Discrete Mathematics

MH1812

Topic 7.1 - Set Theory I

Dr. Guo Jian

Topic Overview

What's in store…

- **I**ntroduction to Set
- **V**enn Diagram
- **C**artesian Product
- **P**artition

By the end of this lesson, you should be able to…

- Explain the concepts of sets.
- Use Venn diagrams to show the relationship between sets.
- Explain what is cartesian product.
- Explain what is a partition of a set.

Introduction to Set

Introduction to Set: Definition

A **set** is a collection of abstract objects (e.g., prime numbers, domain in predicate logic).

- Determined by (distinct) elements/members:
  - E.g., $\{1, 2, 3\} = \{3, 1, 2\} = \{1, 3, 2\} = \{1, 1, 1, 2, 3, 3, 3\}$
- Two common ways to specify a set:
  - **Explicit**: enumerate the members
    - E.g., $A = \{2, 3\}$
  - **Implicit**: description using predicates $\{x \mid P(x)\}$
    - E.g., $A = \{x \mid x$ is a prime number$\}$

Introduction to Set: Membership

We write $x \in S$ iff x is an element (member) of S.

- **E.g.,** $\{1, 2, 3\} = \{3, 1, 2\} = \{1, 3, 2\} = \{1, 1, 1, 2, 3, 3, 3\}$

- **E.g.,** $A = \{x \mid x$ **is a prime number**$\}$ **then** $A = \{2, 3, 5, 7, \ldots\}$
  $2 \in A, 3 \in A, 5 \in A, \ldots, 1 \notin A, 4 \notin A, 6 \notin A, \ldots$

Introduction to Set: Subset

A set A **is a subset of the set** B, denoted by $A \subseteq B$ iff every element of A is also an element of B.

I.e.,:

- $A \subseteq B \triangleq \forall x(x \in A \rightarrow x \in B)$
- $A \not\subset B \triangleq \neg (A \subseteq B)$

$$\equiv \neg \forall x(x \in A \rightarrow x \in B)$$

$$\equiv \exists x(x \in A \wedge x \notin B)$$

- E.g., $B = \{1, 2, 3\}$, $A = \{1, 2\} \subseteq B$

Introduction to Set: Empty Set

The set that contains no element is called the **empty set** or **null set**.

- The empty set is denoted by $\varnothing$ or by $\{\ \}$.
- Note: $\varnothing \neq \{\varnothing\}$

Introduction to Set: Set Equality

$$A = B \triangleq \forall x(x \in A \leftrightarrow x \in B)$$

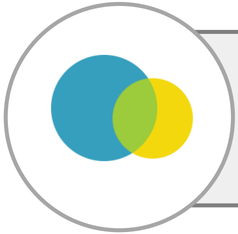
What is equal w triangle symbol

- Two sets A and B are equal iff they have the same elements.

$$A \neq B \triangleq \neg\forall x(x \in A \leftrightarrow x \in B)$$
$$\equiv \exists x\,[(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)]$$

- Two sets are not equal if they do not have identical members, i.e., there is at least one element in one of the sets which is absent in the other.
  - E.g., $\{1, 2, 3\} = \{3, 1, 2\} = \{1, 3, 2\} = \{1, 1, 1, 2, 3, 3, 3\}$

Introduction to Set: Cardinality



The **cardinality** $|S|$ of S is the number of elements in S.
(E.g., for $S = \{1, 3\}$, $|S| = 2$)

- If $|S|$ is finite, S is a finite set; otherwise S is infinite.
  - The set of **positive** integers is an infinite set.
  - The set of **prime** numbers is an infinite set.
  - The set of **even prime** numbers is a finite set.

- Note: $|\varnothing| = 0$

Introduction to Set: Power Set

The **power set** $P(S)$ of a given set S is the set of all subsets of S: $P(S) = \{A \mid A \subseteq S\}$.

