EE2007 Engineering Mathematics II

Linear Algebra

K.V. Ling

ekvling@ntu.edu.sg
Rm: S2-B2a-22, School of EEE
Nanyang Technological University

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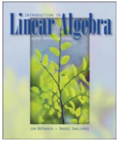
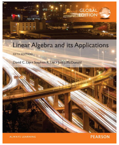
Scope and Textbook

- **Topic 1**: Systems of Linear Equations and Matrices
- **Topic 2**: Linear Combinations and Linear Independence
- **Topic 3**: Vector Spaces
- **Topic 4**: Eigenvalues and Eigenvectors
- **References**:



DeFranza and Gagliardi
Introduction to Linear Algebra with Applications
McGraw-Hill, 2009
NTU LWNL: QA184.2.D316



David C. Lay, Steven R. Lay and Judi J. McDonald
Linear Algebra and Its Applications, Global Edition, 5/E
Pearson, 2016
NTU LWNL QA184.2.L426

Free Resources on the Web

Some examples:

- Professor Gilbert Strang's Video Lectures on Linear Algebra (ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/)
- Free Linear Algebra textbook (joshua.smcvt.edu/linearalgebra)
- Linear Algebra at UC Davis (www.math.ucdavis.edu/∼linear/)

What you will learn

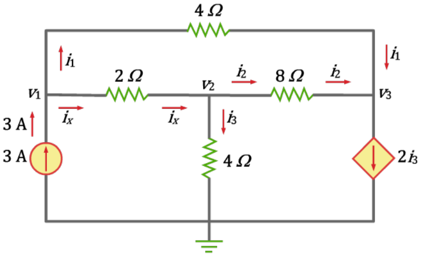
Many diverse applications are modeled by systems of equations. In this module we will:

- develop systematic methods for solving **systems of linear equations**, and
- determine whether the equations will have (a) unique, (b) many, or (c) no solution.

Example: Electrical Networks

Find the node voltages v_1, v_2 and v_3 of the circuit

$$\begin{aligned}
\tfrac{3}{4}v_1 \quad -\tfrac{1}{2}v_2 \quad -\tfrac{1}{4}v_3 &= 3 \\
-\tfrac{1}{2}v_1 \quad +\tfrac{7}{8}v_2 \quad -\tfrac{1}{8}v_3 &= 0 \\
\tfrac{1}{4}v_1 \quad -\tfrac{3}{8}v_2 \quad +\tfrac{1}{8}v_3 &= 0
\end{aligned}$$



Example: Resource allocation

A florist offers three sizes of flower arrangements containing roses, daisies, and chrysanthemums. Each small arrangement contains one rose, three daisies, and three chrysanthemums. Each medium arrangement contains two roses, four daisies and six chrysanthemums. Each large arrangement contains four roses, eight daisies and six chrysanthemums. One day, the florist noted that she used a total of 24 roses, 50 daisies and 48 chrysanthemums in filling orders for these three types of arrangements. How many arrangements of each type did she make?

Solution: Let there be x_1, x_2 and x_3 orders of small, medium and large arrangements respectively. Thus

$$\begin{array}{rrrcl} x_1 & +2x_2 & +4x_3 & = & 24 \\ 3x_1 & +4x_2 & +8x_3 & = & 50 \\ 3x_1 & +6x_2 & +6x_3 & = & 48 \end{array}$$

Example: Photosynthesis

The chemical equation is

$$aCO_2 + bH_2O \rightarrow cO_2 + dC_6H_{12}O_6$$

where a, b, c and d are some positive whole numbers. Balance the chemical equation. This gives us the *system of three linear equations in four variables*

$$\left\{\begin{array}{lrcrcrcrl} C: & a & & & & & - & 6d & = 0 \\ O: & 2a & + & b & - & 2c & - & 6d & = 0 \\ H: & & & 2b & & & - & 12d & = 0 \end{array}\right.$$

Any positive integers a, b, c and d that satisfy all three equations are a solution. For example, $a = 6, b = 6, c = 6$ and $d = 1$. The general solution is $a = b = c = 6d$

Gaussian Elimination

- a systematic way to solve **systems of linear equations**
- determine whether the equations will have (a) unique, (b) many, or (c) no solution.

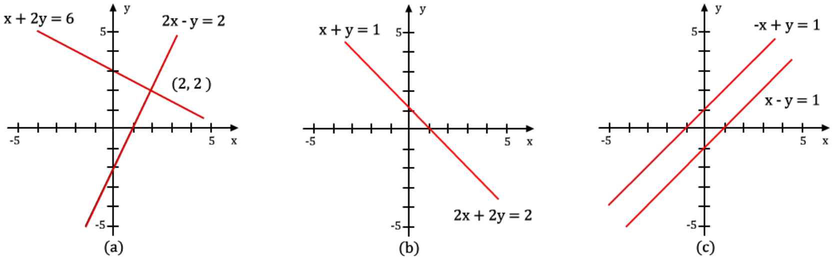
Example: Two Equations with Two Unknowns

To develop this idea, consider the set of equations

$$\begin{cases} x + 2y = 6 \\ 2x - y = 2 \end{cases}$$

Since both equations represent straight lines, a solution exists provided that the lines intersect. In this case, the lines intersect at the unique point $(2, 2)$.

Of course, two lines need not intersect at a single point. They could be parallel (no solution), or they could coincide and hence "intersect" at every point on the line (many solutions).



System of Linear Equations

We generalise to the case of m linear equations in n unknowns.

Given a system of m linear equations in n variables (This is also refers to as an $m \times n$ linear system)

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}\right.$$

There are **three** possible outcomes:

- unique solution
- many solutions
- no solution

Matrix Notation

Coefficient matrix and augmented matrix

For example, the system

$$\begin{array}{rrrcl} x_1 & -2x_2 & +x_3 & = & 0 \\ x_1 & & -x_3 & = & 2 \\ & x_2 & -4x_3 & = & 4 \end{array} \quad \text{can be recorded as} \quad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & -4 & 4 \end{array}\right]$$

The matrix $\left[\begin{array}{rrr} 1 & -2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -4 \end{array}\right]$ is called the **coefficient matrix** and $\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & -4 & 4 \end{array}\right]$ is called the **augmented matrix** of the system.

Elementary Row Operations (EROs)

There are three types ERO:

- Interchanging any two rows, $R_i \leftrightarrow R_j$
- Multiplying any row by a nonzero constant, $R_j \leftarrow \alpha R_j, \ \alpha \neq 0$
- Adding a multiple of one row to another, $R_j \leftarrow R_j + \beta R_i$

Performing any one of the EROs will not change the solution of the linear system.

DEFINITION

Two linear systems are **equivalent** if they have the same solution.

Gaussian Elimination Method

This is an algorithm used to solve linear systems.

The method essentially convert a $m \times n$ linear system into an equivalent system which has a **triangular form**, and then use the **back substitution** method to obtain the solution.

For example

$$\left\{\begin{array}{rrrcr} x_1 & -2x_2 & +x_3 & = & -1 \\ 2x_1 & -3x_2 & -x_3 & = & 3 \\ x_1 & -2x_2 & +2x_3 & = & 1 \end{array}\right. \Rightarrow \left\{\begin{array}{rrrcr} x_1 & -2x_2 & +x_3 & = & -1 \\ & x_2 & -3x_3 & = & 5 \\ & & x_3 & = & 2 \end{array}\right.$$

giving $x_1 = 19, x_2 = 11$ and $x_3 = 2$.

Solving a Linear System / Recording GE Steps

$$\begin{array}{rrrcl} x_1 & -2x_2 & +x_3 & = & 0 \\ x_1 & & -x_3 & = & 2 \\ & x_2 & -4x_3 & = & 4 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & -4 & 4 \end{array}\right]$$

Eliminating x_1 in equation (2) gives $\qquad R_2 \leftarrow R_2 - R_1$

$$\begin{array}{rrrcl} x_1 & -2x_2 & +x_3 & = & 0 \\ & 2x_2 & -2x_3 & = & 2 \\ & x_2 & -4x_3 & = & 4 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -2 & 2 \\ 0 & 1 & -4 & 4 \end{array}\right]$$

Eliminating x_2 in equation (3) gives $\qquad R_3 \leftarrow R_3 - \frac{1}{2}R_2$

$$\begin{array}{rrrcl} x_1 & -2x_2 & +x_3 & = & 0 \\ & 2x_2 & -2x_3 & = & 2 \\ & & -3x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & -3 & 3 \end{array}\right]$$

By **back substitution**,

$$x_3 = -1, \;\; x_2 = \frac{1}{2}(2 + 2x_3) = 0, \;\; x_1 = -x_3 + 2x_2 = 1.$$

The Key Questions

Given a system of linear equations, how can we tell, **systematically**, whether the system has

- unique solution,
- infinitely many solutions, or
- no solution?

Gaussian Elimination, Unique Solution

$$\begin{cases} x + y + z = 4 \\ -x - y + z = -2 \\ 2x - y + 2z = 2 \end{cases} \qquad \left[\begin{array}{rrr|r} 1 & 1 & 1 & 4 \\ -1 & -1 & 1 & -2 \\ 2 & -1 & 2 & 2 \end{array}\right]$$

$$\begin{array}{l} R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 - 2R_1 \end{array} \quad \left[\begin{array}{rrr|r} 1 & 1 & 1 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & -3 & 0 & -6 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \quad \left[\begin{array}{rrr|r} 1 & 1 & 1 & 4 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 2 & 2 \end{array}\right] \qquad \begin{array}{l} R_2 \leftarrow \frac{-1}{3} R_2 \\ R_3 \leftarrow \frac{1}{2} R_3 \end{array} \quad \left[\begin{array}{rrr|r} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Answer: $x = 1, y = 2, z = 1$

Gaussian Elimination, No Solution

Write the row operations in the steps shown

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & 5 \\ 2 & 1 & 0 & 2 \\ 1 & 8 & -1 & 3 \\ -1 & -5 & -12 & 4 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -1 & 2 & 5 \\ 0 & 3 & -4 & -8 \\ 0 & 9 & -3 & -2 \\ 0 & -6 & -10 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1 & 2 & 5 \\ 0 & 3 & -4 & -8 \\ 0 & 0 & 9 & 22 \\ 0 & 0 & -18 & -7 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -1 & 2 & 5 \\ 0 & 3 & -4 & -8 \\ 0 & 0 & 9 & 22 \\ 0 & 0 & 0 & 37 \end{array}\right]$$

The last row essentially says that

$$0x_1 + 0x_2 + 0x_3 = 37$$

which indicates that there is no solution.

