EE2001 Circuit Theory

Applications of the Laplace Transform

Dr. Justin Dauwels

Application of the Laplace Transform

- Circuit Element Models
- Circuit Analysis
- Mutual Inductance
- Transfer Functions

Circuit Analysis in the S-Domain

Steps in Applying the Laplace Transform:

1. Transform the circuit from the time domain to the s-domain

2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar.

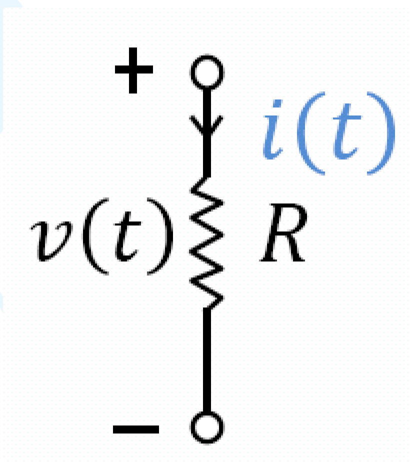
   All the circuit theorems and relationships developed for dc circuits are perfectly valid in the s-domain

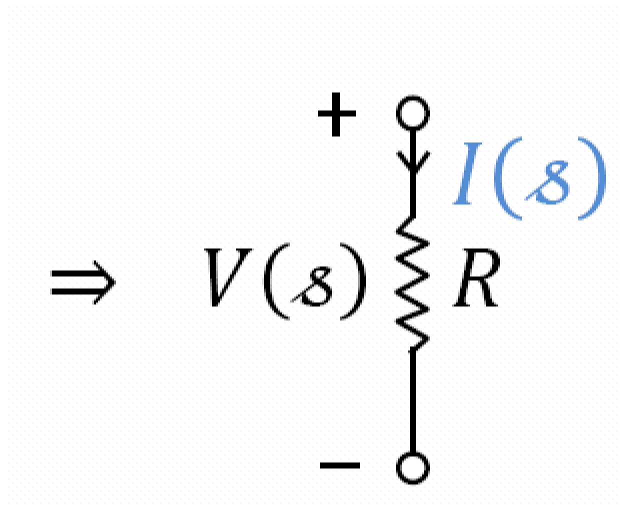
3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

