and $x_2 \in B - C$ which shows that $x \in A \times (B - C)$ and we have the reverse inclusion:

$$(A \times B) - (A \times C) \subseteq A \times (B - C).$$

Note that it is also possible to do a membership table, but then the membership table needs to reflect the cartesian product.

Exercises for Chapter 8

Exercise 70. Consider the sets $A = \{1, 2\}$, $B = \{1, 2, 3\}$ and the relation $(x, y) \in R \iff (x - y)$ is even. Compute the inverse relation R^{-1}. Compute its matrix representation.

Solution. The relation R is

$$(1, 1), (1, 3), (2, 2),$$

therefore the relation R^{-1} is

$$(1, 1), (3, 1), (2, 2).$$

Its matrix representation is obtained by representing B as rows, that is row 1 is $b_1 = 1$, row 2 is $b_2 = 2$, row 3 is $b_3 = 3$, while column 1 is $a_1 = 1$ and column 2 is $a_2 = 2$:

$$\begin{pmatrix} T & F \\ F & T \\ T & F \end{pmatrix}$$

Exercise 71. Consider the sets $A = \{2, 3, 4\}$, $B = \{2, 6, 8\}$ and the relation $(x, y) \in R \iff x \mid y$. Compute the matrix of the inverse relation R^{-1}.

Solution. The relation R is

$$(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)$$

thus the inverse relation R^{-1} is

$$(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)$$

that is $(x, y) \in R^{-1} \iff x$ is a multiple of y, and the corresponding matrix is

$$\begin{pmatrix} T & F & F \\ T & T & F \\ T & F & T \end{pmatrix}$$

Exercise 72. Let R be a relation from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $xRy \leftrightarrow 2|(x-y)$. Show that if n is odd, then n is related to 1.

Solution. Any odd number n can be written of the form $n = 2m+1$ for some integer m. Therefore $n-1 = 2m$ which is divisible by 2 and n is related to 1.

Exercise 73. This exercise is about composing relations.

1. Consider the sets $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2, c_3\}$ with the following relations R from A to B, and S from B to C:

$$R = \{(a_1, b_1), (a_1, b_2)\},\ S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}.$$

   What is the matrix of $R \circ S$?

2. In general, what is the matrix of $R \circ S$?

Solution. 1. Let us write the matrices of R and S first:

$$\begin{pmatrix} T & T \\ F & F \end{pmatrix} \circ \begin{pmatrix} T & F & T \\ T & T & F \end{pmatrix}.$$

Next we have that $(a, c) \in R \circ S$ whenever $aRb \wedge bSc$ for some $b \in B$. So to know, for example, whether (a_1, c_1) is in $R \circ S$, we have to check if we can find a b_i such that $(a_1, b_i) \wedge (b_i, c_1)$, that is whether

$$[(a_1, b_1) \wedge (b_1, c_1)] \vee [(a_1, b_2) \wedge (b_2, c_1)]$$

is true. But (a_i, b_j) means that the coefficient r_{ij} of the matrix R is true, and similarly (b_i, a_j) means that the coefficient s_{ij} of the matrix S is true. So we may rephrase the coefficient of the 1st row, 1st column of the matrix of $R \circ S$ as

$$(r_{11} \wedge s_{11}) \vee (r_{12} \wedge s_{21}).$$

Notice that this is almost like doing the scalar product of the first row of R with the first column of S, except that multiplication is replaced by $\wedge$, and addition by $\vee$. Therefore, we have that the matrix of $R \circ S$ is

$$\begin{pmatrix} T & T \\ F & F \end{pmatrix} \circ \begin{pmatrix} T & F & T \\ T & T & F \end{pmatrix} = \begin{pmatrix} T & T & T \\ F & F & F \end{pmatrix}$$

2. In general, we have a relation R from A to B, and a relation S from B to C, where the size of the set B is n. Denote the coefficients of the matrix of the relation R by r_{ij}, and that of the matrix of the relation S by s_{ij}. Then the matrix of $R \circ S$ will have coefficients t_{ij} given by

$$t_{ij} = (r_{i1} \wedge s_{1j}) \vee (r_{i2} \wedge s_{2j}) \vee \ldots \vee (r_{in} \wedge s_{nj}).$$

Exercise 74. Consider the relation R on $\mathbb{Z}$, given by $aRb \iff a - b$ divisible by n. Is it symmetric?

Solution. Yes it is symmetric. Suppose aRb, then $a - b$ is divisible by n. Thus $-(a - b) = b - a$ is divisible by n, and bRa holds.

Exercise 75. Consider a relation R on any set A. Show that R symmetric if and only if $R = R^{-1}$.

Solution. Consider a relation R. The relation R^{-1} is defined by pairs (y, x) such that $(x, y) \in R$. If R is symmetric, it has the property that $(x, y) \Rightarrow (y, x)$, therefore $(y, x) \in R$ and $R = R^{-1}$. Conversely, if $R = R^{-1}$, then if $(x, y) \in R$, it must be that $(y, x) \in R$ and R is symmetric.

