Vector Calculus

EE2007 – Engineering Mathematics II

Scalars are quantities that only have a magnitude like mass, speed, and electric field strength.

Many times it is often useful to have a quantity that has not only a magnitude but also a direction; such a quantity is called a vector.

Examples of quantities represented by vectors include velocity, acceleration, and virtually any type of force (frictional, gravitational, electric, magnetic, etc.).

Let us look at an example as to why vectors are useful. Suppose a plane traveling at 300 mph to the north with no wind present encounters a westerly crosswind of 50 mph. Then, the resultant velocity of the plane is the sum of the velocities of the wind and the plane.

At the end of these series of lectures, you should be able to:

- Describe the vector fundamentals. (1 hr)
- Perform vector differentiation. (1 hr)
- Work the "∇" (del) operator. (2 hrs)
  - grad, div, curl
- Perform Line Integral. (3 hrs)
- Perform Surface Integral. (2 hrs)
- Perform Volume Integral. (1 hr)
- Apply Divergence, Stokes and Green's Theorems. (2 hrs)

At the end of this lesson, you should be able to:

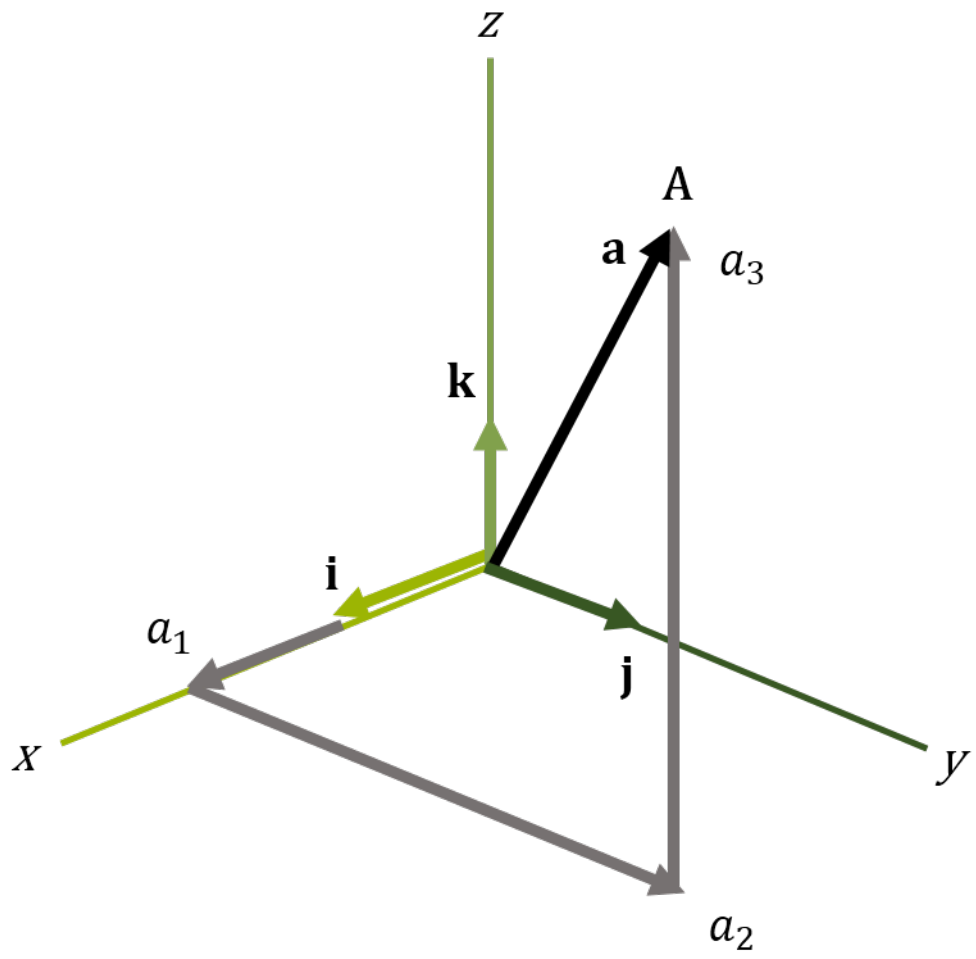
- Describe the fundamentals of vector calculus.

A vector is a quantity that has both magnitude and direction.

For example, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are standard Cartesian unit vectors along *x, y, z* directions.

A vector has both direction and magnitude. It points from tail to head. The tail is always at the origin of the coordinate system. The head is at point A with coordinates (a_1, a_2, a_3).



Given below are the coordinates of unit vectors in *x*, *y* and *z* directions respectively.

$\mathbf{k}$ is a unit vector in the *z* direction, it has coordinates (0, 0, 1). So,

$$\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\mathbf{i}$ is a unit vector in the *x* direction, it has coordinates (1, 0, 0). So,

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbf{j}$ is a unit vector in the *y* direction, it has coordinates (0, 1, 0). So,

$$\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

In the equation $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{a}$ is a 3 x 1 column vector as shown below.

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

$$= a_1\begin{bmatrix}1\\0\\0\end{bmatrix} + a_2\begin{bmatrix}0\\1\\0\end{bmatrix} + a_3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}a_1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\a_2\\0\end{bmatrix} + \begin{bmatrix}0\\0\\a_3\end{bmatrix}$$

$$= \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$$

A 3D (column) vector $\mathbf{v} = [2 \quad 3 \quad -5]^T$ can be expressed in the given form.

$$\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$$

$$\mathbf{v} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 5\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$$

The transpose of a column vector becomes a row vector. Therefore, it can also be written as:

$$\mathbf{v}^{\mathrm{T}} = [2 \quad 3 \quad -5]$$

Let us look at an example to understand the vector representation.

A vector which represents the position of a particle with coordinates (x, y, z) that is, position vector, can be written as:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

A vector which represents the force on the particle can be expressed as $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ where, F_1, F_2, and F_3 are components of $\mathbf{F}$ in the directions of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ respectively.

Vectors are denoted either by boldfaced alphabets or alphabets with an arrow above. For example: $F, f, \text{or } \vec{f}$.

Let us look at an example to find out the magnitude of the vector and unit vector.

Example 1

If $\mathbf{a} = 2\mathbf{i} + 1\mathbf{j} + 2\mathbf{k}$, then

$$\|\mathbf{a}\| = \sqrt{2^2 + 1^2 + 2^2} \quad = \sqrt{9} = 3$$

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{3}(2\mathbf{i} + 1\mathbf{j} + 2\mathbf{k})$$

$$= \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right) = [2/3 \quad 1/3 \quad 2/3]^{\mathrm{T}}$$ is a unit vector.

In general, given any non-zero vector $\mathbf{v}$, $\hat{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|$, is a unit vector in the same direction as $\mathbf{v}$.

Let us look at an example to find out the magnitude of the vector and unit vector.

Example 2

Alternatively $\mathbf{v} = \|\mathbf{v}\|\,\hat{\mathbf{v}}$

Find $\mathbf{v}$ if $\mathbf{v}$ has a length of 5 units in the direction of $2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$

$$\hat{\mathbf{v}} = \frac{1}{\sqrt{2^2 + 3^2 + 5^2}}(2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})$$

$$\hat{\mathbf{v}} = \frac{1}{\sqrt{38}}(2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})$$

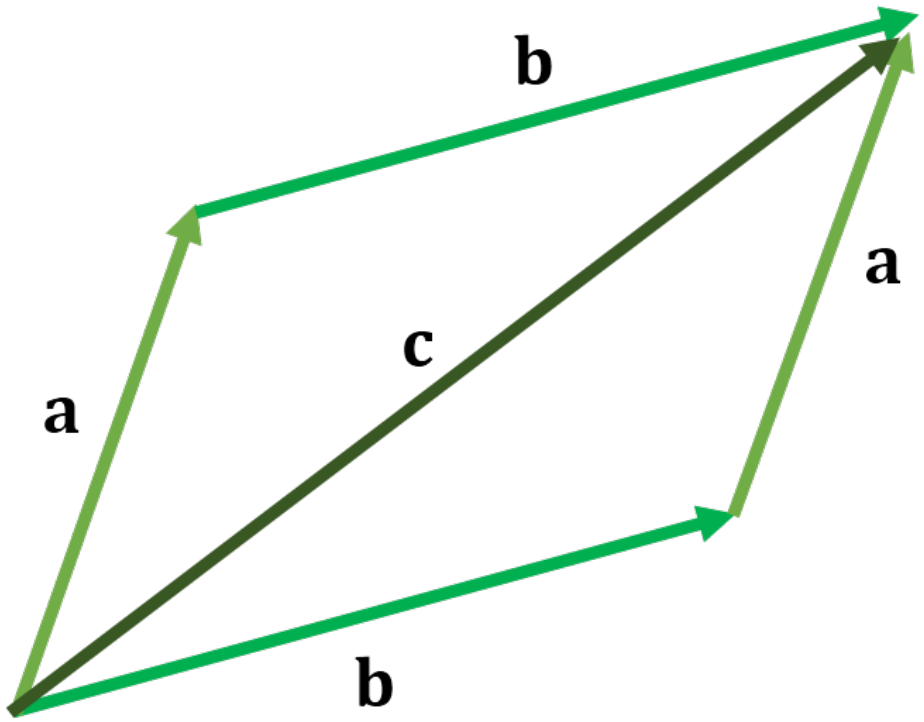
$$\mathbf{v} = 5\left[\frac{1}{\sqrt{38}}(2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})\right]$$

$$\mathbf{v} = \frac{5}{\sqrt{38}}(2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})$$

Let us now understand the concept of vector algebra.

For these vectors, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$



Vector Addition

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}$$
$$= \mathbf{b} + \mathbf{a}$$

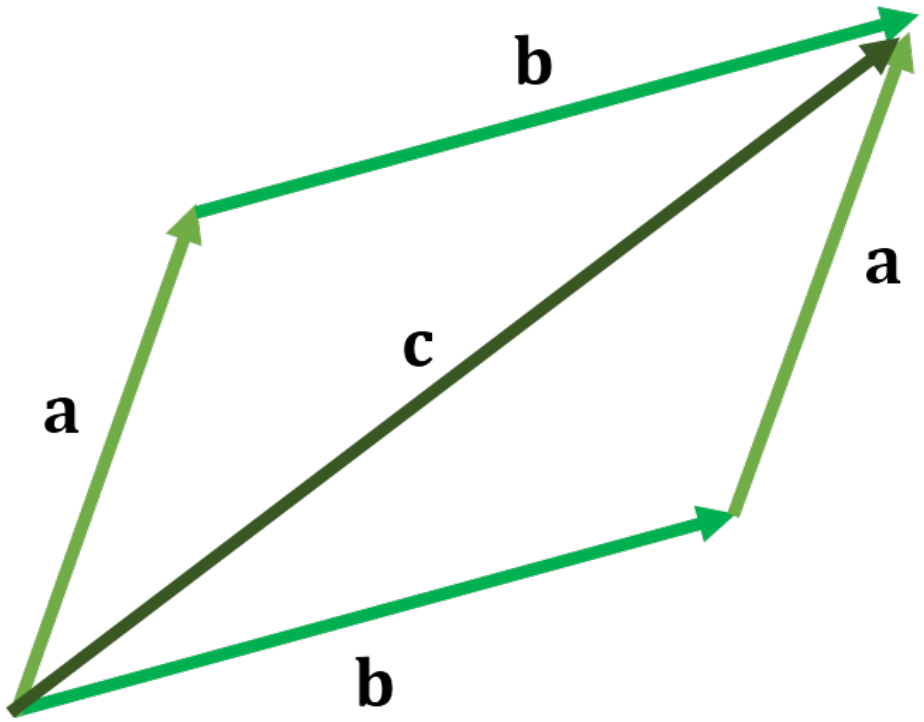
Vector Subtraction

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

Let us now understand the concept of vector algebra.

For these vectors, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$



Scalar Multiplication by a constant *c*

$$c\,\mathbf{a} = c\,a_1\mathbf{i} + c\,a_2\mathbf{j} + c\,a_3\mathbf{k}$$

$$c(\mathbf{a} + \mathbf{b}) = (c\,\mathbf{a} + c\,\mathbf{b})$$

Vector Product

Dot (scalar) product, $\mathbf{a} \cdot \mathbf{b}$ = **scalar quantity**

Cross (vector) product, $\mathbf{a} \times \mathbf{b}$ = **vector quantity**

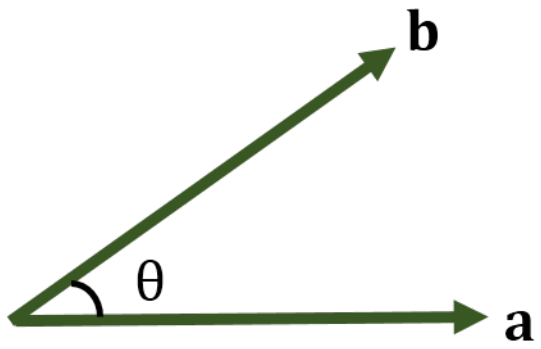
Let us now become familiar with the concept of dot (scalar) product.

The dot product can be expressed as:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta \qquad \text{- (1)}$$

$$= (a_1 b_1) + (a_2 b_2) + (a_3 b_3) \qquad \text{- (2)}$$

$$= \mathbf{b} \cdot \mathbf{a} = \text{scalar}$$



Equation (1) gives the geometric meaning of dot product.

Equation (2) shows the algebraic way of computing the dot product.

Let us now become familiar with the concept of dot (scalar) product.

$$\mathbf{a} \cdot \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

The result of a dot product is a scalar (real value). Thus, the dot product is commutative.

$$\mathbf{a} \cdot \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = {a_1}^2 + {a_2}^2 + {a_3}^2$$

$$\sqrt{\mathbf{a} \cdot \mathbf{a}} = \|\mathbf{a}\|$$

Some peculiarities for $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$ are displayed below.

$\mathbf{a} \| \mathbf{b}$ parallel, $\theta = 0$, $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|$

Anti-parallel, $\theta = \pi$, $\mathbf{a} \cdot \mathbf{b} = -\|\mathbf{a}\|\|\mathbf{b}\|$

$\mathbf{a} \perp \mathbf{b}$ perpendicular, $\theta = {}^{\pi}/_{2}$, $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$ that is, **a** is orthogonal to **b**.

The third property is extremely important. *To show or test that the two vectors are orthogonal to each other, take their dot product. If the result is zero, they are orthogonal.*

Important:

1. $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = \mathbf{1}$ ($\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors)
2. $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = \mathbf{0}$ ($\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually orthogonal).

Let us now calculate the component of vector **a** in another direction **b**.

The component of $\mathbf{a}$ in the direction of $\mathbf{b}$ in the given figure is V.

Geometrically,

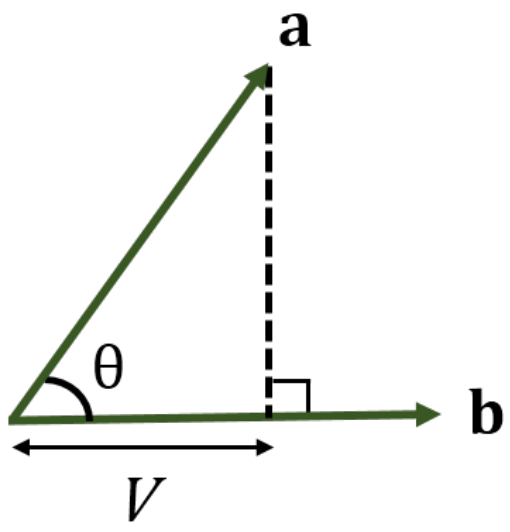
$V = \|\mathbf{a}\| \cos\theta$

Now, $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$

Dividing both sides by $\|\mathbf{b}\|$,

$$\mathbf{a} \cdot \frac{\mathbf{b}}{\|\mathbf{b}\|} = \|\mathbf{a}\| \cos\theta = V$$

$$V = \mathbf{a} \cdot \hat{\mathbf{b}}$$



Therefore, to obtain the component of $\mathbf{a}$ in the direction of $\mathbf{b}$, just take the dot product of $\mathbf{a}$ and the unit vector in the direction of $\mathbf{b}$.

Let us take a look at an example explaining the dot product concept.

Example 3

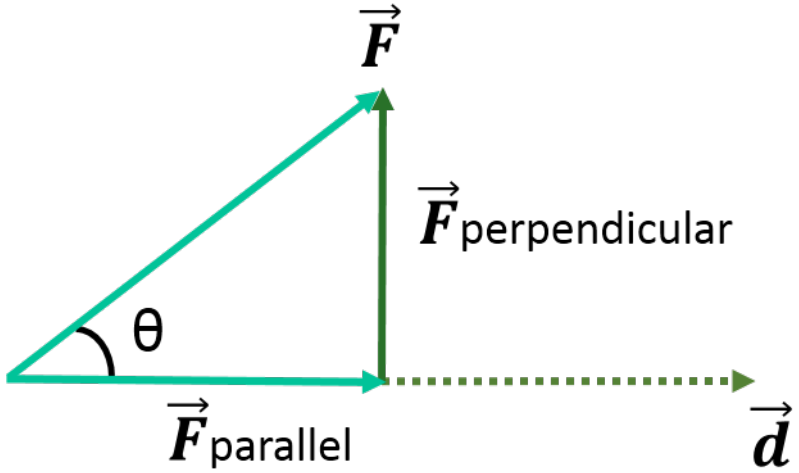
Show that force $\mathbf{F}$ is applied on an object which is moved by $\mathbf{d}$ when work is done.

$$W = \|\mathbf{F}_{\mathbf{parallel}}\|\|\mathbf{d}\|$$

$$= (\|\mathbf{F}\|\cos\theta)\|\mathbf{d}\|$$

$$= \|\mathbf{F}\|\|\mathbf{d}\|\cos\theta$$

$$= \mathbf{F} \cdot \mathbf{d}$$



Recall that the work done is equal to the distance multiplied by the component of the applied (constant) force, in the same direction as the distance moved.

Let us solve the following questions.

1. Compute the dot product of $[1 \quad 2 \quad 3]^{\mathbf{T}}$ and $\mathbf{k}$.

2. What is the length of column vector $[3 \quad 4 \quad 5]^{\mathbf{T}}$? What about for row vector $[1 \quad 2 \quad 3]$?

3. Find the angle between the vectors $[1 \quad 0 \quad 1]^{\mathbf{T}}$ and $[-2 \quad 3 \quad 2]^{\mathbf{T}}$.

4. What is the resultant vector of the sum of vectors $[1 \quad 0 \quad 1]^{\mathbf{T}}$, $\mathbf{j}$, and $[-2 \quad 3 \quad 2]^{\mathbf{T}}$?

5. Suppose $\mathbf{x} \cdot \mathbf{y} = 0$. What is the significance of this?

Let us solve the following questions.

6. What is wrong with this statement : $\mathbf{a} \cdot \mathbf{b} = [1 \quad 1 \quad 0]$?

7. If a constant force $\mathbf{F} = [1 \quad 1 \quad 1]^{\mathbf{T}}$ is applied to an object moving it by $[3 \quad 0 \quad 0]$, what is the work done?

8. Find the component of $[1 \quad 0 \quad 1]^{\mathbf{T}}$ in the direction of $[-2 \quad 3 \quad 2]^{\mathbf{T}}$.

9. What is wrong with the statement: $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c} = 7$?

Let us now learn about the cross (vector) product.

For the given vectors,

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

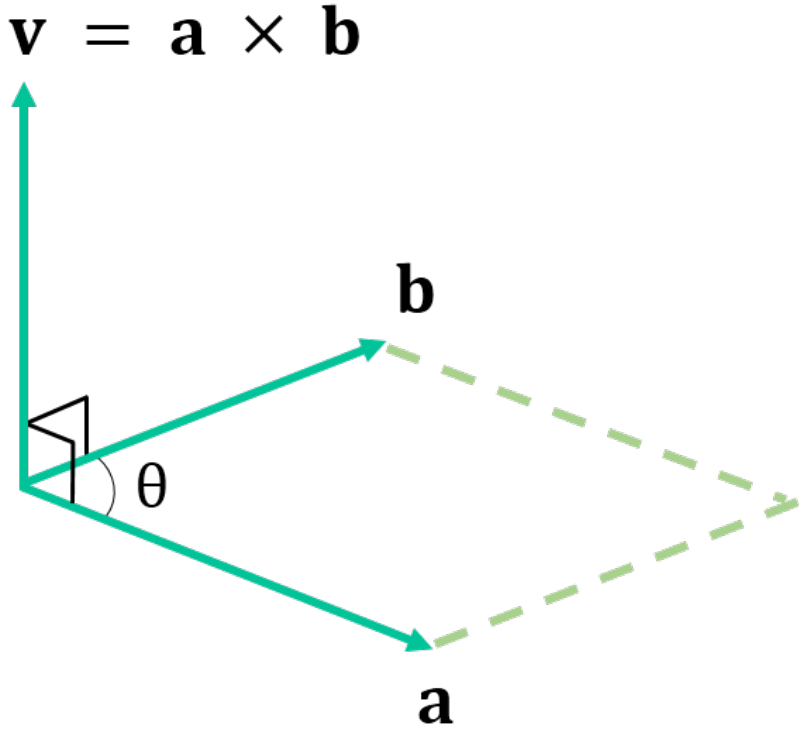
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

$$\mathbf{a} \text{ x } \mathbf{b} \text{ (= vector)} = \mathbf{v},$$

The cross (vector) product can be expressed as:

$$\|\mathbf{v}\| = \|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\| \sin\theta \qquad \text{- (3)}$$

The direction of $\mathbf{v}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.
$\boldsymbol{\theta}$ is the angle between $\mathbf{a}$ and $\mathbf{b}$.



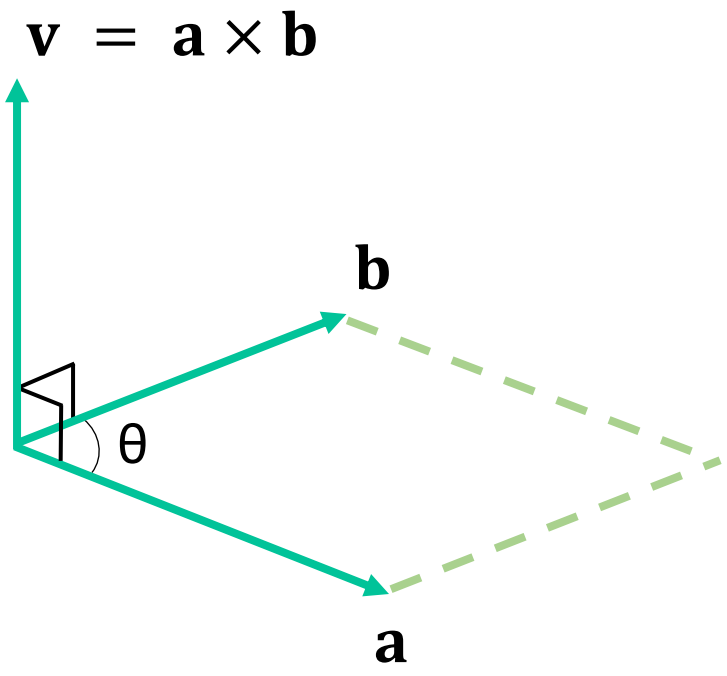
Cross product is valid only for 3D vectors.

Let us define $\hat{\mathbf{n}} = \mathbf{v}/\|\mathbf{v}\|$, where $\hat{\mathbf{n}}$ is the unit vector in the direction of $\mathbf{v}$.

The above relationship can be consolidated by the formula given below.

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta \, \hat{\mathbf{n}} \qquad \text{- (3a)}$$

Where, $\hat{\mathbf{n}}$ is the unit vector that is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. This also means that $\hat{\mathbf{n}}$ acts as the *normal* to the plane defined by $\mathbf{a}$ and $\mathbf{b}$.



