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EE3001 Engineering Electromagnetics

This session is about

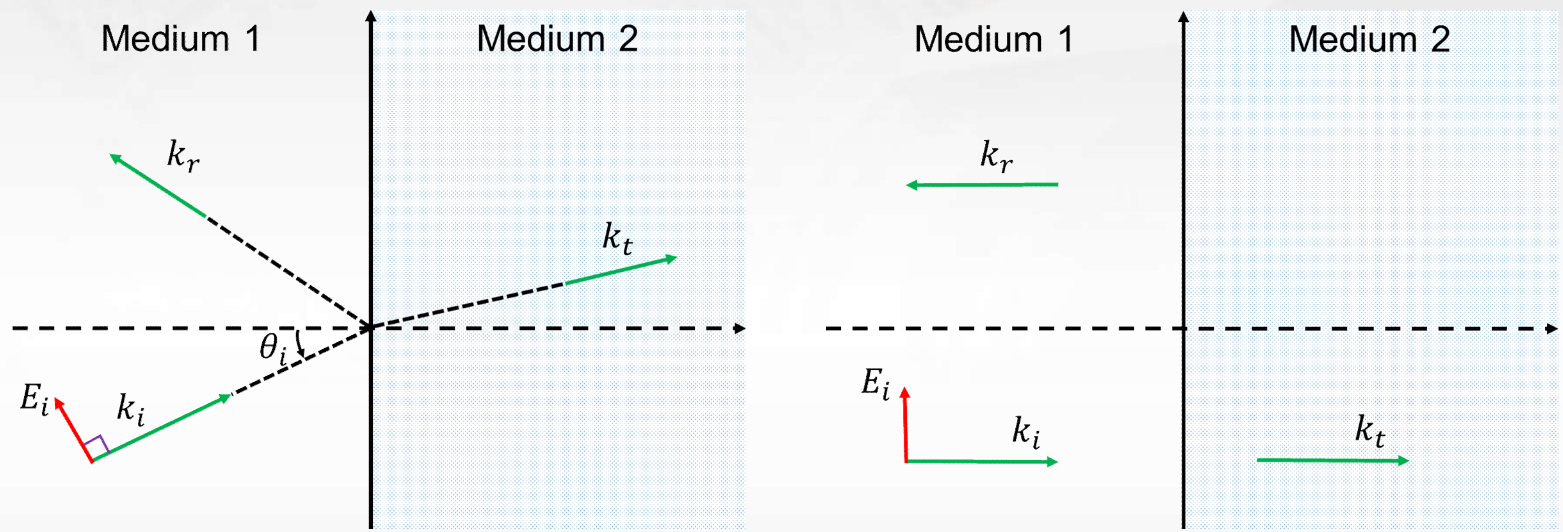
8. Reflection and Transmission of UPW at Normal Incidence

Learning Objectives

- Identify the direction of propagation, direction of E and H fields of the reflected and transmitted waves for a UPW incident normally at the plane boundary;
- Apply boundary conditions to determine the reflection and transmission coefficients, i.e. Γ and τ; and
- Determine the reflected and transmitted E and H fields.

Oblique and Normal Incidence

- A **UPW** travels in Medium 1 is **incident** onto a second medium [Medium 2] as shown.



(a) Oblique Incidence $(\theta_i \neq 0)$ **(b) Normal Incidence** $(\theta_i = 0)$

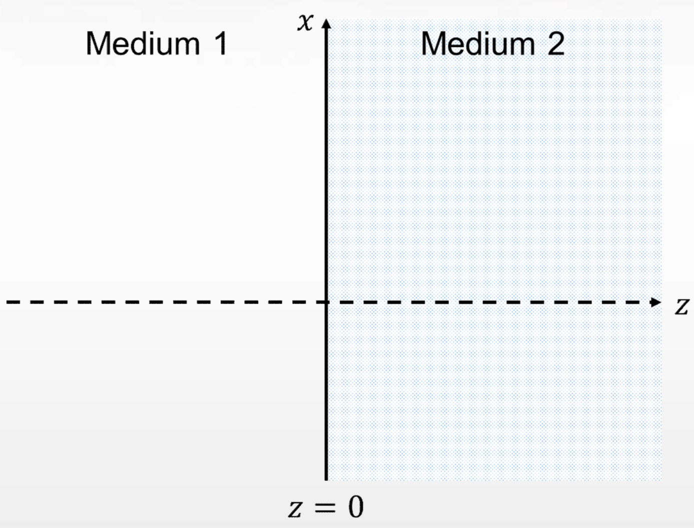
UPW at Normal Incidence

- To describe the incident, reflected and transmitted waves, we need to define $x-$, $y-$, and $z-$axis. Two of the axes can be chosen arbitrarily and the third is determined by using RHR.

- Let Medium 1 occupy the region $z \leq 0$ and Medium 2 $z \geq 0$. That is, Boundary is fixed @ $z = 0$ & $z-$axis $\equiv \rightarrow$.

- Without loss of generality, let $x-$axis $\equiv \uparrow$

  - **RHR**: $y-$axis $\equiv \odot$



UPW at Normal Incidence

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Normal Incidence $(\theta_i = 0)$:

Problem: Given $\vec{E}_i$, Medium 1 and Medium 2, find $\vec{E}_r$ and $\vec{E}_t$ etc

Solution: From $\vec{E}_i$, identify $\vec{a}_{ki}$, $\vec{a}_{Ei}$, E_{oi} [Page 3-6]

1. $\vec{a}_{ki} = +\vec{a}_z \rightarrow \vec{a}_{kr} = -\vec{a}_z$ and $\vec{a}_{kt} = +\vec{a}_z$ $[k_i = k_r = k_1, k_t = k_2]$
2. If $\vec{a}_{Ei} = +\vec{a}_x \rightarrow \vec{a}_{Er} = +\vec{a}_x$ and $\vec{a}_{Et} = +\vec{a}_x$ $\left[\vec{E}_i \uparrow \Rightarrow \vec{E}_r \uparrow \text{ and } \vec{E}_t \uparrow\right]$

   If $\vec{a}_{Ei} = +\vec{a}_y \rightarrow \vec{a}_{Er} = +\vec{a}_y$ and $\vec{a}_{Et} = +\vec{a}_y$ $\left[\vec{E}_i\odot \Rightarrow \vec{E}_r\odot \text{ and } \vec{E}_t\odot\right]$
3. $E_{oi} \longrightarrow E_{or} = \Gamma\, E_{oi}$ and $E_{ot} = \tau\, E_{oi}$

