Tutorial 1 (Solutions) (Tutorial 7)

1a) Let $y = \ln(i^{1/2})$

$$e^y = i^{1/2}$$

$$= \left[e^{i\left(\frac{\pi}{2}+2n\pi\right)}\right]^{1/2} \qquad n = 0, \pm 1, \pm 2, \dots$$

$$= e^{i\left(\frac{\pi}{4}+n\pi\right)}$$

$$\therefore \quad \underline{\underline{y = i\left(\frac{\pi}{4}+n\pi\right) \qquad n = 0, \pm 1, \pm 2, \dots}}$$

b) Let $y = i^i$

$$\ln y = i \ln i$$

$$= i \ln e^{i\left(\frac{\pi}{2}+2n\pi\right)} \qquad n = 0, \pm 1, \pm 2, \dots$$

$$= i\left[i\left(\frac{\pi}{2}+2n\pi\right)\right]$$

$$= -\left(\frac{\pi}{2}+2n\pi\right)$$

$$\therefore \quad \underline{\underline{y = e^{-\left(\frac{\pi}{2}+2n\pi\right)} \qquad n = 0, \pm 1, \pm 2, \dots}}$$

<u>which is real-valued.</u>

(b) (Cont'd).

Let $y = z^i$

$\ln y = i \ln z$

$= i \ln re^{i(\theta + 2n\pi)}$ $\quad n = 0, \pm 1, \pm 2, \ldots$

$= i \ln r - (\theta + 2n\pi)$

$y = z^i = e^{i \ln r} \cdot e^{-(\theta + 2n\pi)}$

$e^{i \ln r} = \cos(\ln r) + i \sin(\ln r)$.

For $y = z^i$ to be real,

$\sin(\ln r) = 0$.

$\ln r = \pm k\pi \quad k = 0, 1, 2, \ldots$

$\underline{r = e^{\pm k\pi}}$.

$\therefore z^i = (re^{i\theta})^i$

$= (e^{\pm k\pi} \cdot e^{i\theta})^i$ is real.

The values of z for real z^i are

$$\underline{\underline{z = e^{\pm k\pi} \cdot e^{i\theta} \quad k = 0, 1, 2, \ldots}}$$

2 a)

$$f(z) = \frac{x^2 y}{x^3 + y^3} + i x y$$

For the limit to exist, $\lim_{z \to z_0} f(z)$ need to be unique and independent of the directions in which z approaches z_0.

Let the direction be given by $y = kx$, k is a constant.

$$\lim_{z \to 0} f(z) = \lim_{x \to 0,\, y = kx} f(z)$$

$$= \lim_{x \to 0} \frac{kx^3}{x^3 + k^3 x^3} + i k x^2$$

$$= \frac{k}{1 + k^3}$$ which depends on k

(the direction in which x, y approach zero)

$\Rightarrow$ the limit does not exist

2b) Let $z = re^{i\theta}$

$$\lim_{z \to 0} f(z) = \lim_{r \to 0} f(z)$$

$$= \lim_{r \to 0} \left[\frac{re^{-i\theta}}{re^{i\theta}} - \frac{re^{i\theta}}{re^{-i\theta}} - \frac{r^2 e^{i2\theta}}{r^2 e^{-i2\theta}} \right].$$

$$= e^{-i2\theta} - e^{i2\theta} - e^{i4\theta}$$

$\Rightarrow$ the limit does not exist

3 a). A function $f(z)$ is continuous at $z = z_0$

if (a) $f(z_0)$ is defined, and

(b) $\lim\limits_{z \to z_0} f(z) = f(z_0)$.

$$f(z) = \begin{cases} Re\left[\dfrac{z}{|z|}\right] & z \neq 0 \\ 0 & z = 0 \end{cases}.$$

$$\lim_{z \to 0} f(z) = \lim_{r \to 0} Re\left[\frac{re^{i\theta}}{|re^{i\theta}|}\right]$$

$$= \lim_{r \to 0} Re\left[\cos\theta + i\sin\theta\right]$$

$$= \cos\theta.$$

$\Rightarrow$ limit does not exist

$\Rightarrow$ $f(z)$ is not continuous at $z = 0$.

b) $$\lim_{z \to 0} f(z) = \lim_{r \to 0} Im\left[\frac{re^{i\theta}}{1 + |re^{i\theta}|}\right]$$

$$= \lim_{r \to 0} Im\left[\frac{re^{i\theta}}{1+r}\right].$$

$$= \lim_{r \to 0} \frac{r}{1+r}. \sin\theta$$

$$= 0.$$

At $z = 0$, $f(z) = 0$

$\Rightarrow$ $f(z)$ is continuous at $z = 0$.

4) $$f(z) = \begin{cases} \text{Im}\left[\frac{z}{|z|}\right] & z \neq 0 \\ 0 & z = 0 \end{cases}.$$

$f(z)$ is continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

<u>For $z = 0$</u>

$$\lim_{z \to 0} f(z) = \lim_{r \to 0} \text{Im}\left[\frac{\cancel{r}e^{i\theta}}{|\cancel{r}e^{i\theta}|}\right]$$

$$= \sin\theta$$

$\Rightarrow$ limit does not exist

$\Rightarrow$ $f(z)$ is <u>not</u> continuous at <u>$z = 0$</u>.

<u>For $z = 5$</u>

$$\lim_{z \to z_0} f(z) = \lim_{r \to 0} \text{Im}\left[\frac{z_0 + re^{i\theta}}{|z_0 + re^{i\theta}|}\right], \quad z_0 = 5.$$

$$= \lim_{r \to 0} \frac{r\sin\theta}{5}$$

$$= 0$$

$$f(5) = \text{Im}\left[\frac{5}{|5|}\right] = 0.$$

$\Rightarrow$ $$\lim_{z \to 5} f(z) = f(5)$$

$\Rightarrow$ <u>$f(z)$ at $z = 5$ is continuous</u>

4) (Cont'd).