Circuit Element Models

Resistor (R) in the s-domain

$$v(t) = Ri(t) \qquad \Rightarrow V(s) = RI(s)$$



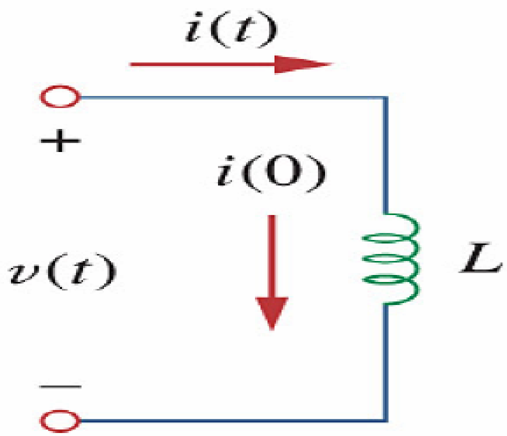


Time-domain **s-domain**

Inductor (L) in the s-domain

Time-domain

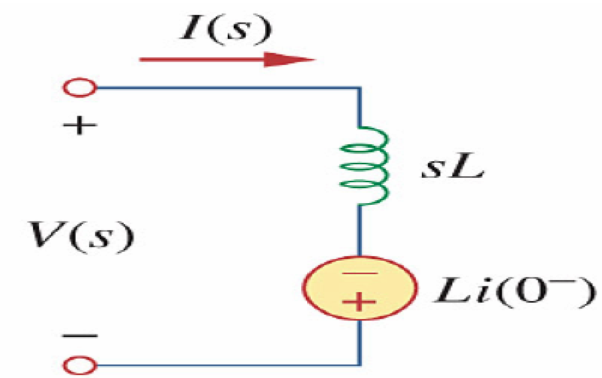
$$v(t) = L\frac{di(t)}{dt}$$



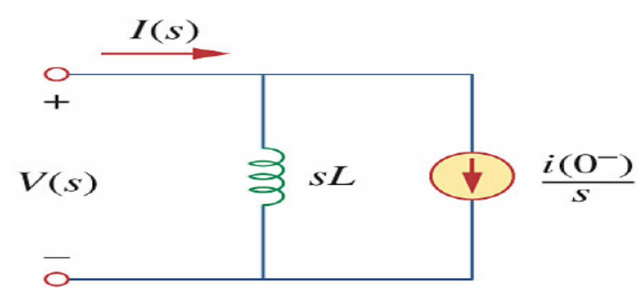
s-domain

$$V(s) = sLI(s) - Li(0^-)$$

$\Rightarrow$



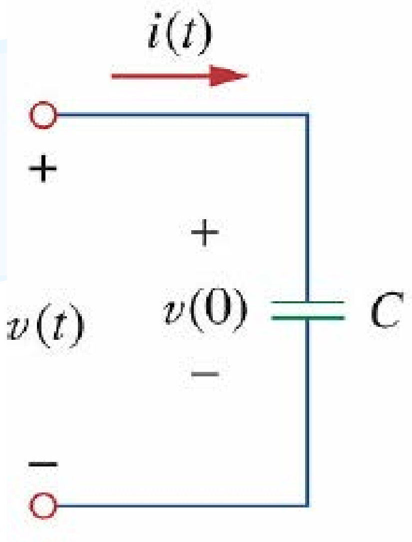
or $\quad I(s) = \dfrac{1}{sL}V(s) + \dfrac{i(0^-)}{s}$



Capacitor (C) in the s-domain

Time-domain

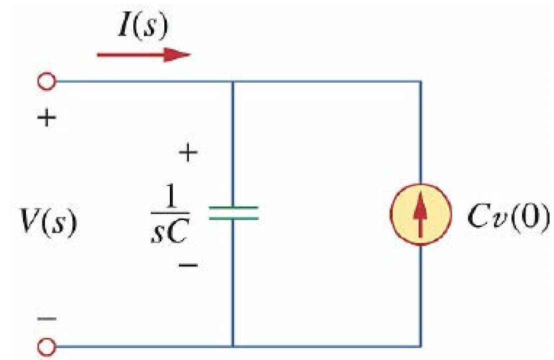
$$i(t) = C\frac{dv(t)}{dt}$$



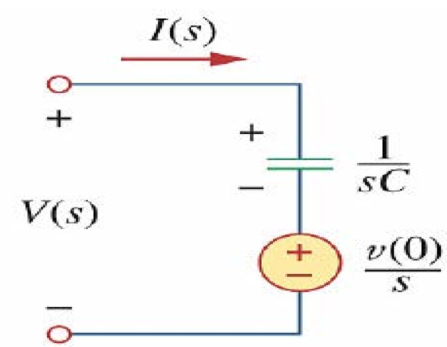
$\Rightarrow$

s-domain

$$I(s) = sCV(s) - Cv(0^-)$$



or $$V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}$$



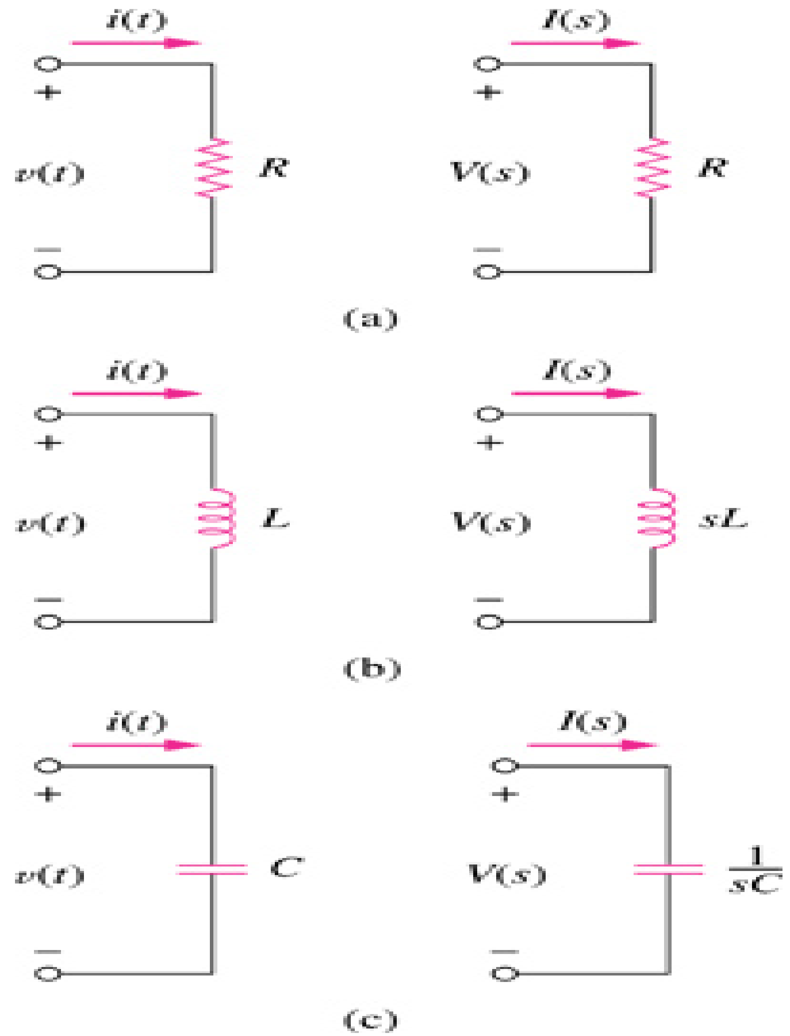
Circuit Element Models (Zero Initial Conditions)

