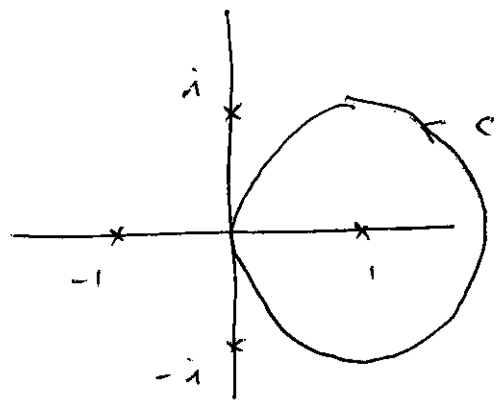
Tutorial 3 (Solutions) (Tutorial 9).

1 (a) $z^4 - 1 = 0$.

$z^4 = 1$

$z^2 = \pm 1$

$z = \pm 1, \pm i$



$|z-1| = 1 \Rightarrow$ path is circle
radius $= 1$
centre at $(1, 0)$.

$\Rightarrow$ encloses singular point at $z = 1$.

$$\oint_c \frac{1}{z^4 - 1}\, dz = \oint_c \frac{1}{(z-1)(z^3+z^2+z+1)}\, dz.$$

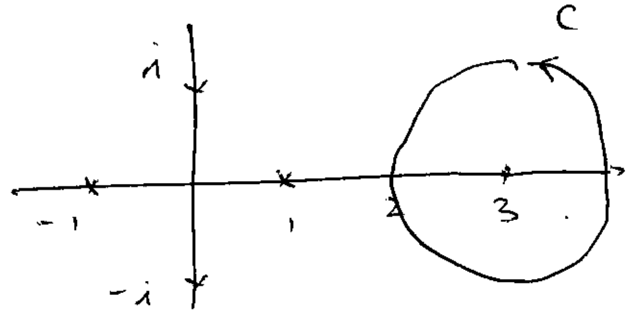
$$= \oint_c \frac{\frac{1}{z^3+z^2+z+1}}{z-1}\, dz.$$

$$= 2\pi i \left(\frac{1}{z^3+z^2+z+1} \right)\Bigg|_{z=1}$$

$$= \frac{\pi i}{2}$$

(b). $|z-3| = 1 \Rightarrow$ path is circle of radius $= 1$
center at $z = 3$.

$$\oint_c \frac{1}{z^4-1}\, dz = 0$$



Q2 (a). $\oint_C \frac{5z}{z^2+4}\,dz$.

$$z^2+4=0$$
$$z^2=-4$$
$$z=\pm 2i$$

$$= \oint_C \frac{5z}{(z-2i)(z+2i)} \cdot dz.$$

$$\Rightarrow z^2+4 = (z-2i)(z+2i).$$

$$= \oint_{C_1} \frac{\frac{5z}{z-2i}}{z+2i}\,dz + \oint_{C_2} \frac{\frac{5z}{z+2i}}{z-2i}\,dz.$$

$$= 2\pi i \left.\frac{5z}{z-2i}\right|_{z=-2i} + 2\pi i \left.\frac{5z}{z+2i}\right|_{z=2i}$$

$$= 2\pi i \frac{5(-2i)}{-4i} + 2\pi i \frac{5(2i)}{4i}$$

$$= \underline{10\pi i}.$$

(b) $\oint_C \frac{z+e^z}{z^3-z}\,dz$.

$$= \oint_C \frac{z+e^z}{z(z-1)(z+1)} \cdot dz.$$

$$= 2\pi i \left[\left.\frac{z+e^z}{z^2-1}\right|_{z=0} + \left.\frac{z+e^z}{z(z+1)}\right|_{z=1} + \left.\frac{z+e^z}{z(z-1)}\right|_{z=-1} \right]$$

$$= 2\pi i \left[-1 + \frac{1+e}{2} + \frac{-1+e^{-1}}{2} \right].$$

$$= \underline{2\pi i \left[-1 + \frac{e+e^{-1}}{2} \right]}$$

3a) Let $z = e^{i\theta}$ $\quad 0 \le \theta \le 2\pi$.

$$\left.\begin{aligned} z &= \cos\theta + i\sin\theta . \\ \bar{z} = \frac{1}{z} &= \cos\theta - i\sin\theta . \end{aligned}\right\} .$$

$$\sin\theta = \frac{z - \frac{1}{z}}{2i} = \frac{z^2 - 1}{2iz} .$$

$$dz = e^{i\theta} . i\, d\theta . \Rightarrow d\theta = \frac{1}{iz} dz .$$

$$\int_0^{2\pi} \frac{d\theta}{5 - 3\sin\theta} = \oint_C \frac{1}{5 - 3\frac{z^2-1}{2iz}} . \frac{1}{iz} dz .$$

$$= -2 \oint_C \frac{1}{3z^2 - 10zi - 3} dz .$$

$$= -2 \oint_C \frac{1}{(z - 3i)(3z - i)} dz .$$

$$= -2 . \oint_C \frac{\frac{1}{3(z-3i)}}{(z - \frac{i}{3})} dz .$$

$$= -\frac{2}{3} . 2\pi i \left. \frac{1}{z - 3i} \right|_{z = \frac{i}{3}}$$

$$= \frac{\pi}{2}$$

3b). Let $z = e^{i\theta}$ $\quad 0 \le \theta \le 2\pi$.