For $z = 5+i$

$$\lim_{z \to z_0} f(z) = \lim_{r \to 0} \operatorname{Im}\left[\frac{z_0 + re^{i\theta}}{|z_0 + re^{i\theta}|} \right] \qquad z_0 = 5+i$$

$$= \lim_{r \to 0} \operatorname{Im} \frac{5+i+re^{i\theta}}{|5+i+\underbrace{re^{i\theta}}_{\to 0}|}.$$

$$= \lim_{r \to 0} \left[\frac{1 + r\sin\theta}{|5+i|} \right]$$

$$= \frac{1}{|5+i|} = \frac{1}{\sqrt{26}}.$$

$$f(5+i) = \operatorname{Im}\left[\frac{5+i}{|5+i|} \right]$$

$$= \frac{1}{\sqrt{26}}.$$

$$\therefore \lim_{z \to z_0} f(z) = f(z_0), \quad z_0 = 5+i.$$

$\Rightarrow$ <u>function is continuous at $z = 5+i$</u>.

Tutorial 2 (Solutions) (Tutorial 8)

1 a). $f(z) = (2x-y) + i(ax+by)$.

Using the C-R equations,

$u_x = 2 \qquad v_x = a$.

$u_y = -1 \qquad v_y = b$.

To satisfy the C-R equations.

$u_x = v_y \ , \quad u_y = -v_x$.

$\therefore \quad \underline{a = 1}$

$\underline{b = 2}$

The function is differentiable for all z.

$f'(z) = u_x + i v_x$

$= 2 + i$

b) (i) $f(z) = \mathrm{Re}[z^2]$

$= \mathrm{Re}[(x+iy)^2]$

$= \mathrm{Re}[x^2 - y^2 + i2xy]$.

$= x^2 - y^2$.

$u_x = 2x \qquad v_x = 0$

$u_y = -2y \qquad v_y = 0$.

The C-R equations are only satisfied at $z = 0$

$\Rightarrow$ $\underline{f(z) \text{ is nowhere analytic}}$.

$\cos(90^\circ - \theta) = \sin\theta$

$\sin(90^\circ - \theta) = \cos\theta$

1 b) (ii) $f(z) = \dfrac{i}{z^4}$

$= \dfrac{1}{r^4} e^{i\left(\frac{\pi}{2} - 4\theta\right)}$

$= \dfrac{1}{r^4}\left[\sin 4\theta + i\cos 4\theta\right].$

$u_r = -\dfrac{4}{r^5}\sin 4\theta \qquad v_r = -\dfrac{4}{r^5}\cos 4\theta.$

$u_\theta = \dfrac{4}{r^4}\cos 4\theta. \qquad v_\theta = -\dfrac{4}{r^4}\sin 4\theta.$

$\because$ $u_r = \frac{1}{r}v_\theta$ and $v_r = -\frac{1}{r}u_\theta$

the C-R equations are satisfied everywhere except at $z=0$ (where the functions u, v are not continuous).

(iii) $f(z) = z - \bar{z} = (x+iy) - (x-iy)$

$= i\,2y.$

$u_x = 0 \qquad v_x = 0.$

$u_y = 0 \qquad v_y = 2.$

C-R. equations are not satisfied

$\Rightarrow$ Not analytic

2-2

1 b) (iv) $f(z) = e^x(\sin y - i\cos y) = \underbrace{e^x \sin y}_{u(x,y)} + i\underbrace{(-e^x \cos y)}_{v(x,y)}$

$u_x = e^x \sin y.$ $\quad v_x = -e^x \cos y.$

$u_y = e^x \cos y.$ $\quad v_y = e^x \sin y.$

$\because$ $u_x = v_y$ and $v_x = -u_y$

$f(z)$ is analytic everywhere in the complex plane.

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2 a). $f(z) = \underbrace{2xy}_{u} - ix^2$ $\quad v(x,y) = -x^2$

$u_x = 2y$ $\quad v_x = -2x.$

$u_y = 2x$ $\quad v_y = 0.$

~~for the~~ C-R equations to be satisfied,

$u_x = v_y. \Rightarrow y = 0.$ (x-axis).

$v_x = -u_y$

$\Rightarrow$ C-R equations are satisfied only at x-axis

$\Rightarrow$ $f'(z)$ exists only on x-axis

$f'(z) = u_x + iv_x$

$= 2y - i2x$

$= -i2x.$ //.

2b) $f(z) = z^2 - 2z + 3$.

$= (x+iy)^2 - 2(x+iy) + 3$.

$= \left(x^2 - y^2 - 2x + 3\right) + i\left(+2xy - 2y\right)$.

$u_x = 2x - 2$ $\qquad v_x = +2y$.

$u_y = -2y$ $\qquad v_y = 2x - 2$.

$\therefore$ $u_x = v_y$ and $v_x = -u_y$

$\therefore$ CR equations are satisfied for all z

$\Rightarrow$ $f(z)$ is analytic for all z.

$f'(z) = u_x + i v_x$

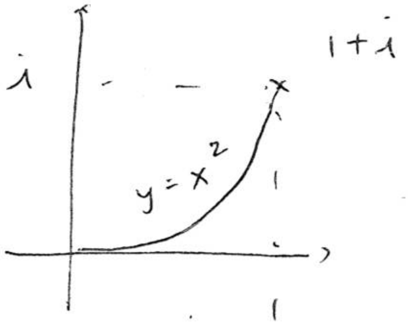
$= 2x - 2 + i\,2y$.

$= 2z - 2$

Note - Polynomials in z are analytic in the entire z plane and the usual differentiation applies.