Resistor : $V(s)=RI(s)$

Assume <u>zero initial condition</u> for the inductor and capacitor,

Inductor: $V(s)=sLI(s)$

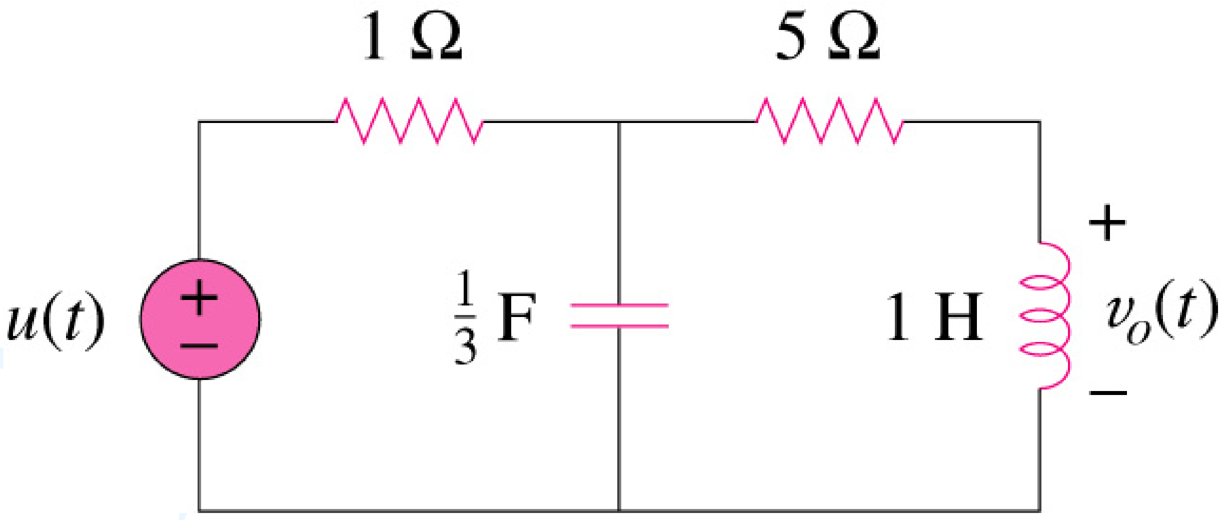
Capacitor: $V(s) = I(s)/sC$



Circuit Analysis *With Zero Initial Conditions*

Example 1:

Find $v_0(t)$ in the circuit shown below, assuming zero initial conditions.



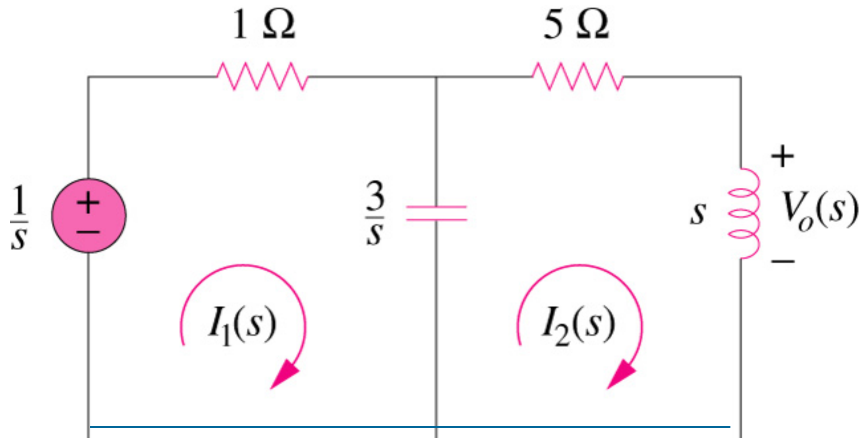
Solution:

