EE2010/IM2004
Signals and Systems

Part II

(Weeks 6 – 11)

Contact Details

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Roadmap for This Part

Topics to be covered in the final exams of this part are as follows:

- ❑ Sinusoids (~ 1 week)
- ❑ Fourier Series (~ 2 weeks)
- ❑ Fourier Transform (~ 2 weeks)
- ❑ Sampling (~1 week)

Sinusoids

Leonard Euler

$$\boxed{\begin{aligned} \cos\theta &= \frac{e^{j\theta} + e^{-j\theta}}{2} \\ \sin\theta &= \frac{e^{j\theta} - e^{-j\theta}}{j2} \\ e^{j\theta} &= \cos\theta + j\sin\theta \end{aligned}}$$

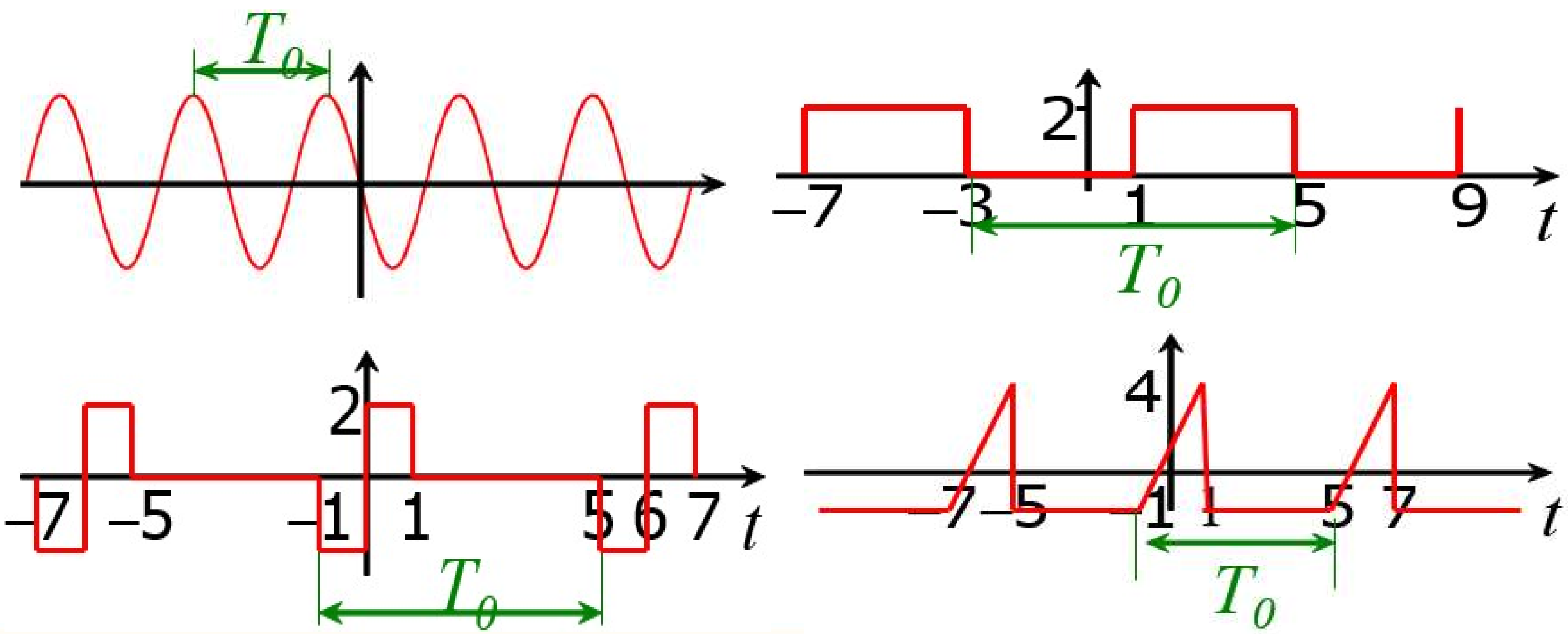
Leonhard Euler (1707-1783)

Source of photo - 1849: given by Rudolf Bischoff-Merian

Periodic Signals

A signal is called **periodic** with a *fundamental period* T_0, if it repeats a signal pattern every T_0 unit times; i.e., $f(t + n\,T_0) = f(t)$, where n is an integer.

Examples:



Some Important Notes

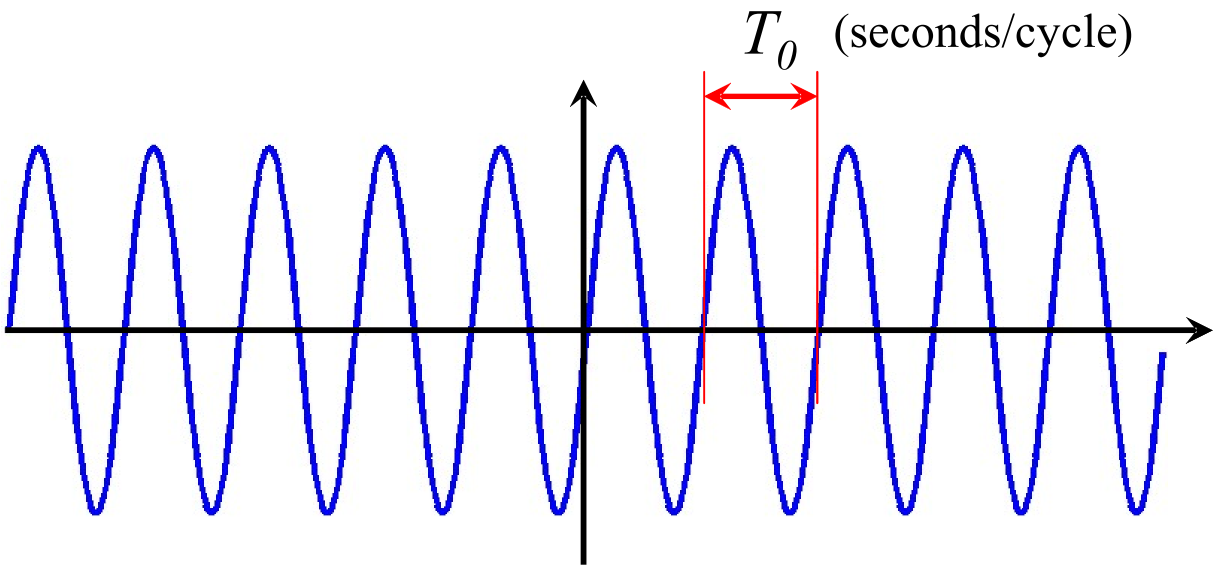
- If T_0 is a *fundamental period* of function $f(t)$, then the numbers $2T_0$, $3T_0$, $4T_0$, … are also *periods*; that is, $f(t + 2T_0) = f(t)$, $f(t + 3T_0) = f(t)$, …, $f(t + nT_0) = f(t)$, where n is an arbitrary integer (positive or negative).

  ➔ Thus, the fundamental period T_0 is the **minimum** time period, such that $f(t + T_0) = f(t)$.

- Oftentimes, the word "period" already implies "fundamental period"; thus, either T_0 or T will be used to denote the (fundamental) period; or, stated clearly, otherwise.

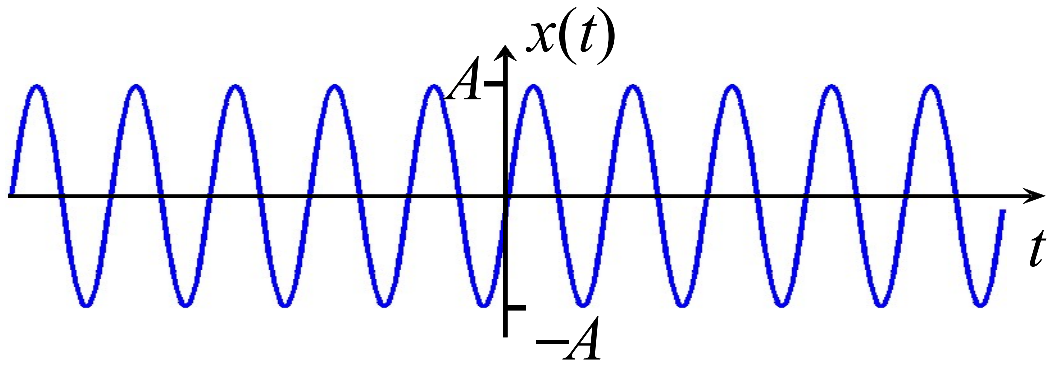
Periodic Signals

The time for one complete cycle is called the ***fundamental period***, T_0



$$T_0 = 1/f_0$$

Sinusoids: The "Particle" of Fourier Analysis



$$\boxed{x(t) = A\sin(\omega_0 t + \theta) = A\sin(2\pi f_0 t + \theta)}$$

$$x(t) = A\cos(\omega_0 t + \phi) = A\cos(2\pi f_0 t + \phi),$$

where $\theta = \phi + 90^\circ$

A sinusoid has **three** key parameters:

(i) Magnitude (or Amplitude), A

Frequency, $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$

(ii) (Initial) Phase, θ

➔ One **frequency** means one **harmonic** (or one **tone** in audio)

$$x(t) = A\sin(\omega_0 t + \theta) = A\sin(\omega_0 t + \theta + 2\pi)$$

$$x(t+T_0) = A\sin\left[\omega_0\left(t+T_0\right)+\theta\right] = A\sin\left[\omega_0 t + \omega_0 T_0 + \theta\right]$$

Since $x(t+T_0) = x(t)$, $\omega_0 T_0 = 2\pi \quad \Rightarrow \quad \boxed{\omega_0 = \frac{2\pi}{T_0}}$

Since one complete cycle corresponds to an angle of 2π radians, f_0 cycles per second (often denoted as **Hz**) corresponds to: $2\pi \times f_0$ radians/second; hence,

$$\omega_0 t = 2\pi f_0 t$$

$$\text{radians} = \frac{\text{radians}}{\text{cycle}} \bullet \frac{\text{cycles}}{\text{second}} \bullet \text{seconds}$$

Examples:

(1). Find the fundamental *frequency* (in Hz) for each of the following signals:

(a). $x(t) = 3\sin(10\pi t)$

(b). $y(t) = 1 + 3\sin(10\pi t)$

(c). $z(t) = \cos(t)$

(d). $v(t) = \pi\cos(\pi t + \pi)$

(e). $w(t) = 3\sin(-10\pi t)$

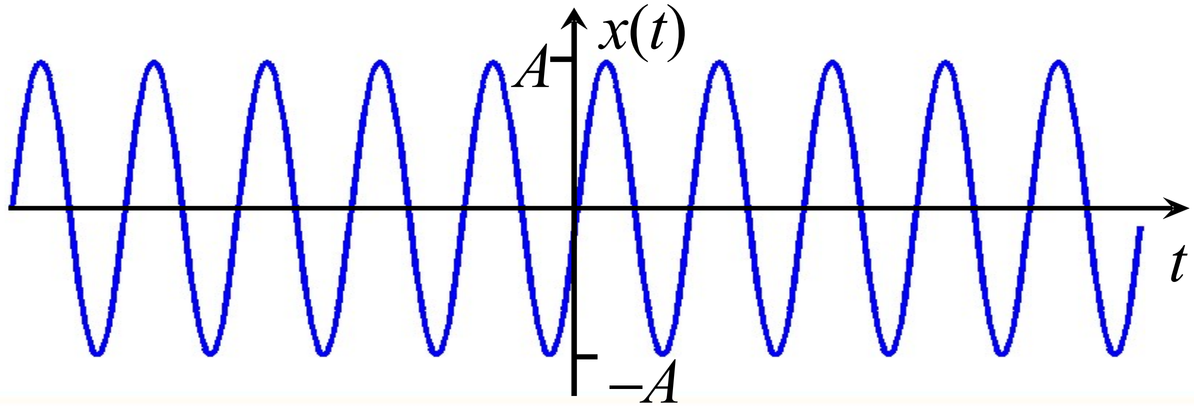
(2). Further find their fundamental *periods* T_0 (in seconds), respectively.

How to draw waveform properly?

➔ No one can draw perfect sine wave; hence, proper labeling of your drawing is absolutely required!

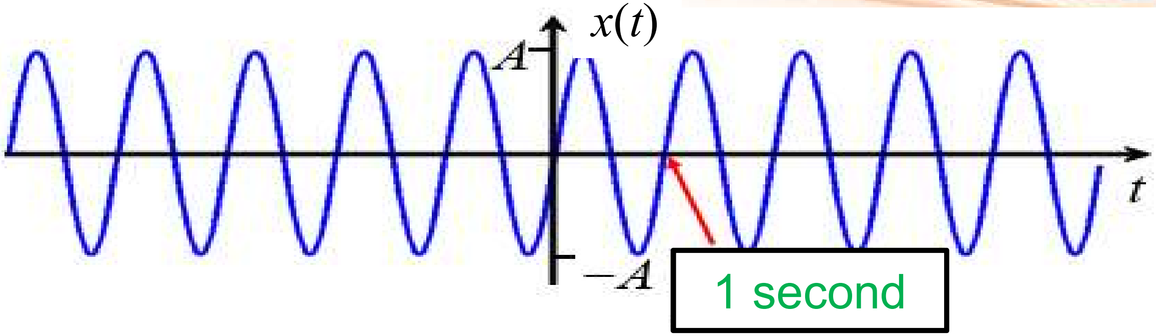
Example: Plot the waveforms in the previous example.

Example: Use the same graph as shown below to denote a 1 Hz, 3 Hz, and 1 kHz sinusoid, respectively, by simply re-labeling the time axis (i.e., the *t*-axis).

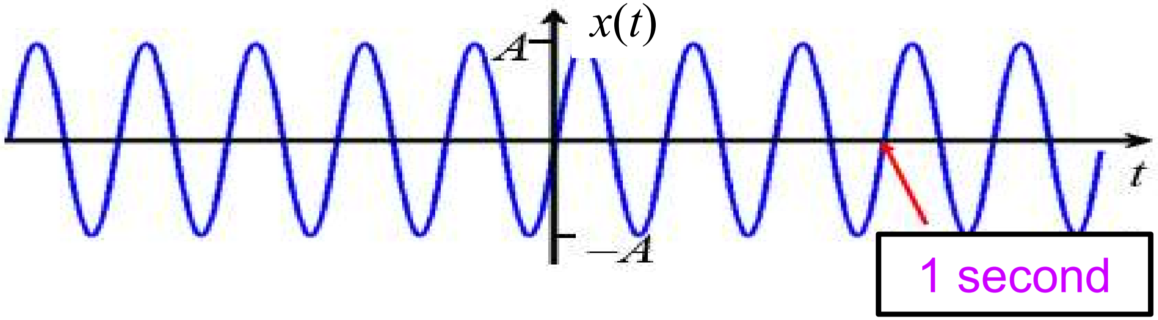


Solution:

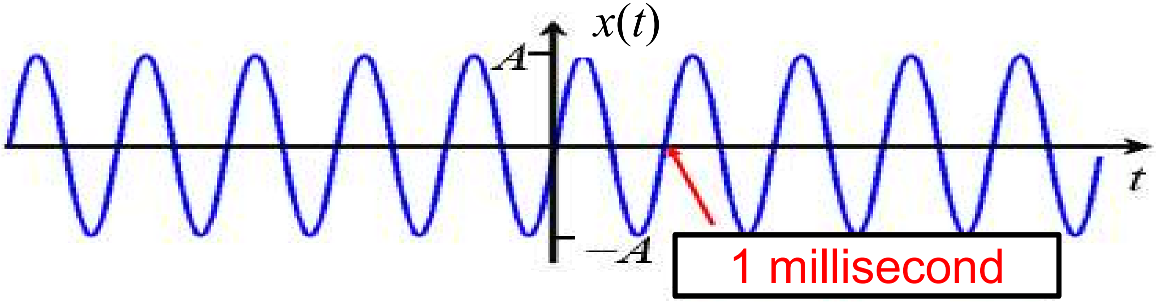
1 Hz



3 Hz



1 KHz



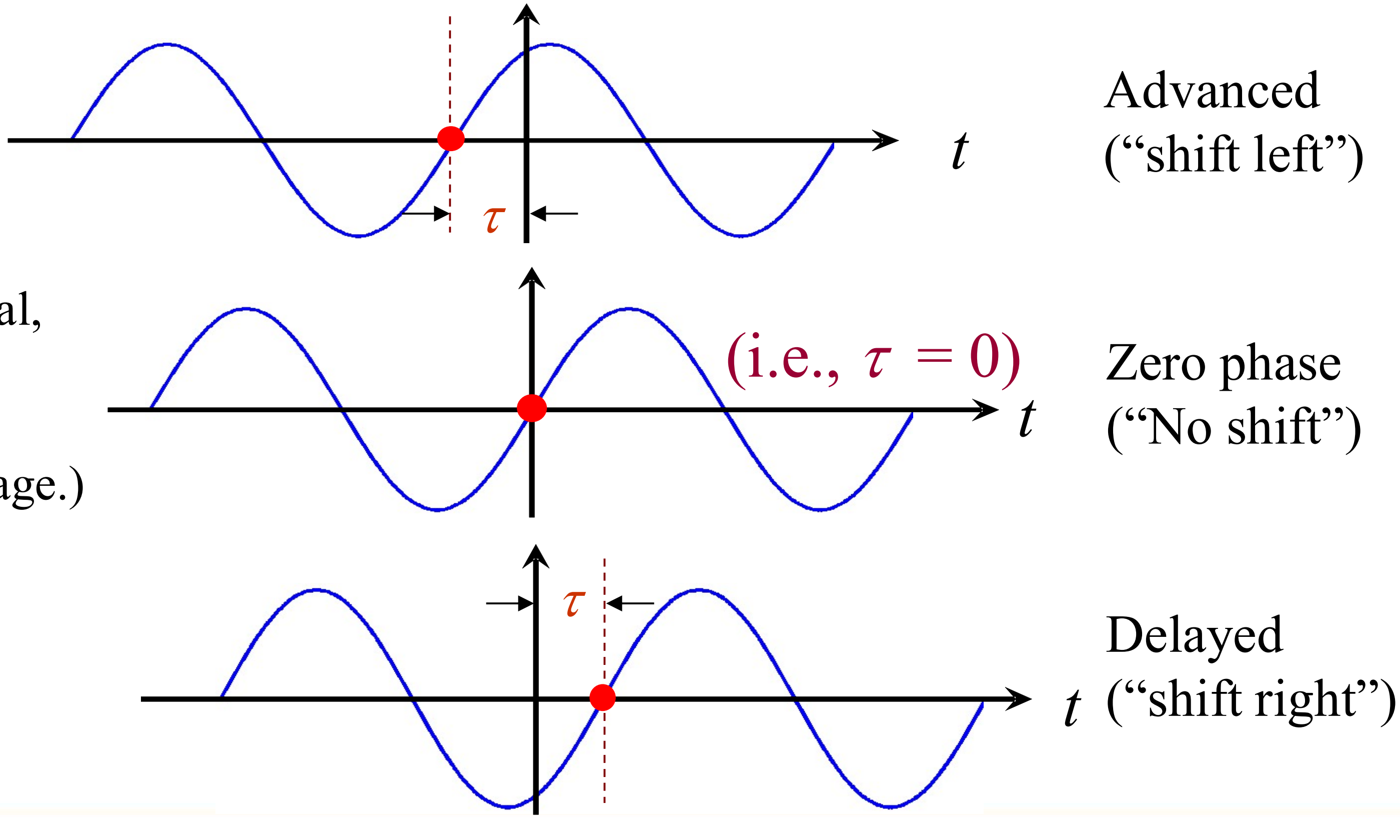
Phase

A *phase change* (by $\pm\theta$) means a *time shift* (by $\pm\tau$), and vice versa!

For the sinusoidal signal,

$$\boxed{\theta = \omega_0 \tau}$$

(Justified in the next page.)



For a sinusoid signal $x(t) = A\sin(\omega_0 t)$, one period is equal to 2π. Therefore,

$$\theta : 2\pi = \tau : T_0 \;\Rightarrow\; \frac{\theta}{2\pi} = \frac{\tau}{T_0} \;\Rightarrow\; \boxed{\theta = \omega_0 \tau}$$

For a <u>positive</u> phase shift,

$$y(t) = A\sin(\omega_0 t + \theta) = A\sin(\omega_0 t + \omega_0 \tau) = A\sin\left[\omega_0 (t + \tau)\right]$$

$$= x(t + \tau) \quad \text{(i.e., "shift left")}$$

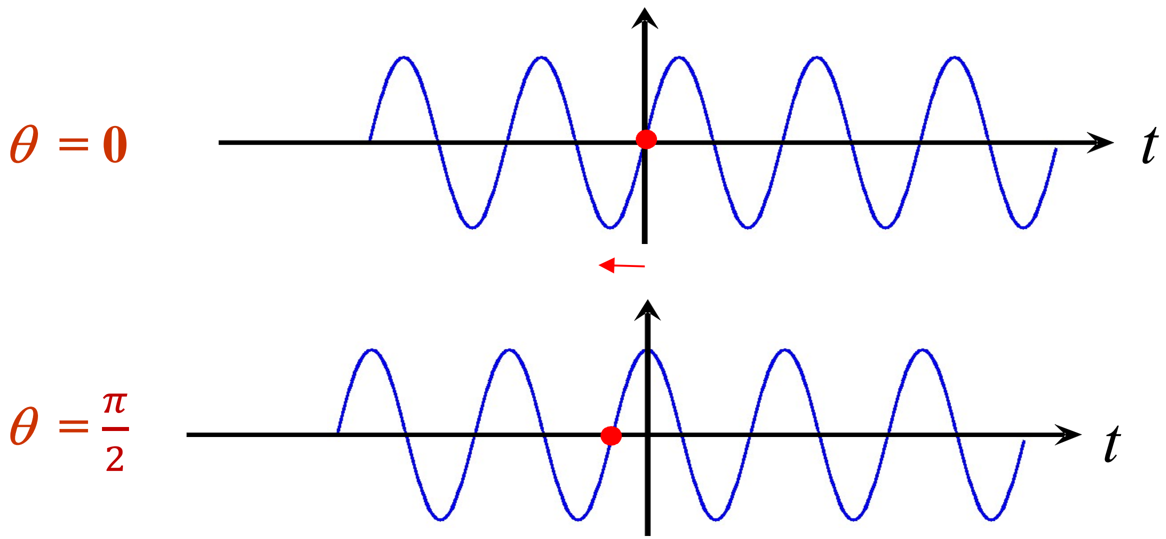
For a <u>negative</u> phase shift,

$$z(t) = A\sin(\omega_0 t - \theta) = A\sin(\omega_0 t - \omega_0 \tau) = A\sin\left[\omega_0 (t - \tau)\right]$$

$$= x(t - \tau) \quad \text{(i.e., "shift right")}$$

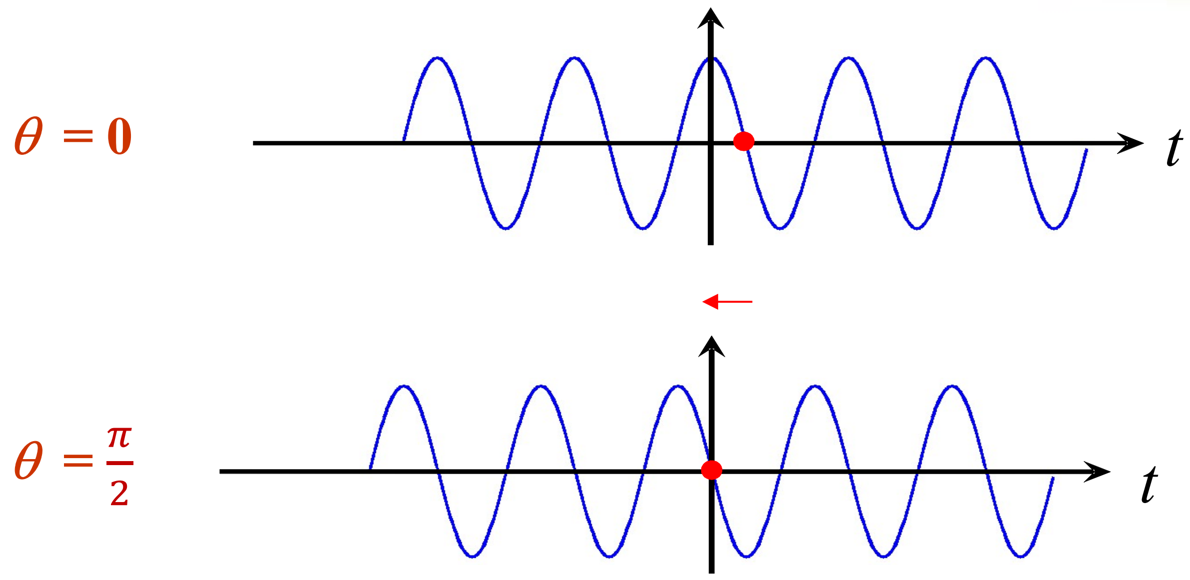
➔ In closing, a <u>*phase change*</u> (by $\pm\theta$) means a <u>*time shift*</u> (by $\pm\tau$), and vice versa!

Example: Given a sinusoidal signal $\sin(\omega_0 t + \theta)$, if



- From the *time shift* perspective, once the sine wave is shifted to the **left** by an amount of π/2 (which is exactly ¼ of one period), it becomes a cosine wave; that is, $\cos(\omega_0 t)$.
- From the *phase change* perspective, $sin\left(\omega_0 t + \frac{\pi}{2}\right) = \cos(\omega_0 t)$.

Example: Likewise, given a cosine signal $\cos(\omega_0 t + \theta)$, if



- From the *time shift* perspective, once the cosine wave shifts to the **left** by an amount of π/2 (which is exactly ¼ of one period), it becomes a "flipped" (upside down) version of sine wave; that is, $-\sin(\omega_0 t)$.
- From the *phase change* perspective, $\cos\left(\omega_0 t + \frac{\pi}{2}\right) = -\sin(\omega_0 t)$.

Real Sinusoid and Complex Sinusoid

- [**Real** Sinusoid]: $x(t) = A\ cos(\omega_0 t + \theta)$ or $x(t) = A\ sin(\omega_0 t + \theta)$
- [**Complex** Sinusoid]: $x(t) = A\ e^{\,j(\omega_0 t + \theta)}$

$$A\cos(\omega_0 t + \theta) = A\left(\frac{e^{j(\omega_0 t+\theta)} + e^{-j(\omega_0 t+\theta)}}{2}\right) = \frac{A}{2}e^{j(\omega_0 t+\theta)} + \frac{A}{2}e^{-j(\omega_0 t+\theta)}$$

$$A\sin(\omega_0 t + \theta) = A\left(\frac{e^{j(\omega_0 t+\theta)} - e^{-j(\omega_0 t+\theta)}}{2j}\right) = \frac{A}{2j}e^{j(\omega_0 t+\theta)} - \frac{A}{2j}e^{-j(\omega_0 t+\theta)}$$

$$\boxed{\begin{aligned} \cos\phi &= \frac{e^{j\phi} + e^{-j\phi}}{2} \\ \sin\phi &= \frac{e^{j\phi} - e^{-j\phi}}{2j} \\ e^{j\phi} &= \cos\phi + j\sin\phi \end{aligned}}$$

[Euler's identities]

Real Signals and Complex Signals*

- Provide some examples of **real** signals and **complex** signals, respectively, and deduce a 'rule-of-thumb' on judging whether a given signal is a real signal or a complex signal.

Examples:

(1). Find the fundamental *frequency* (in Hz) for each of the following signals:

(a). $f(t) = 2\,e^{j\,40\pi t}$

(b). $g(t) = 2\,e^{j\,40t}$

(c). $h(t) = 2\,e^{j\left(t+40^{\circ}\right)}$

(2). Further find their fundamental *periods* T_0 (in seconds), respectively.

Equivalent Forms

Amplitude-phase form

Since $\cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y$

$$\underline{A\cos(\omega t+\theta)} = A\cos\omega t\cdot\cos\theta - A\sin\omega t\cdot\sin\theta$$
$$= \underbrace{A\cos\theta}_{=a}\cdot\cos\omega t + \underbrace{(-A\sin\theta)}_{=b}\cdot\sin\omega t$$
$$= \underline{a\cos\omega t + b\sin\omega t}$$
$$= a\cos(\omega t+0) + b\sin(\omega t+0)$$

← Trigonometric form →

Since $\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y$

$$\underline{A\sin(\omega t+\theta)} = A\sin\omega t\cdot\cos\theta + A\cos\omega t\cdot\sin\theta$$
$$= \underbrace{A\cos\theta}_{=\hat{b}}\cdot\sin\omega t + \underbrace{A\sin\theta}_{=\hat{a}}\cdot\cos\omega t$$
$$= \underline{\hat{a}\cos\omega t + \hat{b}\sin\omega t}$$
$$= \hat{a}\cos(\omega t+0) + \hat{b}\sin(\omega t+0)$$

➔ A *single* harmonic (i.e., *one* frequency f_0) with a **non-zero** phase θ , be it in the cosine form (left-side box above) or the sine form (right-side box above), is a composition of *two* sinusoidal terms, both in **zero-phase**. For these two terms, one must be in sine and the other must be in cosine, and both have the identical frequency f_0.

This is a foundation of Fourier Series that you are going to encounter later on.

Converting from Trig. Form to Amplitude-Phase Form

Use cosine for our further discussion: $\boxed{A\cos(\omega t+\theta) \quad \Leftrightarrow \quad a\cos\omega t+b\sin\omega t}$

➔ How are the amplitude A and phase θ related to constant coefficients a and b?

Usefulness: When ($a\cos\omega t + b\sin\omega t$) is given, how do you combine these **two** zero-phase terms into **one** term $A\cos(\omega t+\theta)$, with non-zero phase?

$$a^2+b^2=\left(A\cos\theta\right)^2+\left(-A\sin\theta\right)^2=A^2\left(\cos^2\theta+\sin^2\theta\right)=A^2$$

$$\Rightarrow \quad \boxed{A=\sqrt{a^2+b^2}}$$ (and select positive value)

$$\frac{b}{a}=\frac{-A\sin\theta}{A\cos\theta}\Rightarrow\frac{-b}{a}=\frac{A\sin\theta}{A\cos\theta}=\tan\theta \;\Rightarrow\; \boxed{\theta=\tan^{-1}\left(\frac{-b}{a}\right)}$$

Example: Consider $x(t) = sin(\omega_0 t)$+$\cos(\omega_0 t)$. Find the resultant signal in terms of: (i) sine form only; and (ii) cosine form only, respectively.

Solution:

(i). The goal is to find the A and θ of $A\ sin(\omega_0 t + \theta)$.

$$x_3(t) = A\sin\left(\omega_0 t + \theta\right) = A\sin\omega_0 t \cdot \cos\theta + A\cos\omega_0 t \cdot \sin\theta = \underbrace{A\cos\theta}_{=a=1} \cdot \sin\omega_0 t + \underbrace{A\sin\theta}_{=b=1} \cdot \cos\omega_0 t$$

$$\therefore A = \sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2} = \sqrt{2}\ ; \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

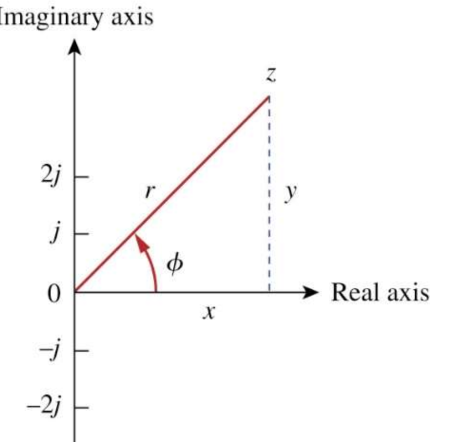
Hence, $x(t) = \sqrt{2}\sin\left(\omega_0 t + \frac{\pi}{4}\right)$

(ii). Based on (i), $x(t) = \sqrt{2}\ cos\left(\omega_0 t + \frac{\pi}{4} - \frac{\pi}{2}\right)$=$\sqrt{2}\ cos\left(\omega_0 t - \frac{\pi}{4}\right)$

PS: You can go through the same way as did in (i) for cosine case.

