EE2007 Engineering Mathematics II
Tutorial 7

1. a. Find all the values of $\ln\left(i^{1/2}\right)$.

   b. Find all the values of $(i)^i$ and show that they are all real. Hence, or otherwise, find all z such that z^i is real.

[Ans: $i\left(\dfrac{\pi}{4}+n\pi\right), n=0,\pm 1,\pm 2,\ldots;\ \exp\left[-\left(\dfrac{\pi}{2}+2n\pi\right)\right], n=0,\pm 1,\pm 2,\ldots;$

$z=e^{\pm k\pi}e^{i\theta}, k=0,1,2,\ldots$]

2. In the following, does the limit of $f(z)$ at the origin exists?

   a. $f(z)=\dfrac{x^2 y}{x^3+y^3}+ixy$

   b. $f(z)=\dfrac{\bar{z}}{z}-\dfrac{z}{\bar{z}}-\dfrac{z^2}{\bar{z}^2}$

[Ans: No; No]

3. Determine whether $f(z)$ is continuous at the origin.

   a. $f(z)=\begin{cases}\operatorname{Re}\left[\dfrac{z}{|z|}\right] & z\neq 0\\ 0 & z=0\end{cases}$

   b. $f(z)=\begin{cases}\operatorname{Im}\left[\dfrac{z}{1+|z|}\right] & z\neq 0\\ 0 & z=0\end{cases}$

[Ans: No; Yes]

4. For $f(z)=\begin{cases}\operatorname{Im}\left[\dfrac{z}{|z|}\right] & z\neq 0\\ 0 & z=0\end{cases}$, determine if $f(z)$ is continuous at z = 0, z = 5 and z = 5 + i.

[Ans: No; Yes; Yes]

EE2007 Engineering Mathematics II
Tutorial 8

1. a. Find the constants a and b such that $f(z) = (2x - y) + i(ax + by)$ is differentiable for all z. Hence, find $f'(z)$.

   b. Are the following functions analytic?

   (i) $f(z) = \mathrm{Re}\left[z^2\right]$

   (ii) $f(z) = \dfrac{i}{z^4}$

   (iii) $f(z) = z - \bar{z}$

   (iv) $f(z) = e^x\left(\sin y - i\cos y\right)$

2. a. Find $f'(z)$, the derivative of $f(z) = 2xy - ix^2$. State clearly the point (or points) where $f'(z)$ exists.

   b. Using the Cauchy-Riemann equations, determine the analyticity of the function $f(z) = z^2 - 2z + 3$ and find its derivative.

3. Evaluate $\int_C f(z)dz$ where

   a. $f(z) = \mathrm{Re}[z]$, C the parabola $y = x^2$ from 0 to 1 + i.

   b. $f(z) = 4z - 3$, C the straight line segment from i to 1 + i.

   c. $f(z) = e^z$, C the boundary of the square with vertices 0, 1, 1 + i, and i (clockwise).

   d. $f(z) = \mathrm{Im}\left[z^2\right]$, C the boundary of the square with vertices 0, 1, 1 + i, and i (clockwise).

Answers:

1. a. $a = 1$, $b = 2$, $f'(z) = 2 + i$

b. No; Yes for $z \neq 0$; No; Yes for all z

2. a. on x-axis, $f'(z) = -i2x$

   b. $\forall z$, $f'(z) = 2z - 2$

3. a. $\dfrac{1}{2} + i\dfrac{2}{3}$

   b. $-1 + i4$

   c. 0

   d. $1 - i$

EE2007 Engineering Mathematics II
Tutorial 9

1. Integrate $\frac{1}{z^4 - 1}$ counterclockwise around the circle:
   a. $|z-1| = 1$
   b. $|z-3| = 1$

[Ans: $\frac{\pi i}{2}$, 0]

2. Evaluate the following integrals where C is any simple closed path such that all the singularities lie inside C (CCW).
   a. $\int_C \frac{5z}{z^2+4} dz$
   b. $\int_C \frac{z+e^z}{z^3-z} dz$

[Ans: $10\pi\, i$, $2\pi\, i\left(-1+\frac{e+e^{-1}}{2}\right)$]

3. Evaluate the following real integrals using the complex integration method.
   a. $\int_0^{2\pi} \frac{d\theta}{5-3\sin\theta}$
   b. $\int_0^{2\pi} \frac{\cos\theta}{13-12\cos 2\theta} d\theta$

[Ans: $\frac{\pi}{2}$, 0]

4. Evaluate the following improper integrals using the complex integration method.
   a. $\int_{-\infty}^{\infty} \frac{x}{\left(x^2-2x+2\right)^2} dx$
   b. $\int_{-\infty}^{\infty} \frac{1}{\left(4+x^2\right)^2} dx$

