EE2007/IM2007 Tutorial 1
Linear Algebra
Systems of Linear Equations, Gaussian Elimination Method

EXERCISE 1. Solve the linear system using the Gaussian elimination method.

(a) $\left\{\begin{array}{rcrcrcr} -3x & - & 2y & + & 2z & = & -2 \\ -x & - & 3y & + & z & = & -3 \\ x & - & 2y & + & z & = & -2 \end{array}\right.$

(c) $\left\{\begin{array}{rrrrcr} -x_1 & & +3x_3 & +x_4 & = & 2 \\ 2x_1 & +3x_2 & -3x_3 & +x_4 & = & 2 \\ 2x_1 & -2x_2 & -2x_3 & -x_4 & = & -2 \end{array}\right.$

(b) $\left\{\begin{array}{rcrcrcr} -2x & - & 2y & + & 2z & = & 1 \\ x & & & + & 5z & = & -1 \\ 3x & + & 2y & + & 3z & = & -2 \end{array}\right.$

(d) $\left\{\begin{array}{rrrrcr} 3x_1 & -3x_2 & +x_3 & +3x_4 & = & -3 \\ x_1 & +x_2 & -x_3 & -2x_4 & = & 3 \\ 4x_1 & -2x_2 & & +x_4 & = & 0 \end{array}\right.$

[$(x=0,\ y=1,\ z=0)$;
$\{x=-1-5t,\ y=6t+\frac{1}{2},\ z=t)$;
$(x_1=1-\frac{1}{2}x_4,\ x_2=1-\frac{1}{2}x_4,\ x_3=1-\frac{1}{2}x_4,\ x_4\in\mathcal{R})$;
$(x_1=1+\frac{1}{3}x_3+\frac{1}{2}x_4,\ x_2=2+\frac{2}{3}x_3+\frac{3}{2}x_4,\ x_3,x_4\in\mathcal{R})$]

EXERCISE 2. Give restrictions on a, b, and c such that the linear system is consistent, i.e., has solution(s).

(a) $\left\{\begin{array}{rcrcrcr} x & - & 2y & + & 4z & = & a \\ 2x & + & y & - & z & = & b \\ 3x & - & y & + & 3z & = & c \end{array}\right.$

(b) $\left\{\begin{array}{rcrcrcr} x & - & y & + & 2z & = & a \\ 2x & + & 4y & - & 3z & = & b \\ 4x & + & 2y & + & z & = & c \end{array}\right.$

[$c-a-b=0$;
$c-2a-b=0$]

EXERCISE 3. Consider the system

$$\left\{\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & 3x_3 & - & x_4 & = & 1 \\ & & x_2 & - & x_3 & + & 2x_4 & = & 2 \end{array}\right.$$

(a) Described the solution set where the variables x_3 and x_4 are free.

(b) Described the solution set where the variables x_2 and x_4 are free.

(c) Show that the two solution sets obtained are equivalent

[$\{(3-2s-t, 2+s-2t, s, t)|s,t\in\mathcal{R}\}$;
$\{(7-2u-5v, u, -2+u+2v, v)|u,v\in\mathcal{R}\}$;
set $u=2+s-2t$, $v=t$, the solution set in part(b) is identical to the solution set in part (a)]

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Exercise 4. Determine the values of k such that the linear system

$$\begin{cases} kx + y + z = 0 \\ x + ky + z = 0 \\ x + y + kz = 0 \end{cases}$$

has

(a) a unique solution

(b) a one-parameter family of solutions (i.e., one free variable)

(c) a two-parameter family of solutions (i.e., two free variables)

[Unique solution if $k \neq 1$ and $k \neq -2$;
one-parameter family of solutions if $k = -2$;
two-parameter family of solutions if $k = 1$;]

Exercise 5. Find the interpolating polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ that passes through the points $(1, 12),\ (2, 15),\ (3, 16)$.

[$a_0 = 7,\ a_1 = 6,\ a_2 = -1$]

Exercise 6. The augmentation matrix of a linear system has the form

$$\left[\begin{array}{rrr|c} -2 & 3 & 1 & a \\ 1 & 1 & -1 & b \\ 0 & 5 & -1 & c \end{array}\right]$$

(a) Determine the values of a, b and c for which the linear system is consistent.

(b) Determine the values of a, b and c for which the linear system is inconsistent.

(c) When it is consistent, does the linear system have a unique solution or infinitely many solutions?

(d) Give a specific consistent linear system and find one particular solution.

