NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2017-2018

EE2007 / IM2007 – ENGINEERING MATHEMATICS II

November / December 2017 Time Allowed: 2½ hours

INSTRUCTIONS

1. This paper contains 4 questions and comprises 4 pages.
2. Answer ALL questions.
3. All questions carry equal marks.
4. This is a closed book examination.
5. Unless specifically stated, all symbols have their usual meanings.
6. A list of useful formulae is given in the Appendix A on page 4.

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1. Given the matrices

$$A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}, \quad B = \left[\begin{array}{c|c} A & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \hline F & G \end{array}\right] \text{ and } C = \left[\begin{array}{c|c} A^{-1} & X \\ \hline Y & Z \end{array}\right],$$

where G is a 2-by-2 invertible matrix, F, X, Y and Z are matrices of appropriate dimensions.

(a) Use elementary row operations to find the inverse of A.

(10 Marks)

(b) What should be the dimensions of X, Y and Z so that the product of B and C, i.e., BC makes sense?

(5 Marks)

(c) What should X, Y and Z be so that C is the inverse of B?

(10 Marks)

2. (a) Write the vector $\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

(7 Marks)

(b) For which value of k will the vector $\begin{bmatrix} 1 \\ -2 \\ k \end{bmatrix}$ be a linear combination of the vectors $\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$?

(3 Marks)

(c) Find the rank and the null space of the matrix $\begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$.

(5 Marks)

(d) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be linearly independent vectors in a vector space.

(i) Show that $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, $\mathbf{u}-2\mathbf{v}+\mathbf{w}$ are also linearly independent.

(ii) Determine the condition(s) for the scalars a and b so that the vectors $\mathbf{u}+\mathbf{v}+a\mathbf{w}$, $\mathbf{u}+b\mathbf{v}-\mathbf{w}$, $\mathbf{v}+\mathbf{w}$ are linearly independent? Justify your answer.

(10 Marks)

3. (a) Using only the principal value(s) in the function, express

$$y = 2\sqrt{3}\exp\left(-i\frac{\pi}{6}\right)\ln\left(\cos\sqrt{3}+i\sin\sqrt{3}\right)$$

in the form $y = a \pm ib$, where a, b are real.

(5 Marks)

Note: Question No. 3 continues on page 3.

(b) Is the function $f(z)=e^{iz}$ continuous at the point $z=2$? Also, comment on the differentiability and analyticity of $f(z)=e^{iz}$ by using the Cauchy-Riemann equations.

(8 Marks)

(c) Evaluate $\int_C \frac{z}{(z-1)^2(z^2-2z+5)}dz$ along the path C, where C is the circle described by $|z-1-i|=2$ counter-clockwise.

(12 Marks)

4. (a) Given $f(x,y,z)=ye^x\cos z$ and $\mathbf{F}(x,y,z)=x^2y\,\mathbf{i}+y\cos z\,\mathbf{j}+ye^z\,\mathbf{k}$, evaluate the following:

(i) ∇f

(ii) $\nabla\cdot\mathbf{F}$

(iii) $\nabla\times\mathbf{F}$

(iv) $\nabla\cdot\nabla\times\mathbf{F}$

(v) $\nabla\times\nabla f$

(5 Marks)

(b) A particle moves along a straight line from $\left(0,\frac{\pi}{2},e\right)$ to $\left(\pi,-\frac{3\pi}{2},e^2\right)$ under a force

$$\mathbf{F}(x,y,z)=\left(-\sin x\ln z+e^x\sin y\right)\mathbf{i}+e^x\cos y\,\mathbf{j}+\frac{\cos x}{z}\mathbf{k}.$$

Find the work done by the force in moving the particle.

(9 Marks)

(c) The plane $2x+2y+z=2$ cuts the x-, y-, and z- axes at P (1, 0, 0), Q (0, 1, 0) and R (0, 0, 2) respectively. Path C is defined as straight-line segments from P to Q to R and back to P. Find the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ along the path C, where $\mathbf{F}(x,y,z)=x^2y\,\mathbf{i}+\cos y\,\mathbf{j}+5e^{3z}\,\mathbf{k}$.