3 a)

$$f(z) = Re[z]$$



$$z(t) = t + i t^2 \qquad 0 \le t \le 1$$

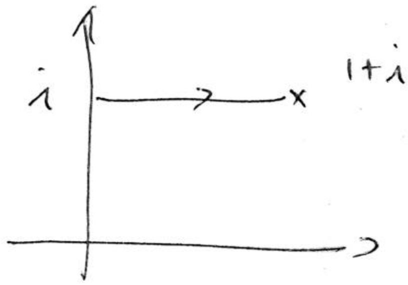
$$dz = (1 + i2t) \cdot dt$$

$$\therefore \int_C f(z) dz = \int_0^1 t \cdot (1 + i 2t) dt$$

$$= \int_0^1 (t + i 2t^2) dt$$

$$= \left[\frac{t^2}{2} + i \frac{2t^3}{3} \right]_0^1 = \frac{1}{2} + i \frac{2}{3}$$

b) $$f(z) = 4z - 3$$



$$z(t) = t + i \quad , 0 \le t \le 1$$

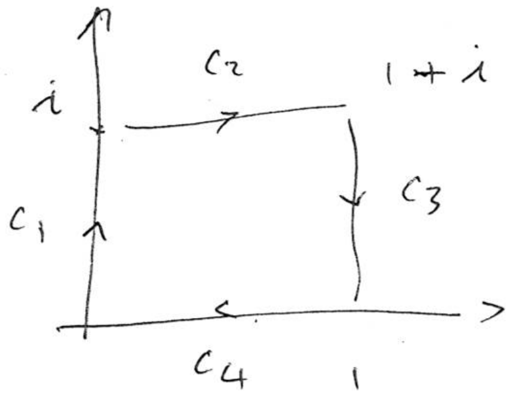
$$dz = dt$$

$$\int_C f(z) dz = \int_0^1 [4(t+i) - 3] \cdot dt$$

$$= \left[2t^2 + (4i - 3)t \right]_0^1$$

$$= 2 + 4i - 3 = -1 + 4i$$

3(b) $f(z) = e^{z}$



c_1 : $z(t) = it \quad 0 \le t \le 1$

$dz = i\,dt$

c_2 : $z(t) = t+i \quad 0 \le t \le 1$

$dz = dt$.

c_3 : $z(t) = 1+it \quad \cancel{0 \le t \le 1}. \ 1 \geq t \geq 0$

$dz = i\,dt$.

c_4 : $z(t) = t \quad \cancel{0 \le t \le 1} \ 1 \geq t \geq 0$

$dz = dt$.

$$\int_c f(z)dz = \int_0^1 e^{it} \cdot i\,dt + \int_0^1 e^{t+i} \cdot dt$$

$$+ \int_1^0 e^{1+it} \cdot i\,dt + \int_1^0 e^{t}\,dt.$$

$$= \left[i \cdot \frac{e^{it}}{i} \right]_0^1 + \left[\frac{e^{t+i}}{1} \right]_0^1$$

$$+ \left[i \cdot \frac{e^{1+it}}{i} \right]_1^0 + \left[e^{t} \right]_1^0$$

$$= (e^{i} - e^{0}) + (e^{1+i} - e^{i}) + (e^{1} - e^{1+i}) + (e^{0} - e^{1})$$

$$= 0$$

3d) $f(z) = \mathrm{Im}\,[z^2]$

Referencing Q3c solution,

C_1 : $f(z) = \mathrm{Im}\,[z^2] = \mathrm{Im}\,[t^2] = 0$

C_2 : $f(z) = \mathrm{Im}\,[(t+i)(t+i)]$

$= \mathrm{Im}\,[t^2 - 1 + i2t] = 2t$

C_3 : $f(z) = \mathrm{Im}\,[(1+it)(1+it)]$

$= \mathrm{Im}\,[1 - t^2 + i2t] = 2t$

C_4 : $f(z) = \mathrm{Im}\,[t^2] = 0$

$$\int_C f(z)\,dz = \int_0^1 0 \cdot i\,dt + \int_0^1 2t\,dt + \int_1^0 2t \cdot i\,dt + \int_1^0 0\,dt$$

$$= [t^2]_0^1 + i[t^2]_1^0$$

$$= 1 - i$$

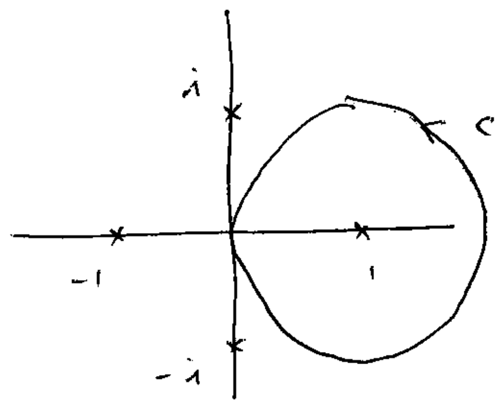
Tutorial 3 (Solutions) (Tutorial 9).

1 (a) $z^4 - 1 = 0$.

$z^4 = 1$

$z^2 = \pm 1$

$z = \pm 1, \pm i$



$|z-1| = 1 \Rightarrow$ path is circle
radius $= 1$
centre at $(1,0)$.

$\Rightarrow$ encloses singular point at $z=1$.

$$\oint_C \frac{1}{z^4-1}\,dz = \oint_C \frac{1}{(z-1)(z^3+z^2+z+1)}\,dz.$$

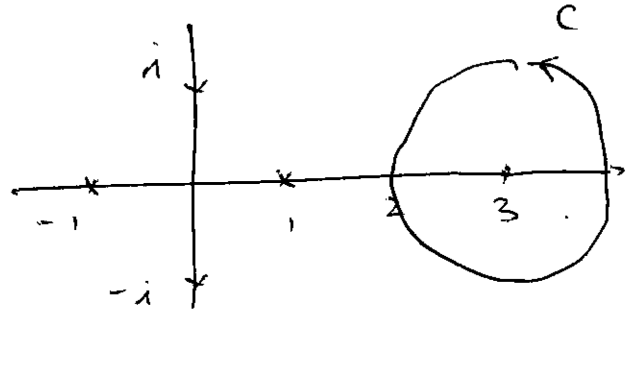
$$= \oint_C \frac{\frac{1}{z^3+z^2+z+1}}{z-1}\,dz.$$

$$= 2\pi i \left(\frac{1}{z^3+z^2+z+1}\right)\Big|_{z=1}$$

$$= \frac{\pi i}{2}$$

(b). $|z-3| = 1 \Rightarrow$ path is circle of radius $= 1$
center at $z = 3$.

$$\oint_C \frac{1}{z^4-1}\,dz = 0$$



Q2 (a). $\oint_C \frac{5z}{z^2+4}\,dz$.

