Tutorial 5 (Tutorial 11) Solutions.

1). a).
$$\underset{\sim}{r} = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \qquad 0 \le t \le 1$$

$$\vec{F} = \begin{pmatrix} 3x^2 + 6y \\ -14yz \\ 20xz^2 \end{pmatrix} = \begin{pmatrix} 3t^2 + 6t^2 \\ -14(t^2)(t^3) \\ 20t(t^3)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 9t^2 \\ -14t^5 \\ 20t^7 \end{pmatrix}$$

$$d\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} dt$$

$$\int_C \vec{F} \cdot d\underset{\sim}{r} = \int_{t=0}^{t=1} \begin{pmatrix} 9t^2 \\ -14t^5 \\ 20t^7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} dt$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\, dt$$

$$= \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1$$

$$= \underline{\underline{5}}$$

b). $C_1: \quad (0,0,0) \to (1,0,0)$.

$C_2: \quad (1,0,0) \to (1,1,0)$

$C_3: \quad (1,1,0) \to (1,1,1)$.

$$\int_C \vec{F} \cdot d\underset{\sim}{r} = \int_{C_1} \vec{F} \cdot d\underset{\sim}{r} + \int_{C_2} \vec{F} \cdot d\underset{\sim}{r} + \int_{C_3} \vec{F} \cdot d\underset{\sim}{r}$$

$$\int_{C_1} \vec{F} \cdot d\underset{\sim}{r} = \int_{x=0}^{x=1} \left[(3x^2+6y)\underset{\sim}{i} - 14yz\,\underset{\sim}{j} + 20xz^2\,\underset{\sim}{k}\right] \cdot (dx\,\underset{\sim}{i} + \underset{=0}{dy}\,\underset{\sim}{j} + \underset{=0}{dz}\,\underset{\sim}{k})$$

$$= \int_{x=0}^{x=1} (3x^2 + 6\underset{=0}{y})\,dx = \left[x^3\right]_0^1 = 1$$

$$\int_{C_2} \vec{F} \cdot d\underset{\sim}{r} = \int \vec{F} \cdot (\underset{=0}{dx}\,\underset{\sim}{i} + dy\,\underset{\sim}{j} + \underset{=0}{dz}\,\underset{\sim}{k}).$$

$$= \int_{y=0}^{y=1} -14y\underset{=0}{z}\;dy. = 0$$

$$\int_{C_3} \vec{F} \cdot d\underset{\sim}{r} = \int \vec{F} \cdot (\underset{=0}{dx}\,\underset{\sim}{i} + \underset{=0}{dy}\,\underset{\sim}{j} + dz\,\underset{\sim}{k})$$

$$= \int_{z=0}^{z=1} 20xz^2\,dz \,\Bigg|_{\substack{x=1\\y=1}}$$

$$= \left[\frac{20z^3}{3}\right]_{z=0}^{z=1} = \frac{20}{3}.$$

$$\therefore \int_C \vec{F} \cdot d\underset{\sim}{r} = 1 + 0 + \frac{20}{3} = \frac{23}{3}$$ //.

c).
$$\underset{\sim}{r} = \begin{pmatrix} t \\ t \\ t \end{pmatrix} \quad 0 \le t \le 1$$

$$\vec{F} = \begin{pmatrix} 3x^2 + 6y \\ -14yz \\ 20xz^2 \end{pmatrix} = \begin{pmatrix} 3t^2+6t \\ -14t^2 \\ 20t^3 \end{pmatrix}$$

$$d\underset{\sim}{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dt.$$

$$\int_c \vec{F} \cdot d\underset{\sim}{r} = \int_{t=0}^{1} \begin{pmatrix} 3t^2+6t \\ -14t^2 \\ 20t^3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dt.$$

$$= \int_0^1 (3t^2+6t-14t^2+20t^3)\,dt.$$

$$= \int_0^1 (20t^3 - 11t^2 + 6t)\,dt$$

$$= \left[5t^4 - \frac{11}{3}t^3 + 3t^2\right]_0^1$$

$$= 5 - \frac{11}{3} + 3 \qquad = \underline{\underline{\frac{13}{3}}}$$

2). $\underset{\sim}{r} = \begin{pmatrix} t^2+1 \\ 2t^2 \\ t^3 \end{pmatrix} \quad 1 \leq t \leq 2.$

$$\vec{F} = \begin{pmatrix} 3xy \\ -5z \\ 10x \end{pmatrix} = \begin{pmatrix} 3(2t^2)(t^2+1) \\ -5t^3 \\ 10(t^2+1) \end{pmatrix}$$