Show how Equation (3) or Equation (3a) can be used to calculate the area of the parallelogram defined by $\mathbf{a}$ and $\mathbf{b}$?

Let us look at an example to learn how to calculate cross (vector) product.

Example 4

$$(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = -3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

For calculating cross (vector) product, note that:

- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$
- $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$
- $\mathbf{i} \times \mathbf{j} = \mathbf{k},\ \mathbf{j} \times \mathbf{k} = \mathbf{i},\ \mathbf{k} \times \mathbf{i} = \mathbf{j}$

Let us now go through the concept of scalar triple product.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot[(b_2c_3-b_3c_2)\mathbf{i}+(b_3c_1-b_1c_3)\mathbf{j}+(b_1c_2-b_2c_1)\mathbf{k}]$$

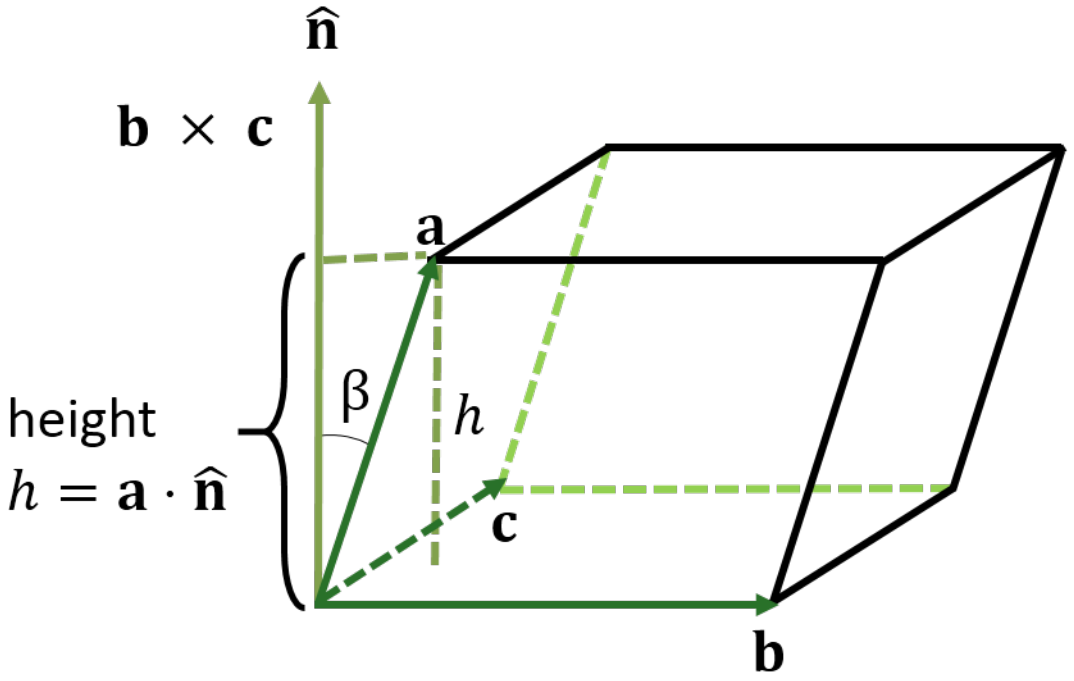
$$= a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Let us understand the concept of scalar triple product with an example.

Show that one may interpret the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) =$ volume of parallelepiped with $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as the edges.

Hint: Recall that $\|\mathbf{b} \times \mathbf{c}\|$ gives the area of the parallelogram and note that the component of $\mathbf{a}$ in the direction of $\mathbf{n}$, the normal to the b-c plane, is h.



$$\text{Volume} = \text{height} \times \text{base area}$$

$$= \mathbf{a} \cdot \hat{\mathbf{n}} \quad \|\mathbf{b} \times \mathbf{c}\| = \mathbf{a} \cdot \underbrace{\|\mathbf{b} \times \mathbf{c}\| \hat{\mathbf{n}}}$$

$$= \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$$

It can be seen that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$ interchanging rows 1 and 3

$$= -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$$

It can also be shown in the following form:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

Summary

Key points discussed in this lesson:

- A vector is a quantity that has both magnitude and direction.
- The transpose of a column vector can become a row vector.
- Vectors are denoted either by boldfaced alphabets or alphabets with an arrow above. For example: $\mathrm{F}, \mathrm{f}, \text{or } \vec{\mathrm{f}}$.
- For a vector, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, the magnitude (length) of the vector is given as: $\|\mathbf{a}\| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$.

Key points discussed in this lesson:

- For a vector, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, the unit vector can be expressed as:

$$\hat{\mathbf{a}} = \frac{1}{\|\mathbf{a}\|}\mathbf{a} = \frac{1}{\sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}}(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).$$

- Geometrically, the dot (scalar) product is expressed as: $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$. The result of a dot product is a scalar (real number). Thus, the dot product is commutative.

- Some useful properties from dot product $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$ are:
  - $\mathbf{a} \| \mathbf{b}$ parallel, $\theta = 0$, $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|$
  - Anti-parallel, $\theta = \pi$, $\mathbf{a} \cdot \mathbf{b} = -\|\mathbf{a}\|\|\mathbf{b}\|$
  - $\mathbf{a} \perp \mathbf{b}$ perpendicular, $\theta = \pi/2$, $\mathbf{a} \cdot \mathbf{b} = 0$ that is, **a** is orthogonal to **b**.

Key points discussed in this lesson:

- From the dot product property, following can be derived:
  - $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$ [$\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors].
  - $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$ [$\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually orthogonal].

- For vectors, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, the cross (vector) product is expressed as: $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\| \sin\theta$.

- For calculating cross (vector) product, note that:
  - $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$
  - $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} \neq \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$
  - $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$
  - $\mathbf{i}\times\mathbf{j} = \mathbf{k},\ \mathbf{j}\times\mathbf{k} = \mathbf{i},\ \mathbf{k}\times\mathbf{i} = \mathbf{j}$

Key points discussed in this lesson:

- The scalar triple product of three vectors **a**, **b**, and **c** is given as:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

- From scalar triple product you get:
  - $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -\,\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a})$.
  - $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$.

Vector Calculus

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At the end of this lesson, you should be able to:

- Perform vector differentiation.
- Describe scalar and vector fields.
- Work with the “∇“ (del) operator - grad, div, curl.

[redacted]

[redacted]

Velocity $\mathbf{v}$ of the point $\boldsymbol{r}$ is expressed as shown below.

$$\mathbf{v}(t) = \boldsymbol{r}' = \frac{d\boldsymbol{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{[x(t + \Delta t)\mathbf{i} + y(t + \Delta t)\mathbf{j} + z(t + \Delta t)\mathbf{k}] - [x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{[x(t + \Delta t) - x(t)]\mathbf{i} + [y(t + \Delta t) - y(t)]\mathbf{j} + [z(t + \Delta t) - z(t)]\mathbf{k}]}{\Delta t}$$

Velocity $\mathbf{v}$ of the point $\boldsymbol{r}$ is expressed as shown below.

$$= \lim_{\Delta t \to 0} \left[\frac{\Delta x}{\Delta t}\mathbf{i} + \frac{\Delta y}{\Delta t}\mathbf{j} + \frac{\Delta z}{\Delta t}\mathbf{k} \right]$$

$$= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

The velocity $v = \boldsymbol{r}'$ is a tangent vector to the curve at the tip of $\boldsymbol{r}$, and points in the direction of increasing t.

$$\boxed{\hat{\mathbf{u}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}}$$

Similarly, acceleration $\mathbf{a}$ of the point vector $\boldsymbol{r}$ is expressed as shown below.

$$\mathbf{a}(t) = \mathbf{v}' = \frac{d\mathbf{v}}{dt}$$

$$= \frac{d}{dt}\frac{dx}{dt}\,\mathbf{i} + \frac{d}{dt}\frac{dy}{dt}\,\mathbf{j} + \frac{d}{dt}\frac{dz}{dt}\,\mathbf{k}$$

$$= \frac{d^2x}{dt^2}\,\mathbf{i} + \frac{d^2y}{dt^2}\,\mathbf{j} + \frac{d^2z}{dt^2}\,\mathbf{k}$$

$$= \boldsymbol{r}''$$

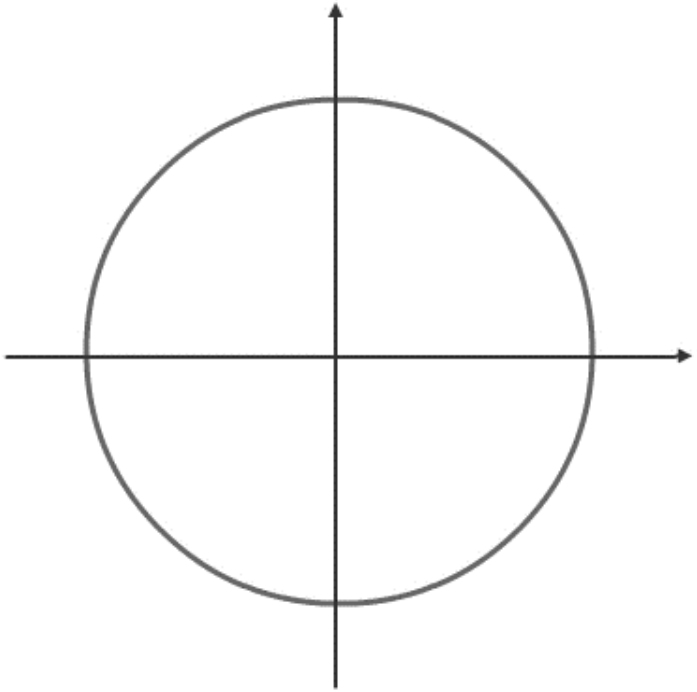
Let's see how to calculate the velocity and acceleration of a particle from its position vector.

Solution (contd.):

$$\text{Velocity}, \mathbf{v}(t) = \frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}$$

$$= \frac{d(cost)}{dt}\,\mathbf{i} + \frac{d(sin\,t)}{dt}\,\mathbf{j}$$

$$= -sint\,\mathbf{i} + cost\,\mathbf{j}$$



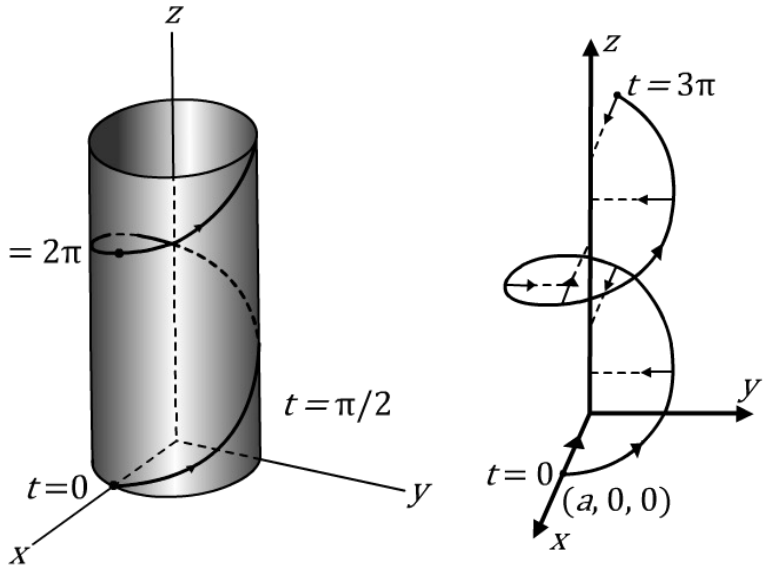
Let's see how to calculate the velocity and acceleration of a particle from its position vector.

Sample Problem 2

Find the velocity and acceleration for the helical path given by:

$\mathbf{r}(t) = acost\ \mathbf{i} + asint\ \mathbf{j} + bt\ \mathbf{k}$; where a and b are constants.

Solution:



Let's see how to calculate the velocity and acceleration of a particle from its position vector.

Solution (contd.):

Velocity, $\mathbf{v} = -asint\ \mathbf{i} + acost\ \mathbf{j} + b\ \mathbf{k}$

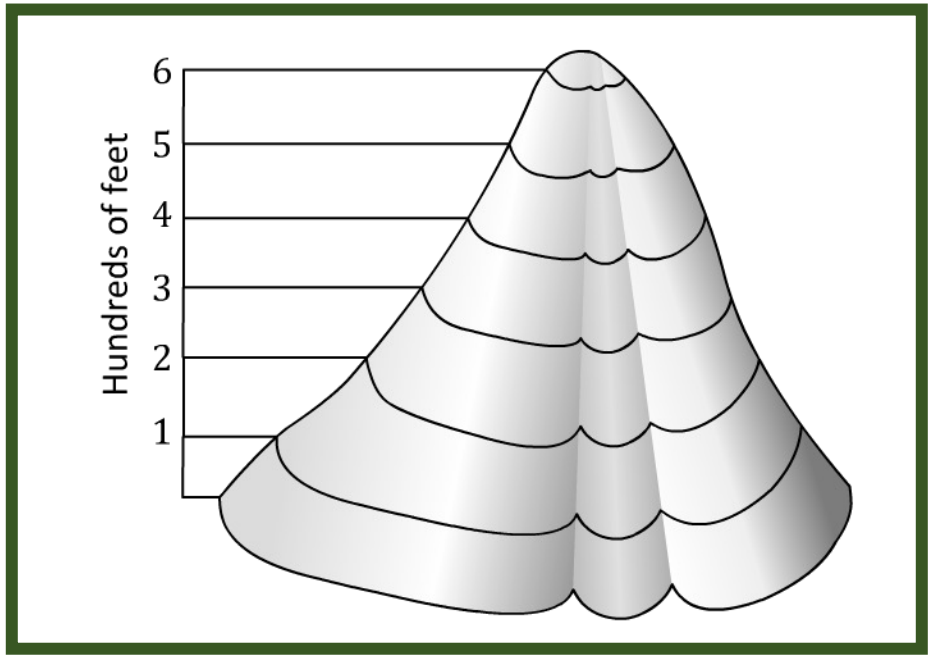
Acceleration, $\mathbf{a} = -acost\ \mathbf{i} - asint\ \mathbf{j}$

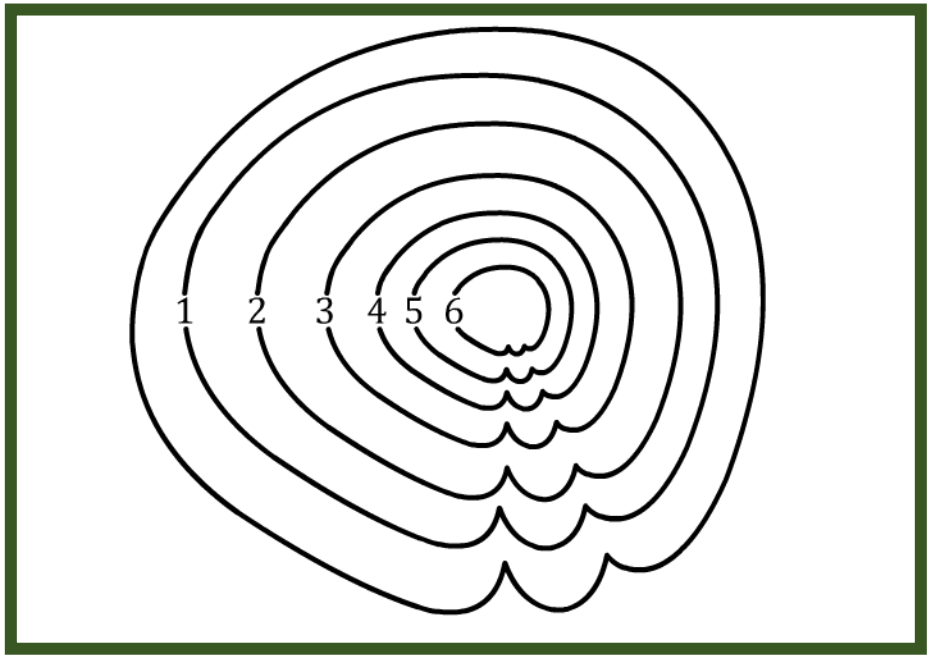
And,

Speed, $\|\mathbf{v}\| = \sqrt{[(-asint)^2 + (acost)^2 + b^2]} = \sqrt{a^2 + b^2}$

Note: The velocity vector is always a tangent vector to the path.

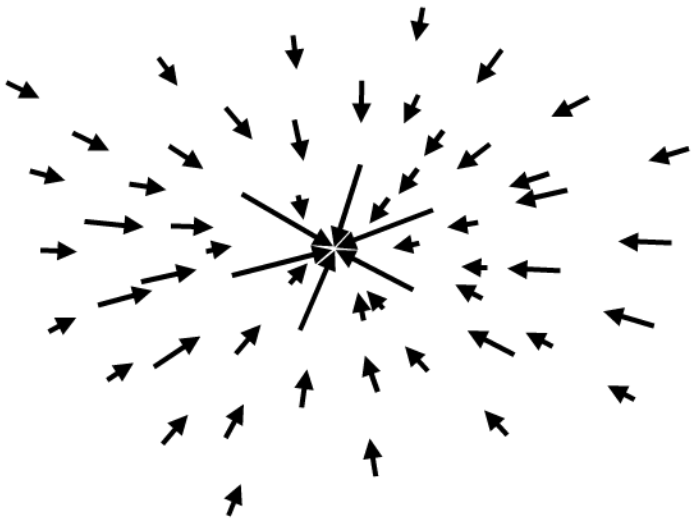
A point in a scalar field possesses a magnitude or value, but no direction. For example, the temperature at a point in a room has a magnitude (say, 20°C). The temperature measured by the thermometer in any direction will have the same magnitude.





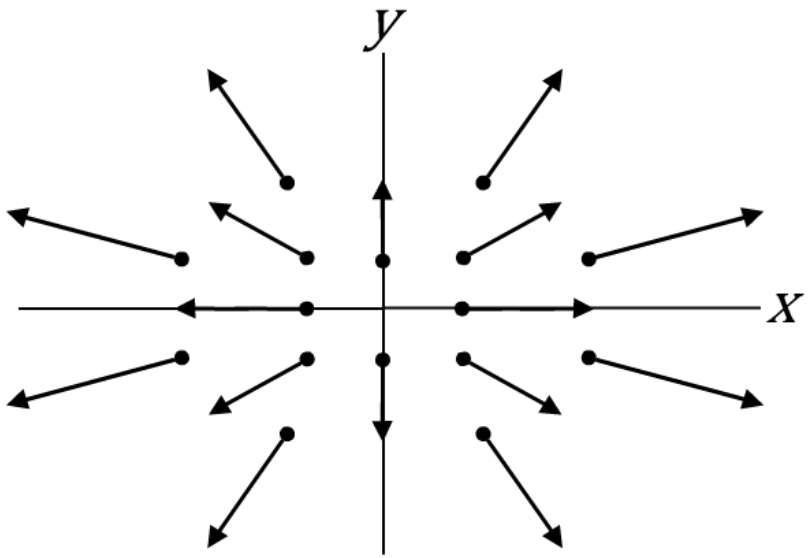
Can you think of more scalar fields?

Let's see a few examples of vector fields.



Gravitational field

$$\mathbf{F} = \frac{GMm}{\|\mathbf{r}^2\|}\left[\frac{-\mathbf{r}}{\|\mathbf{r}\|}\right]$$

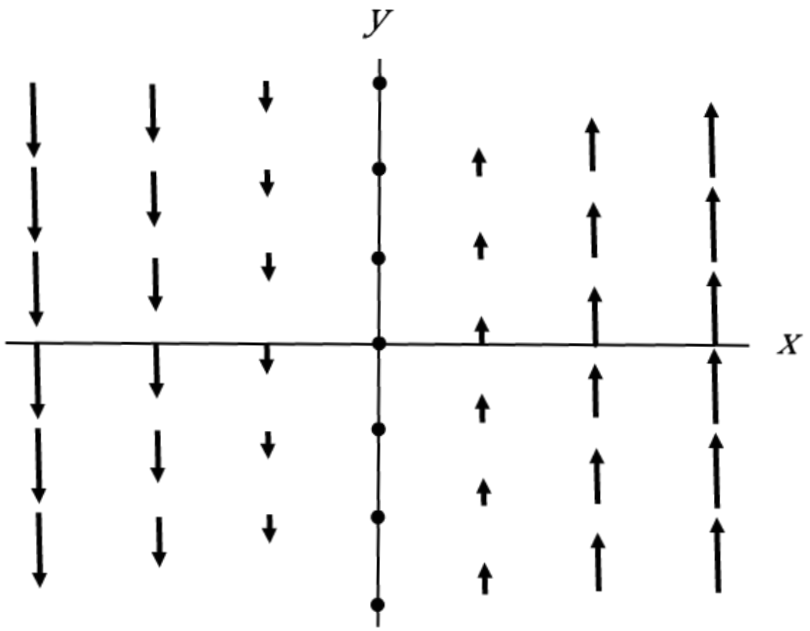


Forces on a plane

$$\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$$

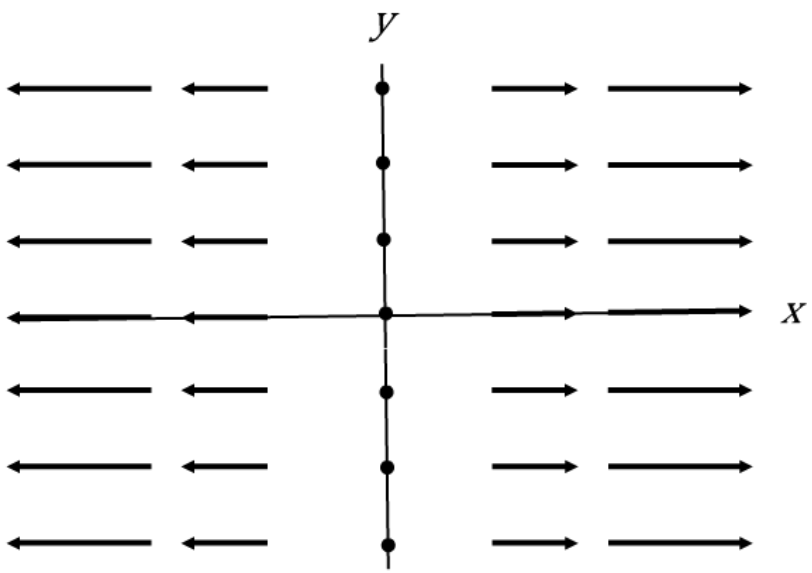
Some examples of vector fields are the earth's gravitational field, force fields, electro-magnetic fields, flow of water in a pipeline, etc.

If the vector field lines flow in the vertical direction, it is known as a vertical flow field, whereas, if the field lines flow in the horizontal direction, it is known as a horizontal flow field.