[consistent when $a + 2b - c = 0$; inconsistent when $a + 2b - c \neq 0$; when consistent, there is a free variables, so infinitely many solutions; for example, choose $a = b = c = 0$, then if the variables are denoted by x, y and z, then one solution is by setting $z = 1$, then $x = \frac{4}{5}$ and $y = \frac{1}{5}$.]

EE2007/IM2007 Tutorial 2
Linear Algebra
RRE, Matrix Algebra, Properties of Determinant, ERO

Exercise 7. [Reduced Row Echelon Form] Slides 24-25
Find the reduced row echelon form of the matrices

(a) $\begin{bmatrix} -2 & 2 & -1 & 2 \\ 0 & 3 & 3 & -3 \\ 1 & -4 & 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & -3 & -4 & -2 \\ -4 & 2 & 1 & -4 \\ -1 & -3 & 1 & -4 \end{bmatrix}$

Can you write a computer program to convert a $m \times n$ matrix to reduced row echelon form?

$$[\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & \frac{6}{5} \\ 0 & 0 & 1 & \frac{8}{5} \end{bmatrix} \quad]$$

Exercise 8. [Matrix Operations] Slides 31 - 42
Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 1 & 1 \\ -3 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -1 \\ -1 & -3 \end{bmatrix}$$

Whenever possible, perform the following operations. If a computation cannot be made, explain why. [Note: A^T means the transpose of A.]

(a) $2A^T - B^T$, (b) $B^T - 2A$, (c) AB^T, (d) BA^T,
(e) $(A^T + B^T)C$, (f) $C(A^T + B^T)$, (g) $(A^TC)B$, (h) $(A^TB^T)C$

Exercise 9. [Matrix Operations] Slides 31 - 42

(a) If A is 3-by-5, B is 2-by-3, and if you stack the matrices A, B, C and D into a bigger P matrix as $P = \begin{bmatrix} A & C \\ D & B \end{bmatrix}$, what should be the dimensions of matrix C and matrix D?

(b) If A is 3-by-2 and G is 3-by-4, what should the dimensions of matrices B and E be so that the matrix multiplication $\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} E \\ G \end{bmatrix}$ make sense?

(c) If A is 3-by-2 and B is 3-by-4, what should the dimensions of matrices E and G be so that the matrix multiplication $\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} E \\ G \end{bmatrix}$ make sense?

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[C is 3-by-3 and D is 2-by-5;
B is 3-by-3 and E is 2-by-4;
E is 2-by-n and G is 4-by-n, n=1,2,3, ...]

EXERCISE 10. [Properties of Determinant] Slides 52 - 57, 59 - 60, 62

(a) If A is a 3×3 matrix and $\det(A) = 10$, find $\det(3A)$, $\det(2A^{-1})$, and $\det[(2A)^{-1}]$.

(b) If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $\det(A) = 10$, find $\det \begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix}$

(c) List the properties of determinant that you have used in parts (a) and (b).

[270;4/5;1/80]

EXERCISE 11. [Determinant via ERO] Slide 61

Use the method of ERO, find the determinant of the matrix

$$\begin{bmatrix} -6 & 4 & 5 \\ 2 & 8 & 2 \\ -1 & -4 & 2 \end{bmatrix}$$

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EE2007/IM2007 Tutorial 3
Linear Algebra
Elementary Matrices, LU Factorization

Exercise 12. **[Elementary Matrices]** Slides 63 - 65

Let E and F be elementary matrices. If E adds row 1 to row 2, and F adds row 2 to row 1, does EF equal FE? Justify your answer.

[No, $EF \neq FE$]

Exercise 13. **[Elementary Matrices]** Slides 63 - 65

(a) Find the matrix P that will re-arrange the column vector $\mathbf{u} = \begin{bmatrix} a & b & c & d & e & f \end{bmatrix}^T$ to $\mathbf{v} = \begin{bmatrix} a & c & e & b & d & f \end{bmatrix}^T$.

(b) Find the matrix Q that will re-arrange the row vector $\mathbf{p} = \begin{bmatrix} a & b & c \end{bmatrix}$ to $\mathbf{q} = \begin{bmatrix} a & c & b \end{bmatrix}$.

