The initial conditions tell us that

$$b_0 = u + v = 2,\ b_1 = u(2+\sqrt{3}) + v(2-\sqrt{3}) = 4.$$

Thus $u = 2 - v$ and

$$4 = (2-v)(2+\sqrt{3})+v(2-\sqrt{3}) = 4+2\sqrt{3}-2v-v\sqrt{3}+2v-v\sqrt{3} = 4+2\sqrt{3}-2v\sqrt{3}$$

showing that $2\sqrt{3} = 2v\sqrt{3}$ that is $v = 1$ and thus $u = 1$. The final solution is then

$$b_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n.$$

## Exercises for Chapter 7

**Exercise 52.** 1. Show that

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

for $1 \leq k \leq l$, where by definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},\ n! = n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1.$$

2. Prove by mathematical induction that

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.$$

You will need 1. for this!

3. Deduce that the cardinality of the power set $P(S)$ of a finite set $S$ with $n$ elements is $2^n$.

*Solution.* To prove

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k},$$

we first expand the left hand side:

$$\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!(n-k+1)}{k!(n-k+1)!} + \frac{n!k}{k!(n-k+1)!}$$

which is equal to

$$\frac{n!(n+1)}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}.$$

To prove the binomial theorem by mathematical induction, we set

$$P(n) = \text{“}(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\text{”},$$

and we want to prove that $\forall n,\ P(n)$. The basis step is to prove that $P(1)$ holds, which is given by

$$(x+y) = \sum_{k=0}^{1} \binom{1}{k} x^{1-k} y^k = x + y.$$

Next for the inductive step, we suppose that $P(l)$ is true, namely

$$(x+y)^l = \sum_{k=0}^{l} \binom{l}{k} x^{l-k} y^k \tag{11.1}$$

and we want to prove $P(l+1)$.

$$\begin{aligned}
(x+y)^{l+1} &= (x+y)(x+y)^l \\
&= (x+y)\sum_{k=0}^{l} \binom{l}{k} x^{l-k} y^k \text{ (using (11.1))} \\
&= x\sum_{k=0}^{l} \binom{l}{k} x^{l-k} y^k + y\sum_{k=0}^{l} \binom{l}{k} x^{l-k} y^k \\
&= \sum_{k=0}^{l} \binom{l}{k} x^{l-k+1} y^k + \sum_{k=0}^{l} \binom{l}{k} x^{l-k} y^{k+1}.
\end{aligned}$$

At this point, it is probably a good idea to remember what we want to prove, namely

$$(x+y)^{l+1} = \sum_{k=0}^{l+1} \binom{l+1}{k} x^{(l+1)-k} y^k.$$

From our aim, we notice that the first sum has already right exponents, namely $x^{l-k+1}y^k$ is a term we want. So we first work on the other sum to get a similar right term present, by doing a change of variable $j = k+1$, to get

$$\sum_{k=0}^{l} \binom{l}{k} x^{l-k} y^{k+1} = \sum_{j=1}^{l+1} \binom{l}{j-1} x^{l-j+1} y^j.$$

We next combine this derivation:

$$\begin{aligned} & (x+y)^{l+1} \\ = & \sum_{k=0}^{l} \binom{l}{k} x^{l-k+1} y^k + \sum_{j=1}^{l+1} \binom{l}{j-1} x^{l-j+1} y^j \\ = & \sum_{k=1}^{l} \binom{l}{k} x^{l-k+1} y^k + \binom{l}{0} x^{l+1} + \sum_{j=1}^{l} \binom{l}{j-1} x^{l-j+1} y^j + \binom{l}{l} y^{l+1} \\ = & \sum_{k=1}^{l} \left[ \binom{l}{k} + \binom{l}{k-1} \right] x^{l-k+1} y^k + y^{l+1} + x^{l+1} \end{aligned}$$

and now is the point where we recognize the formula that we derived in 1., thus

$$\begin{aligned} & (x+y)^{l+1} \\ = & \sum_{k=1}^{l} \binom{l+1}{k} x^{l-k+1} y^k + y^{l+1} + x^{l+1} \\ = & \sum_{k=0}^{l+1} \binom{l+1}{k} x^{l-k+1} y^k \end{aligned}$$

Finally, evaluate the binomial theorem in $x = y = 1$. The only thing left to be seen is the interpretation of $\binom{n}{k}$ as "$n$ choose $k$", which will be discussed into more details in the next chapter, namely $\binom{n}{k}$ counts the number of ways of picking $k$ elements out of $n$. Therefore to count the number of elements in $P(S)$ we just count how many subsets we have with 1 element, with 2 elements, ..., and we sum these numbers up!