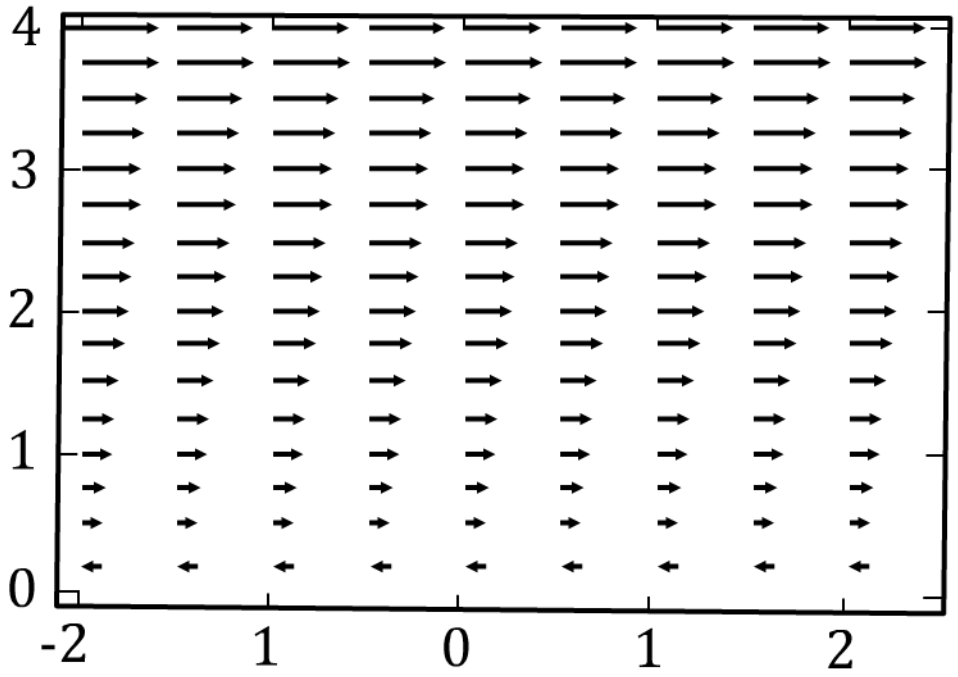
$$\mathbf{F} = x\mathbf{j}$$

Vertical flow field



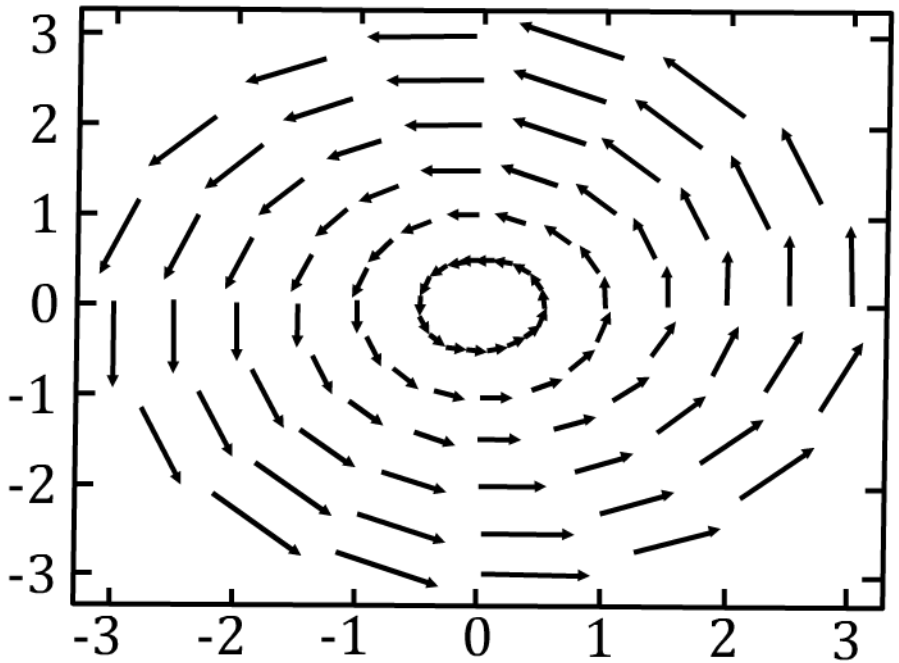
$$\mathbf{F} = x\mathbf{i}$$

Horizontal flow field



$$\mathbf{F}(x, y) = \frac{1}{5}\sqrt{y}\mathbf{i}$$

Horizontal flow with magnitudes as a function of the vertical axis



$$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$$

2D vortex

The del operator is represented as ∇. This operator can be used to operate on scalar fields and vector fields.

The gradient of a scalar field: Grad $f = \nabla f$

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

When applied to a scalar field *f*, yields

$$\nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) f$$

$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Note that *f* is a scalar and the resulting operation ∇f is a vector.

Do not confuse ∇ with Δ. Δf is usually considered to be a small change in f.

Let us look at an example to understand the del operator.

Example 1

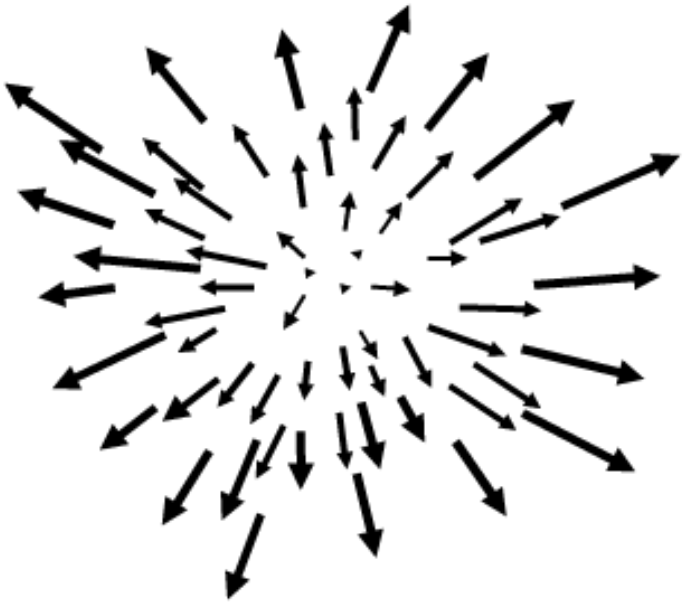
Given that $f(x, y, z)$ is a 3D scalar field :

$$f = x^2 + y^2 + z^2$$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$= 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$$

$$= 2r$$



$\mathbf{F}=2r$

∇f is a vector field, and is called the gradient of f, or grad f.

Let us now understand some properties of del operator ∇f (grad f).

Consider a scalar field f, differentiable at the point (a, b, c), and $\nabla f(a, b, c)$ is not zero.

- $\nabla f(a, b, c)$ is in the direction of the greatest rate of change of f.

- $\nabla f(a, b, c)$ is perpendicular to the level surface of f at (a, b, c). That is, it is a surface normal at (a, b, c).

- $\|\nabla f(a, b, c)\|$ is the maximum rate of change of f at (a, b, c).

Let us now understand some properties of del operator ∇f (grad f).

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$\nabla f(a, b$ t (a, b, c). That is, it is a su

Level surface (or equipotential surface) of f are points such that $f(x, y, z) = \text{c}$, where c is a constant.

$\|\nabla f(a, b, c)\|$ is the maximum rate of change of f at (a, b, c).

Let us look at another example to understand the concept of del operator.

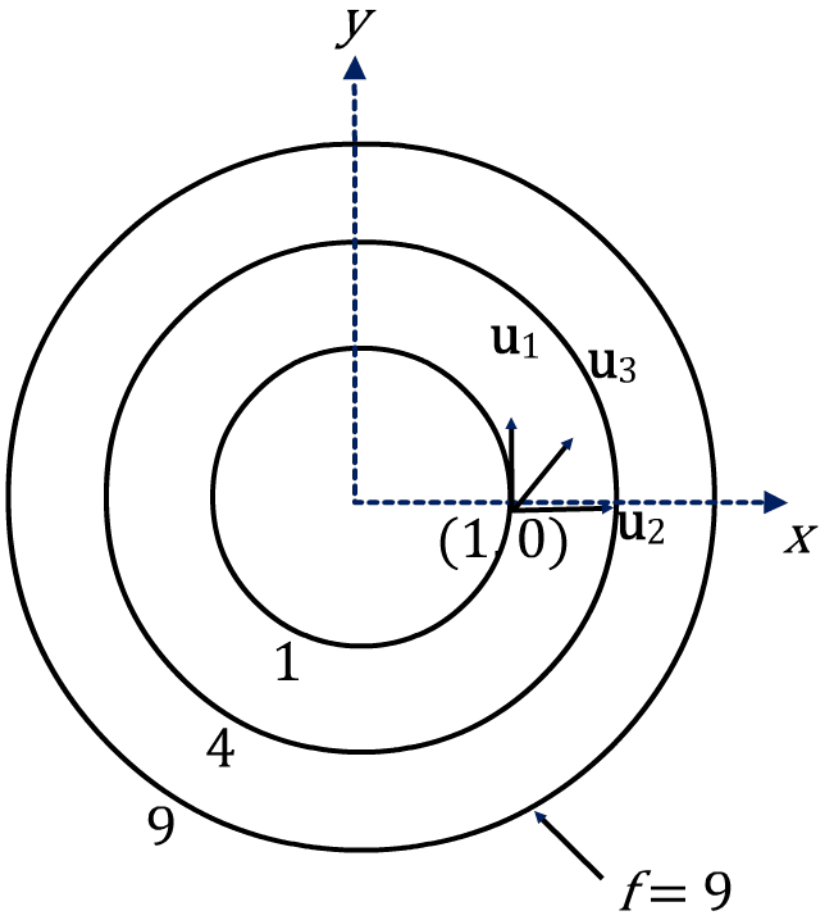
Example 2

Find the height of the stadium.

$$f = x^2 + y^2$$

$$\nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) x^2 + y^2$$

$$= 2x\,\mathbf{i} + 2y\,\mathbf{j}$$



Let us now become familiar with the concept of directional derivative.

The magnitude of the rate of change or slope of f in a particular direction, say $\mathbf{u}$, is called the directional derivative.

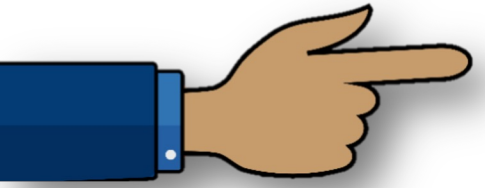
The magnitude of this rate of change is calculated in the following way:

$\nabla f \cdot \mathbf{u}$ (directional derivative of f in the direction $\mathbf{u}$)

where $\mathbf{u}$ is taken to be a unit vector. If $\mathbf{u}$ is not a unit vector, replace it with $\mathbf{u}/\|\mathbf{u}\|$.

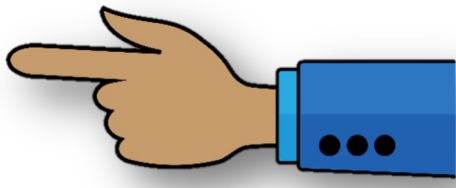
The directional derivative formula has the same use as what you have learnt before: finding the component of one vector in the direction of another vector.

Let us now become familiar with the concept of directional derivative.



In the previous example, you have seen that $\nabla f(x, y) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$. Thus, $\nabla f(1, 0) = 2\mathbf{i}$ is a vector, perpendicular to the contour of f at the point $(1, 0)$, and this direction is the direction of maximum increase of f at the point $(1, 0)$.

Also, this maximum rate of change at the point $(1, 0)$ can be calculated by $\|\nabla f(1,0)\| = \|2\mathbf{i}\| = 2$.
This is the directional derivative of f in the direction of the gradient of f and is the maximum at the point (1, 0).



Here is how you calculate the directional directive for different directions of $\mathbf{u}$ at different point (x, y) to verify the properties of del operator.

At point $(1, 0)$, consider directions $\mathbf{u}_1 = \mathbf{j}, \mathbf{u}_2 = \mathbf{i}, \mathbf{u}_3 = (\mathbf{i} + \mathbf{j})$

Directional derivative along $\mathbf{u}_1$

$$= \nabla f \cdot \mathbf{u}_1 = 2y = 0$$

Directional derivative along $\mathbf{u}_2$

$$= \nabla f \cdot \mathbf{u}_2 = 2x = 2$$

Directional derivative along $\mathbf{u}_3$

$$= \nabla f \cdot \mathbf{u}_3 / \|\mathbf{u}_3\|$$

$$= (2x + 2y)\frac{1}{\sqrt{2}} = \sqrt{2}$$

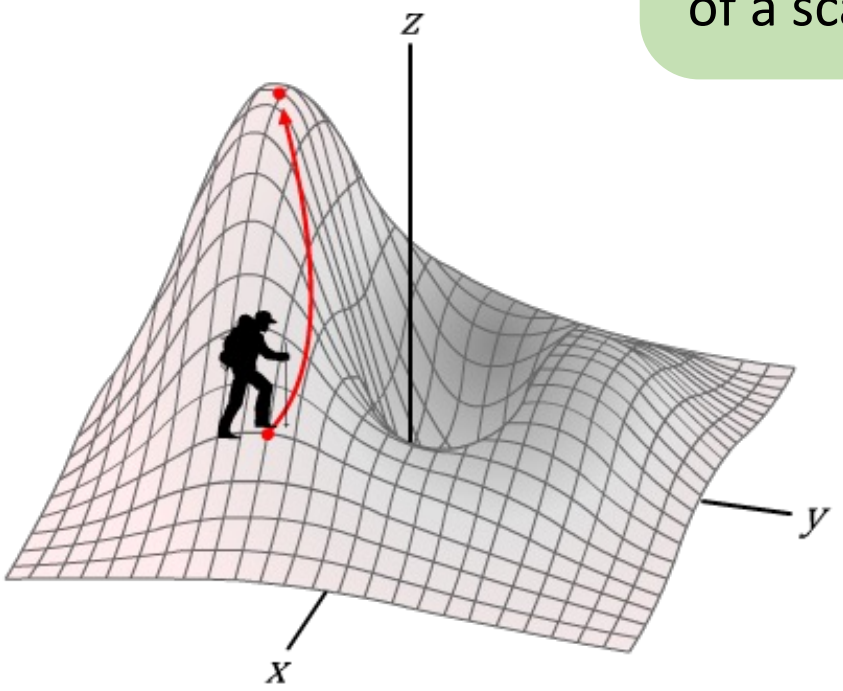
The unit vector perpendicular to the surface of f at (x, y, z), that is, the normal direction to the surface at a point on the surface is given below.

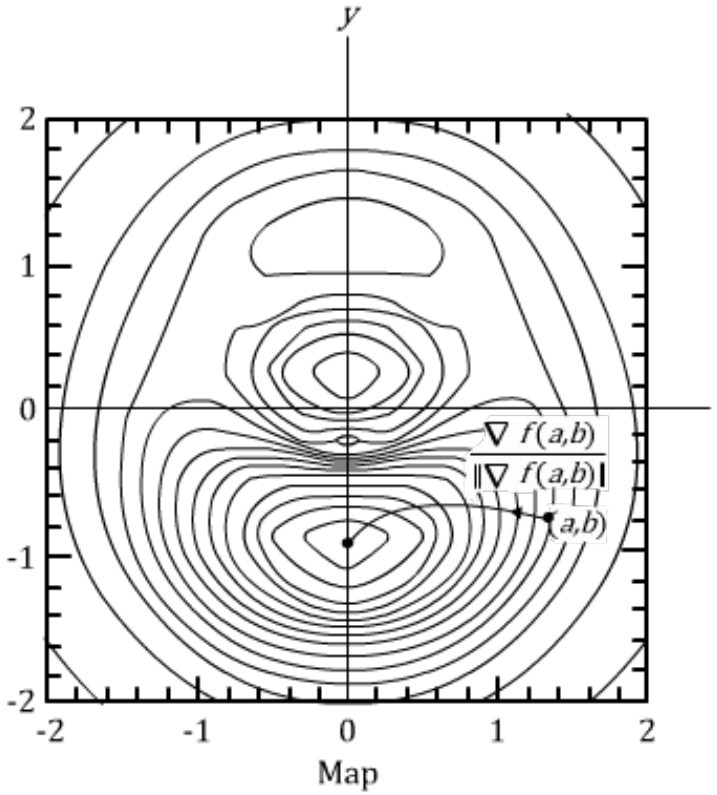
$$\frac{\nabla f\ (x, y, z)}{\|\nabla f\ (x, y, z)\|}$$

Another property of ∇f is that it is normal to the (equipotential) surface of f.

Let us look at an example on how to reach the top of the "hill" by following the path of grad f, that is, the path of maximum rate of change of $\mathbf{f}$.

This technique is known as "hill climbing" or "gradient ascent" method to find the maximum of a scalar function.



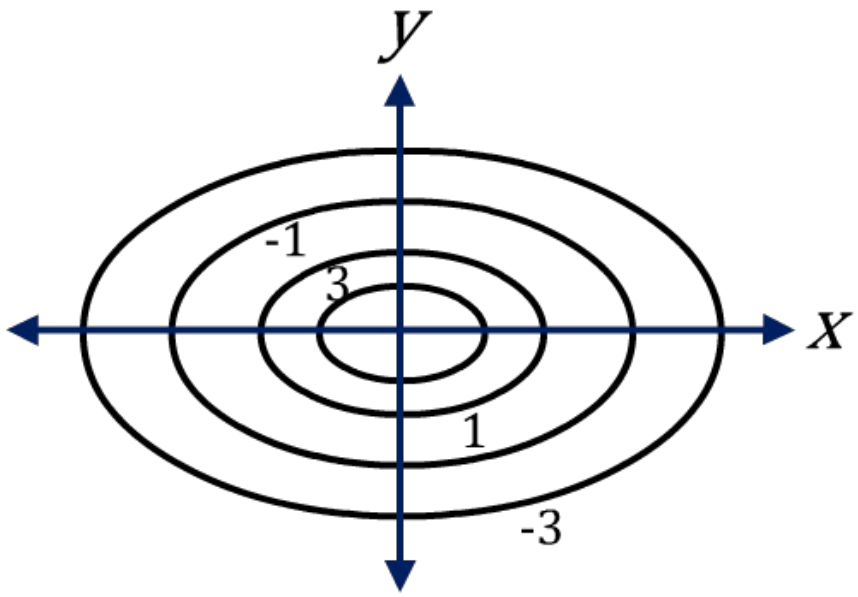


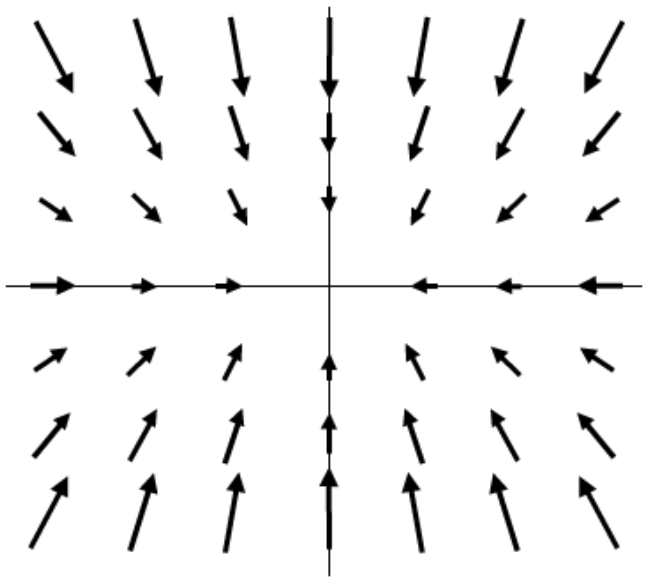
Let us look at an example to understand how del operator works.

Example 3

Given $g(x, y) = 5 - x^2 - 2y^2$

$$\nabla g(x, y) = -2x\mathbf{i} - 4y\mathbf{j}$$



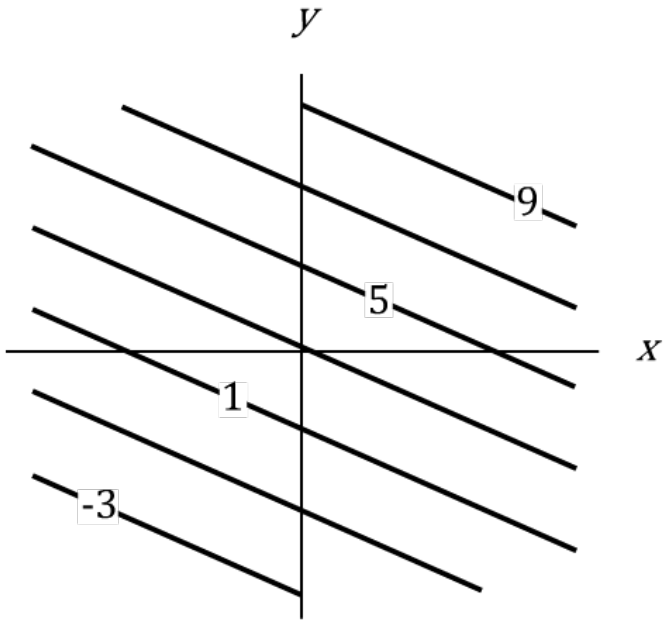


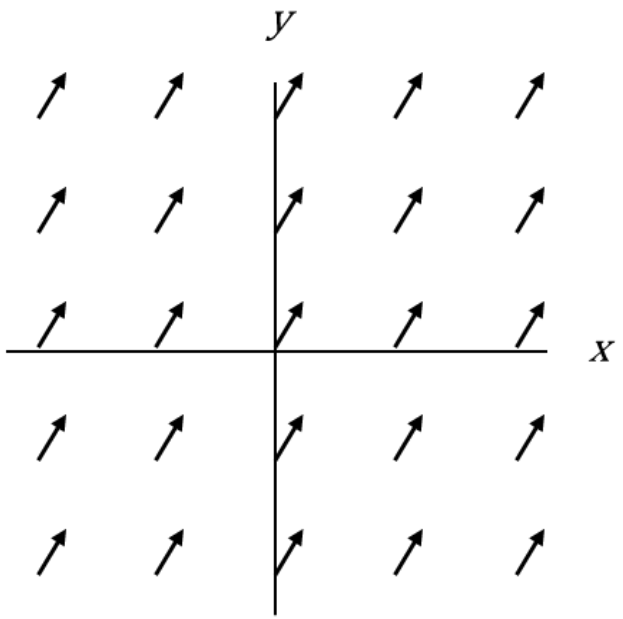
Let us look at another example to understand how del operator functions.

Example 4

Given $h(x, y) = x + 2y + 3$

$$\nabla h(x, y) = \mathbf{i} + 2\mathbf{j}$$





Here is another example to understand how del operator works.

Example 5

Find the directional derivative of $f = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, and the unit normal to the surface at this point.

$$f = x^2yz + 4xz^2 = 6 \text{ at } (1, -2, -1).$$

$$\nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(x^2yz + 4xz^2)$$

$$= (2xyz + 4z^2)\mathbf{i} + (x^2z)\mathbf{j} + (x^2y + 8xz)\mathbf{k}$$

$$= 8\mathbf{i} - \mathbf{j} - 10\mathbf{k} \text{ at } (1, -2, -1)$$

Here is another example to understand how del operator works.

Example 5 (contd.)

The unit vector in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is:

$$\hat{\mathbf{u}} = \frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{\sqrt{2^2 + (-1)^2 + (-2)^2}} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

The required directional derivative is:

$$\nabla f \cdot \hat{\mathbf{u}} = (8\mathbf{i} - \mathbf{j} - 10\mathbf{k}) \cdot \left(\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}$$

Here is another example to understand how del operator works.

Example 5 (contd.)

The unit normal at $(1, -2, -1)$ is:

$$\frac{\nabla f\ (x, y, z)}{\|\nabla f\ (x, y, z)\|} = \frac{(8\mathbf{i} - \mathbf{j} - 10\mathbf{k})}{\sqrt{8^2 + (-1)^2 + (-10)^2}}$$

$$= \frac{1}{\sqrt{165}}(8\mathbf{i} - \mathbf{j} - 10\mathbf{k})$$

Below are the two possible ways for ∇ to operate on vector field $\mathbf{F}$.

$\nabla . \mathbf{F}$: This is called the divergence of $\mathbf{F}$, and the result is scalar.

$\nabla \times \mathbf{F}$: This is called the curl of $\mathbf{F}$, and the result is vector.

For a vector field $\mathbf{F}$, the divergence is given as below.

If the vector field $\mathbf{F}$ is given as $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$

Then, the divergence of $\mathbf{F}$ is given as:

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

The result is a scalar field and is also written as div $\mathbf{F}$.

