda_2^n) \\ 1 - \lambda_2^n & 1 + \lambda_2^n \frac{q}{p} \end{bmatrix}$$

Let $\mathbf{x}$ be the steady state vector. Then

$$\mathbf{x} = A\mathbf{x}$$

Thus, the steady state vector is the eigenvector corresponding to $\lambda = 1$.

Exercise 34

Exercise 35.

$$\begin{bmatrix} \text{Plan A} \\ \text{Plan B} \\ \text{Plan C} \end{bmatrix} : \mathbf{x}_{k+1} = Ax_k, \text{ where } A = \begin{bmatrix} 0.75 & 0.25 & 0.2 \\ 0.15 & 0.45 & 0.4 \\ 0.1 & 0.3 & 0.4 \end{bmatrix}$$

Let $\mathbf{v}$ be the long term percentage enrollment of the plans, then

$$\mathbf{v} = A\mathbf{v} \Rightarrow (I - A)\mathbf{v} = 0$$

In other words, $\mathbf{v}$ is the eigenvector of A corresponding with eigenvalues 1,

$\mathbf{v}$ can be found as follows

$$I - A = \begin{bmatrix} 0.25 & -0.25 & -0.2 \\ -0.15 & 0.55 & -0.4 \\ -0.1 & -0.3 & 0.6 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \quad & \begin{bmatrix} 0.25 & -0.25 & -0.2 \\ -0.15 & 0.55 & -0.4 \\ -0.1 & -0.3 & 0.6 \end{bmatrix} \sim \begin{bmatrix} 25 & -25 & -20 \\ -15 & 55 & -40 \\ -10 & -30 & 60 \end{bmatrix} \sim \begin{bmatrix} 5 & -5 & -5 \\ 3 & -11 & 8 \\ 2 & 6 & 12 \end{bmatrix} \\ \sim & \begin{bmatrix} 30 & -30 & -24 \\ 30 & -110 & 80 \\ 30 & 90 & -180 \end{bmatrix} \sim \begin{bmatrix} 30 & -30 & -24 \\ 0 & -80 & 104 \\ 0 & 120 & -156 \end{bmatrix} \sim \begin{bmatrix} 5 & -5 & -4 \\ 0 & -20 & 26 \\ 0 & 20 & -26 \end{bmatrix} \\ \sim & \begin{bmatrix} 20 & -20 & -16 \\ 0 & -20 & 26 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 20 & 0 & -42 \\ 0 & -20 & 26 \\ 0 & 0 & 0 \end{bmatrix} \sim \end{aligned}$$

Thus

$$\mathbf{v} = \begin{bmatrix} 2.1 \\ 1.3 \\ 1.0 \end{bmatrix} \frac{1}{2.1 + 1.3 + 1.0} \times 100\% = \begin{bmatrix} 47.73\% \\ 29.54\% \\ 22.73\% \end{bmatrix}$$

Exercise 35