EE2001
Circuit Analysis

First-Order Circuits

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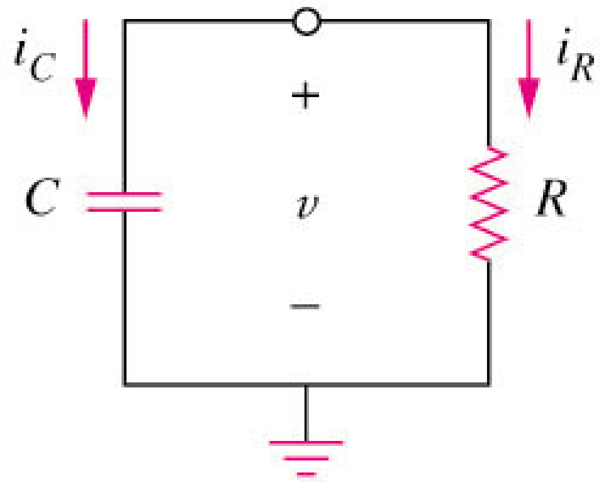
First-Order Circuits

1 The Source-Free RC Circuit
2 The Source-Free RL Circuit
3 Unit-step Function
4 Step Response of an RC Circuit
5 Step Response of an RL Circuit

The Source-Free RC Circuit

- A **first-order circuit** is characterized by a first-order differential equation.



By KCL

$$i_R + i_C = 0 \rightarrow \frac{v}{R} + C\frac{dv}{dt} = 0$$

Ohms law

Capacitor law

- Apply Kirchhoff's laws to <u>purely resistive circuit</u> results in <u>algebraic equations</u>.
- Apply the laws to RC and RL circuits produces <u>differential equations</u>.

- The **natural response** of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation. We can solve for the *natural response* with the *initial condition* $v(0)=V_0$

$$\frac{dv}{dt}+\frac{1}{RC}v=0 \Rightarrow \frac{dv}{v}=-\frac{1}{RC}dt \Rightarrow \ln|v|=-\frac{1}{RC}t+K_1$$

$$\Rightarrow v = e^{-\frac{1}{RC}t+K_1}=K\,e^{-\frac{1}{RC}t}$$

Using $v(0)=V_0$, $K=V_0$

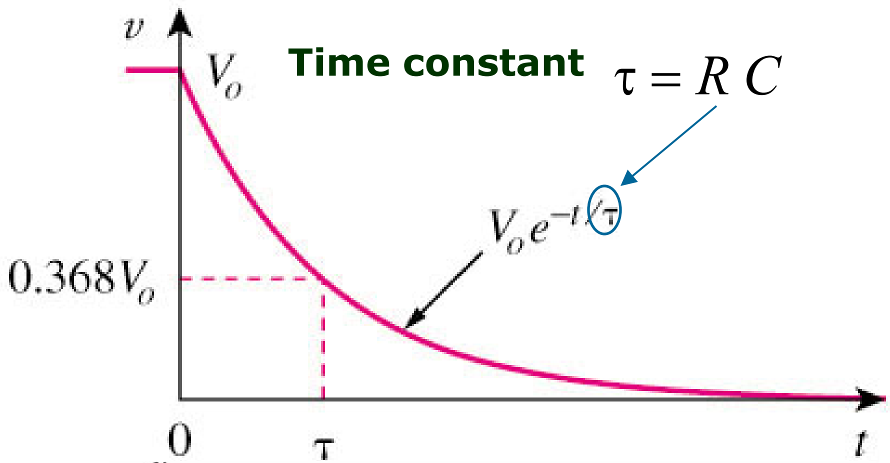
$$v=V_0\,e^{-\frac{1}{RC}t}$$

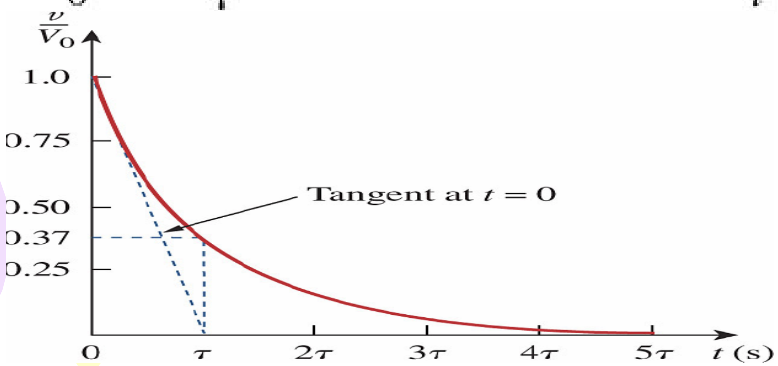
Define $\tau=RC$

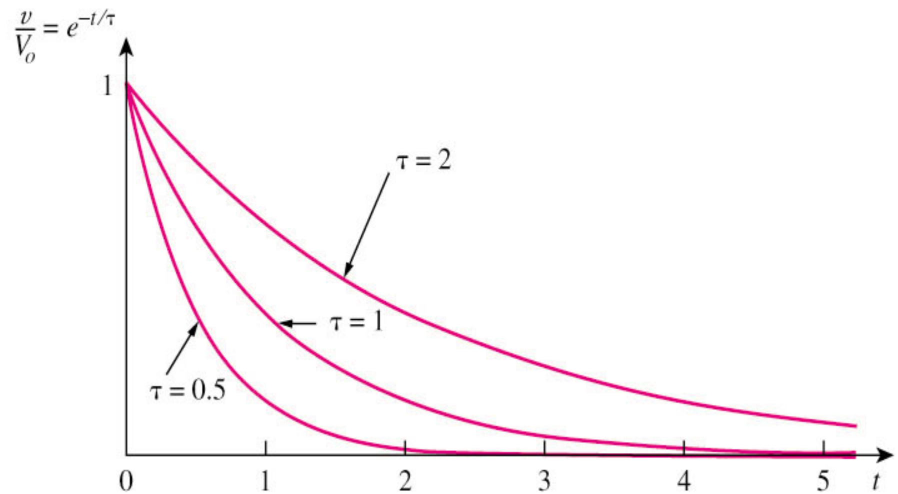
τ is called the time constant.

$$\boxed{v(t)=V_0\,e^{-t/\tau}}$$

- The **time constant** τ of a circuit is the time required for the response to decay by a factor of 1/*e* or 36.8% of its initial value.
- v decays faster for small τ and slower for large τ.



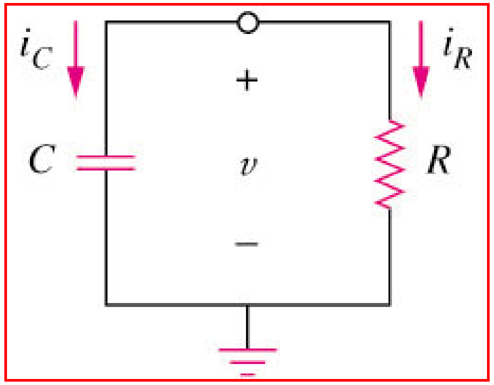




Summary: Calculation of the natural response of an RC circuit

The key to working with a source-free RC circuit is finding:

1. The initial voltage $v(0) = V_0$ across the capacitor.
2. The time constant τ = RC.

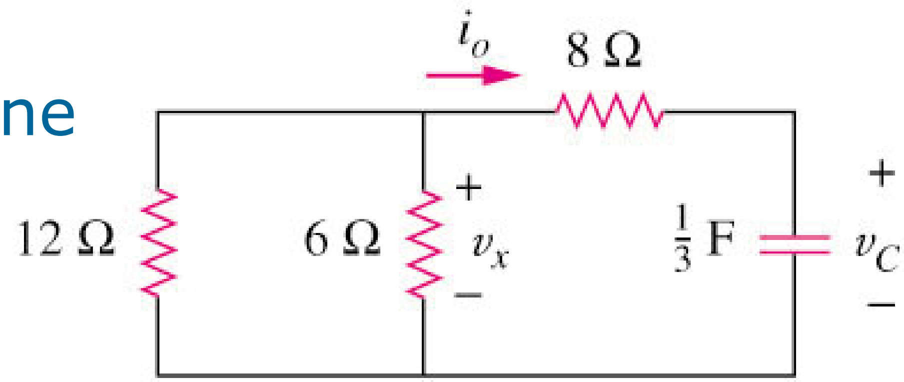


$$v(t) = V_0 \, e^{-t/\tau}$$ where $$\tau = R\,C$$

Example 1

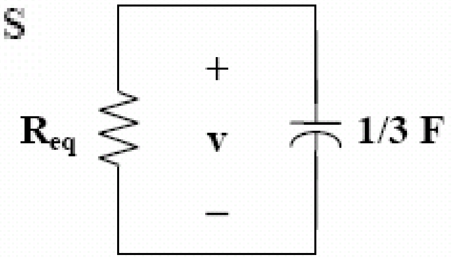
Refer to the circuit, determine v_C, v_x, and i_o for $t \geq 0$.

Assume that $v_C(0) = 45$ V.



The circuit is equivalent to

$$R_{eq} = 8 + 12 \,||\, 6 = 12\,\Omega \qquad \tau = R_{eq}C = (12)(1/3) = 4\text{ s}$$



$$v_c = v_c(0)\,e^{-t/\tau} = 45\,e^{-t/4} = \underline{\mathbf{45e^{-0.25t}\ V}}$$

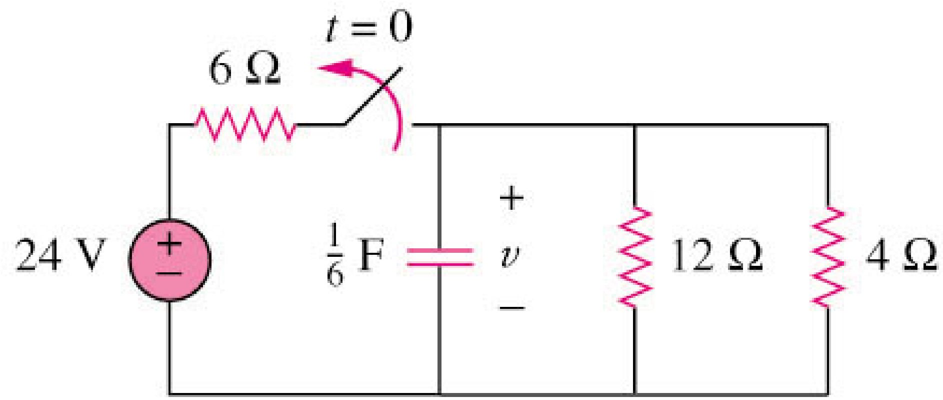
$$v_x = \frac{4}{4+8}v_c \quad = \underline{\mathbf{15e^{-0.25t}\ V}}$$

$$v_x = v_o + v_c \quad \longrightarrow \quad v_o = v_x - v_c = -30\,e^{-0.25t}\ \text{V}$$

$$i_o = \frac{v_o}{8} = \underline{\mathbf{-3.75\,e^{-0.25t}\ A}}$$

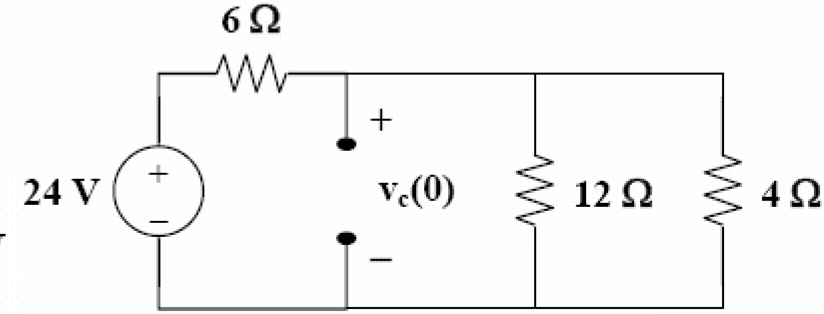
Example 2

The switch in circuit is opened at t = 0, find v(t) for t ≥ 0.