Complex Numbers

- A complex number z can be written in rectangular form as



$$z = x + j\,y \quad (\text{where } j = \sqrt{-1}\ )$$

$x = \mathrm{Re}(z)$ is the **real** part of z;

$y = \mathbf{Im}(z)$ is the **imaginary** part of z.

Note: $j^2 = j \bullet j = = -1$; $j^3 = j \bullet j^2 = -j$; $j^4 = j^2 \bullet j^2 = 1$; $j^{n+4} = j^n$

- Three *equivalent* forms of the complex number z:
  - [Rectangular form]: $z = x + j\,y = r\cos\theta + j\,r\sin\,\theta$
  - [Polar form]: $z = r\angle\theta$; (We call "r" the *magnitude* of z; "θ" the *phase*.)
  - [Exponential form]: $z = re^{j\theta}$

Complex Numbers

Given x and y, we can get r and θ as

$$|z| = r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

On the other hand, if we know r and θ, we can obtain x and y as

$$x = r\cos\theta, \qquad y = r\sin\theta$$

Thus, z may be written as

$$\boxed{z = x + jy = r\angle\theta = r\cos\theta + j\,r\sin\theta \qquad (\text{where } r = |z|)}$$

Polar Form Conversion*

- Given two points with different points with coordinates (6, 3) and (-6, -3):

$$\frac{-3}{-6} = \frac{3}{6}, \text{ but } \tan^{-1}\left(\frac{-3}{-6}\right) \neq \tan^{-1}\left(\frac{3}{6}\right)$$

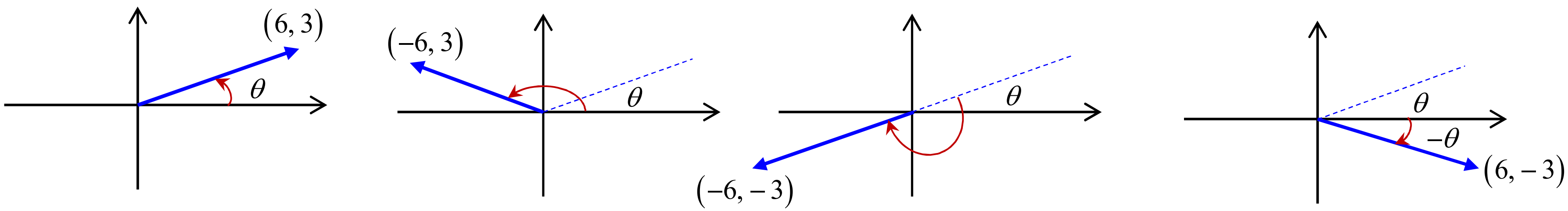
- Likewise, given another two points with coordinates (-6, 3) and (6, -3),

$$\frac{3}{-6} = \frac{-3}{6}, \text{ but } \tan^{-1}\left(\frac{3}{-6}\right) \neq \tan^{-1}\left(\frac{-3}{6}\right)$$

Magnitude is the same for any quadrant; i.e., $\sqrt{(\pm 6)^2 + (\pm 3)^2} = \sqrt{(6)^2 + (3)^2}$

Phase (or angle) depends on the quadrant the coordinate lies!

How to compute the phase or angle?



Conventionally, we show the angle in the *principal* range; i.e., $-\pi \leq \varphi \leq \pi$.

Step 1: Ignore all negative signs (if any); i.e., treating the number, as if it were in the 1st quadrant always.

Step 2: Compute the angle in that case and denote it as θ.

Step 3: Identify which quadrant of the complex number lies in for adjusting θ to the actual angle. For example, in this case, $\theta = 30^0$, therefore, for the 2nd quadrant, the actual angle is $180^0 - 30^0 = 150^0$.

Summation of two sinusoids

Let $x_1(t)$ and $x_2(t)$ be periodic **continuous-time** signals with the fundamental periods T_1 and T_2, respectively. The sum of these two signals; i.e., $x(t) = x_1(t) + x_2(t)$ is *not* always periodic, unless the following condition is met:

$$\boxed{\frac{T_1}{T_2} = \frac{k}{m} = \frac{\text{Integer}}{\text{Integer}} = \text{rational number}}$$

How to derive this condition? Try few specific cases and generalize your 'experiments'*.

If $x(t)$ is periodic, then the fundamental period T of the signal $x(t)$ can be obtained by:

$$\boxed{T = \text{LCM}(T_1, T_2)}$$

Equivalent Checking Methods

Since the T, f, and ω are intimately related to each other and $\boxed{\dfrac{T_1}{T_2} = \dfrac{k}{m} = \dfrac{\text{Integer}}{\text{Integer}}}$

you can also check wither $x(t)$ is periodic based on the frequency, without finding the fundamental period T_0 first. That is,

$$\boxed{\frac{\omega_2}{\omega_1} = \frac{f_2}{f_1} = \frac{\text{Integer}}{\text{Integer}}}$$

If $x(t)$ is periodic, then the fundamental frequency f of the signal $x(t)$ can be obtained by:

$$\boxed{f = \text{HCF}(f_1, f_2) \quad \text{or} \quad \omega = \text{HCF}(\omega_1, \omega_2)}$$

Example: Identify whether each signal is periodic. If yes, determine the fundamental period.

(a). $x(t) = \cos(t) + \sin(5t)$
(b). $y(t) = 1 + 3\sin(10\pi t)$
(c). $z(t) = \cos(t) - 3\sin(10\pi t)$
(d). $v(t) = 2e^{j3t} + 4\cos(2t + \pi)$

EE2010
Signals and Systems

Part II

Fourier Series

with Instructor:

Prof. Ma, Kai-Kuang

Jean Baptiste Joseph Fourier

Jean Baptiste Joseph Fourier
(1768-1830)

Source of photo - "Portraits et Histoire des Hommes Utiles, Collection de Cinquante Portraits," Societe Montyon et Franklin, 1839-1840.

Topics to be covered for the Fourier Series

1. Fourier analysis — What is this about and what for?
2. What is the ‘framework’ to construct the Fourier series (FS) representation
3. Three mathematically-equivalent Fourier series forms
4. What is the significance of the Fourier series coefficients?
5. How to calculate the Fourier series coefficients?
6. Important Fourier series properties

Topics to be covered for the Fourier Series

Important Acronyms & Symbols:

LTI: Linear Time Invariant
CT: Continuous Time
DT: Discrete Time
FS: Fourier Series
FT: Fourier Transform

I: Integer set
R: Real-number set
Z: Complex-number set

Fourier Analysis

Fourier analysis is about a way to represent or analyze a given signal (be it periodic or non-periodic) to reveal its frequency content.

It has two cases:

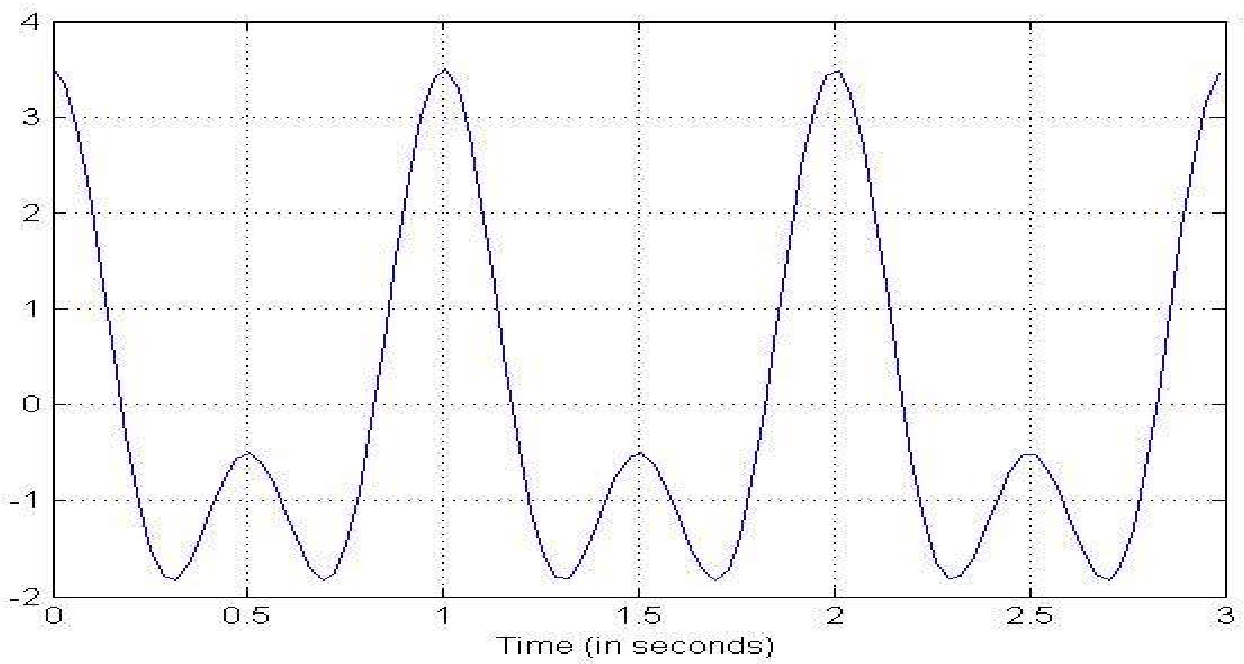
- **Fourier Series (FS)** — The Fourier series of a **periodic** signal $x(t)$ is a representation that decomposes $x(t)$ into a dc component and an ac component, which comprises an infinite number of harmonic sinusoidal terms; all these terms are linearly combined and *harmonically*-related.
- **Fourier Transform (FT)** — representing an **arbitrary** (non-periodic or periodic) signal. In fact, FT is a limiting case of the FS, which is to be discussed later on.

➔Learn this subject beyond its mathematics!

➔ Computing is not just for numbers; in fact, more for insights!

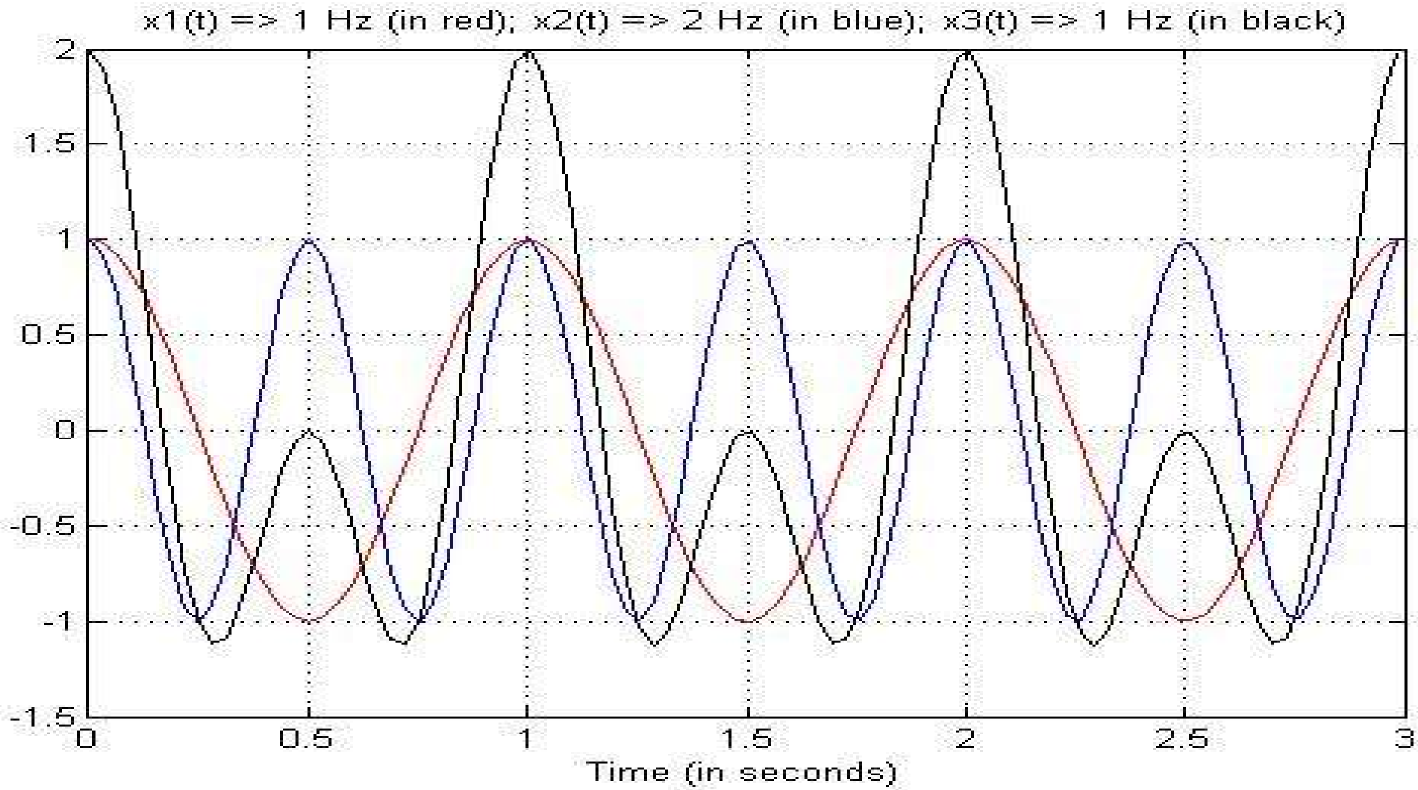
Intuitions First … (1/2)

- What are the constituent frequency components?
- What is the amount of contribution coming from each component?



Intuitions First … (2/2)

$x_3(t) = x_1(t) + x_2(t)$; where $x_1(t) = \cos(2\pi t)$ and $x_2(\mathrm{t}) = \cos(4\pi t)$



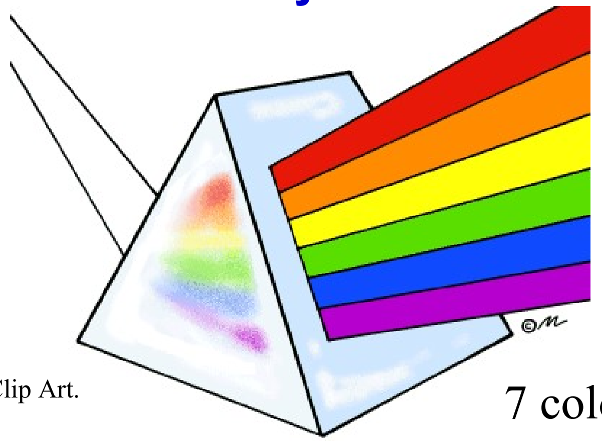
Fourier Analysis is About Spectrum Representation

Analysis

$$x(t) \rightleftarrows$$

Synthesis

White light



Source of photo – Clip Art.

7 colors

Time Domain	Frequency Domain

Trigonometric Fourier Series:

$$\boxed{x(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=0}^{\infty} b_n \sin(n\omega_0 t)}$$

$$= a_0 + a_1\cos(\omega_0 t) + a_2\cos(2\omega_0 t) + \cdots$$

$$+ b_1\sin(\omega_0 t) + b_2\sin(2\omega_0 t) + \cdots$$

$$= \underset{\text{term}}{\text{dc}} + \underbrace{\underset{\text{frequency terms}}{\text{fundamental}} + \underset{\text{terms}}{\text{harmonics}}}_{=\ \text{ac term}}$$

Even-part and Odd-part FS Representations

Consider an arbitrary signal $x(t)$, it can be decomposed as:

$$x(t) = x_e(t) + x_o(t) = \underbrace{\frac{1}{2}\left[x(t) + x(-t)\right]}_{= x_e(t)} + \underbrace{\frac{1}{2}\left[x(t) - x(-t)\right]}_{= x_o(t)}$$

$$x(t) = \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t}_{\text{Even}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin n\omega_0 t}_{\text{Odd}}$$

$$\therefore \quad x_e(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t \quad \text{and} \quad x_o(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

Is FS representation still periodic?

Since $x(t)$ is periodic, $x(t+T_0)=x(t)$

$$\boxed{x(t+T_0)}=\sum_{n=0}^{\infty}a_n\cos\left[n\omega_0(t+T_0)\right]+\sum_{n=0}^{\infty}b_n\sin\left[n\omega_0(t+T_0)\right]$$

$$=\sum_{n=0}^{\infty}a_n\cos\left[n\omega_0 t+n\omega_0 T_0\right]+\sum_{n=0}^{\infty}b_n\sin\left[n\omega_0 t+n\omega_0 T_0\right]$$

$$\because \quad n\omega_0 T_0=n\cdot\frac{2\pi}{T_0}\cdot T_0=n\cdot 2\pi$$

Try to answer such fundamental question in **math**, not in English!

$$=\sum_{n=0}^{\infty}a_n\cos\left(n\omega_0 t+n\cdot 2\pi\right)+\sum_{n=0}^{\infty}b_n\sin\left(n\omega_0 t+n\cdot 2\pi\right)$$

$$=\sum_{n=0}^{\infty}a_n\cos\left(n\omega_0 t\right)+\sum_{n=0}^{\infty}b_n\sin\left(n\omega_0 t\right)=\boxed{x(t)}$$

All AC Sub-terms are *harmonically related* !

- Fourier series frequency spectrum is ***discrete***! $\boxed{\omega_n = n\,\omega_0}$.
- When $n = 1,\ \omega_1 = 1 \cdot \omega_0$ is called the *1st* harmonic (i.e., the *fundamental frequency*).
- When $n = 2,\ \omega_2 = 2 \cdot \omega_0$ is called the *2nd* harmonic.
- When $n = 3,\ \omega_3 = 3 \cdot \omega_0$ is called the *3rd* harmonic.

  …

- When $n = k,\ \omega_k = k \cdot \omega_0$ is called the *k*-th harmonic.

➔ All these terms are called ***harmonically related***, as they share a common factor ω_0!

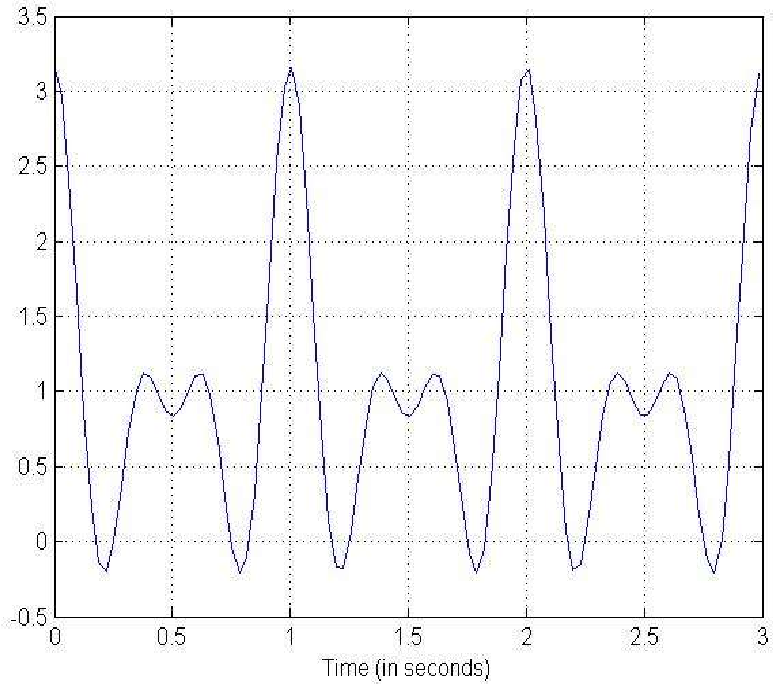
Your First FS Example ☺

$$x(t) = 1 + \frac{1}{2}\cos\omega_0 t + \cos 2\omega_0 t + \frac{2}{3}\cos 3\omega_0 t$$

(a). What are the frequency components contained in x(t)?

(b). Find the Fourier series coefficients, a_0, a_n, and b_n?

(c). How much ***contribution*** from each frequency component?



Re-do parts (a), (b), and (c) for the following signals:

(d). $y(t) = 1 + \frac{1}{2}\sin\omega_0 t + \sin 2\omega_0 t + \frac{2}{3}\sin 3\omega_0 t$

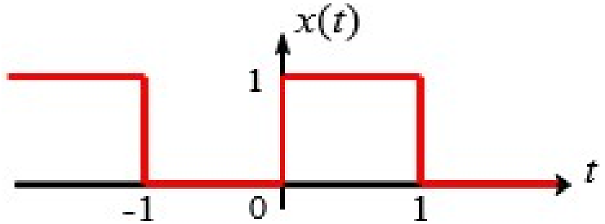
(e). $w(t) = \frac{1}{2}\cos\omega_0 t + \frac{2}{3}\cos 3\omega_0 t$

(f). $r(t) = 1 + \cos 2\omega_0 t + \frac{2}{3}\cos 3\omega_0 t$ (What is the 1st harmonic?)

(g). How many harmonic **terms**, and their **contributions**, contained within each of the above-mentioned signal, respectively?

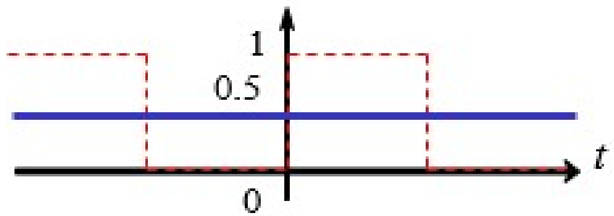
Further Intuition …

Example: A periodic square wave



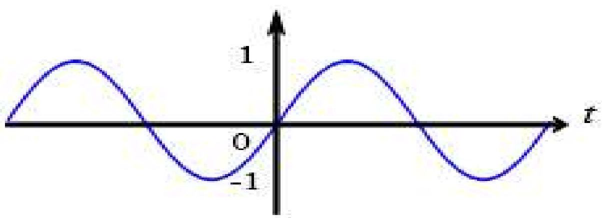
Just get intuition at this stage! We will show how to compute each term later.

- The dc captures the "big picture" (or average) of the signal $x(t)$:



$$x_{\mathrm{dc}}(t) = 0.5$$

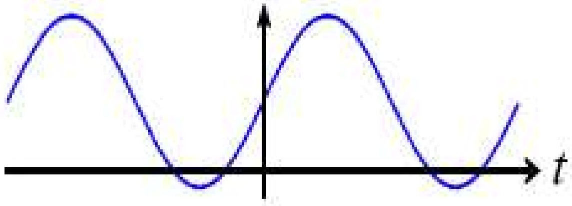
- The first ac harmonic terms fine tunes the details of the signal $x(t)$:



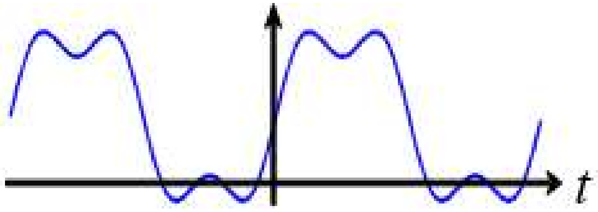
$$x_{\text{ac-1}}(t) = \frac{2}{\pi}\sin\left(\omega_0 t\right)$$

(Continue)

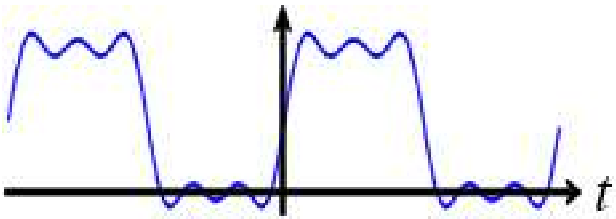
$x_N(t)$: there are N terms are added together.



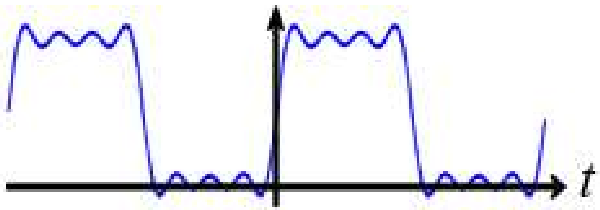
$$x(t) \approx x_2(t) = x_{dc}(t) + x_{ac\text{-}1}(t) = 0.5 + \frac{2}{\pi}\sin(\omega_0 t)$$



$$x(t) \approx x_3(t) = x_{dc}(t) + x_{ac\text{-}1}(t) + x_{ac\text{-}2}(t)$$
$$= 0.5 + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t)$$



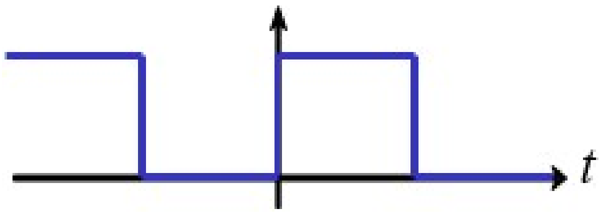
$$x(t) \approx x_4(t) = x_{dc}(t) + x_{ac\text{-}1}(t) + x_{ac\text{-}2}(t) + x_{ac\text{-}3}(t)$$
$$= 0.5 + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \frac{2}{5\pi}\sin(5\omega_0 t)$$



$$x(t) \approx x_5(t) = x_{dc}(t) + x_{ac\text{-}1}(t) + x_{ac\text{-}2}(t) + x_{ac\text{-}3}(t) + x_{ac\text{-}4}(t)$$

$$= 0.5 + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \frac{2}{5\pi}\sin(5\omega_0 t)$$

$$+ \frac{2}{7\pi}\sin(7\omega_0 t)$$



$x(t) = x_\infty(t)$, after all terms being added together.

Basis Set: A Framework to Build $x(t)$

Trigonometric Fourier Series Form

$$\boxed{x(t) = \sum_{n=0}^{\infty} a_n \cos\left(n\,\omega_0\, t\right) + \sum_{n=0}^{\infty} b_n \sin\left(n\,\omega_0\, t\right)}$$

Its basis set (i.e., a set of bases) is:

$$\left\{\, 1,\ \cos\omega_0 t,\ \cos 2\omega_0 t,\ \cos 3\omega_0 t, \ldots,\ \sin\omega_0 t,\ \sin 2\omega_0 t,\ \sin 3\omega_0 t, \ldots \,\right\}$$

*Think about the rectangular (i.e., Cartesian) coordinates that you had learnt before!

Amplitude-Phase (A.-P.) Fourier Series Form

$$x(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n) \qquad \text{[Synthesis Equation]}$$

$$\text{where } A_n = \sqrt{a_n^{\;2} + b_n^{\;2}} \;;\;\; \theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) \qquad \text{[Analysis Equations]}$$

- This can be derived from the trigonometric FS form (refer to p. 21)

$$x(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=0}^{\infty} b_n \sin(n\omega_0 t)$$

- Coefficients a_n and b_n are real numbers!
- You need to get a_n and b_n in order to arrive at A.-P. FS form
- All n are positive integers only!
- The A.-P. FS form has only **one**-sided plot!

Complex-Exponential (C.-E.) Fourier Series Form

We can build this from the amplitude-phase form

Recall that, by Euler's identities, we have

$$A\cos(\omega t+\theta)=A\left(\frac{e^{j(\omega t+\theta)}+e^{-j(\omega t+\theta)}}{2}\right)=\left(\frac{A}{2}e^{j\theta}\right)\cdot e^{j\omega t}+\left(\frac{A}{2}e^{-j\theta}\right)\cdot e^{-j\omega t}$$

By looking at the AC component of amplitude-phase FS : (for *n* positive)

$$A_n\cos(n\omega_0 t+\theta_n)=A_n\left(\frac{e^{j(n\omega_0 t+\theta_n)}+e^{-j(n\omega_0 t+\theta_n)}}{2}\right)$$

$$=\underbrace{\left(\frac{A_n}{2}e^{j\theta_n}\right)}_{=c_n}\cdot e^{j(n)\omega_0 t}+\underbrace{\left(\frac{A_n}{2}e^{-j\theta_n}\right)}_{=c_{-n}}\cdot e^{j(-n)\omega_0 t}$$

Complex-Exponential (C.-E.) Fourier Series Form

The **bases** of the complex-exponential FS form:

$$\left\{ \ldots, e^{-j3\omega_0 t}, e^{-j2\omega_0 t}, e^{-j1\omega_0 t}, e^{j0\omega_0 t} (=1), e^{j1\omega_0 t}, e^{j2\omega_0 t}, e^{j3\omega_0 t}, \ldots \right\}$$

which can be expressed more compactly as $\left\{ e^{jn\omega_0 t} \right\}$ for $n \in$ *all* integer.

The Basis Set - A Framework to Build Periodic x(t)

Complex-Exponential (C.-E.) Fourier Series Form

$$x(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n) \quad \text{[Amplitude-Phase form]}$$

$$= c_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} c_n e^{jn\omega_0 t}, \text{ where } c_n = \frac{A_n}{2} e^{j(\theta_n)} \text{ and } c_{-n} = \frac{A_n}{2} e^{j(-\theta_n)}$$

The Basis Set - A Framework to Build Periodic x(t)

You can also derive this complex exponential form from the trigonometric form as

$$x(t) = \sum_{n=0}^{\infty} a_n \cos\left(n\omega_0 t\right) + \sum_{n=0}^{\infty} b_n \sin\left(n\omega_0 t\right)$$

$$= \sum_{n=0}^{\infty} a_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right) + \sum_{n=0}^{\infty} b_n \left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}\right)$$

$$= \sum_{n=0}^{\infty} a_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right) + \sum_{n=0}^{\infty} j\, b_n \left(\frac{-e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right)$$

$$= \sum_{n=0}^{\infty} \underbrace{\left(\frac{a_n - j\, b_n}{2}\right)}_{=c_n} \cdot e^{j(n)\omega_0 t} + \sum_{n=0}^{\infty} \underbrace{\left(\frac{a_n + j\, b_n}{2}\right)}_{=c_{-n}} \cdot e^{j(-n)\omega_0 t}$$

Note: n positive integer only!

The Basis Set - A Framework to Build Periodic x(t)

- In such **complex exponential Fourier Series** representation, the Fourier Series (FS) coefficients are **complex** numbers, including real numbers, which you should view them as $a + j0$; hence zero phase, if $a > 0$; or, 180-degrees, if $a < 0$. (Why?)