$z^2+4=0$

$z^2=-4$

$z=\pm 2i$

$= \oint_C \frac{5z}{(z-2i)(z+2i)}\cdot dz$.

$\Rightarrow z^2+4=(z-2i)(z+2i)$.

$= \oint_{C_1} \frac{\frac{5z}{z-2i}}{z+2i}\,dz + \oint_{C_2} \frac{\frac{5z}{z+2i}}{z-2i}\,dz$.

$= 2\pi i \left.\frac{5z}{z-2i}\right|_{z=-2i} + 2\pi i \left.\frac{5z}{z+2i}\right|_{z=2i}$

$= 2\pi i\,\frac{5(-2i)}{-4i} + 2\pi i\,\frac{5(2i)}{4i}$

$= \underline{10\pi i}$.

(b) $\oint_C \frac{z+e^z}{z^3-z}\,dz$.

$= \oint_C \frac{z+e^z}{z(z-1)(z+1)}\cdot dz$.

$= 2\pi i\left[\left.\frac{z+e^z}{z^2-1}\right|_{z=0} + \left.\frac{z+e^z}{z(z+1)}\right|_{z=1} + \left.\frac{z+e^z}{z(z-1)}\right|_{z=-1}\right]$

$= 2\pi i\left[-1+\frac{1+e}{2}+\frac{-1+e^{-1}}{2}\right]$.

$= \underline{2\pi i\left[-1+\frac{e+e^{-1}}{2}\right]}$

3a) Let $z = e^{i\theta}$ $\quad 0 \leq \theta \leq 2\pi$.

$$\left.\begin{aligned} z &= \cos\theta + i\sin\theta . \\ \bar{z} = \frac{1}{z} &= \cos\theta - i\sin\theta . \end{aligned}\right\} .$$

$$\sin\theta = \frac{z - \frac{1}{z}}{2i} = \frac{z^2 - 1}{2iz} .$$

$$dz = e^{i\theta} . i\, d\theta . \Rightarrow d\theta = \frac{1}{iz} dz .$$

$$\int_0^{2\pi} \frac{d\theta}{5 - 3\sin\theta} = \oint_c \frac{1}{5 - 3\frac{z^2-1}{2iz}} . \frac{1}{iz} dz .$$

$$= -2 \oint_c \frac{1}{3z^2 - 10zi - 3} dz .$$

$$= -2 \oint_c \frac{1}{(z - 3i)(3z - i)} dz .$$

$$= -2 . \oint_c \frac{\frac{1}{3(z-3i)}}{(z - \frac{i}{3})} dz .$$

$$= -\frac{2}{3} . 2\pi i \left. \frac{1}{z - 3i} \right|_{z = \frac{i}{3}}$$

$$= \frac{\pi}{2}$$

3 b). Let $z = e^{i\theta}$ $\quad 0 \le \theta \le 2\pi$.

$$\cos\theta = \frac{z + \frac{1}{z}}{2} = \frac{z^2+1}{2z}.$$

$$z^2 = e^{i2\theta} = \cos 2\theta + i\sin 2\theta.$$

$$\bar{z}^2 = \frac{1}{z^2} = e^{-i2\theta} = \cos 2\theta - i \sin 2\theta$$

$$\cos 2\theta = \frac{z^2 + \frac{1}{z^2}}{2} = \frac{z^4+1}{2z^2}.$$

$$dz = e^{i\theta} \cdot i \, d\theta \Rightarrow d\theta = \frac{1}{iz} dz.$$

$$\int_0^{2\pi} \frac{\cos\theta}{13 - 12\cos 2\theta} d\theta = \oint_C \frac{\frac{z^2+1}{2z}}{13 - 12 \cdot \frac{z^4+1}{2z^2}} \frac{1}{iz} dz$$

$$= \oint_C \frac{z^2+1}{\cancel{2z}} \cdot \frac{\cancel{2z^2}}{-12z^4 + 26z^2 - 12} \cdot \frac{1}{i\cancel{z}} dz.$$

$$= \frac{i}{2} \oint_C \frac{z^2+1}{6z^4 - 13z^2 + 6} dz.$$

$$= \frac{i}{2} \oint_C \frac{z^4+1}{6\left(z^2 - \frac{2}{3}\right)\left(z^2 - \frac{3}{2}\right)} dz.$$

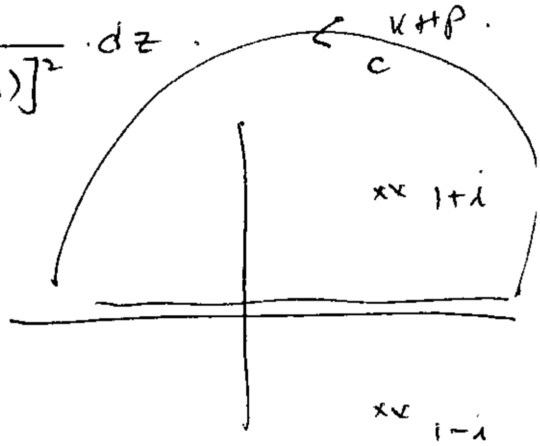
$$= \frac{i}{12}\left[\oint_{C_1} \frac{\frac{z^4+1}{\left(z^2-\frac{3}{2}\right)\left(z - \sqrt{\frac{2}{3}}\right)}}{z + \sqrt{\frac{2}{3}}} dz + \oint_{C_2} \frac{\frac{z^4+1}{\left(z^2-\frac{3}{2}\right)\left(z+\sqrt{\frac{2}{3}}\right)}}{z - \sqrt{\frac{2}{3}}} dz \right].$$