When t < 0, the switch is closed as shown in Fig. (a).

$$R_{eq} = 4 \,||\, 12 = 3\ \Omega \qquad v_c(0) = \frac{3}{3+6}(24) = 8\ \text{V}$$

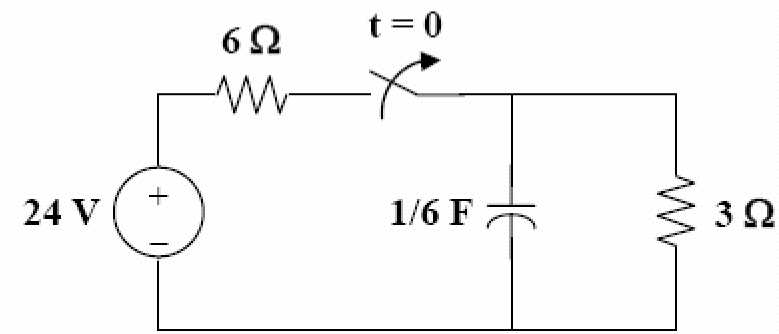


(a)

When t > 0, the switch is open as shown in Fig. (b).

$$\tau = R_{eq}C = (3)(1/6) = 1/2\ \text{s}$$

$$v(t) = v_c(0)e^{-t/\tau} = \underline{\mathbf{8e^{-2t}\ V}}$$



(b)

Example 3

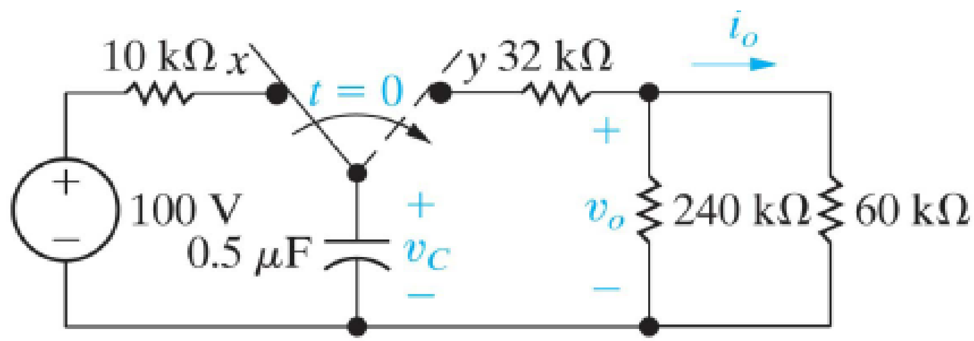
The switch in the circuit shown has been in position x for a long time. At $t = 0$, the switch moves instantaneously to position y. Find

(a) $v_C(t)$ for $t \geq 0$

(b) $v_o(t)$ for $t \geq 0^+$

(c) $i_o(t)$ for $t \geq 0^+$

(d) the total energy dissipated in the 60kΩ resistor.



(a) At $t = 0^-$, switch is at position x, and therefore, the capacitor, acting like an open-circuit, is charged to 100V. Therefore, $v_C(0^-) = 100\text{V}$.

At $t = 0^+$ after the switch moves to position y, $v_C(0^+) = v_C(0^-) = 100\text{V}$, and

$$R_{eq} = 32\text{k} + (240\text{k} \,||\, 60\text{k}) = 80\text{k}\Omega \quad \Rightarrow \quad \tau = R_{eq}C = (80\text{k})(0.5\mu) = 40\text{ms}$$

$$\therefore \quad v_C(t) = v_C(0^+)e^{-\frac{t}{\tau}} = 100e^{-25t}\text{V}, \quad t \geq 0$$

(b) By voltage division,

$$v_o = v_C \frac{(240\text{k} \parallel 60\text{k})}{32\text{k} + (240\text{k} \parallel 60\text{k})} = \frac{48}{80} v_C \quad \Rightarrow \quad v_o(t) = 60e^{-25t}\,\text{V}, \quad t \geq 0^+$$

(c) From Ohm's law,

$$i_o = \frac{v_o}{60\text{k}} \qquad \Rightarrow \quad i_o(t) = e^{-25t}\,\text{mA}, \quad t \geq 0^+$$

(d) Power dissipated in the 60kΩ resistor,

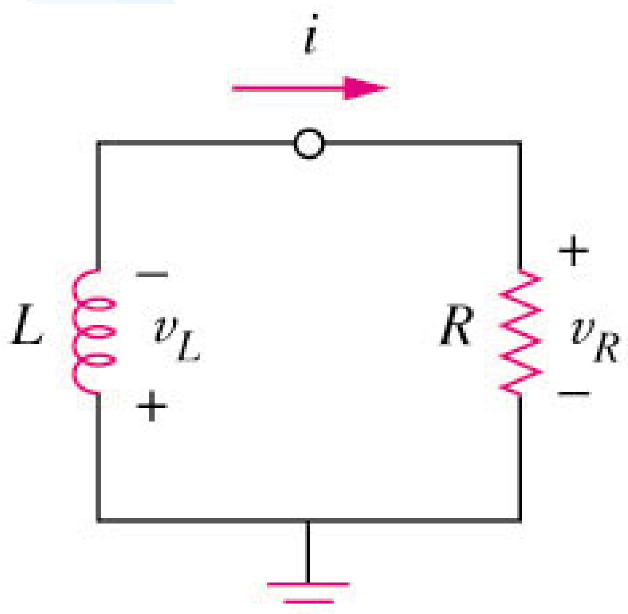
$$p_{60\text{k}\Omega} = i_o^2 (60\text{k}\Omega) \Rightarrow \quad p_{60\text{k}\Omega}(t) = 60e^{-50t}\,\text{mW}, \quad t \geq 0^+$$

Total energy dissipated in the 10Ω resistor,

$$w_{60\text{k}\Omega} = \int_0^\infty p_{60\text{k}\Omega}(t)dt = \int_0^\infty 60e^{-50t}dt = 1.2\text{mJ}$$

The Source-Free RL Circuit

- A **first-order RL circuit** consists of an inductor L (or its equivalent) and a resistor (or its equivalent)

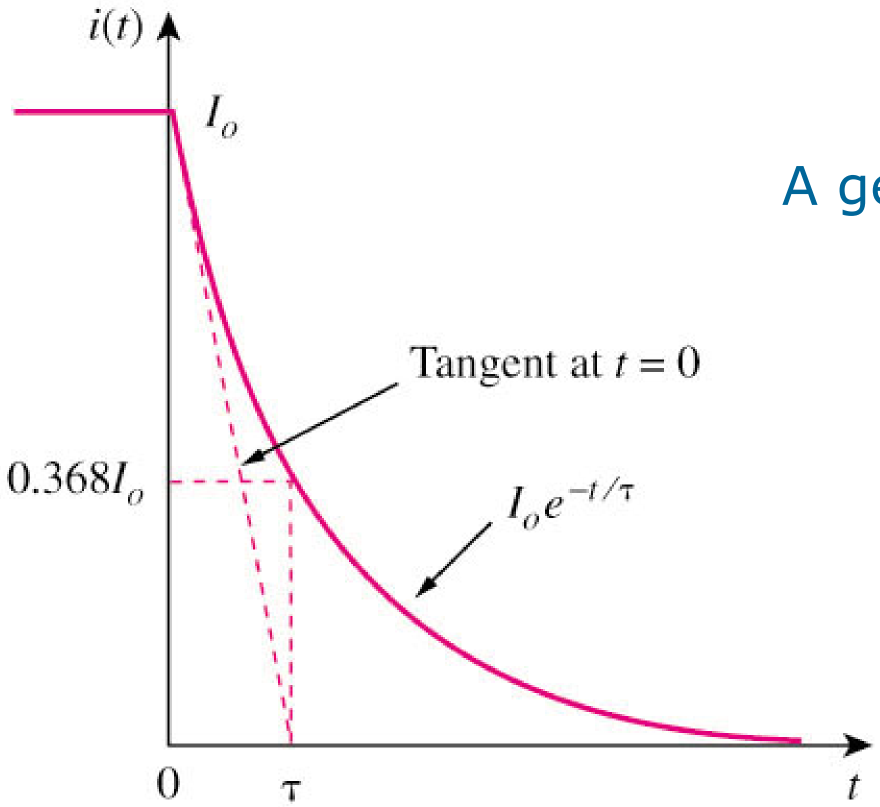


By KVL $$v_L + v_R = 0$$

$$\rightarrow \quad L\frac{di}{dt} + iR = 0$$

Inductors law Ohms law

$$\rightarrow \quad \frac{di}{i} = -\frac{R}{L}dt \quad \rightarrow \quad i(t) = I_0\, e^{-Rt/L}$$



A general form representing an RL

$$i(t) = I_0\, e^{-t/\tau}$$

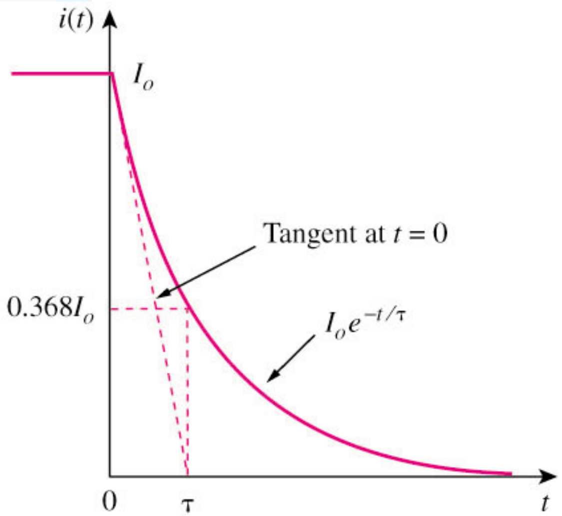
where $\tau = \dfrac{L}{R}$

- The **time constant** τ of a circuit is the time required for the response to decay by a factor of 1/*e* or 36.8% of its initial value.
- $i(t)$ decays faster for small τ and slower for large τ.
- The general form is very similar to a RC source-free circuit.

Comparison between a RL and RC circuit

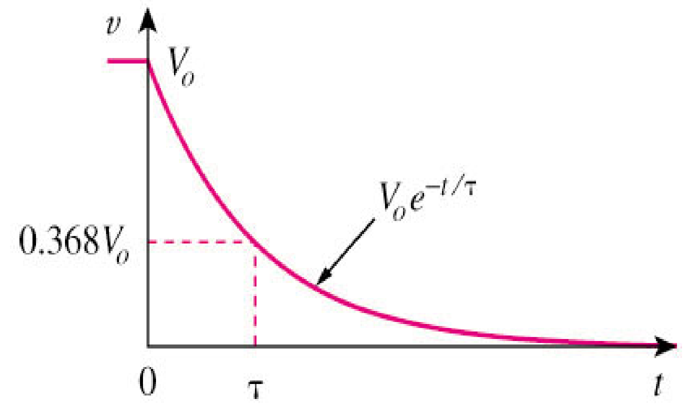
A RL source-free circuit

$$i(t) = I_0\, e^{-t/\tau}$$ **where** $$\tau = \frac{L}{R}$$



A RC source-free circuit

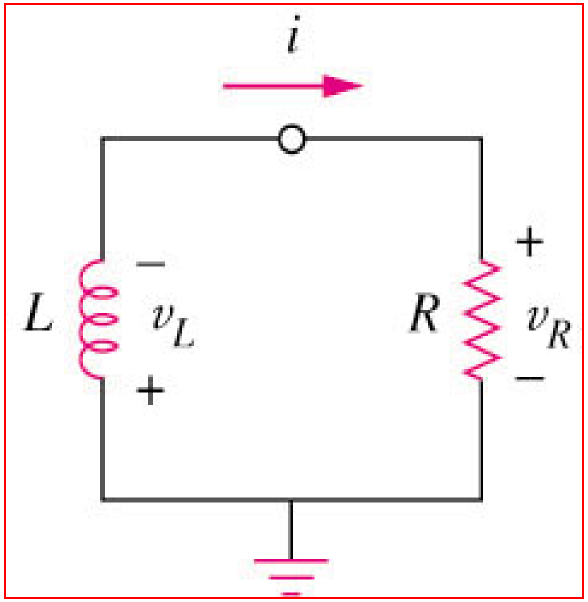
$$v(t) = V_0\, e^{-t/\tau}$$ **where** $$\tau = RC$$



Summary:
Calculation of the natural response of an RL circuit

1. The initial current $i(0) = I_0$ through the inductor.
2. The time constant τ = L/R.

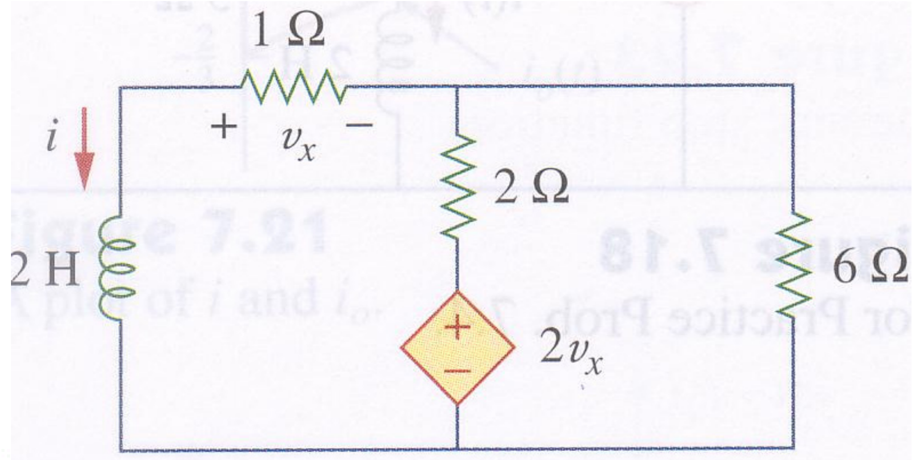
$$i(t) = I_0\, e^{-t/\tau}$$ where $$\tau = \frac{L}{R}$$