- Due to the complex number, we can have its magnitude and phase information. Hence, we can plot them separately; thus,
  - The magnitude (or amplitude) spectrum ➡ *even* function
  - The phase spectrum ➡ *odd* function

  Together, they are referred to as discrete frequency spectra or line spectra

- For c_n, it is a **two**-sided plot

Complex Exponential Form ➔ Trigonometric Form

$$x(t)=\sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{-1} c_n e^{jn\omega_0 t} + c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t}$$

$$= c_0 + \sum_{n=1}^{\infty}\left(c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t}\right)$$

Using Euler's identity: $e^{\pm jn\omega_0 t} = \cos n\omega_0 t \pm j \sin n\omega_0 t$

$$x(t) = c_0 + \sum_{n=1}^{\infty}\left[\left(c_n + c_{-n}\right)\cos n\omega_0 t + j\left(c_n - c_{-n}\right)\sin n\omega_0 t\right]$$

Complex Exponential Form ➔ Trigonometric Form

Compare with the trigonometric FS expansion form:

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right]$$

$$\Rightarrow \boxed{a_0 = c_0 \qquad a_n = c_n + c_{-n} \qquad b_n = j\left(c_n - c_{-n}\right)}$$

[Check]: Obviously, $c_0 = a_0$, as constant can only be equaled to constant.

$$a_n - jb_n = \left(c_n + c_{-n}\right) - j\left[j\left(c_n - c_{-n}\right)\right] = \left(c_n + c_{-n}\right) + \left(c_n - c_{-n}\right) = 2c_n$$

$$\Rightarrow \boxed{c_n = \frac{a_n - jb_n}{2}}$$

(Note: n positive integer only!)

Three Mathematically-equivalent Fourier Series Forms

- Complex-exponental:

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}, \quad \text{where } \omega_0 = \frac{2\pi}{T_0} \qquad \text{[Synthesis Equation]}$$

$$c_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt \qquad \text{[Analysis Equation]}$$

- Trigonometric Form:

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right\} \qquad \text{[Syn. Eq.]}$$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\, dt \qquad \text{[Analysis Equations]}$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\, dt$$

- Amplitude-phase:

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n) \qquad \text{[Synthesis Equation]}$$

$$A_n = \sqrt{a_n^2 + b_n^2}\ ; \quad \theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) \qquad \text{[Analysis Equations]}$$

$\boxed{\text{DC}: a_0 = A_0 = c_0}$; $\boxed{\text{AC}: \quad c_n = \frac{a_n - jb_n}{2}; \quad c_n = \frac{A_n e^{j\theta_n}}{2} \quad (\text{where } n \in \mathbb{N}+)}$; $\boxed{c_{-n} = c_n^*}$

Three Mathematically-equivalent Fourier Series Forms

Example: Find the FS coefficients of $x_1(t) = \cos\omega_0 t$ and $x_2(t) = \sin\omega_0 t$.

By Euler's identities, we have

$$x_1(t) = \cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$= \underbrace{\left(\frac{1}{2}\right)}_{\text{FS coefficient}} \cdot e^{j(\omega_0)t} + \underbrace{\left(\frac{1}{2}\right)}_{\text{FS coefficient}} \cdot e^{j(-\omega_0)t}$$

$$\boxed{\cos\phi = \frac{e^{j\phi} + e^{-j\phi}}{2} \qquad \sin\phi = \frac{e^{j\phi} - e^{-j\phi}}{j2}}$$

[Euler's identities]

By looking up the complex exponential form, the coefficients are 1/2 contributing from $-\omega_0$ and another 1/2 from ω_0.

Note: The (real) cosine signal, in fact, has two components, one is at the <u>postive</u> frequency ω_0 and the other at the <u>negative</u> frequency $-\omega_0$ with contribution 1/2 (in magnitute) from each.

Three Mathematically-equivalent Fourier Series Forms

Likewise, for the sine signal, we have

$$x_2(t) = \sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{j2}$$

$$= \underbrace{\left(\frac{1}{j2}\right)}_{\text{FS coefficient}} \cdot e^{j(\omega_0)t} + \underbrace{\left(\frac{-1}{j2}\right)}_{\text{FS coefficient}} \cdot e^{j(-\omega_0)t}$$

$$\boxed{\cos\phi = \frac{e^{j\phi} + e^{-j\phi}}{2} \qquad \sin\phi = \frac{e^{j\phi} - e^{-j\phi}}{j2}}$$

[Euler's identities]

Note: The (real) sine signal also has two components, one is at the <u>postive</u> frequency ω_0 and the other is at the <u>negative</u> frequency $-\omega_0$. But, its FS coefficients are different, compared with that of the cosine case.

Three Mathematically-equivalent Fourier Series Forms

Example: Find the FS representation of $x_3(t) = A\cos\left(\omega_0 t + \theta\right)$ and $x_4(t) = A\sin\left(\omega_0 t + \theta\right)$.

By Euler's identities, we have

$$x_3(t) = A\cos\left(\omega_0 t + \theta\right) = A\left(\frac{e^{j\left(\omega_0 t+\theta\right)} + e^{-j\left(\omega_0 t+\theta\right)}}{2}\right)$$

$$= \frac{A}{2}e^{j\theta}\cdot e^{j\omega_0 t} + \frac{A}{2}e^{-j\theta}\cdot e^{-j\omega_0 t}$$

$$= \underbrace{\left(\frac{A}{2}e^{j\theta}\right)}_{\text{FS coefficient}}\cdot e^{j\left(\omega_0\right)t} + \underbrace{\left(\frac{A}{2}e^{-j\theta}\right)}_{\text{FS coefficient}}\cdot e^{j\left(-\omega_0\right)t}$$

$$\boxed{\begin{aligned}\cos\phi &= \frac{e^{j\phi}+e^{-j\phi}}{2}\\ \sin\phi &= \frac{e^{j\phi}-e^{-j\phi}}{j2}\end{aligned}}$$

[Euler's identities]

Three Mathematically-equivalent Fourier Series Forms

Likewise, for the sine signal, we have

$$x_4(t) = A\sin\left(\omega_0 t + \theta\right) = A\left(\frac{e^{j\left(\omega_0 t+\theta\right)} - e^{-j\left(\omega_0 t+\theta\right)}}{j2}\right)$$

$$= \frac{A}{j2}e^{j\theta}\cdot e^{j\omega_0 t} - \frac{A}{j2}e^{-j\theta}\cdot e^{-j\omega_0 t} = \underbrace{\left(\frac{A}{j2}e^{j\theta}\right)}_{\text{FS coefficient}}\cdot e^{j\left(\omega_0\right)t} + \underbrace{\left(\frac{-A}{j2}e^{-j\theta}\right)}_{\text{FS coefficient}}\cdot e^{j\left(-\omega_0\right)t}$$

Fourier Series Exercises

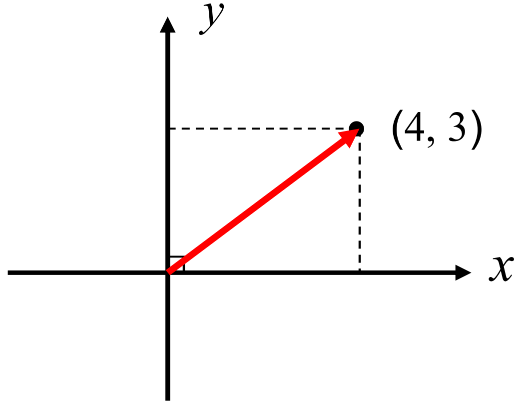
Examples: Revisit the previous examples and identify the FS coefficients (i.e., contribution) according to the **trigonometric** form:

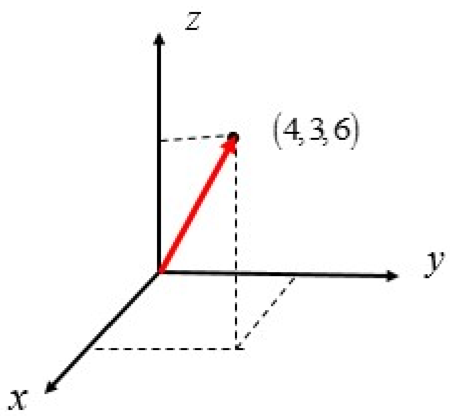
(i). $x(t) = 1 + \frac{1}{2}\cos\omega_0 t + \cos 2\omega_0 t + \frac{2}{3}\cos 3\omega_0 t$

(ii). $x(t) = 0.5 + \frac{2}{\pi}\sin\left(\omega_0 t\right) + \frac{2}{3\pi}\sin\left(3\omega_0 t\right) + \frac{2}{5\pi}\sin\left(5\omega_0 t\right) + \frac{2}{7\pi}\sin\left(7\omega_0 t\right) + \ldots$

Orthgonality

- Simply put, “orthogonal” = “perpendicular” (i.e., 90^{o})
- To convince us about the FS representation, we need to recap a concept which we are already quite familiar with — the rectangular coordinate, as follows:





➔ …

2-D 3-D ➔ n-D

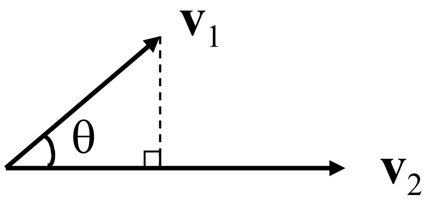
- Each coordinate component 4, 3, and 6 is the **<u>projection</u>** amount projected onto its corresponding axis. These magnitudes bear the same ‘sense’ and meaning of “<u>Fourier series coefficients</u>.” We can extend this concept further, as follows.

Vectors vs Signals

Vectors

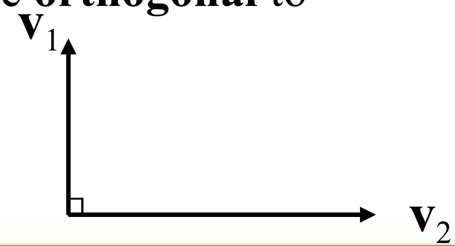
Dot product measurement:

$$\boxed{v_1 \bullet v_2 = |v_1||v_2|\cos\theta}$$



➔ Represent how similar of $\mathbf{v}_1$ with respect to $\mathbf{v}_2$.

If $\theta =$ **90º**, then two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are **orthogonal** to each other.



Signals

Cross-correlation measurement:

$$\boxed{\int_a^b \phi_m(t)\, \phi_k^*(t)\, dt = \begin{cases} 0, & m \neq k \\ \alpha, & m = k \end{cases}}$$

where ϕ_m and ϕ_k are taken from

$$\left\{ e^{\pm j n \omega_0 t}, n \in I \right\}.$$

If cross-correlation is **zero**, then two bases ϕ_m and ϕ_k are **orthogonal** (i.e., not correlated) to each other, over an interval [a, b].

➔ The complex-valued "Fourier series coefficients" are the projected amounts onto axis ϕ_n, indicating how much contribution amount are coming from harmonics ($n\ \omega_0 = \omega_n$).

The "Building Blocks" of Fourier Series Representation

- The (real) sinusoidal signal set

  $$\left\{ 1, \cos\left(\omega_0 t\right), \cos\left(2\omega_0 t\right), \ \ldots, \sin\left(\omega_0 t\right), \sin\left(2\omega_0 t\right), \ \ldots \right\}$$

  or, expressed more compactly as $\left\{ \cos(n\,\omega_0\,t), \sin(n\,\omega_0\,t) \right\}$,

  for $n \in$ *positive* integer (including $n = 0$).

  $\Rightarrow$ These sinusoidals are *harmonically related*.

- The complex exponential signal set

  $$\left\{ \ldots, e^{-j3\omega_0 t}, e^{-j2\omega_0 t}, e^{-j1\omega_0 t}, e^{j0\omega_0 t} (= 1), e^{j1\omega_0 t}, e^{j2\omega_0 t}, e^{j3\omega_0 t}, \ldots \right\}$$

  which can be expressed more compactly as $e^{\pm jn\omega_0 t}$, for $n \in I$.

  $\Rightarrow$ These complex exponentials are *harmonically related*.

➔ We shall start with the **complex exponential** "building-block" set as our mathematical **basis** to build Fourier series representation.

All Basis Functions Are Orthogonal to Each Other!

- For the complex exponential harmonically related set:

$$\int_0^{T_0} \left(e^{jm\omega_0 t}\right)\left(e^{jk\omega_0 t}\right)^* dt = \begin{cases} 0, & m \neq k \\ T_0, & m = k \end{cases}$$

The complex exponential signal set is orthogonal over one period T_0.

- For the real sinusoidal harmonically related set, it is also orthogonal over one period T_0, since

$$\int_0^{T_0} (1)\cdot\left(\cos m\omega_0 t\right) dt = \int_0^{T_0} \cos m\,\omega_0 t\; dt = 0,\ \text{for all } m \text{ (except } m \neq 0)$$

$$\int_0^{T_0} (1)\cdot\left(\sin m\omega_0 t\right) dt = \int_0^{T_0} \sin m\,\omega_0 t\; dt = 0,\ \text{for all } m$$

All Basis Functions Are Orthogonal to Each Other!

$$\int_0^{T_0} \left(\sin m\,\omega_0 t\right)\cdot\left(\cos k\,\omega_0 t\right) dt = 0, \text{ for all } m \text{ and } k \text{ (including when } m = k)$$

$$\int_0^{T_0} \left(\cos m\,\omega_0 t\right)\cdot\left(\cos k\,\omega_0 t\right) dt = \begin{cases} 0, & m \neq k \\ \dfrac{T_0}{2}, & m = k \end{cases}$$

$$\int_0^{T_0} \left(\sin m\,\omega_0 t\right)\cdot\left(\sin k\,\omega_0 t\right) dt = \begin{cases} 0, & m \neq k \\ \dfrac{T_0}{2}, & m = k \end{cases}$$

How to determine the Fourier coefficients c_n ?

$\Rightarrow$ Projecting $x(t)$ on the kth "axis" (or basis):

$$\text{Projection amount} = \int_{T_0} x(t)\left(e^{jk\omega_0 t}\right)^* dt = \int_{T_0} x(t)\, e^{-jk\omega_0 t} dt$$

$$= \int_{T_0} \left(\sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} \right) \cdot e^{-jk\omega_0 t} dt$$

$$= \sum_{n=-\infty}^{+\infty} c_n \cdot \left(\int_{T_0} e^{j(n-k)\omega_0 t} dt \right)$$

- Case (1): If $\boxed{n \neq k}$, then $n-k$ is just an integer (say, $m = n-k$), and thus,

$$\int_{T_0} e^{jm\omega_0 t} dt = \int_{T_0} \cos(m\ \omega_0 t)\, dt + j \int_{T_0} \sin(m\ \omega_0 t)\, dt = 0 + j0 = 0.$$

How to determine the Fourier coefficients c_n ?

- Case (2): If $\boxed{n = k}$, then

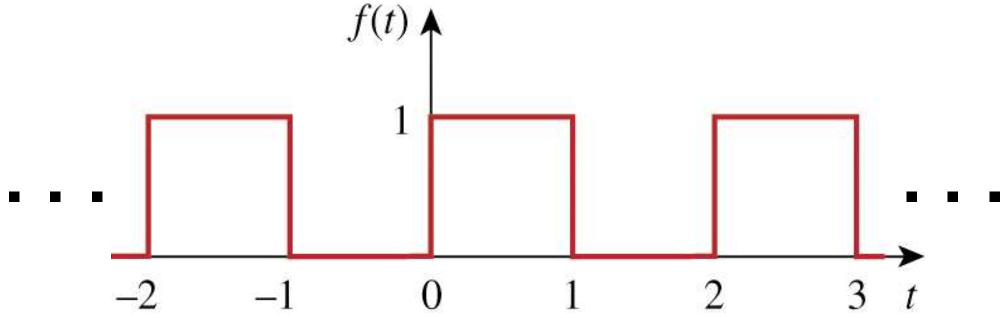
$$\int_{T_0} e^{j(0)\,\omega_0\,t}\,dt = \int_{T_0} dt = T_0$$

$$\sum_{n=-\infty}^{+\infty} c_n \cdot \int_{T_0} e^{j(n-k)\,\omega_0\,t}\,dt = \ldots + c_{k-2}\cdot 0 + c_{k-1}\cdot 0 + c_k \cdot T_0 + c_{k+1}\cdot 0 + c_{k+2}\cdot 0 + \ldots = c_k T_0 = c_n T_0$$

$$\Rightarrow \quad \boxed{c_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-j\,n\,\omega_0\,t}\,dt}$$

[Analysis Equation]

Fourier Series Example



Determine the Fourier series of the rectangular pulse train waveform shown above using:

(a). the **complex exponential** form;

(b). the **trigonometric** form;

(c). the **amplitude-phase** form;

(d). Plot the amplitude and phase spectra;

(e). Plot its approximated square waveform using the first few terms (i.e., the truncated FS representation).

Solution

(a). $T_0 = 2 \;\Rightarrow\; \omega_0 = 2\pi/2 = \pi$

DC: $c_0 = \dfrac{1}{T_0}\displaystyle\int_0^{T_0} f(t)\,dt = \frac{1}{2}\left[\int_0^1 1\,dt + \int_1^2 0\,dt\right] = \frac{1}{2};$

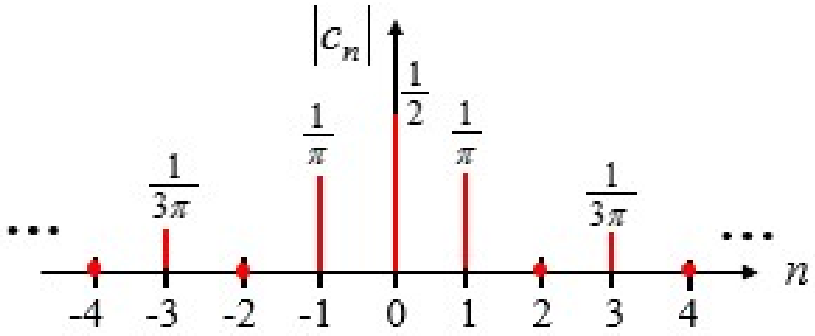
AC: $c_n = \dfrac{1}{2}\displaystyle\int_0^1 1\cdot e^{-jn\pi t}dt = \left(\frac{1}{2}\right)\left(\frac{1}{-jn\pi}\right)\left[e^{-jn\pi t}\Big|_0^1\right] = \frac{1}{-j2n\pi}\left(e^{-jn\pi}-1\right) = \begin{cases} j\dfrac{-1}{n\pi}, & n = \text{odd;} \\ 0, & n = \text{even.}\end{cases}$

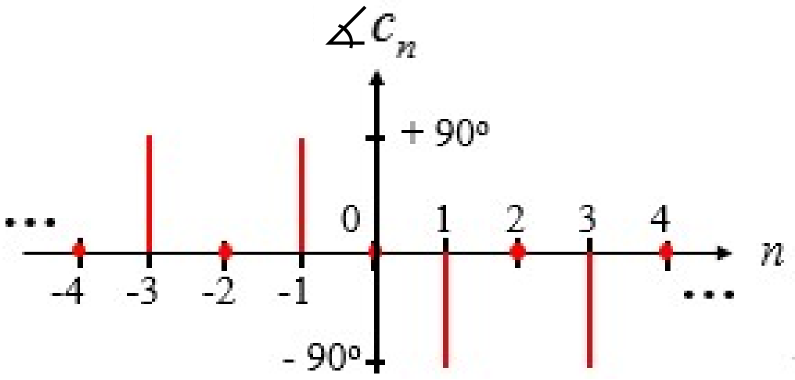
Note that $e^{-jn\pi} = 1$, for n = even. For n = odd, $e^{-jn\pi} = -1$.

$\Rightarrow |c_n| = \begin{cases} \dfrac{1}{|n|\pi}, & n = \text{odd;} \\ 0, & n = \text{even.}\end{cases} \qquad \angle c_n = \tan^{-1}\dfrac{\left(\dfrac{-1}{n\pi}\right)}{0} = \begin{cases} -90^\circ, & n = \text{positive odd;} \\ +90^\circ, & n = \text{negative odd;} \\ 0^\circ, & n = \text{even.}\end{cases}$

Solution

$$f(t) = \frac{1}{2} - j\frac{1}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0;\ n=\text{odd}}}^{\infty} \frac{1}{n} e^{j n \pi t} \quad \text{or} \quad f(t) = \frac{1}{2} - j\frac{1}{\pi} \sum_{m=-\infty}^{\infty} \frac{1}{(2m+1)} e^{j(2m+1)\pi t}$$





Solution

(b). $T_0 = 2$, $\omega_0 = \dfrac{2\pi}{T_0} = \pi$, and $a_0 = \dfrac{1}{2}$ (via inspection of rectangular area) or

DC: $$a_0 = \frac{1}{T_0}\int_{T_0} f(t)\,dt = \frac{1}{2}\left[\int_0^1 1\,dt + \int_1^2 0\,dt\right] = \frac{1}{2}t\Big|_0^1 = \frac{1}{2}$$

AC: $$a_n = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(n\,\omega_0 t)\,dt = \frac{2}{2}\left[\int_0^1 1\cdot\cos(n\,\pi t)\;dt + \int_1^2 0\cdot\cos(n\,\pi t)\;dt\right]$$

$$= \frac{1}{n\pi}\sin n\,\pi t\,\Big|_0^1 \quad = \frac{1}{n\pi}\sin n\,\pi = 0$$

Solution

(b). $b_n = \frac{2}{T_0}\int_0^{T_0} f(t)\sin(n\,\omega_0 t)\,dt = \frac{2}{2}\left[\int_0^1 1\cdot\sin(n\pi t)\,dt + \int_1^2 0\cdot\sin(n\pi t)\,dt\right]$

$$= \frac{-1}{n\pi}\cos n\pi t\,\Big|_0^1 \quad = \frac{-1}{n\pi}(\cos n\pi - 1)$$

Since $\cos n\pi = (-1)^n$, $b_n = \frac{1}{n\pi}(1-\cos n\pi) = \begin{cases} \frac{2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{1}{n}\sin(n\pi t) \qquad \text{or} \qquad f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)}\sin\left[(2m-1)\pi t\right]$$

Solution

(c).

$$A_n = \sqrt{a_n^{\,2} + b_n^{\,2}} = \begin{cases} \dfrac{2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

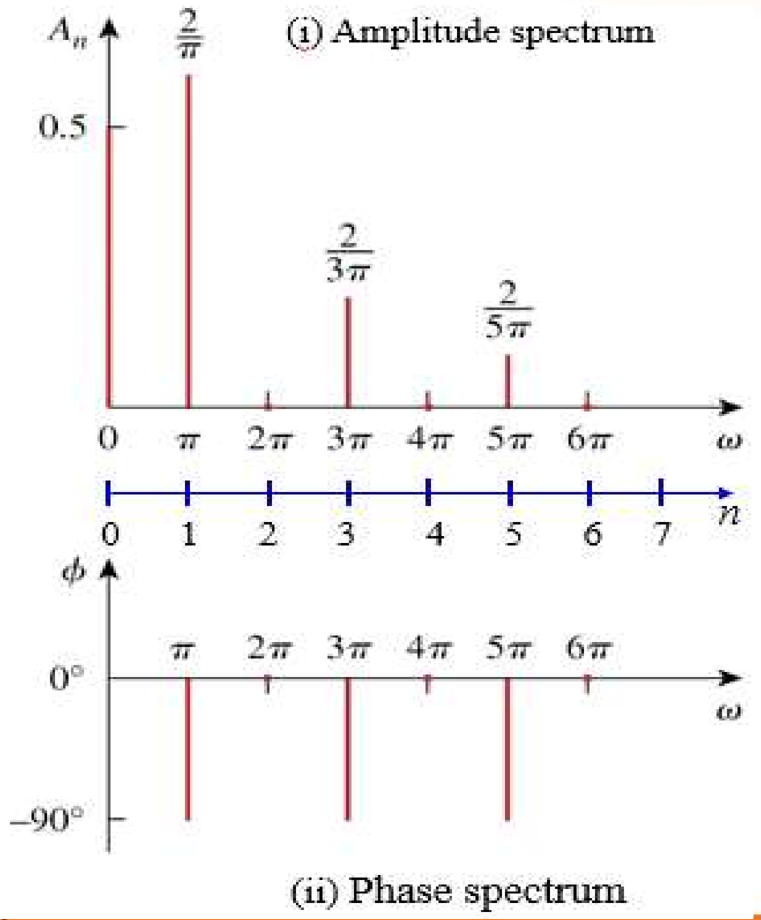
$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = \tan^{-1}\left(\frac{-b_n}{0}\right)$$

$$= \begin{cases} -90°, & n = \text{odd} \\ 0°, & n = \text{even} \end{cases}$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{\substack{n=1 \\ n = \text{odd}}}^{\infty} \frac{1}{n}\cos(n\pi t - 90°)$$

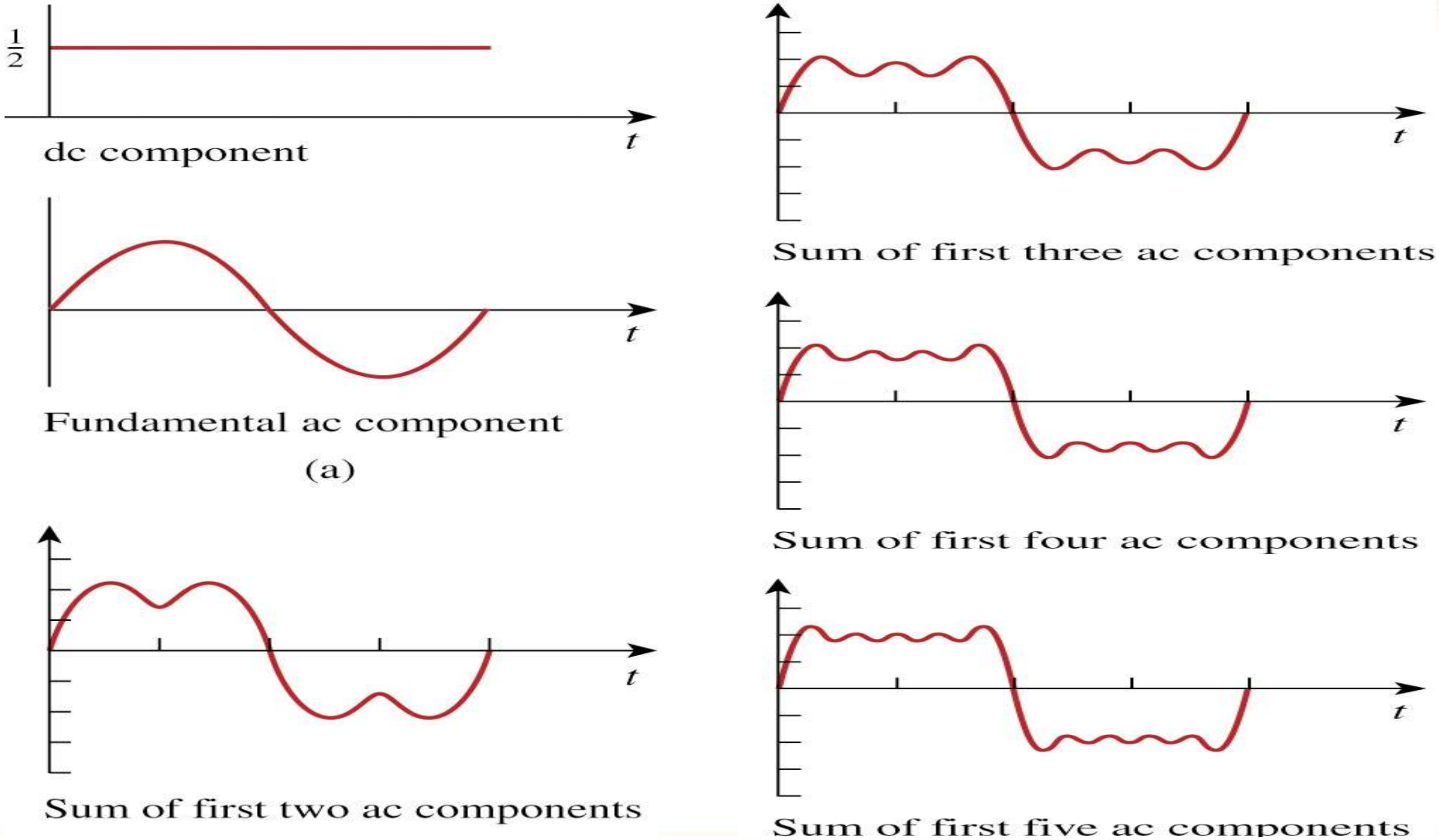
$$= \frac{1}{2} + \frac{2}{\pi}\sum_{\substack{n=1 \\ n = \text{odd}}}^{\infty} \frac{1}{n}\sin(n\pi t)$$

or $$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{1}{(2m-1)}\sin\left[(2m-1)\pi t\right]$$



Solution

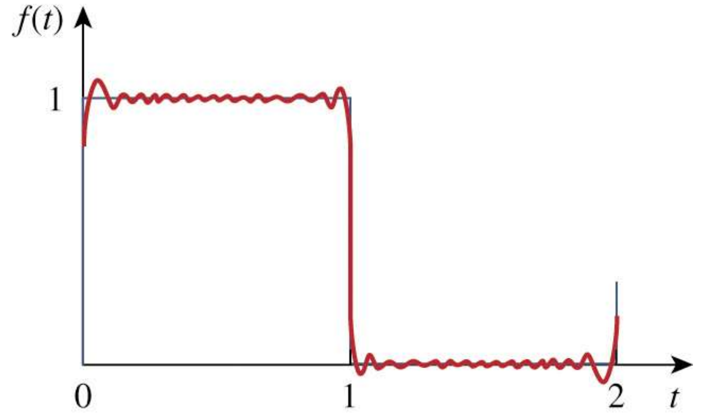
(d).



$\frac{1}{2}$
dc component
t
Fundamental ac component
t
(a)
Sum of first two ac components
t
Sum of first three ac components
t
Sum of first four ac components
t
Sum of first five ac components
t

Solution

(e).



N = 11

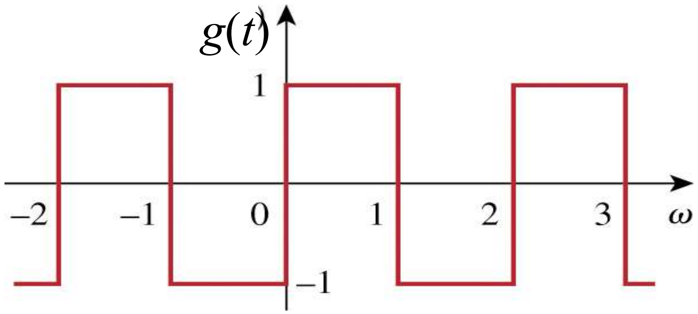
(for one period plot only)

Remarks:

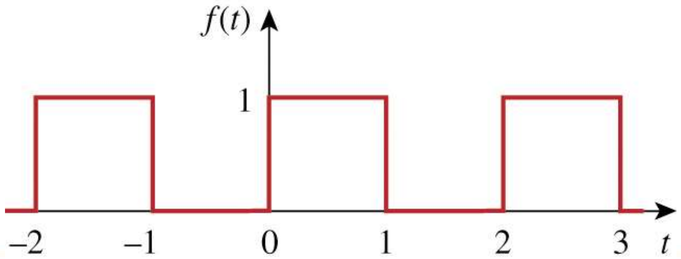
- We observe how superposition of the terms (from N =1 to N = 6, and to N=11) can evolve into the original square wave. The more terms you add, the better approximation towards the square wave.
- We notice that the partial sum oscillates above and below the actual value of $f(t)$. At the discontinuity (t = 0, 1, 2, …), overshoot (about 9% of the peak value) and damped oscillation present. This is called the ***Gibbs*' phenomenon**, incurred at the points of discontinuity (i.e., with amplitude jump).