$$\cos\theta = \frac{z + \frac{1}{z}}{2} = \frac{z^2+1}{2z}.$$

$$z^2 = e^{i2\theta} = \cos 2\theta + i\sin 2\theta.$$

$$\bar{z}^2 = \frac{1}{z^2} = e^{-i2\theta} = \cos 2\theta - i\sin 2\theta$$

$$\cos 2\theta = \frac{z^2 + \frac{1}{z^2}}{2} = \frac{z^4+1}{2z^2}.$$

$$dz = e^{i\theta}\cdot i\, d\theta \Rightarrow d\theta = \frac{1}{iz}dz.$$

$$\int_0^{2\pi} \frac{\cos\theta}{13 - 12\cos 2\theta}d\theta = \oint_C \frac{\frac{z^2+1}{2z}}{13 - 12\cdot\frac{z^4+1}{2z^2}} \frac{1}{iz}dz$$

$$= \oint_C \frac{z^2+1}{\cancel{2z}} \cdot \frac{\cancel{2z^2}}{-12z^4+26z^2-12} \cdot \frac{1}{i\cancel{z}} dz.$$

$$= \frac{i}{2}\oint_C \frac{z^2+1}{6z^4-13z^2+6}dz.$$

$$= \frac{i}{2}\oint_C \frac{z^4+1}{6(z^2-\frac{2}{3})(z^2-\frac{3}{2})}dz.$$

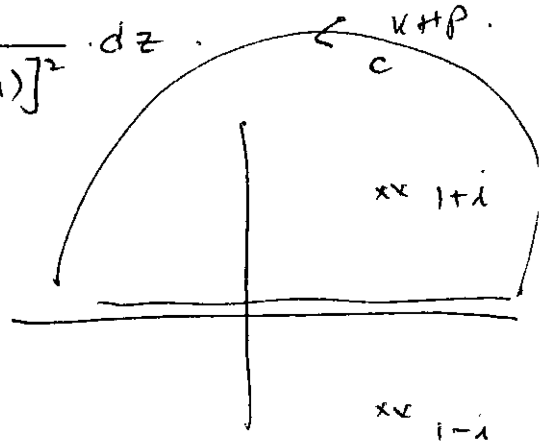
$$= \frac{i}{12}\left[\oint_{C_1} \frac{\frac{z^4+1}{(z^2-\frac{3}{2})(z-\sqrt{\frac{2}{3}})}}{z+\sqrt{\frac{2}{3}}}dz + \oint_{C_2} \frac{\frac{z^4+1}{(z^2-\frac{3}{2})(z+\sqrt{\frac{2}{3}})}}{z-\sqrt{\frac{2}{3}}}dz\right].$$

$$= \frac{i}{12}\left[\frac{z^4+1}{(z^2-\frac{3}{2})(z-\sqrt{\frac{2}{3}})}\bigg|_{z=-\sqrt{\frac{2}{3}}} + \frac{z^4+1}{(z^2-\frac{3}{2})(z+\sqrt{\frac{2}{3}})}\bigg|_{z=+\sqrt{\frac{2}{3}}}\right].$$

$$= \underline{\underline{0}}\,.$$

4a). $$\int_{-\infty}^{\infty} \frac{x}{(x^2-2x+2)^2}\,dx = \oint_{UHP} \frac{z}{(z^2-2z+2)^2}\,dz.$$

$$= \oint_{UHP} \frac{z}{[z-(1+i)]^2[z-(1-i)]^2}\,dz.$$



$$= \oint_{UHP} \frac{\frac{z}{[z-(1-i)]^2}}{[z-(1+i)]^2}\,dz$$

$$= 2\pi i \cdot \frac{d}{dz}\frac{z}{[z-(1-i)]^2}\Bigg|_{z=1+i}$$

$$= 2\pi i\left(\frac{-i}{4}\right) = \frac{\pi}{2}$$

4b). $$\int_{-\infty}^{\infty} \frac{1}{(4+x^2)^2}\,dx = \oint_{UHP} \frac{1}{(4+z^2)^2}\,dz.$$

$$= \oint_{UHP} \frac{1}{(z+2i)^2(z-2i)^2}\,dz.$$

$$= \oint_{UHP} \frac{\frac{1}{(z+2i)^2}}{(z-2i)^2}\,dz.$$

$$= 2\pi i \cdot \frac{d}{dz}\frac{1}{(z+2i)^2}\Bigg|_{z=2i}$$

$$= 2\pi i\left(\frac{-2i}{4^3}\right) = \frac{\pi}{16}$$