Tutorial 2 (Solutions) (Tutorial 8)

1 a). $f(z) = (2x-y) + i(ax+by)$.

Using the C-R equations,

$u_x = 2$ $\quad v_x = a$.

$u_y = -1$ $\quad v_y = b$.

To satisfy the C-R equations.

$u_x = v_y$, $u_y = -v_x$.

$\therefore$ $\underline{a = 1}$

$\underline{b = 2}$

The function is differentiable for all z.

$$f'(z) = u_x + i v_x$$
$$= 2 + i$$

b) (i) $f(z) = \mathrm{Re}[z^2]$

$$= \mathrm{Re}[(x+iy)^2]$$
$$= \mathrm{Re}[x^2 - y^2 + i\,2xy].$$
$$= x^2 - y^2.$$

$u_x = 2x$ $\quad v_x = 0$

$u_y = -2y$ $\quad v_y = 0$.

The C-R equations are only satisfied at $z = 0$

$\Rightarrow$ $\underline{f(z) \text{ is nowhere analytic}}$.

$\cos(90^\circ - \theta) = \sin\theta$

$\sin(90^\circ - \theta) = \cos\theta$

1 b (ii) $\quad f(z) = \dfrac{i}{z^4}$

$$= \frac{1}{r^4} e^{i\left(\frac{\pi}{2} - 4\theta\right)}$$

$$= \frac{1}{r^4}\left[\sin 4\theta + i\cos 4\theta\right].$$

$u_r = -\dfrac{4}{r^5}\sin 4\theta \qquad v_r = -\dfrac{4}{r^5}\cos 4\theta.$

$u_\theta = \dfrac{4}{r^4}\cos 4\theta. \qquad v_\theta = -\dfrac{4}{r^4}\sin 4\theta.$

$\because\ u_r = \frac{1}{r}v_\theta$ and $v_r = -\frac{1}{r}u_\theta$

the C-R equations are satisfied everywhere except at $z = 0$.(where the functions u, v are not continuous).

(iii) $\quad f(z) = z - \bar{z} = (x + iy) - (x - iy)$

$$= i\,2y.$$

$u_x = 0 \qquad v_x = 0.$

$u_y = 0 \qquad v_y = 2.$

C-R. equations are not satisfied

$\Rightarrow$ Not analytic

1 b) (iv) $f(z) = e^x(\sin y - i\cos y) = \underbrace{e^x \sin y}_{u(x,y)} + i(\underbrace{-e^x \cos y}_{v(x,y)})$

$u_x = e^x \sin y$. $\quad v_x = -e^x \cos y$.

$u_y = e^x \cos y$. $\quad v_y = e^x \sin y$.

$\because \quad u_x = v_y$ and $v_x = -u_y$

$f(z)$ is analytic everywhere in the complex plane.

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2 a). $f(z) = \underbrace{2xy}_{u} - ix^2 \quad v(x,y) = -x^2$

$u_x = 2y \quad v_x = -2x$.

$u_y = 2x \quad v_y = 0$.

~~for~~ the C-R equations to be satisfied,

$u_x = v_y$. $\Rightarrow y = 0$. (x-axis).

$v_x = -u_y$

$\Rightarrow$ C-R equations are satisfied only at x-axis

$\Rightarrow f'(z)$ exists only on x-axis

$$f'(z) = u_x + i v_x$$
$$= 2y - i2x$$
$$= -i2x$$

2b) $f(z) = z^2 - 2z + 3$.

$= (x+iy)^2 - 2(x+iy) + 3$.

$= \left(x^2 - y^2 - 2x + 3\right) + i\left(+2xy - 2y\right)$.

$u_x = 2x - 2 \qquad v_x = +2y$.

$u_y = -2y \qquad v_y = 2x - 2$.

$\therefore \quad u_x = v_y$ and $v_x = -u_y$

$\therefore$ CR equations are satisfied for all z

$\Rightarrow$ $f(z)$ is analytic for all z.

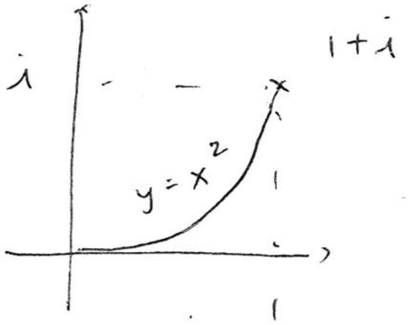
$f'(z) = u_x + i v_x$

$= 2x - 2 + i\,2y$.