**Exercise 53.** Consider the set $A = \{1, 2, 3\}$, $P(A)$ =power set of $A$.

- Compute the cardinality of $P(A)$ using the binomial theorem approach.

- Compute the cardinality of $P(A)$ using the counting approach.

*Solution.* • To compute the cardinality of $P(A)$, we need to count the empty set (1), the number of subset of size 1 ($\binom{3}{1}$), the number of sets of size 2 ($\binom{3}{2}$) and the whole set (1), therefore:

$$|P(A)| = 1 + \binom{3}{1} + \binom{3}{2} + 1 = 2^3$$

where the last equality follows from the binomial theorem.

- In a counting approach, we write binary strings to identify whether a given element belongs to a subset, for example 000 corresponds to the empty set, and 000 is read as 1 is not in this subset, 2 is not, and 3 is not either. Now for every subset, each element either belongs, or does not belong, therefore we get the 8 possible binary strings, and the cardinality is $2^3$.

**Exercise 54.** Let $P(C)$ denote the power set of $C$. Given $A = \{1, 2\}$ and $B = \{2, 3\}$, determine:

$$P(A \cap B),\ P(A),\ P(A \cup B),\ P(A \times B).$$

*Solution.* • $A \cap B = \{2\}$, therefore $P(A \cap B) = \{\varnothing, \{2\}\}$.

- $P(A) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.
- $A \cup B = \{1, 2, 3\}$, therefore $P(A \cup B) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.
- $A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$, therefore $P(A \times B)$ contains
  - $\varnothing$, $\{(1, 2)\}$,$\{(1, 3)\}$,$\{(2, 2)\}$,$\{(2, 3)\}$,
  - $\{(1, 2), (1, 3)\}$, $\{(1, 2), (2, 2)\}$, $\{(1, 2), (2, 3)\}$,$\{(1, 3), (2, 2)\}$, $\{(1, 3), (2, 3)\}$, $\{(2, 2), (2, 3)\}$,
  - $\{(1, 2), (1, 3), (2, 2)\}$,$\{(1, 2), (1, 3), (2, 3)\}$,$\{(1, 2), (2, 3), (2, 2)\}$,$\{(1, 3), (2, 2), (2, 3)\}$,
  - $\{(1, 2), (1, 3), (2, 2), (2, 3)\}$.

**Exercise 55.** Prove by contradiction that for two sets $A$ and $B$

$$(A - B) \cap (B - A) = \varnothing.$$

*Solution.* Suppose by contradiction that $(A - B) \cap (B - A)$ is not empty. Then there exists an element $x$ which belongs to both $(A-B)$ and $(B-A)$. This means that $x$ belongs to $A$ (since $x \in A - B$), and $x$ does not belong to $A$ (since $x \in B - A$), which is a contradiction! Therefore the assumption was false, and $(A - B) \cap (B - A)$ is empty.

Note that from a propositional logic point of view, what we did is set $p$="$(A - B) \cap (B - A) = \varnothing$", $q$="$x \in A$", and we prove that

$$\neg p \to (q \wedge \neg q)$$

which turns out to be equivalent to $p$.

**Exercise 56.** Let $P(C)$ denote the power set of $C$. Prove that for two sets $A$ and $B$

$$P(A) = P(B) \iff A = B.$$

*Solution.* We need to show that $P(A) = P(B) \to A = B$ and $A = B \to P(A) = P(B)$.

- suppose $P(A) = P(B)$: then all sets containing one element are the same for $P(A)$ and $P(B)$, and $A = B$.