$$= \begin{pmatrix} 6t^2(t^2+1) \\ -5t^3 \\ 10(t^2+1) \end{pmatrix}.$$

$$d\underset{\sim}{r} = \begin{pmatrix} 2t \\ 4t \\ 3t^2 \end{pmatrix} dt.$$

$$W.D. = \int \vec{F} \cdot d\underset{\sim}{r}$$

$$= \int_{t=1}^{t=2} \begin{pmatrix} 6t^2(t^2+1) \\ -5t^3 \\ 10(t^2+1) \end{pmatrix} \cdot \begin{pmatrix} 2t \\ 4t \\ 3t^2 \end{pmatrix} dt.$$

$$= \int_1^2 \left[12t^3(t^2+1) - 20t^4 + 30t^2(t^2+1)\right] dt.$$

$$= \int_1^2 \left[12t^5 + 10t^4 + 12t^3 + 30t^2\right] dt.$$

$$= \left[2t^6 + 2t^5 + 3t^4 + 10t^3\right]_1^2$$

$$= \underline{\underline{303}} \,//.$$

3).

$$\underset{\sim}{r} = \begin{pmatrix} t \\ 2t^2 \end{pmatrix} \qquad 0 \le t \le 1$$

$$\vec{F} = \begin{pmatrix} 3xy \\ -y^2 \end{pmatrix} = \begin{pmatrix} 3t(2t^2) \\ -4t^4 \end{pmatrix}$$

$$d\underset{\sim}{r} = \begin{pmatrix} 1 \\ 4t \end{pmatrix} dt.$$

$$\int \vec{F} \cdot d\underset{\sim}{r} = \int_{t=0}^{t=1} \begin{pmatrix} 6t^3 \\ -4t^4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4t \end{pmatrix} dt.$$

$$= \int_0^1 (6t^3 - 16t^5)\, dt.$$

$$= \left[\frac{3t^4}{2} - \frac{16}{6} t^6 \right]_0^1$$

$$= -\frac{7}{6}.$$

4 a).

$$\text{curl } \vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}.$$

$$= 0\underset{\sim}{i} + (3z^2 - 3z^2)\underset{\sim}{j} + (2x - 2x)\underset{\sim}{k}.$$

$$= \vec{0} \Rightarrow \vec{F} \text{ is conservative}$$

4b). $\vec{F} = \nabla V = \begin{pmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial V}{\partial z} \end{pmatrix} = \begin{pmatrix} 2xy + z^3 \\ x^2 \\ 3xz^2 \end{pmatrix}$.

$\frac{\partial V}{\partial x} = 2xy + z^3 \Rightarrow V(x,y,z) = x^2y + xz^3 + g_1(y,z)$.

$\frac{\partial V}{\partial y} = x^2 \Rightarrow V = x^2y + g_2(x,z)$.

$\frac{\partial V}{\partial z} = 3xz^2 \Rightarrow V = xz^3 + g_3(x,y)$.

$\Rightarrow \underline{V(x,y,z) = x^2y + xz^3 + C}$.

4c). W.D. $= \int_C \vec{F} \cdot d\underset{\sim}{r} = V(3,1,4) - V(1,-2,1)$.

$= [3^2(1) + 3(4)^4] - [(1)^2(-2) + (1)(1)^3]$.

$= \underline{\underline{202}}$.

5) $\oiint_S \vec{F} \cdot d\vec{A}$

For the spherical surface, let.

$$\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a\cos u \sin v \\ a \sin u \sin v \\ a\cos v \end{pmatrix} \qquad \begin{matrix} 0 \le u \le 2\pi \\ 0 \le v \le \pi \end{matrix}.$$

$$\underset{\sim}{r_u} = \begin{pmatrix} -a\sin u \sin v \\ a\cos u \sin v \\ 0 \end{pmatrix} \quad \underset{\sim}{r_v} = \begin{pmatrix} a\cos u \cos v \\ a\sin u \cos v \\ -a\sin v \end{pmatrix}.$$

$$\vec{N} = \underset{\sim}{r_u} \times \underset{\sim}{r_v} = -\begin{pmatrix} a^2\cos u \sin^2 v \\ a^2 \sin u \sin^2 v \\ a^2 \sin v \cos v \end{pmatrix}.$$

$$= -a\sin v \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -a\sin v \,\underset{\sim}{r}.$$

Redefine outward normal as

$$\vec{N} = a\sin v\, \underset{\sim}{r}$$

$$\oiint_S \vec{F} \cdot d\vec{A} = \iint a\sin v \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} du\, dv.$$

$$= \iint a\sin v\,(z^2)\, du\, dv$$

$$= \int_{v=0}^{v=\pi} \int_{u=0}^{u=2\pi} a\sin v \,.\, a^2\cos^2 v \; du\, dv$$

$$= a^3 \int_{v=0}^{\pi} \sin v \cos^2 v\, dv \,.\, \int_{u=0}^{2\pi} du$$

$$= a^3 \left[-\frac{\cos^3 v}{3} \right]_0^{\pi} \left[u \right]_0^{2\pi}$$

$$= a^3 \left[\frac{2}{3} \right] \left[2\pi \right] = \underline{\underline{\frac{4}{3}\pi a^3}} \;//.$$