Fourier Series Example

Based on the CTFS results of the signal $f(t)$ obtained in the previous example, find the Fourier series of the following square wave $g(t)$ and plot its amplitude and phase spectra.



PS: Re-plot $f(t)$ from the previous example here for comparison



Solution

Through their graphs, it is easy to observe that the relationship between the signal $g(t)$ and the signal $f(t)$ is: $g(t) = [\, f(t) - ½ \,] \times 2$. Therefore, you can simply manipulate the CTFS result of $f(t)$ obtained via two simple steps:

Step 1):

$$\text{Since } f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left[(2n-1)\pi t\right]$$

$$f(t) - \frac{1}{2} = \left\{\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left[(2n-1)\pi t\right]\right\} - \frac{1}{2}$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left[(2n-1)\pi t\right]$$

Solution

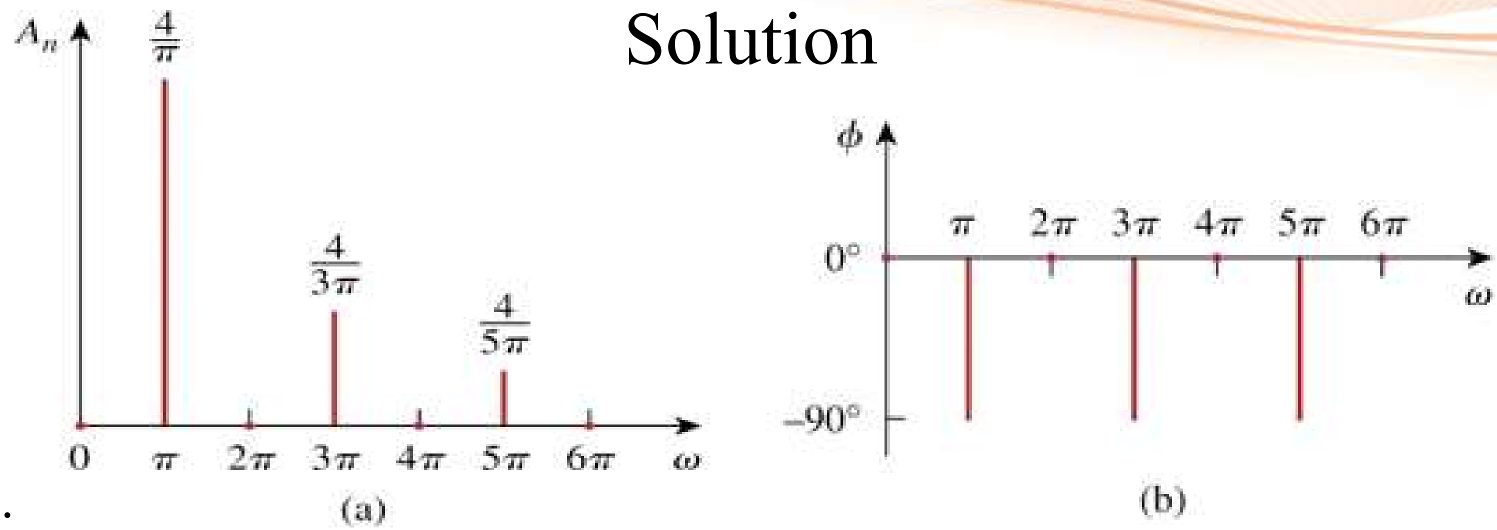
Through their graphs, it is easy to observe that the relationship between the signal $g(t)$ and the signal $f(t)$ is: $g(t) = [f(t) - ½] \times 2$. Therefore, you can simply manipulate the CTFS result of $f(t)$ obtained via two simple steps:

Step 2):

$$g(t) = 2 \times \left\{ f(t) - \frac{1}{2} \right\} = 2 \times \left\{ \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin\left[(2n-1)\pi t \right] \right\}$$

$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin\left[(2n-1)\pi t \right]$$

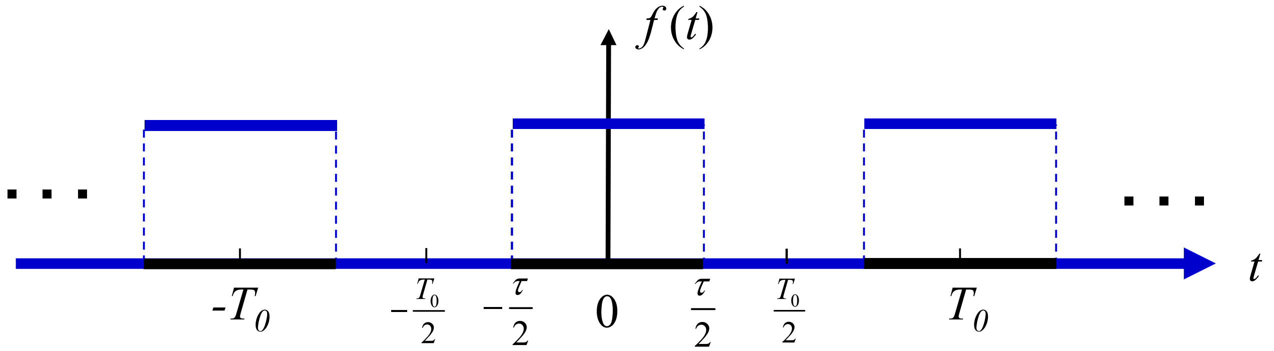
Solution



Remarks:

- An addition of a constant amount (say, a fixed voltage) onto a signal only results in the same amount being added to the dc term, as expected. (PS: "addition" could be a negative amount.)
- An amplitude scaling by a constant factor onto a signal will affect the amplitude of all the dc and ac terms in the same way. However, the phase part is unchanged!
- Since $g(t) = [\,f(t) - ½\,] \times 2 = 2\,f(t) - 1$, you can do "× 2" first and then by adding "– 1".

Duty Cycle



The pulse train has a pulse width τ, over one fundamental period T_0.

In electronics, their ratio is defined as the *duty cycle* (often in %); i.e.,

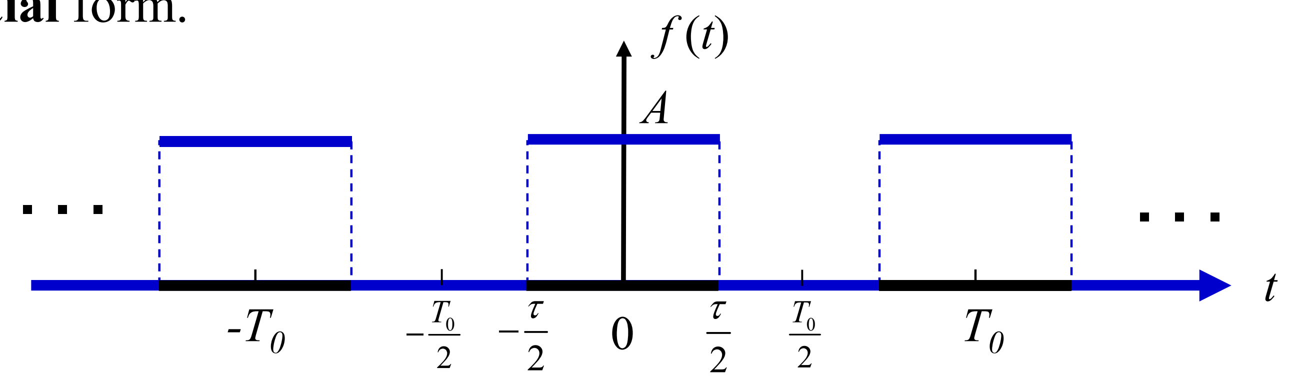
$$\boxed{\text{duty cycle} = \frac{\tau}{T_0}} \qquad \left(\text{or} \quad \frac{100 \times \tau}{T_0} \% \right)$$

For example,

(a). $\frac{\tau}{T_0} = \frac{1}{2} = 50\%$; and (b). $\frac{\tau}{T_0} = \frac{1}{4} = 25\%$.

Example

Determine the Fourier series of the rectangular pulse train $f(t)$ using the **complex-exponential** form.



Plot the amplitude spectrum and the phase specturm of the following cases, respectively:

(a). $A = 1,\ T_0 = 1,\ \ \tau = 0.5$

(b). $A = 1,\ T_0 = 1,\ \ \tau = 0.25$

Solution

$$f(t)=\sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}, \text{ where } \omega_0=\frac{2\pi}{T_0}$$

$$c_0=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,dt=\frac{A}{T_0}\int_{-\tau/2}^{\tau/2} 1\,dt=\frac{A\tau}{T_0} \quad (\text{as expected! } \because \text{ rectangular pulse area} = A\tau)$$

$$c_n=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jn\omega_0 t}\,dt=\frac{A}{T_0}\int_{-T_0/2}^{T_0/2} e^{-jn\omega_0 t}\,dt$$

$$=\frac{A}{T_0}\int_{-\tau/2}^{\tau/2} e^{-jn\omega_0 t}dt=\frac{A}{T_0}\cdot\frac{1}{-jn\omega_0}\cdot e^{-jn\omega_0 t}\Big|_{-\tau/2}^{\tau/2}$$

$$=\frac{A}{\cancel{T}_0}\cdot\frac{\cancel{T}_0}{\cancel{-j2}\pi n}\cdot\underbrace{\left(e^{-jn\omega_0\frac{\tau}{2}}-e^{jn\omega_0\frac{\tau}{2}}\right)}_{=-j2\sin n\omega_0\frac{\tau}{2}=\cancel{-j2}\sin n\frac{\cancel{2}\pi}{T_0}\frac{\tau}{\cancel{2}}}=\boxed{\frac{A}{n\pi}\cdot\sin\frac{n\pi\tau}{T_0}}$$

Solution

$$= \frac{A\tau}{T_0} \cdot \frac{\sin\left(\frac{n\pi\tau}{T_0}\right)}{\left(\frac{n\pi\tau}{T_0}\right)} = \boxed{\frac{A\tau}{T_0} \cdot \text{sinc}\,\frac{n\pi\tau}{T_0}}$$ (A sinc function representation, equivalently)

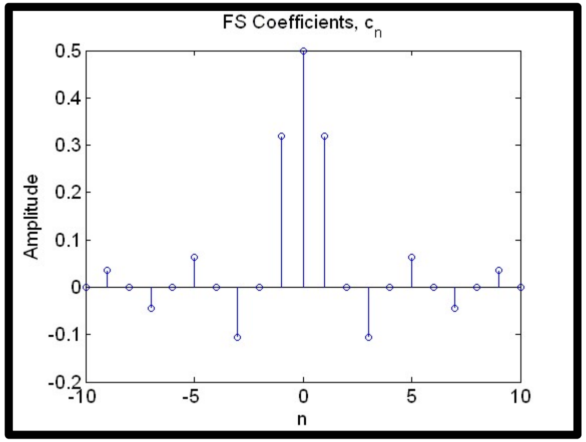
$$f(t) = \frac{A\tau}{T_0} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \left(\frac{A}{n\pi} \cdot \sin\frac{n\pi\tau}{T_0}\right) e^{jn\omega_0 t}$$ or $$f(t) = \frac{A\tau}{T_0} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \left(\frac{A\tau}{T_0} \cdot \text{sinc}\frac{n\pi\tau}{T_0}\right) e^{jn\omega_0 t}$$

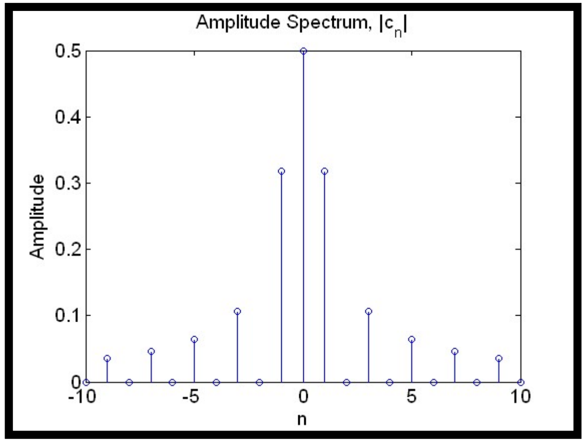
Solution

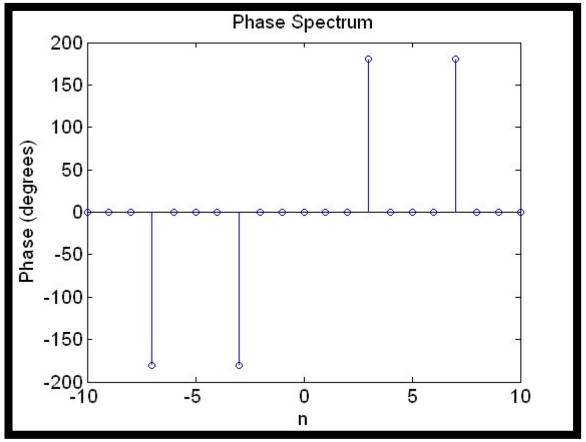
(a). $\tau = 0.5$

$T_0 = 1$

$A = 1$





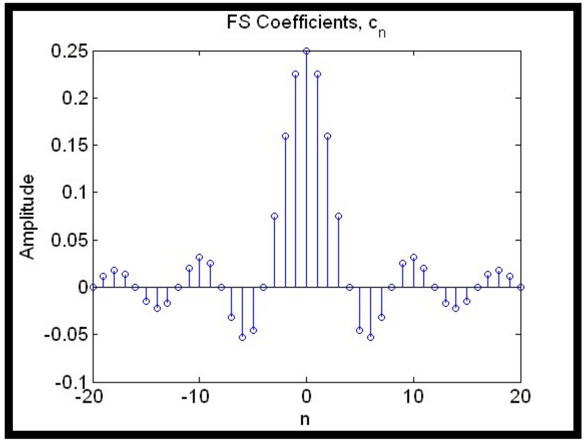


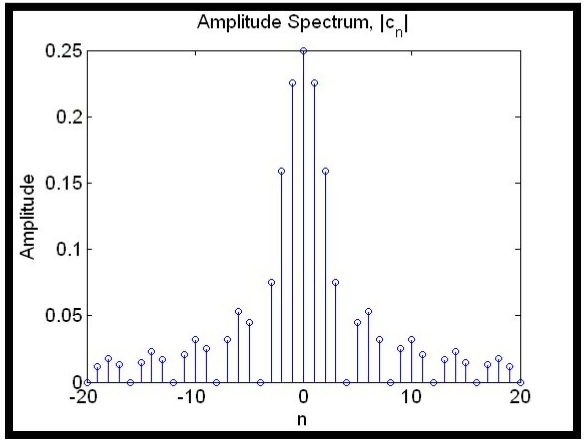
Solution

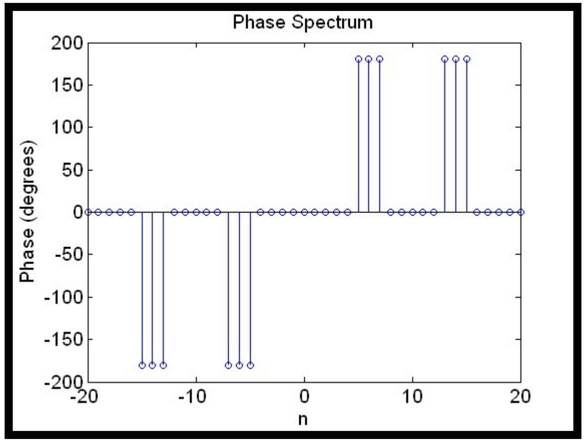
(b). $\tau = 0.25$

$T_0 = 1$

$A = 1$

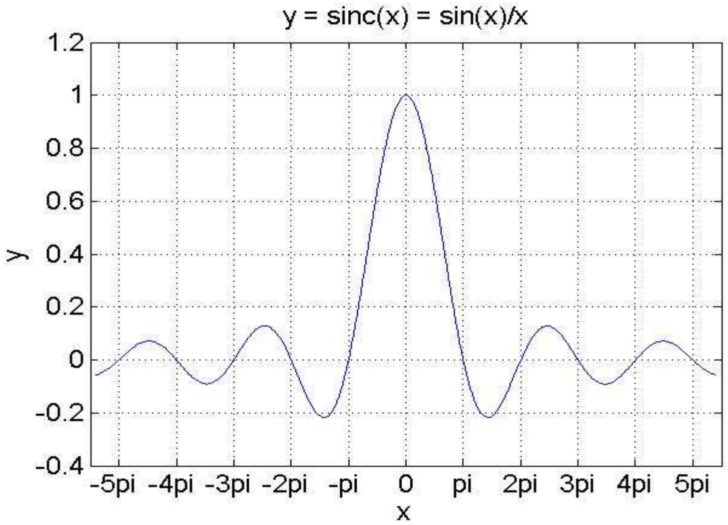


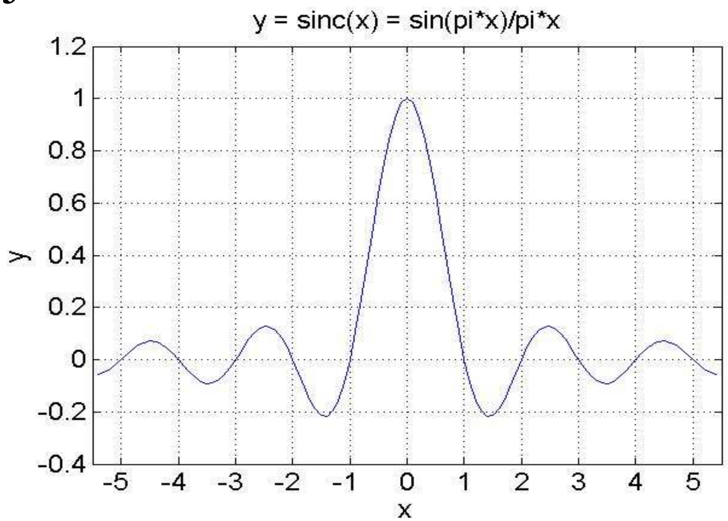




Sinc (or Sampling) Function, *Sa*(*x*)

The form of *sinc function* (or the *sampling function*, denoted as *Sa*(*x*), is defined and graphed below. Note that there are two versions of sinc function with a different scaling by a factor of π on the *x*-axis. This is a very important function that you will encounter in the various subjects.





$$\boxed{\operatorname{sinc} x = \frac{\sin x}{x}} \quad \text{for } x \in R \text{ (real number)}$$

$$\boxed{\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}} \quad \text{for } x \in R \text{ (real number)}$$

Sinc (or Sampling) Function, $Sa(x)$

The following functions are often encountered in the computation of the Fourier analysis. Thus, you should mathematically verify each entry of the table — ***Don't*** simply memorize them without developing your quick derivation ability!

Values of cosine, sine, and exponential functions for integral multiples of π.

Function	Value
$\cos 2n\pi$	1
$\sin 2n\pi$	0
$\cos n\pi$	$(-1)^n$
$\sin n\pi$	0
$\cos \frac{n\pi}{2}$	$\begin{cases}(-1)^{n/2}, & n = \text{even} \\ 0, & n = \text{odd}\end{cases}$
$\sin \frac{n\pi}{2}$	$\begin{cases}(-1)^{(n-1)/2}, & n = \text{odd} \\ 0, & n = \text{even}\end{cases}$
$e^{j2n\pi}$	1
$e^{jn\pi}$	$(-1)^n$
$e^{jn\pi/2}$	$\begin{cases}(-1)^{n/2}, & n = \text{even} \\ j(-1)^{(n-1)/2}, & n = \text{odd}\end{cases}$

Overview: Properties of Continuous-Time Fourier Series

- Symmetry (even, odd, half-wave)
- Conjugation (Conjugate Symmetry)
- Parseval's theorem (relation, identity)
- Linearity (superposition)
- Time reversal
- Time shifting
- Time scaling
- Differentiation
- Integration
- Multiplication
- Periodic convolution

Symmetry Considerations on Computing CTFS

Don't always simply plug in a set of FS equations for calculating the FS coefficients. You might be able to exploit some useful, applicable properties (if present) to avoid tedious calculation works.

- Even symmetry
- Odd symmetry
- Half-wave symmetry

Symmetry is based on the even and odd properties

- A signal $x(t)$ is said to be *even*, if $x(-t) = x(t)$.
- A signal $x(t)$ is said to be *odd*, if $-x(-t) = x(t)$.

There are two important properties:

$$\int_{-T/2}^{T/2} x_e(t)\,dt = 2\int_0^{T/2} x_e(t)\,dt$$

$$\int_{-T/2}^{T/2} x_o(t)\,dt = 0$$

Consider an arbitrary signal $x(t)$, it can be decomposed as:

$$x(t) = x_e(t) + x_o(t) = \underbrace{\frac{1}{2}\left[x(t) + x(-t)\right]}_{Even} + \underbrace{\frac{1}{2}\left[x(t) - x(-t)\right]}_{Odd}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\,\omega_0 t$$

Even Symmetry

$$f(-t) = f(t)$$

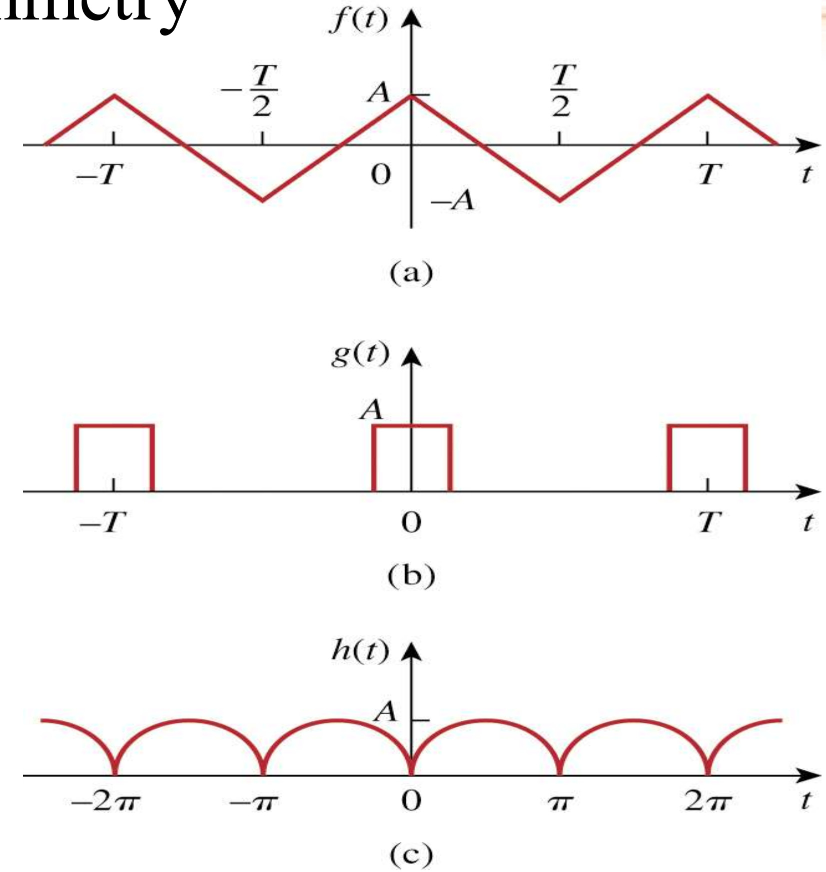
Fourier Cosine Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$a_0 = \frac{2}{T_0}\int_0^{\frac{T_0}{2}} x(t)\,dt$$

$$a_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} x(t)\cos n\omega_0 t\,dt$$

Note: $b_n = 0$, for all n.



Typical examples of even periodic waveforms

Odd Symmetry

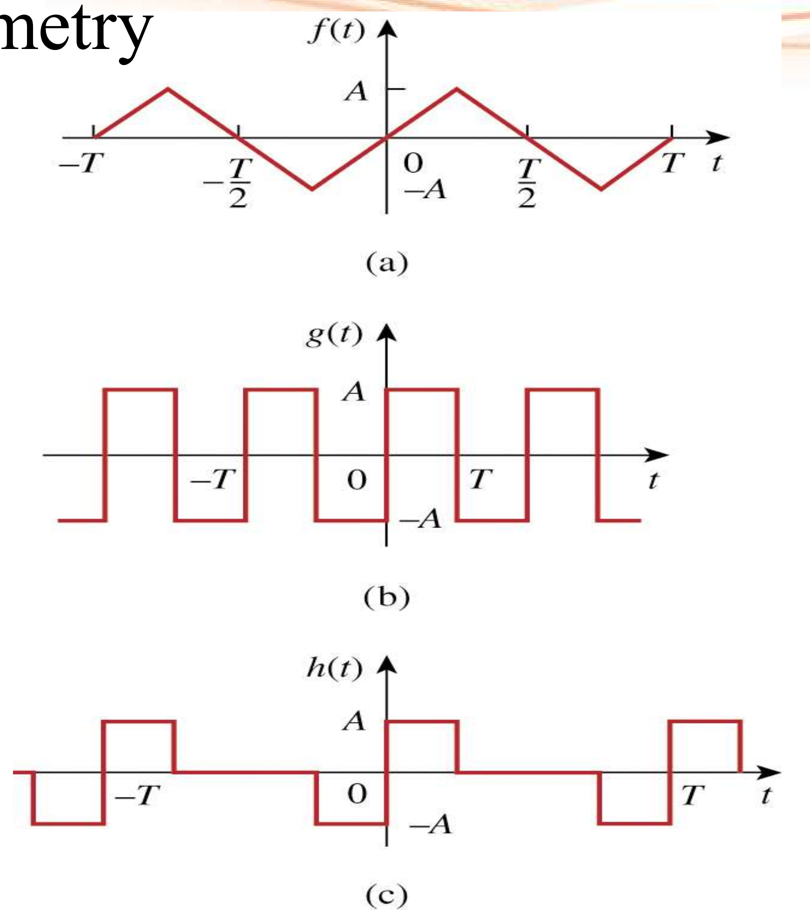
$$-f_{\mathrm{AC}}(-t) = f_{\mathrm{AC}}(t)$$

Fourier Sine Series

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(n\,\omega_0\, t)$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\,\omega_0 t)\,dt$$

Note: $a_n = 0$, for all n.



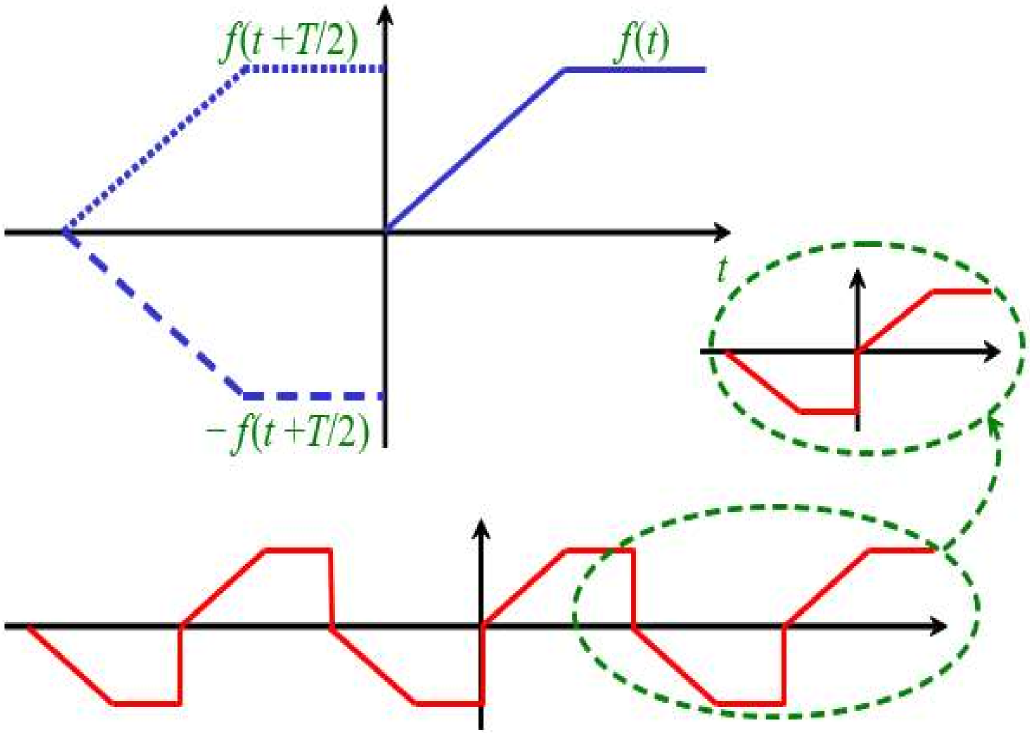
Typical examples of odd periodic waveforms

Half-Wave Symmetry

$$-f_{\mathrm{AC}}\left(t-\frac{T}{2}\right)=f_{\mathrm{AC}}(t) \quad \text{or} \quad -f_{\mathrm{AC}}\left(t+\frac{T}{2}\right)=f_{\mathrm{AC}}(t)$$

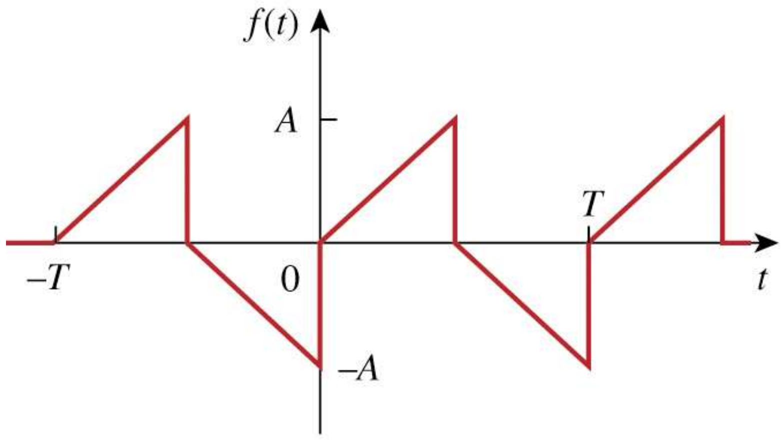
- Half-wave symmetry means that each half-cycle is the mirror image (with respect to x-axis) of the next half-cycle.