$= 2z - 2$

Note - Polynomials in z are analytic in the entire z plane and the usual differentiation applies.

3 a).



$$f(z) = Re[z]$$

$$z(t) = t + i t^2 \qquad 0 \le t \le 1$$

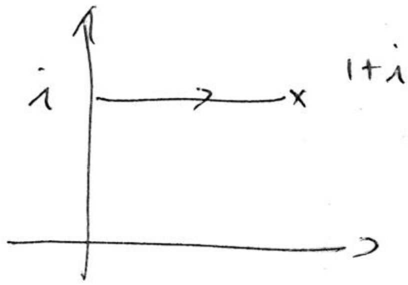
$$dz = (1 + i\,2t) \cdot dt$$

$$\therefore \int_C f(z)\,dz = \int_0^1 t \,.\, (1 + i\,2t)\,dt$$

$$= \int_0^1 (t + i\,2t^2)\,dt$$

$$= \left[\frac{t^2}{2} + i\,\frac{2t^3}{3}\right]_0^1 = \frac{1}{2} + i\,\frac{2}{3}$$

b)



$$f(z) = 4z - 3$$

$$z(t) = t + i \quad , 0 \le t \le 1$$

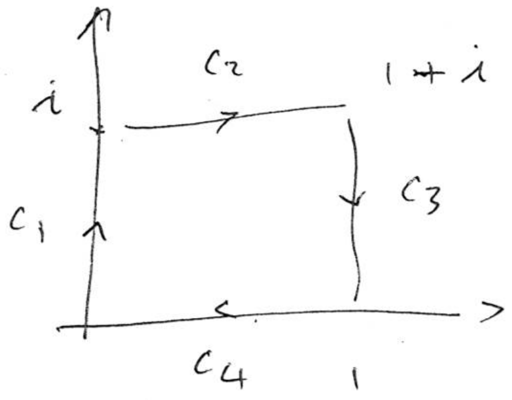
$$dz = dt$$

$$\int_C f(z)\,dz = \int_0^1 [4(t+i) - 3]\,.\,dt$$

$$= \left[2t^2 + (4i - 3)t\right]_0^1$$

$$= 2 + 4i - 3 = -1 + 4i$$

3(b) $f(z) = e^{z}$



C_1: $z(t) = it \quad 0 \le t \le 1$

$dz = i\,dt$

C_2: $z(t) = t+i \quad 0 \le t \le 1$

$dz = dt$.

C_3: $z(t) = 1+it \quad \cancel{0 \le t \le 1}.\ 1 \ge t \ge 0$

$dz = i\,dt$.

C_4: $z(t) = t \quad \cancel{0 \le t \le 1}\ 1 \ge t \ge 0$

$dz = dt$.

$$\int_c f(z)dz = \int_0^1 e^{it} \cdot i\,dt + \int_0^1 e^{t+i} \cdot dt + \int_1^0 e^{1+it} \cdot i\,dt + \int_1^0 e^t\,dt.$$

$$= \left[i \cdot \frac{e^{it}}{i} \right]_0^1 + \left[\frac{e^{t+i}}{1} \right]_0^1 + \left[i \cdot \frac{e^{1+it}}{i} \right]_1^0 + \left[e^t \right]_1^0$$

$$= (e^{i} - e^{0}) + (e^{1+i} - e^{i}) + (e^{1} - e^{1+i}) + (e^{0} - e^{1})$$

$$= \underline{\underline{0}} //.$$

3d) $f(z) = \mathrm{Im}\left[z^2\right]$.

Referencing Q3c solution,

C_1 : $f(z) = \mathrm{Im}\left[z^2\right] = \mathrm{Im}\left[t^2\right] = 0$.

C_2 : $f(z) = \mathrm{Im}\left[(t+i)(t+i)\right]$

$= \mathrm{Im}\left[t^2 - 1 + i2t\right] = 2t$.

C_3 : $f(z) = \mathrm{Im}\left[(1+it)(1+it)\right]$

$= \mathrm{Im}\left[1 - t^2 + i2t\right] = 2t$.

C_4 : $f(z) = \mathrm{Im}\left[t^2\right] = 0$.

$$\int_c f(z)\,dz = \int_0^1 0 \,.\, i\,dt + \int_0^1 2t \,.\, dt + \int_1^0 2t \,.\, i\,dt + \int_1^0 0 \,.\, dt$$

$$= \left[t^2\right]_0^1 + i\left[t^2\right]_1^0$$

$$= 1 - i$$