$$= \frac{i}{12}\left[\left. \frac{z^4+1}{\left(z^2-\frac{3}{2}\right)\left(z-\sqrt{\frac{2}{3}}\right)} \right|_{z=-\sqrt{\frac{2}{3}}} + \left. \frac{z^4+1}{\left(z^2-\frac{3}{2}\right)\left(z+\sqrt{\frac{2}{3}}\right)} \right|_{z=+\sqrt{\frac{2}{3}}} \right].$$

$$= \underline{\underline{0}}$$

4a). $$\int_{-\infty}^{\infty} \frac{x}{(x^2-2x+2)^2}\,dx = \oint_{UHP} \frac{z}{(z^2-2z+2)^2}\,dz.$$

$$= \oint_{UHP} \frac{z}{[z-(1+i)]^2[z-(1-i)]^2}\,dz.$$



$$= \oint_{UHP} \frac{\dfrac{z}{[z-(1-i)]^2}}{[z-(1+i)]^2}\,dz$$

$$= 2\pi i \, \frac{d}{dz} \frac{z}{[z-(1-i)]^2}\Bigg|_{z=1+i}$$

$$= 2\pi i \left(\frac{-i}{4}\right) = \frac{\pi}{2}$$ //.

4b). $$\int_{-\infty}^{\infty} \frac{1}{(4+x^2)^2}\,dx = \oint_{UHP} \frac{1}{(4+z^2)^2}\,dz.$$

$$= \oint_{UHP} \frac{1}{(z+2i)^2(z-2i)^2}\,dz.$$

$$= \oint_{UHP} \frac{\dfrac{1}{(z+2i)^2}}{(z-2i)^2}\,dz.$$

$$= 2\pi i \, \frac{d}{dz} \frac{1}{(z+2i)^2}\Bigg|_{z=2i}$$

$$= 2\pi i \left(\frac{-2i}{4^3}\right) = \frac{\pi}{16}$$ //.

Tutorial 4 Solutions (Tutorial 10)

1. $$f(x,y,z) = 3x^2y - y^3z^2.$$

$$\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}$$

$$= \left.\begin{pmatrix} 6xy \\ 3x^2 - 3y^2z^2 \\ -2y^3z \end{pmatrix}\right|_{\substack{x=1 \\ y=-2 \\ z=-1}}$$

$$= \begin{pmatrix} -12 \\ -9 \\ -16 \end{pmatrix} \quad \text{or} \quad -12\underset{\sim}{i} - 9\underset{\sim}{j} - 16\underset{\sim}{k}.$$

∇f is a vector that gives a direction of maximum rate of change of f at a given point.
The magnitude $\|\nabla f\|$ is the maximum rate of change.

2. $f(x,y,z) = x^2 y + 2xz = 4.$

$$\vec{N} = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} 2xy + 2z \\ x^2 \\ 2x \end{pmatrix}$$

At point $(2,-2,3)$

$$\vec{N} = \begin{pmatrix} -2 \\ 4 \\ 4 \end{pmatrix}$$

unit normal,

$$\hat{n} = \frac{1}{\|\vec{N}\|}\vec{N} = \frac{1}{\sqrt{2^2 + 4^2 + 4^2}} \begin{pmatrix} -2 \\ 4 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix}.$$

OR $$\hat{n} = \begin{pmatrix} 1/3 \\ -2/3 \\ -2/3 \end{pmatrix}.$$

3). "Directional derivative"

- derivative in a particular direction.

$$f(x,y,z) = x^2 e^y .$$

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x e^y \\ x^2 e^y \\ 0 \end{pmatrix}$$

$$D_{-\hat{j}} f = \nabla f \cdot (-\hat{j}) = \begin{pmatrix} 2x e^y \\ x^2 e^y \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$= -x^2 e^y$$

At $(-2,0,0)$,

$$D_{-\hat{j}} f = -x^2 e^y \Big|_{\substack{x=-2 \\ y=0}}$$

$$= -4$$

$$\text{Max } D_{\hat{j}} f = \|\nabla f\| = \left\| \begin{pmatrix} 2x e^y \\ x^2 e^y \\ 0 \end{pmatrix} \Big|_{\substack{x=-2 \\ y=0}} \right\|$$

$$= \left\| \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} \right\|$$

$$= \sqrt{4^2 + 4^2 + 0^2}$$

$$= 5.6569$$

4).
$$r = \|\underset{\wedge}{r}\| = \sqrt{x^2+y^2+z^2}$$
$$= (x^2+y^2+z^2)^{1/2}.$$
$$r^n = (x^2+y^2+z^2)^{n/2}$$
$$LHS = \nabla r^n = \begin{pmatrix} \frac{n}{2}(x^2+y^2+z^2)^{\frac{n-2}{2}} \cdot 2x \\ \frac{n}{2}(x^2+y^2+z^2)^{\frac{n-2}{2}} \cdot 2y \\ \frac{n}{2}(x^2+y^2+z^2)^{\frac{n-2}{2}} \cdot 2z \end{pmatrix}$$
$$= n\left(\sqrt{x^2+y^2+z^2}\right)^{n-2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= n r^{n-2} \underset{\sim}{r} = RHS \text{ (shown)}$$
//.

5). $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\underset{\sim}{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$.

$$\underset{\sim}{v} = \underset{\sim}{w} \times \underset{\sim}{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$= \begin{pmatrix} w_2 z - w_3 y \\ w_3 x - w_1 z \\ w_1 y - w_2 x \end{pmatrix}.$$

$$\frac{1}{2} \nabla \times \underset{\sim}{v} = \frac{1}{2} \begin{vmatrix} \underline{i} & \underline{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_2 z - w_3 y & w_3 x - w_1 z & w_1 y - w_2 x \end{vmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \frac{\partial}{\partial y}(w_1 y - w_2 x) - \frac{\partial}{\partial z}(w_3 x - w_1 z) \\ \frac{\partial}{\partial z}(w_2 z - w_3 y) - \frac{\partial}{\partial x}(w_1 y - w_2 x) \\ \frac{\partial}{\partial x}(w_3 x - w_1 z) - \frac{\partial}{\partial y}(w_2 z - w_3 y) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} w_1 - (-w_1) \\ w_2 - (-w_2) \\ w_3 - (-w_3) \end{pmatrix}$$

$$= \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \underset{\sim}{w}. \quad \text{(Shown)}$$

