Exercise 45. How many ternary strings of length 4 have zero ones?

Solution. We are looking at strings of length 4, and ternary means that the symbols are 0,1 and 2. How if 1 is forbidden, in the first position of the string, we have only 2 choices, and 2 choices for the 2nd, 3rd, and 4rth position. Then a total of 2^4 choices.

Exercise 46. How many permutations are there of the word "repetition"?

Solution. It is a word of length 10. Suppose we want to permute R, we have 10 choices. Now that R is fixed, we are left with 9 slots to fill. Let us try to put the E. There are two E. Thus we have $C(9, 2)$ ways to put them, since we do not distinguish between the two of them. Then we have $C(7, 1) = 7$ for P, $C(6, 2)$ for T, $C(4, 2)$ for I, 2 choices for O, and 1 spot left for N. The total is thus

$$10 \cdot C(9,2) \cdot 7 \cdot C(6,2) \cdot C(4,2) \cdot 2 = 10 \cdot \frac{9!}{7!2} \cdot 7 \cdot \frac{6!}{4!2} \cdot \frac{4!}{4} \cdot 2.$$

We can simplify this expression to get

$$10 \cdot 36 \cdot 7 \cdot 15 \cdot 6 \cdot 2.$$

Alternatively, we can use the formula

$$\frac{10!}{2!2!2!} = 5 \cdot 9 \cdot 4 \cdot 7 \cdot 3 \cdot 5 \cdot 4 \cdot 3 \cdot 2$$

and both of them give the same solution!

Exercises for Chapter 6

Exercise 47. Consider the linear recurrence $a_n = 2a_{n-1} - a_{n-2}$ with initial conditions $a_1 = 3$, $a_0 = 0$.

- Solve it using the backtracking method.
- Solve it using the characteristic equation.

Solution. • We have $a_n = 2a_{n-1} - a_{n-2}$, thus $a_{n-1} = 2a_{n-2} - a_{n-3}$, $a_{n-2} = 2a_{n-3} - a_{n-4}$, $a_{n-3} = 2a_{n-4} - a_{n-5}$, etc therefore

$$\begin{aligned} a_n &= 2a_{n-1} - a_{n-2} \\ &= 2(2a_{n-2} - a_{n-3}) - a_{n-2} = 3a_{n-2} - 2a_{n-3} \\ &= 3(2a_{n-3} - a_{n-4}) - 2a_{n-3} = 4a_{n-3} - 3a_{n-4} \\ &= 4(2a_{n-4} - a_{n-5}) - 3a_{n-4} = 5a_{n-4} - 4a_{n-5} \\ &= \dots \end{aligned}$$

We see that a general term is $(i+1)a_{n-i} - ia_{n-(i+1)}$. Therefore the last term is when $n - i - 1 = 0$ that is $i = n - 1$, for which we have $na_1 - (n-1)a_0$, therefore with initial condition $a_0 = 0$ and $a_1 = 3$, we get

$$a_n = 3n.$$

Optional. Now if one wants to be sure that this is indeed the right answer, this can be checked using a proof by mathematical induction! However here, the mathematical induction is slightly different from our usual one! We have

$$P(n) = \text{“}a_n = 3n\text{”},$$

so the basis step which is $P(0) =$ “$a_0 = 0$” holds. However we will also need a second basis step, which is $P(1) =$ “$a_1 = 3$”, which still holds. Now suppose $P(k) =$ “$a_k = 3k''$ and $P(k-1) =$ “$a_{k-1} = 3(k-1)''$ are both true. Then

$$\begin{aligned} a_{k+1} &= 2a_k - a_{k-1} \\ &= 6k - 3(k-1) \\ &= 6k - 3k + 3 = 3k + 3 = 3(k+1) \end{aligned}$$

as needed, where we used both our induction hypotheses!