Gaussian Elimination, Many Solutions

$$\left[\begin{array}{rrrr|r} 4 & -8 & -3 & 2 & 13 \\ 3 & -4 & -1 & -3 & 5 \\ 2 & -4 & -2 & 2 & 6 \end{array}\right]$$

$$R_1 \leftrightarrow R_3 \left[\begin{array}{rrrr|r} 2 & -4 & -2 & 2 & 6 \\ 3 & -4 & -1 & -3 & 5 \\ 4 & -8 & -3 & 2 & 13 \end{array}\right]$$

$$R_1 \leftarrow \frac{1}{2} R_1 \left[\begin{array}{rrrr|r} 1 & -2 & -1 & 1 & 3 \\ 3 & -4 & -1 & -3 & 5 \\ 4 & -8 & -3 & 2 & 13 \end{array}\right]$$

$$\begin{array}{l} R_2 \leftarrow R_2 - 3R_1 \\ R_3 \leftarrow R_3 - 4R_1 \end{array} \left[\begin{array}{rrrr|r} 1 & -2 & -1 & 1 & 3 \\ 0 & 2 & 2 & -6 & -4 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

Answer: $\{(3t - 2, t - 3, 2t + 1, t) | t \in \mathbb{R}\}$

Gaussian Elimination, Many Solutions

Parametric Description of Solution Sets

The augmented matrix $\left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 3 \\ 0 & 2 & 2 & -6 & -4 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right]$ represents the equations

$$\begin{array}{rrrrcr} x_1 & -2x_2 & -x_3 & +x_4 & = & 3 \\ & 2x_2 & +2x_3 & -6x_4 & = & -4 \\ & & x_3 & -2x_4 & = & 1 \end{array}$$

So we have 3 equations with 4 unknowns. If we chose x_4 to be a **free** variable, then by **back substitution**, we can write

$$\begin{aligned} x_4 &= \text{free} \stackrel{\text{let}}{=} t \\ x_3 &= 1 + 2x_4 = 1 + 2t \\ x_2 &= \tfrac{1}{2}(-4 + 6x_4 - 2x_3) = -3 + t \\ x_1 &= 3 - x_4 + x_3 + 2x_2 = -2 + 3t \end{aligned}$$

Gaussian Elimination, Many Solutions

Parametric Description of Solution Sets

Alternatively, we could have chosen x_3 as free variable, giving

$$\begin{aligned}
x_3 &= \text{free} \stackrel{\text{let}}{=} s \\
x_4 &= \tfrac{1}{2}(x_3 - 1) = \tfrac{1}{2}s - \tfrac{1}{2} \\
x_2 &= \tfrac{1}{2}(-4 + 6x_4 - 2x_3) = \tfrac{-7}{2} + \tfrac{1}{2}s \\
x_1 &= 3 - x_4 + x_3 + 2x_2 = \tfrac{-7}{2} + \tfrac{3}{2}s
\end{aligned}$$

We can also write the solution set as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7/2 \\ -7/2 \\ 0 \\ -1/2 \end{bmatrix} + s \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \\ 1/2 \end{bmatrix}, \quad s \in \mathbb{R}$$

Note that choosing either x_3 or x_4 as free variable gives the same solution set.
This can be verified by the substitution $s = 1 + 2t$.

Use software to perform Gaussian Elimination

$$\begin{cases} x_1 & -6x_2 & -4x_3 & = & -5 \\ 2x_1 & -10x_2 & -9x_3 & = & -4 \\ -x_1 & +6x_2 & +5x_3 & = & 3 \end{cases}$$

Solve using MS Excel:

	A	B	C	D	E
1	R1	1	-6	-4	-5
2	R2	2	-10	-9	-4
3	R3	-1	6	5	3
4					
5	R1	1	-6	-4	-5
6	R2<-R2-2R1	0	2	-1	6
7	R3<-R3+R1	0	0	1	-2

Solve using MATLAB:

```
>> M = [1 -6 -4 -5; 2 -10 -9 -4; -1 6 5 3]

M =

     1    -6    -4    -5
     2   -10    -9    -4
    -1     6     5     3

>> M(2,:) = M(2,:)-2*M(1,:);
>> M(3,:) = M(3,:)+M(1,:)

M =

     1    -6    -4    -5
     0     2    -1     6
     0     0     1    -2
```

Answer: $x_1 = -1, x_2 = 2, x_3 = -2$

Practice

- Is the system consistent?

$$\left[\begin{array}{ccc|c} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{array}\right]$$

- Is $(3, 4, -2)$ a solution of the following system?

$$\left[\begin{array}{ccc|c} 5 & -1 & 2 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & -7 \end{array}\right]$$

- For what values of h and k is the system consistent?

$$\left[\begin{array}{cc|c} 2 & -1 & h \\ -6 & 3 & k \end{array}\right]$$

Row Echelon (RE) Form

The leading nonzero term in each (non-zero) row is called a **pivot**.
A matrix is in row echelon form if **all entries in a column below a pivot are zeros**.
For example, these matrices are in RE form:

$$\begin{bmatrix} \blacktriangle & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ 0 & 0 & \blacktriangle & \blacksquare & \blacksquare \\ 0 & 0 & 0 & \blacktriangle & \blacksquare \end{bmatrix} \quad \begin{bmatrix} \blacktriangle & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacktriangle & \blacksquare & \blacksquare \\ 0 & 0 & \blacktriangle & \blacksquare \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \blacktriangle & \blacksquare & \blacksquare & \blacksquare & 0 & 0 \\ 0 & \blacktriangle & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ 0 & 0 & \blacktriangle & \blacksquare & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & \blacktriangle & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon (RRE) Form

The matrix is in **reduced row echelon form (RRE)** if, in addition, **each pivot is a 1** and **all other entries in this column are 0**.

For example, the following matrices are in RRE form:

$$\begin{bmatrix} 1 & \blacksquare & 0 & 0 & \blacksquare \\ 0 & 0 & 1 & 0 & \blacksquare \\ 0 & 0 & 0 & 1 & \blacksquare \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & \blacksquare \\ 0 & 1 & 0 & \blacksquare \\ 0 & 0 & 1 & \blacksquare \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & \blacksquare & 0 & 0 \\ 0 & 1 & 0 & \blacksquare & 0 & \blacksquare \\ 0 & 0 & 1 & \blacksquare & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 1 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the RRE form of a matrix is **unique**

The process of transforming a matrix to RRE form is called **Gauss-Jordan Elimination**.

Example: Reduce the matrix to RE and RRE

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\sim \cdots \sim \quad \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

1. Start with the leftmost nonzero column. This is the pivot column.
2. Interchange rows, if necessary, such that the top of the pivot column is nonzero.
3. Use ERO to reduce the matrix to a RE form.
4. Start with the rightmost pivot and work to the left. Use ERO to reduce the matrix to a RRE form.

Gauss-Jordan Elimination

Reduce the coefficient matrix to RRE form

$$\begin{cases} x_1 & -x_2 & -2x_3 & +x_4 & = & -3 \\ 2x_1 & -x_2 & -3x_3 & +2x_4 & = & -1 \\ -x_1 & +2x_2 & +x_3 & +3x_4 & = & 18 \\ x_1 & +x_2 & -x_3 & +2x_4 & = & 8 \end{cases}$$

R1	1	-1	-2	1	-3
R2	2	-1	-3	2	-1
R3	-1	2	1	3	18
R4	1	1	-1	2	8
R1	1	-1	-2	1	-3
R2<-R2-2R1	0	1	1	0	5
R3<-R3+R1	0	1	-1	4	15
R4<-R4-R1	0	2	1	1	11
R1	1	-1	-2	1	-3
R2	0	1	1	0	5
R3<-R3-R2	0	0	-2	4	10
R4<-R4-2R2	0	0	-1	1	1
R1	1	-1	-2	1	-3
R2	0	1	1	0	5
R3<-R3/(-2)	0	0	1	-2	-5
R4	0	0	-1	1	1

R1	1	-1	-2	1	-3
R2	0	1	1	0	5
R3	0	0	1	-2	-5
R4<-R4+R3	0	0	0	-1	-4
R1	1	-1	-2	1	-3
R2	0	1	1	0	5
R3	0	0	1	-2	-5
R4<--R4	0	0	0	1	4
R1<-R1-R4	1	-1	-2	0	-7
R2	0	1	1	0	5
R3<-R3+2*R4	0	0	1	0	3
R4	0	0	0	1	4
R1<-R1+2*R3	1	-1	0	0	-1
R2<-R2-R3	0	1	0	0	2
R3	0	0	1	0	3
R4	0	0	0	1	4
R1<-R1+R2	1	0	0	0	1
R2	0	1	0	0	2
R3	0	0	1	0	3
R4	0	0	0	1	4

Answer: $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4$

Many Solutions

$$\begin{cases} 3x_1 & -x_2 & +x_3 & -8x_4 & = & 4 \\ x_1 & +2x_2 & -x_3 & -5x_4 & = & 3 \\ -x_1 & -3x_2 & +2x_3 & +3x_4 & = & -2 \end{cases}$$

R1	3	-1	1	-8	4
R2	1	2	-1	-5	3
R3	-1	-3	2	3	-2
R1<->R3, R1<- -R1	1	3	-2	-3	2
R2	1	2	-1	-5	3
R1<->R3	3	-1	1	-8	4
R1	1	3	-2	-3	2
R2<-R2-R1	0	-1	1	-2	1
R3<-R3-3R1	0	-10	7	1	-2
R1	1	3	-2	-3	2
R2	0	-1	1	-2	1
R3<-R3-10*R2	0	0	-3	21	-12
R1	1	3	-2	-3	2
R2<- -R2	0	1	-1	2	-1
R3 <- R3/(-3)	0	0	1	-7	4

R1<-R1+2*R3	1	3	0	-17	10
R2<-R2+R3	0	1	0	-5	3
R3	0	0	1	-7	4
R1<-R1-3*R2	1	0	0	-2	1
R2	0	1	0	-5	3
R3	0	0	1	-7	4

Answer: $S = \{(1+2t,\ 3+5t,\ 4+7t,\ t) | t \in \mathbb{R}\}$

Inconsistent Linear System, No Solution

$$\begin{cases} x_1 & +x_2 & +x_3 & = & 2 \\ 3x_1 & -x_2 & -x_3 & = & 2 \\ x_1 & +3x_2 & +3x_3 & = & 5 \end{cases}$$

R1	1	1	1	2
R2	3	-1	-1	2
R3	1	3	3	5
R1	1	1	1	2
R2<-R2-3R1	0	-4	-4	-4
R3<-R3-R1	0	2	2	3
R1	1	1	1	2
R2	0	-4	-4	-4
R3<-R3+R2/2	0	0	0	1

The third row of the RE form corresponds to the equation $0.x_1 + 0.x_2 + 0.x_3 = 1$, hence the system is inconsistent and has no solution.