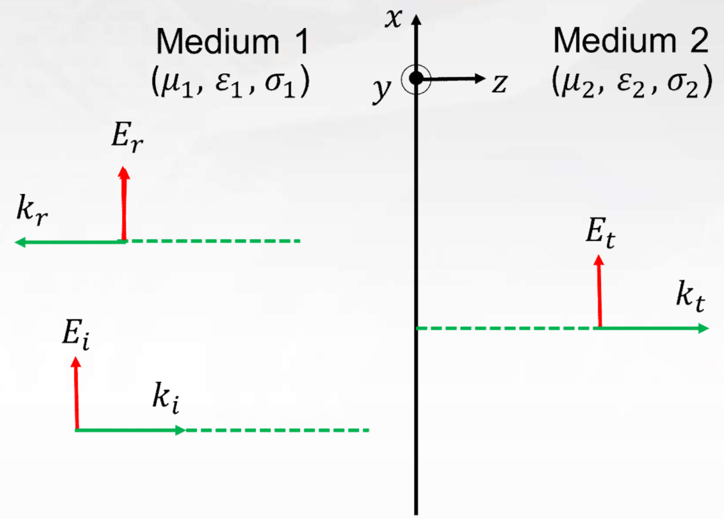
   Γ and τ depend on intrinsic impedance in Medium 1 and Medium 2, i.e. η_1 and η_2

UPW at Normal Incidence

❑ Normal Incidence $(\theta_i = 0)$

Given $\vec{E}_i$

1. $\vec{a}_{k_i} \longrightarrow \vec{a}_{k_r}$ & $\vec{a}_{k_t}$
2. $\vec{a}_{E_i} \longrightarrow \vec{a}_{E_r}$ & $\vec{a}_{E_t}$
3. $E_{oi} \longrightarrow E_{or} = \Gamma E_{oi}$
   $E_{ot} = \tau E_{oi}$



	$z < 0$	$z = 0$	$z > 0$
RHR:	$H_i \odot$	$H_r \otimes$	$H_t \odot$

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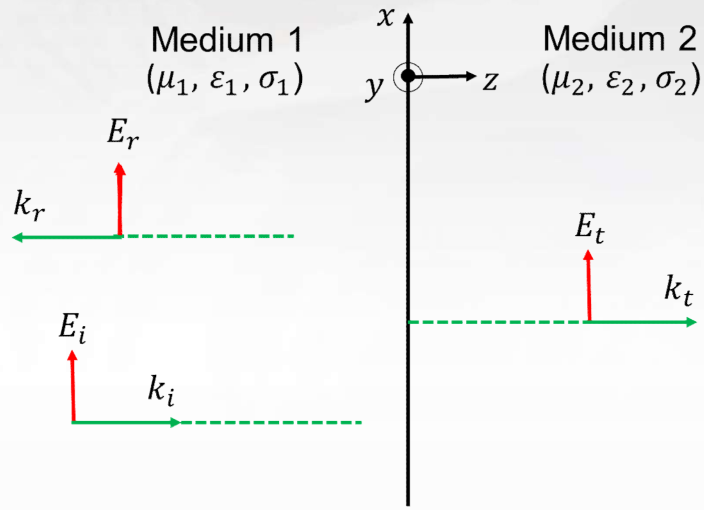
8. Reflection and Transmission of UPW at Normal Incidence (2)

UPW at Normal Incidence

- **Normal Incidence** $(\theta_i = 0)$

Given $\vec{E}_i$

1. $\vec{a}_{k_i} \longrightarrow \vec{a}_{k_r}$ & $\vec{a}_{k_t}$
2. $\vec{a}_{E_i} \longrightarrow \vec{a}_{E_r}$ & $\vec{a}_{E_t}$
3. $E_{oi} \longrightarrow E_{or} = \Gamma E_{oi}$
   $E_{ot} = \tau E_{oi}$



	$z < 0$	$z = 0$	$z > 0$
RHR:	$H_i \odot$	$H_r \otimes$	$H_t \odot$

UPW at Normal Incidence

- To simplify the discussion, we will first assume that both media are **lossless** $(\sigma_1 = \sigma_2 = 0)$.

- We may extend our analysis to **lossy media** $(\sigma \neq 0)$ by merely replacing

$$\boxed{\varepsilon \to \varepsilon_c = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}; \quad \eta \to \eta_c = \sqrt{\frac{\mu}{\varepsilon_c}}; \quad k \to k_c = \omega\sqrt{\mu\,\varepsilon_c}$$

- The incident $(\vec{E}_i,\ \vec{H}_i)$, reflected $(\vec{E}_r,\ \vec{H}_r)$ and transmitted $(\vec{E}_t,\ \vec{H}_t)$ waves can be written as

Incident, Reflected and Transmitted Waves

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- **Medium 1** $(\mu_1,\ \varepsilon_1,\ \sigma_1 = 0)$: $k_1 = \omega\sqrt{\mu_1\,\varepsilon_1}$; $\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$

Incident Wave

$$\vec{E}_i = E_{oi}\, e^{-jk_1 z}\ [\vec{a}_x]$$

$$\vec{H}_i = \frac{E_{oi}}{\eta_1}\, e^{-jk_1 z}\ [\vec{a}_y]$$

Reflected Wave

$$\vec{E}_r = E_{or}\, e^{+jk_1 z}\ [\vec{a}_x]$$

$$\vec{H}_r = \frac{E_{or}}{\eta_1}\, e^{+jk_1 z}\ [-\vec{a}_y]$$

E_{oi} and E_{or} denote the amplitude of E_i and E_r at $z = 0$ respectively.

- **Total Field in Medium 1:**
  - $\vec{E}_1 = \vec{E}_i + \vec{E}_r = \vec{a}_x\, E_{oi}\, e^{-jk_1 z} + \vec{a}_x\, E_{or}\, e^{+jk_1 z}$
  - $\vec{H}_1 = \vec{H}_i + \vec{H}_r = \vec{a}_y\, \frac{E_{oi}}{\eta_1}\, e^{-jk_1 z} - \vec{a}_y\, \frac{E_{or}}{\eta_1}\, e^{+jk_1 z}$

Incident, Reflected and Transmitted Waves

- **<u>Medium 2</u>** $(\mu_2,\ \varepsilon_2,\ \sigma_2 = 0)$: $k_2 = \omega\sqrt{\mu_2\,\varepsilon_2}$; $\eta_2 = \sqrt{\dfrac{\mu_2}{\varepsilon_2}}$

  <u>Transmitted Wave</u>:

  $$\vec{E}_t = E_{ot}\, e^{-jk_2 z}\,[\vec{a}_x]$$

  $$\vec{H}_t = \frac{E_{ot}}{\eta_2}\, e^{-jk_2 z}\,[\vec{a}_y]$$

- **Total Field in Medium 2:**
  - $\vec{E}_2 = \vec{E}_t = \vec{a}_x\, E_{ot}\, e^{-jk_2 z}$
  - $\vec{H}_2 = \vec{H}_t = \vec{a}_y\, \dfrac{E_{ot}}{\eta_2}\, e^{-jk_2 z}$

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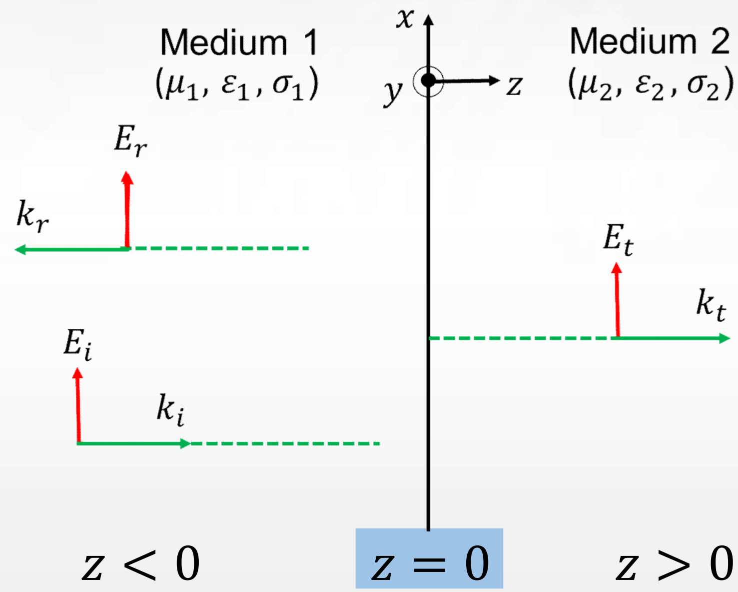
8. Reflection and Transmission of UPW at Normal Incidence (3)

Boundary Conditions

Tangential Components of E and H fields are **continuous** across the boundary @ $z = 0$. That is, **magnitude and direction of tangential E and H** fields are **exactly the same** at $z = 0$.