Let us look at an example to understand the concept of divergence of a vector.

Example 6

Given: $\mathbf{v} = 3xz\mathbf{i} + 2xy\mathbf{j} - yz^2\mathbf{k}$

The divergence of vector $\mathbf{v}$ is given as:

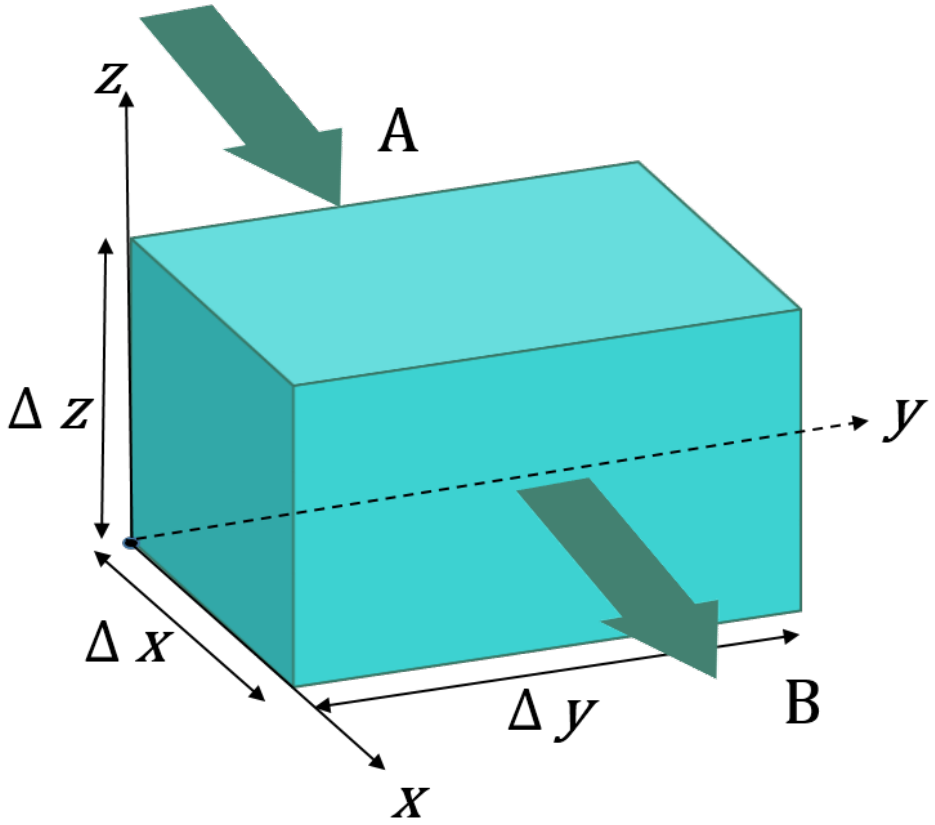
$$\nabla \cdot \mathbf{v} = \frac{\partial(3xz)}{\partial x} + \frac{\partial(2xy)}{\partial y} + \frac{\partial(-yz^2)}{\partial z}$$

$$= 3z + 2x - 2yz$$

The physical interpretation of divergence can be shown in the given figure below.

The divergence (of a vector field) tells the extent to which the field explodes or diverges.

Consider the movement of a fluid, so that its velocity at any point is $\mathbf{v}(x, y, z)$, then the net rate of gain in fluid per unit volume at any point is given by $\nabla \cdot \mathbf{v}$.



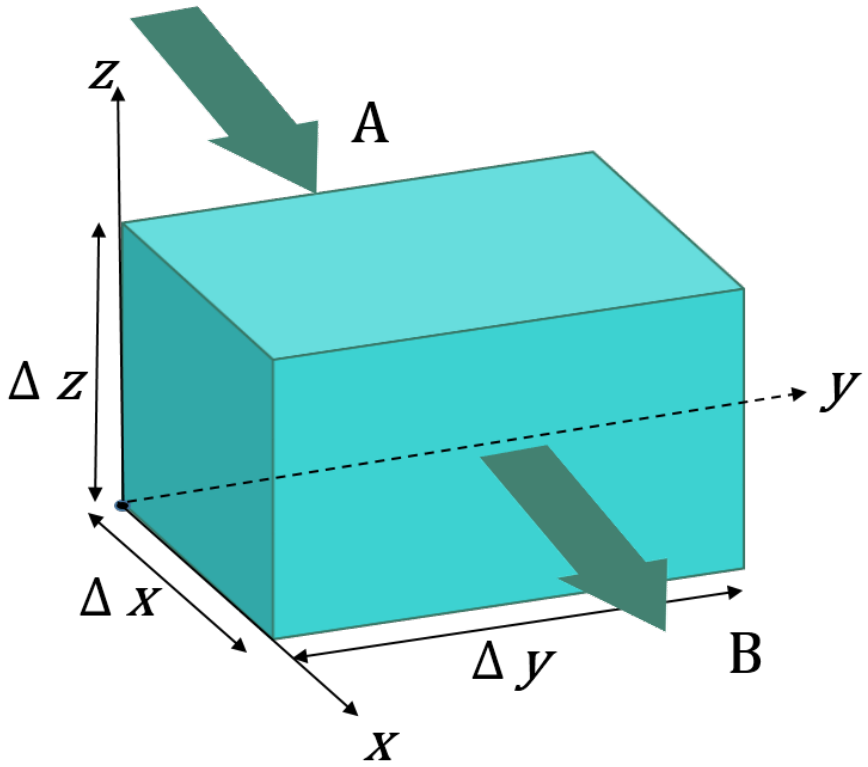
The physical interpretation of divergence can be shown in the given figure below.

If $\nabla \cdot \mathbf{v} = 0$ at a point, then the rate at which the fluid is flowing into that point is equal to the rate at which the fluid is flowing out.

If $\nabla \cdot \mathbf{v} = 0$ everywhere, fluid is incompressible (or solenoidal or divergence free).

If $\nabla \cdot \mathbf{v} > 0$ at a point, more fluid is flowing out than in. For example, the existence of a 'source'.

If $\nabla \cdot \mathbf{v} < 0$ at a point, more fluid is flowing in than out.
For example, the existence of a 'sink'.



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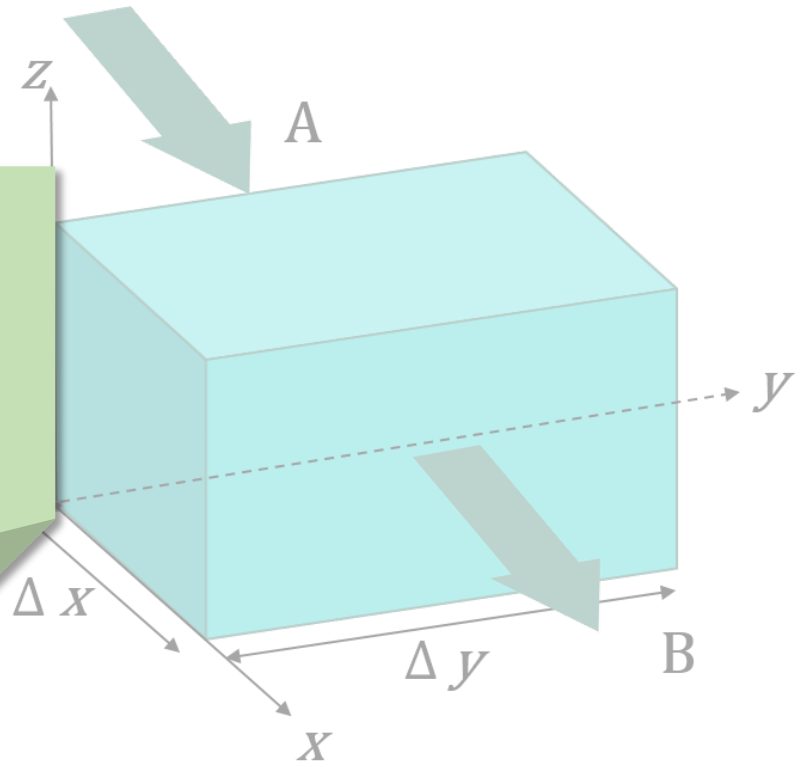
For example, the existence of a 'sink'.

The notion of divergence can be extended to any vector field such as magnetic, electric, etc.



Let us look at an example to understand the concept of divergence of a vector.

Example 7

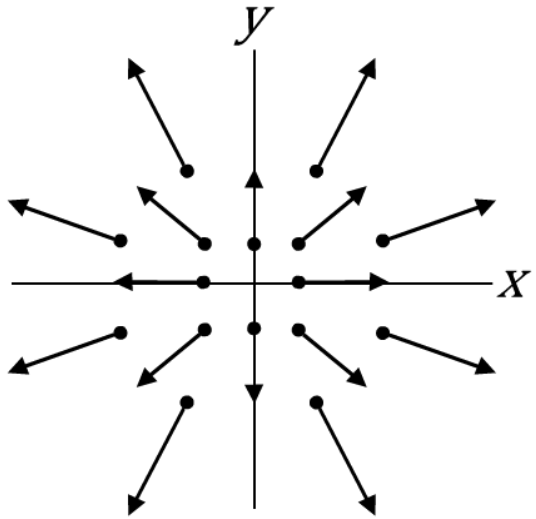
i) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

The divergence of vector $\boldsymbol{F}$ is given as:

$$\nabla \cdot \mathbf{F} = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j}$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}$$

$$= 1 + 1 = 2$$



ii) Similarly if, $F = -x\mathbf{i} - y\mathbf{j}$

$$\nabla \cdot \mathbf{F} = -2$$

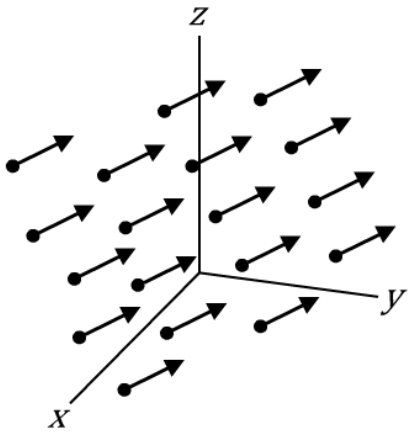
Let us look at an example to understand the concept of divergence of a vector.

Example 8

i) $\mathbf{F} = a$ (constant vector)

The divergence of vector $\mathbf{F}$ is given as:

$$\nabla \cdot \mathbf{F} = 0$$

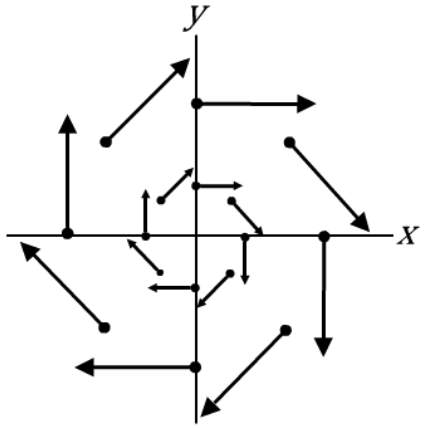


ii) $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$

The divergence of vector $\mathbf{F}$ is given as:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\mathbf{i} \cdot \mathbf{F} + \frac{\partial}{\partial y}\mathbf{j} \cdot \mathbf{F}$$

$$= \frac{\partial y}{\partial x} + \frac{\partial(-x)}{\partial y} = 0$$



For a vector field $\mathbf{F}$, the curl is represented as shown below.

If the vector field $\mathbf{F}$ is expressed as $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$

Then, the curl of $\mathbf{F}$ is expressed as:

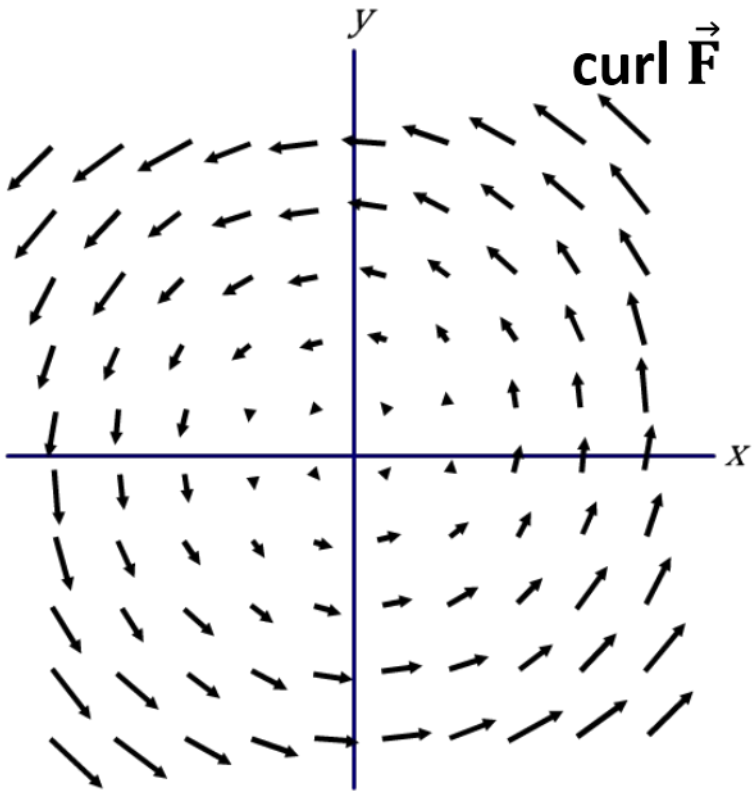
$$\nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k})$$

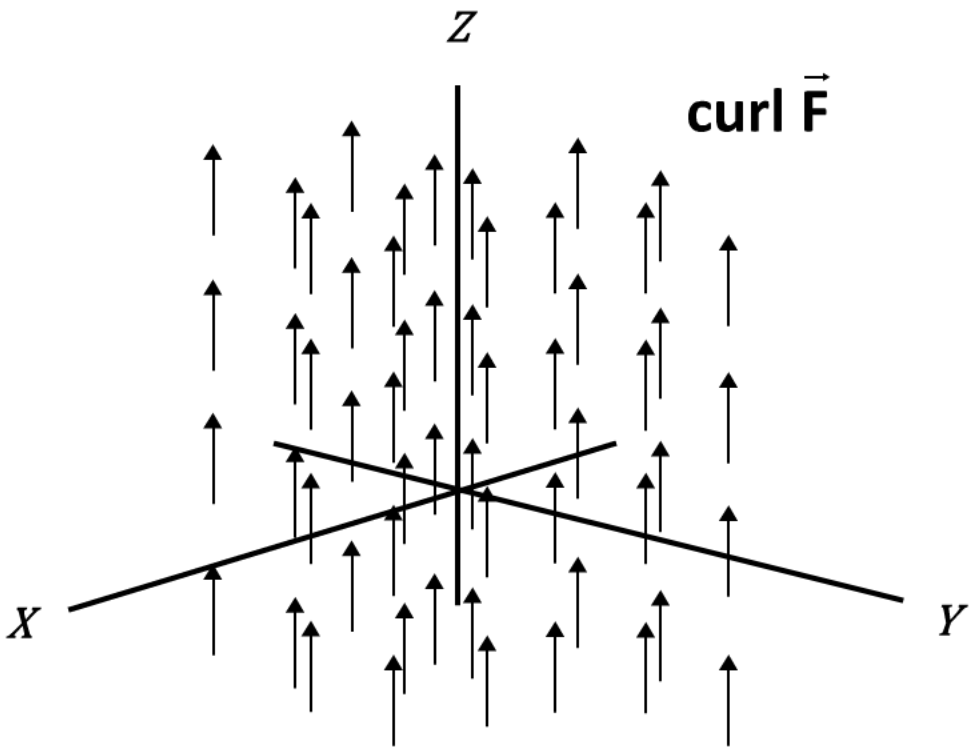
$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$ which is a vector field.

Let us understand the concept of curl by observing the examples given below.

The curl of a vector is related to the ‘circulation’ at each point of the vector field.





A curl exists when there is a component of a field whose value is changing in a direction orthogonal to this component.

The result shows that the curl is in the direction orthogonal to both the component and the direction of change.

For example, $\mathbf{F} = 2y^2\mathbf{i}$

$\mathbf{F}$ has a component in the $\mathbf{i}$ direction and its value $(= 2y^2)$ is changing in the $\mathbf{j}$ direction ($\mathbf{j}$ is orthogonal to $\mathbf{i}$). Thus, a curl exists and is equal to $-4\mathbf{yk}$, ($\mathbf{k}$ is orthogonal to both $\mathbf{i}$ and $\mathbf{j}$).

Let us look at an example to understand the curl of a vector.

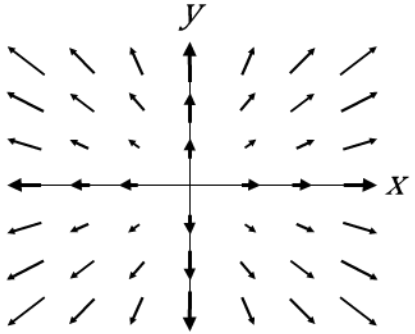
Example 9

i) $\mathbf{F} = -2y\mathbf{i} + 2x\mathbf{j}$

The curl of vector $\boldsymbol{F}$ is given as:

$$\nabla\times\mathbf{F} = \left(\frac{\partial(0)}{\partial y} - \frac{\partial(2x)}{\partial z}\right)\mathbf{i} - \left(\frac{\partial(0)}{\partial x} - \frac{\partial(-2y)}{\partial z}\right)\mathbf{j} + \left(\frac{\partial(2x)}{\partial x} - \frac{\partial(-2y)}{\partial y}\right)\mathbf{k}$$

$$= 4\mathbf{k}$$



ii) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

The curl of vector $\mathbf{F}$ is given as:

$$\nabla\times\mathbf{F} = \left(\frac{\partial(0)}{\partial y} - \frac{\partial y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial(0)}{\partial x} - \frac{\partial x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\mathbf{k} = 0$$

Let us look at an example to understand the curl of a vector.

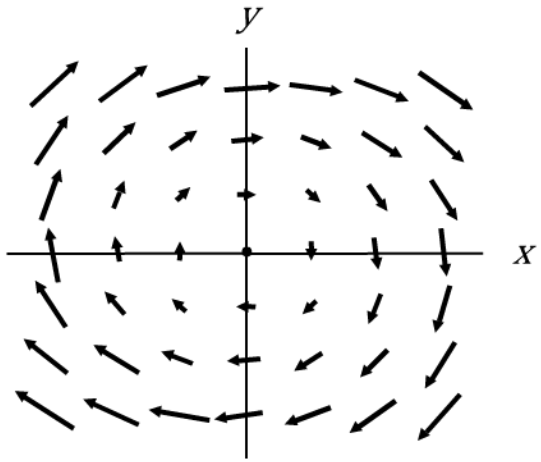
Example 10

Given: $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$

The curl of vector $\mathbf{F}$ is given as:

$$\nabla\times\mathbf{F} = \left(\frac{\partial(0)}{\partial y} - \frac{\partial(-x)}{\partial z}\right)\mathbf{i} + \left(\frac{\partial y}{\partial z} - \frac{\partial(0)}{\partial x}\right)\mathbf{j} + \left(\frac{\partial(-x)}{\partial x} - \frac{\partial(y)}{\partial y}\right)\mathbf{k}$$

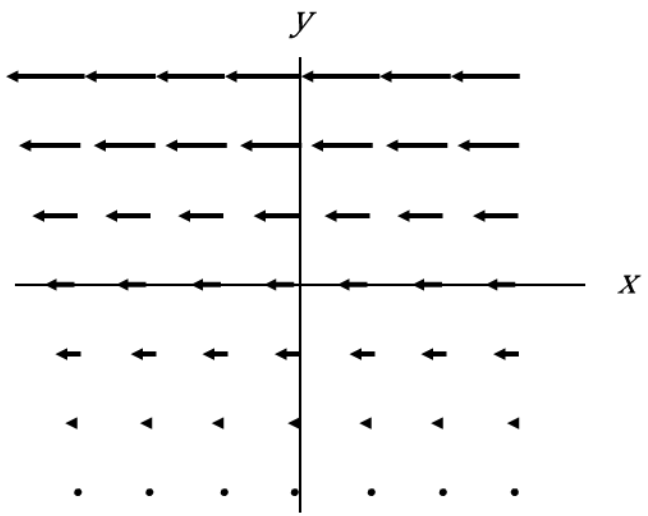
$$= -2\mathbf{k}$$



Let us look at another example to understand the curl of a vector.

Example 11

Given: $\mathbf{F} = -(y+1)\mathbf{i}$



The curl of vector $\mathbf{F}$ is given as:

$$= \nabla \times \mathbf{F}$$

$$= \left(\frac{\partial(0)}{\partial y} - \frac{\partial(0)}{\partial z}\right)\mathbf{i} + \left(\frac{\partial(-y-1)}{\partial z} - \frac{\partial(0)}{\partial x}\right)\mathbf{j} + \left(\frac{\partial(0)}{\partial x} - \frac{\partial(-y-1)}{\partial y}\right)\mathbf{k}$$

$$= \mathbf{k}$$

Let us now get familiar with the concept of Laplacian.

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \nabla^2 f$$

$$\operatorname{div}(\nabla f) = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right).\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)$$

$$= \frac{\partial^2 f}{x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$$

Let us look at an example to understand the concept of Laplacian.

Example 12

Let $f = x^4y^3 + 1/z$

$$\nabla^2 f = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 z}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(4x^3y^3) = 12x^2y^3$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^4 3y^2) = 6x^4y$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(-z^{-2}) = 2z^{-3}$$

$$\therefore \nabla^2 f = 12x^2y^3 + 6x^4y + 2z^{-3}$$

Some other useful formulae to be remembered in del operator.

$$\nabla(f\,g) = f\,\nabla g + g\,\nabla f$$

$$\nabla \cdot (f\,\mathrm{v}) = f\,\nabla \cdot \mathrm{v} + \mathrm{v} \cdot \nabla f$$

$$\nabla \times (f\,\mathrm{v}) = f\,\nabla \times \mathrm{v} + \nabla f \times \mathrm{v}$$

$$\nabla \times (\nabla f) = 0$$

$$\nabla \cdot (\nabla \times \mathrm{v}) = 0$$

In the given formulae:
f and g must denote scalar functions and **v** must denote a vector function.

Summary

Key points discussed in this lesson:

- The velocity $\mathbf{v}$ of the point $\boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is given by:
  $$\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

- The acceleration $\mathbf{a}$ of the point $\boldsymbol{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is given by:
  $$\frac{d^2x}{dt^2}\,\mathbf{i} + \frac{d^2y}{dt^2}\,\mathbf{j} + \frac{d^2z}{dt^2}\,\mathbf{k}$$

- A point in a scalar field possesses a magnitude or value, but no direction.

- The del operator is represented as ∇. This operator can be used to operate on scalar fields and vector fields.

Key points discussed in this lesson:

- The gradient of a scalar field: Grad $f = \nabla f$ where, f is a scalar and the resulting operation ∇f is a vector.

- The magnitude of the rate of change or slope of f in a particular direction, say $\mathbf{u}$, is called the directional derivative.

- The two possible ways for ∇ to operate on a vector field $\mathbf{F}$ are:
  - $\nabla \cdot \mathbf{F}$: This is called the divergence of $\mathbf{F}$, and the result is scalar.
  - $\nabla \times \mathbf{F}$: This is called the curl of $\mathbf{F}$, and the result is vector.

Key points discussed in this lesson:

- The divergence (of a vector field) tells the extent to which the field explodes or diverges.
- The curl of a vector is related to the ‘circulation’ at each point of the vector field.
- A curl exists when there is a component of a field whose value is changing in a direction orthogonal to this component.

Vector Calculus

EE2007 – Engineering Mathematics II

At the end of this lesson, you should be able to:

- Perform the vector line integral.
- Perform the vector surface integral.