(11 Marks)

Appendix A

1. Complex Analysis

(a) Complex Power: $z^c = e^{c \ln z}$

(b) De Moivre's Formula: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(c) Cauchy-Riemann equations:

$$u_x = v_y,\ v_x = -u_y,\ \text{or } u_r = \frac{1}{r}v_\theta,\ v_r = \frac{-1}{r}u_\theta$$

(d) Cauchy Integral Formula:

$$\int_C \frac{f(z)}{(z-z_o)^m}dz = \frac{2\pi i}{(m-1)!}\ \frac{d^{(m-1)}}{dz^{(m-1)}}f(z)\Big|_{z=z_o}$$

2. Vector Analysis. Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$.

(a) Scalar Triple Product: $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$

(b) Gradient: $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$

(c) Divergence: $\nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

(d) Curl: $\nabla\times\mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

(e) Gauss Theorem: $\iiint_T \nabla\cdot\mathbf{F}\,dv = \oiint_S \mathbf{F}\cdot\mathbf{n}\,dA$

(f) Stokes Theorem: $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dA = \oint_C \mathbf{F}\cdot d\mathbf{r}$

END OF PAPER

1. (a)

$$\left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 2 & -3 & 1 & 0 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 + R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -3 & 1 & 1 & 0 \\ -1 & 2 & -3 & 1 & 0 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & -3 & 1 & 1 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 4R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -3 & 1 & 1 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & -14 & 17 & -4 & -4 & 1 \end{array}\right] \xrightarrow[R_2 \leftarrow \frac{1}{5}R_2]{R_1 \leftarrow R_1 - \frac{3}{5}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & -6/5 & \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & -14 & 17 & -4 & -4 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + 14R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & -\frac{6}{5} & \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{8}{5} & -\frac{6}{5} & 1 \end{array}\right] \xrightarrow[R_2 \leftarrow R_2 + 6R_3]{R_1 \leftarrow R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 4 & -3 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & \frac{1}{5} & \frac{8}{5} & -\frac{6}{5} & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow 5R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 4 & -3 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right]$$

1. (b) B has 5 columns → c has 5 rows → $Y: 2 \times 3$ $Z: 2 \times m$
$X: 3 \times m$ m is integer number

(c)

$$\left[\begin{array}{c|c} A & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \hline F & G \end{array}\right] \times \left[\begin{array}{c|c} A^{-1} & X \\ \hline Y & Z \end{array}\right] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

(not sure)

$$\Rightarrow \begin{bmatrix} I & AX \\ A^{-1}F + GY & XF + GZ \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \Rightarrow \begin{array}{l} X = 0 \\ Z = G^{-1} \\ Y = -A^{-1}FG^{-1} \end{array}$$

2 (a)

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} \iff$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 2 \end{bmatrix}$$

2(b) $c_1\begin{bmatrix}3\\0\\-2\end{bmatrix}+c_2\begin{bmatrix}2\\-1\\5\end{bmatrix}=\begin{bmatrix}1\\-2\\k\end{bmatrix}\Rightarrow\begin{cases}3c_1+2c_2=1\\-c_2=-2\end{cases}$

$\Rightarrow\begin{cases}c_1=-1\\c_2=2\end{cases}\Rightarrow k=-2\times(-1)+5\times2=12$

(c) Null space $:=\{X\mid AX=0\}$ ~~X:~~

$X=\begin{bmatrix}c_1\\c_2\\c_3\\c_4\\c_5\end{bmatrix}$ $\begin{bmatrix}1&3&1&-2&-3\\1&4&3&-1&-4\\2&3&-4&-7&-3\\3&8&1&-7&-8\end{bmatrix}\xrightarrow[\substack{R_3\leftarrow R_3-2R_1\\R_4\leftarrow R_4-3R_1}]{R_2\leftarrow R_2-R_1}\begin{bmatrix}1&3&1&-2&-3\\0&1&2&1&-1\\0&-3&-6&-3&3\\0&-1&-2&-1&1\end{bmatrix}$

$\xrightarrow[\substack{R_3\leftarrow R_3+3R_2\\R_4\leftarrow R_4+R_2}]{}\begin{bmatrix}1&3&1&-2&-3\\0&1&2&1&-1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\Rightarrow\begin{cases}c_1+3c_2+c_3-2c_4-3c_5=0\\c_2+2c_3+c_4-c_5=0\end{cases}$

$\Rightarrow\begin{cases}c_1=5c_3+5c_4\\c_2=-2c_3-c_4+c_5\end{cases}$

$X=\begin{bmatrix}5c_3+5c_4\\-2c_3-c_4+c_5\\c_3\\c_4\\c_5\end{bmatrix}$ Rank $=2$