Condition for consistent Linear System

Re-do the previous example, but now with unknown RHS

$$\begin{cases} x_1 & +x_2 & +x_3 & = & a \\ 3x_1 & -x_2 & -x_3 & = & b \\ x_1 & +3x_2 & +3x_3 & = & c \end{cases}$$

R1	1	1	1	a
R2	3	-1	-1	b
R3	1	3	3	c
R1	1	1	1	a
R2<-R2-3R1	0	-4	-4	b-3a
R3<-R3-R1	0	2	2	c-a
R1	1	1	1	a
R2	0	-4	-4	b-3a
R3<-R3+R2/2	0	0	0	c-a+(b-3a)/2

So we need $c - a + \frac{b-3a}{2} = 0$ for consistency.

Summary: Conditions for Unique, Many, and No Solution

Represent a linear system with m equations and n unknowns as an augmented matrix $[A|b]$. Transform $[A|b]$ to RE form and denote the result as $[A_R|b_R]$, i.e.,

$$[A|b] \quad \sim \quad \cdots \quad \sim \quad [A_R|b_R]$$

1. $[A_R|b_R]$ and A_R has the same number of non-zero rows
   1.1 **[Unique solution]** The number of non-zero rows is equal to n [$\text{rank}(A|b) = \text{rank}(A) = n$]
   1.2 **[Many solutions]** The number of non-zero rows is less than n [$\text{rank}(A|b) = \text{rank}(A) < n$]
2. **[No solution]** A_R has a row of zeros, and the corresponding term in b_R is not zero. [$\text{rank}(A|b) > \text{rank}(A)$].

where we denote $\text{rank}(A) =$ number of non-zero rows in A_R. A more formal definition of rank will come later.

Introduction to Matrix Algebra

Notations and Definitions

Let A be an $m \times n$ matrix. Then each entry of A can be uniquely specified by using the row and column indices of its location, as shown below:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

We can define addition and multiplication so that algebra can be performed with matrices, just like numbers. This extends the application of matrices beyond just a means for representing a linear systems.

Row and Column Vectors of a Matrix

A **vector** is just an $n \times 1$ matrix. For a given matrix A, it is convenient to refer to its **row** vectors and its **column** vectors. For example, let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & -1 & 2 \end{bmatrix}$$

Then the column vectors of A are

$$\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

while the row vectors of A, written vertically, are

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

Addition, Scalar Multiplication and Equality of Matrices

Consider two $m \times n$ matrices A and B.

$$A + B \quad \text{means} \quad a_{ij} + b_{ij}, \quad 1 \le i \le m \text{ and } 1 \le j \le n.$$

$$cA \quad \text{means} \quad ca_{ij}, \quad 1 \le i \le m \text{ and } 1 \le j \le n, \quad \text{where } c \text{ is a real number}$$

$$A = B \quad \text{means} \quad a_{ij} = b_{ij}, \quad 1 \le i \le m \text{ and } 1 \le j \le n.$$

Example: Addition and Scalar Multiplication of Matrices

Perform the operations: (a) $A+B$ and (b) $2A-3B$ if

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 3 & -1 \\ -3 & 6 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 3 & -1 \\ 3 & 5 & 6 \\ 4 & 2 & 1 \end{bmatrix}$$

```
>> A = [2 0 1; 4 3 -1; -3 6 5]

A =

     2     0     1
     4     3    -1
    -3     6     5

>> B = [-2 3 -1; 3 5 6; 4 2 1]

B =

    -2     3    -1
     3     5     6
     4     2     1
```

```
>> A+B

ans =

     0     3     0
     7     8     5
     1     8     6

>> 2*A-3*B

ans =

    10    -9     5
    -1    -9   -20
   -18     6     7
```

Properties of Matrix Addition and Scalar Multiplication

Let A, B, and C be $m \times n$ matrices and c and d be real numbers. Then

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $c(A + B) = cA + cB$
4. $(c + d)A = cA + dA$
5. $c(dA) = (cd)A$
6. $A + 0 = 0 + A = A$, where 0 denotes a $m \times n$ matrix with all zero entries
7. $-A = (-1)A$, thus $A + (-A) = (-A) + A = 0$

Review: Dot Products of Vectors

Given two vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

The **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is defined by $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

Matrix Multiplication – the Dot Product View

Let A be an $m \times n$ matrix and B an $n \times p$ matrix, then the product AB is an $m \times p$ matrix. The ij term of AB is the dot product of the ith row vector of A with the jth column vector of B, so that

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Example

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & -3 \\ -4 & 6 & 2 \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1)(3) + (3)(-1) + (0)(1) & -2 + 12 + 0 & 5 - 6 + 0 \\ 6 - 1 - 3 & -4 + 4 + 0 & 10 - 2 - 9 \\ -12 - 6 + 2 & 8 + 24 + 0 & -20 - 12 + 6 \end{bmatrix}$$

Matrix Multiplication is not Commutative, $AB \neq BA$

For example, let $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 3 & -1 \\ -3 & 6 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & -1 \\ 3 & 5 & 6 \\ 4 & 2 & 1 \end{bmatrix}$.

It can be verified that $AB = \begin{bmatrix} 0 & 8 & -1 \\ -3 & 25 & 13 \\ 44 & 31 & 44 \end{bmatrix} \neq BA = \begin{bmatrix} 11 & 3 & -10 \\ 8 & 51 & 28 \\ 13 & 12 & 7 \end{bmatrix}$

Sometimes, it could happen that AB is defined but BA is not. For example

$$AB = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & -2 \\ 1 & 0 & 3 \end{bmatrix} \quad \text{is defined}$$

$$BA = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & -2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & -3 \end{bmatrix} \quad \text{is NOT defined}$$

If $AB = BA$, we say the two matrices A and B **commute**.

Example

Find all 2×2 matrices that commute with the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Let $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $SA = AS$. Then ...

Answer: $S = \left\{ \begin{bmatrix} a & 0 \\ c & a \end{bmatrix} | a, c \in \mathbb{R} \right\}$

Properties of Matrix Multiplication

Let A, B and C be matrices with sizes so that the given expression are all defined, and let c be a real number.

1. $A(BC) = (AB)C$
2. $c(AB) = (cA)B = A(cB)$
3. $A(B + C) = AB + AC$
4. $(B + C)A = BA + CA$

Transpose of a Matrix

If A is an $m \times n$ matrix, the **transpose** of A, denoted by A^T, is the $n \times m$ matrix with ij term

$$(A^T)_{ij} = a_{ji} \quad \text{where } 1 \leq i \leq n \text{ and } 1 \leq j \leq m.$$

For example, the transpose of the matrix

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ -1 & 2 & 1 \end{bmatrix} \text{ is } A^T = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \\ -3 & 4 & 1 \end{bmatrix}$$

In some textbook, the notation A' is also used to denote the transpose of A.

Using the transpose notation, the dot product of two (column) vectors can be written as

$$u \cdot v = u^T v = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

Properties of Matrix Transpose

Suppose A and B are $m \times n$ matrices, C is an $n \times p$ matrix, and c is a scalar.

1. $(A + B)^T = A^T + B^T$
2. $(AC)^T = C^T A^T$
3. $(A^T)^T = A$
4. $(cA)^T = cA^T$
5. If $A^T = A$, then A is known as a **symmetric** matrix.

Inverse of a Square Matrix

Definition:
Let A be an $n \times n$ matrix. The inverse of A, denoted as A^{-1}, if exists, is such that

$$AA^{-1} = I = A^{-1}A$$

When the inverse of a matrix A exists, we say A is **invertible or non-singular**. Otherwise, we say A is **non-invertible or singular**.

The inverse of a matrix, if it exists, is unique.

Properties of Inverse

Theorem

Let A and B be $n \times n$ invertible matrices. Then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof:

$$B^{-1}A^{-1}AB = I \Rightarrow (AB)^{-1} = B^{-1}A^{-1}$$

$$ABB^{-1}A^{-1} = I \Rightarrow (AB)^{-1} = B^{-1}A^{-1}$$

The above result can be extended to $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ and, in general, $(A_1A_2\ldots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\ldots A_1^{-1}$.

Inverse of 2×2 Matrix via definition

You already knew and memerised the result, but can you construct it?

Theorem

The inverse of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ exists if and only if $ad - bc \neq 0$. In this case, the inverse is the matrix

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Proof :

Let $B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$. We want $B = A^{-1}$. By definition of inverse matrix, we want to find w, x, y, z such that $AB = I$. Thus

$$AB = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and use ERO to solve the resulting 4×4 system.

Inverse of 2×2 Matrix, cont'd

$$\begin{cases} aw \quad\quad +by \quad\quad = 1 \\ \quad\quad ax \quad\quad +bz = 0 \\ cw \quad\quad +dy \quad\quad = 0 \\ \quad\quad cx \quad\quad +dz = 1 \end{cases} \Rightarrow \left[\begin{array}{cccc|c} a & 0 & b & 0 & 1 \\ 0 & a & 0 & b & 0 \\ c & 0 & d & 0 & 0 \\ 0 & c & 0 & d & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & b/a & 0 & 1/a \\ 0 & 1 & 0 & b/a & 0 \\ c & 0 & d & 0 & 0 \\ 0 & c & 0 & d & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & b/a & 0 & 1/a \\ 0 & 1 & 0 & b/a & 0 \\ 0 & 0 & d-bc/a & 0 & -c/a \\ 0 & 0 & 0 & d-bc/a & 1 \end{array}\right]$$

$$z = \frac{a}{ad-bc}; y = \frac{-c}{ad-bc}; x = \frac{-bz}{a} = \frac{-b}{ad-bc}; w = \ldots = \frac{d}{ad-bc}$$

Inverse of $n \times n$ Matrix via Method of Augmented Matrix

Instead of finding the inverse of an $n \times n$ matrix A by definition, an alternative is to form the augmented matrix $[A|I]$ and then row reducing the $n \times 2n$ augmented matrix. If, in the reduction process, A is transformed to the identity matrix, then the resulting augmented part of the matrix is the inverse.

$$[A|I] \sim \quad \ldots \quad \sim [I|A^{-1}]$$

Why this method works?