Boundary Conditions

- Tangential means parallel to the interface i.e. x and y.
  [Tangential component $\equiv \vec{a}_x, \vec{a}_y$; normal component $\equiv \vec{a}_z$]

- Total Field in Medium 1 ($z \leq 0$) [Page 8-8]
  - $\vec{E}_1 = \vec{E}_i + \vec{E}_r = \vec{a}_x E_{oi}\, e^{-jk_1 z} + \vec{a}_x E_{or}\, e^{+jk_1 z}$
  - $\vec{H}_1 = \vec{H}_i + \vec{H}_r = \vec{a}_y \frac{E_{oi}}{\eta_1} e^{-jk_1 z} - \vec{a}_y \frac{E_{or}}{\eta_1} e^{+jk_1 z}$

- Total Field in Medium 2 ($z \geq 0$) [Page 8-9]
  - $\vec{E}_2 = \vec{E}_t = \vec{a}_x E_{ot} e^{-jk_2 z}$
  - $\vec{H}_2 = \vec{H}_t = \vec{a}_y \frac{E_{ot}}{\eta_2} e^{-jk_2 z}$

Boundary Condition @ $z = 0$

$\vec{E}_1 = \vec{E}_2 \rightarrow E_{oi} + E_{or} = E_{ot}$

Reflection and Transmission Coefficients

Boundary Conditions: $\boldsymbol{z} = \mathbf{0}$

- ❑ $\vec{E}_1 = \vec{E}_2 \rightarrow E_{oi} + E_{or} = E_{ot}$ (1)
- ❑ $\vec{H}_1 = \vec{H}_2 \rightarrow \frac{E_{oi}}{\eta_1} - \frac{E_{or}}{\eta_1} = \frac{E_{ot}}{\eta_2}$ (2)

- ❑ Assume E_{oi}, η_1 and η_2 are known, the two unknowns, E_{or} and E_{ot}, can be solved in terms of E_{oi}, η_1 and η_2:

$$E_{or} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{oi} \quad \& \quad E_{ot} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{oi}$$

- ❑ Define the <u>reflection coefficient</u> $\Gamma = \frac{E_{or}}{E_{oi}}$ and <u>transmission coefficient</u> $\tau = \frac{E_{ot}}{E_{oi}}$.

Reflection and Transmission Coefficients

$$\boxed{\Gamma = \frac{E_{or}}{E_{oi}} = \frac{\eta_2-\eta_1}{\eta_2+\eta_1}} \qquad \left[\frac{P_r}{P_i} = |\Gamma|^2\right] \qquad (3) \qquad [E_{or} = \Gamma E_{oi}]$$

$$\boxed{\tau = \frac{E_{ot}}{E_{oi}} = \frac{2\eta_2}{\eta_2+\eta_1}} \qquad (4) \qquad [E_{ot} = \tau E_{oi}]$$

- Γ and τ only depend on η_1 and η_2 in these media.
- $P_r = |\Gamma|^2 P_i$: The larger the difference between η_2 and η_1, the larger the proportion of power reflected from the surface.
- If $\eta_2 = \eta_1, \Gamma = 0$ i.e. no wave reflected from the surface (all the incident power is transmitted into medium 2).
- For Lossy Medium $(\sigma \neq 0)$: $\eta \to \eta_c = \sqrt{\frac{\mu}{\varepsilon_c}}$; $\quad \boxed{\varepsilon_c = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}$

Summary

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- Boundary conditions: Tangential Components of E and H fields are continuous, i.e., magnitude and direction of tangential E and H fields are exactly the same at the interface.

- The reflection and transmission coefficients are
  - $\Gamma = \frac{E_{or}}{E_{oi}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$
  - $\tau = \frac{E_{ot}}{E_{oi}} = \frac{2\eta_2}{\eta_2 + \eta_1}$

- The power reflected from the interface is $P_r = |\Gamma|^2 P_i$. The larger the difference between η_2 and η_1, the larger the proportion of power reflected from the surface.

- For Lossy Medium $(\sigma \neq 0)$: $\eta \rightarrow \eta_c = \sqrt{\frac{\mu}{\varepsilon_c}}$; $\varepsilon_c = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)$

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9. Examples of Reflection and Transmission of UPW at Plane Boundary

Example 1

Given $\eta_1 = 377\ \Omega$ and $\eta_{2c} = 200\ \angle 20° = 188 + j\ 68\ \Omega$, find $\%P_r$ and $\%P_t$

- $\Gamma = \frac{\eta_{2c}-\eta_1}{\eta_{2c}+\eta_1} = \frac{-189 + j\ 68}{565 + j\ 68} = \frac{201\ \angle 160°}{565\ \angle 7°} = 0.35\ \angle 153°$

- $P_r = |\Gamma|^2 P_i = 0.35^2 P_i = 0.12 P_i; \quad P_t = (1 - |\Gamma|^2) P_i = 0.88 P_i$

Given $\tilde{\vec{E}}_i(z,t) = \vec{a}_x 100 \cos(\omega t - k_1 z + 10°)$ V/m, find $\tilde{\vec{E}}_r(z,t)$.

- $E_{oi} = 100\ \angle 10° \rightarrow E_{or} = \Gamma\ E_{oi} = 35\ \angle 163°$

- $\vec{E}_r = \vec{a}_{E_r}\ E_{or}\ e^{+jk_1 z} = \vec{a}_x\ 35\ \angle 163°\ e^{+jk_1 z}$

- $\tilde{\vec{E}}_r = \vec{a}_x\ 35\ \cos(\omega t + k_1 z + 163°)\ \text{V/m}$

$\boxed{\tilde{\vec{E}}_r = \text{Re}\left(\vec{E}\ e^{j\omega t}\right)}$

Example 2

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A UPW in air $(z \leq 0)$ with $\vec{E}_i = \vec{a}_y\, 50\, e^{-j(2\pi z- \ /6)}$ V/m is **incident normally** on a lossless medium $(\mu_r = 1,\ \varepsilon_r = 9)$ occupying the region $z \geq 0$.