6). curl $\left(xy^2z\,\underset{\sim}{i} + 2x^3y\,\underset{\sim}{j} + 4x^2y^2\,\underset{\sim}{k}\right)$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z & 2x^3y & 4x^2y^2 \end{vmatrix}.$$

$$= \begin{pmatrix} 8x^2y \\ xy^2 - 8xy^2 \\ 6x^2y - 2xyz \end{pmatrix}\Bigg|_{\substack{x=1 \\ y=1 \\ z=-1}} = \begin{pmatrix} 8 \\ -7 \\ 8 \end{pmatrix}$$

---

curl $\left(yz^3\,\underset{\sim}{i} + xz\,\underset{\sim}{j} + 2x\,\underset{\sim}{k}\right)$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^3 & xz & 2x \end{vmatrix}$$

$$= \begin{pmatrix} -x \\ 3yz^2 - 2 \\ z - z^3 \end{pmatrix}\Bigg|_{\substack{x=1 \\ y=1 \\ z=-1}}$$

$$= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Tutorial 5 (Tutorial 11) Solutions

1). a). $\underset{\sim}{r} = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$ $\quad 0 \le t \le 1$

$$\vec{F} = \begin{pmatrix} 3x^2 + 6y \\ -14yz \\ 20xz^2 \end{pmatrix} = \begin{pmatrix} 3t^2 + 6t^2 \\ -14(t^2)(t^3) \\ 20t(t^3)^2 \end{pmatrix}.$$

$$= \begin{pmatrix} 9t^2 \\ -14t^5 \\ 20t^7 \end{pmatrix}.$$

$$d\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} dt.$$

$$\int_C \vec{F} \cdot d\underset{\sim}{r} = \int_{t=0}^{t=1} \begin{pmatrix} 9t^2 \\ -14t^5 \\ 20t^7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} dt.$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\, dt.$$

$$= \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1$$

$$= \underline{\underline{5}}.$$

b). $C_1 : (0,0,0) \to (1,0,0)$.

$C_2 : (1,0,0) \to (1,1,0)$

$C_3 : (1,1,0) \to (1,1,1)$.

$$\int_C \vec{F} \cdot d\underset{\sim}{r} = \int_{C_1} \vec{F} \cdot d\underset{\sim}{r} + \int_{C_2} \vec{F} \cdot d\underset{\sim}{r} + \int_{C_3} \vec{F} \cdot d\underset{\sim}{r}$$

$$\int_{C_1} \vec{F} \cdot d\underset{\sim}{r} = \int_{x=0}^{x=1} \left[(3x^2+6y)\underset{\sim}{i} - 14yz\,\underset{\sim}{j} + 20xz^2\,\underset{\sim}{k}\right] \cdot (dx\,\underset{\sim}{i} + \underbrace{dy}_{=0}\,\underset{\sim}{j} + \underbrace{dz}_{=0}\,\underset{\sim}{k})$$

$$= \int_{x=0}^{x=1} (3x^2 + \underbrace{6y}_{=0})\,dx = \left[x^3\right]_0^1 = 1$$

$$\int_{C_2} \vec{F} \cdot d\underset{\sim}{r} = \int \vec{F} \cdot (\underbrace{dx}_{=0}\,\underset{\sim}{i} + dy\,\underset{\sim}{j} + \underbrace{dz}_{=0}\,\underset{\sim}{k}).$$

$$= \int_{y=0}^{y=1} -14y\underbrace{z}_{=0}\,dy = 0$$

$$\int_{C_3} \vec{F} \cdot d\underset{\sim}{r} = \int \vec{F} \cdot (\underbrace{dx}_{=0}\,\underset{\sim}{i} + \underbrace{dy}_{=0}\,\underset{\sim}{j} + dz\,\underset{\sim}{k})$$

$$= \int_{z=0}^{z=1} 20xz^2\,dz \Big|_{\substack{x=1 \\ y=1}}$$

$$= \left[\frac{20z^3}{3}\right]_{z=0}^{z=1} = \frac{20}{3}.$$

$$\therefore \int_C \vec{F} \cdot d\underset{\sim}{r} = 1 + 0 + \frac{20}{3} = \frac{23}{3}$$ //

c). $\underset{\sim}{r} = \begin{pmatrix} t \\ t \\ t \end{pmatrix} \quad 0 \le t \le 1$

$$\vec{F} = \begin{pmatrix} 3x^2 + 6y \\ -14yz \\ 20xz^2 \end{pmatrix} = \begin{pmatrix} 3t^2 + 6t \\ -14t^2 \\ 20t^3 \end{pmatrix}$$

$$d\underset{\sim}{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dt .$$

$$\int_C \vec{F} \cdot d\underset{\sim}{r} = \int_{t=0}^{1} \begin{pmatrix} 3t^2+6t \\ -14t^2 \\ 20t^3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dt .$$

$$= \int_0^1 (3t^2 + 6t - 14t^2 + 20t^3)\, dt .$$

$$= \int_0^1 (20t^3 - 11t^2 + 6t)\, dt$$

$$= \left[5t^4 - \frac{11}{3}t^3 + 3t^2 \right]_0^1$$

$$= 5 - \frac{11}{3} + 3 \quad = \frac{13}{3}$$

2). $\underset{\sim}{r} = \begin{pmatrix} t^2+1 \\ 2t^2 \\ t^3 \end{pmatrix} \quad 1 \leq t \leq 2.$

$$\vec{F} = \begin{pmatrix} 3xy \\ -5z \\ 10x \end{pmatrix} = \begin{pmatrix} 3(2t^2)(t^2+1) \\ -5t^3 \\ 10(t^2+1) \end{pmatrix}$$

$$= \begin{pmatrix} 6t^2(t^2+1) \\ -5t^3 \\ 10(t^2+1) \end{pmatrix}.$$

$$d\underset{\sim}{r} = \begin{pmatrix} 2t \\ 4t \\ 3t^2 \end{pmatrix} dt.$$

$$W.D. = \int \vec{F} \cdot d\underset{\sim}{r}$$

$$= \int_{t=1}^{t=2} \begin{pmatrix} 6t^2(t^2+1) \\ -5t^3 \\ 10(t^2+1) \end{pmatrix} \cdot \begin{pmatrix} 2t \\ 4t \\ 3t^2 \end{pmatrix} dt.$$

$$= \int_1^2 \left[12t^3(t^2+1) - 20t^4 + 30t^2(t^2+1) \right] dt.$$

$$= \int_1^2 \left[12t^5 + 10t^4 + 12t^3 + 30t^2 \right] dt.$$

$$= \left[2t^6 + 2t^5 + 3t^4 + 10t^3 \right]_1^2$$

$$= \underline{\underline{303}} \;//.$$

3).