Find inverse of a matrix by augmented matrix

Find A^{-1} if $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{rrr|rrr} 1 & 1 & -2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right]$$

$$\cdots \sim \left[\begin{array}{rrr|rrr} 3 & 0 & 0 & 6 & 3 & 12 \\ 0 & 3 & 0 & 3 & 3 & 6 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right] \cdots \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right]$$

Other Ways of Finding Inverse of a Matrix

Akan Datang ...
see slide 58 [Adjoint matrix and inverse]

$Ax = b$, Matrix Form of Linear System

The linear system

$$\left\{\begin{array}{rrrcr} x_1 & -6x_2 & -4x_3 & = & -5 \\ 2x_1 & -10x_2 & -9x_3 & = & -4 \\ -x_1 & +6x_2 & +5x_3 & = & 3 \end{array}\right. \qquad \text{can be written as } Ax = b$$

where

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 2 & -10 & -9 \\ -1 & 6 & 5 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$$

Solving $Ax = b$

via $x = A^{-1}b$

Theorem
If the $n \times n$ matrix A is invertible, then for every vector b, with n components, the linear system $Ax = b$ has the unique solution $x = A^{-1}b$.
Proof:

$$Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b$$

From previous slide, given A, we can compute

$$A^{-1} = \begin{bmatrix} 2 & 3 & 7 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}$$

Then, the solution is $x = A^{-1}b \Rightarrow x_1 = -1,\ x_2 = 2,\ x_3 = -2$

Non-trivial Solution of Homogeneous System[1]

Find all vectors of x such that $Ax = 0$ where $A = \left[\begin{smallmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{smallmatrix}\right]$.

Solution:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $x_2 = x_3$, and $x_1 = -2x_2 - x_3$. Let $x_3 = t$, the solution set in vector form is

$$x = t \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathcal{R}$$

[1]A **homoogeneous** linear system is a system of the form $Ax = 0$. The vector $x = 0$ is always a solution to the homogeneous system $Ax = 0$ and is called the **trivial solution**.

Determinant of a 2×2 Matrix

The **determinant** of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, denoted by $|A|$, is given by

$$|A| = \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Determinant of a 3 × 3 Matrix

The determinant of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinant of a 3×3 Matrix

The computation of the 3×3 determinant takes the form

$$|A| = a_{11} \begin{vmatrix} * & * & * \\ * & a_{22} & a_{23} \\ * & a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} * & * & * \\ a_{21} & * & a_{23} \\ a_{31} & * & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} * & * & * \\ a_{21} & a_{22} & * \\ a_{31} & a_{32} & * \end{vmatrix}$$

In other words, the determinant of a 3×3 matrix is calculated by using an **expansion along the first row**. With an adjustment of signs as shown in figure below, the determinant can be computed by using an expansion along any row or column.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Hint: To reduce computation, expand along a row/column which has many zeros.

Example: Determinant of a 3×3 Matrix

Compute $\det(A)$ where $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 4 \\ 5 & -3 & 3 \end{bmatrix}$.

Solution:

Expand along row 1

$$\det(A) = |A| = 2\begin{vmatrix} 1 & 4 \\ -3 & 3 \end{vmatrix} - 1\begin{vmatrix} 3 & 4 \\ 5 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 1 \\ 5 & -3 \end{vmatrix} = 55$$

Expand along row 2

$$\det(A) = |A| = -3\begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 5 & 3 \end{vmatrix} - 4\begin{vmatrix} 2 & 1 \\ 5 & -3 \end{vmatrix} = 55$$

Expand along row 3

$$\det(A) = |A| = 5\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} - (-3)\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 55$$

Determinant of a $n \times n$ Matrix

The generalisation of the 3×3 determinant to the $n \times n$ case can be expressed as follows:

1. Expand along the ith row of A

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

2. Expand along the jth column of A

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

The **minor** M_{ij} is defined to be the determinant of the $(n-1) \times (n-1)$ matrix that results from A by removing the ith row and the jth column.

The expression $(-1)^{i+j} M_{ij}$ is known as **cofactor**.

Thus, for the 3×3 matrix $\begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 4 \\ 5 & -3 & 3 \end{bmatrix}$, several minors are

$$M_{11} = \begin{vmatrix} 1 & 4 \\ -3 & 3 \end{vmatrix}, \qquad M_{12} = \begin{vmatrix} 3 & 4 \\ 5 & 3 \end{vmatrix}, \qquad M_{13} = \begin{vmatrix} 3 & 1 \\ 5 & -3 \end{vmatrix}$$

Adjoint Matrix and Inverse

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A) \quad \text{where} \quad \mathrm{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \ddots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}^T$$

is the transpose of the cofactor matrix and is called the **adjoint** matrix of A.

Example Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\mathrm{adj}(A) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$

$$= \begin{bmatrix} (+1)M_{11} & (-1)M_{12} \\ (-1)M_{21} & (+1)M_{22} \end{bmatrix}^T = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinant of a Triangular Matrix

Theorem: Determinant of a triangular matrix

If A is an $n \times n$ triangular matrix, then the determinant of A is the product of the terms on the diagonal. That is

$$\det(A) = a_{11}a_{22}\dots a_{nn}$$

Proof: Proof by induction.

Let $n = 1, A = a_{11}$. Then $|A| = a_{11}$. Next, assume the case n is true, i.e.,

$\det(A_{n\times n}) = a_{11}a_{22}\dots a_{nn}$, where $A_{n\times n} = \begin{bmatrix} a_{11} & \dots & \dots & \dots \\ & a_{22} & \dots & \dots \\ & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix}$. Now, consider

$$A_{(n+1)\times(n+1)} = \begin{bmatrix} A_{n\times n} & \vdots \\ & a_{(n+1)(n+1)} \end{bmatrix} \quad \Rightarrow \quad \det(A_{(n+1)\times(n+1)})$$

$$= (-1)^{(n+1)+(n+1)} a_{(n+1)(n+1)} \det(A_{n\times n}) = a_{11}a_{22}\dots a_{nn}a_{(n+1)(n+1)}.$$

Properties of Determinants

Let A be a square matrix.

1. If two rows(columns) of A are interchanged to produce a matrix B, then $\det(B) = -\det(A)$.
2. If a multiple of one row(column) of A is added to another row(column) to produce a matrix B, then $\det(B) = \det(A)$.
3. If a row(column) of A is multiplied by a real number α to produce a matrix B, then $\det(B) = \alpha \det(A)$.

With the above properties, an alternative method to find the determinant of a matrix is to row(column)-reduce the matrix to triangular form and at each step, record the changes to the determinant due to the row (column)-reduce operation. See example in next slide

Find determinant via ERO

Find the determinant of the matrix $A = \begin{bmatrix} 0 & 1 & 3 & -1 \\ 2 & 4 & -6 & 1 \\ 0 & 3 & 9 & 2 \\ -2 & -4 & 1 & -3 \end{bmatrix}$

$$\det(A) \overset{R_4 \leftarrow R_4 + R_2}{=} \begin{vmatrix} 0 & 1 & 3 & -1 \\ 2 & 4 & -6 & 1 \\ 0 & 3 & 9 & 2 \\ 0 & 0 & -5 & -2 \end{vmatrix} \overset{R_1 \leftrightarrow R_2}{=} (-1) \begin{vmatrix} 2 & 4 & -6 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 3 & 9 & 2 \\ 0 & 0 & -5 & -2 \end{vmatrix}$$

$$\overset{R_3 \leftarrow R_3 - 3R_2}{=} (-1) \begin{vmatrix} 2 & 4 & -6 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & -5 & -2 \end{vmatrix} \overset{R_3 \leftrightarrow R_4}{=} (-1)(-1) \begin{vmatrix} 2 & 4 & -6 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -5 & -2 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

$$= (-1)(-1)(2)(1)(-5)(5) = -50$$

More Properties of Determinant

Let A and B be $n \times n$ matrices and α a real number.

1. $\det(AB) = \det(A)\det(B)$
2. $\det(\alpha A) = \alpha^n \det(A)$
3. $\det(A^T) = \det(A)$
4. If A has a row (or column) of all zeros, then $\det(A) = 0$.
5. If A has two equal rows (or columns), then $\det(A) = 0$.
6. If A has a row (or column) that is a multiple of another row (or column) then $\det(A) = 0$.
7. A square matrix A is invertible if and only if $\det(A) \neq 0$

Elementary Matrices

DEFINITION

An **elementary matrix** is any matrix that can be obtained from the identity matrix by performing a single elementary row operation.