(d)
(i) We ~~want to show~~ know that the only solution to $c_1\vec{u}+c_2\vec{v}+c_3\vec{w}=0$ is $c_1=c_2=c_3=0$ ①

We want to show the only solution to $c_4(\vec{u}+\vec{v})+c_5(\vec{u}-\vec{v})+c_6(\vec{u}-2\vec{v}+\vec{w})=0$ is $c_4=c_5=c_6=0$

$\Leftrightarrow$ ~~want~~ Want to show only solution to $(c_4+c_5+c_6)\vec{u}+(c_4-c_5-2c_6)\vec{v}+c_6\vec{w}=0$ is $c_4=c_5=c_6=0$ ~~$\begin{cases}c_4+c_5+c_6=0\\c_4-c_5-2c_6=0\\c_6=0\end{cases}$~~

~~$\Rightarrow$ which is c_4~~

according to ①, we know that $\begin{cases}c_4+c_5+c_6=0\\c_4-c_5-2c_6=0\\c_6=0\end{cases}$

continue (i) $\therefore c_4=c_5=c_6=0$ ※

(ii) We want to find a, b s.t the only solution to

$c_7(\vec{u}+\vec{v}+a\vec{w})+c_8(\vec{u}+b\vec{v}-\vec{w})+c_9(\vec{v}+\vec{w})=0$ ②

is $c_7=c_8=c_9=0$

② $\Leftrightarrow \underbrace{(c_7+c_8)}_{=0}\vec{u}+\underbrace{(c_7+bc_8+c_9)}_{=0}\vec{v}+\underbrace{(ac_7-c_8+c_9)}_{=0}\vec{w}=0$

according to ①

$$\begin{cases} c_7+c_8=0 \\ c_7+bc_8+c_9=0 \\ ac_7-c_8+c_9=0 \end{cases} \Rightarrow a+b=0$$

3 (a) $\ln(\cos\sqrt{3}+i\sin\sqrt{3})=\ln e^{i\sqrt{3}}=i\sqrt{3}$

$y=6i\,e^{i(-\frac{\pi}{6})}=6i(\cos(-\frac{\pi}{6})+i\sin(-\frac{\pi}{6}))$

$=6i(\frac{\sqrt{3}}{2}-i\frac{1}{2})=3+3\sqrt{3}i$

(b) $z=x+iy$

$\lim\limits_{\substack{x=2\\y\to0}} f(z)=\cancel{\cos(}\lim\limits_{y\to0}[\cos(2+iy)+i\sin(2+iy)]=\cancel{x}\,e^{i2}$

$\lim\limits_{\substack{x\to2\\y=0}} f(z)=\lim\limits_{x\to2}(\cos x+i\sin x)=e^{i2}$

$\therefore \lim\limits_{\substack{x=2\\y\to0}} f(z)=\lim\limits_{\substack{x\to2\\y=0}} f(z)=f(2)=e^{i2}$ $\Rightarrow$ continuous at $z=2$

$f(z)=f(x,y)=\cos(x+iy)+i\sin(x+iy)$

$u=\cos(x+iy)$ $\quad v=\sin(x+iy)$

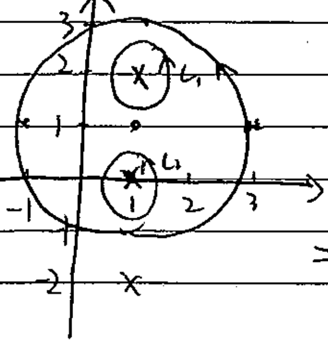
$u_x=-\sin(x+iy)$ $\quad v_x=\cos(x+iy)$

$u_y=-i\sin(x+iy)$ $\quad v_y=i\cos(x+iy)$

$u_x\neq v_y$

$\therefore$ not differentiable not analytic

(c) $z^2-2z+5=(z-(1+2i))(z-(1-2i))$



$$\int_C \frac{z}{(z-1)^2(z^2-2z+5)}\,dz = \int_{C_1}\frac{\frac{z}{(z-1)^2(z-(1-2i))}}{(z-(1+2i))}\,dz+\int_{C_2}\frac{\frac{z}{(z^2-2z+5)}}{(z-1)^2}\,dz$$

$$= 2\pi i \left. \frac{z}{(z-1)^2(z-(1-2i))} \right|_{z=1+2i} + 2\pi i \left. \frac{d}{dz}\left(\frac{z}{z^2-2z+5}\right)\right|_{z=1}$$

$$= 2\pi i\left(\frac{i-2}{16}+\frac{1}{4}\right) = \frac{\pi i(i+2)}{8} = \frac{-\pi+2\pi i}{8}$$