Examples:

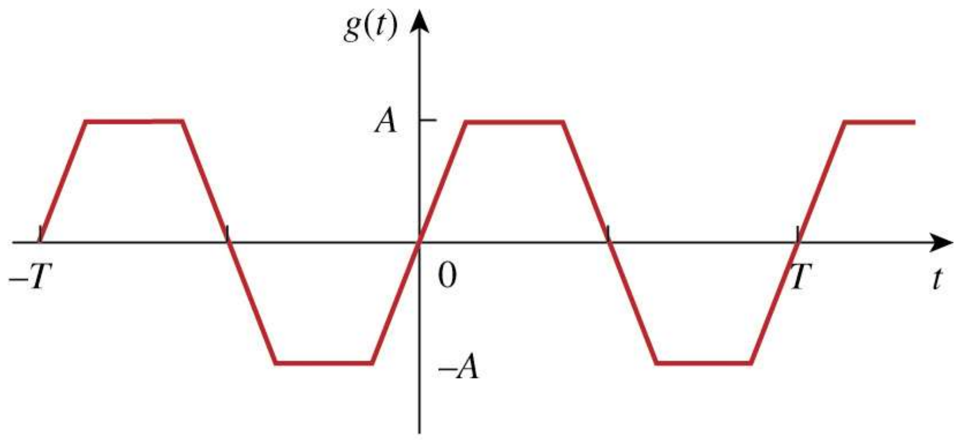


Half-Wave Symmetry

More Examples:



(a)



(b)

Half-Wave Symmetry

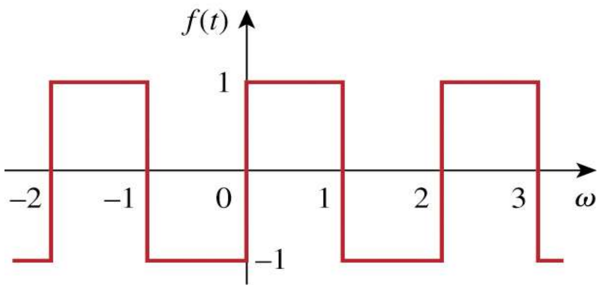
$$-f\left(t-\frac{T}{2}\right)=f(t) \quad \text{or} \quad -f\left(t+\frac{T}{2}\right)=f(t)$$

$$\boxed{\begin{aligned}
a_0 &= 0 \\
a_n &= \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt, \quad \text{for } n \text{ odd} \\
b_n &= \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega_0 t)\,dt, \quad \text{for } n \text{ odd} \\
a_n &= b_n = 0, \quad \text{for } n \ \underline{\text{even}}
\end{aligned}}$$

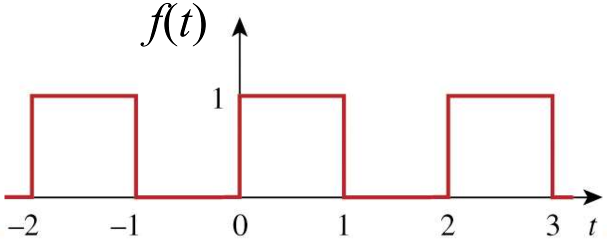
Half-Wave Symmetry

Examples:

(a). Re-visit an example studied earlier, you will notice that due to its odd symmetry. Thus, it only has sine terms; i.e., all $a_n = 0$, for all n.



(b). Now, you should realize that even its scaled version also contains only sine terms, too, except $a_0 \neq 0$ (for taking care of dc part), while the rest $a_n = 0$.



In Summary: (of Symmetry Properties)

TABLE 17.2 Effects of symmetry on Fourier coefficients.

Symmetry	a_0	a_n	b_n	Remarks
Even	$a_0 \neq 0$	$a_n \neq 0$	$b_n = 0$	Integrate over $T/2$ and multiply by 2 to get the coefficients.
Odd	$a_0 = 0$	$a_n = 0$	$b_n \neq 0$	Integrate over $T/2$ and multiply by 2 to get the coefficients.
Half-wave	$a_0 = 0$	$a_{2n} = 0$ $a_{2n+1} \neq 0$	$b_{2n} = 0$ $b_{2n+1} \neq 0$	Integrate over $T/2$ and multiply by 2 to get the coefficients.

Conjugation

The complex conjugate of the signal $x(t)$, which is denoted as $x^*(t)$, has the FS coefficients

$$x^*(t) \leftrightarrow \boxed{c_{-n}^*} \qquad \left[\text{i.e., } x^*(t) = \sum_{n=-\infty}^{+\infty} c_{-n}^* e^{jn\omega_0 t}\right]$$

Proof): Since $x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}$, take complex conjugate on both sides

$$x^*(t) = \left(\sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}\right)^* = \sum_{n=-\infty}^{+\infty} \left(c_n e^{jn\omega_0 t}\right)^* = \sum_{n=-\infty}^{+\infty} \left(c_n\right)^* \left(e^{jn\omega_0 t}\right)^* = \sum_{n=-\infty}^{+\infty} c_n^* e^{-jn\omega_0 t}$$

Conjugation

Let $m = -n$, we have $x^*(t) = \sum_{m=-\infty}^{+\infty} c_{-m}^* e^{jm\omega_0 t} \left(\Rightarrow \sum_{n=-\infty}^{+\infty} c_{-n}^* e^{jn\omega_0 t} \right)$

Note: For the *real*-valued signal, $x^*(t) = x(t)$, and this leads to

$c_{-n}^* = c_n \qquad \left(\text{or, } c_{-n} = c_n^* \right) \quad \Rightarrow$ called "Congjugate Symmetric"

Parseval's Theorem

$$\boxed{P_{ave} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 \, dt = \sum_{n=-\infty}^{+\infty} |c_n|^2}$$

$\Rightarrow$ Parseval's Theorem (relation, identity) simply states that:

"The total average power of a periodic signal $x(t)$ is equal to the sum of the average powers in all of its harmonic components."

Proof):

Consider an arbitrary complex-valued periodic signal $x(t)$, the average power is the energy per period.

Parseval's Theorem

$$P_{ave} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 \, dt = \frac{1}{T_0}\int_{T_0} x(t) \cdot x^*(t) \, dt$$

$$= \frac{1}{T_0}\int_{T_0} x(t) \cdot \left[\sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}\right]^* dt = \frac{1}{T_0}\int_{T_0} x(t) \cdot \left[\sum_{n=-\infty}^{+\infty} c_n^* e^{-jn\omega_0 t}\right] dt$$

$$= \sum_{n=-\infty}^{+\infty} c_n^* \cdot \underbrace{\left[\frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt\right]}_{=c_n} = \sum_{n=-\infty}^{+\infty} c_n^* \cdot c_n = \sum_{n=-\infty}^{+\infty} |c_n|^2 \qquad \text{Q.E.D.}$$

Note:

- $|c_n|^2$ denotes the average power in the n-th harmonic component of $x(t)$.

  The plot of $|c_n|^2$ (versus n) is called the *power spectrum*.
- The total average power is preserved in the time domain and in the frequency domain.

Example: For the cosine signal, we have

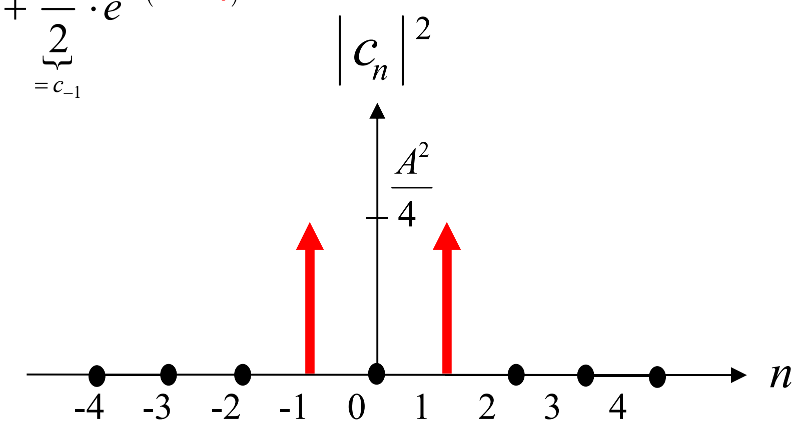
$$x(t) = A\cos\omega_0 t = A\left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) = \underbrace{\frac{A}{2}}_{=c_1}\cdot e^{j(1\cdot\omega_0)t} + \underbrace{\frac{A}{2}}_{=c_{-1}}\cdot e^{j(-1\cdot\omega_0)t}$$

$$\Rightarrow c_1 = c_{-1} = \frac{A}{2},\ c_n = 0,\ \text{for all } n \neq \pm 1.$$

The power spectrum is given by $|c_1|^2 = |c_{-1}|^2 = \frac{A^2}{4}$

and $|c_n|^2 = 0$, for all $n \neq \pm 1$.

The total average power $= \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2}$.



Even function!

Parseval's Theorem

Alternatively, from the time domain, we can compute the signal energy of the cosine signal,

$$\frac{1}{T_0}\int_{T_0}|x(t)|^2 dt = \frac{1}{T_0}\int_{T_0}|A\cos\omega_0 t|^2 dt = \frac{A^2}{T_0}\int_{T_0}\cos^2\omega_0 t\, dt$$

$$= \frac{A^2}{T_0}\int_{T_0}\frac{1+\cos 2\omega_0 t}{2}dt = \frac{A^2}{2T_0}\underbrace{\int_{T_0} 1\, dt}_{=T_0} + \frac{A^2}{2T_0}\underbrace{\int_{T_0}\cos 2\omega_0 t\, dt}_{=0}$$

$$= \frac{A^2}{2T_0}\cdot T_0 = \frac{A^2}{2}$$

In conclusion, the power spectrum computed in the time domain is the same as that in the frequency domain, based on the line spectra.

PS: Likewise, you can do the same for the sine signal in its time domain.

Example

Plot the power spectrum of the sine signal $A\sin\omega_0 t$.

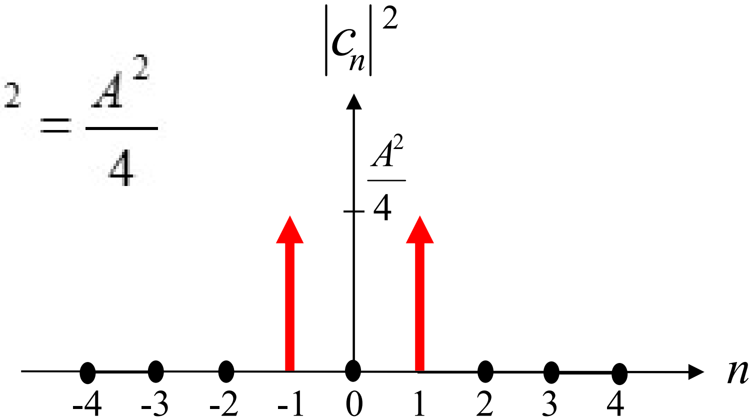
$$x(t) = A\sin\omega_0 t = A\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{j2}\right) = \underbrace{\frac{A}{j2}}_{=c_1}\cdot e^{j(1\cdot\omega_0)t} - \underbrace{\frac{A}{j2}}_{=c_{-1}}\cdot e^{j(-1\cdot\omega_0)t}$$

$$\Rightarrow c_1 = -j\frac{A}{2}\ ,\ \ c_{-1} = j\frac{A}{2}\ ,\ c_n = 0,\ \text{for all } n \neq \pm 1.$$

The power spectrum is given by $|c_1|^2 = |c_{-1}|^2 = \dfrac{A^2}{4}$

and $|c_n|^2 = 0$, for all $n \neq \pm 1$.

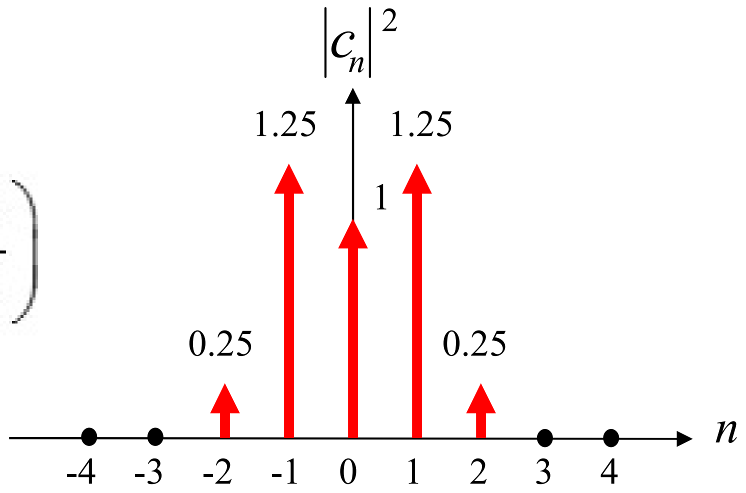
The total average power $= \dfrac{A^2}{4} + \dfrac{A^2}{4} = \dfrac{A^2}{2}$.



Example

Plot the power spectrum of $x(t)$:

$$x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$$



Linearity

Let $x(t)$ and $y(t)$ denote two periodic signals with period T and with FS coefficients c_n and d_n, resepectively. That is,

$$x(t) \leftrightarrow c_n \qquad \text{and} \qquad y(t) \leftrightarrow d_n$$

If the signal $z(t)$ is a linear combination of $x(t)$ and $y(t)$,

$$z(t) = A\, x(t) + B\, y(t)$$

then the FS coefficients e_n of $z(t)$ can be found as

$$z(t) \leftrightarrow \boxed{e_n = A\, c_n + B\, d_n} \qquad \left[\text{i.e., } z(t) = \sum_{n=-\infty}^{+\infty} \underbrace{\left(A c_n + B d_n\right)}_{=e_n} e^{j n \omega_0 t} \right]$$

Linearity

Proof):

$$z(t) = A\,x(t) + B\,y(t) = A \sum_{n=-\infty}^{+\infty} c_n e^{j n \omega_0 t} + B \sum_{n=-\infty}^{+\infty} d_n e^{j n \omega_0 t}$$

$$= \sum_{n=-\infty}^{+\infty} A c_n e^{j n \omega_0 t} + \sum_{n=-\infty}^{+\infty} B d_n e^{j n \omega_0 t} = \sum_{n=-\infty}^{+\infty} \underbrace{\left(A c_n + B d_n\right)}_{=e_n} e^{j n \omega_0 t}$$

Time Shifting

If the signal $x(t)$ has a time shift by an amount of τ , then

$$x(t-\tau) \leftrightarrow \boxed{e^{-jn\omega_0\tau}\, c_n} \quad \left[\text{i.e., } x(t-\tau) = \sum_{n=-\infty}^{+\infty} \left(e^{-jn\omega_0\tau}\, c_n\right) e^{jn\omega_0 t}\right]$$

$\Rightarrow$ *Time shift* introduces *phase delay* (if $\tau > 0$) or *advance* (if $\tau < 0$).

Proof 1): Use the *synthesis* equation,

$$x(t-\tau) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0(t-\tau)} = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} e^{-jn\omega_0\tau} = \sum_{n=-\infty}^{+\infty} \left(c_n e^{-jn\omega_0\tau}\right) e^{jn\omega_0 t}$$

Time Shifting

Proof 2): Use the *analysis* equation, let $y(t) = x(t-\tau)$ and $u = t-\tau$;

hence, $t = u+\tau$ and $dt = du$. The FS coefficients of $y(t)$ is

$$d_n = \frac{1}{T_0}\int_{T_0} y(t)e^{-jn\omega_0 t}dt = \frac{1}{T_0}\int_{T_0} x(t-\tau)e^{-jn\omega_0 t}dt$$

$$= \frac{1}{T_0}\int_{T_0} x(u)e^{-jn\omega_0 (u+\tau)}du = e^{-jn\omega_0\tau}\underbrace{\left[\frac{1}{T_0}\int_{T_0} x(u)e^{-jn\omega_0 u}du\right]}_{=\,c_n}$$

Time Reversal

If the signal $x(t)$ is time reversed; i.e., $x(-t)$, then

$$x(-t) \leftrightarrow \boxed{c_{-n}} \qquad \left[\text{i.e., } x(-t) = \sum_{n=-\infty}^{+\infty} c_{-n}\, e^{j n\omega_0 t}\right]$$

Proof):

Let $y(t) = x(-t)$. Using the synthesis equation of $y(t)$:

$$y(t) = x(-t) = \sum_{n=-\infty}^{+\infty} c_n e^{j n\omega_0(-t)} = \sum_{n=-\infty}^{+\infty} c_n e^{j(-n)\omega_0 t}$$

$$= \sum_{m=-\infty}^{+\infty} c_{-m}\, e^{j m\omega_0 t} \quad (\text{let } n = -m)$$

That is, c_{-n} is the FS coefficients of $x(-t)$.

Note:

- If $x(-t) = x(t)$ as *even* function, then $c_{-n} = c_n$.
- If $-x(-t) = x(t)$ as *odd* function, then $-c_{-n} = c_n$.

Time Scaling

If the signal $x(t)$ is scaled in time by a factor of a (>0), then

$$x(at) \leftrightarrow \boxed{c_n} \quad (Unchanged!) \quad \left[\text{i.e., } x(at) = \sum_{n=-\infty}^{+\infty} c_n e^{jn(a\omega_0)t}\right]$$

$Note:$ The subscript n of c_n denotes the line spectra at $\omega = n\omega_0$.

If you view c_n as a discrete-time signal, $c_n \Rightarrow c[n] \Rightarrow c[n\omega_0]$

Proof): $x(at) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0(at)} = \sum_{n=-\infty}^{+\infty} c_n e^{jn(a\omega_0)t}$

Time Scaling

One can clearly see that c_n is unchanged, but the fundamental angular frequency is changed from ω_0 to $a\,\omega_0$ (thus, the spacing of line spectra drawing is $a\,\omega_0$).

Example : If $a = 2$, $x(2t)$ is 'compressed' version of $x(t)$; on the other hand, the line spectra spacing is now 'expanded', since ω_0 becomes $2\omega_0$.

Differentiation

If the signal $x(t)$ is differentiated with respect to time variable t, then

$$\frac{d\,x(t)}{d\,t} \leftrightarrow \boxed{j\,n\,\omega_0\,c_n} \quad \left[\text{i.e., } \frac{d\,x(t)}{d\,t} = \sum_{n=-\infty}^{+\infty} \left(j\,n\,\omega_0\,c_n\right) e^{j\,n\,\omega_0\,t}\right]$$

Proof):

$$\frac{d\,x(t)}{d\,t} = \frac{d}{d\,t}\left\{\sum_{n=-\infty}^{+\infty} c_n\,e^{j\,n\,\omega_0\,t}\right\} = \sum_{n=-\infty}^{+\infty} c_n\,\frac{d}{d\,t}\left\{e^{j\,n\,\omega_0\,t}\right\} = \sum_{n=-\infty}^{+\infty} c_n\left(j\,n\,\omega_0\right)e^{j\,n\,\omega_0\,t}$$

Integration

If the signal $x(t)$ is integrated with respect to time variable t, then

$$\int_{-\infty}^{t} x(t)\,dt \leftrightarrow \boxed{\frac{c_n}{j n \omega_0}} \quad \left[\text{i.e., } \int_{-\infty}^{t} x(t)\,dt = \sum_{n=-\infty}^{+\infty}\left(\frac{c_n}{j n \omega_0}\right) e^{j n \omega_0 t}\right]$$

Proof):

$$\int_{-\infty}^{t} x(t)\,dt = \int_{-\infty}^{t}\left\{\sum_{n=-\infty}^{+\infty} c_n e^{j n \omega_0 t}\right\} dt = \sum_{n=-\infty}^{+\infty} c_n \int_{-\infty}^{t} e^{j n \omega_0 t}\,dt$$

$$= \sum_{n=-\infty}^{+\infty} c_n \left(\frac{1}{j n \omega_0}\right) e^{j n \omega_0 t}$$

Multiplication

Let $x(t)$ and $y(t)$ denote two periodic signals with period T_0 and with FS coefficients c_n and d_n, resepectively. That is,

$$x(t) \leftrightarrow c_n \qquad \text{and} \qquad y(t) \leftrightarrow d_n$$

If the signal $z(t)$ is a product of $x(t)$ and $y(t)$,

$$z(t) = x(t)\, y(t)$$

then the FS coefficients f_n of $z(t)$ can be found as

$$z(t) \leftrightarrow \boxed{f_n = \sum_{k=-\infty}^{+\infty} c_k\, d_{n-k}}$$

$$\left[\text{i.e., } z(t) = \sum_{n=-\infty}^{+\infty} \underbrace{\left(\sum_{k=-\infty}^{+\infty} c_k\, d_{n-k} \right)}_{=f_n} e^{jn\omega_0 t} \right]$$

Multiplication in the *time* domain $\Leftrightarrow$ Convolution in the *frequency* domain

Multiplication

Proof):

$$f_n = \frac{1}{T_0}\int_0^{T_0}\left[x(t)\,y(t)\right]e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0}\underbrace{\left(\sum_{k=-\infty}^{+\infty} c_k\, e^{jk\omega_0 t}\right)}_{=x(t)} y(t)\,e^{-jn\omega_0 t}\,dt$$

$$= \sum_{k=-\infty}^{+\infty} c_k\,\underbrace{\left(\frac{1}{T_0}\int_0^{T_0} y(t)\,e^{-j(n-k)\omega_0 t}dt\right)}_{=d_{n-k}}$$

$$= \sum_{k=-\infty}^{+\infty} c_k\, d_{n-k}$$

Periodic Convolution

$$\left(\begin{array}{l}\text{PS: "}\textit{Periodic}\text{" here means that the convolution integral operation} \\ \qquad \text{is performed over one period only, rather than over } (-\infty, +\infty)!\end{array}\right)$$

Let $x(t)$ and $y(t)$ denote two periodic signals with period T_0 and with FS coefficients c_n and d_n, resepectively. That is,

$$x(t) \leftrightarrow c_n \qquad \text{and} \qquad y(t) \leftrightarrow d_n$$

If the signal $z(t)$ is a periodic convolution of $x(t)$ and $y(t)$,

$$\boxed{z(t) = x(t) \otimes y(t) \ = \int_0^{T_0} x(\tau)\, y(t-\tau)\, d\tau}$$

then the FS coefficients f_n of $z(t)$ can be found as

Periodic Convolution

then the FS coefficients f_n of $z(t)$ can be found as

$$z(t) \leftrightarrow \boxed{f_n = T_0 c_n d_n} \qquad \left[\text{i.e., } z(t) = \sum_{n=-\infty}^{+\infty} \underbrace{\left(T_0 c_n d_n \right)}_{=f_n} e^{j n \omega_0 t} \right]$$

Convolution in the *time* domain $\Leftrightarrow$ Multiplication in the *frequency* domain

Periodic Convolution

In the periodic convolution, imposing the integration's lower and upper limits over **one** fundamental period is the only difference, compared with that in the ordinary convolution that you learnt earlier, integrating from -∞ to + ∞.

Proof):

$$z(t) = x(t) \otimes y(t) = \int_0^{T_0} x(\tau)\, y(t-\tau)\, d\tau$$

$$= \int_0^{T_0} x(\tau) \underbrace{\left(\sum_{n=-\infty}^{+\infty} d_n\, e^{jn\omega_0 (t-\tau)} \right)}_{=y(t-\tau)} d\tau = \sum_{n=-\infty}^{+\infty} d_n\, e^{jn\omega_0 t} \underbrace{\left(\int_0^{T_0} x(\tau)\, e^{-jn\omega_0 \tau} d\tau \right)}_{=T_0 c_n}$$

$$= \sum_{n=-\infty}^{+\infty} \underbrace{\left(T_0\, c_n d_n \right)}_{=f_n} e^{jn\omega_0 t} = \sum_{n=-\infty}^{+\infty} f_n\, e^{jn\omega_0 t}$$

EE2010
Signals and Systems

Part II

Fourier Transform

with Instructor:

Prof. Ma, Kai-Kuang

Introduction

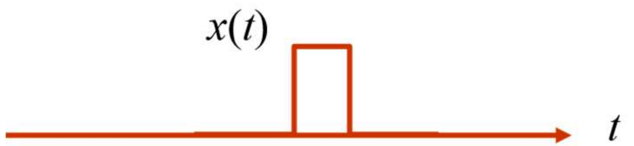
- **Fourier series** (FS) is applicable only for **periodic** signals. For **aperiodic** (or **non-periodic**) signals, we need to exploit the **Fourier transform** (FT) to conduct spectral analysis.

- Fourier transform is, in fact, a limiting case of Fourier series (in the sense when the signal's fundamental period $T_0 \to \infty$).

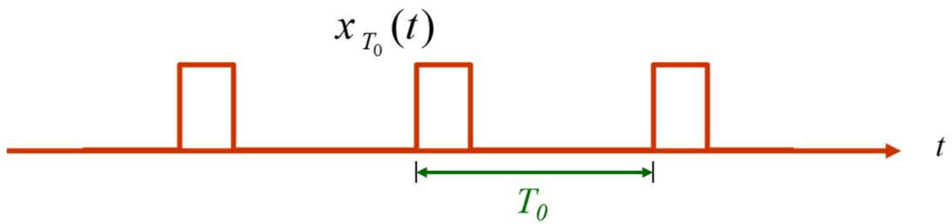
- Note that the Fourier transform can be used to represent both periodic **and** aperiodic (or non-periodic) signals.

How do we arrive at FT?

- For ease of drawing, consider an "one-shot" rectangular pulse $x(t)$ to be one of any possible finite-duration waveforms.

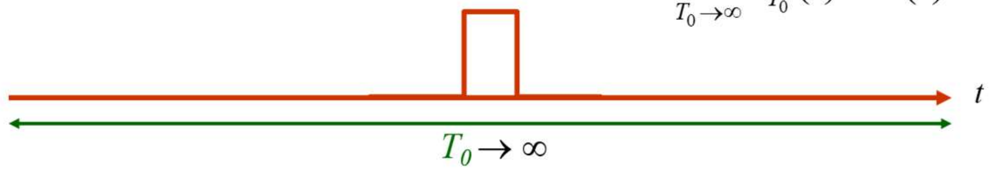


- Let us duplicate this pulse along the time line and make up a periodic rectangular pulse train, with the fundamental period T_0.



- Then, let $T_0 \to \infty$. It equals to $x(t)$! That is,

$$\lim_{T_0 \to \infty} x_{T_0}(t) = x(t).$$



How do we arrive at FT?

- We can view any finite-duration, non-periodic signal as a "period" of a make-up periodic signal with the fundamental period T_0. Now, the **FS** can be directly applied to.

- By letting $T_0 \to \infty$, the FS-derived result becomes the spectral analysis of $x(t)$.

➔ The **FT** is indeed a limiting (via $T_0 \to \infty$) case of the **FS**!

Derivation of the FT

Since $x_{T_0}(t)$ is periodic, it can be represented by the FS, as follows.

$$x_{T_0}(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} \qquad \text{[Synthesis Eq.]}$$

where $c_n = \dfrac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{-jn\omega_0 t}\, dt \qquad$ [Analysis Eq.]

The spacing of the two adjacent line spectra: $\Delta\omega = (n+1)\omega_0 - n\omega_0 = \omega_0 = \dfrac{2\pi}{T_0}$.

Derivation of the FT

Substitute the analysis equation c_n and $\Delta\omega$ into the synthesis equation, $x_{T_0}(t)$,

$$x_{T_0}(t) = \sum_{n=-\infty}^{+\infty} \left[\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{-jn\omega_0 t} dt \right] e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{+\infty} \left[\frac{\Delta\omega}{2\pi} \int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{-jn\omega_0 t} dt \right] e^{jn\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left[\int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{-jn\omega_0 t} dt \right] e^{jn\omega_0 t}\, \Delta\omega$$

$$\underbrace{\lim_{T_0\to\infty} x_{T_0}(t)}_{=x(t)} = \lim_{T_0\to\infty} \left\{ \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left[\int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{-jn\omega_0 t} dt \right] e^{jn\omega_0 t}\, \Delta\omega \right\}$$

Derivation of the FT

As $T_0 \to \infty$,

$$\boxed{\begin{array}{ccc} \displaystyle\sum_{n=-\infty}^{\infty} & \Rightarrow & \displaystyle\int_{-\infty}^{\infty} \\ \Delta\omega & \Rightarrow & d\omega \\ n\omega_0 & \Rightarrow & \omega \end{array}}$$

$$\underbrace{\lim_{T_0 \to \infty} x_{T_0}(t)}_{=x(t)} = \frac{1}{2\pi} \lim_{T_0 \to \infty} \left\{ \sum_{n=-\infty}^{+\infty} \left[\int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{-jn\omega_0 t} dt \right] e^{jn\omega_0 t}\, \Delta\omega \right\}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt \right]}_{=X(\omega)} e^{j\omega t} d\omega$$

Derivation of the FT

In summary,

$$X(\omega)=\mathscr{F}\left[x(t)\right] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad \text{[Fourier Transform]}$$

$$x(t) = \mathscr{F}^{-1}\left[X(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \qquad [\textit{Inverse}\ \text{Fourier Transform}]$$

We say that $x(t)$ and $X(\omega)$ form a Fouier transform pair:

$$\boxed{x(t) \leftrightarrow X(\omega)}$$

Important Notes about the FT (1/2)

- $X(\omega)$ is the **frequency**-domain representation of the **time**-domain signal $x(t)$. Both functions $x(t)$ and $X(\omega)$ represent the same thing!

- The transformation pair is <u>unique</u>, <u>one-to-one</u>, and <u>invertible</u>.

- Like the c_n of the FS, the $X(\omega)$ of the FT is a **complex-valued** function; thus,

$$X(\omega) = |X(\omega)| e^{j\theta(\omega)}, \quad \text{where} \quad \theta(\omega) = \measuredangle X(\omega)$$

$$\Rightarrow \begin{cases} |X(\omega)|: \text{ the magnitude spectrum (or amplitude spectrum)} \\ \theta(\omega): \text{ the phase spectrum} \end{cases}$$

Important Notes about the FT (2/2)

- If $x(t)$ is a *real*-valued signal, then $X(-\omega) = X^*(\omega)$.

Proof): $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

By taking complex conjugation on both sides of the FT,

$$X^*(\omega) = \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \right]^*$$

$$= \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt = \int_{-\infty}^{\infty} x^*(t)\, e^{-j(-\omega)t}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-j(-\omega)t}\, dt = X(-\omega)$$

Existing condition of the Fourier Transform*

The Fourier transform $X(\omega)$ of $x(t)$ exists, if the following Dirichlet conditions are satisfied.

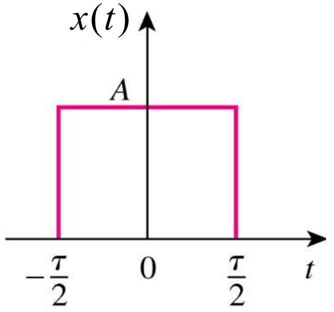
1. $x(t)$ is absolutely integrable; that is, $\int_{-\infty}^{\infty} |x(t)| dt < \infty$.
2. $x(t)$ has a finite number of maxima and minima within any finite interval.
3. $x(t)$ has a finite number of discountinuities within any finite interval, and each of these discontinuities is finite.

$\Rightarrow$ Note that this set of conditions together are *sufficient* to guarantee the existence of the Fourier transform; however, they are not necessary!
For examples, $\cos\omega_0 t$, $\sin\omega_0 t$, and $u(t)$ are not absolutely integrable, but they do have Fourier transform!