Thus, by performing the three types of elementary row operations on a 3×3 identity matrix, we obtain the corresponding 3×3 elementary matrices, and their determinants

$$R_1 \leftrightarrow R_3 \qquad R_2 \leftarrow \alpha R_2, \alpha \neq 0 \qquad R_3 \leftarrow R_3 + \beta R_1$$

$$E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad E_2(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{13}(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix}$$

$$\det(E_{13}) = -1 \qquad \det(E_2(\alpha)) = \alpha \qquad \det(E_{13}(\beta)) = 1$$

Elementary Matrices and Elementary Row Operations

Elementary matrices can be used to perform or record row operations.
In other words, if E is the elementary matrix corresponding to the elementary row operation $\mathcal{R}$, then

$$A \overset{\mathcal{R}}{\sim} B \Leftrightarrow B = EA$$

Thus, if we perform a series of Elementary Row Operations $\mathcal{R}_i$ represented by E_i, then

$$A \overset{\mathcal{R}_1}{\sim} B_1 \overset{\mathcal{R}_2}{\sim} B_2 \cdots \overset{\mathcal{R}_k}{\sim} B \Leftrightarrow B = E_k \ldots E_2 E_1 A$$

Example: Elementary Matrices and ERO

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 0 \\ -1 & 1 & 1 \end{bmatrix} \overset{E_1 : R_2 \leftarrow R_2 - 3R_1}{\sim} [\cdot] \overset{E_2 : R_3 \leftarrow R_3 + R_1}{\sim} [\cdot] \overset{E_3 : R_3 \leftarrow R_3 + 3R_2}{\sim} B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

and it can be verified that

$$B = E_3 E_2 E_1 A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

Determinant via Elementary Matrices

The example in the previous slide suggest that we can also compute the determinant of a $n \times n$ matrix by ERO and Elementary Matrices

To see this, continue with the example, where after ERO we arrived at

$$E_3 E_2 E_1 A = B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\Rightarrow \det(E_3 E_2 E_1 A) = \det \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\Rightarrow \det(E_3)\det(E_2)\det(E_1)\det(A) = (1)(-1)(9)$$

Thus

$$\Rightarrow \det(A) = \frac{-9}{\det(E_3)\det(E_2)\det(E_1)} = -9$$

since, from the properties of Elementary Matrices, $\det(E_1) = \det(E_2) = \det(E_3) = 1$.

Elementary Matrices are Invertible

Let E be an $n \times n$ elementary matrix. Then E is invertible. Moreover, its inverse is also an elementary matrix.

As an illustration, consider the row operation $R_1 \leftarrow R_1 + 2R_2$. The corresponding elementary matrix is

$$E = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, E is invertible and is given by (how to find E^{-1}?)

$$E = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

LU Factorization

In certain cases, a $m \times n$ matrix A can be written as $A = LU$ where L is a lower triangular matrix and U is an upper triangular matrix. We call this an LU **factorization** of A.

For example

$$\begin{bmatrix} -3 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$$

When such a factorization of A exists, we can solve the linear system $Ax = b$ using forward and back substitution

$$Ax = b \Rightarrow LUx = b \Rightarrow \begin{cases} Ly = b, & (forward) \\ Ux = y & (back\ substitution) \end{cases}$$

Example: LU factorization of A

$$A = \begin{bmatrix} 3 & 6 & -3 \\ 6 & 15 & -5 \\ -1 & -2 & 6 \end{bmatrix} \overset{E_1:R_1 \leftarrow R_1/3}{\sim} [\cdot] \overset{E_2:R_2 \leftarrow R_2 - 6R_1}{\sim} [\cdot] \overset{E_3:R_3 \leftarrow R_3 + R1}{\sim} U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

Thus, $E_3 E_2 E_1 A = U \Rightarrow A = \underbrace{(E_3 E_2 E_1)^{-1}}_{L} U$ where

$$L = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 3 & 0 & 0 \\ 6 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

In other words, A can be factorised as

$$A = \begin{bmatrix} 3 & 6 & -3 \\ 6 & 15 & -5 \\ -1 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 6 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

Solving $Ax = b$ by LU Factorization

With $b = \begin{bmatrix} 3 & 11 & 9 \end{bmatrix}^T$, we have

$$Ax = \begin{bmatrix} 3 & 6 & -3 \\ 6 & 15 & -5 \\ -1 & -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 9 \end{bmatrix}$$

which can be equivalently written as

$$LUx = \begin{bmatrix} 3 & 0 & 0 \\ 6 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{y} = \begin{bmatrix} 3 \\ 11 \\ 9 \end{bmatrix} \Rightarrow \begin{cases} Ly = b \\ Ux = y \end{cases}$$

So that we can first solve for y by forward substitution and then x by back substitution.

Solving $Ax = b$ by LU Factorization

continue ...

Let $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, so with $Ly = b$

$$\begin{bmatrix} 3 & 0 & 0 \\ 6 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 9 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

and

$$Ux = y \Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Is LU Factorization Unique?

Consider the matrix $A = \begin{bmatrix} -3 & -2 \\ 3 & 4 \end{bmatrix}$. Let's carry out the following operations on A:

$$A = \begin{bmatrix} -3 & -2 \\ 3 & 4 \end{bmatrix} \overset{E_1: R_2 \leftarrow R_2 + R_1}{\sim} U_1 = \begin{bmatrix} -3 & -2 \\ 0 & 2 \end{bmatrix} \overset{E_2: R_1 \leftarrow R_1/3}{\sim} U_2 = \begin{bmatrix} -1 & -2/3 \\ 0 & 2 \end{bmatrix}$$

Thus

$$U_1 = E_1 A \Rightarrow A = E_1^{-1} U_1 = L_1 U_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ 0 & 2 \end{bmatrix}$$

and

$$U_2 = E_2 E_1 A \Rightarrow A = (E_2 E_1)^{-1} U_2 = L_2 U_2 = \begin{bmatrix} 3 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2/3 \\ 0 & 2 \end{bmatrix}$$

which demonstates that LU factorization is not unique.

PLU Factorization

Sometimes, we may need to interchange rows in order to reduce A to a LU form, as illustrated by the following example:

$$A = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 4 & 3 \\ 1 & 2 & 0 \end{bmatrix} \overset{E_1:R_1 \leftrightarrow R_3}{\sim} [\cdot] \overset{E_2:R_2 \leftarrow R_2 - R_1}{\sim} [\cdot] \overset{E_3:R_3 \leftarrow R_3 - R_2}{\sim} U = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & -5 \end{bmatrix}$$

Notice that E_1 is a permuatation matrix while E_2 and E_3 are lower triangular. So,

$$A = (E_3 E_2 E_1)^{-1} U = E_1^{-1} \underbrace{(E_2^{-1} E_3^{-1})}_{L} U = PLU.$$

Vectors in $\mathbb{R}^n$

Euclidean n-space, denoted by $\mathbb{R}^n$, or simply n-space, is defined by

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} | x_i \in \mathbb{R}, \text{ for} i = 1, 2, \ldots, n \right\}$$

The entries of a vector are called the **components** of the vector.

For example, $\mathbb{R}^2$ and $\mathbb{R}^3$.

Addition and Scalar Multiplication of Vectors

You should already know this

Example: Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$$

Find $(2\mathbf{u} + \mathbf{v}) - 3\mathbf{w}$.

$$\left(2\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}\right) - 3\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 9 \end{bmatrix} - \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix} = \begin{bmatrix} -11 \\ -6 \\ -9 \end{bmatrix}$$

Algebraic Properties of Vectors in $\mathbb{R}^n$

You should already know this

Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c and d be scalars. The following algebraic properties hold.

1. Commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. Associative: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. Additive identity: The vector $\mathbf{0}$ satisfies $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$
4. Additive inverse: For every $\mathbf{u}$, the vector $-\mathbf{u}$ satisfies $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$
8. $(1)\mathbf{u} = \mathbf{u}$

Linear Combination

Linear Combination:
Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a set of vectors in $\mathbb{R}^n$ and let $c_1, c_2, \ldots, c_k$ be scalars. An expression of the form

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \sum_{i=1}^{k} c_i\mathbf{v_i}$$

is called a **linear combination** of the vectors of S. Any vector $\mathbf{v}$ that can be written in this form is also called a linear combination of the vector of S.

Linear Combinations

Determine whether the vector $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 10 \end{bmatrix}$ is a linear combination of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}.$$

Solution:

We just need to check whether $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}$ has solution or not. In other words, whether

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{v} \quad \text{or equivalently} \quad \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 10 \end{bmatrix}$$

has a solution or not? Yes, $c_1 = 1, c_2 = -2, c_3 = 1$.

Linear Combinations

Determine whether the vector $\mathbf{v} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$ is a linear combination of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}.$$

Solution:

We just need to check whether $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}$ has solution or not.

$$\begin{bmatrix} 1 & 0 & 2 \\ -2 & 5 & 0 \\ 2 & 5 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

which has no solution.

Spanning Set

The set of all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ is called the **span** of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ and is denoted by $span(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k})$ or $span(S)$.

In other words

$$span(S) = \{c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} | c_1, c_2, \ldots, c_k \in \mathbb{R}\}$$

Example: Spanning Set

Show that *span*$\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$ is $\mathbb{R}^2$.

Solution:
We need to show that

$$c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

has solution for all x and y.

$$\left[\begin{array}{cc|c} 2 & 1 & x \\ -1 & 3 & y \end{array}\right] \sim \left[\begin{array}{cc|r} 2 & 1 & x \\ 0 & 7 & x+2y \end{array}\right]$$

has solution for all x and y.

Example: Spanning Set

Let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$. What is $span(\mathbf{u}, \mathbf{v})$?

Solution:
Let $\begin{bmatrix} x & y & z \end{bmatrix}^T$ be a vector in the $span(\mathbf{u}, \mathbf{v})$, then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 1 & y \\ 3 & -3 & z \end{array}\right] \sim \left[\begin{array}{cc|r} 1 & -1 & x \\ 0 & 1 & y \\ 0 & 0 & z-3x \end{array}\right]$$

has solution when $z - 3x$. Thus the $span(\mathbf{u}, \mathbf{v})$ is the plane $z - 3x = 0$.

Geometrically, the set of all linear combination of $\mathbf{u}$ and $\mathbf{v}$ is just the plane through the origin with $\mathbf{u}$ and $\mathbf{v}$ as direction vectors.

Example: Spanning Set

Let

$$\mathbf{v_1} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

Show that $\mathbf{v} = \begin{bmatrix} -4 \\ 4 \\ -6 \end{bmatrix}$ is in $span(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$.

Solution: We have to show that $\mathbf{v}$ can be written as a linear combination of $\mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$, i.e.,

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} = \mathbf{v}$$

Solving this linear system, we obtain

$$c_1 = -2, \quad c_2 = 1, \quad c_3 = -1$$

Hence $\mathbf{v}$ is in $span(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$.

Example: Span

Show that $span(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}) = \mathbb{R}^3$

Solution:

A vector in $\mathbb{R}^3$ has the form $\begin{bmatrix} x & y & z \end{bmatrix}^T$. We need to show that

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{has solution for all } x, y \text{ and } z.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 1 & 0 & 1 & y \\ 1 & 2 & 0 & z \end{array}\right] \sim \left[\begin{array}{ccc|r} 1 & 1 & 1 & x \\ 0 & -1 & 0 & y - x \\ 0 & 0 & -1 & z + y - 2x \end{array}\right]$$

has solution for all x, y, and z.