Vector line integral is the operation of integrating a vector field along a curve in space.

- The curve is expressed in a parametric form by writing x, y, z as functions of a single parameter 't'.

- The parameter 't' can then be used to specify points $(x(t), y(t), z(t))$ along the curve.

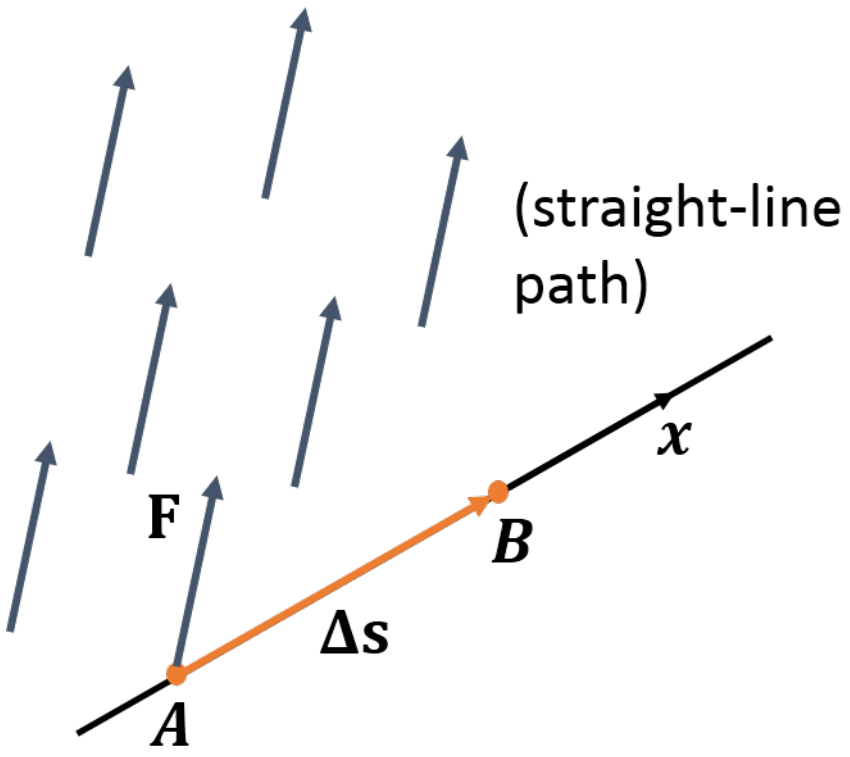
- Such integrals have considerable utility in many areas of Engineering and Physics.

The work done W by a force F moving an object is one such example of vector line integrals.

From the given figure, for a constant force field F pushing an object in a straight line, is expressed as:

$$W = \mathbf{F} \cdot \mathbf{r}$$

$$W = \|\mathbf{F}\| \|\mathbf{r}\| \cos\theta$$

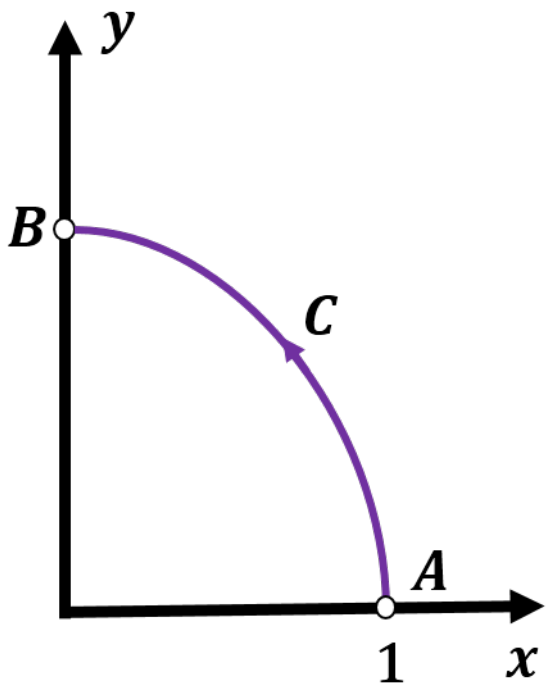


If $\mathbf{F}$ is not a constant force and if the object is pushed along a curve instead of a straight line, then use a line integral to calculate the work done.

Let us look at an example to understand the vector line integral.

Example 1

Given $\mathbf{F} = -y\,\mathbf{i} - xy\,\mathbf{j}$, and C is the circular arc from A to B, calculate the line integral $\mathbf{F} \cdot d\mathbf{r}$.



The points on the arc can be written as $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$

Where, $x^2 + y^2 = 1$

Let us define t such that

$x = \cos t, y = \sin t, 0 \leq t \leq \pi/2$

Then, $\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$

$$d\mathbf{r} = (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\,dt$$

Let us look at an example to understand the vector line integral.

Example 1 (contd.)

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_0^{\pi/2} (-\sin t\,\mathbf{i} + \cos t \sin t\,\mathbf{j}) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})dt$$

$$= \int_0^{\pi/2} (\sin^2 t + \cos^2 t \sin t)\mathrm{d}t = \frac{\pi}{4} + \frac{1}{3}$$

Let us look at another example to understand the vector line integral.

Example 2

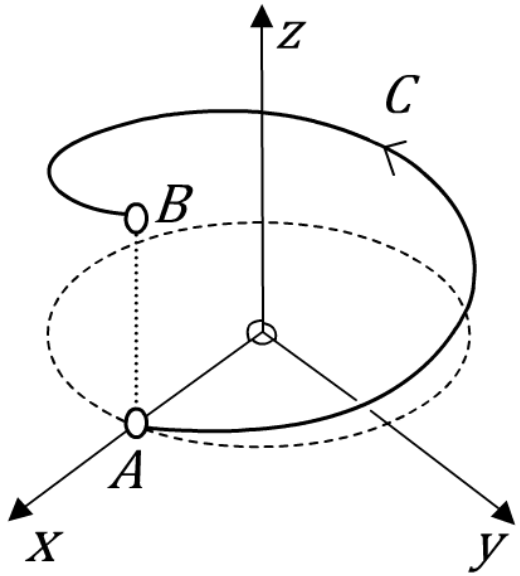
Given $\mathbf{F} = -z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$, and C is the helix given by

$x = \cos t, y = \sin t, z = 3t, 0 \leq t \leq 2\pi$

The points on the helix can be written as:

$\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 3t\,\mathbf{k}$

$d\mathbf{r} = (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 3\mathbf{k})dt$



Let us look at another example to understand the vector line integral.

Example 2 (contd.)

$$W = \int_C \mathbf{F} \cdot d\boldsymbol{r}$$

$$= \int_P^Q (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \cdot (x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k})dt$$

$$= \int_0^{2\pi} (-3t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 3\mathbf{k})dt$$

Let us look at another example to understand the vector line integral.

Example 2 (contd.)

$$W = \int_0^{2\pi} (3t\sin t + \cos^2 t + 3\sin t)dt$$

$$= -6\pi + \pi + 0$$

$$= -5\pi$$

Here is another example to understand the concept of vector line integral.

Example 3

Given the force $\mathbf{F} = x^2\mathbf{i} + y\mathbf{j} + (xz - y)\mathbf{k}$, find the work done by $\mathbf{F}$ from $(0,0,0)$ to $(1,2,4)$ in:

(i) C_1 a straight line joining the points,

(ii) C_2 along the curve given parametrically by $x = t^2, y = 2t, z = 4t^3$

(i) Parametric equations for the straight line path C_1 are:

$$x = t, y = 2t, z = 4t, 0 \leq t \leq 1$$

$$\mathrm{d}\boldsymbol{r} = (\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})dt$$

So, W_1 is given by: $\displaystyle\int_{\mathrm{C}_1} \mathbf{F} \cdot d\mathbf{r} = 11/3$

Here is another example to understand the concept of vector line integral.

Example 3 (contd.)

(ii) Along Path C_2 :

$$d\mathbf{r} = (2t\,\mathbf{i} + 2\,\mathbf{j} + 12t^2\,\mathbf{k})dt$$

So, W_2 is given by:

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 7/3$$

In the previous example, it is to note that $W_1 \neq W_2$, which means that work done is dependent on the path C.

Q

Is there a vector field $\mathbf{F}$ such that W is independent of path taken? What is the necessary condition of such a field to yield this path independence property?

A

The answer is YES. Such a vector field $\mathbf{F}$ is called conservative.

Let us understand the meaning of conservative field and prove that the line integration is path independent.

A vector field $\mathbf{F}$ is called conservative if there is a scalar field V such that $\mathbf{F} = \nabla V$ or equivalently if $\nabla \times \mathbf{F} = 0$.

Suppose there exists a scalar field V such that:

$$\mathbf{F} = \nabla V = \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}.$$

Then, $\displaystyle\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{\mathrm{A}}^{\mathrm{B}} \left(\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$.

Let us understand the meaning of conservative field and prove that the line integration is path independent.

$$\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathrm{A}}^{\mathrm{B}} \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right) = \int_{\mathrm{A}}^{\mathrm{B}} dV = V\Big|_{A}^{B} = V(B) - V(A).$$

Since $dV = \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right).$

The line integral is independent of path.

The line integral is given by the values of the scalar field V at the end points, $V(B) - V(A)$.
V is called the potential function of $\mathbf{F}$ and $\mathbf{F}$ is a conservative field.

For a conservative field $\mathbf{F}$, $\mathbf{F} = \nabla V$ and its curl can be shown as $\nabla \times \mathbf{F} = \nabla \times (\nabla V) = 0$ which is derived below.

$$\nabla \times \mathbf{F} = \nabla \times (\nabla V) = \nabla \times \left(\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k} \right)$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial y} & \dfrac{\partial V}{\partial z} \end{vmatrix}$$

$$\nabla \times \mathbf{F} = \left[\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y} \right] \mathbf{i} + \left[\frac{\partial^2 V}{\partial z \partial x} - \frac{\partial^2 V}{\partial x \partial z} \right] \mathbf{j} + \left[\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x} \right] \mathbf{k} = 0$$

The concept of conservative field can be summarised as below.

- There are two ways to check if $\mathbf{F}$ is conservative.
  - First, find a scalar field V such that $\mathbf{F} = \nabla V$.
  - Second, equivalently check whether $\nabla \times \mathbf{F} = 0$.

Let us look at an example to understand the conservative field.

Example 4

Given $\mathbf{F} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 4z\,\mathbf{k}$, check whether $\mathbf{F}$ is conservative, hence or otherwise calculate the line integrals along the two paths from (0,0,0) to (1,1,1).

$\mathrm{C}_1: \boldsymbol{r}(t) = t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}, 0 \le t \le 1$

$\mathrm{C}_2: \boldsymbol{r}(t) = t\,\mathbf{i} + t^3\,\mathbf{j} + t^2\,\mathbf{k}, 0 \le t \le 1$

To see if $\mathbf{F}$ is conservative, check the curl:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x & 2y & 4z \end{vmatrix} = 0.$$ Hence, $\mathbf{F}$ is conservative.

Let us look at an example to understand the conservative field.

Example 4 (contd.)

Along C_1: $\boldsymbol{r}(t) = t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}$

That is, $x = t, y = t, z = t, 0 \le t \le 1$

$$d\boldsymbol{r}(t) = dt\,\mathbf{i} + dt\,\mathbf{j} + dt\,\mathbf{k}$$

$$d\boldsymbol{r}(t) = (\mathbf{i} + \mathbf{j} + \mathbf{k})dt$$

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\boldsymbol{r} = \int_0^1 (2t\,\mathbf{i} + 2t\,\mathbf{j} + 4t\,\mathbf{k})\,.\,(\mathbf{i} + \mathbf{j} + \mathbf{k})dt$$

$$= \int_0^1 (2t + 2t + 4t\,)dt \;= \frac{8}{2}t^2\Big|_0^1 = 4$$

Let us look at an example to understand the conservative field.

Example 4 (contd.)

Along C_2: $\boldsymbol{r}(t) = t\,\mathbf{i} + t^3\,\mathbf{j} + t^2\,\mathbf{k}$

That is, $x = t, y = t^3, z = t^2, 0 \le t \le 1$

$$d\boldsymbol{r}(t) = dt\,\mathbf{i} + 3t^2 dt\,\mathbf{j} + 2t dt\,\mathbf{k}$$

$$d\boldsymbol{r}(t) = (\mathbf{i} + 3t^2\mathbf{j} + 2t\mathbf{k})dt$$

$$\int_{C_2} \mathbf{F} \cdot \mathrm{d}\boldsymbol{r} = \int_0^1 (2t\,\mathbf{i} + 2t^3\,\mathbf{j} + 4t^2\,\mathbf{k}) \cdot (\mathbf{i} + 3t^2\,\mathbf{j} + 2t\,\mathbf{k})dt$$

$$= \int_0^1 (2t + 6t^5 + 8t^3)dt = t^2 + \frac{6}{6}t^6 + \frac{8}{4}t^4\Big|_0^1 = 4$$

Let us look at an example to understand the conservative field.

Example 4 (contd.)

Once it is found that $\mathbf{F}$ is conservative, the other way to deduce V such that $\mathbf{F} = \nabla V$ is as follows:

$$\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k} = \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}$$

$$\frac{\partial V}{\partial x} = F_1 = 2x, \Rightarrow V = x^2 + f(y,z),$$

$$\frac{\partial V}{\partial y} = F_2 = 2y, \Rightarrow V = y^2 + g(x,z),$$

$$\frac{\partial V}{\partial z} = F_3 = 4z, \Rightarrow V = 2z^2 + h(x,y)$$

Let us look at an example to understand the conservative field.

Example 4 (contd.)

$$V = x^2 + y^2 + 2z^2 + c$$

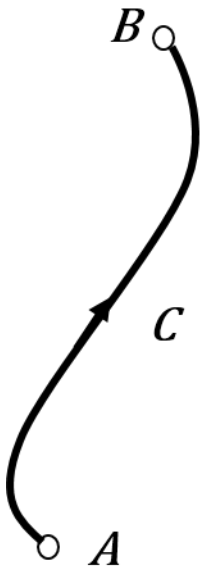
$$\int_C \mathbf{F} \cdot d\boldsymbol{r} = \int_A^B dV = V(B) - V(A)$$

$$= V(1,1,1) - V(0,0,0)$$

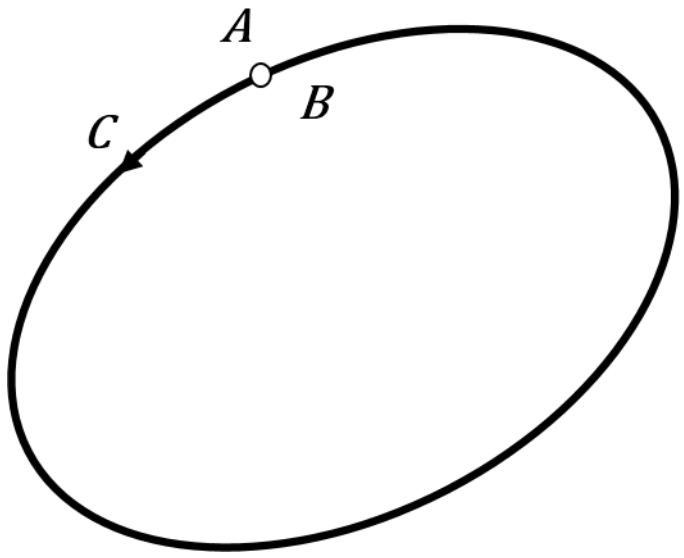
$$= (4 + c) - (0 + c) = 4$$

Where, c is a constant.

The concept of conservative field can be summarised by the given diagrams.



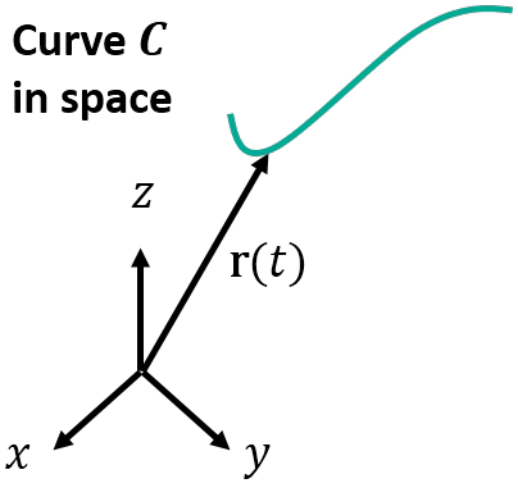
$$\int_A^B \mathbf{F} \cdot d\boldsymbol{r} = V(B) - V(A)$$



$$\oint_C \mathbf{F} \cdot d\boldsymbol{r} = 0$$

Non-conservative fields are also called as dissipative which can be expressed as $\oint_C \mathbf{F} \cdot d\boldsymbol{r} \neq 0$

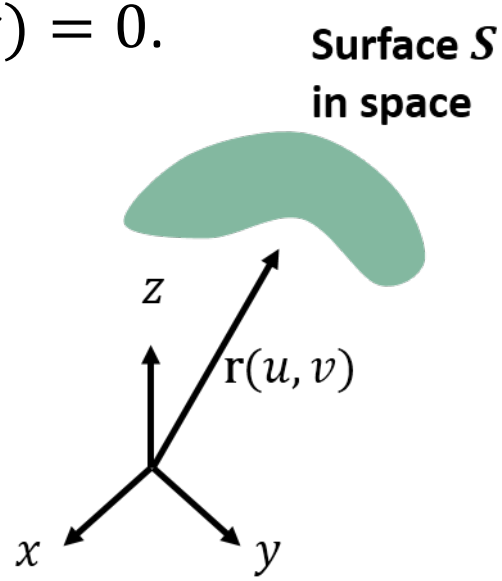
A surface S in the xyz-space can be represented as $z = f(x, y)$ or $g(x, y, z) = 0$.

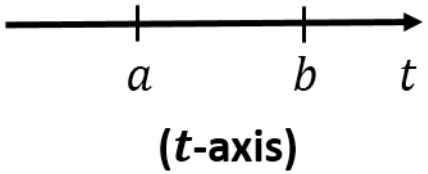


For example, $z = \sqrt{(a^2 - x^2 - y^2)}$

Or $x^2 + y^2 + z^2 - a^2 = 0\,;\, z \geq 0$

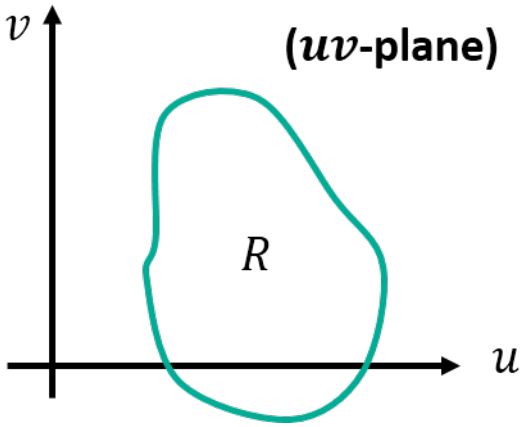
It represents a hemisphere of radius a and centre at the origin.





Just like the parametric representation of a curve in space, a surface can also be represented parametrically.

Surfaces are two-dimensional, hence two parameters are required, which are denoted as u and v.



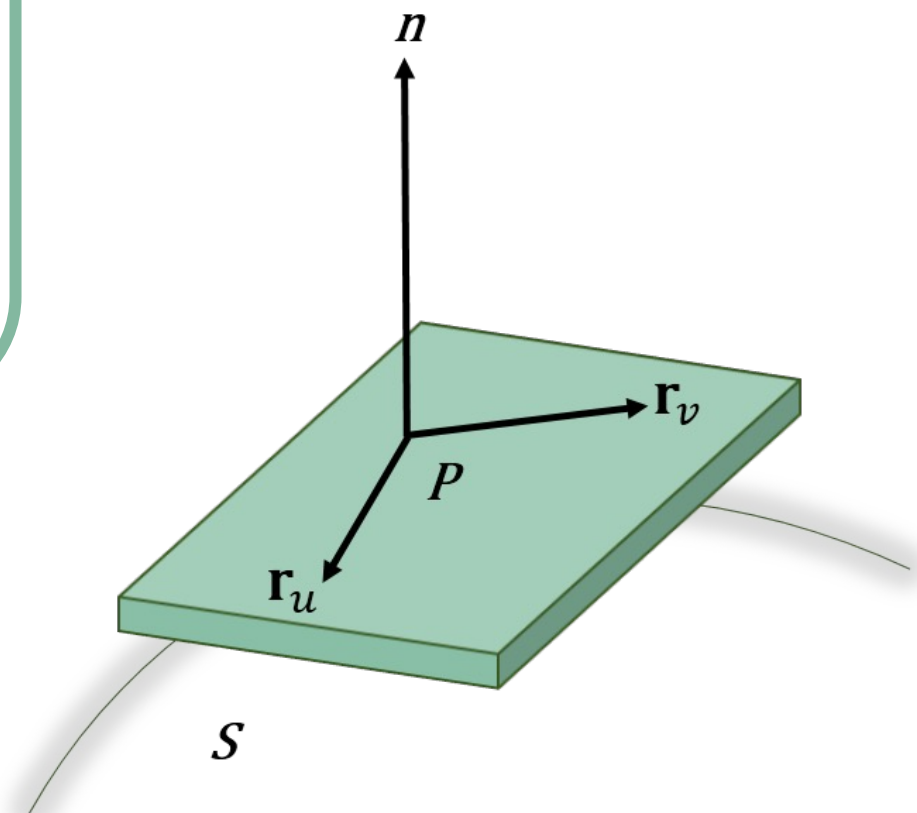
If curve C is represented by $r(t)$, then the tangent to the curve is the derivative $dr(t)/dt$.

From the given parametric representation of the surface S: $\mathbf{r}(u, v)$, the surface normal can be obtained as:

$$\mathbf{N} = \mathbf{r}_\mathrm{u} \times \mathbf{r}_\mathrm{v}$$

Where, $\mathbf{r}_\mathrm{u}$ and $\mathbf{r}_\mathrm{v}$ are partial derivatives of $\mathbf{r}(u, v)$ with respect to u and v respectively.

The unit normal vector is $\mathbf{n} = \mathbf{N}/\|\mathbf{N}\|$.



The two ways derived for computing normal to a surface is summarised below.

1. If the surface is specified by $z = f(x, y)$ or $f(x, y, z) = 0$, then grad f is a normal vector.

2. If the surface is specified parametrically, that is, $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, then $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector.

Unit normal is obtained in both cases by dividing the normal to its length.

Let us look at an example to understand the parametric representation of a circular cylinder.

Example 5

The surface S of a circular cylinder can be shown as:

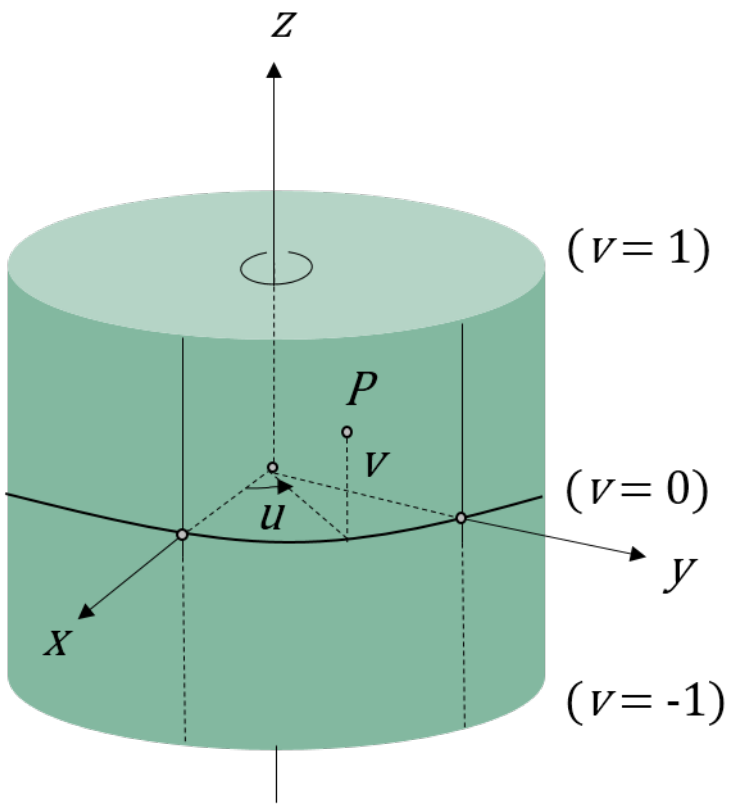
$$x^2 + y^2 = a^2, -1 \leq z \leq 1$$

$$\boldsymbol{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

The parameters u and v can be defined as:

$$x = a\cos u;\ y = a\sin u;\ z = v;$$

$$0 \leq u \leq 2\pi;\ -1 \leq v \leq 1$$



Let us look at an example to understand the parametric representation of a circular cylinder.