Example 4

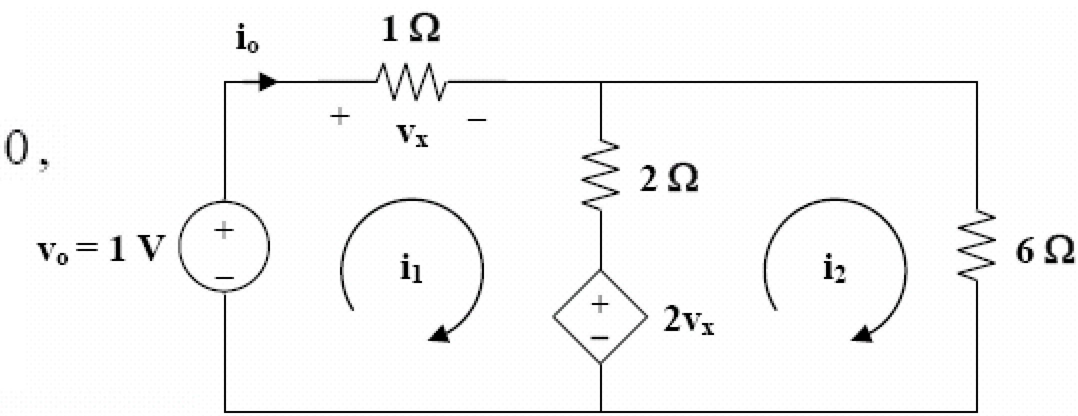
Find i and v_x in the circuit.

Assume that i(0) = 5 A.



Find R_{th} at the inductor terminals by inserting a voltage source.

Applying mesh analysis gives



Loop 1: $-1 + 3i_1 - 2i_2 + 2v_x = 0$,

where $v_x = 1i_1$

$\Rightarrow 5i_1 - 2i_2 = 1$

Loop 2: $8i_2 - 2i_1 - 2v_x = 0 = 8i_2 - 2i_1 - 2i_1$

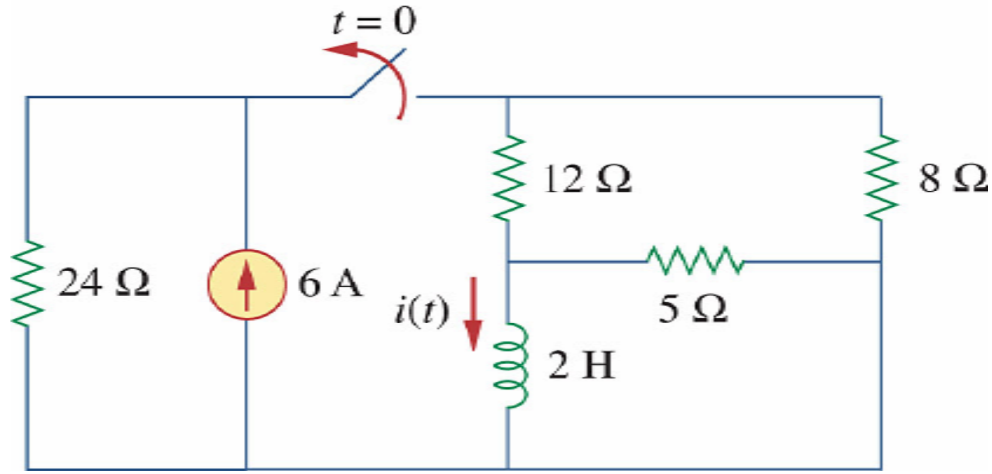
$\Rightarrow i_2 = \frac{1}{2} i_1 \Rightarrow 5i_1 - 1i_1 = 1 \Rightarrow i_o = i_1 = (1/4)$ A

$R_{th} = \frac{v_o}{i_o} = 4\,\Omega$, $\Rightarrow \tau = \frac{L}{R} = \frac{2}{4} = \frac{1}{2}$ s $\Rightarrow i(t) = 5e^{-2t}$ A

Example 5

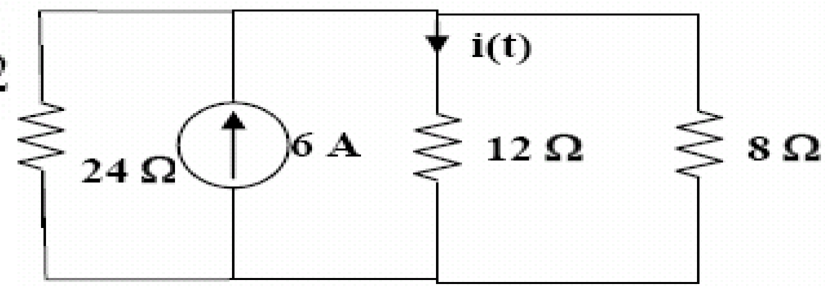
For the circuit,

find i(t) for t > 0.



For t < 0, the equivalent circuit is shown in Fig. (a).

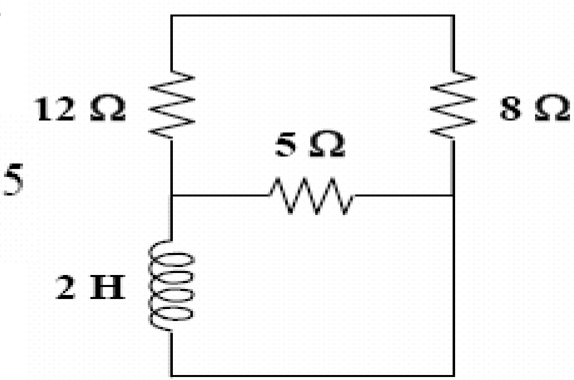
$$i(0) = 6[1/\{(1/24) + (1/12) + (1/8)\}]/12$$
$$= (6x24/6)/12 = 2 \text{ A}$$



(a)

For t > 0, the current source and 24-ohm is cut off

RL circuit is shown in Fig. (b).



(b)

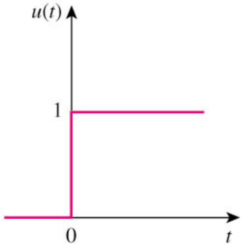
$$R_{eq} = (12+8) \,||\, 5 = 20 \,||\, 5 = 4\,\Omega, \quad \tau = \frac{L}{R_{eq}} = \frac{2}{4} = 0.5$$

$$i(t) = i(0)\,e^{-t/\tau} = \underline{\mathbf{2\,e^{-2t}\ A, \quad t > 0}}$$

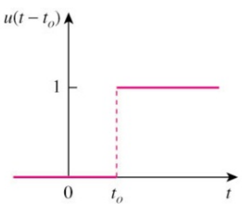
Unit-Step Function

- The **unit step function** $u(t)$ is 0 for negative values of t and 1 for positive values of t.

$$u(t)=\begin{cases}0, & t<0\\ 1, & t\geq 0\end{cases}$$

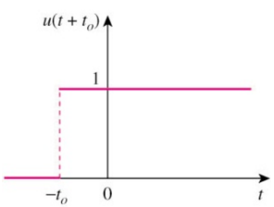


$$u(t-t_o)=\begin{cases}0, & t<t_o\\ 1, & t\geq t_o\end{cases}$$



(a)

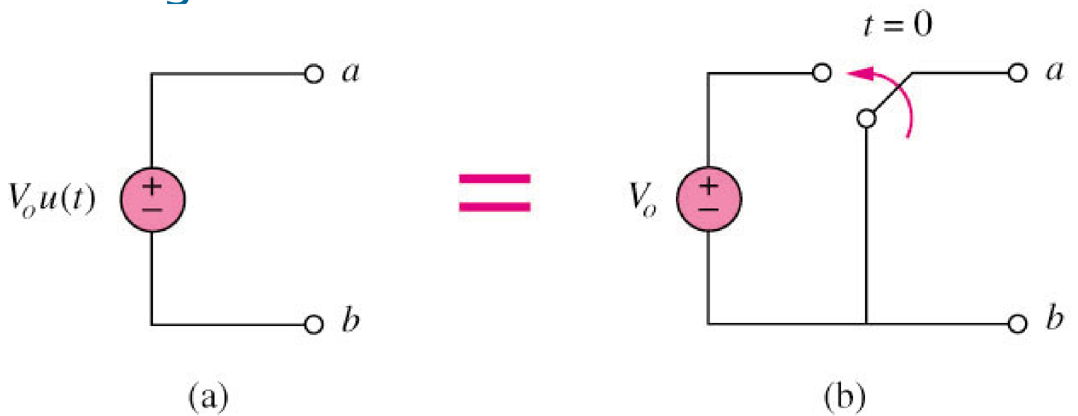
$$u(t+t_o)=\begin{cases}0, & t<-t_o\\ 1, & t\geq -t_o\end{cases}$$



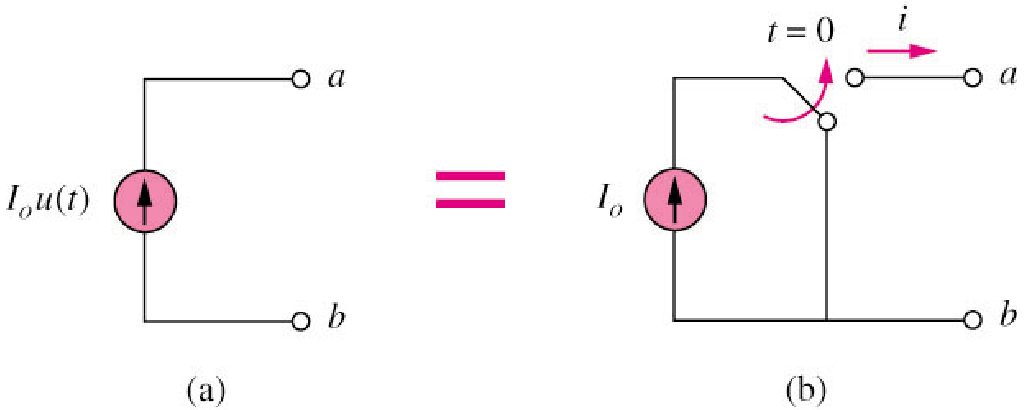
(b)

Represent an abrupt change for:

1. voltage source.

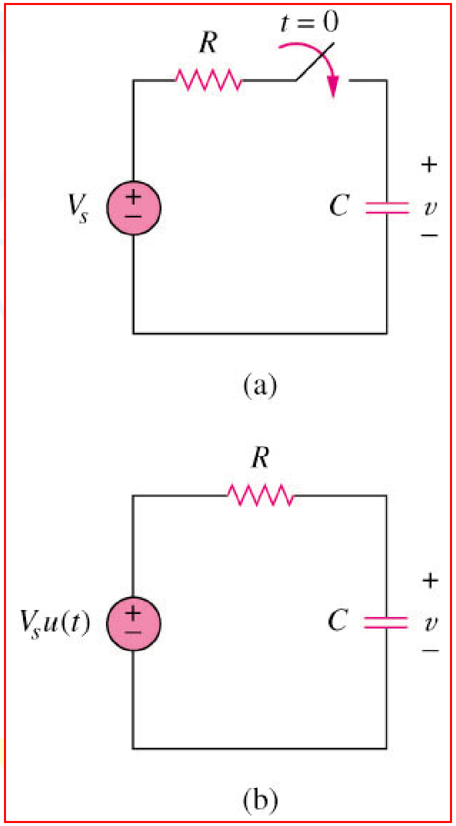


2. for current source:



The Step-Response of a RC Circuit

- The **step response** of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.