Find the following:

(a) The frequency of the UPW

(b) The propagation constant γ and intrinsic impedance η in air and in the lossless medium

(c) $\vec{E}_r,\ \vec{H}_r,\ \vec{E}_t\ and\ \vec{H}_t$

Solution to Example 2

Medium 1 $\left(\mu_1 = \mu_o,\ \varepsilon_1 = \varepsilon_{o,}\ \sigma_1 = 0\right)$ and Medium 2 $\left(\mu_2 = \mu_o,\ \varepsilon_2 = 9\varepsilon_{o,}\ \sigma_2 = 0\right)$

- $\vec{E}_i = \vec{a}_y\ 50\ e^{j\pi/6}\ e^{-j2\pi}\ \ \mathrm{V/m} \equiv \vec{a}_{E_i}\ E_{oi}\ e^{-jk_1 z}$
- $\vec{a}_{k_i} = \vec{a}_z,\ \vec{a}_{E_i} = \vec{a}_y,\ \ E_{oi} = 50\angle\frac{\pi}{6},\ k_1 = 2\pi$

(a) $k_1 = 2\pi = \omega\sqrt{\mu_1\ \varepsilon_1} = 2\pi f\sqrt{\mu_o\ \varepsilon_o} \rightarrow f = 3 \times 10^8\ \mathrm{Hz}$

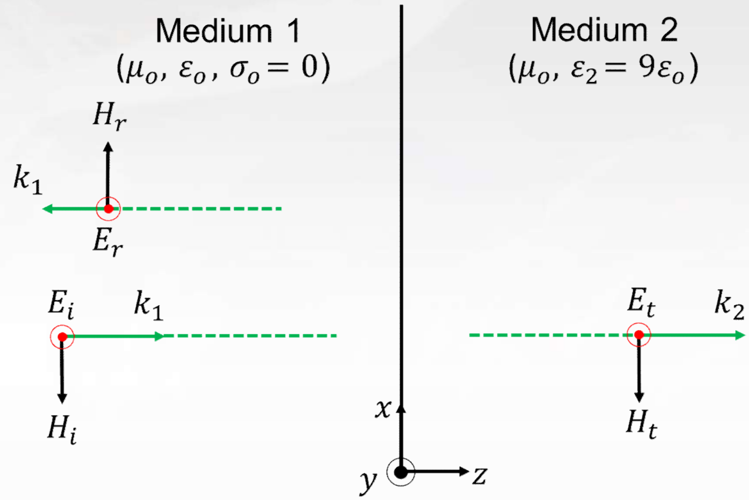
(b) $\gamma_1 = j\ k_1 = j\ 2\pi,\qquad \gamma_2 = j\ k_2 = j\ \omega\sqrt{\mu_2\ \varepsilon_2} = j\ 2\pi f\sqrt{\mu_o\ 9\varepsilon_o} = j\ 6\pi$

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = 120\pi\ \Omega,\ \ \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = 40\pi\ \Omega$$

Solution to Example 2 (cont.)

(c) $k_1 = 2\pi$, $\eta_1 = 120\pi$, $k_2 = 6\pi$, $\eta_2 = 40\pi$

1. $\vec{a}_{k_i} = \vec{a}_z$
2. $\vec{a}_{E_i} = \vec{a}_y$
3. $E_{oi} = 50\angle\frac{\pi}{6}$



- $\Gamma = \frac{E_{or}}{E_{oi}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -0.5 = 0.5\angle\pi \rightarrow E_{or} = \Gamma E_{oi} = 25\angle 7\pi/6$
- $\tau = \frac{E_{ot}}{E_{oi}} = \frac{2\eta_2}{\eta_2 + \eta_1} = 0.5 = 0.5\angle 0 \rightarrow E_{ot} = \tau E_{oi} = 25\angle\pi/6$

Solution to Example 2 (cont.)

- $\vec{E}_i = \vec{a}_y \, E_{oi} \, e^{-jk_1 z}$ $\qquad \left[E_{oi} = 50 \angle \frac{\pi}{6}; \;\; k_1 = 2\pi\right]$

- $\vec{E}_r = \vec{a}_y \, \Gamma \, E_{oi} \, e^{+jk_1 z} = \vec{a}_y \, 25 \angle \frac{7\pi}{6} \; e^{+j2\pi z};$ $\qquad [\Gamma = 0.5 \angle \pi]$

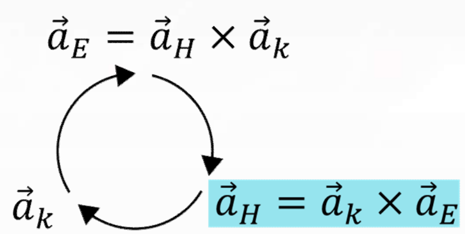
- $\vec{E}_t = \vec{a}_y \, \tau \, E_{oi} \, e^{-jk_2 z} = \vec{a}_y \, 25 \angle \frac{\pi}{6} \; e^{-j6\pi z};$ $\qquad [\tau = 0.5 \angle 0, \;\; k_2 = 6\pi]$

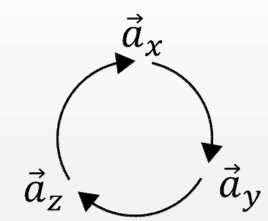
- $\vec{H}_r = \vec{a}_{H_r} \frac{\Gamma \, E_{oi}}{\eta_1} \, e^{+jk_1 z} = \vec{a}_x \, \frac{5}{24\pi} \angle \frac{7\pi}{6} \; e^{+j2\pi z} \;\; \mathrm{A/m}$

  $\left[\vec{a}_{H_r} = \vec{a}_{k_r} \times \vec{a}_{E_r} = -\,\vec{a}_z \times \vec{a}_y = +\,\vec{a}_x\right]$

- $\vec{H}_t = \vec{a}_{H_t} \frac{\tau \, E_{oi}}{\eta_2} \, e^{-jk_2 z} = -\,\vec{a}_x \, \frac{5}{8\pi} \angle \frac{\pi}{6} \; e^{-j6\pi z} \;\; \mathrm{A/m}$

  $\left[\vec{a}_{H_t} = \vec{a}_{k_t} \times \vec{a}_{E_t} = \vec{a}_z \times \vec{a}_y = -\,\vec{a}_x\right]$





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10. Power Incident, Reflected and Transmitted at the Interface

Learning Objectives

- Define the power incident, reflected and transmitted at the interface for UPW at normal incidence; and
- Apply power conservation at the interface to simplify the calculation of the reflected and transmitted power.