$$\underset{\sim}{r} = \begin{pmatrix} t \\ 2t^2 \end{pmatrix} \qquad 0 \le t \le 1$$

$$\vec{F} = \begin{pmatrix} 3xy \\ -y^2 \end{pmatrix} = \begin{pmatrix} 3t(2t^2) \\ -4t^4 \end{pmatrix}$$

$$d\underset{\sim}{r} = \begin{pmatrix} 1 \\ 4t \end{pmatrix} dt .$$

$$\int \vec{F} \circ d\underset{\sim}{r} = \int_{t=0}^{t=1} \begin{pmatrix} 6t^3 \\ -4t^4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4t \end{pmatrix} dt .$$

$$= \int_0^1 (6t^3 - 16t^5)\, dt .$$

$$= \left[\frac{3t^4}{2} - \frac{16}{6} t^6 \right]_0^1$$

$$= -\frac{7}{6} .$$

//

4 a).

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix} .$$

$$= 0\underset{\sim}{i} + (3z^2 - 3z^2)\underset{\sim}{j} + (2x - 2x)\underset{\sim}{k} .$$

$$= \vec{0} \quad \Rightarrow \vec{F} \text{ is conservative}$$

//.

4b). $\vec{F} = \nabla V = \begin{pmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial V}{\partial z} \end{pmatrix} = \begin{pmatrix} 2xy + z^3 \\ x^2 \\ 3xz^2 \end{pmatrix}.$

$\frac{\partial V}{\partial x} = 2xy + z^3 \Rightarrow V(x,y,z) = x^2y + xz^3 + g_1(y,z).$

$\frac{\partial V}{\partial y} = x^2 \Rightarrow V = x^2y + g_2(x,z).$

$\frac{\partial V}{\partial z} = 3xz^2 \Rightarrow V = xz^3 + g_3(x,y).$

$\Rightarrow \underline{V(x,y,z) = x^2y + xz^3 + C}.$

4c). W.D. $= \int_C \vec{F} \cdot d\underset{\sim}{r} = V(3,1,4) - V(1,-2,1).$

$= \left[3^2(1) + 3(4)^4\right] - \left[(1)^2(-2) + (1)(1)^3\right].$

$= \underline{\underline{202}}.$

5) $\oiint_S \vec{F} \cdot d\vec{A}$

For the spherical surface, let

$$\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a\cos u \sin v \\ a \sin u \sin v \\ a\cos v \end{pmatrix} \qquad \begin{array}{l} 0 \le u \le 2\pi \\ 0 \le v \le \pi . \end{array}$$

$$\underset{\sim}{r_u} = \begin{pmatrix} -a\sin u \sin v \\ a\cos u \sin v \\ 0 \end{pmatrix} \quad \underset{\sim}{r_v} = \begin{pmatrix} a\cos u \cos v \\ a\sin u \cos v \\ -a\sin v \end{pmatrix}.$$

$$\vec{N} = \underset{\sim}{r_u} \times \underset{\sim}{r_v} = -\begin{pmatrix} a^2\cos u \sin^2 v \\ a^2 \sin u \sin^2 v \\ a^2 \sin v \cos v \end{pmatrix}.$$

$$= -a\sin v \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -a\sin v \underset{\sim}{r}.$$

Redefine outward normal as

$$\vec{N} = a\sin v \underset{\sim}{r}$$

$$\oiint_S \vec{F} \cdot d\vec{A} = \iint a\sin v \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} du\,dv$$

$$= \iint a\sin v\,(z^2)\,du\,dv$$

$$= \int_{v=0}^{v=\pi} \int_{u=0}^{u=2\pi} a\sin v \,.\, a^2\cos^2 v\,du\,dv$$

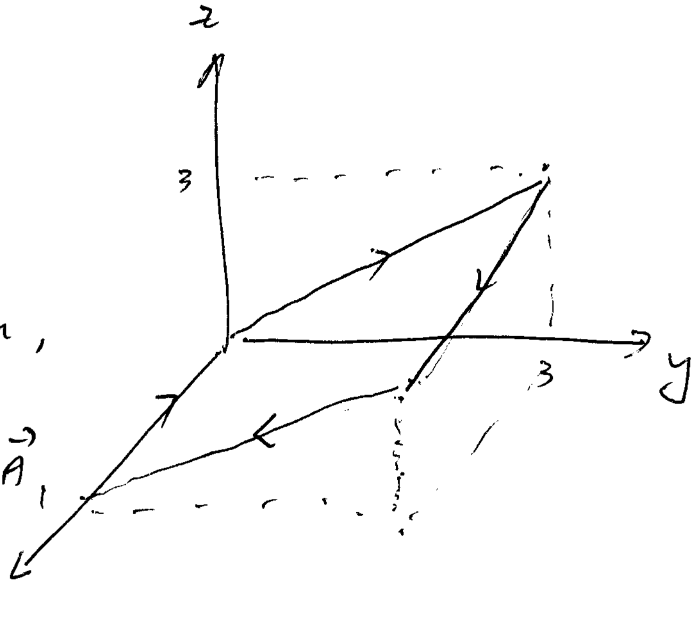
$$= a^3 \int_{v=0}^{\pi} \sin v \cos^2 v\,dv \,.\, \int_{u=0}^{2\pi} du$$

$$= a^3 \left[-\frac{\cos^3 v}{3}\right]_0^{\pi} \left[u\right]_0^{2\pi}$$

$$= a^3 \left[\frac{2}{3}\right]\left[2\pi\right] = \underline{\underline{\frac{4}{3}\pi a^3}}$$

Tutorial 6 (Tutorial 12) Solutions

1) $W.D. = \oint_C \vec{F} \cdot d\underset{\sim}{r}$.