(c) If, in (b), the dimensions of a, b, and c are 2-by-2, 2-by-3, and 2-by-4 respectively. What will be the dimensions of Q? Write down the matrix Q. How would your answer change if the dimensions of a, b, and c are n-by-2, n-by-3, and n-by-4 respectively?

$$[\quad \text{(a) } v = Pu \text{ where } P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{(b) } q = pQ \text{ where } Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(c) Q is 9-by-9. Form Q as follows: Q = eye(9,9); Q = Q(:,[1 2 6 7 8 9 3 4 5]; no change]

Exercise 14. [Determinants and Inverses of Elementary Matrices] Slide 63, 67
Write down the elementary matrix, its determinant and inverse for each of the following elementary row operation on a 4×4 matrix?

(a) Interchange rows 1 and 3

(b) Multiply row 3 by a factor of 5

(c) Add eight times of row 2 to row 1

[The determinants are -1, 5 and 1; The inverses are elementary matrices that would "undo" the corresponding elementary row operations]

Exercise 15. [Determinant via Elementary Matrices] Slide 66

Use the method of Elementary Matrices, find the determinant of the matrix

$$\begin{bmatrix} -6 & 4 & 5 \\ 2 & 8 & 2 \\ -1 & -4 & 2 \end{bmatrix}$$

[-168]

Exercise 16. [Elementary Matrices] Slides 63 - 67

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad |A| = 5, \quad \text{and}$$

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^n A \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^n$$

where n is a positive integer.

(a) With $n = 1$, explain, in terms of row and column operations, what operations were performed on matrix A.

(b) Write down the resulting matrix B when n is old, i.e., $n = 1, 3, 5, \ldots$, and the determinant of matrix B. Justify your answer.

(c) Write down the resulting matrix B when n is even, i.e., $n = 2, 4, 6, \ldots$, and the determinant of matrix B. Justify your answer.

Exercise 17. [**Determinant of Block Diagonal Matrix**]

(a) Show that the determinant of the block partitioned matrix $\begin{bmatrix} I_n & 0 \\ 0 & B \end{bmatrix}$ is equal to $\det(B)$.

(b) Show that $\det(\begin{bmatrix} A & C \\ 0 & I_m \end{bmatrix}) = \det(A)$

(c) Hence, or otherwise, show that $\det(P) = \det(A)\det(B)$ where P is the block partitioned matrix $P = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and the sub-matrices A, B and C have dimensions $n \times n$, $m \times m$ and $n \times m$ respectively.

(d) Given that

$$A = \begin{bmatrix} 3 & 2 & 0 & 2 \\ 6 & 8 & 2 & 1 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -6 & 4 & 5 \\ 2 & 8 & 2 \\ -1 & -4 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 12 & 6 & 1 \\ 4 & 4 & 1 \\ 7 & 4 & 3 \\ 8 & 8 & 3 \end{bmatrix}$$

Determine the determinant of the matrix $P = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$.

[(a)By expanding the determinant along the first row or first column;
(b)By expanding the determinant along the last row;
(c)By factorizing $P = \begin{bmatrix} I_n & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} A & C \\ 0 & I_m \end{bmatrix}$;
$\det(P) = -12096$]

Exercise 18. [**LU factorization**]

Solve the linear systems using LU factorization.

(a) $\left\{ \begin{array}{rrrrcr} 2x & -4y & +7z & +3w & = & 14 \\ x & -y & +5z & +3w & = & 6 \\ x & -2y & +3z & +w & = & 5 \\ -x & +y & -5z & -2w & = & -8 \end{array} \right.$ (b) $\left\{ \begin{array}{rrrrcr} 2x & +2y & +2z & +2w & = & 1 \\ x & +2y & +2z & -w & = & 5 \\ -x & -2y & -z & +4w & = & 1 \\ & y & +z & -w & = & -2 \end{array} \right.$

[$L = \begin{bmatrix} 1 & & & \\ 0.5 & 1 & & \\ 0.5 & 0 & 1 & \\ -0.5 & -1 & 0 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & -4 & 7 & 3 \\ & 1 & 1.5 & 1.5 \\ & & -0.5 & -0.5 \\ & & & 1 \end{bmatrix}$;
$x = -25, y = -7, z = 6, w = -2$;

$L = \begin{bmatrix} 1 & & & \\ 0.5 & 1 & & \\ -0.5 & -1 & 1 & \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 2 & 2 & 2 \\ & 1 & 1 & -2 \\ & & 1 & 3 \\ & & & 1 \end{bmatrix}$, $x = 15.5, y = -34, z = 25.5, w = -6.5$
]

EE2007/IM2007 Tutorial 3A
Linear Algebra

Additional Questions for Tutorial 3
Matrix Inverse via Elementary Row Operations

Abstract

The aims of this tutorial are to illustrate (1) the method of finding matrix inverse via Elementary Row Operations (EROs), and (2) that EROs can be applied to block matrices. The tutorial first starts with asking students to use EROs to re-derive the familiar formula for the inverse of a 2-by-2 matrix. Then, by applying the EROs to block matrices, one could derive the formula for the inverse of a block partition matrix.