Example 2 (Page 9-6)

- $\vec{E}_i = \vec{a}_y\, E_{oi}\, e^{-jk_1 z}$ $\quad$ $\left[E_{oi} = 50\angle\frac{\pi}{6};\ \ k_1 = 2\pi\right]$

- $\vec{E}_r = \vec{a}_y\, \Gamma\, E_{oi}\, e^{+jk_1 z} = \vec{a}_y\, 25\angle\frac{7\pi}{6}\, e^{+j2\pi z};$ $\quad$ $[\Gamma = 0.5\angle\pi]$

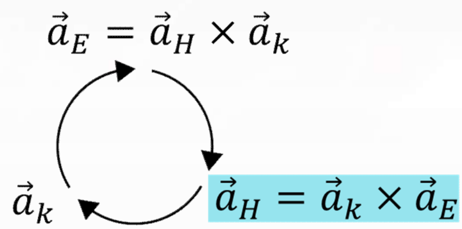
- $\vec{E}_t = \vec{a}_y\, \tau\, E_{oi}\, e^{-jk_2 z} = \vec{a}_y\, 25\angle\frac{\pi}{6}\, e^{-j6\pi}\ ;$ $\quad$ $[\tau = 0.5\angle 0,\ \ k_2 = 6\pi]$

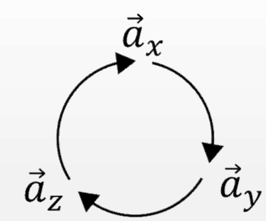
- $\vec{H}_r = \vec{a}_{H_r} \frac{\Gamma\, E_{oi}}{\eta_1}\, e^{+jk_1 z} = \vec{a}_x\, \frac{5}{24}\,\angle\frac{7\pi}{6}\, e^{+j2\pi z}$ A/m

  $\left[\vec{a}_{H_r} = \vec{a}_{k_r} \times \vec{a}_{E_r} = -\,\vec{a}_z \times \vec{a}_y = +\,\vec{a}_x\right]$

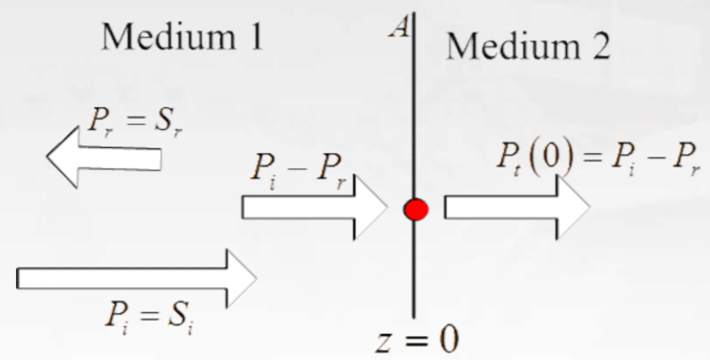
- $\vec{H}_t = \vec{a}_{H_t} \frac{\tau\, E_{oi}}{\eta_2}\, e^{-jk_2 z} = -\,\vec{a}_x\, \frac{5}{8\pi}\,\angle\frac{\pi}{6}\, e^{-j6\pi}$ A/m

  $\left[\vec{a}_{H_t} = \vec{a}_{k_t} \times \vec{a}_{E_t} = \vec{a}_z \times \vec{a}_y = -\,\vec{a}_x\right]$





Incident, Reflected and Transmitted Powers



Consider per-unit area of the interface i.e. $A = 1\ \mathrm{m}^2$ @ $z = 0$

- $P_i = S_i = \left|\frac{1}{2}\ \mathrm{Re}\left(\vec{E}_i \times \vec{H}_i^*\right)\right| \xrightarrow{\sigma_1 = 0} \frac{|E_{oi}|^2}{2\eta_1} = 3.32\ \mathrm{W/m^2}$, $P_r = S_r = \frac{|E_{or}|^2}{2\eta_1} = 0.83\ \mathrm{W/m^2}$
- $P_i - P_r = 2.49\ \mathrm{W/m^2} = P_t$ @ $z = 0$
- $P_t = S_t = \left|\frac{1}{2}\ \mathrm{Re}\left(\vec{E}_t \times \vec{H}_t^*\right)\right| \xrightarrow{\sigma_2 = 0} \frac{|E_{ot}|^2}{2\eta_2} = 2.49\ \mathrm{W/m^2}$

Power Conservation

Power Conservation: $P_t = P_i - P_r$ @ $z = 0$

- $P_r = \dfrac{|E_{or}|^2}{2\eta_1} = \dfrac{|\Gamma|^2\,|E_{oi}|^2}{2\eta_1} = |\Gamma|^2\, P_i$ $\quad [|\Gamma| \le 1]$
- $P_t = (1 - |\Gamma|^2)\, P_i$ @ $z = 0$, $\quad P_t(z) = P_t(0)\, e^{-2\alpha z}$ $\quad z \ge 0$

$$\vec{S}_t(z) = \vec{S}_t(0)\, e^{-2\alpha z}$$

- **REMARKS:**

Γ and τ apply to **normal incidence** of UPW at Interface between **any two media.**

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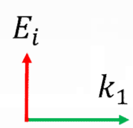
10. Power Incident, Reflected and Transmitted at the Interface (2)

Example 3

If Medium 2 is conducting $(\sigma \neq 0)$:

Medium 1
$(\mu_1, \varepsilon_1, \sigma_1 = 0)$

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$



Medium 2
$(\mu_2, \varepsilon_2, \sigma_2 \neq 0)$

$$\eta_{2c} = \sqrt{\frac{\mu_2}{\varepsilon_{2c}}}$$

$$\varepsilon_{2c} = \varepsilon_2 \left(1 - j\,\frac{\sigma_2}{\omega \varepsilon_2}\right)$$

$$\vec{S}_t(z) = \vec{S}_t(0)\, e^{-2\alpha z}$$

- ❑ $\Gamma = \dfrac{\eta_{2c} - \eta_1}{\eta_{2c} + \eta_1} = |\Gamma| \angle \theta_o$ $\qquad [P_r = |\Gamma|^2\, P_i]$ (3)

- ❑ $\tau = \dfrac{2\,\eta_{2c}}{\eta_{2c} + \eta_1} = |\tau| \angle \phi_o$ $\qquad [P_t = (1 - |\Gamma|^2)\, P_i]$ @ $z = 0$ (4)

Example 4

A 1 MHz UPW in air is normally incident on a copper plate (μ_o, ε_o, $\sigma = 5.8 \times 10^7$ S/m), determine Γ and τ, and the proportion of incident power reflected from the copper plate.

Solution:

$$\left[\Gamma = \frac{\eta_{2c} - \eta_1}{\eta_{2c} + \eta_1}, \qquad \tau = \frac{2\eta_{2c}}{\eta_{2c} + \eta_1}\right]$$

Medium 1: $\boxed{\eta_1 = 377\ \Omega}$, Medium 2: $\eta_{2c} = \sqrt{\frac{\mu_2}{\varepsilon_{2c}}}$

Medium 2: $\frac{\sigma_2}{\omega\, \varepsilon_2} = \frac{5.8 \times 10^7}{2\pi \times 10^6 \times \frac{1}{36\pi} \times 10^{-9}} \approx 10^{12} > 20$ (Good Conductor)

Solution to Example 4

- $\frac{\sigma_2}{\omega\,\varepsilon_2} > 20$: $\boxed{\delta_2 = \sqrt{\frac{2}{\omega\,\mu_o\,\sigma_2}} = \frac{1}{\sqrt{\pi\,f\,\mu_o\,\sigma_2}} = 66\ \mu\text{m}} = 66 \times 10^{-6}\ \text{m}$
- $\eta_{2c} = \sqrt{\frac{\mu_2}{\varepsilon_{2c}}} \simeq \frac{1+j}{\sigma\,\delta} = 0.000261(1+j) = 0.000369\,\angle 45°\ \Omega,$ $\quad \boxed{\eta_1 = 377\ \Omega}$
- $\Gamma = \frac{\eta_{2c}-\eta_1}{\eta_{2c}+\eta_1} = 0.9999986\,\angle 179.999972°$
- $\tau = \frac{2\eta_{2c}}{\eta_{2c}+\eta_1} = 0.00000196\,\angle 45°$
- $P_r = \frac{|E_{or}|^2}{2\,\eta_1} = \frac{|\Gamma|^2\,|E_{oi}|^2}{2\,\eta_1} = |\Gamma|^2\,P_i = 0.9999972\,P_i$

Summary (UPW at Normal Incidence)

- Incident power per unit area, $P_i = S_i = \left|\frac{1}{2}\,\mathrm{Re}\left(\vec{E}_i \times \vec{H}_i^*\right)\right| = \frac{|E_{oi}|^2}{2\eta_1}$ (if $\sigma_1 = 0$).