- suppose $A = B$: subsets of $A$ and subsets of $B$ are the same, and $P(A) = P(B)$.

**Exercise 57.** Let $P(C)$ denote the power set of $C$. Prove that for two sets $A$ and $B$

$$P(A) \subseteq P(B) \iff A \subseteq B.$$

*Solution.* We need to show that $P(A) \subseteq P(B) \to A \subseteq B$ and $A \subseteq B \to P(A) \subseteq P(B)$.

- suppose $P(A) \subseteq P(B)$: then $A \subseteq P(B)$, from which $A \subseteq B$.

- suppose $A \subseteq B$: then for any $X \in P(A)$, $X \subseteq A$, $X \subseteq B$, therefore $X \in P(B)$.

**Exercise 58.** Show that the empty set is a subset of all non-null sets.

*Solution.* Recall the definition of subset: $Y \subseteq X$ means by definition that $\forall x$, $(x \in Y \to x \in X)$. Now take $Y$ to be the empty set $\varnothing$. Since $x \in Y$ is necessarily false (one cannot take any $x$ in the empty set), then the conditional statement is vacuously true.

**Exercise 59.** Show that for two sets $A$ and $B$

$$A \neq B \equiv \exists x[(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)].$$

*Solution.*

$$\begin{aligned}
A \neq B &= \neg\forall x(x \in A \leftrightarrow x \in B) \\
&\equiv \exists x \neg(x \in A \leftrightarrow x \in B) \text{ (negation of universal quantifier)} \\
&\equiv \exists x \neg[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)] \text{(definition)} \\
&\equiv \exists x[\neg(x \in A \rightarrow x \in B) \vee \neg(x \in B \rightarrow x \in A)] \text{(DeMorgan)} \\
&\equiv \exists x[\neg(x \notin A \vee x \in B) \vee \neg(x \notin B \vee x \in A)] \text{ (Conversion)} \\
&\equiv \exists x[(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)] \text{ (DeMorgan)}
\end{aligned}$$

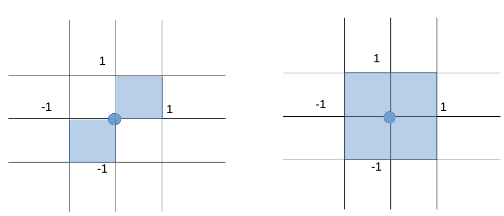
**Exercise 60.** Prove that for the sets $A$, $B$, $C$, $D$

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$$

Does equality hold?

*Solution.* Suppose $x \in (A \times B) \cup (C \times D)$, then $x = (x_1, x_2)$ with $x_1 \in A$, $x_2 \in B$ (or $x_1 \in C$, $x_2 \in D$). But then $x_1 \in A$ (or $C$), and $x_2 \in B$ (or $D$), therefore $x \in (A \cup C) \times (B \cup D)$. The equality does not hold: take $A = [-1, 0]$, $B = [-1, 0]$, $C = [0, 1]$, $D = [0, 1]$ (all the set are intervals, that is $[a, b]$ means the interval from $a$ to $b$). Then

$$([-1, 0] \times [-1, 0]) \cup ([0, 1] \times [0, 1]) \neq [-1, 1] \times [-1, 1].$$



**Exercise 61.** Does the equality

$$(A_1 \cup A_2) \times (B_1 \cup B_2) = (A_1 \times B_1) \cup (A_2 \times B_2)$$

hold?

*Solution.* No it does not. For example, take $A_1 = \{0\}$, $A_2 = \{1\}$, $B_1 = \{0\}$, $B_2 = \{1\}$, then

$$\{0, 1\} \times \{0, 1\} \neq \{(0, 0), (1, 1)\}.$$

**Exercise 62.** How many subsets of $\{1, \ldots, n\}$ are there with an even number of elements? Justify your answer.

*Solution.* One way to solve this is to say that the total number of subsets of $\{1, \ldots, n\}$ is $2^n$. Now this total number counts subsets of odd and even numbers of elements. The way we proved that the total number is $2^n$ is by noting that counting all subsets is adding the choice of $k$ elements out of $n$, that is