- E.g., for $S = \{1,2,3\}$

$$P(S) = \{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$$

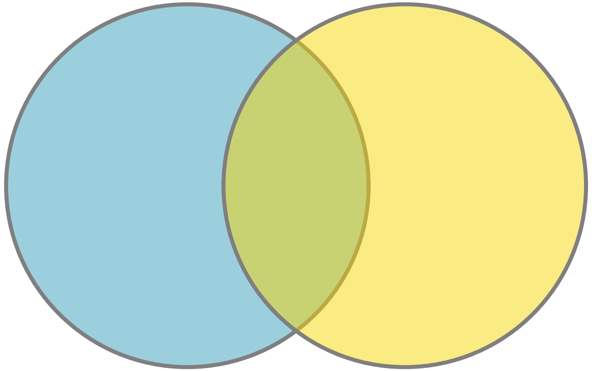
- If a set S has n elements, then $P(S)$ has 2^n elements.
  - Hint: Try to leverage the Binomial theorem.

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n,$$

Venn Diagram

Venn Diagram: Definition

A Venn diagram is used to show/visualise the possible relations among a collection of sets.

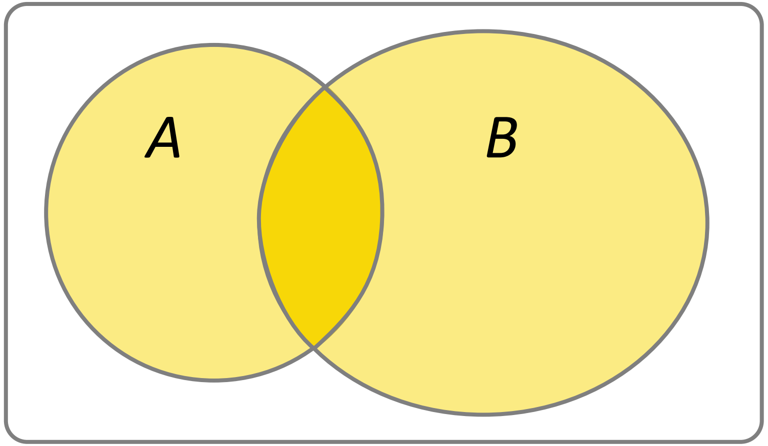


John Venn
(1834 - 1923)

Venn Diagram: Union and Intersection

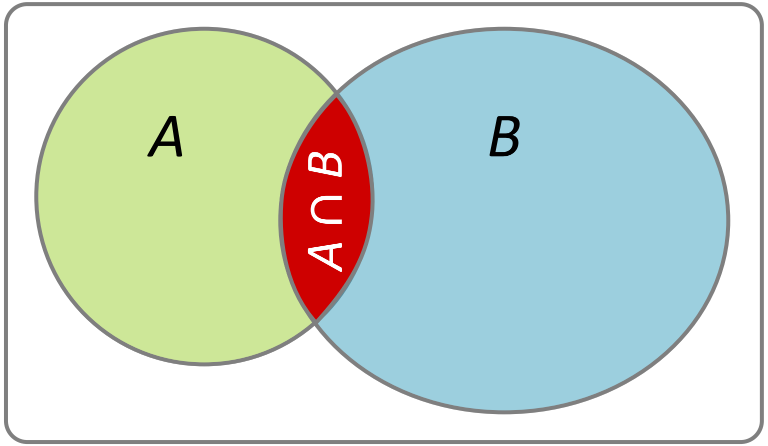
The **union of sets** A **and** B is the set of those elements that are either in A, in B, or both.

$$A \cup B \triangleq \{x \mid x \in A \vee x \in B\}$$



The **intersection of the sets** A **and** B is the set of all elements that are in both A and B.