Exercise 76. Consider the set $A = \{a, b, c, d\}$ and the relation

$$R = \{(a, a), (a, b), (a, d), (b, a), (b, b), (c, c), (d, a), (d, d)\}.$$

Is this relation reflexive? symmetric? transitive?

Solution. It is reflexive since $(a, a), (b, b), (c, c), (d, d) \in R$. It is symmetric since $(a, b), (b, a)$, $(a, d), (d, a) \in R$. It is not transitive, indeed, $(b, a), (a, d) \in R$ but $(b, d) \notin R$.

Exercise 77. Consider the set $A = \{0, 1, 2\}$ and the relation $R = \{(0, 2), (1, 2), (2, 0)\}$. Is R antisymmetric?

Solution. No, since $(0, 2)$ and $(2, 0)$ are in R, but $2 \neq 0$.

Exercise 78. Are symmetry and antisymmetry mutually exclusive?

Solution. There is no connection between symmetry and antisymmetry, so no they are not mutually exclusive. For example, the relation $A = B$ is both symmetric and antisymmetric. Then the relation "A is brother of B" is neither symmetric (if A is a brother of B, it could be that B is a sister of A) nor antisymmetric.

Exercise 79. Consider the relation R given by divisibility on positive integers, that is $xRy \leftrightarrow x|y$. Is this relation reflexive? symmetric? antisymmetric? transitive? What if the relation R is now defined over non-zero integers instead?

Solution. It is reflexive since $x|x$ always. It is not symmetric, since for example $1|y$ but y will never divide 1 if $y > 1$. It is antisymmetric, since if $x|y$ then $y = ax$ while if $y|x$ then $x = by$ and it must be that $y = ax = a(by) = aby$ and $a = b = 1$. It is transitive, since $x|y$ and $y|z$ imply $y = ax$, $z = by$ thus $z = by = b(ax)$ and $x|z$.

If we consider instead non-zero integers, the relation is not antisymmetric, indeed $y = ax = a(by) = aby$ could imply $a = b = -1$ in which case $x|y$ and $y|x$ when $y = -x$ is possible.

Exercise 80. Consider the set $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Show that the relation $xRy \leftrightarrow 2|(x - y)$ is an equivalence relation.

Solution. It is reflexive: $2|(x-x)$. It is symmetric: if $2|(x-y)$ then $(x-y) = 2n$ for some integer n, and thus $(y - x) = -2n$ showing that $2|(y - x)$. It is transitive: if $2|(x-y)$ and $2|(y-z)$, then $(x-y) = 2n$, and $(y-z) = 2m$, for some integers m, n. Therefore $x-z = (x-y)+(y-z) = 2n+2m = 2(n+m)$ and $2|(x - z)$.

Exercise 81. Show that given a set A and an equivalence relation R on A, then the equivalence classes of R partition A.

Solution. Let $a, b \in A$, and $[a], [b]$ denote their equivalence classes. It is possible that $[a] = [b]$. Suppose that this is not the case. Then we will show that $[a]$ and $[b]$ are disjoint. Suppose by contradiction that there exists one element $c \in [a] \cap [b]$. Then aRc and bRc. But R is an equivalence relation, therefore it is symmetric (and cRb) and transitive, implying that aRb. But then $b \in [a]$ and $a \in [b]$ by symmetry, and it must be that $[a] = [b]$. Indeed: we show that $[a] \subseteq [b]$ and $[b] \subseteq [a]$. Take an element x of $[a]$, then aRx, that is xRa (symmetry), and since aRb (b is in $[a]$), it must be that xRb (transitivity) and thus bRx (symmetry again), which shows that x is in $[b]$. The same reasoning will show that $[b]$ belongs to $[a]$.

Since either $[a] = [b]$ or they are disjoint, take the union of the classes $[a]$ that give distinct classes, and this gives a partition of A.

Exercise 82. Consider the set $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the relation

$$xRy \leftrightarrow \exists c \in \mathbb{Z},\ y = cx.$$

Is R an equivalence relation? is R a partial order?

Solution. R is reflexive: $xRx \leftrightarrow \exists c \in \mathbb{Z},\ x = cx$, take $c = 1$. R is not symmetric: xRy means $x = cy$, but then $y = \frac{x}{c}$ so apart if $c = \pm 1$, $\frac{1}{c}$ will not be in $\mathbb{Z}$. For example, $2R4$ since $4 = c2$ with $c = 2$, but $2 = c4$ means that c cannot be an integer. We conclude that R cannot be an equivalence relation.

Let us check antisymmetry and transitivity. Suppose $y = cx$ and $x = c'y$, then $x = c'cx$ and $c'c = 1$. So either $c = c' = -1$, which cannot happen because all elements of A are positive, or $c = c' = 1$, and the relation is antisymmetric. For transitivity, suppose $xRy \iff y = cx$, $yRz \iff z = c'y$. Then $z = c'y = c'cx$ with $cc' \in \mathbb{Z}$ thus xRz as needed. We conclude that R is a partial order.