[Ans: $\frac{\pi}{2}$, $\frac{\pi}{16}$]

EE2007 Engineering Mathematics II
Tutorial 10

1. If $f(x, y, z) = 3x^2 y - y^3 z^2$, find ∇f at the point (1, -2, -1). Give a physical interpretation of ∇f. [Ans: $-12\mathbf{i} - 9\mathbf{j} - 16\mathbf{k}$]

2. Find a unit normal to the surface defined by $x^2 y + 2xz = 4$ at the point (2, -2, 3).

   [Ans: $-\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$]

3. What is meant by the term "directional derivative"? For $f(x, y) = x^2 e^y$, find the directional derivative at (-2, 0, 0) in the direction $-\mathbf{j}$. At point (-2, 0, 0), what is the maximum directional derivative? [Ans: -4, 5.6569]

4. Show that $\nabla r^n = nr^{n-2}\,\mathbf{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$.

5. If $\mathbf{v} = \mathbf{w} \times \mathbf{r}$, where $\mathbf{w}$ is a constant vector, show that $\mathbf{w} = \frac{1}{2}\nabla \times \mathbf{v}$.

6. Determine curl $\left(xy^2 z\mathbf{i} + 2x^3 y\mathbf{j} + 4x^2 y^2\mathbf{k}\right)$ and curl $\left(yz^3\mathbf{i} + xz\mathbf{j} + 2x\mathbf{k}\right)$ at the point (1, 1, -1).

   [Ans: $8\mathbf{i} - 7\mathbf{j} + 8\mathbf{k}$, $-\mathbf{i} + \mathbf{j} + 0\mathbf{k}$]

EE2007 Engineering Mathematics II
Tutorial 11

1. If $\mathbf{F} = (3x^2 + 6y)\mathbf{i} - 14yz\mathbf{j} + 20xz^2\mathbf{k}$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ from (0, 0, 0) to (1, 1, 1) along the following paths C:

   a. $x = t$, $y = t^2$, $z = t^3$.

   b. the straight lines from (0, 0, 0) to (1, 0 0) to (1, 1, 0) and then to (1, 1, 1).

   c. the straight line joint (0, 0, 0) and (1, 1, 1).

[Ans: 5, $\frac{23}{3}, \frac{13}{3}$]

2. Find the work done in moving a particle in a force field given by $\mathbf{F} = 3xy\mathbf{i} - 5z\mathbf{j} + 10x\mathbf{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from t = 1 to t = 2.

[Ans: 303]

3. If $\mathbf{F} = 3xy\mathbf{i} - y^2\mathbf{j}$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve in the xy plane, $y = 2x^2$, from (0, 0) to (1, 2).

[Ans: $-\frac{7}{6}$]

4. a. Show that $\mathbf{F} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$ is a conservative force field.

   b. Find the scalar potential field.

   c. Find the work done in moving an object in this field from (1, -2, 1) to (3,1, 4).

[Ans: $V(x, y, z) = x^2 y + xz^3 + c$, 202]

5. Give a physical meaning to the expression div (**F**) where **F** is a vector field. Find the flux of the vector field **F**(x, y, z) = z **k** across the outward-oriented sphere $x^2 + y^2 + z^2 = a^2$.

[Ans: $\frac{4}{3}\pi a^3$]

EE2007 Engineering Mathematics II
Tutorial 12

1. If the vector field is $\mathbf{F}(x, y, z) = x^2\mathbf{i} + 4xy^3\mathbf{j} + y^2x\mathbf{k}$, find the work performed in bringing a particle along straight-line segments, in the plane $z = y$, from (0, 0, 0) to (0, 3, 3) to (1, 3, 3) to (1, 0, 0) to (0, 0 0).

   [Ans: -90]

2. If the vector field is $\mathbf{F}(x, y, z) = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$, calculate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA$ across the surface S which is the portion of the paraboloid $z = 4 - x^2 - y^2$, $z \geq 0$, with upward orientation and $C:\ x^2 + y^2 = 4$ forms the boundary of S on the xy plane.

   [Ans: 12π]

3. Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{k}$ across the outward-oriented sphere $x^2 + y^2 + z^2 = a^2$.

   [Ans: $\frac{4}{3}\pi a^3$]

4. Find the outward flux of the vector field $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ across the surface of the region enclosed by the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ and the plane $z = 0$.

   [Ans: $\frac{6\pi a^5}{5}$]

5. Find the outward flux of the vector field $\mathbf{F}(x, y, z) = 2x\mathbf{i} + 3y\mathbf{j} + z^2\mathbf{k}$ across the surface of a unit cube bounded by planes at $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$.

   [Ans: 6]