Slides 43 - 52

EXERCISE 19. **[Use ERO to find inverse]**

(a) Use ERO to derive the formula for the inverse of the 2-by-2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. What assumptions do you need for the inverse to exist?

(b) Repeat part (a) for the block partition matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C, D are matrices of compatible dimensions. What assumptions do you need for the inverse to exist?

EE2007/IM2007 Tutorial 4
Linear Algebra
Linear Combination, Independence, Span, Basis

EXERCISE 20. **[Concepts and Examples discussed in lectures]**
The following were discuss in lectures. Review them and ask questions in the tutorial class, if necessary. Generate your own example to test your understanding.

Given $\mathbf{v}, \mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$ in $\mathbb{R}^3$. What do you have to do to

(a) Determine whether $\mathbf{v}$ is a linear combination of $\mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$

(b) Find $span(\mathbf{v_1}, \mathbf{v_2})$ and $span(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$

(c) Determine whether $\mathbf{v}$ is in $span(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$

(d) Determine whether $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent or not.

EXERCISE 21. **[Linear Combinations]** slides 77 - 79

(a) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 4 \\ -3 & 2 & 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$. Write the product $A\mathbf{x}$ as a linear combination of the column vectors of A.

(b) Let $A = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix}$. Write each column vector of AB as a linear combination of the column vectors of A.

(c) Describe all vectors in $\mathbb{R}^3$ that can be written as a linear combination of the vectors

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 7 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

[$[a, b, c]'$ such that $3a - b + c = 0$]

Exercise 22. [Row and column spaces of a matrix] Slides 100 - 103

Consider the matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$.

(a) Is $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the column space of A?

(b) Is $\mathbf{w} = [4\ 5]$ in the row space of A?

(c) Describe $row(A)$ and $col(A)$.

[Yes;Yes;$row(A) = \mathbb{R}^2$;$col(A)$ is the plane spanned by the two column vectors of A and passes through the origin, or equivalently the plane $z - 3x = 0$.]

Exercise 23. [Linearly Indepedence] Slides 86 - 88

Determine whether the following are linearly independent or dependent. Justify your answers.

(a) The vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(b) The matrices $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$.

(c) The polynomials $p(x) = 1 + x, q(x) = 1 - x$ and $h(x) = 1 - x^2$.

[Yes;Yes;Yes]

Exercise 24. [Span] Slides 80 - 85

Answer the following and justify your answers.

(a) Is $\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$ in the space spanned by $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$?

(b) Is $2 - 3x + x^2$ in the space spanned by $1 + x$ and $1 + x^2$?

[Yes; No]

Exercise 25. **[Linear Combination, Linearly Independence]**
If $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are independent vectors. Show that $\mathbf{v}_1 = \mathbf{w}_2 + \mathbf{w}_3, \mathbf{v}_2 = \mathbf{w}_1 + \mathbf{w}_3$ and $\mathbf{v}_3 = \mathbf{w}_1 + \mathbf{w}_2$ are independent. (Hint: write $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$ in terms of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$, and solve for c_1, c_2, c_3.)

Exercise 26. **[Rank and nullity]** Slides 116

If A is a 3×5 matrix, explain why the columns of A must be linearly dependent. What are the possible values of $nullity(A)$?

[$nullity(A) = 2, 3$ or 4]

EE2007/IM2007: Tutorial 4A[1]

Additional Questions for Tutorial 4

Linear Combinations, Independence, Span, Bases as $Ax = b$

Abstract

The aim of this tutorial is to help students connect the mechanism of solving $Ax = b$ problems with concepts discussed from slides 74 onwards, i.e., concepts such as Linear Combinations, Linear Indepdence, Span, Column-, Row-, and Null-spaces of a matrix, etc.

In Tutorial 1, Exercise 1, you were asked to solve $Ax = b$ problems by Gaussian elimination. Here, instead of directly asking you to solve $Ax = b$, we will rephrase the question in various ways such that finding answers to these questions all boil down to solving some $Ax = b$ problems.

