Solution. Consider the domain $D = \{$ SCE students $\}$. Set $P(x)$="x studies discrete mathematics. Then every SCE student studies discrete mathematics becomes

$$\forall x \in D,\ P(x).$$

Now Jackson is a SCE student means Jackson belongs to D. This gives

$$\forall x \in D,\ P(x); \text{Jackson} \in D; \therefore P(\text{Jackson}).$$

Exercise 37. Here is an optional exercise about universal generalization. Consider the following two premises: (1) for any number x, if $x > 1$ then $x - 1 > 0$, (2) every number in D is greater than 1. Show that therefore, for every number x in D, $x - 1 > 0$.

Solution. Set $P(x)$ = "$x > 1$" and $Q(x)$ = "$x - 1 > 0$". Let us formalize what we want to prove:

$$[\forall x\ (P(x) \to Q(x)) \wedge \forall x \in D\ P(x)] \to \forall x \in D,\ Q(x).$$

1. $\forall x\ (P(x) \to Q(x))$, by hypothesis

2. $\forall x \in D\ P(x)$, also by hypothesis

3. $P(y) \to Q(y)$, by universal instantiation on the first hypothesis

4. $P(y)$, by universal instantiation on D in the second hypothesis

5. $Q(y)$, using modus ponens

6. $\forall x \in D,\ Q(x)$, using universal generalization.

Exercises for Chapter 4

Exercise 38. Let q be a positive real number. Prove or disprove the following statement: if q is irrational, then $\sqrt{q}$ is irrational.

Solution. We prove the contrapositive of this statement, namely: if $\sqrt{q}$ is rational, then q is rational. But if $\sqrt{q}$ is rational, then $\sqrt{q} = \frac{a}{b}$, a, b integers, $b \neq 0$, and thus $q = \frac{a^2}{b^2}$ which shows that q is rational.

Exercise 39. Prove using mathematical induction that the sum of the first n odd positive integers is n^2.

Solution. We want to prove that $\forall n,\ P(n)$ where

$$P(n) = \text{“}\sum_{j=1}^{n}(2j-1) = n^2\text{”},\ n \in \mathbb{Z},\ \ \geq 1.$$

- Basis Step: we need to show that $P(1)$ is true.

$$P(1) = (2-1) = 1 = 1^2$$

  which is true.

- Inductive Step: Assume $P(k)$ is true, that is we assume that

$$\sum_{j=1}^{k}(2j-1) = k^2$$

  is true. We now need to prove that $P(k+1)$ is true.

$$\begin{aligned}
&\sum_{j=1}^{k+1}(2j-1) \\
= \ &\sum_{j=1}^{k}(2j-1) + 2(k+1) - 1 \\
= \ &k^2 + 2(k+1) - 1 \text{ using } P(k) \\
= \ &k^2 + 2k + 1 = (k+1)^2.
\end{aligned}$$

  This shows that $P(k+1)$ is true, therefore $P(n)$ is true for all n.

Exercise 40. Prove using mathematical induction that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

Solution. We first set $P(n) = $ “$3 \mid n^3 - n$”, that is 3 divides $n^3 - n$.

- Basis Step: $P(1) =$ "$3|0$" which is true.
- Inductive Step: Assume $P(k)$ is true, that is we assume that
$$3 \mid (k^3 - k).$$
is true. We now need to prove that $P(k+1)$ is true. When $n = k+1$, we get
$$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1) = (k^3 - k) + 3k^2 + 3k$$
which is divisible by 3, since $(k^3 - k)$ is divisible by 3, and so is $3(k^2 + k)$. Therefore $P(k+1)$ is true, and we conclude that $P(n)$ is true for all n.

Exercise 41. Prove by mathematical induction that
$$1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

Solution. We have $P(n) =$"$1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$". The basis step is $P(1)$ which is true, since $1^2 = 1 = \frac{1}{6}1(1+1)(2+1)$. Suppose true for $P(k)$, that is $1^2 + 2^2 + \ldots + k^2 = \frac{1}{6}k(k+1)(2k+1)$ holds. We need to prove that $P(k+1)$ is true, namely we need to prove that
$$1^2 + 2^2 + \ldots + (k+1)^2 = \frac{1}{6}(k+1)(k+2)(2(k+1)+1) = \frac{1}{6}(k+1)(k+2)(2k+3)$$
is true. So let us start by computing the left hand side:
$$\begin{aligned} 1^2 + 2^2 + \ldots + (k+1)^2 &= 1^2 + 2^2 + \ldots + k^2 + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \end{aligned}$$
using the induction hypothesis. We continue to compute
$$\begin{aligned} 1^2 + 2^2 + \ldots + (k+1)^2 &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= \frac{1}{6}(k+1)\left(k(2k+1) + 6(k+1)\right) \\ &= \frac{1}{6}(k+1)\left(2k^2 + k + 6k + 6\right) \end{aligned}$$
Since
$$\frac{1}{6}(k+1)(k+2)(2k+3) = \frac{1}{6}(k+1)\left(2k^2 + k + 6k + 6\right)$$
this concludes the proof.