4 (a)

(i) $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z} k$

$= ye^x\cos z \, i + e^x\cos z \, j + (-ye^x\sin z)\, k$

(ii) $\nabla\cdot\vec{F} = 2xy\, i + \cos z\, j + ye^z\, k$

(iii) $\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & y\cos z & ye^z \end{vmatrix} = (e^z + y\sin z)i + (0-0)j + (0-x^2)k$

$= (e^z + y\sin z)i - x^2 k$

(iv) 0

(v) 0

(b) $\nabla\times\vec{F} = \begin{vmatrix} i & \hat{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\sin x\ln z + e^x\sin y & e^x\cos y & \frac{\cos x}{z} \end{vmatrix} = 0$

$\therefore \vec{F}$ is conservative

$\therefore V = \int \vec{F} = (\cos x\ln z + e^x\sin y)i + (e^x\sin y)j + (\cos x\ln z)k$

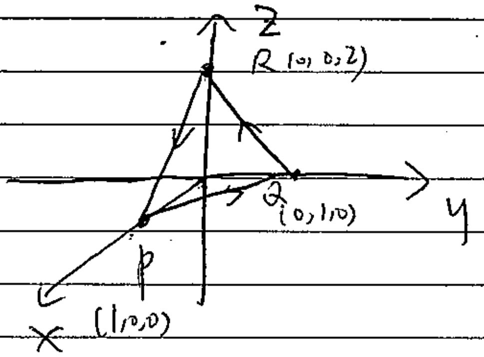
$\therefore W = \int \vec{F}\, d\vec{r} = V_2 - V_1$ #

$V_1 = 2i + j + k$

$V_2 = (-2+e^\pi)i + e^\pi j - 2k$

$V_2 - V_1 = (-4+e^\pi)i + (e^\pi-1)j + (-3)k$

4 (c)



$t : 0 \to 1$

$1-t : 1 \to 0$

$2(1-t) : 2 \to 0$

$\vec{r} = xi + yj + zk$

C_1 $(1,0,0) \to (0,1,0)$

$x = 1-t \quad y = t \quad z = 0 \qquad \vec{r_1} = (1-t)i + tj$

$\vec{F_1} = t(1-t)^2 i + \cos t\, j + 5k \qquad d\vec{r_1} = (-i+j)\,dt$

$W_1 = \int_{C_1} \vec{F_1} \cdot d\vec{r_1} = \int_0^1 (t(1-t)^2 i + \cos t\, j + 5k)(-i+j)\,dt$

$= \int_0^1 (-t(1-t)^2 + \cos t)\,dt = -\frac{1}{12} + \sin 1$

C_2 $(0,1,0) \to (0,0,2)$

$x = 0 \quad y = 1-t \quad z = 2t \qquad \vec{r_2} = (1-t)j + 2tk$

$\vec{F_2} = \cos(1-t)j + 5e^{6t}k \qquad d\vec{r_2} = (-j+2k)\,dt$

$W_2 = \int_{C_2} \vec{F_2} \cdot d\vec{r_2} = \int_0^1 (\cos(1-t)j + 5e^{6t}k)(-j+2k)\,dt$

$= \int_0^1 (-\cos(1-t) + 10e^{6t})\,dt = \frac{5}{3}e^6 - \frac{5}{3} - \sin 1$

C_3 $(0,0,2) \to (1,0,0)$

$x = t \quad y = 0 \quad z = 2(1-t) \qquad \vec{r_3} = ti + 2(1-t)k$

$\vec{F_3} = j + 5e^{6(1-t)}k \qquad d\vec{r_3} = (i-2k)\,dt$

$W_3 = \int_{C_3} \vec{F_3} \cdot d\vec{r_3} = \int_0^1 (j + 5e^{6(1-t)}k)(i-2k)\,dt$

$= \int_0^1 (-10e^{6(1-t)})\,dt = \frac{5}{3} - \frac{5}{3}e^6$

$\therefore \oint \vec{F} \cdot d\vec{r} = W_1 + W_2 + W_3 = \left(-\frac{1}{12} + \sin 1\right) + \left(\frac{5}{3}e^6 - \frac{5}{3} - \sin 1\right) + \left(\frac{5}{3} - \frac{5}{3}e^6\right)$

$= -\frac{1}{12}$