• Suppose now we want to solve the same recurrence using a characteristic equation. We have $x^n = 2x^{n-1} - x^{n-2}$ that is

$$x^n - 2x^{n-1} + x^{n-2} = 0 \iff x^{n-2}(x^2 - 2x + 1) = 0.$$

We have $x^2 - 2x + 1 = (x-1)^2$, therefore

$$a_n = u + vn.$$

Then

$$a_0 = u = 0,\ a_1 = u + v = 3$$

thus $v = 3$, yielding

$$a_n = 3n.$$

Exercise 48. What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Solution. The characteristic equation is $x^2 - x - 2 = 0$. Its roots are $x = -1$ and $x = 2$ since $(x+1)(x-2) = 0$. Therefore $a_n = u2^n + v(-1)^n$ is a solution. We are left with identifying u, v using the initial conditions.

$$a_0 = 2 = u + v,\ a_1 = 2u - v = 7.$$

So $u = 3, v = -1$, therefore

$$a_n = 3 \cdot 2^n - (-1)^n.$$

Exercise 49. Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a linear homogeneous recurrence. Assume both sequences a_n, a'_n satisfy this linear homogeneous recurrence. Show that $a_n + a'_n$ and αa_n also satisfy it, for α some constant.

Solution. We have

$$\begin{aligned} a_n + a'_n &= (c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}) + (c_1 a'_{n-1} + c_2 a'_{n-2} + \ldots + c_k a'_{n-k}) \\ &= c_1(a_{n-1} + a'_{n-1}) + c_2(a_{n-2} + a'_{n-2}) + \ldots + c_k(a_{n-k} + a'_{n-k}). \end{aligned}$$

Thus $a_n + a'_n$ is a solution of the recurrence. Similarly

$$\begin{aligned} \alpha a_n &= \alpha(c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}) \\ &= c_1 \alpha a_{n-1} + c_2 \alpha a_{n-2} + \ldots + c_k \alpha a_{n-k}. \end{aligned}$$

Therefore αa_n is a solution of the recurrence.

Exercise 50. Solve the two following two recurrence relations:

$$a_n = 3a_{n-1},\ a_1 = 4$$

and

$$b_n = 4b_{n-1} - 3b_{n-2},\ b_1 = 0,\ b_2 = 12.$$

Solution. The first one is easier to solve using backtracking:

$$a_n = 3a_{n-1} = 3(3a_{n-2}) = 9(3a_{n-3}) = 3^i a_{n-i} = 3^{n-1}a_1 = 4 \cdot 3^{n-1}.$$

We can check it by mathematical induction: For $n = 1$, we have $a_1 = 4$ as needed. Then suppose $a_n = 4 \cdot 3^{n-1}$.

$$a_{n+1} = 3a_n = 3(4 \cdot 3^{n-1}) = 4 \cdot 3^n$$

as needed.

The second one is easier to solve using the characteristic equation:

$$x^n = 4x^{n-1} - 3x^{n-2} \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0$$

therefore

$$b_n = c3^n + d$$

with

$$3c + d = 0,\ 9c + d = 12.$$

Thus $c = 2$, $d = -6$ and

$$b_n = 2 \cdot 3^n - 6.$$

Exercise 51. Solve the following linear recurrence relation:

$$b_n = 4b_{n-1} - b_{n-2},\ b_0 = 2,\ b_1 = 4.$$

Solution. Since

$$x^n = 4x^{n-1} - x^{n-2} \iff x^{n-2}(x^2 - 4x + 1) = 0$$

The characteristic equation is

$$x^2 - 4x + 1 = 0.$$

The roots are

$$\frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}.$$

The general solution is

$$b_n = u(2+\sqrt{3})^n + v(2-\sqrt{3})^n.$$

The initial conditions tell us that

$$b_0 = u + v = 2,\ b_1 = u(2+\sqrt{3}) + v(2-\sqrt{3}) = 4.$$

Thus $u = 2 - v$ and

$$4 = (2-v)(2+\sqrt{3})+v(2-\sqrt{3}) = 4+2\sqrt{3}-2v-v\sqrt{3}+2v-v\sqrt{3} = 4+2\sqrt{3}-2v\sqrt{3}$$

showing that $2\sqrt{3} = 2v\sqrt{3}$ that is $v = 1$ and thus $u = 1$. The final solution is then

$$b_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n.$$

Exercises for Chapter 7

Exercise 52. 1. Show that

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

for $1 \le k \le l$, where by definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},\ n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

2. Prove by mathematical induction that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

You will need 1. for this!

3. Deduce that the cardinality of the power set $P(S)$ of a finite set S with n elements is 2^n.

Solution. To prove

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k},$$

we first expand the left hand side:

$$\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!(n-k+1)}{k!(n-k+1)!} + \frac{n!k}{k!(n-k+1)!}$$