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = (1+1)^n$$

by using the binomial theorem:

$$(x+y)^n = \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k}.$$

Now use again the binomial theorem but this time with $x = -1$ and $y = 1$:

$$0 = \sum_{k=1}^{n} \binom{n}{k} (-1)^k = \sum_{k\ odd}^{n} \binom{n}{k} (-1) + \sum_{k\ even}^{n} \binom{n}{k}$$

This shows that

$$\sum_{k\ odd}^{n} \binom{n}{k} = \sum_{k\ even}^{n} \binom{n}{k}$$

or in other words the number of subsets with even number of elements is the the same as the number of subsets with odd number of elements, therefore we have

$$2^n/2 = 2^{n-1}$$

subsets of $\{1, \ldots, n\}$ with an even number of elements.

**Exercise 63.** Prove the following set equality:

$$\{12a + 25b, a, b \in \mathbb{Z}\} = \mathbb{Z}.$$

*Solution.* We prove double inclusion. First we note that since $a, b \in \mathbb{Z}$, $12a + 25b \in \mathbb{Z}$ (closure of addition and multiplication of integers), thus

$$\{12a + 25b, a, b \in \mathbb{Z}\} \subseteq \mathbb{Z}.$$

Next we need to prove the reverse inclusion. Take any $x \in \mathbb{Z}$. We need to prove that $x$ can be written of the form $x = 12a + 25b$ for $a, b \in \mathbb{Z}$. One way of doing this is to pick $a = -2x$, and $b = x$ then $x = 12(-2x) + 25x$. This shows that every element $x \in \mathbb{Z}$ is of the form $12a + 25b$ for some $a, b \in \mathbb{Z}$ therefore

$$\mathbb{Z} \subseteq \{12a + 25b, a, b \in \mathbb{Z}\}.$$

and we have equality.

**Exercise 64.** Prove or disprove the following set equality:

$$A - (B \cup C) = (A - B) \cap (A - C).$$

*Solution.* Recall that $A - B = A \cap \bar{B}$, thus

$$A - (B \cup C) = A \cap \overline{(B \cup C)} = A \cap (\bar{B} \cap \bar{C})$$

using De Morgan law for sets. Then since $A \cap A = A$, we further have

$$A \cap (\bar{B} \cap \bar{C}) = A \cap \bar{B} \cap A \cap \bar{C} = (A - B) \cap (A - C).$$

**Exercise 65.** For all sets $A$, $B$, $C$, prove that

$$\overline{(A - B) - (B - C)} = \bar{A} \cup B.$$

using set identities.

*Solution.* We have

$$\begin{aligned}
\overline{(A - B) - (B - C)} &= \overline{(A \cap \bar{B}) - (B \cap \bar{C})} \\
&= \overline{(A \cap \bar{B}) \cap \overline{(B \cap \bar{C})}} \\
&= \overline{(A \cap \bar{B})} \cup (B \cap \bar{C}) \\
&= (\bar{A} \cup B) \cup (B \cap \bar{C}) \\
&= \bar{A} \cup B \cup (B \cap \bar{C}) \\
&= \bar{A} \cup B
\end{aligned}$$

Both the 3rd and 4rth equality follows from De Morgan's Laws for sets, and $\bar{\bar{S}} = S$ for any set $S$. The 5th equality is associativity, while the last equality is true because $(B \cap \bar{C})$ is a subset of $B$.

**Exercise 66.** This exercise is more difficult. For all sets $A$ and $B$, prove $(A \cup B) \cap \overline{A \cap B} = (A - B) \cup (B - A)$ by showing that each side of the equation is a subset of the other.

*Solution.* We have to prove that (1) $(A \cup B) \cap \overline{A \cap B} \subseteq (A - B) \cup (B - A)$ and (2) $(A \cup B) \cap \overline{A \cap B} \supseteq (A - B) \cup (B - A)$.