Here is how we will play the game of translation: by just looking at the answers (just the answers, don't need to work out or refer to the solutions), see if you could answer the following questions, and give reasons to support your answers.

For easy reference, Tutorial 1, Exercise 1 with answers is reproduced here.

Exercise 1. Solve the linear system using the Gaussian elimination method.

(a) $$\begin{cases} -3x - 2y + 2z = -2 \\ -x - 3y + z = -3 \\ x - 2y + z = -2 \end{cases}$$

(c) $$\begin{cases} -x_1 + 3x_3 + x_4 = 2 \\ 2x_1 + 3x_2 - 3x_3 + x_4 = 2 \\ 2x_1 - 2x_2 - 2x_3 - x_4 = -2 \end{cases}$$

(b) $$\begin{cases} -2x - 2y + 2z = 1 \\ x + 5z = -1 \\ 3x + 2y + 3z = -2 \end{cases}$$

(d) $$\begin{cases} 3x_1 - 3x_2 + x_3 + 3x_4 = -3 \\ x_1 + x_2 - x_3 - 2x_4 = 3 \\ 4x_1 - 2x_2 + x_4 = 0 \end{cases}$$

$$[\ (x = 0,\ y = 1,\ z = 0);$$
$$\{(-1 - 5t, 6t + \tfrac{1}{2}, t) | t \in \mathcal{R}\}$$
$$(x_1 = 1 - \tfrac{1}{2}x_4,\ x_2 = 1 - \tfrac{1}{2}x_4,\ x_3 = 1 - \tfrac{1}{2}x_4,\ x_4 \in \mathcal{R});$$
$$(x_1 = 1 + \tfrac{1}{3}x_3 + \tfrac{1}{2}x_4,\ x_2 = 2 + \tfrac{2}{3}x_3 + \tfrac{3}{2}x_4,\ x_3, x_4 \in \mathcal{R})\ \ ]$$

Figure 1: Tutorial 1, Exercise 1

[1] This article was first published on 14Feb2017, as a Valentine day special edition: "In Love with Linear Algebra"

Translate Exercise 1(a)

Denote

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ -3 \\ -2 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -2 \\ -3 \\ -2 \end{bmatrix}, \; A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$$

(a) Is $\mathbf{b}$ a linear combination of $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$?

(b) Are the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ linearly independent?

(c) Is $\mathbf{b}$ in $span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$?

(d) Is the matrix A invertible?

(e) Is $\mathbf{b}$ in the column space of A?

(f) Find a basis for the column space of A.

(g) What are the dimensions for the column-, row- and null spaces of A?

Got it? Let's try the same with Exercise 1(b).

Translate Exercise 1(b)

Denote

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \; \mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}, \; \mathbf{v}_3 = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \; A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$$

and repeat the same translations.

Conclusion

Hope this helps. Generate your own translations for other questions in Tutorial 1. Discuss with your tutors, or me :-)

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EE2007/IM2007 Tutorial 5

Linear Algebra

Diagonalisation, Eigenvalues and Eigenvectors

Exercise 27. **[Diagonalisation of matrix]**

Explain how eigenvalues and eigenvectors are used to diagonalise a $n \times n$ matrix. What condition(s) is(are) needed so that a matrix A can be diagonalised? The next two exercises reinforce the concept.

[see lecture notes]

Exercise 28. **[Diagonalisation of matrix]**

Let $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(a) Find the eigenvalues of A.

(b) From your result in part (a) can you conclude whether A is diagonalizable? Explain.

(c) Find the eigenvectors corresponding to each eigenvalue.

(d) Are the eigenvectors found in part (c) linearly independent? Explain.

(e) From your result in part (d) can you conclude whether A is diagonalizable? Explain.

(f) If your answer to part (e) is yes, find a matrix P that diagonalizes A. Specify the diagonal matrix D such that $D = P^{-1}AP$.

[(a) $0, 2, -1$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

(f) $P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & & \\ & 2 & \\ & & -1 \end{bmatrix}$]

EXERCISE 29. Repeat the previous exercise with

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

[(a)0, 0, 1, 1;

(c) $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$]

EXERCISE 30. **[Eigenvalues and eigenvectors of symmetric matrix]**

Let $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ for some real numbers of a and b.

(a) Show that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A. What is the corresponding eigenvalue?

(b) Find the other eigenvalue of A and the corresponding eigenvector.

(c) Diagonalize the matrix A.