The Fourier Transform of Some Important Functions

- A Gate Function (or A Rectangular Pulse Signal)

$$x(t) = \begin{cases} A, & \text{for } -\frac{\tau}{2} < t < \frac{\tau}{2}; \\ 0, & \text{else.} \end{cases}$$

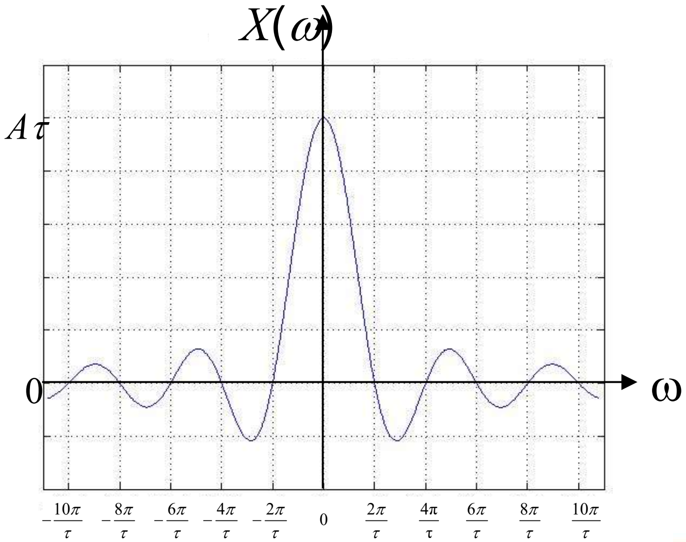


$$\boxed{X(\omega)} = \int_{-\infty}^{\infty} Ae^{-j\omega t}dt = A\int_{-\tau/2}^{\tau/2} e^{-j\omega t}dt$$

$$= -\frac{A}{j\omega}e^{-j\omega t}\Big|_{-\tau/2}^{\tau/2} = \left(-\frac{A}{j\omega}\right)\cdot(-j2)\cdot\sin\left(\frac{\omega\tau}{2}\right)$$

$$= \frac{2A}{\omega}\cdot\sin\left(\frac{\omega\tau}{2}\right) = \frac{2A}{\omega}\cdot\left(\frac{\omega\tau}{2}\right)\cdot\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}$$

$$= \boxed{A\tau\cdot\text{sinc}\left(\frac{\omega\tau}{2}\right)}$$



The Fourier Transform of Some Important Functions

- Single-sided Exponental Signal (for $a > 0$)

$$x(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

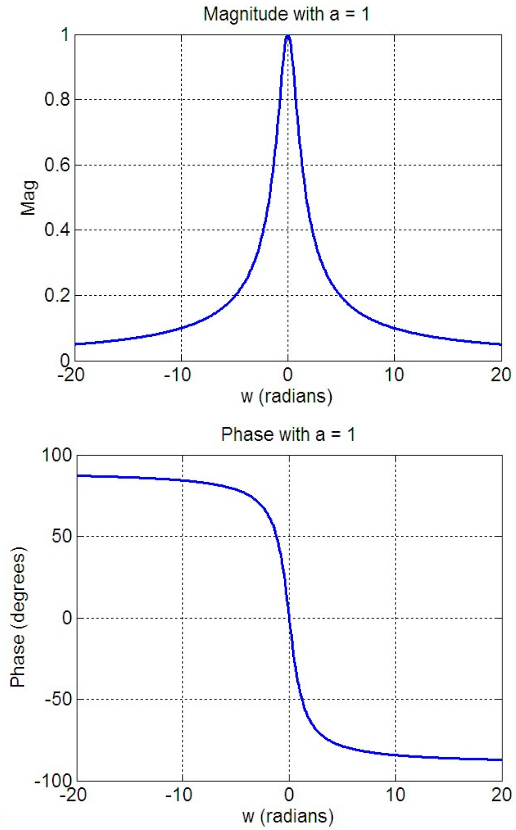
$$\boxed{X(\omega)} = \int_0^{\infty} e^{-at}\, e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$= -\frac{1}{a+j\omega} e^{-(a+j\omega)t}\Big|_0^{\infty} = \boxed{\frac{1}{a+j\omega}}$$

Thus,

$$\begin{cases} |X(\omega)| = \dfrac{1}{\sqrt{a^2+\omega^2}} \\ \angle X(\omega) = -\tan^{-1}\left(\dfrac{\omega}{a}\right) \end{cases}$$

FT Spectrum Plots



Frequency Response

$$\longrightarrow \boxed{h(t)} \longrightarrow$$

The Fourier transform of the **impulse response *h*(*t*)** is called the **frequency response H(ω)**. That is, they form a Fourier transform pair:

That is, $h(t)$ and $H(\omega)$ form a Fouier transform pair:

$$\boxed{h(t) \quad \leftrightarrow \quad H(\omega)}$$

where

$$H(\omega) = \mathscr{F}\left[h(t)\right] = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt \qquad \text{[Fourier Transform]}$$

$$h(t) = \mathscr{F}^{-1}\left[H(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\, e^{j\omega t}\, d\omega \qquad [\textit{Inverse}\ \text{Fourier Transform}]$$

Example: In the gate function case,

$$h(t) = A, \ \text{ for } -\frac{\tau}{2} < t < \frac{\tau}{2} \quad \leftrightarrow \quad H(\omega) = A\tau \cdot \text{sinc}\left(\frac{\omega\tau}{2}\right)$$

Example: In the exponential function case,

$$h(t) = \begin{cases} e^{-at}, & t \geq 0\,; \\ 0, & \text{otherwise.} \end{cases} \quad \leftrightarrow \quad H(\omega) = \frac{1}{a + j\omega} \quad (\text{where } a \ \geq \ 0)$$

- If $h(t)$ is a *filter*, then can you tell what kind of filtering is performed in each case, simply based on the magnitude spectrum plot?
- Quite often, we exploit the existing FT pairs to find out the inverse FT of more complicated ones. See an example in the next page.

An Example of Finding the Inverse FT

Example: Find the impulse response $h(t)$ of an LTI system's freuqncy response $H(\omega)$

$$H(\omega) = \frac{-6 - j7\omega}{2(1 + j\omega)(2 + j\omega)}$$

<u>Solution:</u>

Using the partial fraction expansion method to convert the product term into the summation terms

$$H(\omega) = \frac{-6 - j7\omega}{2(1 + j\omega)(2 + j\omega)} = \frac{A}{(1 + j\omega)} + \frac{B}{(2 + j\omega)}$$

Multiply both sides by $(1 + j\omega)$ and let $j\omega = -1$: $A = \left.\frac{-6 - j7\omega}{2(2 + j\omega)}\right|_{j\omega=-1} = \frac{-6+7}{2(2-1)} = \frac{1}{2}$

Multiply both sides by $(2 + j\omega)$ and let $j\omega = -2$: $B = \left.\frac{-6 - j7\omega}{2(1 + j\omega)}\right|_{j\omega=-2} = \frac{-6-7(-2)}{2(1-2)} = -4$

Frequency Response

$$\Rightarrow \quad H(\omega) = \frac{\left(\frac{1}{2}\right)}{(1 + j\omega)} + \frac{(-4)}{(2 + j\omega)} = \frac{1}{2(1 + j\omega)} - \frac{4}{2 + j\omega}$$

Since $\boxed{h(t) = e^{-at}, \text{ for } t \geq 0 \quad \leftrightarrow \quad H(\omega) = \frac{1}{a + j\omega}}$, the impulse response can be found

by computing the inverse FT of $H(\omega)$, term by term. By inspection, we can get

$$\boxed{h(t) = \frac{1}{2}e^{-t} - 4e^{-2t}, \text{ for } t \geq 0}$$

Important FT Pairs

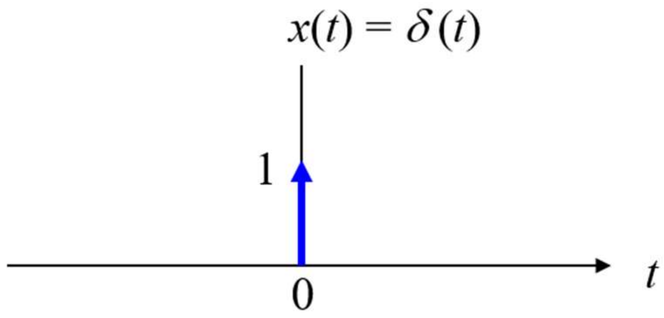
- Impulse Signal
- Constant Signal
- Exponential Signal
- Impulse Train Signal

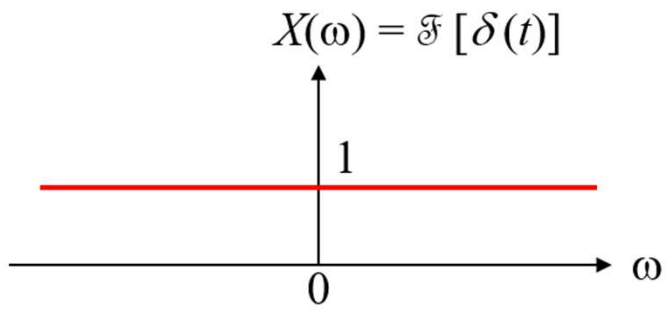
- The Unit-strength Impulse Function $\boxed{\mathcal{F}\left[\delta(t)\right]=1}$

Proof):

$x(t)=\delta(t)$

$$X(\omega)=\mathcal{F}\left[x(t)\right]=\mathcal{F}\left[\delta(t)\right]=\int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty}\delta(t)e^{-j\omega\cdot 0}dt=1$$





Reversely, the *inverse* Fourier transform of 1 is $\delta(t)$. That is,

$$\mathscr{F}^{-1}[1] = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} d\omega = \delta(t)$$

$$\Rightarrow \quad \boxed{\int_{-\infty}^{\infty} e^{j\omega t} d\omega = 2\pi\,\delta(t)} \qquad \left(\text{or,} \ \int_{-\infty}^{\infty} e^{jxy} dx = 2\pi\,\delta(y)\right)$$

This is a very useful integration identity, as the direct integration is difficult.

Therefore, $\int_{-\infty}^{\infty} e^{j\omega t} dt = 2\pi\,\delta(\omega)$

Note: Since $\delta(t)$ is an even function, thus, $\delta(-a) = \delta(a)$ and

$$\int_{-\infty}^{\infty} e^{-jxa}\, dx = 2\pi\,\delta(-a) = 2\pi\,\delta(a) = \int_{-\infty}^{\infty} e^{jxa}\, dx$$

In summary,

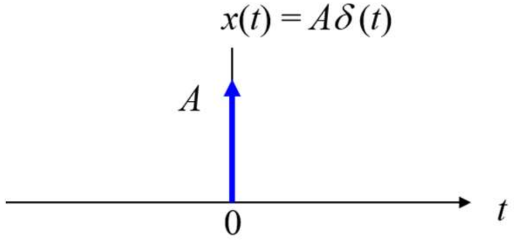
$$\boxed{\int_{-\infty}^{\infty} e^{jxa}\, dx = \int_{-\infty}^{\infty} e^{-jxa}\, dx = 2\pi\,\delta(a)}$$

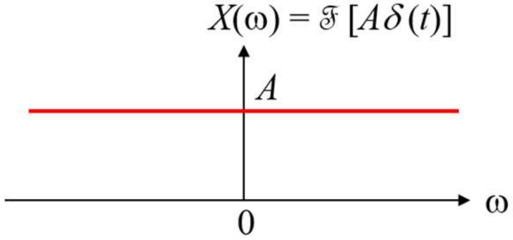
- The Impulse Function with Strength A $\boxed{\mathscr{F}\left[A\delta(t)\right] = A}$

Proof):

$x(t) = A\delta(t)$

$$\mathscr{F}\left[A\delta(t)\right] = \int_{-\infty}^{\infty} A\delta(t)e^{-j\omega t}dt = A\int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = Ae^{-j\omega\cdot 0} = A$$





Reversely,

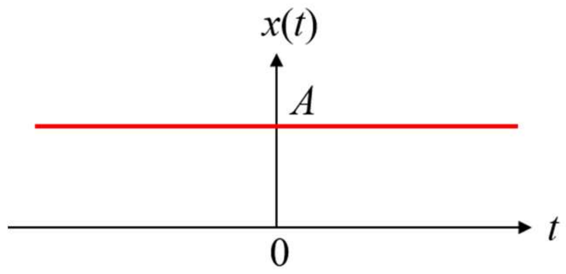
$$\mathscr{F}^{-1}\left[A\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} A\cdot e^{j\omega t}d\omega = A\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}d\omega}_{=\delta(t)} = A\delta(t)$$

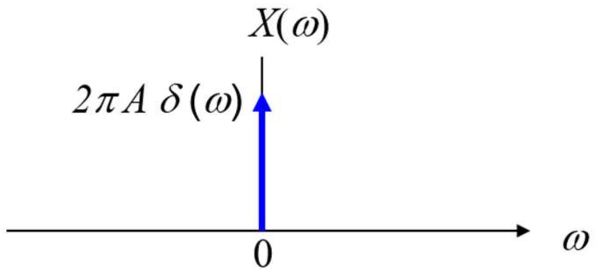
- Fourier Transform of a Constant A $\boxed{\mathscr{F}[A] = 2\pi A\delta(\omega)}$

Proof): $\mathscr{F}[A] = \int_{-\infty}^{\infty} A e^{-j\omega t} dt = 2\pi A\delta(\omega)$

PS: This is obtained by using the identity dervied earlier:

$$\int_{-\infty}^{\infty} e^{jxa}\, dx = \int_{-\infty}^{\infty} e^{-jxa}\, dx = 2\pi\,\delta(-a) = 2\pi\,\delta(a)$$





- Fourier Transform of an Exponential $e^{\pm j\omega_0 t}$

$$\boxed{\mathcal{F}\left[e^{\pm j\omega_0 t}\right] = 2\pi\,\delta(\omega \mp \omega_0)}$$

Proof):

$$\mathcal{F}\left[e^{j\omega_0 t}\right] = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{j(\omega_0-\omega)t} dt$$

By using the following identity dervied earlier:

$$\int_{-\infty}^{\infty} e^{jxa}\,dx = \int_{-\infty}^{\infty} e^{-jxa}\,dx = 2\pi\,\delta(-a) \;= 2\pi\,\delta(a)$$

Thus,

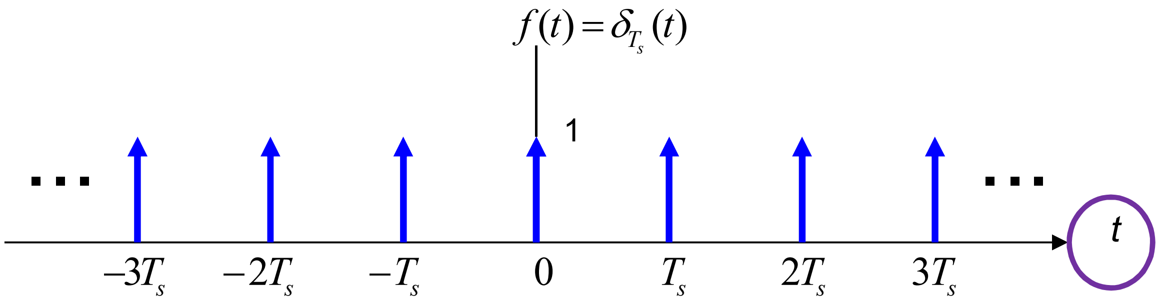
$$\mathcal{F}\left[e^{j\omega_0 t}\right] = \int_{-\infty}^{\infty} e^{j(\omega_0-\omega)t}\,dt = 2\pi\,\delta(\omega_0-\omega) = 2\pi\,\delta(\omega-\omega_0)$$

Similarly, $\mathcal{F}\left[e^{-j\omega_0 t}\right] = 2\pi\,\delta(\omega+\omega_0)$

- Fourier Transform of an impulse train $\delta_{T_0}(t)$

$$\boxed{\mathscr{F}\left[\delta_{T_0}(t)\right] = \omega_0\, \delta_{\omega_0}(\omega)}$$

$$\delta_{T_0}(t) \quad \leftrightarrow \quad \omega_0\, \delta_{\omega_0}(\omega) \qquad \left(\text{where } \ \omega_0 = \frac{2\pi}{T_0}\right)$$



Proof): The equidistance unit-strength impulse train

$$\delta_{T_0}(t) = \ldots + \delta\left(t - 3T_0\right) + \delta\left(t - 2T_0\right) + \delta\left(t - T_0\right) + \delta\left(t\right)$$

$$+ \delta\left(t + T_0\right) + \delta\left(t + 2T_0\right) + \delta\left(t + 3T_0\right) + \ldots = \sum_{n=-\infty}^{+\infty} \delta\left(t - nT_0\right)$$

The FS representation : $\delta_{T_0}(t) = \sum_{n=-\infty}^{+\infty} c_n \, e^{j n \omega_0 t}$

where $c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta_{T_0}(t) e^{-j n \omega_0 t} dt = \frac{1}{T_0}\underbrace{\int_{-T_0/2}^{T_0/2} \delta(t) e^{-j n \omega_0 t}\, dt}_{= e^{-j n \omega_0 \cdot 0} = 1}$

$$\therefore \quad \delta_{T_0}(t) = \frac{1}{T_0}\sum_{n=-\infty}^{+\infty} c_n \, e^{j n \omega_0 t} = \sum_{n=-\infty}^{+\infty}\left(\frac{1}{T_0}\right)\cdot e^{j n \omega_0 t} = \frac{1}{T_0}\sum_{n=-\infty}^{+\infty} e^{j n \omega_0 t}$$

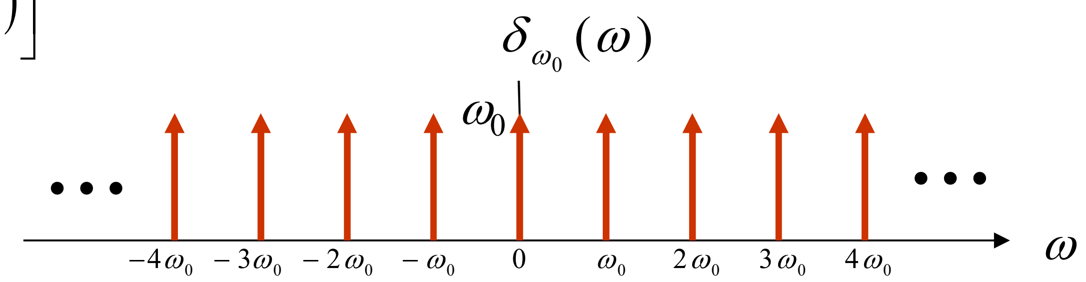
By taking the Fourier transform on both sides of this FS representation,

$$\mathscr{F}\left[\delta_{T_0}(t)\right] = \mathscr{F}\left[\frac{1}{T_0}\sum_{n=-\infty}^{+\infty} e^{j n \omega_0 t}\right] = \frac{1}{T_0}\mathscr{F}\left[\sum_{n=-\infty}^{+\infty} e^{j n \omega_0 t}\right]$$

$$= \frac{1}{T_0}\left[\sum_{n=-\infty}^{+\infty} 2\pi\,\delta\left(\omega - n\,\omega_0\right)\right]$$

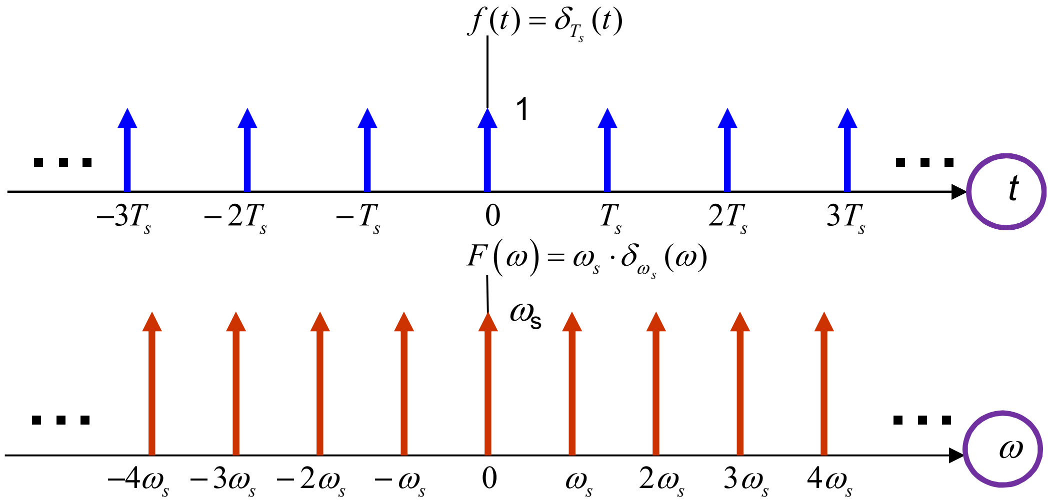
$$= \omega_0\left[\sum_{n=-\infty}^{+\infty} \delta\left(\omega - n\,\omega_0\right)\right]$$

$$= \omega_0 \cdot \delta_{\omega_0}(\omega)$$



The Fourier transform of an impulse train is still an impulse train, *except* the scaling factor on the amplitude by ω_0 and the line spacing; T_0 in the time-domain impulse train and ω_0 in the frequency-domain impulse train, where $\omega_0 = 2\pi / T_0$.

$$\delta_{T_0}(t) \quad \leftrightarrow \quad \omega_0 \cdot \delta_{\omega_0}(\omega) \quad \left(\text{where} \ \ \omega_0 = \frac{2\pi}{T_0} \right)$$



Fourier Transform (FT) Properties

Properties - Linearity

- Linearity (or Superposition)

If $x_1(t) \leftrightarrow X_1(\omega)$ and $x_2(t) \leftrightarrow X_2(\omega)$, then

$$\boxed{a_1 x_1(t) + a_2 x_2(t) \quad \leftrightarrow \quad a_1 X_1(\omega) + a_2 X_2(\omega)}$$

Proof):

$$\mathscr{F}\left[a_1 x_1(t) + a_2 x_2(t)\right] = \int_{-\infty}^{\infty}\left[a_1 x_1(t) + a_2 x_2(t)\right]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty}\left[a_1 x_1(t)\right]e^{-j\omega t}dt + \int_{-\infty}^{\infty}\left[a_2 x_2(t)\right]e^{-j\omega t}dt$$

$$= a_1\int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}dt + a_2\int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}dt$$

$$= a_1 X_1(\omega) + a_2 X_2(\omega)$$

$$\left(\text{or} \quad \mathscr{F}\left[\, a_1 x_1(t) + a_2 x_2(t)\right] = a_1\mathscr{F}\left[x_1(t)\right] + a_2\mathscr{F}\left[x_2(t)\right]\right)$$

Properties – Time Shifting

- Time Shiftng

If $x(t) \leftrightarrow X(\omega)$, then $\boxed{x(t-\tau) \leftrightarrow e^{-j\omega\tau} X(\omega)}$

Proof):

$$\left(\text{Let } u = t-\tau \quad \Rightarrow \quad t = u+\tau,\ du = dt\right)$$

$$\mathscr{F}\left[\, x(t-\tau)\right] = \int_{-\infty}^{\infty} x(t-\tau)\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(u)\, e^{-j\omega(u+\tau)} du$$

$$= e^{-j\omega\tau} \int_{-\infty}^{\infty} x(u)\, e^{-j\omega u} du = e^{-j\omega\tau} X(\omega)$$

Properties – Frequency Shifting

- Frequency Shiftng

If $x(t) \leftrightarrow X(\omega)$, then $\boxed{x(t)\, e^{j\omega_0 t} \leftrightarrow X(\omega-\omega_0)}$

Proof):

$$\mathscr{F}\left[x(t)\, e^{j\omega_0 t} \right] = \int_{-\infty}^{\infty} x(t)\, e^{j\omega_0 t}\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\omega_0)t} dt$$

$$= X(\omega-\omega_0)$$

Properties – Time Scaling

- Time Scaling

If $x(t) \leftrightarrow X(\omega)$, then $\boxed{x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)}$

for a real constant a.

Proof): Let $u = at$ and $a > 0 \Rightarrow t = \frac{u}{a}$

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt = \frac{1}{a}\int_{-\infty}^{\infty} x(u)\, e^{-j\omega\left(\frac{u}{a}\right)} du = \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

For $a < 0$, similarly, this leads to $\mathcal{F}[x(at)] = -\frac{1}{a} X\left(\frac{\omega}{a}\right)$

Combine both together $x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

* Note that if $a = -1$, $\boxed{x(-t) \leftrightarrow X(-\omega)}$ (Time Reverse property)

Properties – Time Differentiation

- Time Differentiation

$$\boxed{\frac{dx(t)}{dt} \leftrightarrow j\omega X(\omega)}$$

Proof):

The inverse Fourier transform of $X(\omega)$ is $x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega.$

$$\Rightarrow \quad \frac{dx(t)}{dt} = \frac{1}{2\pi}\frac{d}{dt}\left[\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{d}{dt}(e^{j\omega t})d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[j\omega X(\omega)\right]e^{j\omega t}d\omega$$

Properties – Time Differentiation

Clearly, $\dfrac{dx(t)}{dt}$ is the *inverse* Fourier transform of $j\omega X(\omega)$.

PS: Continue such differentiation for n times, we can get

$$\boxed{\frac{d^n x(t)}{dt^n} \leftrightarrow (j\omega)^n X(\omega)}$$

Properties – Integration Property

- Integration property

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \;\leftrightarrow\; \pi X(0)\delta(\omega) + \frac{1}{j\omega}X(\omega)}$$

Proof): We have

$$\int_{-\infty}^{t} x(\tau)\,d\tau = x(t) * u(t)$$

Thus, by the time convolution theorem we obtain

$$\mathscr{F}\left[x(t) * u(t)\right] = X(\omega)\cdot\left[\pi\,\delta(\omega) + \frac{1}{j\omega}\right] = \pi X(\omega)\delta(\omega) + \frac{1}{j\omega}X(\omega)$$

$$= \pi X(0)\delta(\omega) + \frac{1}{j\omega}X(\omega)$$

since $X(\omega)\delta(\omega) = X(0)\delta(\omega)$.

Properties – Duality (or Symmetry)

- Duality (or Symmetry)

If $x(t) \leftrightarrow X(\omega)$, then $\boxed{X(t) \leftrightarrow 2\pi\, x(-\omega)}$

Proof):

From the inverse Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\; e^{j\omega t}\, d\omega \;\;\Rightarrow\;\; 2\pi x(t) = \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow\;\; 2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

Exchange the position of variables t and ω in the above equation:

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)\; e^{-j\omega t}\, dt = \mathcal{F}\left[X(t)\right]$$

Properties – Time Convolution

- Time Convolution

If $x(t) \leftrightarrow X(\omega)$, $h(t) \leftrightarrow H(\omega)$, and $y(t) \leftrightarrow Y(\omega)$, then

$$\boxed{Y(\omega) = \mathscr{F}\left[h(t) * x(t)\right] = H(\omega)X(\omega)}$$

A key foundation for "filtering" topic!!

Properties – Time Convolution

Proof):

$$Y(\omega)=\int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty}\left[h(t)*x(t)\right]e^{-j\omega t}dt$$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right]e^{-j\omega t}dt$$

$$=\int_{-\infty}^{\infty} h(\tau)d\tau\left[\int_{-\infty}^{\infty} x(t-\tau)\,e^{-j\omega t}dt\right]$$

Let $\lambda=t-\tau$; hence, $t=\lambda+\tau$ and $d\lambda=dt$. Then,

$$Y(\omega)=\int_{-\infty}^{\infty} h(\tau)d\tau\left[\int_{-\infty}^{\infty} x(\lambda)\,e^{-j\omega(\lambda+\tau)}d\lambda\right]$$

$$=\left[\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau\right]\cdot\left[\int_{-\infty}^{\infty} x(\lambda)\,e^{-j\omega\lambda}d\lambda\right]=H(\omega)X(\omega)$$

$$\boxed{\text{Convolution in the time domain} \;\Leftrightarrow\; \text{Multiplication in the frequency domain}}$$

Properties – Time Multiplication

- Time Multiplication

$$\boxed{x_1(t)x_2(t) \quad \leftrightarrow \quad \frac{1}{2\pi}X_1(\omega)*X_2(\omega)}$$

A key foundation for "sampling" topic!!

Properties – Time Multiplication

Proof): By definitions, we have

$$\mathscr{F}\left[x_1(t)\,x_2(t)\right] = \int_{-\infty}^{\infty} x_1(t)x_2(t)\,e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)e^{j\lambda t}d\lambda\right]x_2(t)\,e^{-j\omega t}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\left[\int_{-\infty}^{\infty} x_2(t)\,e^{-j(\omega-\lambda)t}\,dt\right]d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)X_2(\omega-\lambda)\,d\lambda = \frac{1}{2\pi}X_1(\omega)*X_2(\omega)$$

Multiplication in the time domain $\Leftrightarrow$ Convolution in the frequency domain

Properties – Parseval's Theorem

- Parseval's Theorem

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega}$$

Proof):

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t) x^*(t) \, dt$$

$$= \int_{-\infty}^{\infty} x(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega \right)^* dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) \left(\int_{-\infty}^{\infty} \left[X(\omega) e^{j\omega t} \right]^* d\omega \right) dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) \left(\int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} \, d\omega \right) dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \underbrace{\left(\int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \right)}_{=X(\omega)} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

Summary – Fourier Transform Pairs

$x(t)$	$X(\omega)$
$\delta(t)$	1
$\delta(t-\tau)$	$e^{-j\omega\tau}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$
$\sin\omega_0 t$	$-j\pi\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$
$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$
$u(-t)$	$\pi\delta(\omega)-\dfrac{1}{j\omega}$
$e^{-at}u(t),\ a>0$	$\dfrac{1}{a+j\omega}$
$te^{-at}u(t),\ a>0$	$\dfrac{1}{(a+j\omega)^2}$

Summary – Fourier Transform Pairs

$x(t)$	$\mathrm{X}(\omega)$
$e^{-a\lvert t\rvert},\ a>0$	$\dfrac{2a}{a^2+\omega^2}$
$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$
$\delta_{T_0}(t)$	$\omega_0 \cdot \delta_{\omega_0}(\omega)$
$= \sum_{n=-\infty}^{\infty} \delta(t-nT_0)$	$= \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0),\ \omega_0 = \dfrac{2\pi}{T_0}$
$p_a(t) = \begin{cases} 1, & \lvert t\rvert < a \\ 0, & \lvert t\rvert > a \end{cases}$	$2a\dfrac{\sin a\omega}{a\omega}\left(= 2a\ \mathrm{sinc}(a\omega)\right)$
$\dfrac{\sin at}{\pi t}\left(= \dfrac{a}{\pi}\ \mathrm{sinc}(at)\right)$	$p_a(\omega) = \begin{cases} 1, & \lvert\omega\rvert < a \\ 0, & \lvert\omega\rvert > a \end{cases}$

Summary – Fourier Transform Properties

Property	Signal	Fourier Transform
(Establishing notations)	$x(t)$	$X(\omega)$
	$x_1(t)$	$X_1(\omega)$
	$x_2(t)$	$X_2(\omega)$
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time shifting	$x(t-\tau)$	$e^{-j\omega\tau}X(\omega)$
Frequency shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Time reverse	$x(-t)$	$X(-\omega)$
Time differentiation	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
	$\frac{d^n x(t)}{dt^n}$	$(j\omega)^n X(\omega)$
Frequency differentiation	$(-jt)x(t)$	$\frac{dX(\omega)}{d\omega}$
	$(-jt)^n x(t)$	$\frac{d^n X(\omega)}{d\omega^n}$

Summary – Fourier Transform Properties

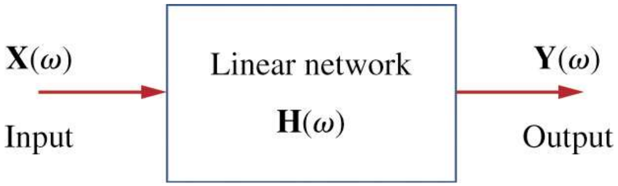
Property	Signal	Fourier Transform
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\pi X(0)\delta(\omega) + \frac{1}{j\omega}X(\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Convolution	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega) * X_2(\omega)$
Real signal	$x(t) = x_e(t) + x_o(t)$	$X(\omega) = A(\omega) + jB(\omega)$
		$X(-\omega) = X^*(\omega)$
Even component	$x_e(t)$	$\mathrm{Re}\{X(\omega)\} = A(\omega)$
Odd component	$x_o(t)$	$j\,\mathrm{Im}\{X(\omega)\} = jB(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$	

The Frequency Response of Continuous-time LTI System

— Filtering —

$$\boxed{Y(\omega) = \mathscr{F}\left[h(t) * x(t)\right] = H(\omega)X(\omega)}$$

Filter



A ***filter*** is a circuit that is designed to pass signals with desired frequencies and reject or attenuate others.