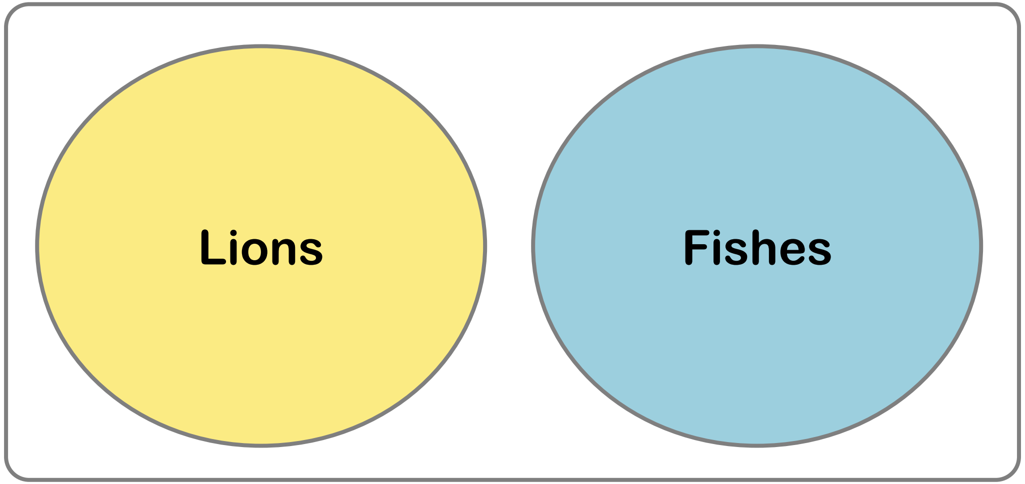
$$A \cap B \triangleq \{x \mid x \in A \wedge x \in B\}$$



Venn Diagram: Disjoint Sets

Sets A and B are **disjoint** iff $A \cap B = \varnothing$

$|A \cap B| = 0$

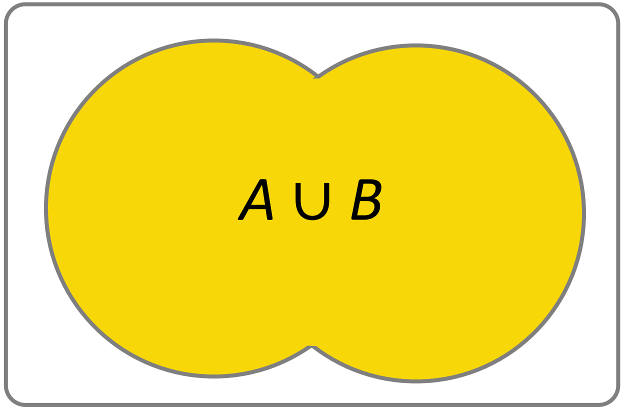


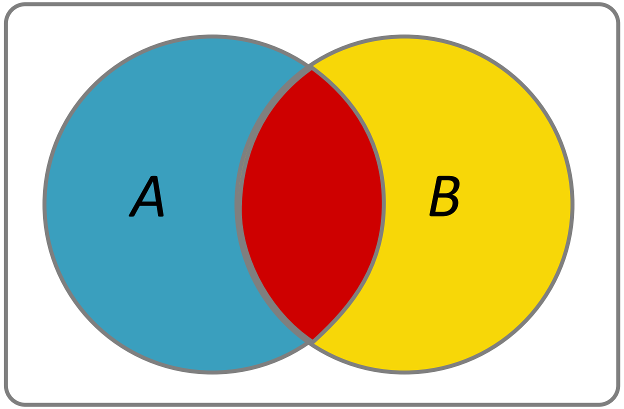
Lions ∩ Fishes = ∅

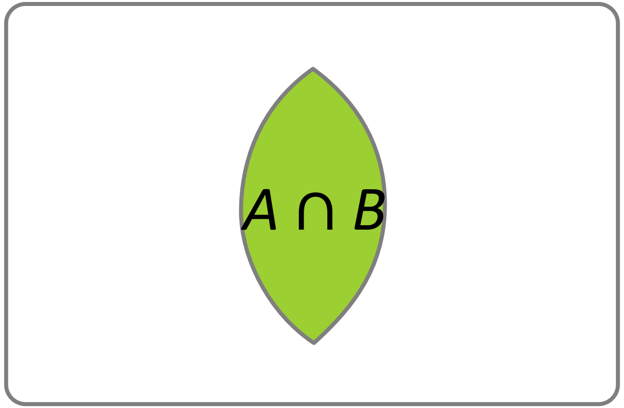
What about the Merlion?