[(a) $a + b$; (b) $a - b$; $[1 \ \ -1]^T$]

EXERCISE 31. **[Eigenvalues and eigenvectors]**
Find the matrix B if B has eigenvalues -2 and 5 and the corresponding eigenvectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ respectively. How can you tell in advance, without computing B, whether B is symmetric? Explain clearly your reasoning.

[$B = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$]

EXERCISE 32. [Eigenvalues and Eigenvectors of Block Diagonal Matrix]

In an earlier Tutorial, we studied the determinant of block diagonal matrices. We showed that $\det(P) = \det(A)\det(B)$ where P is the block partitioned matrix $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and the sub-matrices A, B and C have dimensions $n \times n$, $m \times m$ and $n \times m$ respectively.

What can you say about the eigenvalues (and eigenvectors[2]) of P? State all assumptions made.

With the help of `MATLAB`, verify that your conclusion with the following numerical example (or any numerical example of your own):

$$A = \begin{bmatrix} 3 & 2 & 0 & 2 \\ 6 & 8 & 2 & 1 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -6 & 4 & 5 \\ 2 & 8 & 2 \\ -1 & -4 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 12 & 6 & 1 \\ 4 & 4 & 1 \\ 7 & 4 & 3 \\ 8 & 8 & 3 \end{bmatrix}$$

[Eigenvalues of P are the eigenvalues of A and B;
Eigenvalues of A satisfy $(\lambda_A^2 - 11\lambda_A + 12)(\lambda_A^2 - 9\lambda_A + 6) = 0$;
Eigenvalues of B satisfy $\lambda_B^3 - 4\lambda_B^2 - 39\lambda_B + 168 = 0$;
If $\mathbf{v_A}$ and $\mathbf{v_B}$ are eigenvectors of A and B respectively, then $\begin{bmatrix} \mathbf{v_A} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} (\lambda_B I - A)^{-1} C \mathbf{v_B} \\ \mathbf{v_B} \end{bmatrix}$]

[2] can skip if time does not permit

EE2007/IM2007 Tutorial 6
Linear Algebra
Diagonalisation, Applications

Exercise 33. [Linear System of Differential Equations]

Solve the following linear system of differential equations by reducing it to an uncoupled system.

$$\mathbf{y}'(t) = \begin{bmatrix} -5/50 & 5/100 \\ 5/50 & -5/100 \end{bmatrix} \mathbf{y}(t) \quad \text{where } y_1(0) = 8, y_2(0) = 0$$

$$[\ \mathbf{y}(t) = \tfrac{8}{3} \begin{bmatrix} 2e^{-\frac{3}{20}t} + 1 \\ -2e^{-\frac{3}{20}t} + 2 \end{bmatrix}\ ]$$

Exercise 34. [Markov Chain, nth power 2×2 Transition Matrix]

Derive a general expression for the nth power of the transition matrix

$$T = \begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix}$$

where $0 < p < 1$ and $0 < q < 1$.

Hence find the steady state probability vector.

$$[\ T^n = \tfrac{p}{p+q} \begin{bmatrix} \frac{q}{p} + \lambda_2^n & \frac{q}{p}(1-\lambda_2^n) \\ 1-\lambda_2^n & 1+\lambda_2^n \frac{q}{p} \end{bmatrix} \text{ where } \lambda_2 = 1-p-q;\ \tfrac{1}{p+q} \begin{bmatrix} q \\ p \end{bmatrix}\ ]$$

Exercise 35. **[Markov Chain, 3×3 matrix]**

A group of insurance plan allows three different options for participants, plan A, B, or C. Suppose that the percentages of the total number of participants enrolled in each plan are 25 percent, 30 percent, and 45 percent, respectively. Also, from past experience

(a) 15 percent and 10 percent of the participants who originally enrolled in plan A will switch to plan B and plan C respectively.

(b) 25 percent and 30 percent of the participants who originally enrolled in plan B will switch to plan A and plan C respectively.

(c) 20 percent and 40 percent of the participants who originally enrolled in plan C will switch to plan A and plan B respectively.

Construct a mathematical model for this system, and hence determine, in the long term, the percentage of enrollment in each plan.

$$\left[\ T = \begin{bmatrix} 0.75 & 0.25 & 0.2 \\ 0.15 & 0.45 & 0.4 \\ 0.1 & 0.3 & 0.4 \end{bmatrix}; \text{Plan A } 47.73\%, \text{ Plan B } 29.54\%, \text{ Plan C } 22.73\%\ \right]$$