- Reflected power per unit area, $P_r = S_r = \left|\frac{1}{2}\,\mathrm{Re}\left(\vec{E}_r \times \vec{H}_r^*\right)\right| = \frac{|E_{or}|^2}{2\eta_1}$ (if $\sigma_1 = 0$).

- Transmitted power per unit area, $P_t = S_t = \left|\frac{1}{2}\,\mathrm{Re}\left(\vec{E}_t \times \vec{H}_t^*\right)\right| = \frac{|E_{ot}|^2}{2\eta_2}$ (if $\sigma_2 = 0$).

- Power Conservation:
  - $P_r = |\Gamma|^2\, P_i \qquad [|\Gamma| \leq 1]$
  - $P_t = (1 - |\Gamma|^2)\, P_i \qquad @\, z = 0$
  - $P_t(z) = P_t(0)\, e^{-2\alpha z} \qquad z \geq 0$

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This session is about

11. Examples on Reflection and Transmission of UPW at Normal Incident

Example (Nov/ Dec 2014)

3(a) Medium 1 ($\mu_o,\ \varepsilon_o,\ \sigma_1 = 0$), Medium 2 ($\mu_o,\ 2.25\,\varepsilon_o,\ \sigma_2 = 0.1\ \mathrm{S/m}$), $f = 15\ \mathrm{MHz}$ and $\tilde{\vec{H}}_i(z,t) = \vec{a}_x\, 0.15\ \sin\left(\omega t - k_i z - \frac{\pi}{4}\right)\ \mathrm{A/m}$. **Find** (i) $\omega,\ k_i$ (ii) η_{2c} (iii) α_2 (iv) Γ and (v) position z when $P_t(z) = 0.005\, P_i$

(i) $\omega = 2\pi f = 9.42 \times 10^7\ \mathrm{rad/s},\ k_i = \omega\sqrt{\mu_o\,\varepsilon_o} = 0.314\ \ \mathrm{rad/m}$

(ii) $\frac{\sigma_2}{\omega\,\varepsilon_2} = 53.3 > 20$: $\delta = \frac{1}{\sqrt{\pi\, f\, \mu\, \sigma_2}} = 0.411,\ \eta_{2c} = \frac{1+j}{\sigma_2\,\delta} = 24.3 + j\,24.3\ \Omega$

(iii) $\alpha_2 = \frac{1}{\delta} = 2.434$

(iv) $\Gamma = \frac{\eta_{2c}-\eta_1}{\eta_{2c}+\eta_1} = 0.879\,\angle 172.6°$ $[\eta_1 = 120\pi\ \Omega,\ \ \eta_{2c} = 24.3 + j\,24.3\ \Omega]$

(v) $P_t(z) = P_t(0)\, e^{-2\alpha_2 z} = (1 - 0.879^2)\, P_i\ e^{-4.868z} = 0.005\, P_i,\ z = 0.784\ \ \mathrm{m}$

Example (April/ May 2013)

The electric field of a UPW in **free space** $(z \leq 0)$ is given by

$\tilde{\vec{E}}_i(z,t) = \vec{a}_y\, 20\ \cos(1.6\pi \times 10^9\, t - kz)\ \ \mathrm{V/m}.$

The wave is normally incident on a lossy medium

$(\mu = \mu_o,\ \varepsilon = 2.25\,\varepsilon_o,\ \sigma = 0.09\ \ \mathrm{S/m})$ occupying the region $z \geq 0$.

Determine the following:

(i) The wave number, k, in free space.

(ii) The propagation constant, γ, in the lossy medium.

(iii) The time-average Poynting vector, $\vec{S}_t$, in the lossy medium.

(iv) The percentage of average power dissipated as the transmitted wave travels from $z = 0$ to $z = 0.1$ m.

Example (April/ May 2013) Solution

- $(\mu_1 = \mu_o,\ \varepsilon_1 = \varepsilon_o,\ \sigma_1 = 0), (\mu_2 = \mu_o,\ \varepsilon_2 = 2.25\ \varepsilon_o,\ \sigma_2 = 0.09\ \text{ S/m})$
- $\tilde{\vec{E}}_i(z,t) = \vec{a}_y\, 20\ \cos(1.6\pi \times 10^9\, t - kz)\ \text{ V/m}$
- $\omega = 1.6\pi \times 10^9 \rightarrow f = 800\text{ MHz}$

(i) $k = \omega\sqrt{\mu_1\,\varepsilon_1} = \dfrac{1.6\pi \times 10^9}{3 \times 10^8} = 16.76\ \text{ rad/m}$

(ii) $\dfrac{\sigma_2}{\omega\,\varepsilon_2} = 0.9 < 20$: $\varepsilon_{2c} = 2.25\,\varepsilon_o\,(1 - j\,0.9)$

$$\gamma = j\,\omega\sqrt{\mu_2\,\varepsilon_{2c}} = j\,\omega\sqrt{\mu_o\, 2.25\,\varepsilon_o}\sqrt{1 - j\,0.9}$$

$$= j\,25.13 \times 1.16\angle - 21° = 10.44 + j\,27.2 = \alpha + j\beta$$

$$1 - j\,0.9 = 1.345\angle - 42°$$

$$\sqrt{1 - j\,0.9} = 1.16\angle - 21°$$

Example (April/ May 2013) Solution (cont.)