Venn Diagram: Cardinality of Union

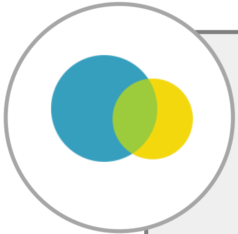
$$|A \cup B| = |A| + |B| - |A \cap B|$$





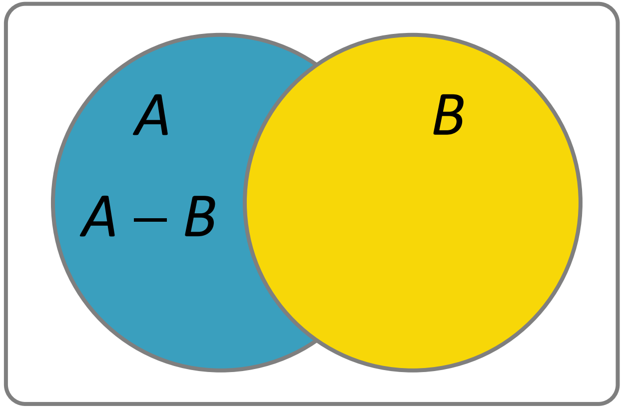


Venn Diagram: Set Difference and Complement



The **difference of A and B** (or **complement of B with respect to A**) is the set containing those elements that are in A but not in B.

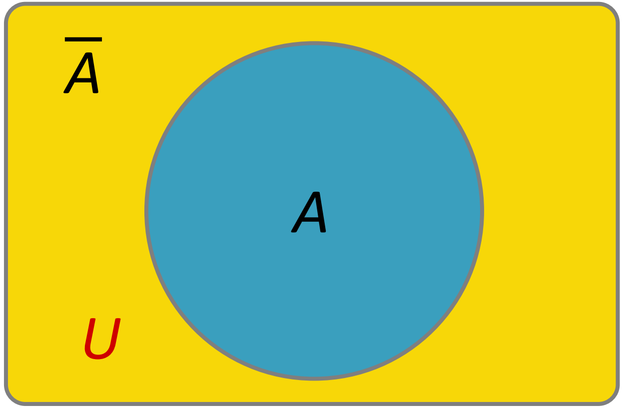
$$A - B \triangleq \{x \mid x \in A \wedge x \notin B\}$$



Venn Diagram: Set Difference and Complement

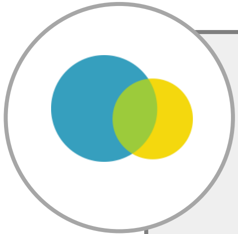
The **complement of A** is the complement of A with respect to U.

$$\overline{A} = U - A \triangleq \{x \mid x \notin A\}$$



Cartesian Product

Cartesian Product: Definition



The **Cartesian product** $A \times B$ of the sets A and B is the set of all **ordered pairs** (a,b) where $a \in A$ and $b \in B$.

$$A \times B \triangleq \{(a,b) \mid a \in A \wedge b \in B\}$$

René Descartes (1596 - 1650)

Portrait of René Descartes by André Hatala under WikiCommons (PD-US)