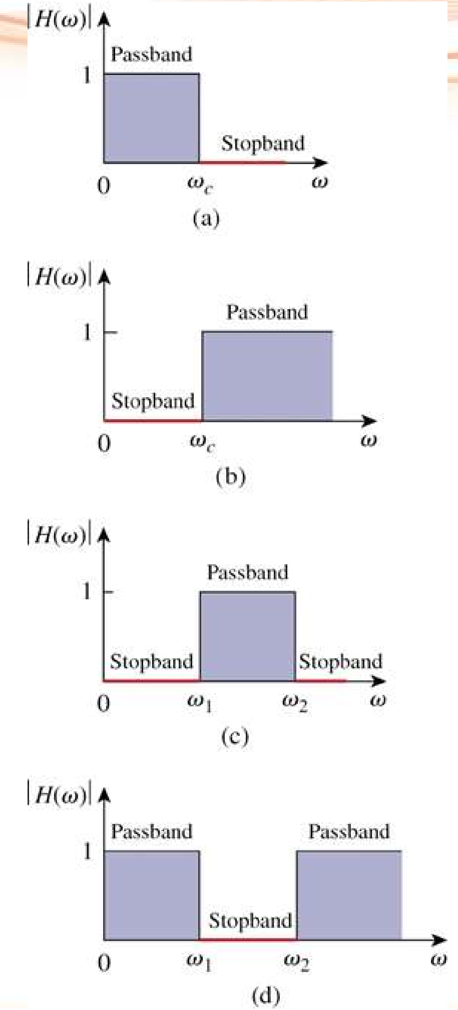
a) **Lowpass** filter — pass only frequencies from dc up to the cutoff frequency ω_c.

b) **Highpass** filter — pass all frequencies above the cutoff frequency ω_c.

c) **Bandpass** filter — pass all frequencies within a band of frequencies, $\omega_1 < \omega < \omega_2$.

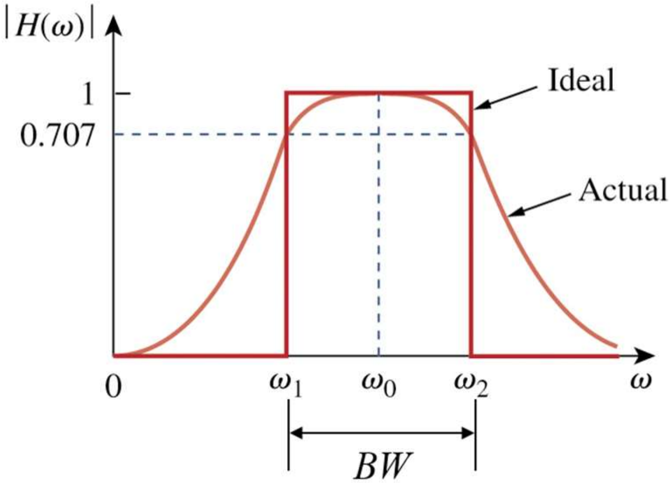
d) **Bandstop** filter — stop or eliminate all frequencies within a band of frequencies, $\omega_1 < \omega < \omega_2$.

PS: The drawing with vertical cut-off is ideal only!



Bandwidth, BW

- An LTI System:



- A signal:

A signal $x(t)$ is called a *band-limited* signal if

$$\left|X(\omega)\right| = 0 \qquad \text{for } \left|\omega\right| > \text{Bandwidth, } BW$$

Frequency Response

Recap:

The Fourier transform of the **impulse response *h(t)*** is called the **frequency response H(ω)**. That is, they form a Fourier transform pair:

That is, $h(t)$ and $H(\omega)$ form a Fouier transform pair:

$$\boxed{h(t) \quad \leftrightarrow \quad H(\omega)}$$

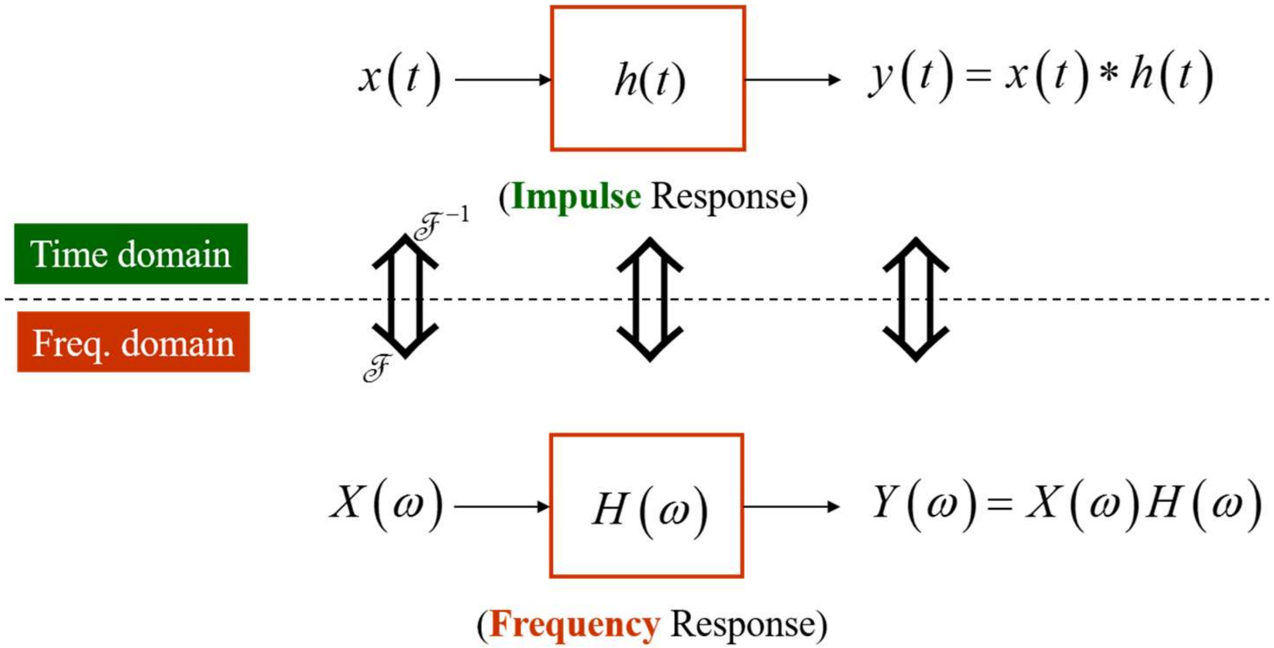
where

$$H(\omega) = \mathscr{F}\left[h(t)\right] = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt \qquad \text{[Fourier Transform]}$$

$$h(t) = \mathscr{F}^{-1}\left[H(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\, e^{j\omega t}\, d\omega \qquad \text{[\textit{Inverse} Fourier Transform]}$$

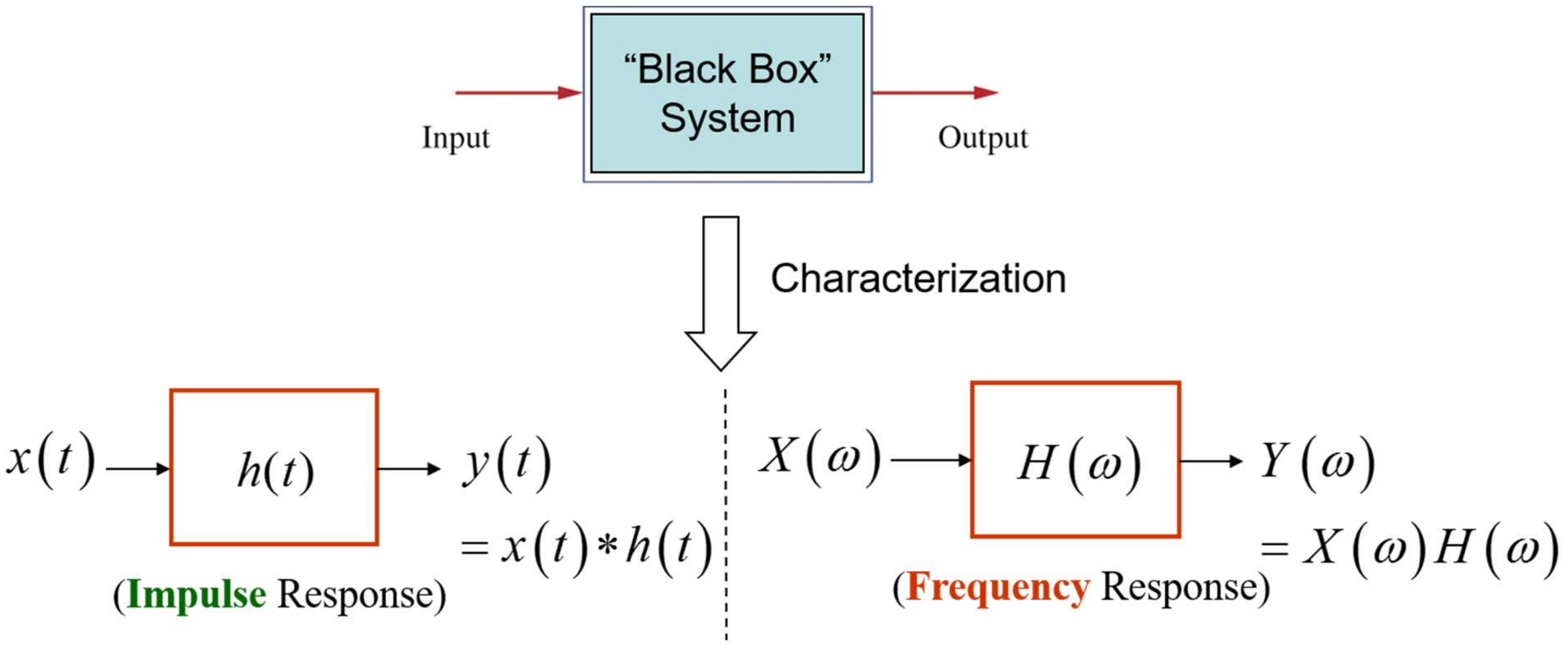
LTI System

Consider an LTI system, and $Y(\omega)$, $X(\omega)$ and $H(\omega)$ are the Fourier transforms of $y(t)$, $x(t)$, and $h(t)$, respectively.

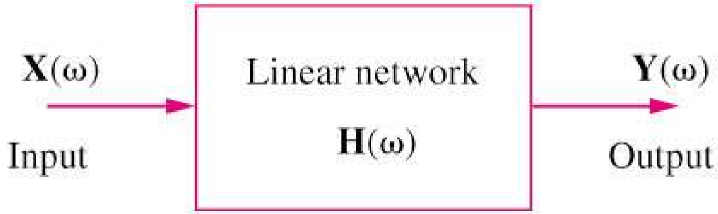


LTI System

Why the impulse response is qualified to fully characterize the LTI system?



Transfer Function



The function $H(\omega)$ is called the transfer function or the frequency response of the system.

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)} = |H(\omega)|\, e^{j\theta_H(\omega)} = |H(\omega)| \angle \theta_H(\omega)}$$

where $|H(\omega)|$ = the magnitude response

$\theta_H(\omega)$ = the phase response

of the system.

Transfer Function

Consider

$$X(\omega) = |X(\omega)|\, e^{j\theta_X(\omega)} \quad \text{and} \quad Y(\omega) = |Y(\omega)|\, e^{j\theta_Y(\omega)}$$

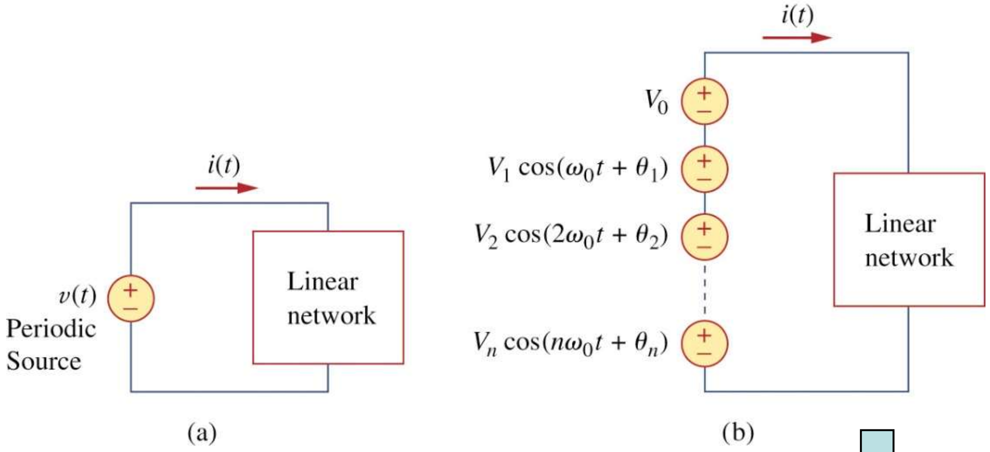
Both are complex-valued functions in polar form, thus we have

$$\boxed{\begin{aligned} |Y(\omega)| &= |X(\omega)|\,|H(\omega)| \\ \theta_Y(\omega) &= \theta_X(\omega) + \theta_H(\omega) \end{aligned}}$$

Hence, the magnitude spectrum $|X(\omega)|$ of the input is multiplied by the magnitude response $|H(\omega)|$ of the system to determine the magnitude spectrum $|Y(\omega)|$ of the output, and the phase response $\theta_H(\omega)$ is added to the phase spectrum $\theta_X(\omega)$ of the input to produce the phase spectrum $\theta_Y(\omega)$ of the output.

$\Rightarrow$ The LTI system $H(\omega)$ "*shapes*" the input signal's spectrum $X(\omega)$ through *gain* $|H(\omega)|$ (via *multiplication*) and phase $\theta_H(\omega)$ (via *addition*).

Filter



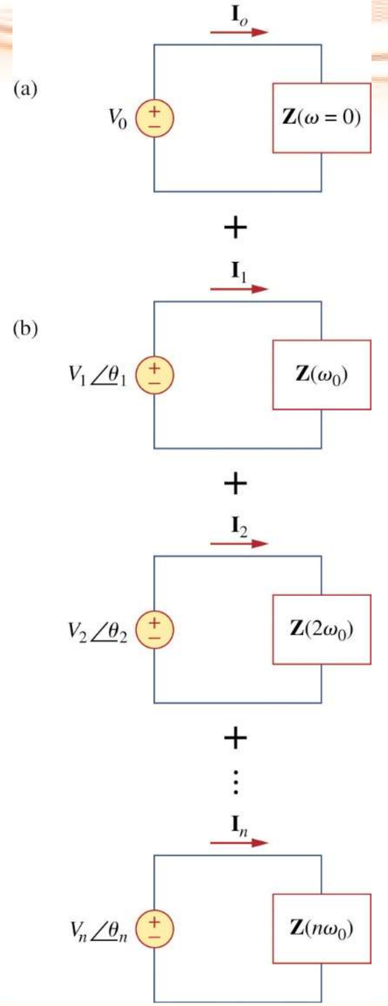
For a periodic voltage signal,

$$v(t) = \sum_{n=-\infty}^{+\infty} c_n \, e^{j n \omega_0 t}$$

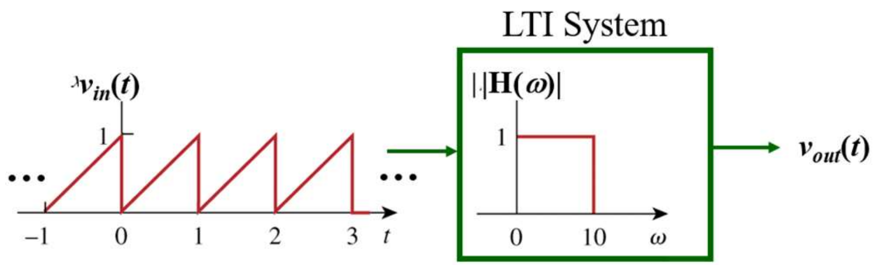
$$= a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos(n \omega_0 t) + b_n \sin(n \omega_0 t) \right\}$$

$$= a_0 + \sum_{n=1}^{\infty} A_n \cos(n \omega_0 t + \theta_n)$$

(Superposition)



Example



A sawtooth voltage input signal, $v_{in}(t)$, is applied to an LTI system with the frequency response as shown. The FS representation in the trigonometric form was previously obtained (from our Tutorial #8 Q2):

$$v_{in}(t) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2n\pi t = \frac{1}{2} - \frac{1}{\pi}\sin 2\pi t - \frac{1}{2\pi}\sin 4\pi t - \frac{1}{3\pi}\sin 6\pi t - \ldots$$

What is the output voltage waveform, $v_{out}(t)$?

In order to keep more harmonics terms, how do you design the cut-off frequency ω_c?

Example

Consider the transfer function of an LTI system:

$$2\frac{dy(t)}{dt} + y(t) = x(t)$$

Find the output response $y(t)$ for the input excitation $x(t) = 2e^{-3t}u(t)$

Solution:

Take the FT on both sides of the ordinary differential equation:

$$2\,(j\omega)Y(\omega) + Y(\omega) = X(\omega)$$

$$(j2\omega + 1)Y(\omega) = X(\omega)$$

$$\Rightarrow\ H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + j2\omega}$$

Based on the FT pair: $x(t) = e^{-at}u(t) \quad \leftrightarrow \quad X(\omega) = \dfrac{1}{a + j\omega}$

we can easily obtain $X(\omega) = \dfrac{2}{3 + j\omega}$

Example

$$\Rightarrow \; Y(\omega) = H(\omega)X(\omega) = \frac{1}{1+j2\omega}\cdot\frac{2}{3+j\omega} = \frac{2}{(1+j2\omega)(3+j\omega)}$$

$$= \frac{A}{1+j2\omega} + \frac{B}{3+j\omega}$$

where constants A and B can be found via partial fraction expansion:

$$A = \left.\frac{2}{3+j\omega}\right|_{j\omega=-0.5} = \frac{2}{2.5} = 0.8$$

$$B = \left.\frac{2}{1+j2\omega}\right|_{j\omega=-3} = \frac{2}{-5} = -0.4$$

$$\Rightarrow \; Y(\omega) = \frac{0.8}{1+j2\omega} + \frac{-0.4}{3+j\omega}$$

Take the inverse Fourier transform on both side of the transfer function:

Example

$$\mathscr{F}^{-1}\{Y(\omega)\} = \mathscr{F}^{-1}\left\{\frac{0.8}{1+j2\omega}+\frac{-0.4}{3+j\omega}\right\} = \mathscr{F}^{-1}\left\{\frac{0.8}{1+j2\omega}\right\}+\mathscr{F}^{-1}\left\{\frac{-0.4}{3+j\omega}\right\}$$

where $\mathscr{F}^{-1}\left\{\dfrac{0.8}{1+j2\omega}\right\} = 0.8\,\mathscr{F}^{-1}\left\{\dfrac{1}{1+j2\omega}\right\} = 0.8\,\mathscr{F}^{-1}\left\{\dfrac{0.5}{0.5+j\omega}\right\}$

$$= 0.4\,\mathscr{F}^{-1}\left\{\frac{1}{0.5+j\omega}\right\} \quad \leftrightarrow \quad 0.4\,e^{-0.5t}u(t)$$

$$\mathscr{F}^{-1}\left\{\frac{-0.4}{3+j\omega}\right\} = -0.4\,\mathscr{F}^{-1}\left\{\frac{1}{3+j\omega}\right\} \quad \leftrightarrow \quad -0.4\,e^{-3t}u(t)$$

Thus, $\boxed{y(t) = 0.4\left(e^{-0.5t}-e^{-3t}\right)u(t)}$

EE2010
Signals and Systems

Part II

Sampling

with Instructor:

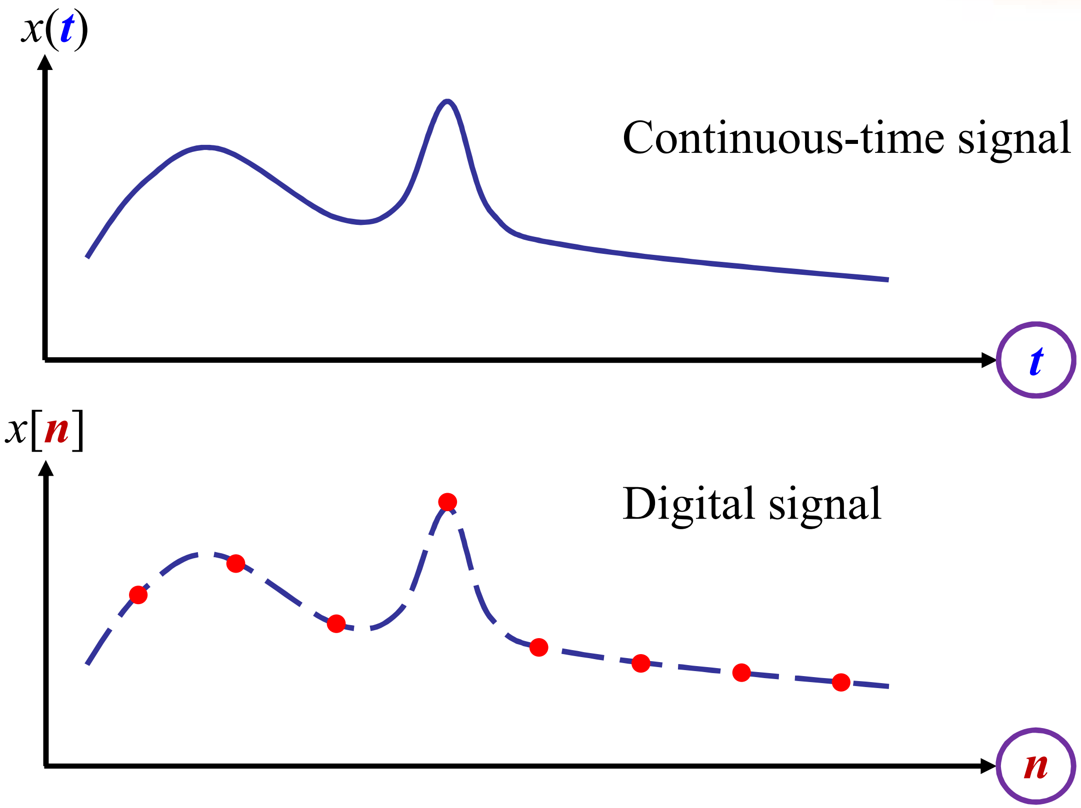
Prof. Ma, Kai-Kuang

Why Sampling?

However, computers can only handle discrete samples (or data). So, what happens if the given signal is not discrete-time, but **continuous**-time?

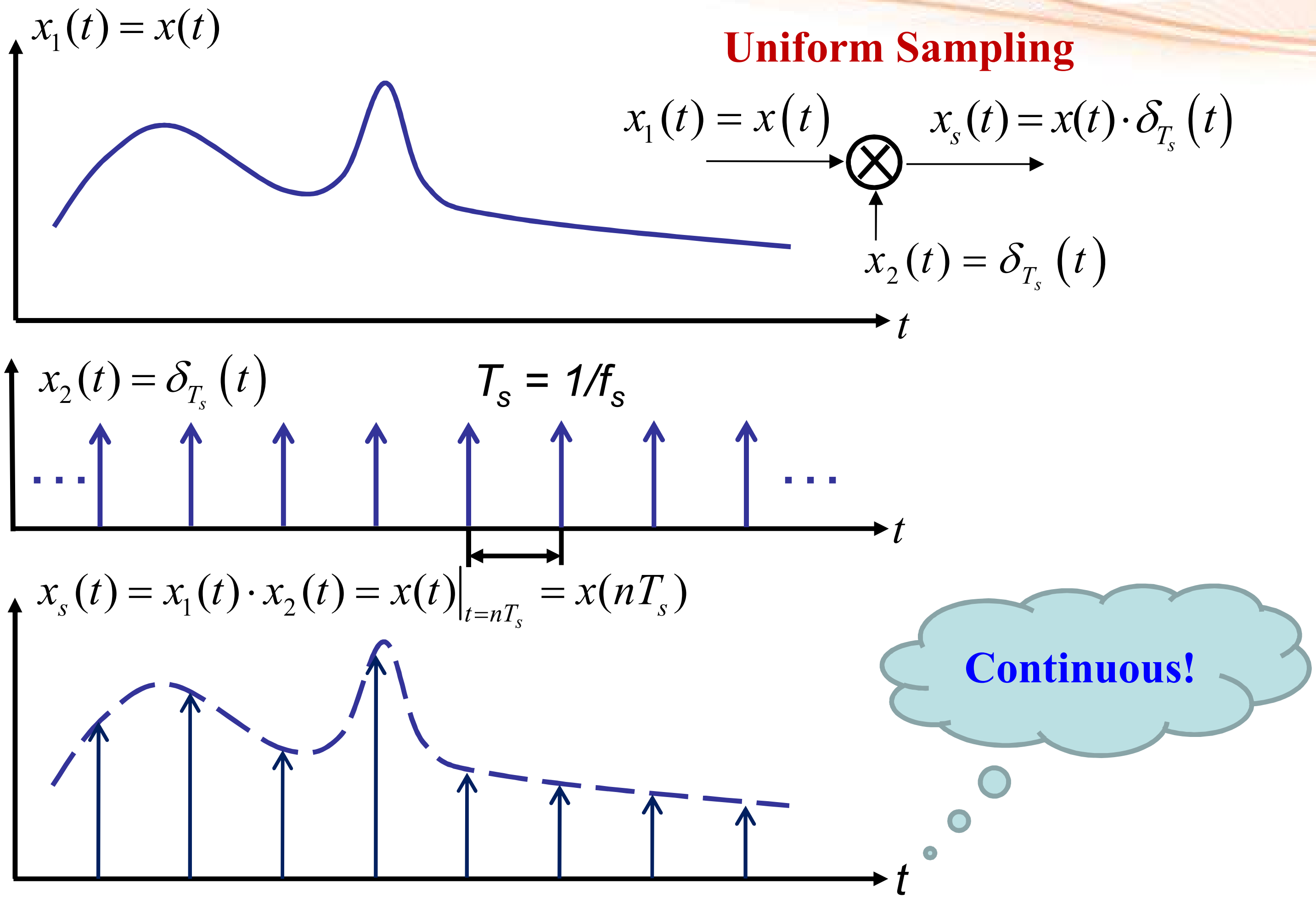
➔ **Solution:** Generate discrete-time signal from continuous-time signal as its approximation so that the computer can deal with these numbers directly. Such conversion, from continuous-time to discrete-time, is called **sampling** or **sampling process.**

Example:

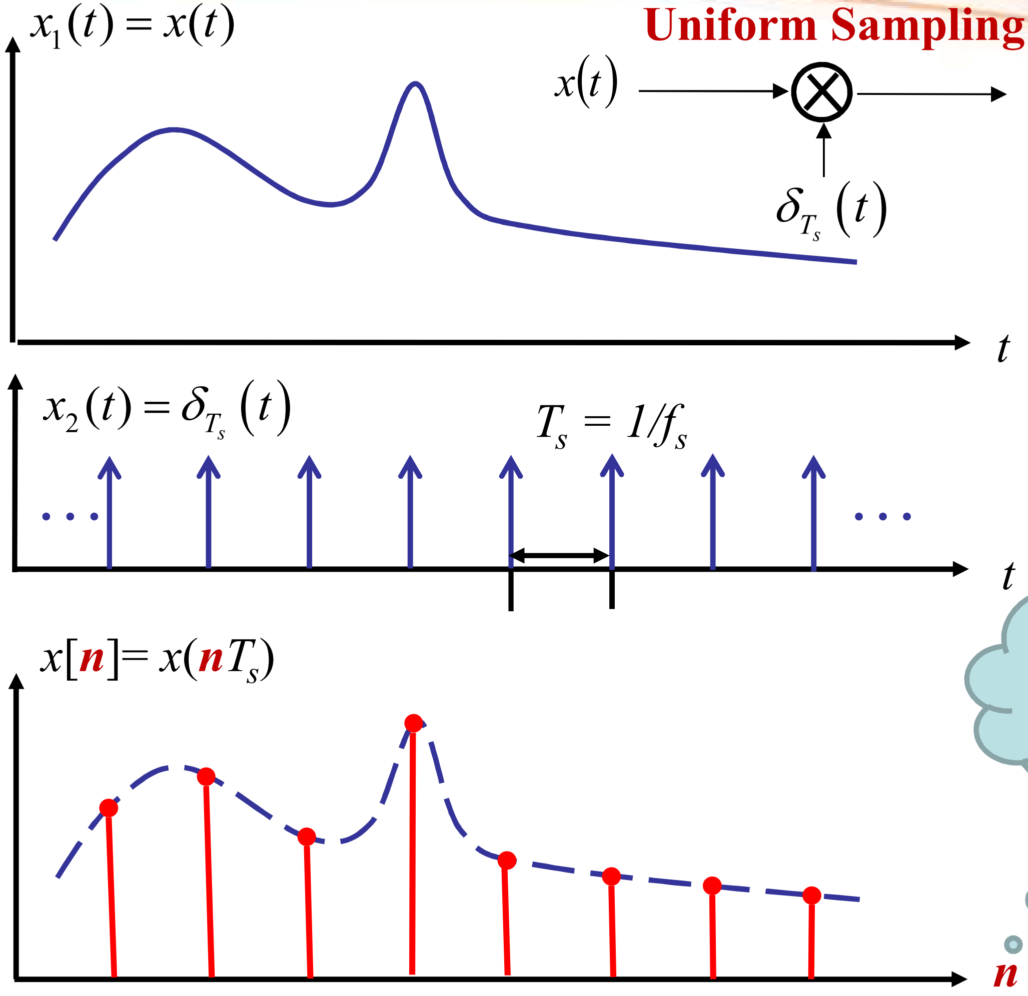


Sampling ➔ ***time***-domain discretization process

Uniform Sampling



Uniform Sampling



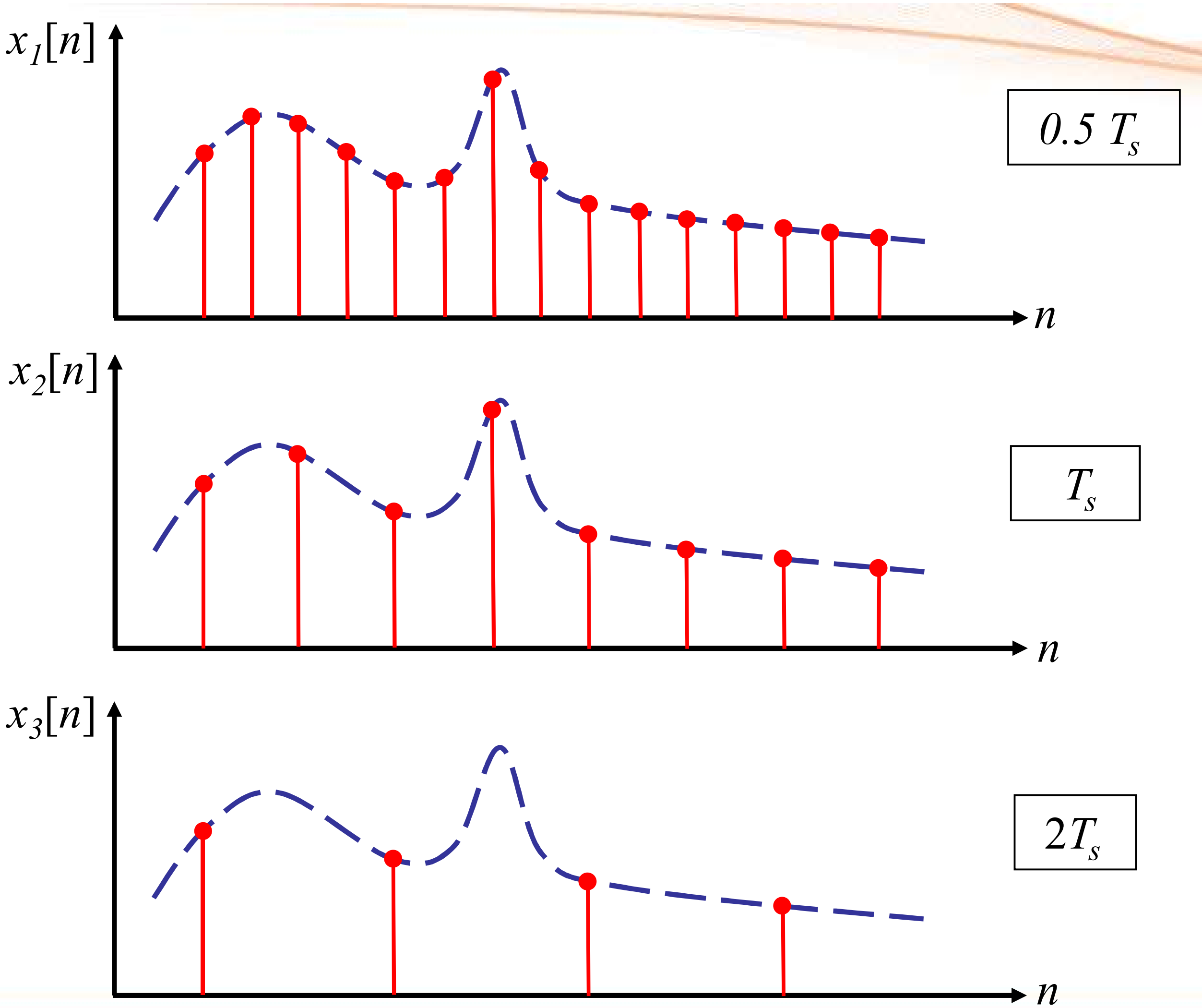
$$x[n] = x(t) \cdot \delta_{T_s}(t)$$
$$= x(t)\big|_{t=nT_s}$$
$$= x(nT_s)$$
$$= x\left(\frac{n}{f_s}\right)$$

In summary, to conduct (uniform) sampling process:

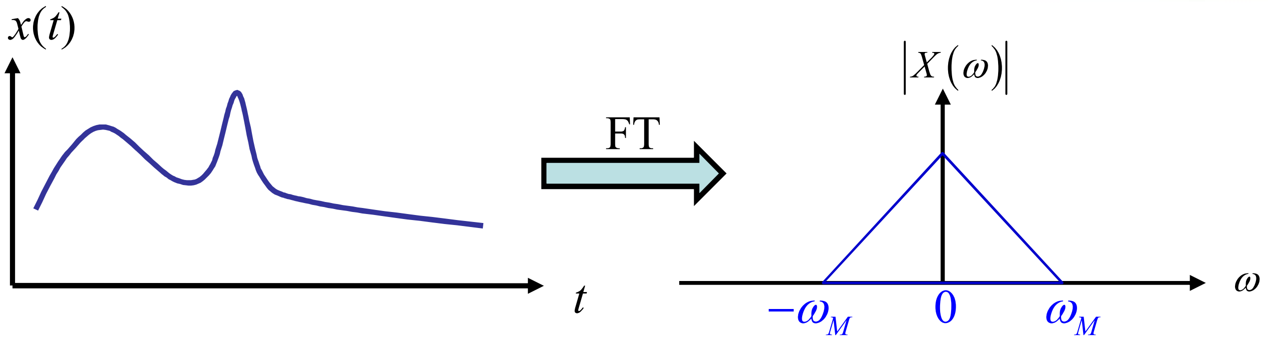
- The impulse train signal $\delta_{T_s}(t)$, with equal intervals of time T_s and unit strength, is exploited for conducting uniform sampling operation; this can be simply achieved by multiplying it to the continuous-time signal $x(t)$ for taking its samples every T_s seconds.