- **Initial condition**:

$$v(0-) = v(0+) = V_0$$

- Applying KCL,

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

or for $t \geq 0$,

$$\frac{dv}{dt} = -\frac{v - V_s}{RC}u(t)$$

where u(t) is the unit-step function

$$\frac{dv}{v-V_s} = -\frac{1}{\tau}u(t)dt$$

For t≥0

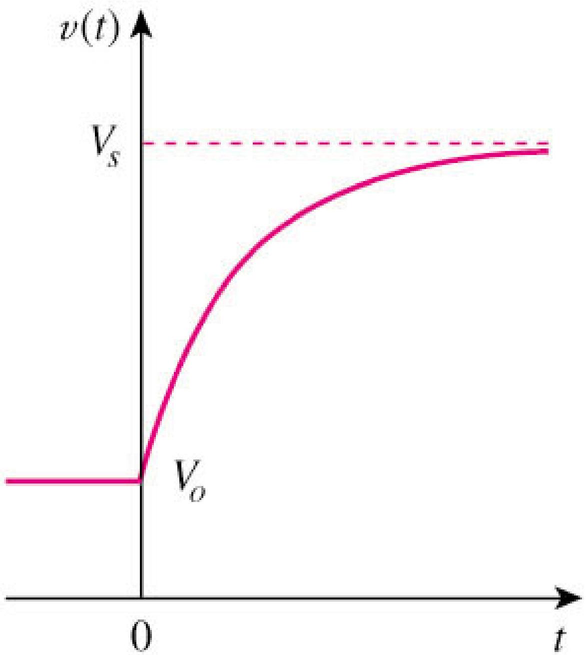
$$\int_0^t \frac{dv}{v-V_s} = -\int_0^t \frac{1}{\tau}u(t)dt$$

$$\ln(v(t)-V_s) - \ln(v(0)-V_s) = -\frac{t}{\tau}$$

$$\ln\frac{v(t)-V_s}{v(0)-V_s} = -\frac{t}{\tau}$$

$$v(t) = V_s + [v(0)-V_s]e^{-\frac{t}{\tau}}$$

The Step-Response of an RC Circuit



$$v(t) = \begin{cases} V_0 & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau} & t \geq 0 \end{cases}$$

Final value at t -> ∞

Initial value at t = 0

Complete Response

$= \quad V_0 e^{-t/\tau} \quad + \quad V_s(1 - e^{-t/\tau})$

Natural response (stored energy)

Forced Response (independent source)

Summary: The Step-Response of an RC Circuit

Three steps to find out the step response of an RC circuit:

1. The initial capacitor voltage v(0).
2. The final capacitor voltage v(∞) — DC voltage across C.
3. The time constant τ.

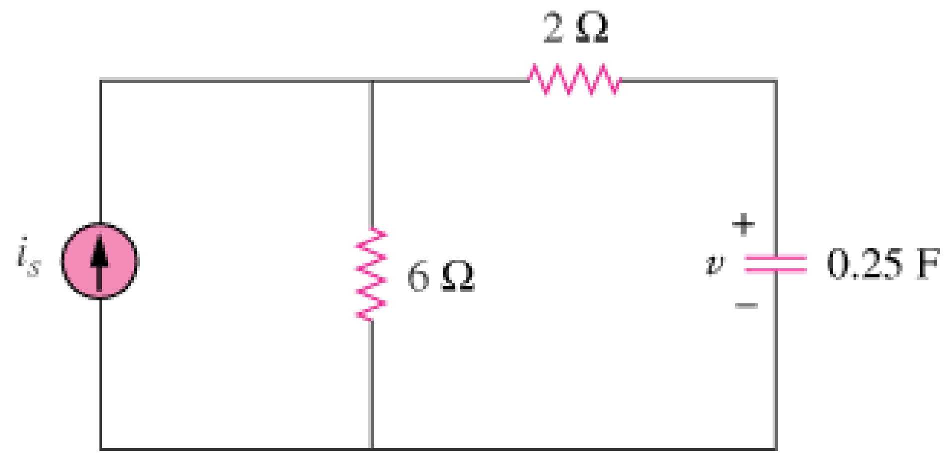
$$v(t) = v(\infty) + [v(0+) - v(\infty)]e^{-t/\tau}$$

Example 6:

For the circuit,

$i_s(t) = 5u(t)$ A, v(0)=0V.

Find $v(t)$.



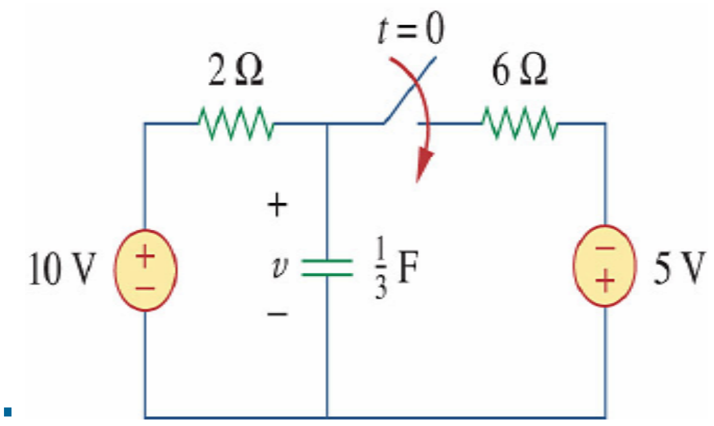
$$\tau = R_{Th}C$$
$$= (2+6)\times 0.25 = 2s$$

$$v(\infty) = 6i_s = 6\times 5 = 30 \text{ V}$$

$$v(t) = \{v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}\}u(t)$$
$$= 30(1 - e^{-t/2})u(t) \text{ V}$$

Example 7

Find v(t) for t > 0 in the circuit. Assume the switch has been open for a long time and is closed at t = 0. Calculate v(t) at t = 0.5s.



For t < 0, the capacitor acts like an open circuit.

$$v(0^-) = v(0^+) = v(0) = 10$$

For t > 0, $[(v(\infty)-10)/2] + [(v(\infty)-(-5))/6] = 0$

$$(4/6)v(\infty) = (30-5)/6 \qquad v(\infty) = 6.25\ \text{V}$$

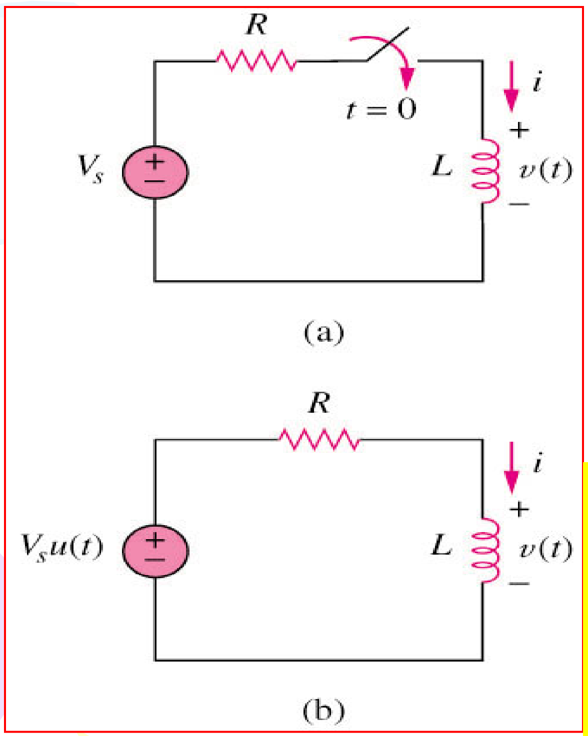
$$R_{th} = 2 \,||\, 6 = \frac{3}{2}\,\Omega, \qquad \tau = R_{th}C = \frac{3}{2}\times\frac{1}{3} = \frac{1}{2}$$

$$v(t) = v(\infty) + \left[v(0) - v(\infty)\right]e^{-t/\tau} = 6.25 + (10-6.25)e^{-2t}$$

$v(t) = \underline{\textbf{(6.25 + 3.75e}^{-2t}\textbf{) V for all t > 0}}$

At t = 0.5, $v(0.5) = 6.25 + 3.75e^{-1} = 6.25 + 1.3795 = \underline{\mathbf{7.63\ V}}$

The Step-response of a RL Circuit

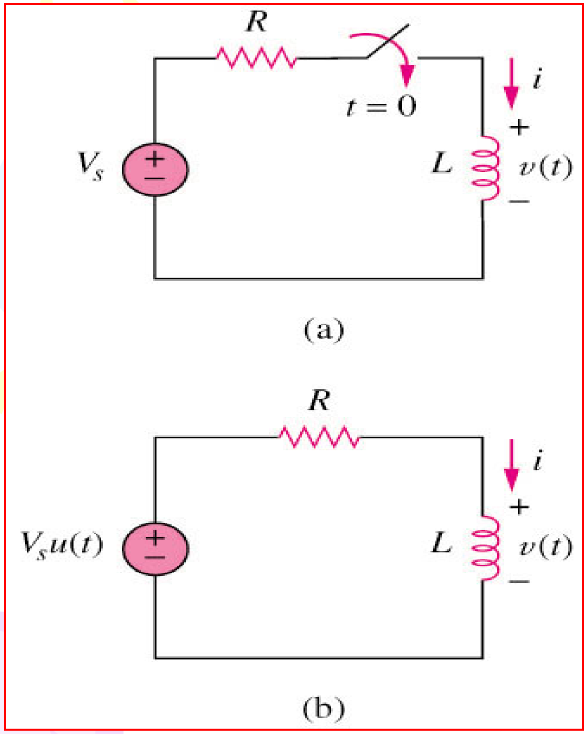


- **Initial current**
  $i(0-) = i(0+) = I_o$
- **Final inductor current**
  $i(\infty) = Vs/R$
- Time constant $\tau = L/R$

$$i(t) = \frac{V_s}{R} + (I_o - \frac{V_s}{R})e^{-\frac{t}{\tau}}u(t)$$

For t>0,

$$V_s = Ri + L\frac{di}{dt}$$



$$\Rightarrow \frac{di}{dt} = \frac{-Ri+V_s}{L} = \frac{-R}{L}\left(i - \frac{V_s}{R}\right)$$

$$\Rightarrow di = \frac{-R}{L}\left(i - \frac{V_s}{R}\right)dt$$

$$\Rightarrow \frac{di}{i-(V_s/R)} = \frac{-R}{L}dt$$

$$\Rightarrow \ln\frac{i(t)-(V_s/R)}{I_0-(V_s/R)} = \frac{-R}{L}t$$

$$\Rightarrow \frac{i(t)-(V_s/R)}{I_0-(V_s/R)} = e^{-(R/L)t}$$

$$\Rightarrow i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-(R/L)t}$$

Summary: The Step-Response of a RL Circuit

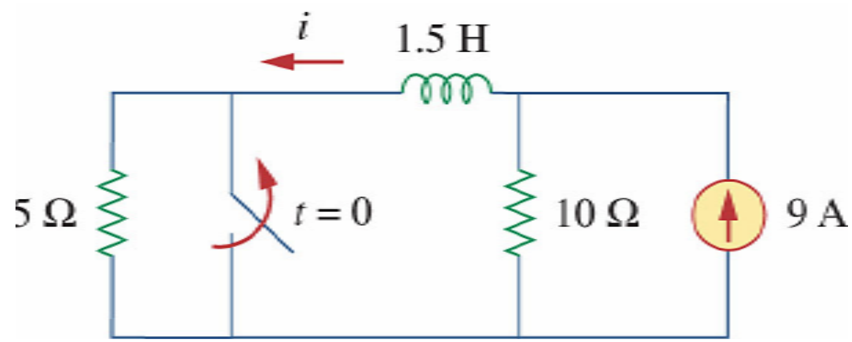
Three steps to find out the step response of an RL circuit:

1. The initial inductor current $i(0)$ at $t = 0+$.
2. The final inductor current $i(\infty)$.
3. The time constant τ.

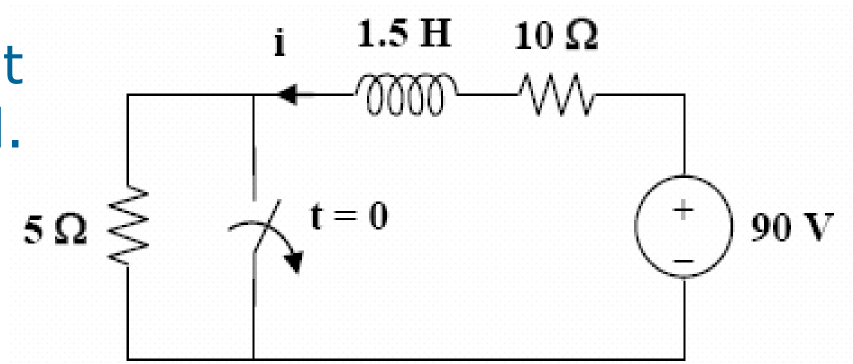
$$i\,(t) \;=\; i\,(\infty) + [i\,(0+) - i\,(\infty)]\,e^{-t/\tau},\;\; t \geq 0$$

Example 8

The switch in the circuit has been closed for a long time. It opens at t = 0. Find *i(t)* for t > 0.