Cartesian Product: Example

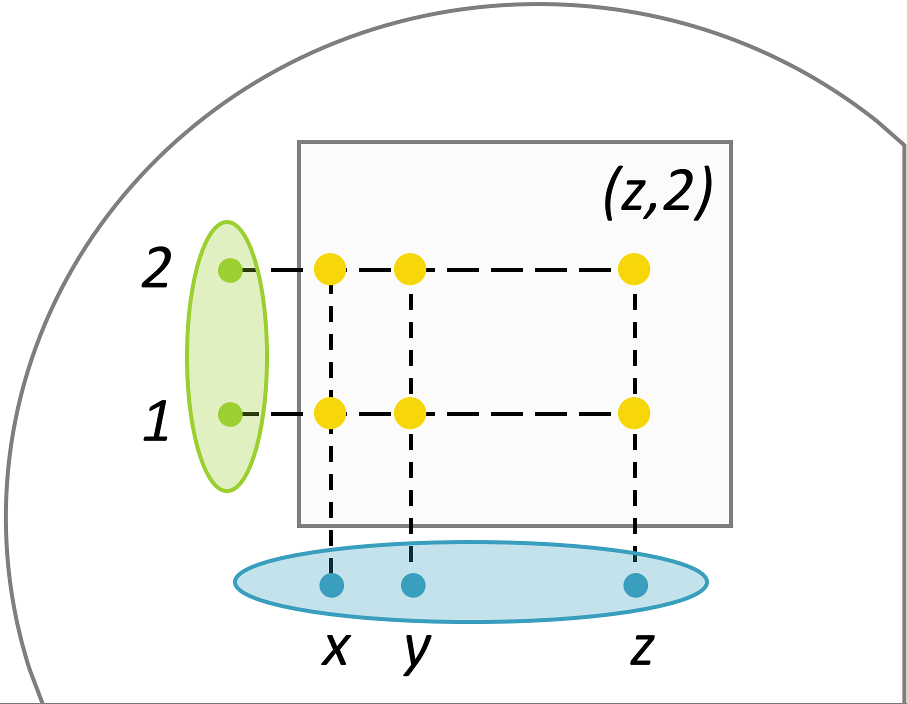
$A = \{1,2\}$, $B = \{x,y,z\}$

$A \times B = \{(1,x), (1,y), (1,z), (2,x), (2,y), (2,z)\}$

$B \times A = \{(x,1), (x,2), (y,1), (y,2), (z,1), (z,2)\}$

In general: $A_1 \times A_2 \times \ldots \times A_n \triangleq \{(a_1,a_2, \ldots, a_n) \mid a_i \in A_i$ **for** $i = 1,2, \ldots, n\}$

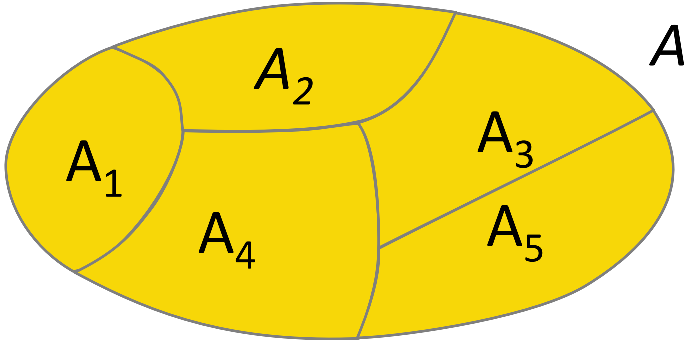
$|A_1 \times A_2 \times \ldots \times A_n| = |A_1|\,|A_2| \ldots |A_n|$



Partition

Partition: Definition

A collection of nonempty sets $\{A_1, A_2, ..., A_n\}$ is a **partition** of a set A, iff $A = A_1 \cup A_2 \cup \dots A_n$ and $A_1, A_2, ..., A_n$ are **mutually disjoint**, i.e., $A_i \cap A_j = \emptyset$ for all $i, j = 1, 2, ..., n$, and $i \neq j$.