Example 5 (contd.)

Then, $\mathbf{r}(u,v) = a\cos u\,\mathbf{i} + a\sin u\,\mathbf{j} + v\,\mathbf{k}$

Let $\mathbf{r}_u = \dfrac{\partial \boldsymbol{r}}{\partial u} = -a\sin u\ \mathbf{i} + a\cos u\ \mathbf{j}$, $\mathbf{r}_v = \dfrac{\partial \boldsymbol{r}}{\partial v} = \mathbf{k}$

A normal to the surface can be given by: $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$

$$\mathbf{N} = a\sin u\ \mathbf{j} + a\cos u\ \mathbf{i} = (x\,\mathbf{i} + y\,\mathbf{j})$$

Check if: $f = x^2 + y^2 = a^2$

Then, $\nabla f = \left(\dfrac{\partial}{\partial x}\mathbf{i} + \dfrac{\partial}{\partial y}\mathbf{j}\right)(x^2 + y^2) = 2(x\mathbf{i} + y\,\mathbf{j})$

Now, let us take a look at an example on the parametric representation of a sphere.

Example 6

Surface S: The position vector $\boldsymbol{r}$ of a point of the sphere S is given by:

$\boldsymbol{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

Subject to the constraint,

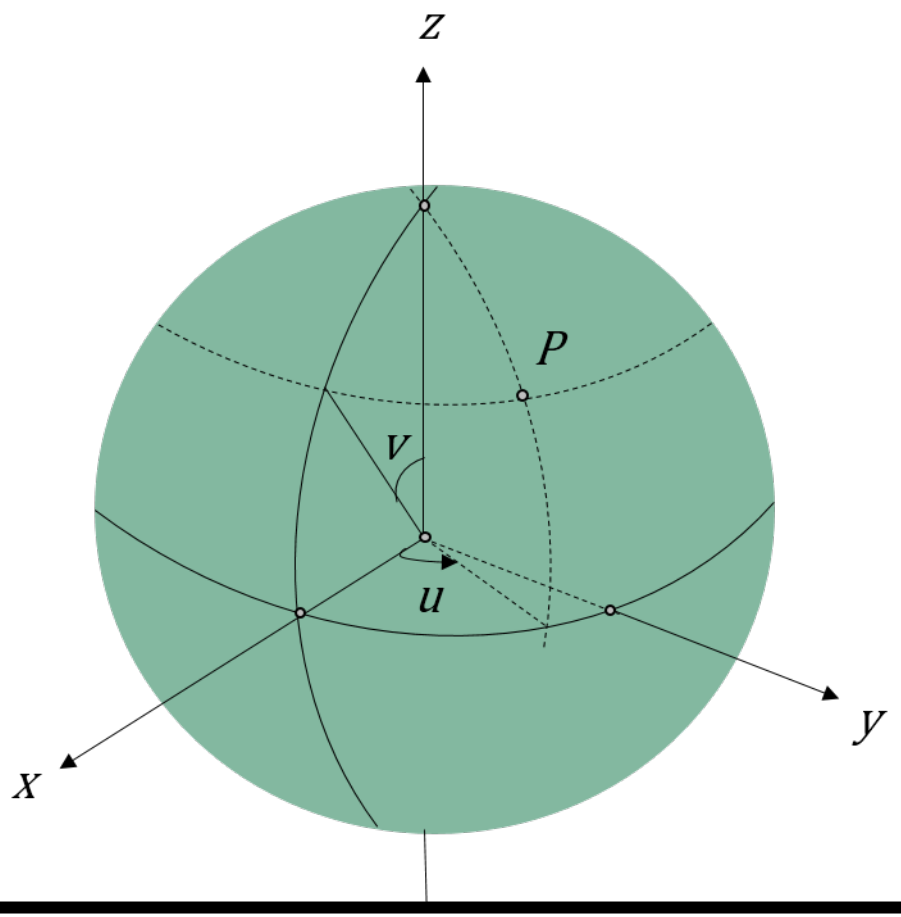
$$x^2 + y^2 + z^2 = a^2,$$

The parameters u and v can be defined as:

$x = a\cos u \sin v\,;\ 0 \le u \le 2\pi$

$y = a\sin u \sin v\,; 0 \le v \le \pi$

$z = a\cos v$



Now, let us take a look at an example on the parametric representation of a sphere.

Example 6 (contd.)

If $\mathbf{r}(u,v) = a\cos u \sin v\,\mathbf{i} + a\sin u \sin v\,\mathbf{j} + a\cos v\,\mathbf{k}$

Then, $\mathbf{r}_u = \dfrac{\partial \boldsymbol{r}}{\partial u} = -a\sin u \sin v\,\mathbf{i} + a\cos u \sin v\,\mathbf{j}$,

$$\mathbf{r}_v = \frac{\partial \boldsymbol{r}}{\partial v} = a\cos u \cos v\,\mathbf{i} + a\sin u \cos v\,\mathbf{j} - a\sin v\,\mathbf{k}$$

A normal to the surface can be found by: $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$

$$\mathbf{N} = -a\sin v(\,a\cos u \sin v\,\mathbf{i} + a\sin u \sin v\,\mathbf{j} \;+ a\cos v\,\mathbf{k})$$

$$= -a\sin v(\,x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

Now, let us take a look at an example on the parametric representation of a sphere.

Example 6 (contd.)

$\mathbf{N}$ is thus in the inward radial direction. For outward pointing normal, take the negative of $\mathbf{N}$. That is, $a \sin v(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$

Check if: $f = x^2 + y^2 + z^2 = a^2$

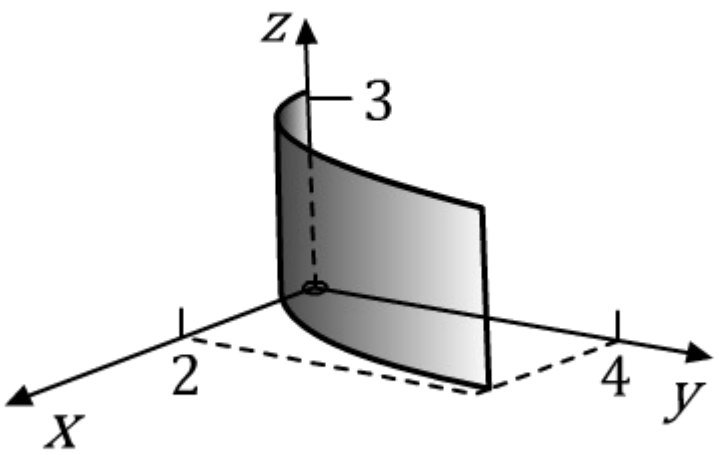
Then, $\nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(x^2 + y^2 + z^2) = 2(x\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$

which is in outward radial direction.

Here is an example on the parametric representation of a parabolic cylinder.

Example 7

The surface S of a parabolic cylinder can be shown as:



$y = x^2, 0 \leq x \leq 2, 0 \leq z \leq 3$

$\boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

If $\mathbf{r}(u, v) = u\,\mathbf{i} + u^2\,\mathbf{j} + v\,\mathbf{k}$

Then,

$$\mathbf{r}_u = \frac{\partial \boldsymbol{r}}{\partial u} = \mathbf{i} + 2u\,\mathbf{j}, \mathbf{r}_v = \frac{\partial \boldsymbol{r}}{\partial v} = \mathbf{k} \begin{cases} x = u; 0 \leq u \leq 2 \\ y = u^2 \\ z = v; 0 \leq v \leq 3 \end{cases}$$

Here is an example on the parametric representation of a parabolic cylinder.

Example 7 (contd.)

The normal vector can be given as: $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = 2u\,\mathbf{i} - \mathbf{j}$

Check if: $f = y - x^2 = 0$

$$\nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(y - x^2)$$

$$= (-2x\,\mathbf{i} - \mathbf{j})$$

Given below is an example to understand the parametric representation of a plane.

Example 8

The surface S of a plane can be shown as:

$$x + y + z = 1, 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$$

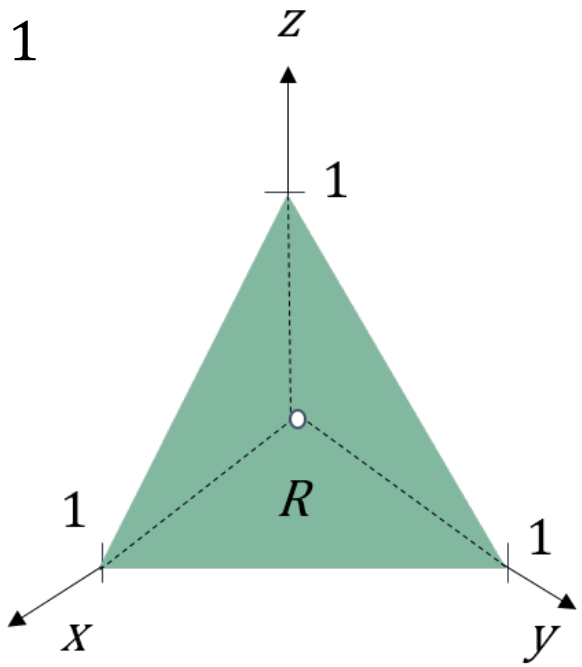
$$\boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

If $\mathbf{r}(u, v) = u\,\mathbf{i} + v\,\mathbf{j} + (1 - u - v)\,\mathbf{k}$

Then,

$$\mathbf{r}_u = \frac{\partial \boldsymbol{r}}{\partial u} = \mathbf{i} - \mathbf{k}, \quad \begin{cases} x = u; 0 \le u \le 1 \\ y = v; 0 \le v \le 1 \\ z = 1 - u - v \end{cases}$$

$$\mathbf{r}_v = \frac{\partial \boldsymbol{r}}{\partial v} = \mathbf{j} - \mathbf{k}$$



Given below is an example to understand the parametric representation of a plane.

Example 8 (contd.)

The normal vector can be given as: $\mathbf{N} = \mathbf{r}_u \text{ x } \mathbf{r}_v = \mathbf{i} + \mathbf{j} + \mathbf{k}$

Check if: $f = x + y + z = 1$

$$\nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(x + y + z)$$

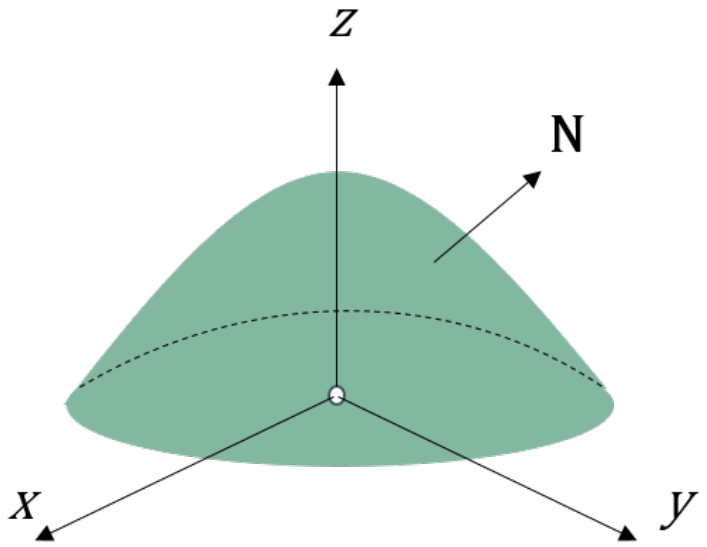
$$= (i + j + k)$$

Let us now look at an example on the parametric representation of a paraboloid.

Example 9

The surface S of a paraboloid can be shown as:

$$z = 1 - (x^2 + y^2),\ z \geq 0$$



From parameterisation and normal vector:

$$\begin{cases} x = \sqrt{1-v}\cos u \\ y = \sqrt{1-v}\sin u \\ z = v;\ 0 \leq v \leq 1 \end{cases}$$

$$\mathbf{r}_u = \frac{\partial \boldsymbol{r}}{\partial u} = -\sqrt{1-v}\sin u\,\mathbf{i} + \sqrt{1-v}\cos u\,\mathbf{j},$$

Let us now look at an example on the parametric representation of a paraboloid.

Example 9 (contd.)

$$\mathbf{r}_v = \frac{\partial \boldsymbol{r}}{\partial v} = -\frac{1}{2\sqrt{1-v}}\cos u\,\mathbf{i} - \frac{1}{2\sqrt{1-v}}\sin u\,\mathbf{j} + \mathbf{k}$$

The normal vector can be given as: $\mathbf{N} = \mathbf{r}_u \text{ x } \mathbf{r}_v$

$$\mathbf{N} = +\frac{1}{2}\sin^2 u\,\mathbf{k} + \sqrt{1-v}\sin u\,\mathbf{j}$$

$$+\frac{1}{2}\cos^2 u\,\mathbf{k} + \sqrt{1-v}\cos u\,\mathbf{j}$$

$$= \sqrt{1-v}\cos u\,\mathbf{i} + \sqrt{1-v}\sin u\,\mathbf{j} + 1/2\ \mathbf{k}$$

Now let's get familiarised with the flux integral or flux of F through surface S.

Let $\mathrm{d}A$ be the area of a small patch in a surface S. Then, it can be seen that $\mathbf{n}\mathrm{d}A = \mathbf{N}\,\mathrm{d}u\,\mathrm{d}v$.

Where, $\mathbf{N}$ is a surface normal given by $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ and $\mathbf{n}$ is the unit normal vector given by $\mathbf{n} = \mathbf{N}/\|\mathbf{N}\|$.

If the notation $\mathbf{dA}$ can be introduced as a vector normal to the small patch with a magnitude $\mathrm{d}A$, then it can be seen that $\mathbf{dA} = \mathbf{n}\mathrm{d}A = \mathbf{N}\,\mathrm{d}u\,\mathrm{d}v$.

Now let's get familiarised with the flux integral or flux of F through surface S.

Double integral of $\mathbf{F} \cdot \mathbf{dA}$ or $\mathbf{F} \cdot \mathbf{n}\mathrm{d}A$ over S is called a flux integral as it gives the flux (Latin for "flow") of $\mathbf{F}$ through the surface S.

If $\mathbf{F}$ is the velocity of a fluid then the integral gives the amount of fluid that flows through S per unit time.

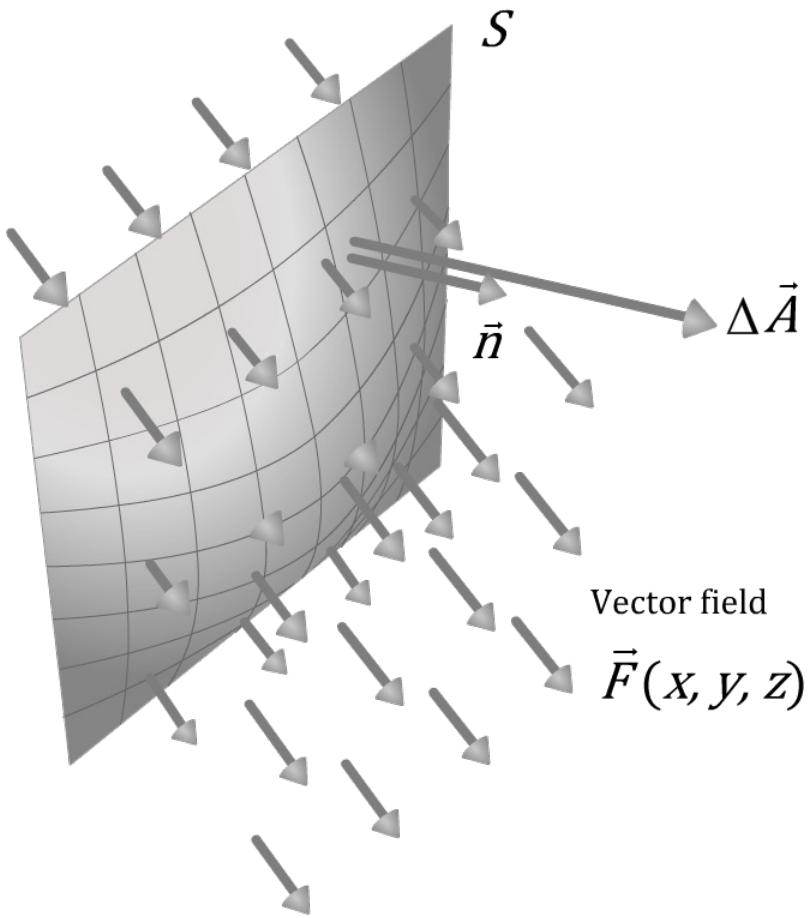
The flux integral of the vector field $\mathbf{F}$ through the oriented surface S is shown below.

$$\iint_S \mathbf{F} \cdot d\mathbf{A} = \lim_{|\Delta \mathrm{A}| \to 0} \sum \mathbf{F} \cdot \Delta \mathbf{A}$$

$$= \iint_S \mathbf{F} \cdot d\mathbf{A}$$

$$= \iint_S \mathbf{F} \cdot \mathbf{n} d\mathbf{A}$$

$$= \iint_R \mathbf{F}(u, v) \cdot \mathbf{N} \, \mathrm{d}u \, \mathrm{d}v$$



Let us take a look at a sample problem to understand the concept of surface integral.

Sample Problem 1

Compute the flux of the vector field $F = \mathbf{i} + xy\,\mathbf{j}$ across the surface given by: $x = u + v$, $y = u - v$, $z = u^2$; $0 \le u \le 1$, $0 \le v \le 1$

Solution:

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, \mathrm{d}u\, \mathrm{d}v = \int_0^1 \int_0^1 \mathbf{i} + xy\,\mathbf{j} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, \mathrm{d}u\, \mathrm{d}v$$

$$\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + u^2\mathbf{k}$$

$$\mathbf{r}_u = \mathbf{i} + \mathbf{j} + 2u\mathbf{k},\ \mathbf{r}_v = \mathbf{i} - \mathbf{j}$$

Let us take a look at a sample problem to understand the concept of surface integral.

Solution (contd.):

$$\mathbf{F}(u,v) = \mathbf{i} + (u^2 - v^2)\mathbf{j}$$

$$\mathbf{F} \cdot \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} 1 & u^2 - v^2 & 0 \\ 1 & 1 & 2u \\ 1 & -1 & 0 \end{vmatrix} = 2u^3 - 2uv^2 + 2u$$

$$\int_0^1 \int_0^1 (2u^3 - 2uv^2 + 2u)\, \mathrm{d}u\, \mathrm{d}v = \int_0^1 \left[\frac{1}{2} - v^2 + 1\right] \mathrm{d}v = \frac{7}{6}$$

Let us take a look at another sample problem to understand the concept of surface integral.

Sample Problem 2

Consider the vector field $\boldsymbol{F} = 2x\,\mathbf{i} + 2y\,\mathbf{j}$.

And, consider the surface S:

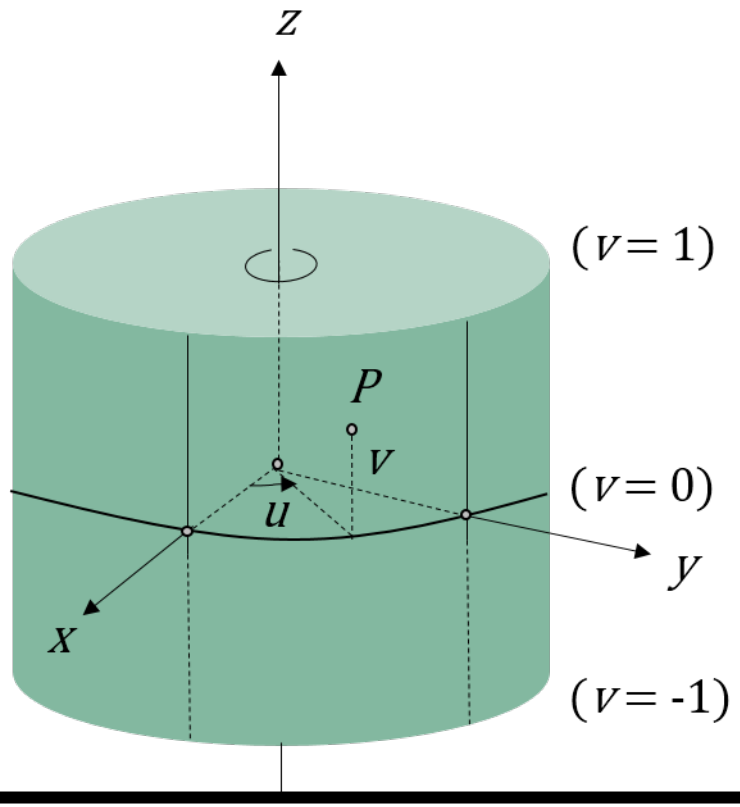
$$x^2 + y^2 = a^2$$

$$-1 \le z \le 1$$

Find $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{A}$

Parameters are: $\begin{cases} x = a\cos u \\ y = a\sin u\,, 0 \le u \le 2\pi \\ z = v \end{cases}$

$\mathbf{N} = a\cos u\,\mathbf{i} + a\sin u\,\mathbf{j}$



Let us take a look at another sample problem to understand the concept of surface integral.

Solution:

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{A} = \iint_R \mathbf{F}(u,v) \cdot \mathbf{N}(u,v)\, \mathrm{d}u\, \mathrm{d}v$$

$$= \int_{-1}^{1} \int_{0}^{2\pi} (2a \cos u\, \mathbf{i} + 2a \sin u\, \mathbf{j}) \cdot (a \cos u\, \mathbf{i} + a \sin u\, \mathbf{j})\, \mathrm{d}u\, \mathrm{d}v$$

$$= \int_{-1}^{1} \int_{0}^{2\pi} (2a^2 \cos^2 u + 2a^2 \sin^2 u)\, \mathrm{d}u\, \mathrm{d}v$$

Let us take a look at another sample problem to understand the concept of surface integral.

Solution (contd.):

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{A} = \int_{-1}^{1} \int_{0}^{2\pi} (2a^2)\, \mathrm{d}u\, \mathrm{d}v$$

$$= 2a^2 \int_{-1}^{1} \mathrm{d}v \int_{0}^{2\pi} \mathrm{d}u$$

$$= 8\pi a^2$$

Here is another sample problem explaining the concept of surface integral.