Applying source transformation, the circuit is equivalent to



At t < 0, the switch is closed so that the 5 ohm resistor is short circuited.

$$i(0^-) = i(0) = \frac{90}{10} = 9 \text{ A}$$

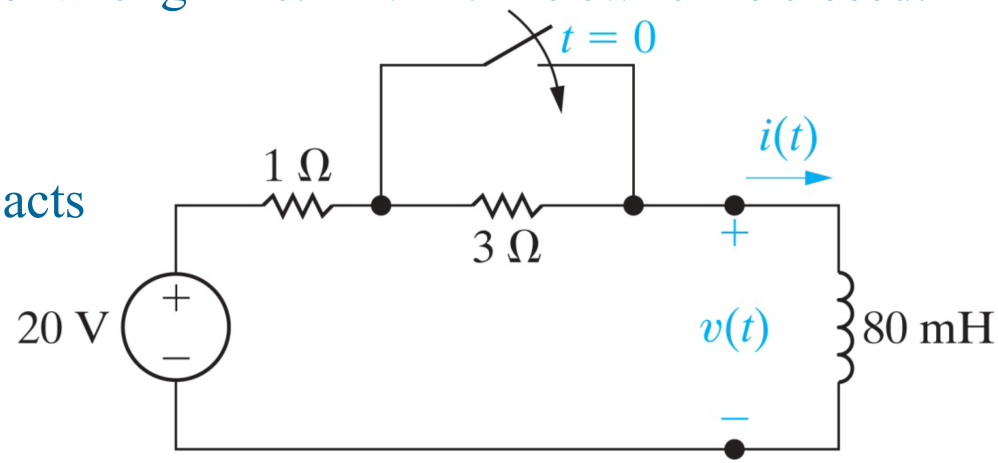
For t > 0, the switch is open.

$$R_{th} = 10 + 5 = 15, \quad \tau = \frac{L}{R_{th}} = \frac{1.5}{15} = 0.1 \quad i(\infty) = \frac{90}{10+5} = 6 \text{ A}$$

$$i(t) = i(\infty) + \left[i(0) - i(\infty) \right] e^{-t/\tau} = 6 + (9-6)\,e^{-10t} = \underline{(6 + 3e^{-10t}) \text{ A}}$$

Example 9

The switch in the circuit has been open for a long time. At $t = 0$ the switch is closed. Find the expression for $i(t)$ when $t \geq 0$

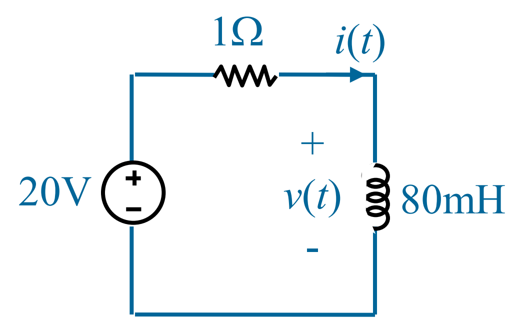


At $t < 0$, switch is open and the inductor acts as an short-circuit.

- The initial value of i,

$$i(0^-) = \frac{20}{1+3} = 5\text{A}$$

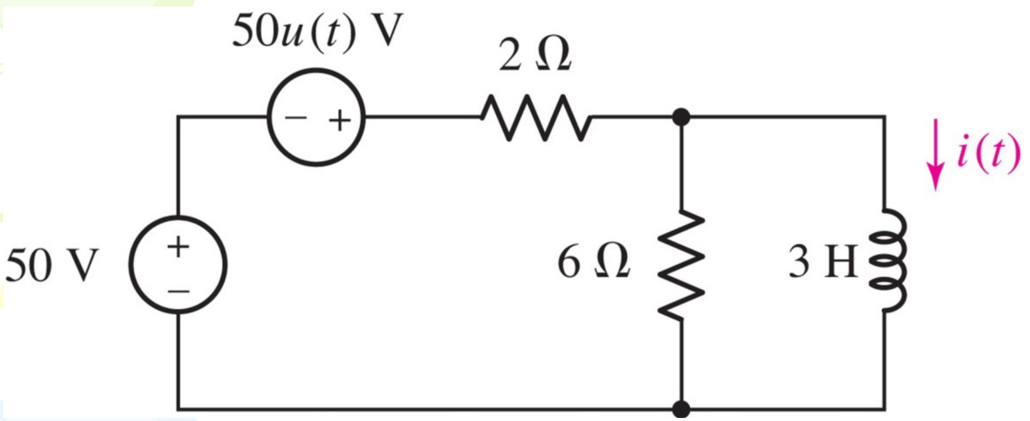
After the switch is closed, the 3Ω resistor is bypassed.



$$\tau = \frac{L}{R} = \frac{80\text{m}}{1} = 0.08\text{s} \qquad i(\infty) = \frac{20}{1} = 20\text{A}$$

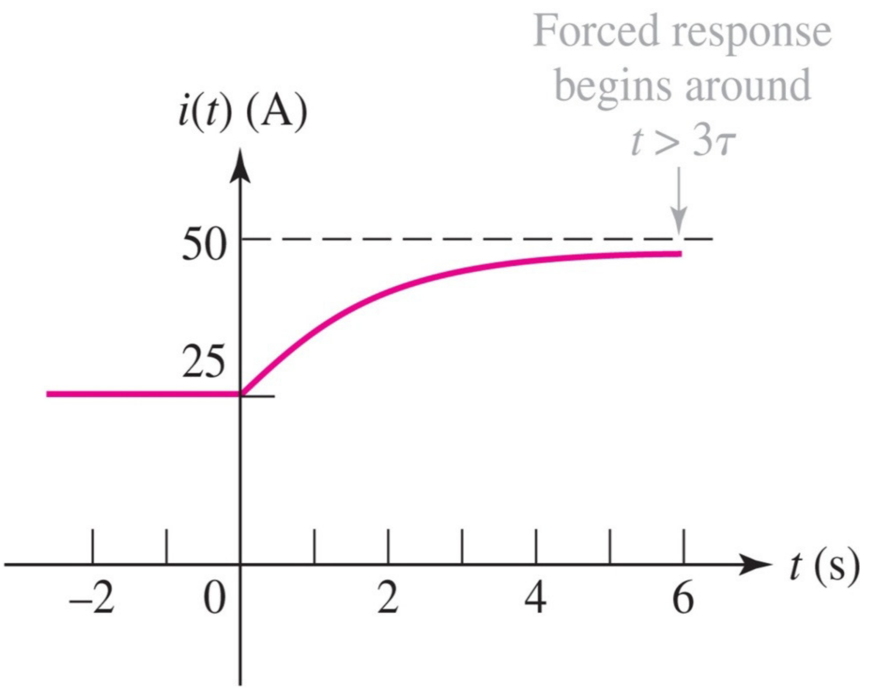
$$i(t) = i(\infty) + \left(i(0^+) - i(\infty)\right)e^{-t/\tau} = 20 + (5-20)e^{-t/0.08} = 20 - 15e^{-12.5t}\text{A}, \quad t \geq 0$$

Exercise



Show that

$i(t)=25+25(1-e^{-t/2})u(t)$ A



Summary: The general solution of a variable:

$$x(t) = \underline{x_f} + [\underline{x(t_0)} - \underline{x_f}]e^{-\frac{(t-t_0)}{\tau}}$$

Final Value

Steady-State

Initial Value

Time Constant

- The general solution in words,

$$\begin{pmatrix}\text{unknown variable}\\ \text{as a function of time}\end{pmatrix} = \begin{pmatrix}\text{final value of}\\ \text{the variable}\end{pmatrix} + \left[\begin{pmatrix}\text{initial value of}\\ \text{the variable}\end{pmatrix} - \begin{pmatrix}\text{final value of}\\ \text{the variable}\end{pmatrix}\right] \times e^{-\left[t-\begin{pmatrix}\text{switching}\\ \text{time}\end{pmatrix}\right] \Big/ \begin{pmatrix}\text{time}\\ \text{constant}\end{pmatrix}}$$

Steps to compute step and natural responses of circuits

- Identify the variable of interest for the circuit.
  - For RC circuits, it is most convenient to choose the capacitive voltage.
  - For RL circuits, it is best to choose the inductive current.
- Determine the initial value of the variable, which is its value at time $t = t_0$.
- Calculate the final value of the variable, which is its value as $t \to \infty$.
- Calculate the time constant τ for the circuit.
- With these quantities, write out the solution using the above general expression

- **Three key values: initial value, final value and time constant τ**

Exercises (Problems in Chapter 7 of the textbook)

7.7 Ans: $[6 + 2e^{-t/20}]$ V for all $t > 0$.

7.10 Ans: $15e^{-16.667t}$ V, 65.92 ms.

7.14 Ans: $0.2083\mu s$

7.44: Ans: $-3e^{-0.25t}$ A

7.48 Ans: $60e^{-t/3}\ V$, $-2e^{-t/3}\ A$

7.50: Ans: $7.5\left(3-e^{-4t}\right)$ mA, $t > 0$

7.56: Ans: $-4e^{-20t}$ V

7.64: Ans: $1.6667\left(1-e^{-t}\right)$V

EE2001
Circuit Analysis

Second-Order Circuits

Dr. Justin Dauwels

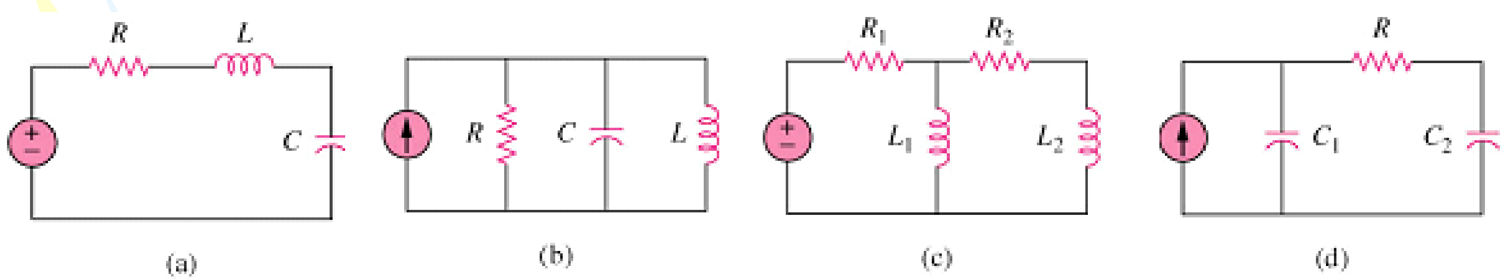
Second-Order Circuits

1 Examples of 2nd order RCL circuit
2 The source-free series RLC circuit
3 The source-free parallel RLC circuit
4 Step response of a series RLC circuit
5 Step response of a parallel RLC

Examples of Second Order RLC circuits

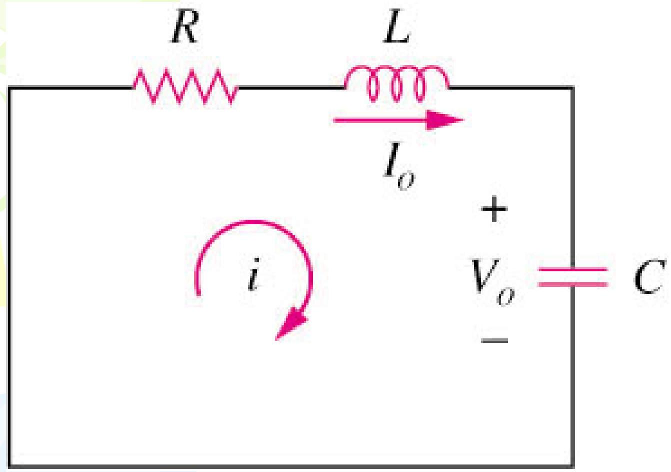
What is a 2nd order circuit?

A second-order circuit is characterized by a <u>second-order differential equation</u>. It consists of <u>resistors</u> and the equivalent of <u>two energy storage elements</u>.



RLC Series   RLC Parallel   RL T-config   RC Pi-config

Source-Free Series RLC Circuits



- The solution of the source-free series RLC circuit is called as the natural response of the circuit.
- The circuit is excited by the energy initially stored in the capacitor and inductor.

$$iR + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,dt + v(0) = 0 \quad\Rightarrow\quad R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{1}{C}i = 0$$

⇒ **The 2nd order of expression**

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

How to solve the equation?