Topic Summary

Let's recap…

- Sets:
  - Membership
  - Subset
  - Null set
  - Equality

- Venn diagram

Let's recap…

- Set operations:
  - Union
  - Intersection
  - Complement
  - Difference

- Cartesian Product

- Partition

NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

Discrete Mathematics

MH1812

Topic 7.2 - Set Theory II

Dr. Guo Jian

Topic Overview

What's in store…

Set Identities

Proving Set Identities

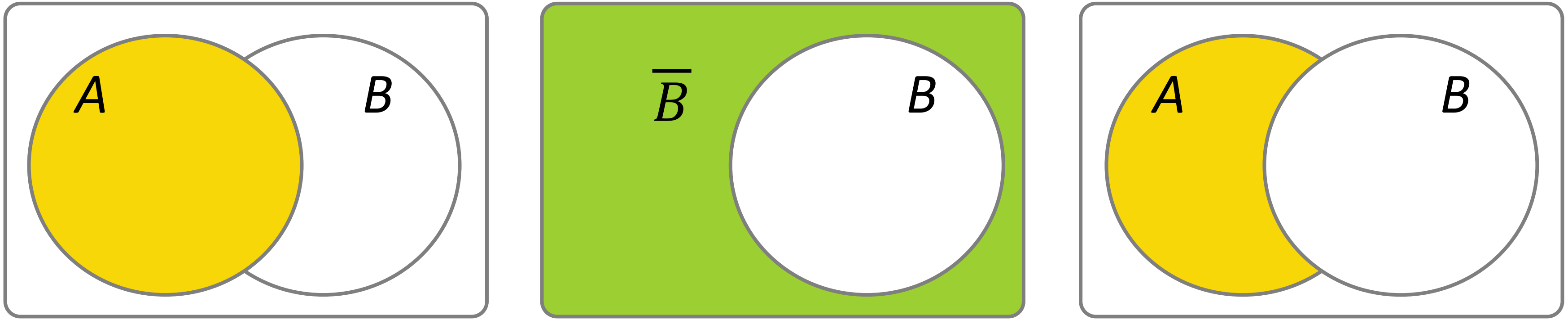
By the end of this lesson, you should be able to…

- Explain the different types of set identities.
- Apply the three methods to prove set identities.

Set Identities

Set Identities: Set Difference

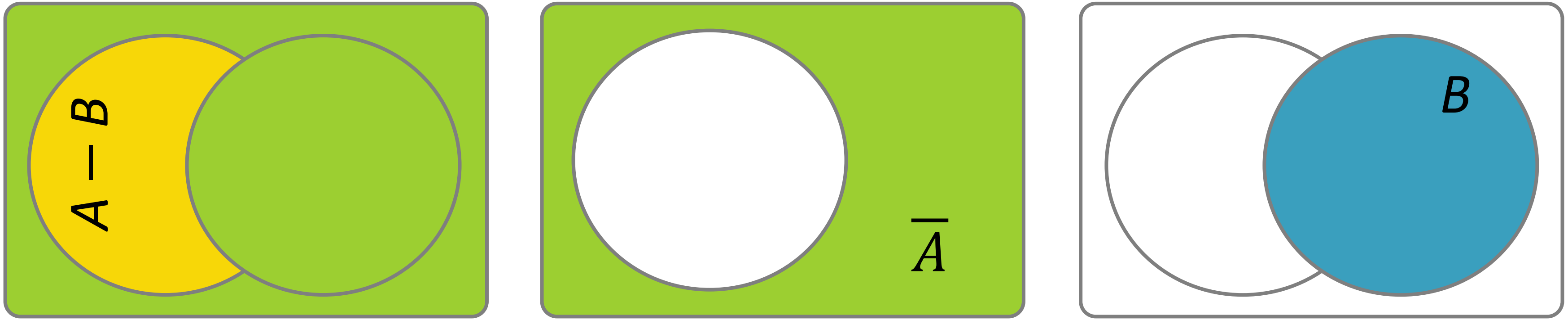
$$A \cap \overline{B} = A - B$$



Compare $A \cap \overline{B}$ **with** $A - B = \{x \mid x \in A \wedge x \notin B\}$

Set Identities: Set Difference

$$\overline{A \cap \overline{B}} = \overline{A} \cup B$$



- Consider $\overline{A - B} = \overline{A \cap \overline{B}}$
- This is De Morgan's Law $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$ with $X = A$ and $Y = \overline{B}$