**Part (1).** Suppose that $x \in (A \cup B) \cap \overline{A \cap B}$, then

$$(x \in (A \cup B)) \wedge (x \in \overline{A \cap B}).$$

Now $(x \in (A \cup B))$ means that $(x \in A) \vee (x \in B)$, that is

$$\begin{aligned} & [(x \in A) \vee (x \in B)] \wedge (x \in \overline{A \cap B}) \\ \equiv \; & [(x \in A) \wedge (x \in \overline{A \cap B})] \vee [(x \in B) \wedge (x \in \overline{A \cap B})] \end{aligned}$$

using distributivity.

Next $(x \in \overline{A \cap B})$ means that $x \notin A \cap B$ ($x$ always lives in the universe $U$, so it is not repeated). Now the negation of $x \in A \cap B$ is $(x \notin A) \vee (x \notin B)$. We thus get that $(x \in A) \wedge (x \in \overline{A \cap B})$ becomes

$$(x \in A) \wedge [(x \notin A) \vee (x \notin B)] \equiv F \vee [(x \in A) \wedge (x \notin B)]$$

using distributivity. Repeating the same procedure by flipping the role of $B$ and $A$ in $(x \in B) \wedge (x \in \overline{A \cap B})$, we finally obtain that

$$[(x \in A) \wedge (x \notin B)] \vee [(x \in B) \wedge (x \notin A)].$$

We have thus shown that $(A \cup B) \cap \overline{A \cap B} \subseteq (A - B) \cup (B - A)$.

**Part (2).** For the second part, we need to show $(A \cup B) \cap \overline{A \cap B} \supseteq (A - B) \cup (B - A)$.

Suppose thus that $x \in (A - B) \cup (B - A)$, that is $x \in (A - B)$ or $x \in (B - A)$. If $x \in (A - B)$ then $x \in A$ and $x \notin B$ by definition. Therefore $x \in A \cup B$ and $x \notin A \cap B$.

Similarly, if $x \in (B - A)$ then $x \in B$ and $x \notin A$ by definition. Therefore $x \in A \cup B$ and $x \notin A \cap B$.

**Exercise 67.** The symmetric difference of $A$ and $B$, denoted by $A \oplus B$, is the set containing those elements in either $A$ or $B$, but not in both $A$ and $B$.

1. Prove that $(A \oplus B) \oplus B = A$ by showing that each side of the equation is a subset of the other.

2. Prove that $(A \oplus B) \oplus B = A$ using a membership table.

*Solution.* It is a good idea to draw a Venn diagram to visualize $(A \oplus B)$, which consists of $A \cup B$ without the intersection $A \cap B$.

1. We have to show that (1) $(A \oplus B) \oplus B \subseteq A$, and (2) $(A \oplus B) \oplus B \supseteq A$.

   - $(A \oplus B) \oplus B \subseteq A$: take $x \in (A \oplus B) \oplus B$. If $x \in B$, then $x \notin A \oplus B$ by definition. But then, it must be that $B \in A \cap B$ that is, $x \in A$ as desired. Next if $x \notin B$, then $x \in A \oplus B$ by definition. But then $x$ is in the union $A \cup B$ though not in the intersection $A \cap B$, and since it is not in $B$, it must be in $A$.
   - $(A \oplus B) \oplus B \supseteq A$: we now start with $x \in A$. If $x \in B$ (that is, $x \in A \cap B$), then $x \notin A \oplus B$, then $x \in (A \oplus B) \oplus B$. Next if $x \notin B$, then $x \in A \oplus B$, and thus $x \in (A \oplus B) \oplus B$.