General solution of second-order differential equation

- Considering a second order differential equation

$$f(t) = \frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0x$$

- The solution will have the form of $x = x_n + x_f$

where x_n is the natural response

x_f is the forced response

- Natural response is for $f(t) = 0$, $\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0x = 0$
  - This equation is known as a homogeneous equation (zero forcing function).

- One possibility is the exponential form, $x = Ae^{st}$ where A and s are constants
- Note:

$$\frac{dx}{dt} = Ase^{st} \quad ; \quad \frac{d^2x}{dt^2} = As^2e^{st}$$

- Therefore, the equation can be rewritten as,

$$As^2e^{st} + a_1Ase^{st} + a_0Ae^{st} = 0$$

$$Ae^{st}\left(s^2 + a_1s + a_0\right) = 0$$

- Since $e^{st} \neq 0$ and for non-zero A, it is followed that the parenthetical term vanishes,

$$s^2 + a_1s + a_0 = 0$$

- This equation is called the **characteristic equation** and the solutions are

$$s_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$$

$$\left(a_1^2 - 4a_0\right) > 0 \quad \Rightarrow \quad \text{Real and distinct roots } \left(s_1 = -\alpha_1, s_2 = -\alpha_2\right)$$

$$\left(a_1^2 - 4a_0\right) < 0 \quad \Rightarrow \quad \text{Complex roots } \left(s_{1,2} = -\alpha \pm j\omega_d\right)$$

$$\left(a_1^2 - 4a_0\right) = 0 \quad \Rightarrow \quad \text{Real and equal roots } \left(s_{1,2} = -\alpha\right)$$

- Real and distinct roots

  → **Overdamped response** $x = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t}$

- Complex roots

  → **Underdamped response** $x = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$

- Equal (repeated) roots

  → **Critically damped response** $x = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$

  – where $A_1, A_2, B_1, B_2, D_1, D_2$ are constants to be determined

Source-Free Series RLC Circuits

Solutions for the 2nd order circuit differential equation:

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \quad \Rightarrow \quad \frac{d^2 i}{dt^2} + 2\alpha\frac{di}{dt} + \omega_0^2 i = 0$$

where $\alpha = \dfrac{R}{2L} \quad and \quad \omega_0 = \sqrt{\dfrac{1}{LC}}$

Characteristic Equation $\quad s^2 + 2\alpha s + \omega_0^2 = 0$

Solutions:

$$s_{1,2} = \frac{-2\alpha \pm \sqrt{(2\alpha)^2 - 4\omega_0^{\,2}}}{2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^{\,2}}$$

The types of solutions for *i(t)* depend on the relative values of α and ω_o.

Corresponding to the solutions, there are three possible responses for the RLC circuit

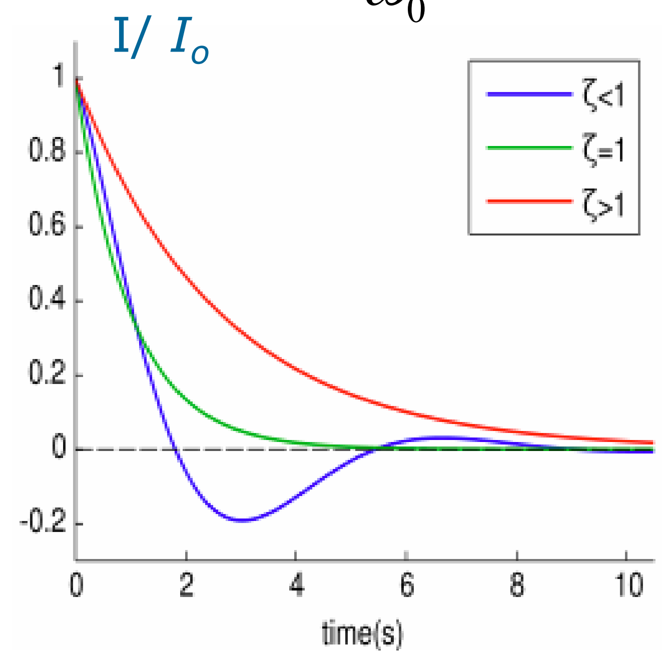
1. If $\alpha > \omega_o$, over-damped case

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$ **where** $$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^{\ 2}}$$

$$\varsigma = \frac{\alpha}{\omega_0}$$

2. If $\alpha = \omega_o$, critical damped case

$$i(t) = (A_2 + A_1 t) e^{-\alpha t}$$ **where** $$s_{1,2} = -\alpha$$



3. If $\alpha < \omega_o$, under-damped case

$$s_{1,2} = -\alpha \pm j\sqrt{\omega_0^{\ 2} - \alpha^2} = -\alpha \pm j\omega_d$$

where $$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

Finding Initial Values

To find the coefficients in the solutions, the initials v(0) [or i(0)] and dv(0)/dt [or di(0)/dt] are required.

Recall two key points:

1) At t = 0, time instant of switching (or change in circuit)
   - Voltage across capacitor cannot change abruptly,

   $$v_C(0^-) = v_C(0) = v_C(0^+)$$

   - Current through inductor cannot change abruptly,

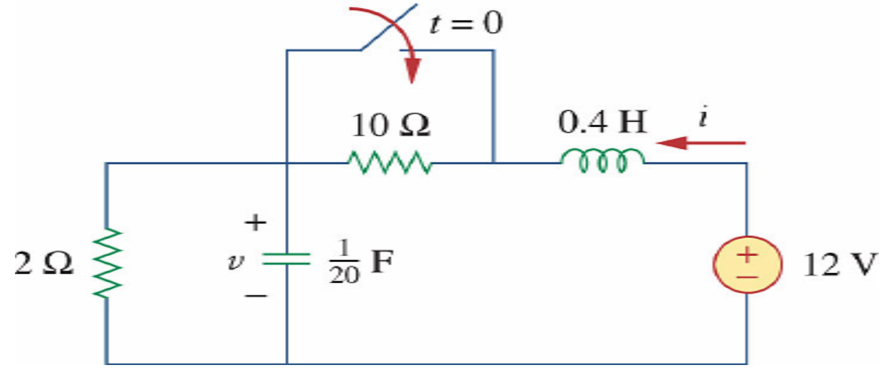
   $$i_L(0^-) = i_L(0) = i_L(0^+)$$

2) At $t = 0^-$, *which is usually corresponding to steady state of the circuit before switching*

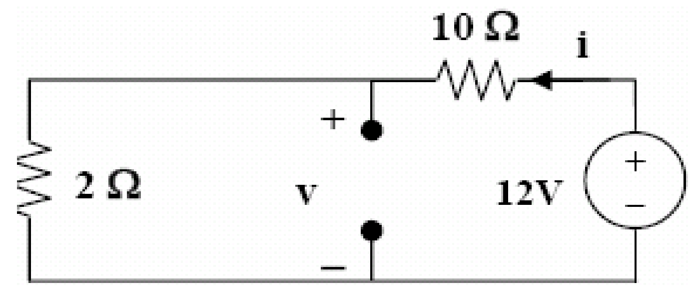
- A capacitor acts like an <u>**open circuit**</u> to a DC voltage
- An inductor acts as a <u>**short circuit**</u> to a DC current

Example 1

The circuit was open for a long time, but closed at t = 0. Determine *i(0⁺), v(0⁺), di(0⁺)/dt, dv(0⁺)/dt*.



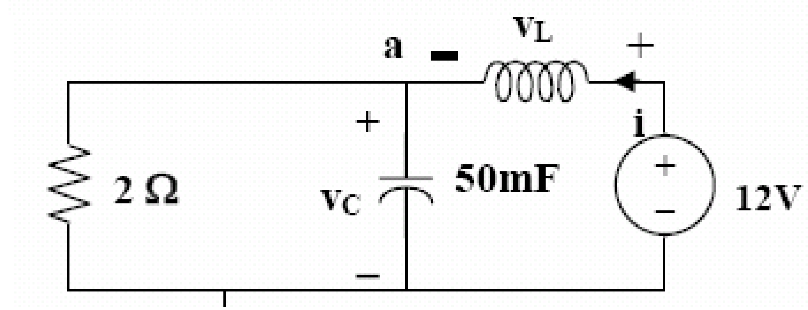
At t = 0-, the equivalent circuit is



$i(0^+)$= $i(0\text{-}) = 12/(2+10) = \underline{\mathbf{1\ A}},$

$v(0^+)$= $v(0\text{-}) = 2i(0\text{-}) = 2\text{ V}$

At t = 0+, the equivalent circuit is .



$L(di/dt) = v_L,$ leads to $(di/dt) = v_L/L$

$v_C(0+) + v_L(0+) = 12 \quad v_L(0+) = 10 \quad (di(0+)/dt) = 10/0.4 = \underline{\mathbf{25\ A/s}}$

$C(dv/dt) = i_C$ leading to $(dv/dt) = i_C/C$

But at node a, KCL gives

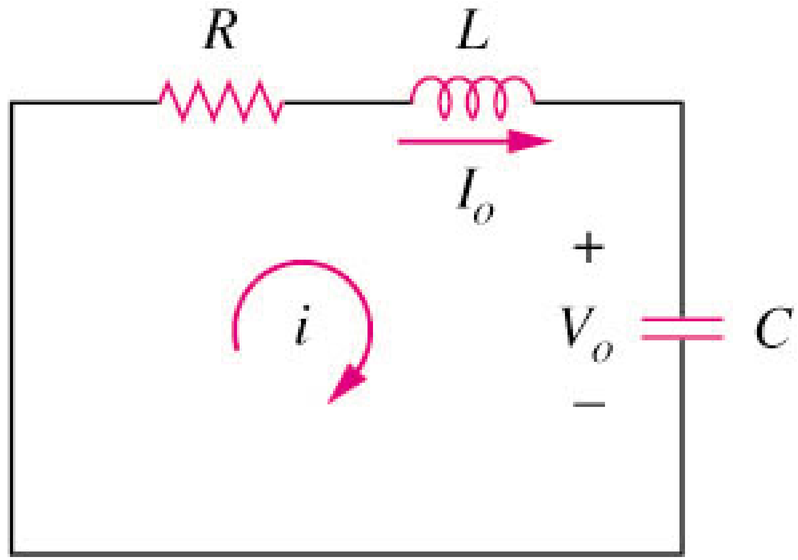
$i(0+) = i_C(0+) + v(0+)/2 \quad 1 = i_C(0+) + 2/2 \quad i_C(0+) = 0,$

$(dv(0+)/dt) = \underline{\mathbf{0\ V/s}}$

Example 2

If R = 10 Ω, L = 5 H, and C = 2 mF, find $\alpha, \omega_0, s_1, s_2$

What type of natural response will the circuit have?



$$\alpha = R/(2L) = 10/(2x5) = \underline{\mathbf{1}},$$

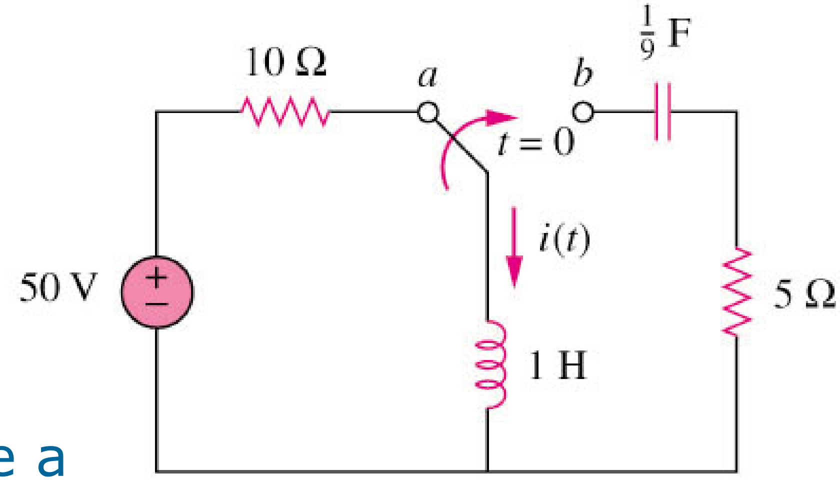
$$\omega_o = 1/\sqrt{LC} = 1/\sqrt{5\times 2\times 10^{-3}} = 10$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2} = -1 \pm \sqrt{1-100} = \underline{\mathbf{-1 \pm j9.95}}.$$

Since $\alpha < \omega_o$, we clearly have an underdamped response.