Set Identities: Laws

Identity	Name
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{\overline{A}} = A$	Double Complement laws

Set Identities: Laws

Identity	Name
$A \cup B = B \cup A$ $A \cap B = B \cap A$	**Commutative laws**
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	**Associative laws**
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	**Distributive laws**
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	**De Morgan's laws**

Set Identities: Laws

Identity	Name
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	**Absorption laws**
$A - B = A \cap \overline{B}$	**Alternate representation for set difference**

Proving Set Identities

Proving Set Identities: Three Methods

- Recall: two sets are **equal if and only if** they contain exactly the same elements, i.e., iff $A \subseteq B$ and $B \subseteq A$.

Three Methods to Prove Set Identities

- Show that each set is a subset of the other
- Apply set identity theorems
- Use membership table

Proving Set Identities: Each Others' Subset

Show that $(B - A) \cup (C - A) = (B \cup C) - A$

For any $x \in \text{LHS}$, $x \in (B - A)$ **or** $x \in (C - A)$ **(or both)**

When $x \in B - A$

$\Rightarrow (x \in B) \wedge (x \notin A)$

$\Rightarrow (x \in B \cup C) \wedge (x \notin A)$

$\Rightarrow x \in (B \cup C) - A$

When $x \in C - A$

$\Rightarrow (x \in C) \wedge (x \notin A)$

$\Rightarrow (x \in B \cup C) \wedge (x \notin A)$

$\Rightarrow x \in (B \cup C) - A$

Therefore LHS $\subseteq$ RHS

Proving Set Identities: Each Others' Subset

Show that $(B - A) \cup (C - A) = (B \cup C) - A$

For any $x \in \text{RHS}$, $x \in (B \cup C)$ **and** $x \notin A$

When $x \in B$ **and** $x \notin A$	$(x \in B) \land (x \notin A)$	$\Rightarrow x \in B - A$
		$\Rightarrow x \in (B - A) \cup (C - A)$
When $x \in C$ **and** $x \notin A$	$(x \in C) \land (x \notin A)$	$\Rightarrow x \in C - A$
		$\Rightarrow x \in (B - A) \cup (C - A)$

Therefore RHS $\subseteq$ LHS

With LHS $\subseteq$ RHS and RHS $\subseteq$ LHS, we can conclude that LHS = RHS.

Proving Set Identities: Using Set Identity Theorems

Show that $(A - B) - (B - C) = A - B$

$$
\begin{aligned}
(A-B)-(B-C) &= (A \cap \overline{B}) \cap (B \cap \overline{C}) && \textbf{(By alternate representation for set difference)} \\
&= (A \cap \overline{B}) \cap (\overline{B} \cup C) && \textbf{(By De Morgan's laws)} \\
&= [(A \cap \overline{B}) \cap \overline{B}] \cup [(A \cap \overline{B}) \cap C] && \textbf{(By Distributive laws)} \\
&= [A \cap (\overline{B} \cap \overline{B})] \cup [A \cap (\overline{B} \cap C)] && \textbf{(By Associative laws)} \\
&= (A \cap \overline{B}) \cup [A \cap (\overline{B} \cap C)] && \textbf{(By Idempotent laws)} \\
&= A \cap [\overline{B} \cup (\overline{B} \cap C)] && \textbf{(By Distributive laws)} \\
&= A \cap \overline{B} && \textbf{(By Absorption laws)} \\
&= A - B && \textbf{(By alternate representation for set difference)}
\end{aligned}
$$

Proving Set Identities: Using Membership Tables

Similar to truth table (in propositional logic):

- Columns for different set expressions
- Rows for all combinations of memberships in constituent sets
- "*1*" = membership, "*0*" = non-membership
- Two sets are equal iff they have identical columns

Proving Set Identities: Using Membership Tables

Prove that $(A \cup B) - B = A - B$

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
0	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

Topic Summary

Let's recap…

- Set identities

- Prove set identities:
  - Each others' subset
  - Set identity theorems
  - Membership table