Example: Span

Show that $span\left(\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ 5 \end{bmatrix}\right) \neq \mathbb{R}^3$

Solution:

A vector in $\mathbb{R}^3$ has the form $\begin{bmatrix} x & y & z \end{bmatrix}^T$. Since

$$c_1 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} -6 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} -1 & 4 & -6 & x \\ 2 & 1 & 3 & y \\ 1 & -3 & 5 & z \end{array}\right] \sim \left[\begin{array}{ccc|r} -1 & 4 & -6 & x \\ 0 & 9 & 9 & y+2x \\ 0 & 0 & 0 & 9z+7x-y \end{array}\right]$$

has solution only if $9z + 7x - y = 0$, the three vectors do not span $\mathbb{R}^3$.

Linear Independence

The set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ in $\mathbb{R}^n$ is **linear independent** provided that the only solution to the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = 0$$

is the trival solution $c_1 = c_2 = \cdots = c_m = 0$. Otherwise, the set S is **linearly dependent**.

In other words, to determine whether the set of vectors $\mathbf{v_1}, \dots, \mathbf{v_n}$ is LI, we form

$$\left[\begin{array}{ccc} \mathbf{v_1} & \dots & \mathbf{v_n} \end{array}\right] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0 \quad \text{and solve for } c_1, \dots, c_n.$$

If $c_1 = \ldots = c_n = 0$, then $\mathbf{v_1}, \dots, \mathbf{v_n}$ are LI; otherwise LD.

Example: Linear Independence

Determine whether the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \text{ and } \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} \text{ are linearly independent or linearly dependent.}$$

Solution: Form the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ and solve for c_1, c_2 and c_3.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the solution is $c_1 = c_2 = c_3 = 0$, the vectors are linearly independent.

Example: Linear Independence

Determine whether the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, \ \mathbf{v}_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ are linearly independent}$$

Solution: Form the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$ and solve for c_1, c_2, c_3 and c_4.

$$\begin{bmatrix} 1 & -1 & -2 & 2 \\ 0 & 1 & 3 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & -1 & -2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -7 & -7 \end{bmatrix}$$

Since there are many (non-trival) solutions the vectors are Linearly Dependent.

Orthogonality

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthongonal if $\mathbf{u}^T\mathbf{v} = 0$

Example

If $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Find a $\mathbf{v}$ that is (i) orthogonal, and (ii) not orthogonal to $\mathbf{u}$.

Vector Form of a Linear System

A linear system with m equations and n variables

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{cases}$$

can be written in matrix form as $A\mathbf{x} = \mathbf{b}$.

The matrix equation can also be written in the equivalent form

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

This last equation is called the **vector form** of the linear system.

Consistency of $Ax = b$

Until now, consistency means $A\mathbf{x} = \mathbf{b}$ has solution (either unique or many solutions).
Here is another way of looking at consistency of $A\mathbf{x} = \mathbf{b}$.

THEOREM
The linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the vector $\mathbf{b}$ can be expressed as a linear combination of the column vectors of A.

$$A\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \ldots & \mathbf{v_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{b}$$

$$\Rightarrow \quad x_1\mathbf{v_1} + x_2\mathbf{v_2} + \ldots + x_n\mathbf{v_n} = \mathbf{b}$$

Linear Systems having Unique Solution

Let $A\mathbf{x} = \mathbf{b}$ be a consistent $m \times n$ linear system. The solution is unique if and only if the column vectors of A are linearly independent.

Summary: Different Views of the Same Coin

If A is a $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^n$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

Summary: Different Views of the Same Coin

Cont'd

Thus, a system of linear equations may be viewed in three different but equivalent ways:

- as a matrix equation
- as a vector equation, or
- as a system of linear equations

Whenever you construct a mathematical model of a problem in real life, you are free to choose whichever viewpoint that is most natural. Then you may switch from one formulation of a problem to another whenever it is convenient.

Photosynthesis Application Revisited

Recall that in slide 7, we set up a system of equations to balance the chemical equation

$$aCO_2 + bH_2O \rightarrow cO_2 + dC_6H_{12}O_6$$

We could instead set up a **vector equation** that describes the number of atoms of each type present in a reaction.

$$\begin{bmatrix} C \\ O \\ H \end{bmatrix} : \quad a\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = c\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + d\begin{bmatrix} 6 \\ 6 \\ 12 \end{bmatrix}$$

which is the same *system of three linear equations in four variables* shown in slide 7

$$\begin{cases} C: & a & & & & & - & 6d & = 0 \\ O: & 2a & + & b & - & 2c & - & 6d & = 0 \\ H: & & & 2b & & & - & 12d & = 0 \end{cases}$$

Application: Resource allocation

A coffee merchant sells three blends of coffee. A bag of the house blend contains 300g of Colombian beans and 200g of French roast beans. A bag of the special blend contains 200g of Colombian beans, 200g of Kenyan beans and 100g of French roast beans. A bag of the gourmet blend contains 100g of Colombians, 200g of Kenyan beans and 200g of French roast beans. The merchant has on hand 30kg of Colombian beans, 15kg of Kenyan beans and 25kg of French roast beans. If he wishes to use up all of the beans, how many bags of each type of blend can be made?

Solution:

Let there be x_1, x_2 and x_3 bags of the house blend, special blend and gourmet blend respectively. Thus

$$\begin{bmatrix} \text{C} \\ \text{F} \\ \text{K} \end{bmatrix} : x_1 \begin{bmatrix} 300 \\ 200 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 200 \\ 100 \\ 200 \end{bmatrix} + x_3 \begin{bmatrix} 100 \\ 200 \\ 200 \end{bmatrix} = \begin{bmatrix} 30000 \\ 25000 \\ 15000 \end{bmatrix}$$

[65 30 40]

Application: Resource allocation

A florist offers three sizes of flower arrangements containing roses, daisies, and chrysanthemums. Each small arrangement contains one rose, three daisies, and three chrysanthemums. Each medium arrangement contains two roses, four daisies and six chrysanthemums. Each large arrangement contains four roses, eight daisies and six chrysanthemums. One day, the florist noted that she used a total of 24 roses, 50 daisies and 48 chrysanthemums in filling orders for these three types of arrangements. How many arrangements of each type did she make?

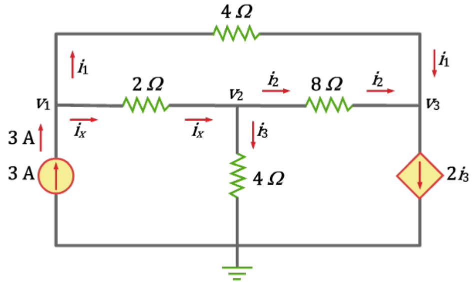
Solution: Let there be x_1, x_2 and x_3 orders of small, medium and large arrangements respectively. Thus

$$\begin{bmatrix} \text{R} \\ \text{D} \\ \text{C} \end{bmatrix} : x_1 \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 24 \\ 50 \\ 48 \end{bmatrix}$$

[2 3 4]

Application of Linear Systems: Electrical Networks

Find the node voltages of the circuit



Summary: Linear System, Linear Independence, Invertibility of Matrices and Determinants

Let A be a square matrix. Then the following statements are equivalent.

1. The matrix A is invertible
2. The determinant of the matrix A is nonzero
3. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b}$
4. The homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
5. The column vectors of A are linearly independent
6. The matrix A is row equivalent to the identity matrix

Null Space, Column Space and Row Space of a Matrix

Definition: Let A be an $m \times n$ matrix.

1. The **null space** of A, denoted by $N(A)$, is the set of all vectors in $\mathbb{R}^n$ such that $A\mathbf{x} = 0$
$$N(A) = \{x | Ax = 0\}$$
2. The **column space** of A, denoted by $col(A)$, is the set of all linear combinations of the column vectors of A
$$col(A) = \{\mathbf{v} | \mathbf{v} = c_1\mathbf{v_1} + \ldots + c_n\mathbf{v_n}\} = span(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$$
3. The **row space** of A, denoted by $row(A)$, is the set of all linear combinations of the row vectors of A
$$row(A) = \{\mathbf{r} | \mathbf{r} = c_1\mathbf{r_1} + \ldots + c_m\mathbf{r_m}\} = span(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m)$$

What are the dimensions of $\mathbf{x}$, $\mathbf{v}_i$ and $\mathbf{r}_i$?

$\mathbf{x} : n \times 1$; $\mathbf{v}_i : m \times 1$; $\mathbf{r}_i : n \times 1$

Example: Column, Row and Null Space of a Matrix

Let $A = \begin{bmatrix} -1 & 0 & 3 & 1 \\ 2 & 3 & -3 & 1 \\ 2 & -2 & -2 & -1 \end{bmatrix}$, then

1. $col(A) = span(\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix})$
2. $row(A) = span(\begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -2 \\ -1 \end{bmatrix})$
3. $N(A) = span(\begin{bmatrix} 1 \\ 1 \\ 1 \\ -2 \end{bmatrix})$

Example: Column, Row and Null Space of a Matrix

cont.

Null space of A can be determined as follows:

$$\begin{bmatrix} -1 & 0 & 3 & 1 \\ 2 & 3 & -3 & 1 \\ 2 & -2 & -2 & -1 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

The last row gives $2x_3 = -x_4$. Let $x_3 = \alpha$, then $x_4 = -2\alpha$, and by back substitution,

$$x_2 = -x_3 - x_4 = \alpha, \quad x_1 = 3x_3 + x_4 = \alpha$$

Hence, the solution for $A\mathbf{x} = 0$ is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

Example: Null Space and Column Space

Let

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \\ 2 & -2 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}$$

Determine whether $\mathbf{b}$ is in $col(A)$, and find $N(A)$.

Solution:

1. Find out whether

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}$$

has solution. If yes, then $\mathbf{b}$ is in $col(A)$.

2. Solve $Ax = 0$, and the solution(s) is the Null Space of A.

[Yes; $N(A) = \{\mathbf{0}\}$]

Subspaces, Basis, Dimension, and Rank

In our study of vectors, we have seen that geometry and algebra are linked, and we often use geometric intuition and reasoning to obtain algebraic results, and the power of algebra will often allow us to extend our findings beyond geometric settings.