Example 3

The circuit has reached steady state at t = 0-. If the switch moves to position b at t = 0, calculate i(t) for t > 0.

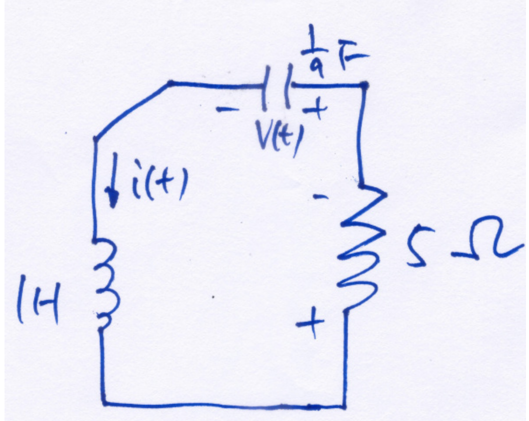


For t < 0, the inductor acts like a short circuit.

i(0-) = 50/10 = 5 = i(0+) = i(0)

The voltage across the capacitor is 0 = v(0-) = v(0+) = v(0).

For t > 0, we have a source-free RLC circuit.



$$L\frac{di}{dt}+Ri+v=0 \quad \Rightarrow \quad \frac{di}{dt}=-\frac{1}{L}(Ri+v)$$

di(0)/dt = -(1/L)[Ri(0) + v(0)] =-1×5×5=-25

$$\omega_o = 1/\sqrt{LC} = 1/\sqrt{1x\frac{1}{9}} = 3 \qquad \alpha = R/(2L) = 5/(2x1) = 2.5$$

Since $\alpha < \omega_o$, we have an underdamped case.

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2} = -2.5 \pm \sqrt{6.25 - 9} = -2.5 \pm j1.6583$$

$$i(t) = e^{-2.5t}[A_1\cos 1.6583t + A_2\sin 1.6583t]$$

We now determine A_1 and A_2. $\quad i(0) = 5 = A_1$

$$di/dt = -2.5\{e^{-2.5t}[A_1\cos 1.6583t + A_2\sin 1.6583t]\} + 1.6583e^{-2.5t}[-A_1\sin 1.6583t + A_2\cos 1.6583t]$$

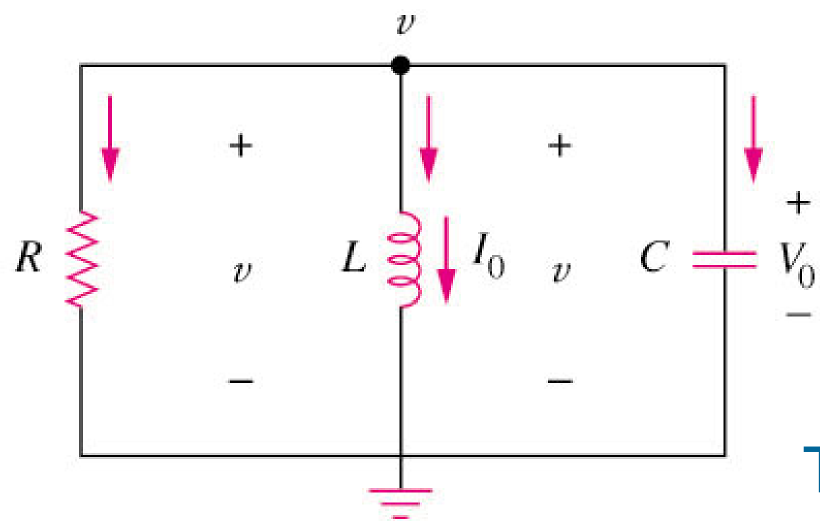
$$di(0^+)/dt = -2.5A_1 + 1.6583A_2$$

$$-25 = -2.5(5) + 1.6583A_2$$

$$A_2 = -7.5378$$

$$i(t) = e^{-2.5t}[5\cos 1.6583t - 7.538\sin 1.6583t]\ A$$

Source-Free Parallel RLC Circuits



Let $i(0) = I_0$ v(0) = V_0

Apply KCL to the top node:

$$\frac{v}{R} + \frac{1}{L}\int_0^t v\,dt + v(0) + C\frac{dv}{dt} = 0$$

Taking the derivative with respect to t and dividing by C gives

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

The 2nd order of expression

That is:

$$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \omega_0^2 v = 0 \quad \text{where} \quad \alpha = \frac{1}{2RC} \quad \text{and} \quad \omega_0 = \sqrt{\frac{1}{LC}}$$

The corresponding characteristic equation is:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

There are three possible solutions for the 2nd order differential equation:

1. If $\alpha > \omega_o$, over-damped case

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$ **where** $$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0{}^2}$$

2. If $\alpha = \omega_o$, critical damped case

$$v(t) = (A_2 + A_1 t)\, e^{-\alpha t}$$ **where** $$s_{1,2} = -\alpha$$

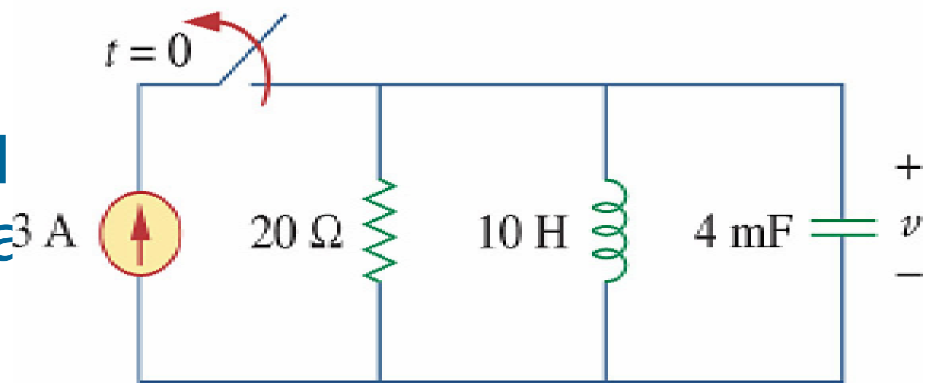
3. If $\alpha < \omega_o$, under-damped case

$$s_{1,2} = -\alpha \pm j\sqrt{\omega_0{}^2 - \alpha^2} = -\alpha \pm j\omega_d$$ **where** $$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$v(t) = e^{-\alpha t}\,(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

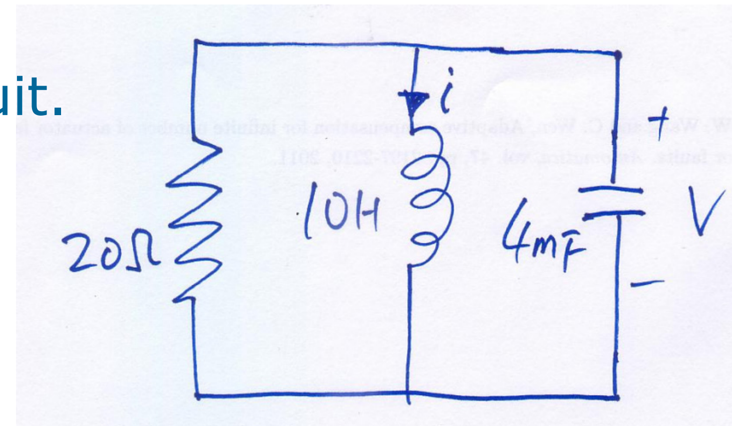
Example 4

The switch has been closed for long time and is open a t=0. Find v(t) for t > 0.



$i(0) = 3A$ and $v(0) = 0$.

For t > 0, a source-free parallel RLC circuit.



$$C\frac{dv}{dt}+\frac{v}{R}+ i=0 \qquad \frac{dv}{dt}=-\frac{1}{C}\left(\frac{v}{R}+i\right)$$

$$\frac{dv(0^+)}{dt}=-\frac{1}{C}\left[\frac{v(0^+)}{R}+i(0^+)\right] = \frac{1}{4\times 10^{-3}}[0+3] = -750$$

$$\alpha = 1/(2RC) = 1/(2x20x4x10^{-3}) = 6.25$$

$$\omega_o = 1/\sqrt{LC} = 1/\sqrt{10x4x10^{-3}} = 5$$

Since $\alpha > \omega_o$, this is an overdamped response.

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o} = -6.25 \pm \sqrt{(6.25)^2 - 25} = -2.5 \text{ and } -10$$

Thus, $v(t) = A_1 e^{-2.5t} + A_2 e^{-10t}$

$v(0) = 0 = A_1 + A_2$, which leads to $A_2 = -A_1$

But, $dv/dt = -2.5 A_1 e^{-2.5t} - 10 A_2 e^{-10t}$

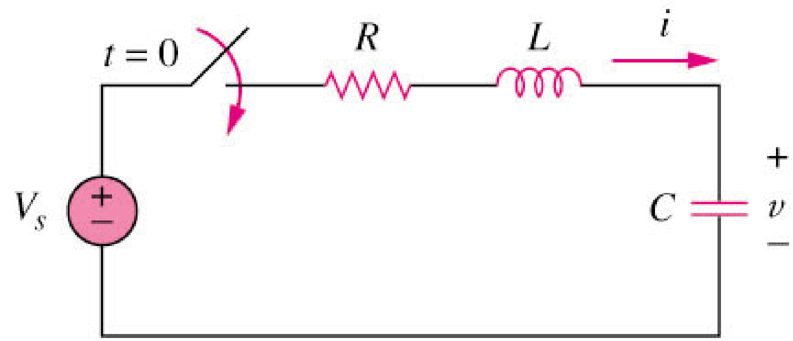
At $t = 0$, $-750 = -2.5A_1 - 10A_2 = 7.5A_1$ since $A_1 = -A_2$

$A_1 = -100$, $A_2 = 100$

$$v(t) = \underline{100(e^{-10t} - e^{-2.5t})\ V}$$

Step-Response Series RLC Circuits

The step response is obtained by the sudden application of a dc source.



$$Ri + L\frac{di}{dt} + v = v_s \qquad i = C\frac{dv}{dt} \Rightarrow RC\frac{dv}{dt} + LC\frac{d^2v}{dt^2} + v = v_s$$

The 2nd order of expression

$$\boxed{\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{v_s}{LC}}$$

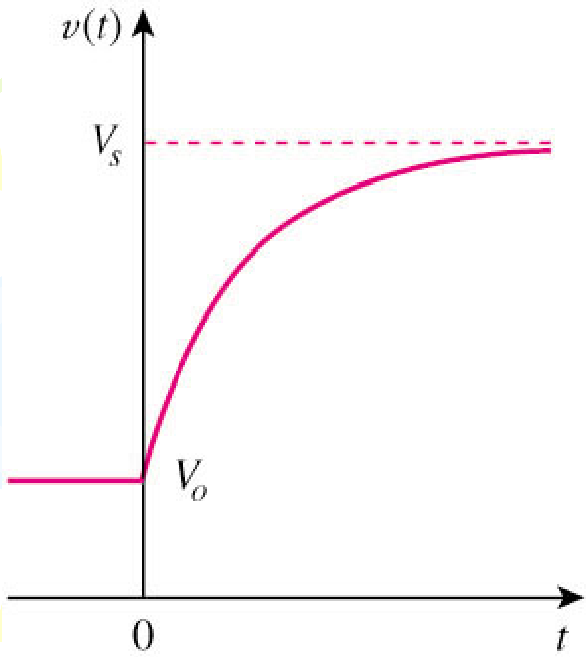
The above equation has the same form as the equation for source-free series RLC circuit.

$$\boxed{\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \omega_0^2 v = \frac{v_s}{LC}}$$

The same coefficients (important in determining the Parameters α and ω_o).

$$\alpha = \frac{R}{2L} \quad and \quad \omega_0 = \sqrt{\frac{1}{LC}}$$

Recall: The Step-Response of an RC Circuit



$$v(t) = \begin{cases} V_0 & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau} & t \geq 0 \end{cases}$$

Final value
at t -> ∞

Initial value
at t = 0

For t≥0,

$$v(t) = \text{Steady-State Response} + \text{Transient Response}$$

The solution of the 2nd equation should have two components: the transient (source-free) response $v_t(t)$ & the steady-state response $v_{ss}(t)$:

$$v(t) = v_t(t) + v_{ss}(t)$$

- The transient response v_t is the same as that for source-free case, i.e depending on the characteristic equation:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

$$v_t(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$ (over-damped)

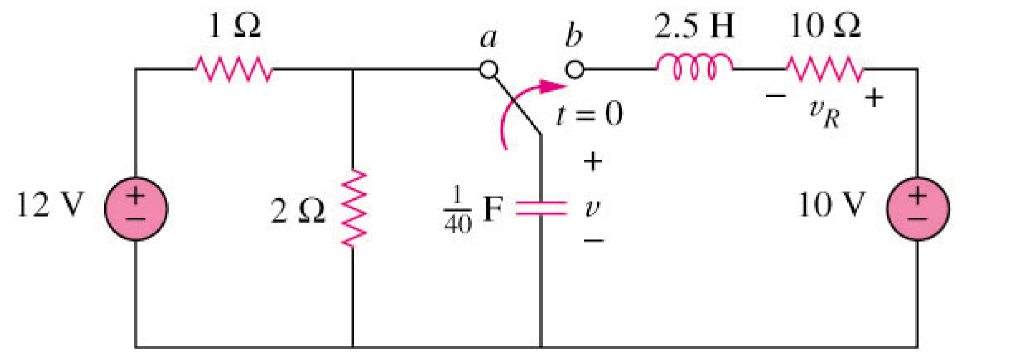
$$v_t(t) = (A_1 + A_2 t) e^{-\alpha t}$$ (critically damped)

$$v_t(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$ (under-damped)

- The steady-state response is the final value of v(t).
  - $v_{ss}(t) = v(\infty)$
- The values of A_1 and A_2 are obtained from the initial conditions:
  - v(0) and dv(0)/dt.