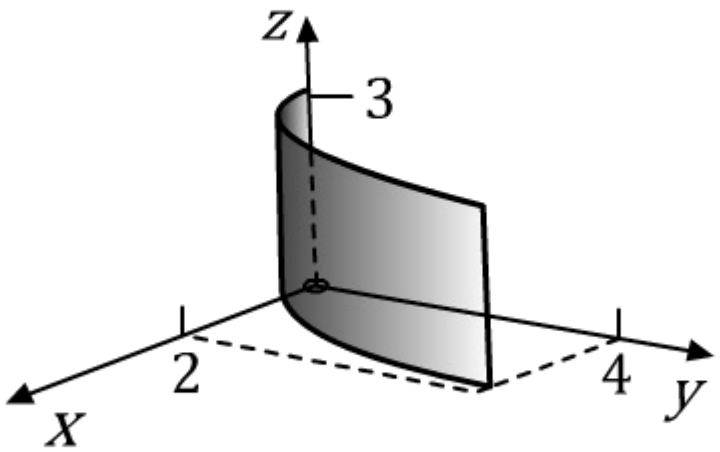
Sample Problem 3

Compute the flux of water through the parabolic cylinder if the velocity vector field is given by $\boldsymbol{F} = 3z^2\mathbf{i} + 6\mathbf{j} + 6xz\mathbf{k}$

And, the surface S is given by:

$y = x^2, 0 \leq x \leq 2, 0 \leq z \leq 3$

$\boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$



Here is another sample problem explaining the concept of surface integral.

Solution:

$$\mathbf{r}(u,v) = u\mathbf{i} + u^2\mathbf{j} + v\mathbf{k};\ 0 \le u \le 2,\ 0 \le v \le 3$$

$$\mathbf{r}_u = \mathbf{i} + 2u\mathbf{j} + 0\mathbf{k};\ r_v = +\mathbf{k}$$

$$\mathbf{N} = 2u\mathbf{i} - 1\mathbf{j} + 0\mathbf{k};\ \boldsymbol{F}(u,v) = 3v^2\mathbf{i} + 6\mathbf{j} + 6uv\mathbf{k}$$

Here is another sample problem explaining the concept of surface integral.

Solution(contd.):

$$\iint_S \mathbf{F}\cdot\mathbf{n}\mathrm{d}\mathbf{A} = \iint_R \mathbf{F}(u,v)\cdot\mathbf{N}(u,v)\mathrm{d}u\,\mathrm{d}v$$

$$= \int_0^3\int_0^2 (3v^2\mathbf{i} + 6\mathbf{j} + 6uv\mathbf{k})\cdot(2u\mathbf{i} - \mathbf{j})\,\mathrm{d}u\,\mathrm{d}v$$

$$= 72$$

Summary

Key points discussed in this lesson:

Vector line integral is the operation of integrating a vector field along a curve in space. Such integrals have considerable utility in many areas of Engineering and Physics.

A vector field $\mathbf{F}$ is called conservative if there is a scalar field V such that $\mathbf{F} = \nabla V$ or equivalently if $\nabla \times \mathbf{F} = 0$.

The line integral is independent of path and is given by the values of the scalar field V at the end points, $V(B) - V(A)$ where, V is called the potential function of $\mathbf{F}$, and $\mathbf{F}$ is a conservative field.

Key points discussed in this lesson:

- Non conservative fields are also called as dissipative as $\oint_C \mathbf{F} \cdot d\boldsymbol{r} \neq 0$.

- From the given parametric representation of the surface S: $\mathbf{r}(u, v)$ you could obtain the surface normal as $\mathbf{N} = \mathbf{r_u} \times \mathbf{r_v}$, where, $\mathbf{r_u}$ and $\mathbf{r_v}$ are partial derivatives of $\mathbf{r}(u, v)$ with respect to u and v respectively. The unit normal vector is given as: $\mathbf{n} = \mathbf{N}/\|\mathbf{N}\|$.

- The two ways derived for computing normal to a surface is summarised below.
  - If the surface is specified by $z = f(x, y)$ or $f(x, y, z) = 0$ then grad f is a normal vector.
  - If the surface is specified parametrically, that is, $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, then $\mathbf{r_u} \times \mathbf{r_v}$ is a normal vector.

  Unit normal is obtained in both cases by dividing the normal to its length.

Key points discussed in this lesson:

- Double integral of $\mathbf{F} \cdot \mathbf{dA}$ or $\mathbf{F} \cdot \mathbf{ndA}$ over S is called a flux integral as it gives the flux of $\boldsymbol{F}$ through the surface S.

- The flux integral of the vector field $\mathbf{F}$ through the oriented surface S is given by:

$$\iint_S \mathbf{F} \cdot \mathbf{dA} = \iint_R \mathbf{F}(u, v) \cdot \mathbf{N} \, \mathrm{du} \, \mathrm{dv}$$

Vector Volume Integral

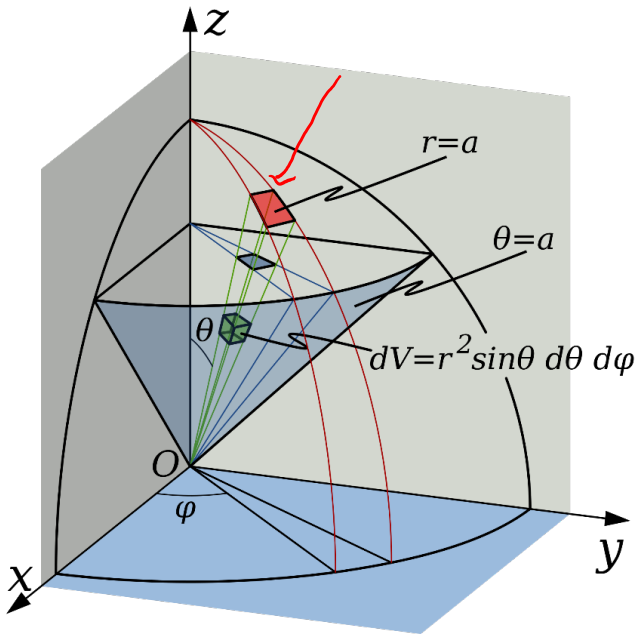
EE2007 – Engineering Mathematics II

At the end of this lesson, you should be able to:

- Perform the volume integrals.

The volume integral of a scalar function f over a prescribed volume V in the Cartesian coordinates is expressed as shown below.

$$\iiint_V f\, dV = \iiint_V f(x, y, z)\, dxdydz$$



Let us look at a sample problem to understand the concept of volume integral.

Sample Problem 1

Let $f = 2(x + y + z)$. Find the volume integral of f over the unit cube.

Solution:

$$\iiint_V f\, dV = \int_0^1 \int_0^1 \int_0^1 2(x + y + z)\, dxdydz$$

$$= \int_0^1 \int_0^1 \left[x^2 + 2x(y + z)\right]_0^1 dydz$$

Let us look at a sample problem to understand the concept of volume integral.

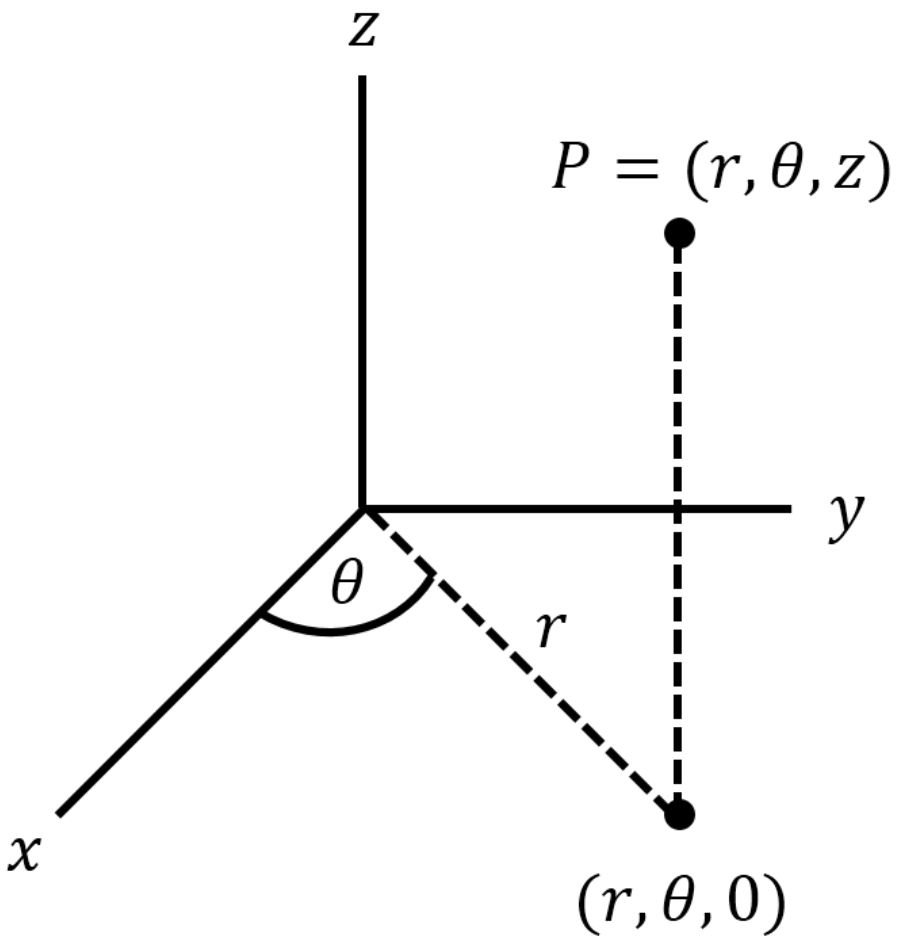
Solution (contd.):

$$\iiint_V f\, dV = \int_0^1 \int_0^1 [1 + 2(y + z)]\, dydz$$

$$= \int_0^1 [y + y^2 + 2yz]_0^1\, dz$$

$$= \int_0^1 [2 + 2z]\, dz = [2z + z^2]_0^1 = 3$$

The Cartesian coordinates (x, y, z) can be converted into cylindrical coordinates (r, θ, z) as shown below.



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$0 \le r < \infty\,;\, 0 \le \theta < 2\pi$$

$$x^2 + y^2 = r^2$$

Using cylindrical coordinates, the terms of the integral can be written in the following form.

$$\int\int\int dV = \int\int\int r\,dr\,d\theta\,dz$$

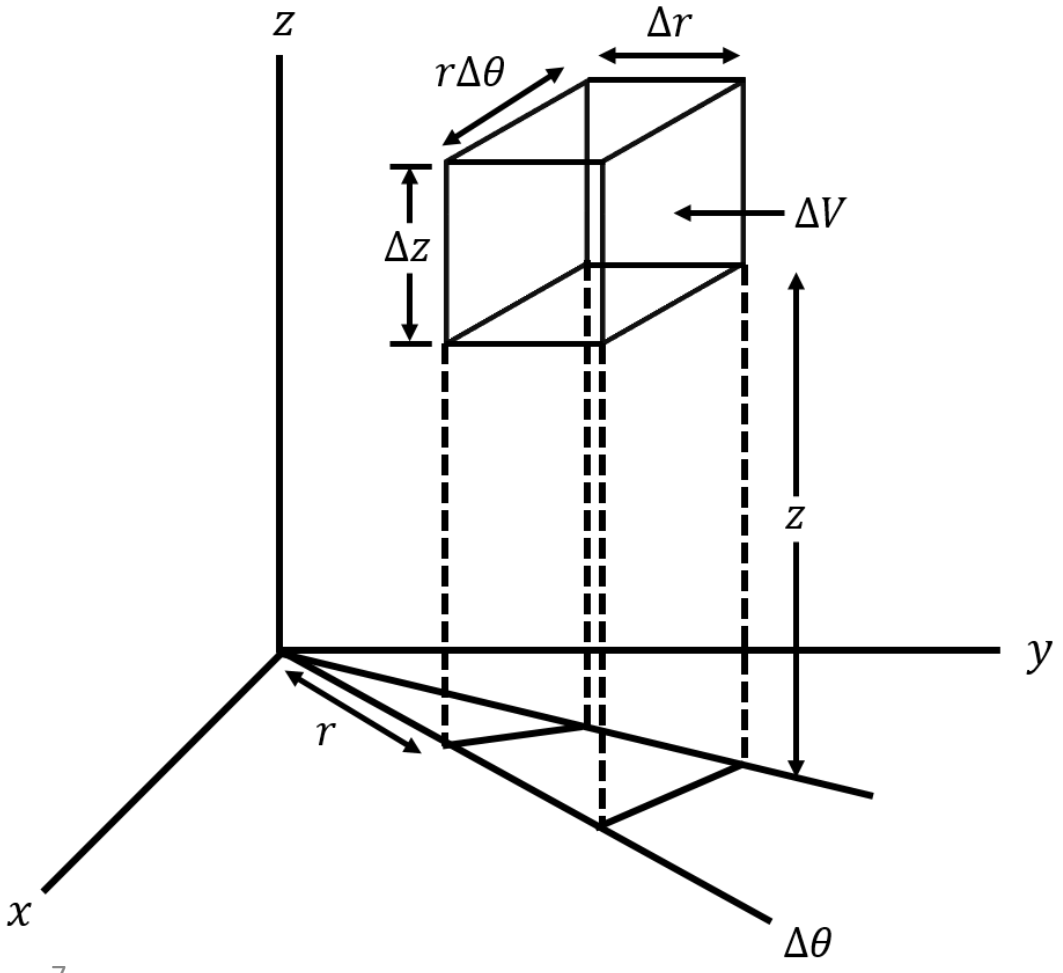
Computing the elemental volume

$$\Delta V \approx (\Delta r)(r\Delta\theta)(\Delta z)$$

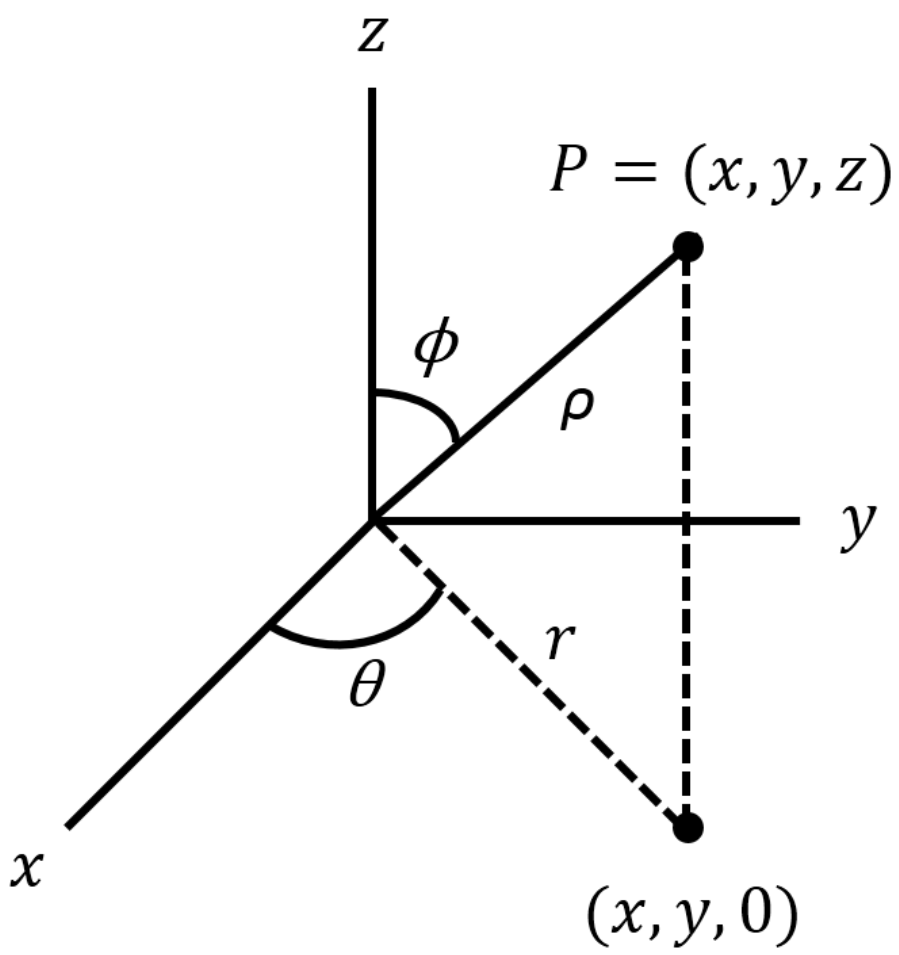
$$\Delta V = \Delta x\,\Delta y\,\Delta z \approx r\,\Delta r\,\Delta\theta\,\Delta z$$

In the limits, it can be written as:

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz$$



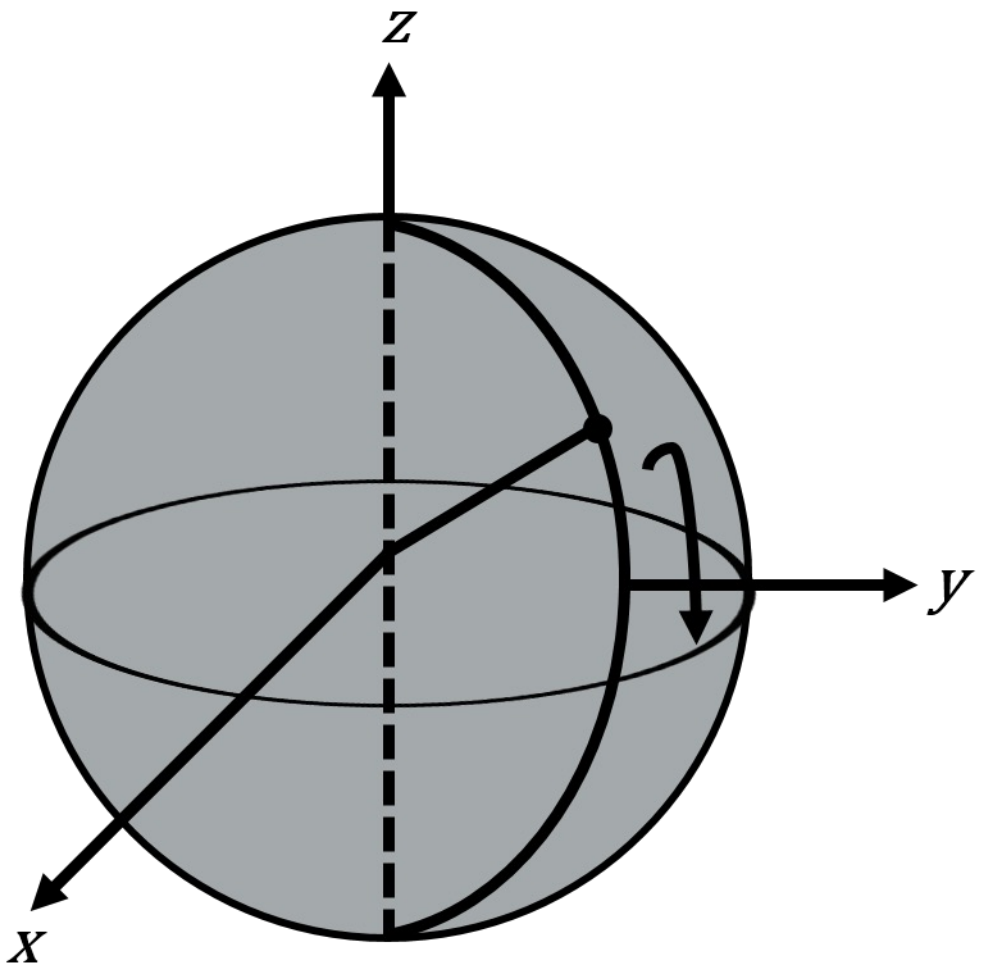
The Cartesian coordinates can be converted into spherical coordinates.

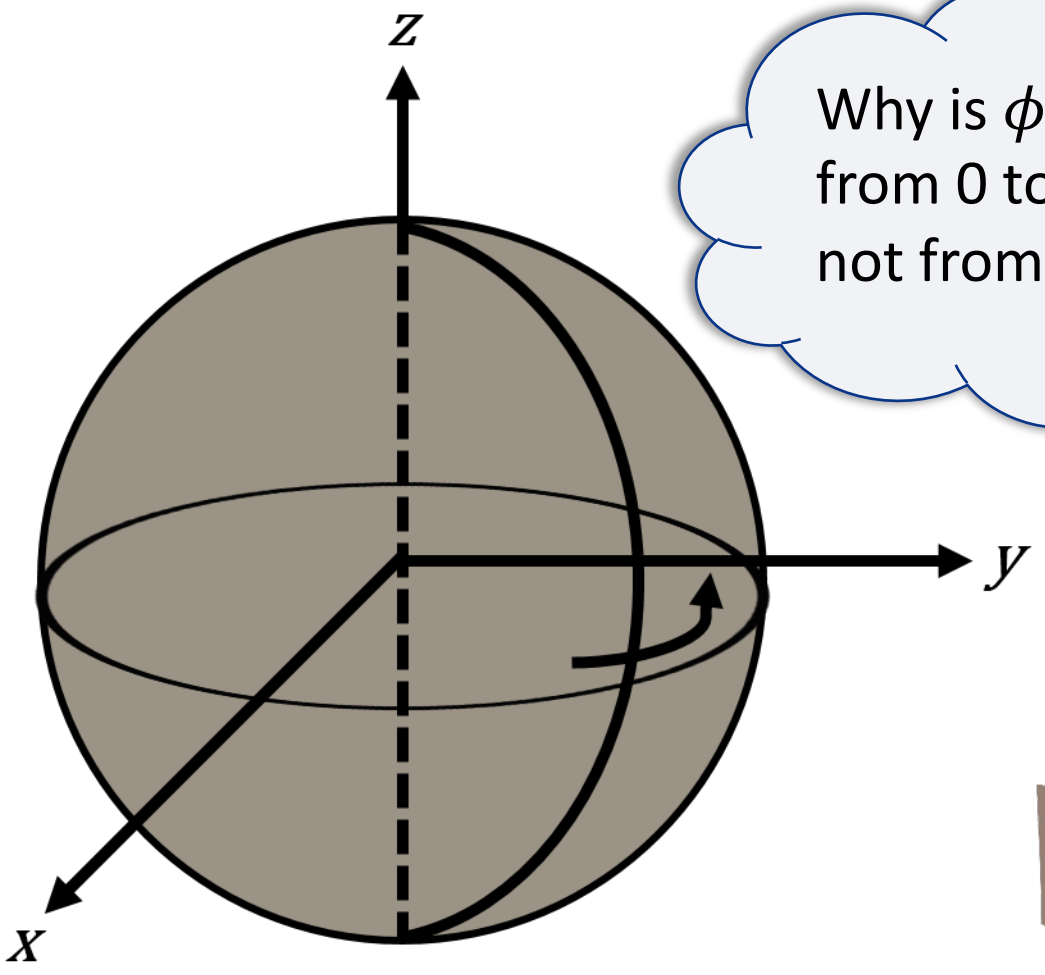


$$\begin{cases} x = \rho\cos\theta\sin\phi \\ y = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{cases}$$

$0 \le \rho < \infty;\ 0 \le \theta < 2\pi$ $\quad$ $0 \le \phi < \pi$

$$x^2 + y^2 + z^2 = \rho^2$$





Why is ϕ ranging from 0 to π, and not from 0 to 2π?

Using spherical coordinates, the terms of the integral can be written in the following form.

$$\int\int\int dV = \int\int\int \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\Delta V \approx (\Delta\rho)(\rho\Delta\phi)(\rho\sin\phi \, \Delta\theta)$$

The following sample problem explains how to evaluate volume integrals in terms of spherical coordinates.

Sample Problem 2

Let $f = z^2(x^2 + y^2 + z^2)^{1/2}$. Find the volume integral of f over the hemisphere given by: $x^2 + y^2 + z^2 = 4$

Solution:

In Cartesian coordinates:

$$\iiint_V f\, dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2 + y^2 + z^2}\, dz\, dy\, dx$$

The following sample problem explains how to evaluate volume integrals in terms of spherical coordinates.