The notion of **subspace** is an algebraic generalisation of the geometric examples of lines and planes through the origin.

The concept of a **basis** for a subspace is linked to direction vectors of such lines and planes. The concept of basis gives us a precise definition of **dimension**.

Subspaces

DEFINITION:
A **subspace** of $\mathbb{R}^n$ is any collection S of vectors in $\mathbb{R}^n$ such that

1. The zero vector is in S
2. If $\mathbf{u}$ and $\mathbf{v}$ are in S, then $\mathbf{u} + \mathbf{v}$ is in S (i.e., S is **closed under vector addition**)
3. If $\mathbf{u}$ is in S and c is a scalar, then $c\mathbf{u}$ is in S (i.e., S is **closed under scalar multiplication**

Note that properties (2) and (3) can equivalently be stated as:
S is **closed under linear combinations**,
i.e., If $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$ are in S and $c_1, c_2, \ldots, c_k$ are scalars,
then $c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_k\mathbf{u_k}$ is in S

Example: Subspaces

Show that $W = \left\{ \begin{bmatrix} a \\ a+1 \end{bmatrix} | a \in \mathbb{R} \right\}$ is not a subspace.

Solution: Consider two vectors in W, e.g.,

$$\mathbf{u} = \begin{bmatrix} a_1 \\ a_1 + 1 \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} a_2 \\ a_2 + 1 \end{bmatrix}$$

Since $\mathbf{u} + \mathbf{v} = \begin{bmatrix} a_1 + a_2 \\ a_1 + a_2 + 2 \end{bmatrix}$ is NOT in W, thus, W is not a subspace.

Alternatively, we could also note that W is not a subspace since the zero vector is not in W.

Example: Subspaces

Every line and plane through the origin in $\mathbb{R}^n$ is a subspace in $\mathbb{R}^n$.

Solution:

The line that passes through the origin and the point with position vector $\mathbf{u}$ in $\mathbb{R}^n$ is

$$\mathcal{L} = \{\mathbf{x} | \mathbf{x} = t\mathbf{u}, \ t \in \mathbb{R}\}$$

Given any $\mathbf{v}, \ \mathbf{w} \in \mathcal{L}$, and any scalar α, we have

$$\mathbf{v} + \mathbf{w} = t_1\mathbf{u} + t_2\mathbf{u} = t_3\mathbf{u} \in \mathcal{L}$$

and $\alpha\mathbf{v} = \alpha t_1\mathbf{u} = (\alpha t_1)\mathbf{u} = t_4\mathbf{u} \in \mathcal{L}$, i.e. $\mathcal{L}$ is closed under vector addition and scalar multiplication.

Similarly, the plane that passes through the origin and two points with position vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in $\mathbb{R}^n$ is

$$\mathcal{P} = \{\mathbf{x} | \mathbf{x} = s\mathbf{u}_1 + t\mathbf{u}_2, \ s, t \in \mathbb{R}\}$$

It can be shown that $\mathcal{P}$ is closed under vector addition and scalar multiplication.

Example: Subspaces

Every line and plane that does not pass through the origin in $\mathbb{R}^n$ is not a subspace in $\mathbb{R}^n$.

Proof:
The zero vector is not included, hence not a subspace.

Basis

Definition: (Basis for a Vector Space)
A subset B of a vector space V is a **basis** for V provided that

1. B is a linearly independent set of vectors in V
2. $span(B) = V$

Theorem: (Non-uniqueness of Bases)
Let $B = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ be a basis for a vector space V and c a nonzero scalar. Then $B_c = \{c\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is a basis.

Example: Show that a given set of vectors is a Basis

Show that the set

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\} \quad \text{is a basis for } \mathbb{R}^3.$$

Solution: We need to show that (1) S spans $\mathbb{R}^3$ and (2) S is LI. Requirement (1) means

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{has a solution for every choice of } a, b, c.$$

Requirement (2) means

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{has solution } c_1 = c_2 = c_3 = 0.$$

Example: Show that a given set of vectors is a Basis

cont.

Combining requirements (1) and (2), we just need to show

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}}_{B} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has solution $c_1 = c_2 = c_3 = 0$. In other words, the matrix B is non-singular or $\det(B) \neq 0$.

Example: Finding a Basis

Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \right\}$. Find a basis for the span of S.

Solution: This means finding a set of independent vectors from $\mathcal{S}$.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & -2 \end{bmatrix} \underset{\sim \cdots \sim}{\text{reduce to RE}} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & -2 & -2 \end{bmatrix}$$

So the basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

Example: Basis for Row Space

Find a basis for the row space of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 10 & 12 & 14 & 16 \end{bmatrix}$$

Solution: Use ERO and reduce A to B.

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 8 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

There are only 2 independent rows. So, the first two rows of A forms a basis for $row(A)$. We can also choose the first two rows of B as a basis for $row(A)$.

Summary: Finding a Basis from a set of vectors

equivalently, find the linearly independent vectors from a set of vectors

Given a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, here are two "*recipes*" to find a basis for $span(S)$:

Method 1:

1. Form a matrix A whose **columns** are the vectors
2. Reduce A to RE form
3. The vectors from S that correspond to the **pivoting columns** of the RE matrix form a basis for $span(S)$.

Method 2:

1. Form a matrix B whose **rows** are the vectors
2. Reduce B to RE form
3. The vectors from S that correspond to the **non-zero rows** of the RE matrix form a basis for $span(S)$.

Dimension

Definition: (Dimension of a Vector Space)
The **dimension** of the vector space V, denoted by $dim(V)$, is the number of vectors in any basis of V.

Note that if a vector space V has a basis with n vectors, then every basis has n vectors. In other words, the dimension is a property of V and does not depend of the choice of basis.

Dimension, Rank and Nullity

Definition:(Dimension)
If S is a subspace of $\mathbb{R}^n$, then the number of vectors in a basis for S is called the **dimension** of S, denoted as $dim(S)$.

$dim(\mathbb{R}^n) = n$, since the standard basis for $\mathbb{R}^n$ has n vectors, and we define $dim(\mathbf{0})$ to be 0.

Definition:(Rank)
The **rank** of a matrix A is the dimension of its row or column space and is denoted as $rank(A)$.

Definition:(Nullity)
The **nullity** of a matrix A is the dimension of its null space and is denoted as $nullity(A)$.

Rank of a Matrix

The row and column space of a matrix A have the same dimension.
Hence we have $rank(A^T) = rank(A)$.

In addition, we have

1. $rank(AB) \leq rank(B)$
2. $rank(AB) = rank(B)$, if A^{-1} exists
3. $rank(AB) = rank(A)$, if B^{-1} exists
4. $rank(A + B) \leq rank(A) + rank(B)$
5. $rank(A) \leq \min(m, n)$, where A is an $m \times n$ matrix

The Rank Theorem

Theorem:
If A is an $m \times n$ matrix, then

$$rank(A) + nullity(A) = n$$

Proof:
Let R be the RRE of A, and suppose $rank(A) = r$. Then R has r leading variables and $(n - r)$ free variables in the solution of $Ax = 0$, i.e. $dim(N(A)) = n - r$. Thus we have

$$rank(A) + nullity(A) = r + (n - r) = n$$

The Rank Theorem

Example: Find the rank and nullity of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 15 & 18 & 21 & 24 \end{bmatrix}$.

Solution:

Use ERO, $A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so $rank(A) = 2$.

Since A is a 3×4 matrix, the Rank Theorem gives

$$rank(A) + nullity(A) = n = 4, \text{ giving } nullity(A) = 2.$$

Alternatively, if we take the $Ax = 0$ method to find nullity, we will have 2 independent equations with 4 unknowns, so there are two free variables, hence $nullity(A) = 2$.

Eigenvalues and Eigenvectors

Definition: Eigenvalues and Eigenvectors
Let A be an $n \times n$ matrix. A number λ is called an **eigenvalue** of A if there exist a nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$ such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

Every nonzero vector $\mathbf{v}$ satisfying this equation is called an **eigenvector** of A **corresponding to the eigenvalue λ.**

Example:
Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$. Since $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 1$. The other eigenvector of A is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda_2 = -1$.

Finding eigenvalues and eigenvectors of $n \times n$ matrix

By definition, if λ is an eigenvalue and $\mathbf{v}$ is the corresponding eigenvector of the matrix A, then

$$A\mathbf{v} = \lambda\mathbf{v} \quad \Rightarrow (\lambda I - A)\mathbf{v} = \mathbf{0}$$

Since $\mathbf{v} \neq 0$, we must have

$$\det(\lambda I - A) = 0, \quad \text{which we can solve to find all the eigenvalues of } A.$$

Once the eigenvalues are found, we then substitute the particular eigenvalue λ_i and solve the following linear system for the corresponding eigenvector $\mathbf{v_i}$

$$(\lambda_i I - A)\mathbf{v_i} = \mathbf{0}$$

In other words, $\mathbf{v_i}$ is in the null space of $(\lambda_i I - A)$, and we sometimes refer it as the **eigenspace** corresponding to λ_i.

Example: Eigenvalues and Eigenvector

Find the eigenvalues and eigenvectors for the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution:

Step 1: Find the eigenvalues

$$\det(\lambda I - A) = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

Step 2: Find the corresponding eigenvectors

Substitute $\lambda_1 = 1$, solve for $\mathbf{v}_1$.

$$(\lambda_1 I - A)\mathbf{v}_1 = 0 \Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 = x_2 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Similarly, substitute $\lambda_2 = -1$, solve for $\mathbf{v}_2$.

$$(\lambda_2 I - A)\mathbf{v}_2 = 0 \Rightarrow \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 = -x_2 \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Geometric Interpretation of Eigenvalues and Eigenvectors

see MATLAB demo: eigshow

Example: Complex Eigenvalues

Find the eigenvalues of $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

Solution:

The characteristic equation is $\det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & -1 & \lambda \end{vmatrix} = \lambda(\lambda^2 + 1) = 0$

Thus the eigenvalues are $\lambda_1 = 0, \lambda_2 = i, \lambda_3 = -i$.

The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}$

Example: one eigenvalue, more than one eigenvectors

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$ and find the corresponding eigenvector.