Example 5

Having been in position for a long time, the switch in the circuit below is moved to position b at t = 0. Find v(t) and $v_R(t)$ for t > 0.



The initial capacitor voltage: v(0) = [2/(2 + 1)]12 =8V

The initial inductor current: *i(0) = 0.*

When the switch is in position b, we have:

The initial capacitor current is the same as the initial inductor current

$$i(0) = C(dv(0)/dt) = 0 \qquad dv(0)/dt = 0$$

$$\alpha = R/(2L) = 10/(2x2.5) = 2 \qquad \omega_o = 1/\sqrt{LC} = 1/\sqrt{(5/2)x(1/40)} = 4$$

Since $\alpha < \omega_o$, we have an underdamped case.

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o} = -2 \pm \sqrt{(2)^2 - 16} = -2 \pm j\,3.464$$

Thus, $v(t) = v_f + [(A_1\cos 3.464t + A_2\sin 3.464t)e^{-2t}]$

where $v_f = v(\infty) = 10$, the final capacitor voltage.

We now impose the initial conditions to get A_1 and A_2.

$v(0) = 8 = 10 + A_1$ leads to $A_1 = -2$

But, $$\begin{aligned} dv/dt &= 3.464[(-A_1\sin 3.464t + A_2\cos 3.464t)e^{-2t}] \\ &\quad -2[(A_1\cos 3.464t + A_2\sin 3.464t)e^{-2t}] \end{aligned}$$

$dv(0)/dt = 0 - 2A_1 + 3.464A_2 \quad A_2 = -4/3.464 = -1.1547$

$v(t) =$ **$\{10 + [(-2\cos 3.464t - 1.1547\sin 3.464t)e-2t]\}$ V**

$i = C(dv/dt)$,

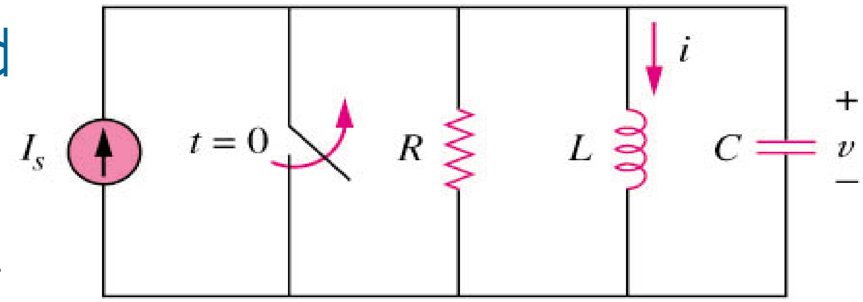
$v_R = Ri = RC(dv/dt) = (1/4)dv/dt$

$= (1/4)[(4 - 4)\cos 3.464t + (2x1.1547 + 2x3.464)\sin 3.464t]e^{-2t}$

$v_R(t) =$ **$[2.31\sin 3.464t]e^{-2t}$ V**

Step-Response Parallel RLC Circuits

The step response is obtained by the sudden application of a dc source.



$$\frac{v}{R} + i + C\frac{dv}{dt} = I_s \qquad v = L\frac{di}{dt}$$

$$\frac{L}{R}\frac{di}{dt} + i + LC\frac{d^2 i}{dt^2} = I_s \qquad \frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}$$

It has the same form as the equation for source-free parallel RLC circuit.

$$\frac{d^2 i}{dt^2} + 2\alpha\frac{di}{dt} + \omega_0^2 i = \frac{I_s}{LC}$$

where $\alpha = \dfrac{1}{2RC}$ and $\omega_0 = \sqrt{\dfrac{1}{LC}}$

The solution of the equation should have two components: the transient (source-free) response $v_t(t)$ & the steady-state response $v_{ss}(t)$:

$$i(t) = i_t(t) + i_{ss}(t)$$

- The transient response i_t is the same as that for source-free case, i.e depending on the characteristic equation

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

$$i_t(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$ (over-damped)

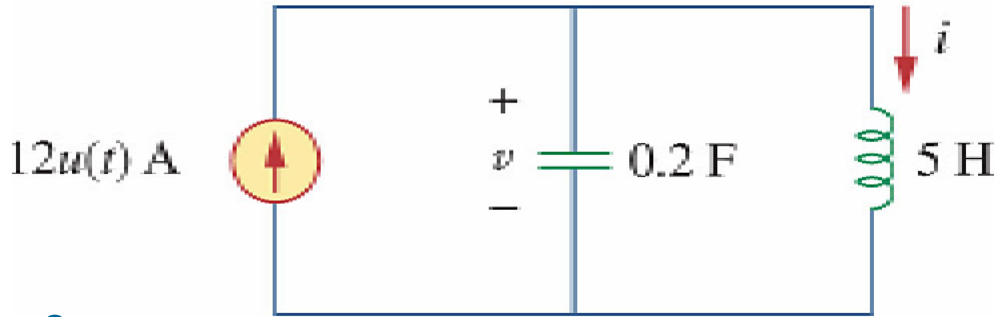
$$i_t(t) = (A_1 + A_2 t) e^{-\alpha t}$$ (critical damped)

$$i_t(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$ (under-damped)

- The steady-state response is the final value of i(t).
  - $i_{ss}(t) = i(\infty) = I_s$
- The values of A_1 and A_2 are obtained from the initial conditions:
  - i(0) and di(0)/dt.

Example 6

Find i(t) and v(t) for $t > 0$ in the circuit



When $t < 0$, $v(0) = 0$, $i(0) = 0$;

for $t > 0$, $\alpha = 0$, $\omega_o = 1/\sqrt{LC} = 1/\sqrt{0.2x5} = 1$

$i(t) = i_s + A_1\cos t + A_2\sin t$ $= 12 + A_1\cos t + A_2\sin t$

$i(0) = 0 = 12 + A_1$ therefore $A_1 = -12$

$L\,di(0)/dt = v(0) = 0$

But $di/dt = -A_1\sin t + A_2\cos t$

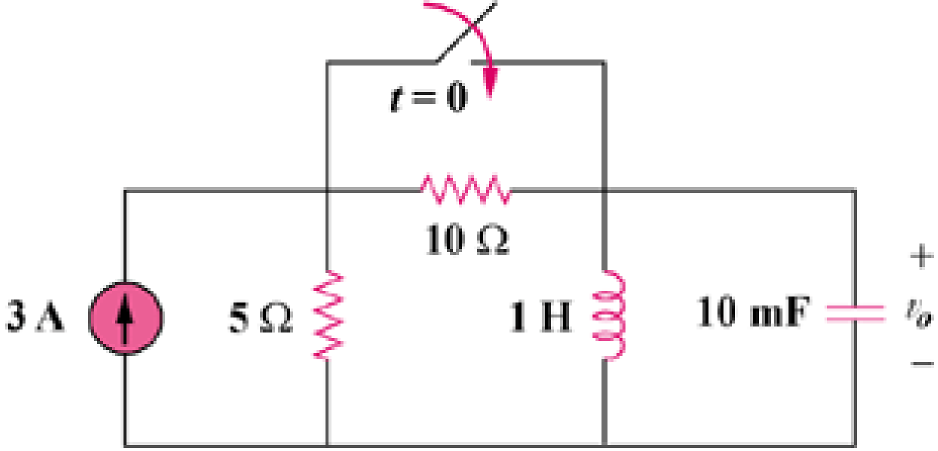
At $t = 0$, $di(0)/dt = 0 + A_2 = 0$

$i(t) = \underline{\mathbf{12(1 - \cos t)\ A}}$ $v(t) = L\,di/dt$

$= 5x12\sin t = \underline{\mathbf{60\sin t\ V}}$

Example 7

The switch has been open for long time and is closed at t=0. Find $v_o(t)$ for t > 0.



At t = 0-,

$i_L(0) = 3x5/(10 + 5) = 1A$ and $v_o(0) = 0.$

For t > 0, we have a parallel RLC circuit with a step input.

$\alpha = 1/(2RC) = (1)/(2x5x0.01) = 10$ $\omega_o = 1/\sqrt{LC} = 1/\sqrt{1x0.01} = 10$

$\alpha = \omega_o \Rightarrow s_{1,2} = -10 \Rightarrow$ a critically damped response

$i(t) = I_s + [(A_1 + A_2 t)e^{-10t}]$, $I_s = 3$ $i(0) = 3 + A_1 = 1$ $A_1 = -2$

$v_o(t) = Ldi/dt = [A_2 e^{-10t}] + [-10(A_1 + A_2 t)e^{-10t}]$

$v_o(0) = A_2 - 10 A_1 = 0$ or $A_2 = -20$

$v_o(t) = [-20 e^{-10t}] + [-10(-2 -20 t)e^{-10t}] = \mathbf{200te^{-10t}}$ V

Summary: Source Free Responses of RLC Circuits

1. Set up the characteristic equation for the circuit $s^2 + 2\alpha s + \omega_0^2 = 0$

1. Find $\alpha = \dfrac{1}{2RC}$; $\omega_0 = \dfrac{1}{\sqrt{LC}}$ or $\alpha = \dfrac{R}{2L}$; $\omega_0 = \dfrac{1}{\sqrt{LC}}$

   For parallel RLC circuit — For series RLC circuit

3. Decide the form of the response (based on the characteristic roots)

$\omega_0^2 < \alpha^2$ $\quad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$ $\quad x(t) = A_1 e^{-s_1 t} + A_2 e^{-s_2 t}$

$\omega_0^2 = \alpha^2$ $\quad s_{1,2} = -\alpha$ $\quad x = (A_2 + A_1 t)e^{-\alpha t}$

$\omega_0^2 > \alpha^2$ $\quad s_{1,2} = -\alpha \pm j\omega_d$, $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

$$x = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

4. Calculate initial values $v_C(0^+)$ and $dv_C(0^+)$ /dt or $i_L(0^+)$ and $di_L(0^+)$ /dt
5. Find the parameters A_1 and A_2, or B_1 and B_2.

Summary: Step Responses of RLC Circuits

The solution should have two components:
the transient (source-free) response $x_t(t)$ & the steady-state response $x_{ss}(t)$:

$$x(t) = x_t(t) + x_{ss}(t)$$

The transient response x_t is the same as that for source-free case

The steady-state response is the final value of $x(t)$.

$$x_{ss}(t) = x(\infty)$$

Calculate initial values $v_C(0^+)$ and $dv_C(0^+)/dt$ or $i_L(0^+)$ and $di_L(0^+)/dt$

Find the parameters A_1 and A_2, or B_1 and B_2.

Exercises (Problems in Chapter 8 of the textbook)

8.4 Ans: 1.25A, 6.25V ; 10 V/s, –40 A/s; –625 mA, 1.875 V

8.21 Ans: $(18e^{-t} - 2e^{-9t})$ V

8.24 Ans: $(12\cos(19.365t) + 3.098\sin(19.365t))\, e^{-5t}$ A

8.31: Ans: 80 V, 40 V

8.42: Ans: $\{-12 + [(4\cos 4t + 3\sin 4t)e^{-3t}]\}$ A

8.48 Ans: $(-4 - 4t)e^{-2t}]$ A, $(4 + 8t)e^{-2t}$ V

8.50: Ans: $(9 + 2e^{-10t} - 8e^{-2.5t})$A