Solution (contd.):

In spherical coordinates:

$$\iiint_V f\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} (\rho\cos\phi)^2 \rho\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} \rho^5 \cos^2\phi\, \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \left[\frac{1}{6}\rho^6\right]_0^2 \cos^2\phi\, \sin\phi\, d\phi\, d\theta$$

The following sample problem explains how to evaluate volume integrals in terms of spherical coordinates.

Solution (contd.):

$$\iiint_V f\, dV = \frac{32}{3}\int_0^{2\pi}\left[-\frac{1}{3}\cos^3\phi\right]_0^{\pi/2} d\theta$$

$$= \frac{32}{9}\int_0^{2\pi} d\theta$$

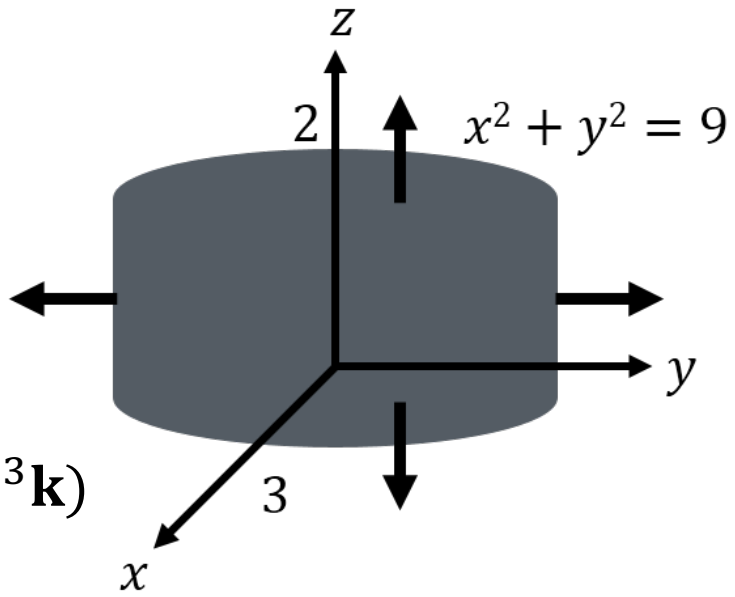
$$= \frac{64}{9}\pi$$

The following sample problem explains how to evaluate volume integrals in terms of cylindrical coordinates.

Sample Problem 3

Let $\mathbf{F} = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^2\,\mathbf{k}$. Find the volume integral of $\nabla \cdot \boldsymbol{F}$ over the cylinder:

$x^2 + y^2 = 9;\ 0 \le z \le 2$



Solution:

$$\nabla \cdot \mathbf{F} = \left[\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right] \cdot (x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k})$$

$$= 3x^2 + 3y^2 + 2z$$

The following sample problem explains how to evaluate volume integrals in terms of cylindrical coordinates.

Solution (contd.):

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \int_0^{2\pi} \int_0^3 \int_0^2 (3x^2 + 3y^2 + 2z) r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^3 \int_0^2 (3r^2 + 2z) r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^3 \left[3r^3 z + r z^2\right]_0^2 dr\, d\theta$$

The following sample problem explains how to evaluate volume integrals in terms of cylindrical coordinates.

Solution (contd.):

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \int_0^{2\pi} \int_0^3 [6r^3 + 4r]\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{6}{4}r^4 + \frac{4}{2}r^2\right]_0^3 d\theta$$

$$= \int_0^{2\pi} \frac{279}{2}\, d\theta = 279\pi$$

Summary

Key points discussed in this lesson:

- The volume integral of a scalar function S over a prescribed volume V in the Cartesian coordinates is given by: $\iiint_V f \cdot dV = f(x, y, z)\, dxdydz$

- The line integral of a conservative vector field is path independent.

Theorems

EE2007 – Engineering Mathematics II

Theorems > Learning Objectives

At the end of this lesson, you should be able to:

- Apply Gauss or Divergence Theorem.
- Apply Stokes Theorem.
- Apply Green's Theorem.

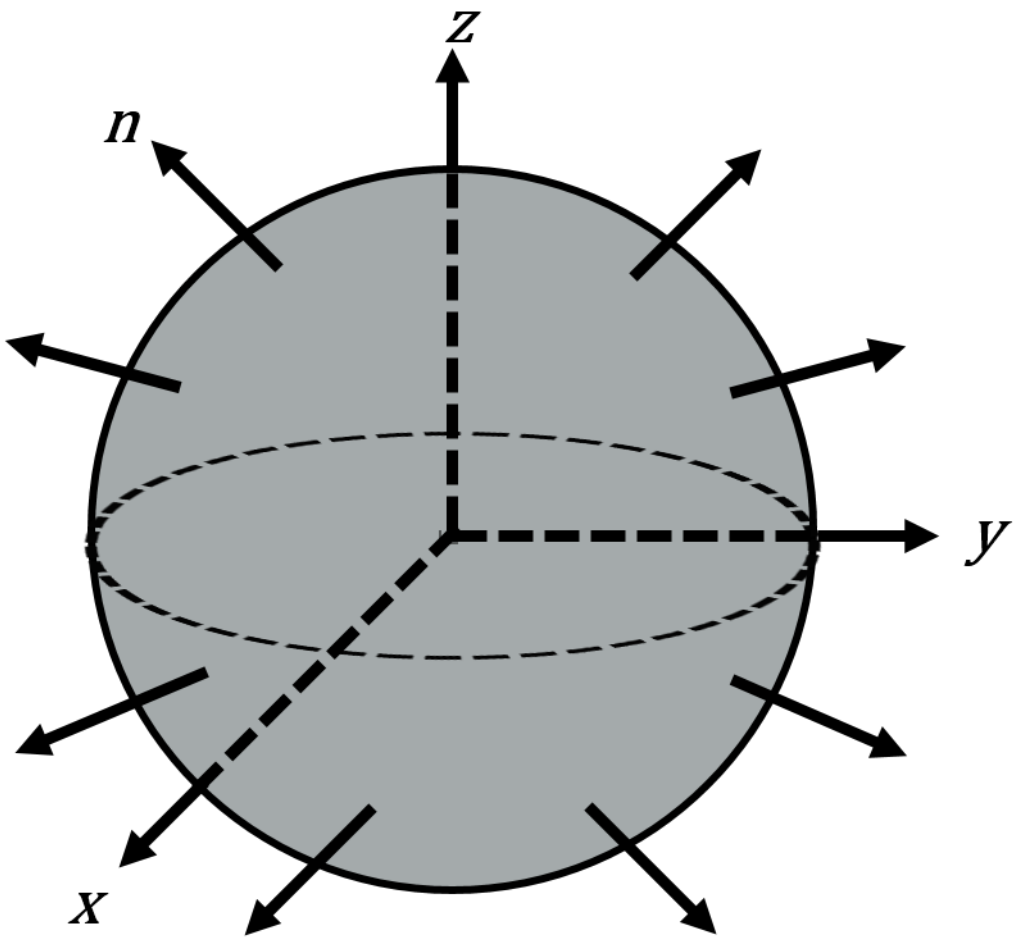
Gauss or Divergence Theorem

Gauss or divergence theorem provides a method to relate surface integrals to triple integrals.

The flux of a vector field across a closed surface with outward orientation is equal to the triple integral of the divergence over the region enclosed by the surface.

$$\oiint_S \mathbf{F} \cdot \boldsymbol{n}\, dA = \iiint_V \nabla \cdot \mathbf{F}\, dV$$

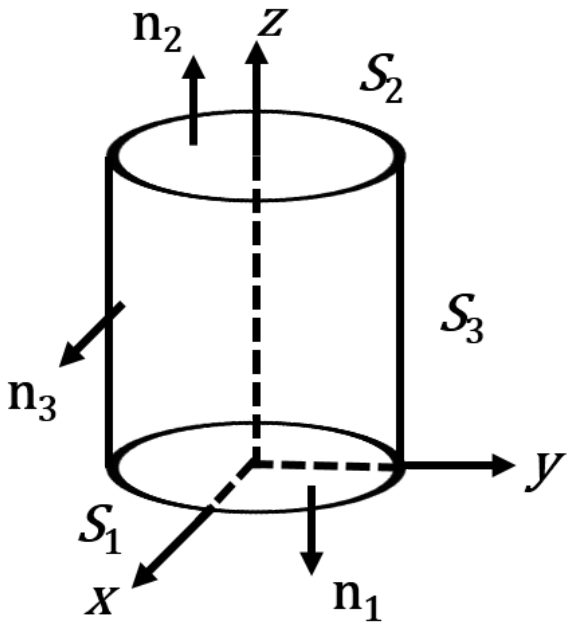
That is, the total divergence of a vector field in a bounded region is equal to the flux of the vector field away from the region.



Now let us try to understand the divergence theorem with the help of the following sample problem.

Sample Problem 4

Let $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Verify the divergence theorem for this vector field and the solid cylinder of radius a, $0 \le z \le b$.



Now let us try to understand the divergence theorem with the help of the following sample problem.

Solution:

$$\oiint_S \mathbf{F}\cdot\boldsymbol{n}\,dA = \iint_{S_1} \mathbf{F}\cdot\boldsymbol{n}\,dA + \iint_{S_2} \mathbf{F}\cdot\boldsymbol{n}\,dA + \iint_{S_3} \mathbf{F}\cdot\boldsymbol{n}\,dA$$

$$\iint_{S_1} \mathbf{F}\cdot\boldsymbol{n}\,dA = \iint_{S_1} (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\cdot(-\mathbf{k})\,dxdy = \iint_{S_1} -z\,dxdy = 0$$

$$\iint_{S_2} \mathbf{F}\cdot\boldsymbol{n}\,dA = \iint_{S_2} (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\cdot(\mathbf{k})\,dxdy = \iint_{S_2} z\,dxdy = b\pi a^2$$

Now let us try to understand the divergence theorem with the help of the following sample problem.

Solution (contd.):

$$\iint_{S_3} \mathbf{F} \cdot \boldsymbol{n}\, dA = \iint_R \mathbf{F}(u,v) \cdot \mathbf{N}(u,v)\, dudv$$

$$= \int_0^b \int_0^{2\pi} (a\cos u\, \mathbf{i} + a\sin u\, \mathbf{j} + v\, \mathbf{k}) \cdot (a\cos u\, \mathbf{i} + a\sin u\, \mathbf{j})\, dudv$$

$$= \int_0^b \int_0^{2\pi} (a^2\cos^2 u + a^2\sin^2 u)\, dudv$$

$$= \int_0^b \int_0^{2\pi} (a^2)\, dudv = 2b\pi a^2$$

Now let us try to understand the divergence theorem with the help of the following sample problem.

Solution (contd.):

Therefore,

$$\oiint_S \mathbf{F}\cdot\mathbf{n}\,dA = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dA + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dA + \iint_{S_3} \mathbf{F}\cdot\mathbf{n}\,dA = 3b\pi a^2$$

On the other hand,

$$\nabla\cdot\mathbf{F} = \left[\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right]\cdot(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = 3$$

$$\iiint_V \nabla\cdot\mathbf{F}\,dV = \iiint_V 3\,dV = 3\iiint_V dV = 3b\pi a^2$$

Now let us try to understand the divergence theorem with the help of the following sample problem.

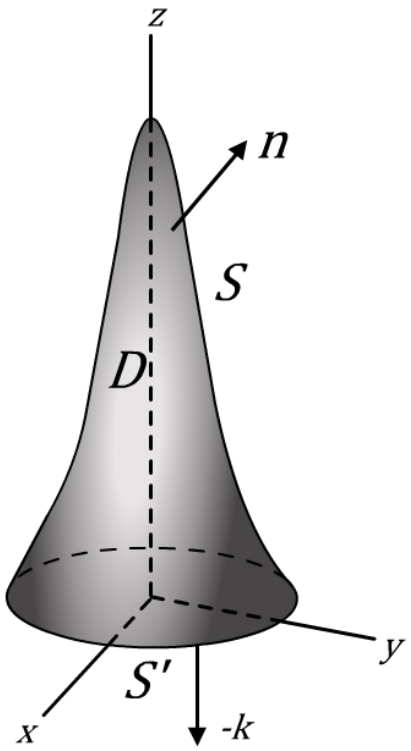
Sample Problem 5

Let $\mathbf{F} = e^{y}\cos z\,\mathbf{i} + \sqrt{x^3+1}\sin z\,\mathbf{j} + (x^2+y^2+3)\mathbf{k}$.

And, S is the surface expressed as:

$$z = (1-x^2-y^2)e^{(1-x^2-3y^2)};\ z \geq 0$$

Calculate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dA$



Now let us try to understand the divergence theorem with the help of the following sample problem.

Solution:

Direct evaluation of this surface integral is not feasible. Therefore, let's try to solve this problem by using an indirect approach. From the divergence theorem,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dA + \iint_{S'} \mathbf{F}\cdot\mathbf{n}\,dA = \oiint_{S+S'} \boldsymbol{F}\cdot\boldsymbol{n}\,dA = \iiint_V \nabla\cdot F\,dV$$

This means that, $\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \iiint_V \nabla\cdot F\,dV - \iint_{S'} \mathbf{F}\cdot\mathbf{n}\,dA$

But, $\nabla\cdot\mathbf{F} = 0 \Rightarrow \iiint_V \nabla\cdot F\,dV = 0$. Therefore, $\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = -\iint_{S'} \mathbf{F}\cdot\mathbf{n}\,dA$

Now let us try to understand the divergence theorem with the help of the following sample problem.

Solution (contd.):

The surface S' is a disk, given by the equation:

$$x^2 + y^2 \leq 1; z = 0$$

Now, using parameterisation, evaluate the surface integral over S'.

$$\begin{cases} x = v\cos u \\ y = v\sin u \end{cases} \quad 0 \leq v \leq 1$$

$$r(u, v) = v\cos u\,\mathbf{i} + v\sin u\,\mathbf{j}$$

Now let us try to understand the divergence theorem with the help of the following sample problem.

Solution (contd.):

$$\mathbf{r}_u = \frac{\partial r}{\partial u} = -v \sin u\, \mathbf{i} + v \cos u\, \mathbf{j}$$

$$\mathbf{r}_v = \frac{\partial r}{\partial v} = \cos u\, \mathbf{i} + \sin u\, \mathbf{j}$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = -(v \sin^2 u + v \cos^2 u)\mathbf{k} = -v\, \mathbf{k}$$

$$\mathbf{F} \cdot \mathbf{N} = \left[e^y \cos z\, \mathbf{i} + \sqrt{x^3 + 1} \sin z\, \mathbf{j} + (x^2 + y^2 + 3)\mathbf{k}\right] \cdot [-v\, \mathbf{k}\,]$$

$$= -v(x^2 + y^2 + 3)$$

Now let us try to understand the divergence theorem with the help of the following sample problem.

Solution (contd.):

$$-\iint_{S'} \mathbf{F}\cdot\mathbf{n}\, dA = -\iint_{S'} -v(x^2+y^2+3)\, dudv$$

$$= \int_0^1 \int_0^{2\pi} v[(v\cos u)^2+(v\sin u)^2+3]\, dudv$$

$$= \int_0^1 \int_0^{2\pi} v[v^2+3]\, dudv$$

Now let us try to understand the divergence theorem with the help of the following sample problem.

Solution (contd.):

$$-\iint_{S'} \mathbf{F} \cdot \mathbf{n}\, dA = 2\pi \left[\frac{1}{4}v^4 + \frac{3}{2}v^2\right]_0^1$$

$$= \frac{7}{2}\pi$$

$$= \iint_{S} \mathbf{F} \cdot \mathbf{n}\, dA$$

Stokes Theorem

Stokes theorem describes the relationship between a line integral and surface integral.

For a given surface S, which is bounded by contour C,

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dA = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

The orientation '$\mathbf{n}$' of S is such that if you walk along the contour C, the surface S would lie on your left.

Stokes Theorem

Stokes theorem describes the relationship between a line integral and surface integral.

For a given surface S, which is bounded by contour C,

Stokes theorem can only be applied to open surfaces in three dimensional space. The surface of a sphere is closed while the surface of a disc is open. Open surfaces have defined boundaries.

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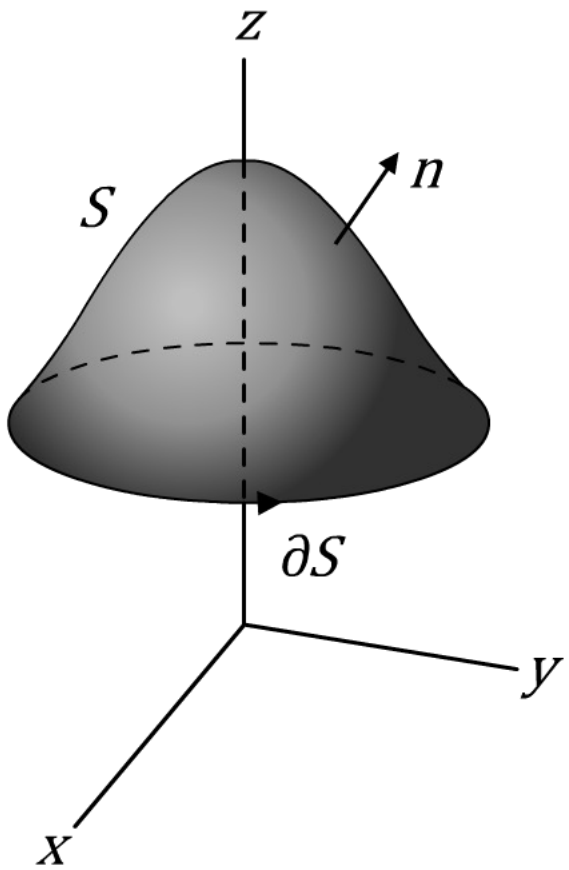
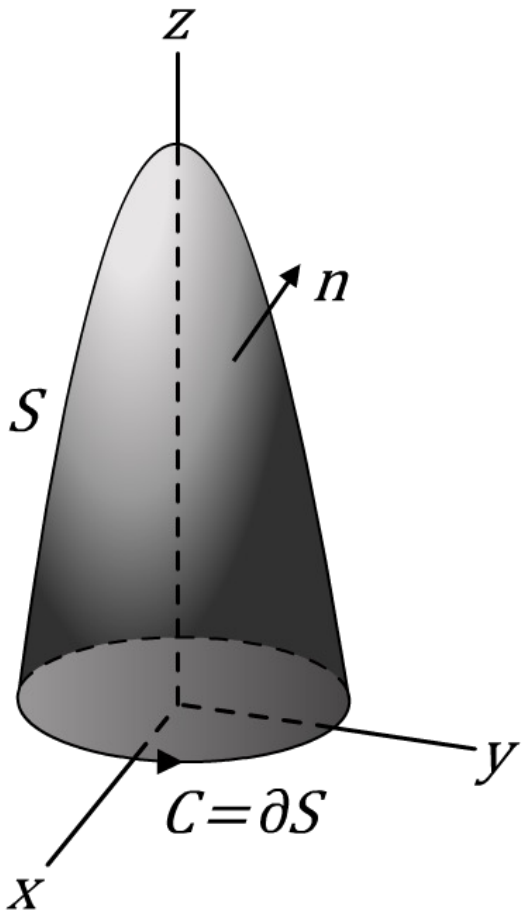
Stokes Theorem





The following sample problem demonstrates the application of Stokes theorem.

Sample Problem 6

Verify Stokes theorem for the surface given by the upper half of the sphere $(x^2 + y^2 + z^2) = 1; z \geq 0$. The vector field is $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$.

Solution:

First, calculate

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dA \text{ and } \oint_C \mathbf{F} \cdot d\mathbf{r}, \text{ and then show that,}$$

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dA = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

The following sample problem demonstrates the application of Stokes theorem.

Solution (contd.):

Let's calculate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ first.

The base of the hemisphere is given by the following equation:

$$C\colon\ x^2 + y^2 + z^2 = 1;\ z = 0$$

And, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

Introduce a parameter t such that, $\begin{cases} x = \cos t \\ y = \sin t \quad 0 \le t \le 2\pi \\ z = 0 \end{cases}$

The following sample problem demonstrates the application of Stokes theorem.

Solution (contd.):

Then, C: $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$

And, $d\mathbf{r}(t) = d(\cos t)\,\mathbf{i} + d(\sin t)\mathbf{j}$

$$= -\sin t\,dt\,\mathbf{i} + \cos t\,dt\,\mathbf{j}$$

$$= (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})dt$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\,dt$$

The following sample problem demonstrates the application of Stokes theorem.

Solution (contd.):

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (\sin t\,\mathbf{i} + 0\,\mathbf{j} + \cos t\,\mathbf{k}) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\,dt$$

$$= \int_0^{2\pi} (-\sin^2 t\,)dt$$

$$= -\pi$$

The following sample problem demonstrates the application of Stokes theorem.

Solution (contd.):

Now, let's calculate the surface integral over the hemisphere.

Parameterisation for a sphere can be given by:

$$\begin{cases} x = a\cos u \sin v & 0 \le u \le 2\pi \\ y = a\sin u \sin v & 0 \le v \le \pi \\ z = a\cos v \end{cases}$$

$$r(u,v) = a\cos u \sin v\,\mathbf{i} + a\sin u \sin v\,\mathbf{j} + a\cos v\,\mathbf{k}$$

The following sample problem demonstrates the application of Stokes theorem.

Solution (contd.):

Outward normal vector is given by:

$$\mathbf{N} = a\sin v\,(a\cos u\,\sin v\,\mathbf{i} + a\sin u\,\sin v\,\mathbf{j} + a\cos v\,\mathbf{k})$$

For the current problem, $a = 1$, and $0 \le v \le \pi/2$

$$\iint_S \nabla \times \mathbf{F} \cdot dA = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{N}(u, v)\, du dv$$

The following sample problem demonstrates the application of Stokes theorem.

Solution (contd.):

$$= \int_0^{\pi/2} \int_0^{2\pi} (-\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot \big(\sin v\,(\cos u \sin v\,\mathbf{i} + \sin u \sin v\,\mathbf{j} + \cos v\,\mathbf{k})\big)\, du dv$$

$$= -\int_0^{\pi/2} \int_0^{2\pi} (\sin^2 v \cos u + \sin^2 v \sin u + \sin v \cos v)\, du dv$$

$$= -\pi$$

Green's Theorem

Green's theorem is the two-dimensional case of Stokes theorem (which is used for three-dimensional space). Green's theorem is represented as shown below.

$$\iint_S \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right] \mathrm{d}A = \oint_C (F_1\,\mathrm{d}x + F_2\,\mathrm{d}y)$$

Green's Theorem

Let us try to prove Green's theorem using Stokes theorem.

Consider a surface on the $x - y$ plane. From Stokes theorem,

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}A = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Now, $\nabla \times \mathbf{F} \cdot \mathbf{n} = \nabla \times \mathbf{F} \cdot \mathbf{k}$

$$= \left[\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right] \mathbf{i} + \left[\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right] \mathbf{j} + \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right] \mathbf{k} \right] \cdot \mathbf{k}$$

$$= \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right]$$

Green's Theorem

Let us try to prove Green's theorem using Stokes theorem.

And,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \oint_C (F_1\,dx + F_2\,dy)$$

Therefore,

$$\boxed{\iint_S \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right] = \oint_C (F_1\,dx + F_2\,dy)}$$

Green's Theorem

The path independence of line integrals can be determined using Stokes theorem.

If $\nabla \times \mathbf{F} = 0$

By Stokes theorem,

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dA = 0 = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

That is, the line integral is independent of the path and $\mathbf{F}$ is a conservative vector field.

Summary

Key points discussed in this lesson:

- Gauss or divergence theorem provides a method to relate surface integrals to triple integrals. It states that the total divergence of a vector field in a bounded region is equal to the flux of the vector field away from the region.
- Stokes theorem describes the relationship between a line integral and surface integral.
- Green's theorem is a two-dimensional case of Stokes theorem (which is used for three-dimensional space).