Tutorial 4 Solutions (Tutorial 10)

1. $f(x,y,z) = 3x^2y - y^3z^2$.

$$\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}$$

$$= \left.\begin{pmatrix} 6xy \\ 3x^2 - 3y^2z^2 \\ -2y^3z \end{pmatrix}\right|_{\substack{x=1\\y=-2\\z=-1}}$$

$$= \begin{pmatrix} -12 \\ -9 \\ -16 \end{pmatrix} \quad \text{or} \quad -12\underset{\sim}{i} - 9\underset{\sim}{j} - 16\underset{\sim}{k}.$$

∇f is a vector that gives a direction of maximum rate of change of f at a given point. The magnitude $\|\nabla f\|$ is the maximum rate of change.

2. $f(x,y,z) = x^2 y + 2xz = 4.$

$$\vec{N} = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} 2xy + 2z \\ x^2 \\ 2x \end{pmatrix}$$

At point $(2,-2,3)$

$$\vec{N} = \begin{pmatrix} -2 \\ 4 \\ 4 \end{pmatrix}$$

unit normal,

$$\hat{n} = \frac{1}{\|\vec{N}\|}\vec{N} = \frac{1}{\sqrt{2^2+4^2+4^2}}\begin{pmatrix} -2 \\ 4 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix}.$$

OR $\hat{n} = \begin{pmatrix} 1/3 \\ -2/3 \\ -2/3 \end{pmatrix}.$

3). "Directional derivative"

- derivative in a particular direction.

$$f(x, y, z) = x^2 e^y.$$

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x e^y \\ x^2 e^y \\ 0 \end{pmatrix}$$

$$D_{-\hat{j}} f = \nabla f \cdot (-\hat{j}) = \begin{pmatrix} 2x e^y \\ x^2 e^y \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$= -x^2 e^y$$

At $(-2, 0, 0)$,

$$D_{-\hat{j}} f = -x^2 e^y \Big|_{\substack{x=-2 \\ y=0}}$$

$$= -4$$

$$\text{Max } D_{\hat{j}} f = \| \nabla f \| = \left\| \begin{pmatrix} 2x e^y \\ x^2 e^y \\ 0 \end{pmatrix} \Big|_{\substack{x=-2 \\ y=0}} \right\|$$

$$= \left\| \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} \right\|$$

$$= \sqrt{4^2 + 4^2 + 0^2}$$

$$= 5.6569$$

4).

$$r = \| \underset{\sim}{r} \| = \sqrt{x^2 + y^2 + z^2}$$

$$= (x^2 + y^2 + z^2)^{1/2}.$$

$$r^n = (x^2 + y^2 + z^2)^{n/2}$$

$$LHS = \nabla r^n = \begin{pmatrix} \frac{n}{2}(x^2+y^2+z^2)^{\frac{n-2}{2}} \cdot 2x \\ \frac{n}{2}(x^2+y^2+z^2)^{\frac{n-2}{2}} \cdot 2y \\ \frac{n}{2}(x^2+y^2+z^2)^{\frac{n-2}{2}} \cdot 2z \end{pmatrix}$$

$$= n \left(\sqrt{x^2+y^2+z^2}\right)^{n-2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= n r^{n-2} \underset{\sim}{r} = RHS \quad (\text{shown})$$

5).

$$\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \underset{\sim}{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}.$$

$$\underset{\sim}{v} = \underset{\sim}{w} \times \underset{\sim}{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$= \begin{pmatrix} w_2 z - w_3 y \\ w_3 x - w_1 z \\ w_1 y - w_2 x \end{pmatrix}.$$

$$\frac{1}{2} \nabla \times \underset{\sim}{v} = \frac{1}{2} \begin{vmatrix} \underline{i} & \underline{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_2 z - w_3 y & w_3 x - w_1 z & w_1 y - w_2 x \end{vmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \frac{\partial}{\partial y}(w_1 y - w_2 x) - \frac{\partial}{\partial z}(w_3 x - w_1 z) \\ \frac{\partial}{\partial z}(w_2 z - w_3 y) - \frac{\partial}{\partial x}(w_1 y - w_2 x) \\ \frac{\partial}{\partial x}(w_3 x - w_1 z) - \frac{\partial}{\partial y}(w_2 z - w_3 y) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} w_1 - (-w_1) \\ w_2 - (-w_2) \\ w_3 - (-w_3) \end{pmatrix}$$

$$= \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \underset{\sim}{w}. \quad \text{(Shown)}$$

6). $\operatorname{curl}\left(xy^2z\,\underset{\sim}{i} + 2x^3y\,\underset{\sim}{j} + 4x^2y^2\,\underset{\sim}{k}\right)$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z & 2x^3y & 4x^2y^2 \end{vmatrix}.$$

$$= \left.\begin{pmatrix} 8x^2y \\ xy^2 - 8xy^2 \\ 6x^2y - 2xyz \end{pmatrix}\right|_{\substack{x=1\\y=1\\z=-1}} = \underline{\underline{\begin{pmatrix} 8 \\ -7 \\ 8 \end{pmatrix}}}.$$

$\operatorname{curl}\left(yz^3\,\underset{\sim}{i} + xz\,\underset{\sim}{j} + 2x\,\underset{\sim}{k}\right)$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^3 & xz & 2x \end{vmatrix}$$

$$= \left.\begin{pmatrix} -x \\ 3yz^2 - 2 \\ z - z^3 \end{pmatrix}\right|_{\substack{x=1\\y=1\\z=-1}}$$

$$= \underline{\underline{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}}.$$