- As a result, $\delta_{T_s}(t)$ decides how many samples to take.
  - The sampling period is T_s (in seconds).
  - The sampling frequency is $f_s = 1/T_s$ (in samples/second).

➔ How often should we take a sample? (i.e., how to choose T_s?) Equivalently to say, how many **samples per second** (i.e., f_s) we need to take from $x(t)$?


x1[n]
n
0.5 Ts
x2[n]
n
Ts
x3[n]
n
2Ts

Band-limited Signal



$\boxed{\text{A signal } x(t) \text{ is called } \textit{band-limited}, \text{ if the bandwidth of signal } x(t) \text{ is } \textit{finite}.}$

Mathematically speaking, let ω_M denote the bandwidth of signal $x(t)$,

A signal $x(t)$ is called *band-limited*, $\underline{\text{if } \omega_M < \infty}$.

Another way to say,

A signal $x(t)$ is called *band-limited*, $\underline{\text{if } |X(\omega)| = 0, \text{ for } |\omega| > \omega_M}$.

Note: The band-limited signal is talking about the signal *itself* as its intrinsic characteristics. It has nothing to do with the filters we discussed earlier at all.

Example: A sinusoidal signal $x(t) = 3\cos(10\pi t)$

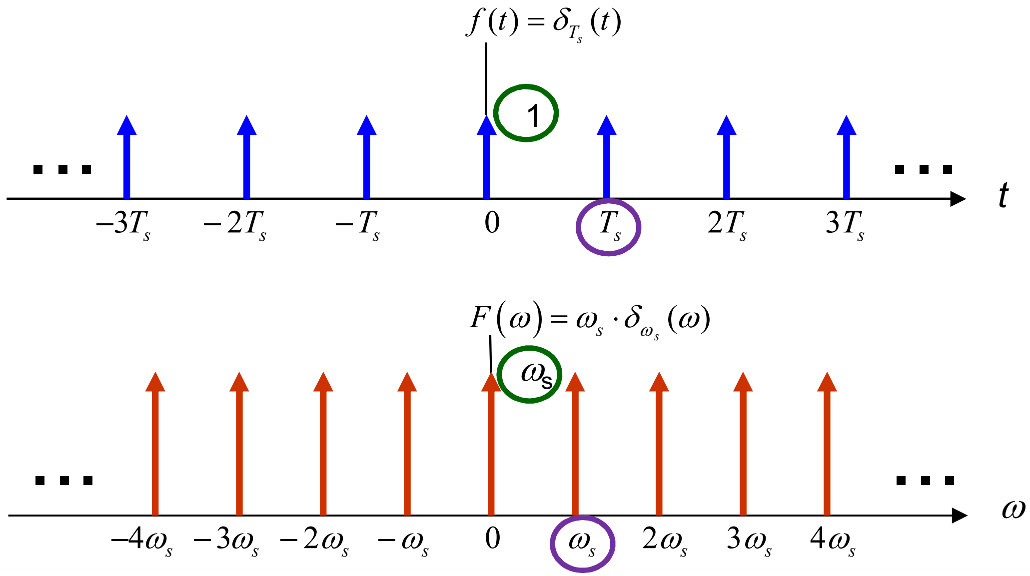
Example: All signals go in and out of our consumer electronics products (say, your handphone, hi-fi stereo system, etc.).

- If the bandwidth of signal $x(t)$ is not finite (i.e., $\omega_M = \infty$), then the $x(t)$ is called a **non-bandlimited** signal.

- How to let a non-bandlimited signal become a bandlimited one?

Two fundamental properties to be used for the uniform sampling theorem:

- $x_1(t) \cdot x_2(t) \quad \leftrightarrow \quad \dfrac{1}{2\pi} X_1(\omega) * X_2(\omega)$
- $\delta_{T_s}(t) \quad \leftrightarrow \quad \omega_s \cdot \delta_{\omega_s}(\omega) \quad \left(\text{where } \omega_s = \dfrac{2\pi}{T_s} \right)$



Recall that:

$$x_1(t) = x(t) \quad \leftrightarrow \quad X_1(\omega) = X(\omega)$$

$$x_2(t) = \delta_{T_s}(t) \quad \leftrightarrow \quad X_2(\omega) = \omega_s \cdot \delta_{\omega_s}(\omega)$$

$$x_s(t) = x_1(t) \cdot x_2(t) \quad \leftrightarrow \quad X_s(\omega) = \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

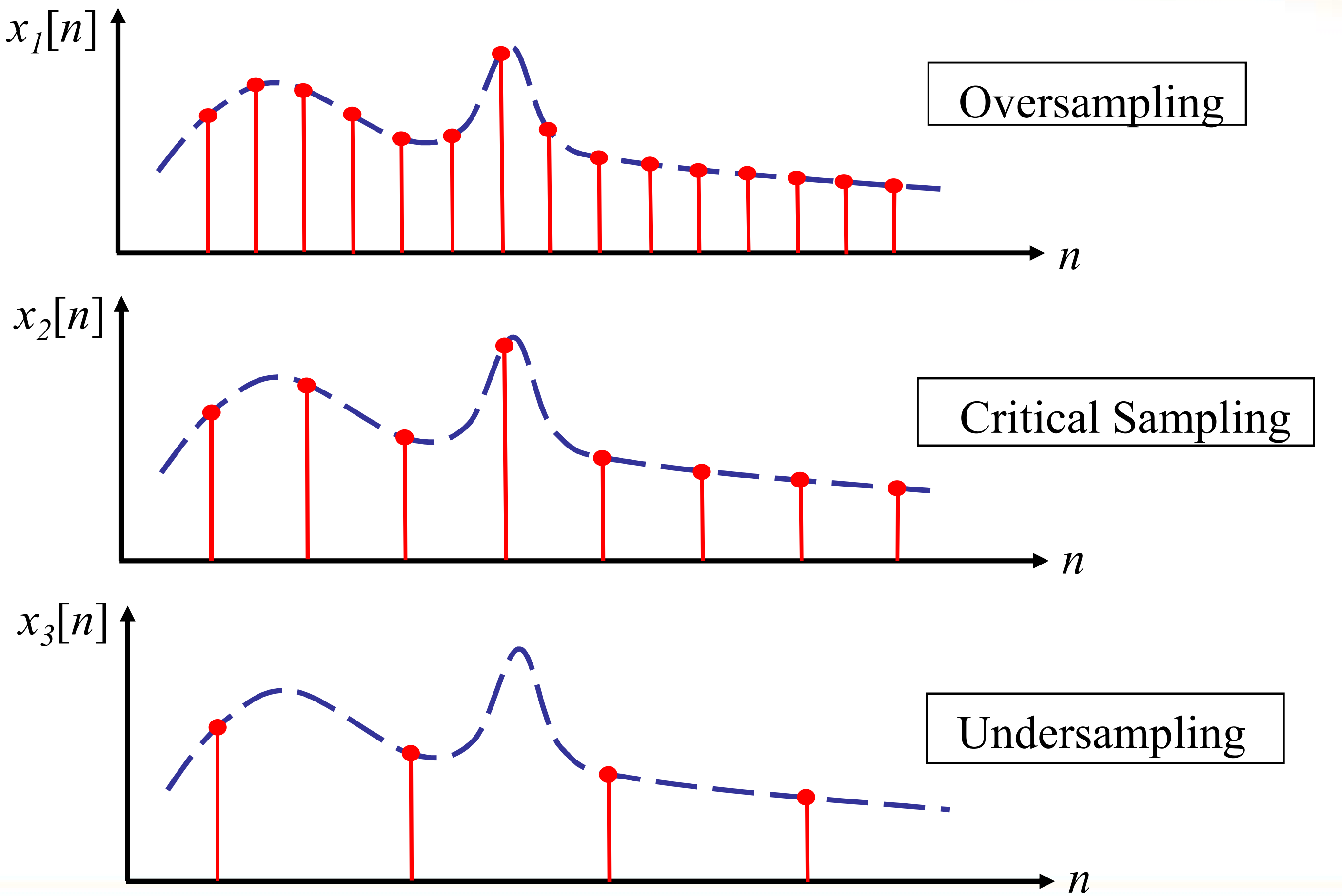
$$= \frac{1}{2\pi} X(\omega) * \left[\omega_s \cdot \delta_{\omega_s}(\omega)\right]$$

$$= \frac{\omega_s}{2\pi} X(\omega) * \delta_{\omega_s}(\omega)$$

$$= \frac{1}{T_s} X(\omega) * \delta_{\omega_s}(\omega)$$

$$\Rightarrow \boxed{X_s(\omega) = \frac{1}{T_s} X(\omega) * \delta_{\omega_s}(\omega)}$$

Three types of sampling processes:



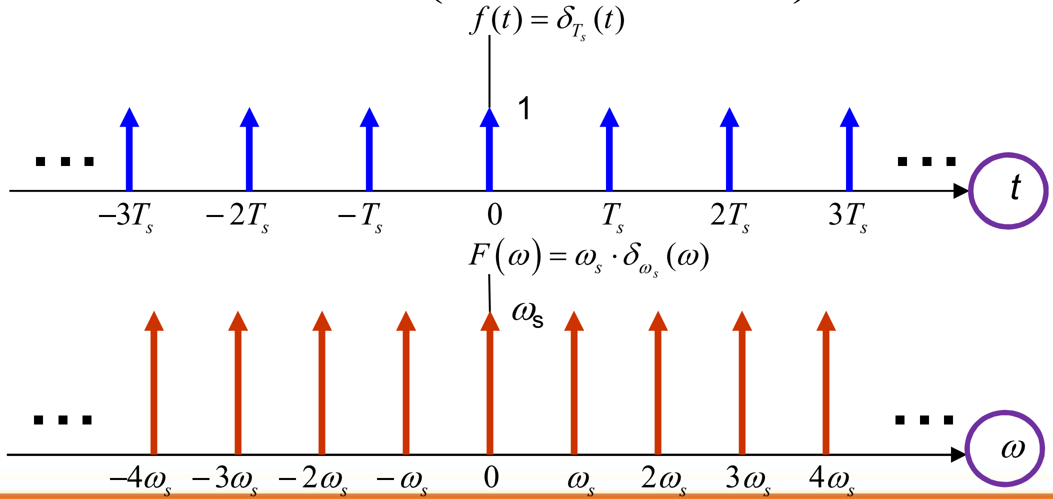
We just completed our study of the 1st fundamental property:

$\Rightarrow$ Multiplication in the time domain is equivalent to convolution in the frequency domain.

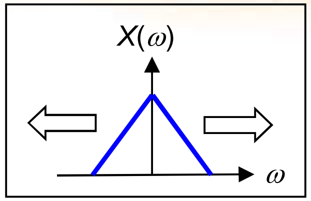
Now, study the 2nd fundamental property:

$\Rightarrow$ The Fourier transform of the impulse train is still an impulse train.

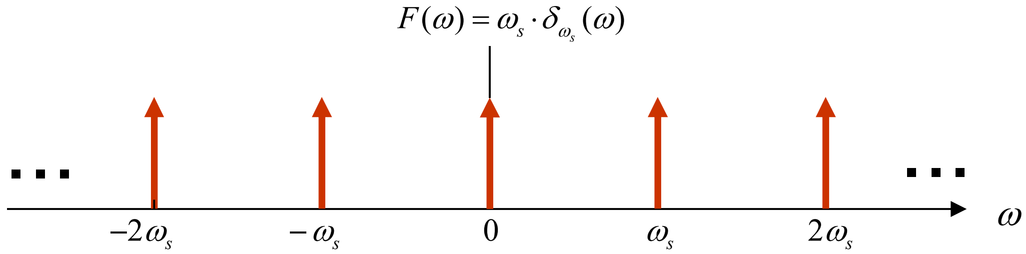
$$\delta_{T_s}(t) \quad \leftrightarrow \quad \omega_s \cdot \delta_{\omega_s}(\omega) \quad \left(\text{where } \omega_s = \frac{2\pi}{T_s}\right)$$



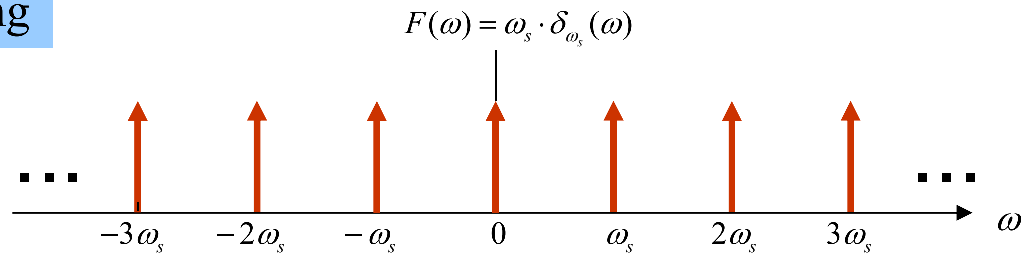
Impulse trains to be convolved with $X(\omega)$ …



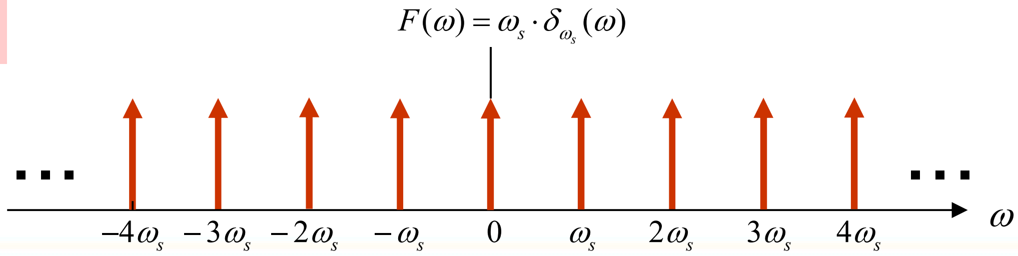
Oversampling



Critical Sampling



Undersampling



$$x_s(t) = x_1(t)\cdot x_2(t) \quad \leftrightarrow \quad X_s(\omega) = \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

$$x_s(t) = x(nT_s) = x(t)\cdot \delta_{T_s}(t) \quad \leftrightarrow \quad X_s(\omega)$$

Derivation of $X_s(\omega)$:

$$X_s(\omega) = \frac{1}{2\pi} X(\omega) * F(\omega) = \frac{1}{2\pi} X(\omega) * \left[\omega_s \cdot \delta_{\omega_s}(\omega)\right]$$

$$= \frac{1}{T_s} X(\omega) * \delta_{\omega_s}(\omega) \qquad \left(\text{Note: } \delta_{\omega_s}(\omega) = \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_s)\right)$$

$$= \frac{1}{T_s} \int_{-\infty}^{+\infty} X(\tau) \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_s - \tau)\, d\tau \qquad \left(\text{Note: } \delta(-t) = \delta(t)\right)$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(\tau)\, \delta\big(\tau - (\omega - n\omega_s)\big)\, d\tau \qquad \left(\text{Note: } \int \leftrightarrow \sum\right)$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} X(\omega - n\omega_s)$$

Interpretation of the derived result, $X_s(\omega)$:

Before sampling :

$X(\omega) \Rightarrow$ the FT of the original signal $x(t)$, also called baseband.

After sampling :

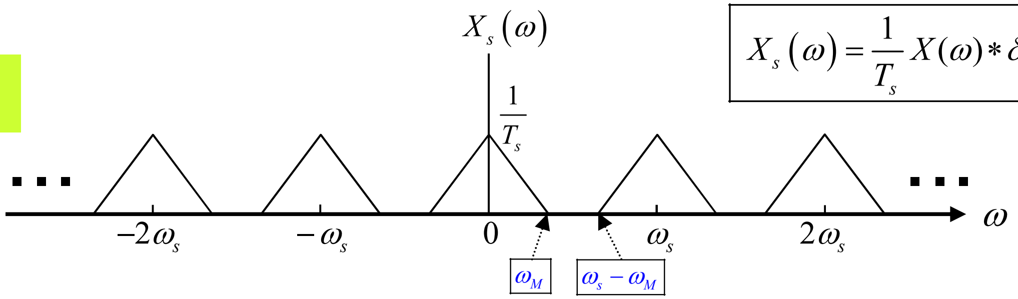
$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} X(\omega - n\omega_s) \Rightarrow \text{ the FT of the } \underline{\text{sampled}} \text{ signal } x(nT_s)$$

produces infinite number of copies of the baseband spectrum in the frequency domain, with each copy centering on the integers of ω_s, besides the scaling factor $\left(\frac{1}{T_s}\right)$ being multiplied on each copy.

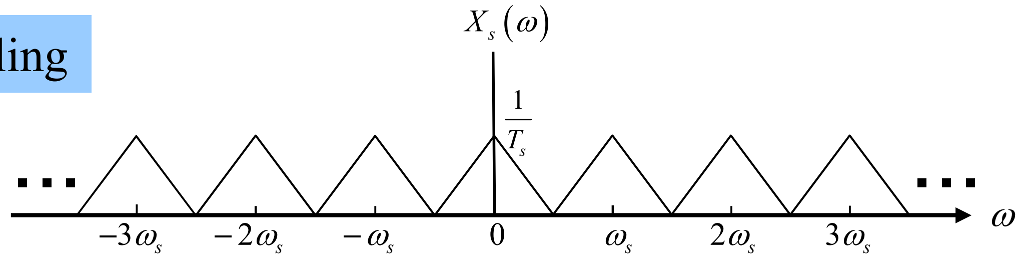
Resulted spectra after convolution …

$$X_s(\omega) = \frac{1}{T_s} X(\omega) * \delta_{\omega_s}(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} X(\omega - n\omega_s)$$

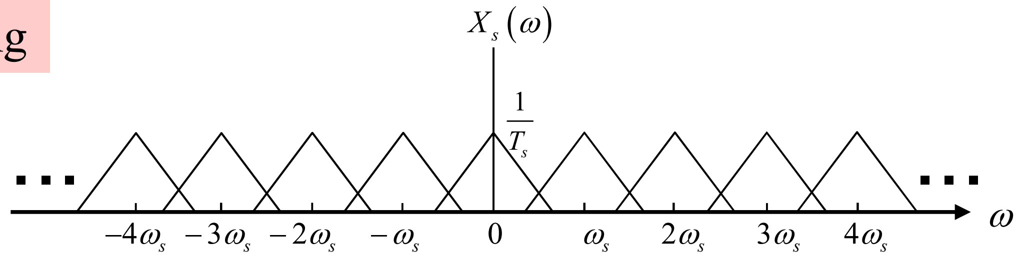
Oversampling



Critical Sampling



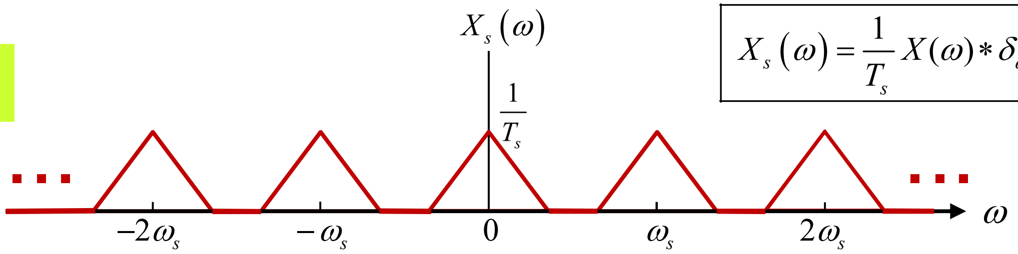
Undersampling



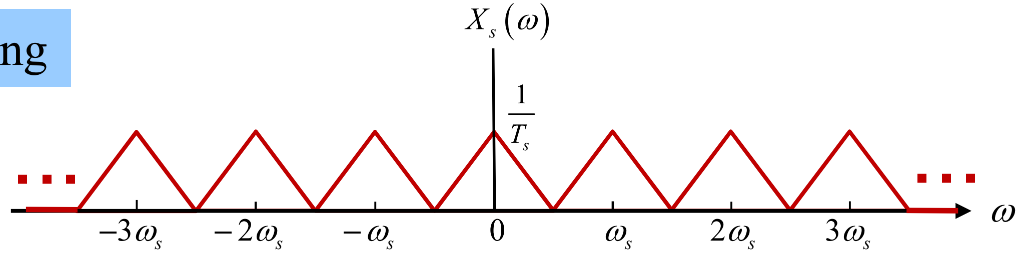
Final result of the resulted spectra …

$$X_s(\omega) = \frac{1}{T_s} X(\omega) * \delta_{\omega_s}(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} X(\omega - n\omega_s)$$

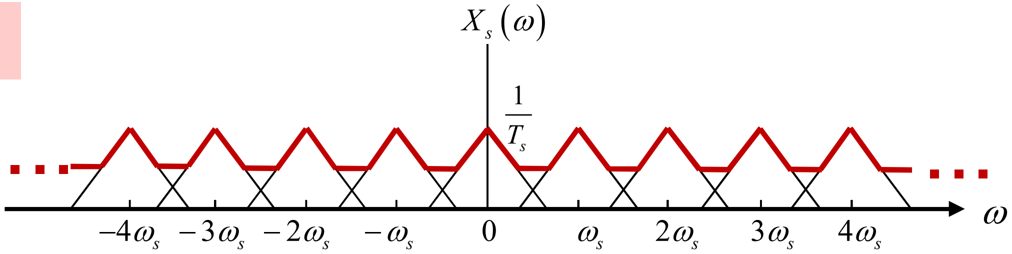
Oversampling



Critical Sampling



Undersampling



Nyquist (Sampling) Rate

$$\boxed{\text{Nyquist rate} = 2 \times \text{the maximum frequency}}$$

$\Rightarrow \quad \boxed{f_N = 2 \times f_M} \quad$ (for Bandlimited Signals)

PS : Nyquist period, $T_N = 1/f_N$

Example:

(a). $x(t) = 10\cos(426\pi t)$

Solution:

The Nyquist rate is $\omega_a = 2\pi f_a = 426\pi$; hence, $f_a = 213$ Hz

$f_M = 213$ Hz and $f_N = 2 \times 213 = 426$ Hz

(b). $x(t) = 10\cos(426\pi t - 60^\circ) - 2.5\sin(1200\pi t - 20^\circ)$

Solution:

$f_M = \max\{213,\ 600\} = 600$ Hz and $f_N = 2 \times 600 = 1200$ Hz

In Summary

- Oversampling: $\omega_s > \omega_N \quad (\text{or } f_s > f_N)$

  $\Rightarrow$ resulting in *more* samples being generated *without* aliasing.

- Critical (or Nyquist) sampling: $\omega_s = \omega_N \quad (\text{or } f_s = f_N)$

  $\Rightarrow$ the required *minimum* sampling rate *without* aliasing!

- Undersampling: $\omega_s < \omega_N \quad (\text{or } f_s < f_N)$

  $\Rightarrow$ resulting in *less* samples being generated *with* aliasing!

(Shannon) Sampling Theorem

Consider a **band-limited** signal $x(t)$ (i.e., $f_M < \infty$), the required sampling frequency (or sampling rate) f_s must satisfy the following condition:

$$\boxed{f_s \geq 2 f_M} \quad \text{or} \quad \frac{f_M}{f_s} \leq \frac{1}{2}$$

in order to preserve the original frequency content (called the ***baseband***) of the signal $x(t)$.

Since $\cos\left(2\pi(-f)\right) = \cos\left(-2\pi f\right) = \cos\left(2\pi f\right)$ and

$\sin\left(2\pi(-f)\right) = \sin\left(-2\pi f\right) = -\sin\left(2\pi f\right)$, we need to consider

negative frequency case as well; thus, the absolute value is incorporated.

$$\left|\frac{f_M}{f_s}\right| \leq \frac{1}{2} \quad \text{or} \quad \boxed{-\frac{1}{2} \leq \frac{f_M}{f_s} \leq \frac{1}{2}}$$

Generation of discrete-time signals via sampling

An analog signal $x(t) = A\cos(\omega_0 t + \theta)$ is under a sampling process to generate its discrete-time version counterpart, $x[n]$.

(In general, we wish $x[n]$ can still keep the frequency content of $x(t)$.)

Notation: Subscripts $a \rightarrow$ analog; $d \rightarrow$ digital

f_a = the analog frequency of $x(t)$ (i.e., $f_a = f_0$)

f_s = the sampling frequency (or sampling rate)

f_d = the *normalized* (digital) frequency of $x[n]$

$$\Rightarrow \boxed{x[n]} = x(t)\big|_{t=nT_s} = A\cos(\omega_0 nT_s + \theta) = A\cos(2\pi f_a nT_s + \theta)$$

$$= A\cos[2\pi \times \underbrace{\left(\frac{f_a}{f_s}\right)}_{=f_d} \times n + \theta] = \boxed{A\cos(2\pi \times f_d \times n + \theta)}$$

Example:

(a). Consider an analog sinusoidal signal $x(t) = 10\cos(426\pi t)$. If the sampling rate is 1 kHz and applied to $x(t)$, what is the resulted discrete-time sinusoidal signal $x[n]$?

(b). If the same sampling frequency 1 kHz is used to recover the analog signal $x(t)$ from $x[n]$, what is the resulted signal analog signal $x(t)$?

<u>Solution</u>: $T_s = 1/f_s = 1/1000$

(a). $$\boxed{x[n]} = x(t)\big|_{t=nT_s} = 10\cos(426\pi t) = 10\cos\left(2\pi \times \underbrace{213}_{=f_a} \times nT_s\right)$$

$$= 10\cos\left(2\pi \times \underbrace{213}_{=f_a} \times n \times \frac{1}{f_s}\right) = 10\cos\left(2\pi \times \underbrace{\frac{213}{1000}}_{=f_d=f_a/f_s} \times n\right) = \boxed{10\cos\left(\frac{213}{500}n\pi\right)}$$

(b). Since $f_s = 1000$ Hz, $f_d = \dfrac{213}{1000} = \dfrac{f_a}{f_s} = \dfrac{f_a}{1000} \quad \Rightarrow \quad \boxed{f_a = 213 \text{ Hz}}$

Note: $f_a = 213$ (in Hz) is the analog frequency of $x(t)$

$f_d = \dfrac{213}{1000}$ (no unit!) is the *normalized* (digital) frequency of $x[n]$

(Con't)

(c). If the sampling frequency used is 200 Hz instead, re-do the whole problem.

Solution: $T_s = 1/f_s = 1/200$

$$\boxed{x[n]} = x(t)\big|_{t=nT_s} = 10\cos\left(2\pi \times \frac{213}{200} \times n\right) = 10\cos\left[2\pi \times \left(1 + \frac{13}{200}\right) \times n\right]$$

$$= 10\cos\left[2\pi n + 2\pi n \times \left(\frac{13}{200}\right)\right] = 10\cos\left[2\pi n \times \left(\frac{13}{200}\right)\right] = \boxed{10\cos\left(\frac{13}{100}\pi n\right)}$$

Since $f_s = 200$ Hz, $f_d = \dfrac{13}{200} = \dfrac{f_a}{f_s} = \dfrac{f_a}{200} \Rightarrow \boxed{f_a = 13 \text{ Hz}}$ (Aliasing!)

Note that the original 213 Hz tone can not be recovered due to undersampling!

End