Solution:

First, find the eigenvalues:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 0 & 0 & 0 \\ 0 & \lambda - 1 & -5 & 10 \\ -1 & 0 & \lambda - 2 & 0 \\ -1 & 0 & 0 & \lambda - 3 \end{vmatrix} = (\lambda - 1)^2(\lambda - 2)(\lambda - 3) = 0$$

Hence, the eigenvalues are $\lambda = 1, 1, 2, 3$.

Example: one eigenvalue, more than one eigenvectors

cont.

Let's find the eigenvector(s) for $\lambda = 1$ through $(\lambda I - A)\mathbf{v_1} = 0$. Solving for the eigenvector by the augmented matrix method

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} = t\mathbf{v}_{11} + s\mathbf{v}_{12}$$

Thus there are two eigenvectors corresponding to $\lambda = 1$:

$$A(t\mathbf{v}_{11} + s\mathbf{v}_{12}) = (t\mathbf{v}_{11} + s\mathbf{v}_{12})$$

Set $s = 0, t = 1, \Rightarrow A\mathbf{v}_{11} = \mathbf{v}_{11}$; set $s = 1, t = 0, \Rightarrow A\mathbf{v}_{12} = \mathbf{v}_{12}$.

Use MATLAB to find eigenvalues and eignevectors

```
>> A = [1 0 0 0; 0 1 5 -10; 1 0 2 0; 1 0 0 3]

A =

     1     0     0     0
     0     1     5   -10
     1     0     2     0
     1     0     0     3

>> [V,D]=eig(A)

V =

         0         0         0    0.6667
    1.0000   -0.9806    0.9806         0
         0         0    0.1961   -0.6667
         0    0.1961         0   -0.3333


D =

     1     0     0     0
     0     3     0     0
     0     0     2     0
     0     0     0     1

>> V1 = [V(:,1) V(:,2)/V(4,2) V(:,3)/V(3,3) V(:,4)/V(4,4)]

V1 =

         0         0         0   -2.0000
    1.0000   -5.0000    5.0000         0
         0         0    1.0000    2.0000
         0    1.0000         0    1.0000
```

Diagonalisation, $A = PDP^{-1}$

An $n \times n$ matrix A is diagonalisable if and only if A has n linearly independent eigenvectors. Moreover, if $D = P^{-1}AP$, with D a diagonal matrix, then the diagonal entries of D are the eigenvalues of A and the column vectors of P are the corresponding eigenvectors.

Proof:
Suppose A has n linearly independent eigenvectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$A\mathbf{v_i} = \lambda_i \mathbf{v_i} \quad \Rightarrow A \underbrace{\begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \ldots & \mathbf{v_n} \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \ldots & \mathbf{v_n} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}}_{D}$$

Mini-Tutorial:

$$\begin{bmatrix} \lambda_1 \mathbf{v}_1 & \dots & \lambda_n \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}$$

Consider $A = \begin{bmatrix} \lambda_1 & 4\lambda_2 & 7\lambda_3 \\ 2\lambda_1 & 5\lambda_2 & 8\lambda_3 \\ 3\lambda_1 & 6\lambda_2 & 9\lambda_3 \end{bmatrix}$. Then we can re-write A as either

$$A = \begin{bmatrix} \lambda_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} & \lambda_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} & \lambda_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \end{bmatrix}$$

or

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Example: Diagonalisation of a matrix

Diagonalise

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & -2 & 0 \\ 7 & -4 & 2 \end{bmatrix}$$

Solution:

It can be shown that the eigenvalues and the corresponding eigenvectors of A are

$$\lambda_1 = 1, \mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \lambda_2 = -2, \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda_3 = 2, \mathbf{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } A = PDP^{-1}.$$

Example: Matrix not diagonalisable

Diagonalise $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Solution:

The matrix has eigenvalues $\lambda = -1, -1, 2$. For eigenvalue $\lambda = -1$, the eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and for eigenvalue $\lambda = 2$, the eigenvector is $\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$. The matrix is not diagonalisable because it does not have a full set of independent eigenvectors.

Mini-Tutorial: Solution of $\dot{x}(t) = \lambda x(t)$

$$
\begin{aligned}
& \tfrac{d}{dt}x(t) = \lambda x(t) \\
\Rightarrow \quad & \int_{x(0)}^{x(t)} \frac{d\,x(t)}{x(t)} = \lambda \int_{t=0}^{t} dt \\
\Rightarrow \quad & \ln \frac{x(t)}{x(0)} = \lambda t \\
\Rightarrow \quad & x(t) = e^{\lambda t} x(0)
\end{aligned}
$$

How to handle coupled differential equations?

$$
\begin{aligned}
\dot{x}_1(t) &= a_1 x_1(t) + a_2 x_2(t) \\
\dot{x}_2(t) &= b_1 x_1(t) + b_2 x_2(t)
\end{aligned}
$$

Application: Solution of Systems of Linear Differential Equations

Find the general solution to the system of differential equations

$$\begin{cases} \frac{d}{dt}y_1(t) & = & \dot{y}_1(t) & = & -y_1(t) \\ \frac{d}{dt}y_2(t) & = & \dot{y}_2(t) & = & 3y_1(t) + 2y_2(t) \end{cases}$$

Solution:

Write the system in the form $Ax = b$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Hence

$$\dot{\mathbf{y}} = A\mathbf{y} = PDP^{-1}\mathbf{y} \quad \Rightarrow \underbrace{P^{-1}\dot{\mathbf{y}}}_{\dot{\mathbf{w}}} = D\,\underbrace{P^{-1}\mathbf{y}}_{\mathbf{w}} \quad \Rightarrow \dot{\mathbf{w}} = D\mathbf{w}$$

The matrix $A = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$ can be diagonalised with $P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$.

The general solution for $\dot{\mathbf{w}} = D\mathbf{w}$ is

$$\mathbf{w}(t) = e^{Dt}\mathbf{w}(0) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \mathbf{w}(0)$$

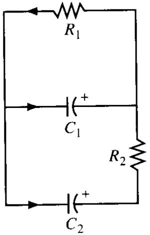
Hence

$$\mathbf{y}(t) = P\mathbf{w}(t) = Pe^{Dt}\mathbf{w}(0) = Pe^{Dt}P^{-1}\mathbf{y}(0) = \begin{bmatrix} e^{-t} & 0 \\ -e^{-t} + e^{2t} & e^{2t} \end{bmatrix} \mathbf{y}(0)$$

Application: Time response of Circuits

The circuit in the figure can be described by the differential equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -(1/R_1 + 1/R_2)/C_1 & 1/(R_2 C_1) \\ 1/(R_2 C_2) & -1/(R_2 C_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



where $\mathbf{x}_1$ and $\mathbf{x}_2$ are voltages across the capacitors C_1 and C_2 respectively; $R_1 = 1$, $R_2 = 2$, $C_1 = 1$ and $C_2 = 0.5$.

The initial voltages on the capacitors were $x_1(0) = 5$ and $x_2(0) = 4$. Show how the voltages across the capacitors change with time.

Application: Time response of Circuits

cont.

Solution:

The system is $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1.5 & 0.5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and the matrix has the corresponding eigenvalue/eigenvector pairs

$$\lambda_1 = -0.5, \ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \lambda_2 = -2, \ \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Thus

$$\dot{\mathbf{x}} = PDP^{-1}\mathbf{x}(0), \text{ where } P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}, \ D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

and ...

Application: Time response of Circuits

cont.

$$
\begin{aligned}
\mathbf{x(t)} &= Pe^{Dt}P^{-1}\mathbf{x}(0) = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2
\end{aligned}
$$

where $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\mathbf{x}(0)$.

The constants c_1 and c_2 can be determined from initial conditions as follows:

$$
\mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}
$$

giving $c_1 = 3$ and $c_2 = -2$.

Application: Long Term Behaviour of Stable Dynamical System

Let $A = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$. Analyse the long-term behaviour of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, with $\mathbf{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$.

Solution:

The long term behavour of $\mathbf{x}$ can be found as follows:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k = A(A\mathbf{x}_{k-1}) = A \dots A\mathbf{x}_1 = A^k\mathbf{x}_1, \quad \text{let } k \to \infty, x_\infty = A^\infty x_1$$

But, how to compute A^∞? The trick is to diagonalise A.

Then

$$\begin{aligned} \mathbf{x}_\infty &= A^\infty \mathbf{x}_1 = (PDP^{-1})^\infty \mathbf{x}_1 \\ &= (PDP^{-1})(PDP^{-1})\dots(PDP^{-1})\mathbf{x}_1 = PD^\infty P^{-1}\mathbf{x}_1 \end{aligned}$$

Application: Long Term Behaviour of Stable Dynamical System

cont.

Matrix A has the following eigenvalue/eigenvector pairs:

$$\lambda_1 = 1, \mathbf{v}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \text{ and } \lambda_2 = 0.92, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence, A can be diagonalised with $P = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$, i.e.,

$$A = PDP^{-1}, \text{ where } D = \begin{bmatrix} 1 & 0 \\ 0 & 0.92 \end{bmatrix}$$

Thus, as $k \to \infty$

$$\mathbf{x}_\infty = P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \mathbf{x}_1 = \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix}$$

MRT Early Bird Incentive Scheme

In July 2013, city S started experimenting with the "early bird" incentive scheme to encourage a change in the morning peak-hour communting pattern on the MRT system. The authority estimated that each month 30% of those who tried the "early bird" scheme would decide to go back to travel during the morning peak-hour, and 20% of those who travelled during the morning peak-hour would switch to the "early bird" scheme.
Suppose the total population of the commuters using the MRT is 100,000 and reamins constant. In July 2013, 20,000 commuters joined the "early bird" scheme.
How many commuters will participate in the "early bird" scheme in the long run?

Solution:

The problem can be modelled by the following dynamical system:

$$\begin{bmatrix} \text{Early Bird} \\ \text{Peak Hour} \end{bmatrix}: \quad \mathbf{x}_{k+1} = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \mathbf{x}_k, \quad \mathbf{x}_1 = \begin{bmatrix} 20000 \\ 80000 \end{bmatrix}$$

The matrix $\begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$ can be diagonalised with

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1.5 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, as $k \to \infty$

$$\begin{bmatrix} \text{Early Bird} \\ \text{Peak Hour} \end{bmatrix}: \quad \mathbf{x}_\infty = P \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} P^{-1}\mathbf{x}_1 = \begin{bmatrix} 40000 \\ 60000 \end{bmatrix}$$