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(iii) Method 1: $\vec{S}_t = \vec{a}_z\, S_t(0)\, e^{-2\alpha z}$, $S_t(0) = S_i\,(1 - |\Gamma|^2)$ $[\alpha = 10.44]$

$$S_i = \frac{|E_{oi}|^2}{2\,\eta_1} = \frac{20^2}{2\,\times 120\pi} = 0.5305\ \text{W/m}^2$$

$$\Gamma = \frac{\eta_{2c} - \eta_1}{\eta_{2c} + \eta_1} = 0.327\,\angle 148.4° \qquad \left[\eta_1 = 120\pi,\ \ \eta_{2c} = \sqrt{\frac{\mu_2}{\varepsilon_{2c}}} = 216.7\,\angle 21°\right]$$

$$S_t(0) = S_i\,(1 - |\Gamma|^2) = 0.4738$$

$$\vec{S}_t = \vec{a}_z\, S_t(0)\, e^{-2(10.44)z} = \vec{a}_z\, 0.4738\, e^{-20.88z}\ \ \text{W/m}^2$$

Method 2: $\vec{S}_t = \frac{1}{2}\,\text{Re}(\vec{E}_t \times \vec{H}_t^*)$

(iv) $S_t(z) \propto e^{-2\alpha z} \rightarrow \%\, P_d = \left(1 - e^{-20.88(0.1)}\right) \times 100\ \% = 87.6\ \%$

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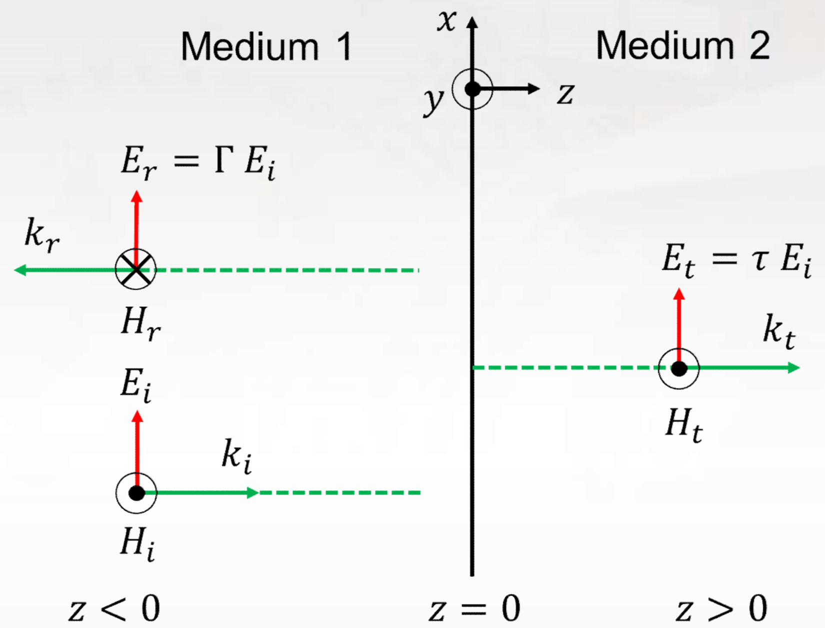
This session is about

12. Standing Waves

Learning Objectives

- List the properties of standing wave;
- Explain the properties of standing wave; and
- Apply standing wave equations to find the values and positions of $|\vec{E}|_{max}$, $|\vec{E}|_{min}$, $|\vec{H}|_{max}$, $|\vec{H}|_{min}$.

Standing Wave of E and H fields



- ❑ Given $\vec{E}_1(z) = \vec{E}_i(z) + \vec{E}_r(z)$, find $\left|\vec{E}_1\right|_{max}$ @ z_{max} and $\left|\vec{E}_1\right|_{min}$ @ z_{min}

Standing Wave Example

Example on Page 9-3 $[\Gamma = 0.5\ \angle\pi \equiv |\Gamma|\ \angle\theta_o]$

- $E_i = 50\ \angle 30°\ e^{-j2\pi}$, $E_r = 25\ \angle 210°\ e^{+j2\pi z}$; $E_1 = E_i + E_r$
- @ $z = 0$:

  $E_i = 50\ \angle 30°$, $E_r = 25\ \angle 210° = 25\ \angle(30° + 180°)$

  $\therefore |E_1|_{min} = 25$ @ $z_{min} = 0$ $[E_1 = E_i + E_r = 25\ \angle 30°]$

- @ $z = -0.25$ m:

  $E_i = 50\ \angle 30°\ e^{+j90°} = 50\ \angle 120°$, $E_r = 25\ \angle 210°\ e^{-j9\ \ °} = 25\ \angle 120°$

  $\therefore |E_1|_{max} = 75$ @ $z_{ma\ \ } = -0.25$ m $[E_1 = E_i + E_r = 75\ \angle 120°]$

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This session is about

12. Standing Waves (2)

Standing Wave: [Incident + Reflected Wave]

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- $\vec{E}_1(z) = \vec{a}_x\, E_{oi}\, e^{-jk_1 z} + \vec{a}_x\, E_{or}\, e^{+jk_1 z}$ $\quad z \leq 0 \quad$ (7)
- $\vec{E}_1(z) = \vec{a}_x\, E_{oi}\, e^{-jk_1 z}\left[1 + \frac{E_{or}}{E_{oi}}\, \frac{e^{+jk_1 z}}{e^{-jk_1 z}}\right]$
- $\Gamma = \frac{E_{or}}{E_{oi}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = |\Gamma|\,\angle\theta_o = |\Gamma|\, e^{j\theta_o}$ $\quad @\, z = 0 \quad$ (3)
- $\vec{E}_1(z) = \vec{a}_x\, E_{oi}\, e^{-jk_1 z}\left[1 + |\Gamma|\, e^{j\theta_o}\, e^{+j2k_1 z}\right]$
- $\vec{E}_1(z) = \vec{a}_x\, E_{oi}\, e^{-jk_1 z}\left[1\,\angle 0 + |\Gamma|\,\angle(\theta_o + 2k_1 z)\right]$
- $\theta_\Gamma = \theta_o + 2\, k_1\, z_{max} = 0, -2\pi, \cdots;$ $\quad \theta_\Gamma = \theta_o + 2\, k_1\, z_{min} = \pi, -\pi, \cdots$

Standing Wave: [Incident + Reflected Wave]

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- $\left|\vec{E}_1(z)\right| = |E_{oi}|\,|1\,\angle 0 + |\Gamma|\,\angle\theta_\Gamma|$; $\quad \theta_\Gamma = 0$: $z = z_{max}$, $|E_1|_{max} = |E_{oi}|\,\big|1 + |\Gamma|\big|$

  $\theta_\Gamma = \pm\pi$: $z = z_{mi}$, $|E_1|_{mi} = |E_{oi}|\,\big|1 - |\Gamma|\big|$

- $\boxed{\left|\vec{H}_1(z)\right| = \frac{|E_{oi}|}{\eta_1}\,|1 - |\Gamma|\,\angle\theta_\Gamma|} \rightarrow \left|\vec{H}_1\right|_{min} = \frac{|E_1|_{min}}{\eta_1}$ @ z_{max}; $\left|\vec{H}_1\right|_{max} = \frac{|E_1|_{max}}{\eta_1}$ @ z_{min}

- $\boxed{\theta_\Gamma = \theta_o + 2\,k_1\,z} = \theta_o + \frac{4\pi}{\lambda_1}\,z \qquad \left[k_1 = \frac{2\pi}{\lambda_1}\right]$

- θ_Γ repeats every $\lambda/2$: $\quad [\triangle\, z \pm \lambda_1/2 \;\rightarrow\; \Delta\,\theta_\Gamma = \pm 2\pi]$

- $\therefore$ $|E_1|$ and $|H_1|$ repeat every $\lambda_1/2$.