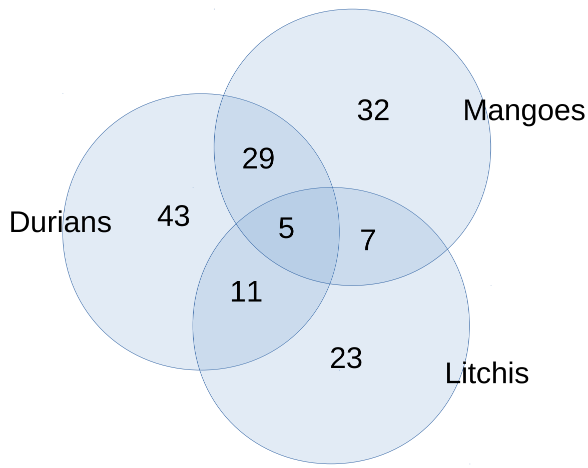
2. We construct a membership table as shown next:

| $A$ | $B$ | $A \oplus B$ | $(A \oplus B) \oplus B$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |

   For example, for the second row, $x \in B$ but not in $A$. Then $x \in A \oplus B$. But then it cannot be in $(A \oplus B) \oplus B$ since $x$ is in both $B$ and $A \oplus B$. We conclude since both the first and last column are the same.

**Exercise 68.** In a fruit feast among 200 students, 88 chose to eat durians, 73 ate mangoes, and 46 ate litchis. 34 of them had eaten both durians and mangoes, 16 had eaten durians and litchis, and 12 had eaten mangoes and litchis, while 5 had eaten all 3 fruits. Determine, how many of the 200 students ate none of the 3 fruits, and how many ate only mangoes?

*Solution.* Let us draw a Venn diagram with 3 sets (one for each of the fruits) and start by identifying the numbers of students who ate all the 3 fruits, namely 5 of them. Then we identify the number of students who ate two types of fruits (for example, 34 ate durians and mangoes, so 34-5=29), and finally only one type of fruit.




We thus get a total of 150, meaning that 50 students ate nothing. 32 students ate only mangoes.

**Exercise 69.** Let $A, B, C$ be sets. Prove or disprove the following set equality:

$$A \times (B - C) = (A \times B) - (A \times C).$$

*Solution.* The first important thing to notice here is that we have a cartesian product of sets. Take $x \in A \times (B - C)$. Then $x = (x_1, x_2)$ with $x_1 \in A$ and $x_2 \in B - C$ (or equivalently $x_2 \in B$ and $x_2 \notin C$). Thus $(x_1, x_2) \in A \times B$ and $(x_1, x_2) \notin A \times C$, which shows that

$$A \times (B - C) \subseteq (A \times B) - (A \times C).$$

Conversely, take $x = (x_1, x_2) \in A \times B$ but not in $A \times C$. Since $x_1 \in A$, it must be that $x_2 \in B$ and also $x_2 \notin C$ for $x$ not to be in $A \times C$. Thus $x_1 \in A$

and $x_2 \in B - C$ which shows that $x \in A \times (B - C)$ and we have the reverse inclusion:

$$(A \times B) - (A \times C) \subseteq A \times (B - C).$$

Note that it is also possible to do a membership table, but then the membership table needs to reflect the cartesian product.

# Exercises for Chapter 8

**Exercise 70.** Consider the sets $A = \{1, 2\}$, $B = \{1, 2, 3\}$ and the relation $(x, y) \in R \iff (x-y)$ is even. Compute the inverse relation $R^{-1}$. Compute its matrix representation.

*Solution.* The relation $R$ is

$$(1, 1), (1, 3), (2, 2),$$

therefore the relation $R^{-1}$ is

$$(1, 1), (3, 1), (2, 2).$$

Its matrix representation is obtained by representing $B$ as rows, that is row 1 is $b_1 = 1$, row 2 is $b_2 = 2$, row 3 is $b_3 = 3$, while column 1 is $a_1 = 1$ and column 2 is $a_2 = 2$:

$$\begin{pmatrix} T & F \\ F & T \\ T & F \end{pmatrix}$$

**Exercise 71.** Consider the sets $A = \{2, 3, 4\}$, $B = \{2, 6, 8\}$ and the relation $(x, y) \in R \iff x \mid y$. Compute the matrix of the inverse relation $R^{-1}$.

*Solution.* The relation $R$ is

$$(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)$$

thus the inverse relation $R^{-1}$ is

$$(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)$$

that is $(x, y) \in R^{-1} \iff x$ is a multiple of $y$, and the corresponding matrix is

$$\begin{pmatrix} T & F & F \\ T & T & F \\ T & F & T \end{pmatrix}$$