Using Stokes' Theorem,

$\oint_C \vec{F} \cdot d\underset{\sim}{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{A}$,

For planar surface S, let

$$\left.\begin{aligned} x &= u \\ y &= v \\ z &= y = v \end{aligned}\right\} \quad \begin{aligned} 0 \le u \le 1 \\ 0 \le v \le 3 \end{aligned}.$$

$$\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ v \end{pmatrix}$$

$$\underset{\sim}{r_u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \underset{\sim}{r_v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \quad \vec{N} = \underset{\sim}{r_u} \times \underset{\sim}{r_v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

For the given path, the normal of the plane should be orientted downwards.

Redefine $\vec{N} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

$$\vec{F} = \begin{pmatrix} x^2 \\ 4xy^3 \\ y^2 x \end{pmatrix}.$$

$$\text{curl } \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 4xy^3 & xy^2 \end{vmatrix}.$$

$$= \begin{pmatrix} 2xy \\ -y^2 \\ 4y^3 \end{pmatrix}.$$

$$\therefore \iint \text{curl } \vec{F} \cdot d\vec{A} = \iint \text{curl } \vec{F} \cdot \vec{N} \, du \, dv$$

$$= \int_{v=0}^{v=3} \int_{u=0}^{u=1} \begin{pmatrix} 2xy \\ -y^2 \\ 4y^3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} du \, dv.$$

$$= \int_{v=0}^{v=3} \int_{u=0}^{u=1} (-y^2 - 4y^3) \, du \, dv$$

$$= \int_{v=0}^{v=3} (-v^2 - 4v^3) dv \cdot \int_{u=0}^{u=1} 1 \, du.$$

$$= \left[-\frac{v^3}{3} - v^4 \right]_0^3 = \underline{\underline{-90 \text{ units}}}$$

2) Using Stokes Theorem,

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{A} = \oint_C \vec{F} \cdot d\underset{\sim}{r}$$

where path c is the boundary of S on the xy plane.

$S: \quad z = 4 - x^2 - y^2, \quad z \geq 0.$

$C: \quad z = 0 \Rightarrow x^2 + y^2 = 4$ (circle of radius $= 2$).

$$\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2\cos t \\ 2\sin t \\ 0 \end{pmatrix} \qquad 0 \leq t \leq 2\pi.$$

$$\vec{F} = \begin{pmatrix} 2z \\ 3x \\ 5y \end{pmatrix} = \begin{pmatrix} 0 \\ 6\cos t \\ 10\sin t \end{pmatrix}.$$

$$d\underset{\sim}{r} = \begin{pmatrix} -2\sin t \\ 2\cos t \\ 0 \end{pmatrix} dt.$$

$$\therefore \oint \vec{F} \cdot d\underset{\sim}{r} = \int_{t=0}^{2\pi} \begin{pmatrix} 0 \\ 6\cos t \\ 10\sin t \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ 2\cos t \\ 0 \end{pmatrix} dt.$$

$$= \int_0^{2\pi} 12\cos^2 t \, dt.$$

$$= \int_0^{2\pi} 6(\cos 2t + 1)\, dt.$$

$$= 6\left[\frac{\sin 2t}{2} + t\right]_0^{2\pi}$$

$$= \underline{\underline{12\pi}}$$

3) Using Divergence Th.

$$\oiint_S \vec{F} \cdot d\vec{A} = \iiint_V \operatorname{div} \vec{F} \, dV$$

$$\operatorname{div} \vec{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = 1$$

$$\therefore \oiint_S \vec{F} \cdot d\vec{A} = \iiint_V 1 \, dv \quad \text{(for sphere).}$$

$$= \frac{4}{3}\pi a^3 \quad \text{(volume of sphere)}$$

4). $$\oiint_S \vec{F} \cdot d\vec{A} = \iiint_V \operatorname{div} \vec{F} \, dv.$$

$$\operatorname{div} \vec{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix} = 3x^2 + 3y^2 + 3z^2.$$

$$\therefore \iiint_V \operatorname{div} \vec{F} \, dv = 3 \iiint \underbrace{(x^2 + y^2 + z^2)}\, \underbrace{dv}$$

$$= 3 \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\rho=0}^{\rho=a} \underbrace{\rho^2} \; \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}.$$

$$= 3 \left[\int_{\rho=0}^{\rho=a} \rho^4 \, d\rho \right] \left[\int_{\phi=0}^{\phi=\frac{\pi}{2}} \sin\phi \, d\phi \right] \left[\int_{\theta=0}^{\theta=2\pi} 1 \, d\theta \right].$$

$$= 3 \left[\frac{\rho^5}{5} \right]_0^a \left[-\cos\phi \right]_0^{\pi/2} \left[2\pi \right].$$

$$= \frac{6}{5}\pi a^5$$

5). $\oint\!\!\oint_S \vec{F} \cdot d\vec{A} = \iiint_V \text{div } \vec{F} \; dV$

$$\text{div } \vec{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} 2x \\ 3y \\ z^2 \end{pmatrix} = 2 + 3 + 2z = 5 + 2z .$$

$$\iiint_V \text{div } \vec{F} \; dV = \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} (5 + 2z) \, dx \, dy \, dz .$$

$$= \left[\int_{x=0}^{1} 1 \, dx \right] \cdot \left[\int_{y=0}^{1} 1 \, dy \right] \cdot \left[\int_{z=0}^{1} (5 + 2z) \, dz \right]$$

$$= (1)(1) \left[5z + z^2 \right]_0^1$$

$$= \underline{\underline{6}} .$$