Summary

- $\left|\vec{E}_1(z)\right| = |E_{oi}|\,|\,1\,\angle 0 + |\Gamma|\,\angle\theta_\Gamma|, \quad \theta_\Gamma = \theta_o + 2\,k_1\,z, \quad z \le 0$
- $\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = |\Gamma|\,\angle\theta_o$ (3)

1. $|E_1|_{max}$ & $|E_1|_{min} \propto |E_{oi}|$ and $|\Gamma|$

$$\boxed{|E_1|_{max} = |E_{oi}|\,(1 + |\Gamma|)} = |E_{oi}| + |E_{or}| \quad (12) \qquad \boxed{|E_{or}| = |\Gamma|\,|E_{oi}|}$$

$$\boxed{|E_1|_{min} = |E_{oi}|\,(\,1 - |\Gamma|)} = |E_{oi}| - |E_{or}| \quad (13)$$

2. z_{max} and z_{min} depend on θ_o

$$\boxed{\theta_\Gamma = \theta_o + 2\,k_1\,z_{max} = 0, -2\pi} \quad (14)$$

$$\boxed{\theta_\Gamma = \theta_o + 2\,k_1\,z_{min} = \pi, -\pi} \quad (15)$$

Summary

3. $\boxed{z_{max} \xleftrightarrow{\lambda_1/4} z_{min},}$ $|E_1|$ and $|H_1|$ repeat every $\lambda_1/2$ (16)

$@\ z_{max}$ $@\ z_{min}$

4. $|E_1|_{max} \Leftrightarrow |H_1|_{min}$ and $|E_1|_{min} \Leftrightarrow |H_1|_{max}$

5. $\boxed{\text{SWR} = \frac{|E_1|_{ma}}{|E_1|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}}$ (17)

$\left|\vec{E}_1\right|_{max} = 75$ and $\left|\vec{E}_1\right|_{min} = 25$: $\text{SWR} = 3 = \frac{1+|\Gamma|}{1-|\Gamma|} \rightarrow |\Gamma| = 0.5$

$P_r = |\Gamma|^2\, P_i = 0.25\, P_i$

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This session is about

13. Examples of Standing Waves

Example

Using the example on Page 9-3, we have

$E_{oi} = 50 \angle \pi/6$; $\Gamma = 0.5 \angle \pi \equiv |\Gamma| \angle \theta_o$ $[E_{or} = 25 \angle 7\pi/6]$

Determine $|E_1|_{max}$, $|E_1|_{min}$, z_{max} and z_{min}.

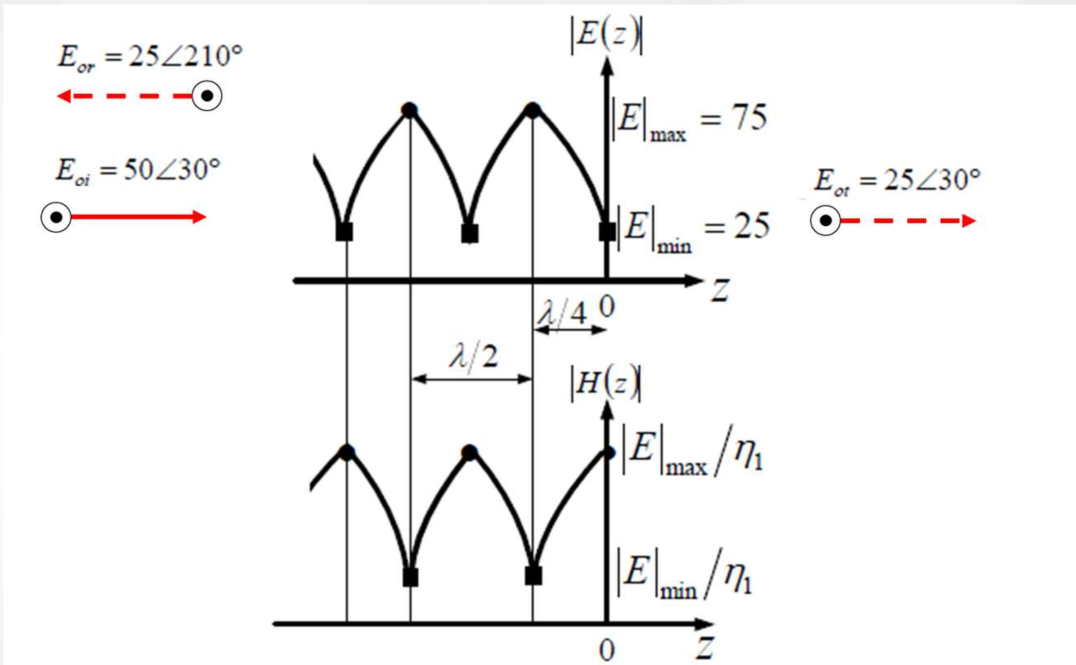
Solution

- $|E_1|_{max} = |E_{oi}|\,(1 + |\Gamma|) = 50\,(1 + 0.5) = 75\ \text{V/m}$
- $|E_1|_{mi} = |E_{oi}|\,(1 - |\Gamma|) = 50\,(1 - 0.5) = 25\ \text{V/m}$
- $\theta_\Gamma = \theta_o + 2\,k_1\,z_{max} = 0$
- $\boxed{z_{max} = -\dfrac{\theta_o}{2\,k_1} = -\dfrac{\lambda_1}{4}}$ $\left[k_1 = \dfrac{2\pi}{\lambda_1};\ \theta_o = \pi\right]$

$\therefore z_{max} = -\dfrac{\lambda_1}{4},\ -\dfrac{3\lambda_1}{4},\ \cdots$ and $z_{min} = 0,\ -\dfrac{2\lambda_1}{4},\ \cdots$

Solution to Example

- Standing Wave



Example (Apr/ May 2015)

3(a) Medium 1 ($\mu_o,\ \varepsilon_o,\ \sigma_1 = 0$), Medium 2 ($\mu_o,\ 2\,\varepsilon_o,\ \sigma_2 = 0.08\ \mathrm{S/m}$),

$$\vec{E}_i(z) = \left(j\,30\,\vec{a}_x - 40\,\vec{a}_y\right) e^{-j4\pi} \quad \mathrm{V/m}$$

Find (i) f (ii) Polarisation (iii) $\eta_1,\ \eta_{2c}$ (iv) z_{max} and (v) $|E_1|_{max}$

(i) $k_i = \omega\sqrt{\mu_o\,\varepsilon_o} = 4\pi \rightarrow f = 600\ \mathrm{MHz}$ $\quad [\lambda = 0.5\ \mathrm{m}]$

(ii) Elliptical

(iii) $\eta_1 = 120\pi\ \Omega,\ \ \eta_{2c} = 213\,\angle 25°\ \Omega$

(iv) $\Gamma = \dfrac{\eta_{2c} - \eta_1}{\eta_{2c} + \eta_1} = 0.355\,\angle 145° = 0.355\,\angle 2.53$ $\quad [|\Gamma| = 0.355,\ \ \theta_o = 2.53]$

$$\theta_\Gamma = \theta_o + 2\,k_1\,z_{max} = 0, -2\pi, \cdots \rightarrow z_{max} = -0.2\lambda, -0.7\lambda, \cdots$$

(v) $|E_1|_{max} = |E_{oi}| \times \big|1 + |\Gamma|\big| = 50 \times 1.355 = 67.7\ \mathrm{V/m}$