Transforming the circuit from the time domain to the s-domain, we have

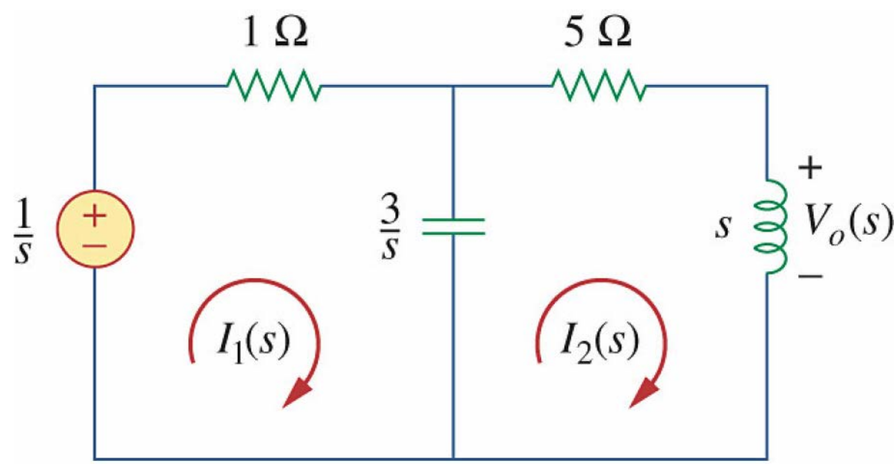
$$u(t) \Rightarrow \frac{1}{s}$$

$$1\,\mathrm{H} \Rightarrow sL = s$$

$$\frac{1}{3}\mathrm{F} \Rightarrow \frac{1}{sC} = \frac{3}{s}$$



- Mesh or nodal analysis can be used.
- Using Mesh Analysis,



$$(1+\frac{3}{s})I_1(s)-\frac{3}{s}I_2(s)=\frac{1}{s} \qquad (1)$$

$$(5+s+\frac{3}{s})I_2(s)-\frac{3}{s}I_1(s)=0 \qquad (2)$$

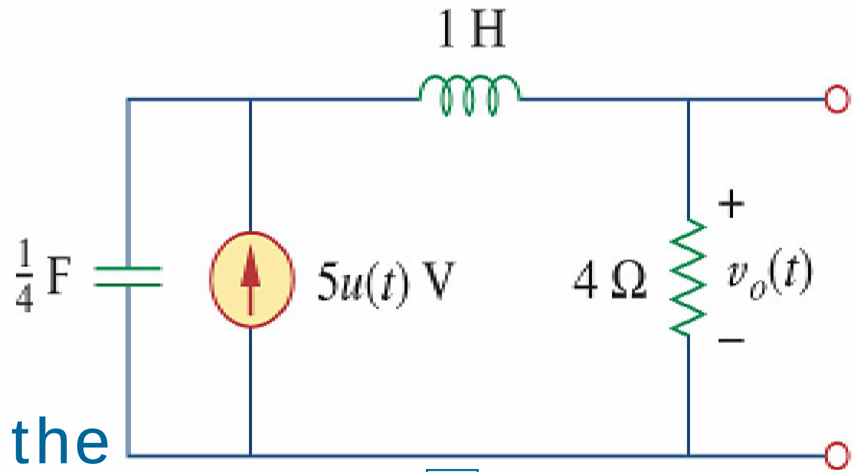
From (2) $I_1(s)=\frac{s}{3}\times(5+s+\frac{3}{s})I_2(s)$

$$(1+\frac{3}{s})\times\frac{s}{3}\times(5+s+\frac{3}{s})I_2(s)-\frac{3}{s}I_2(s)=\frac{1}{s} \Rightarrow s(s^2+8s+18)I_2(s)=3$$

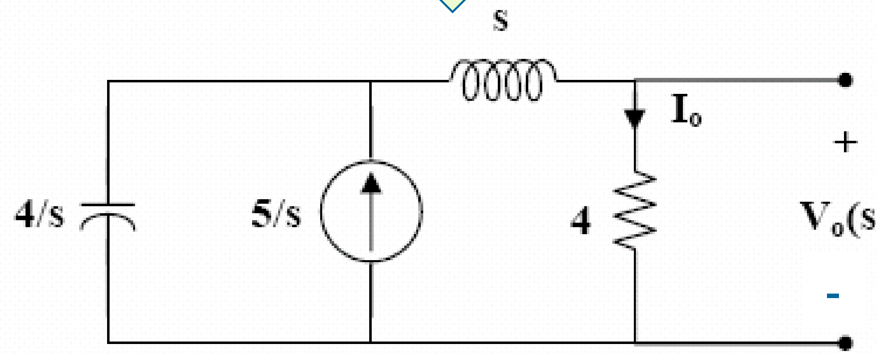
$$I_2(s)=\frac{3}{s(s^2+8s+18)} \Rightarrow V_0(s)=sI_2(s)=\frac{3}{s^2+8s+18}=\frac{3}{(s+4)^2+(\sqrt{2})^2}$$

$$\mathrm{V}_0(s)=\frac{3}{\sqrt{2}}\frac{\sqrt{2}}{(s+4)^2+(\sqrt{2})^2} \qquad \Rightarrow v_0(t)=\frac{3}{\sqrt{2}}e^{-4t}\sin(\sqrt{2}t)\ \ \mathrm{V},\quad t\geq 0$$

Example 2:
Determine $v_0(t)$ in the circuit, assuming zero initial conditions.



Transforming the circuit from the time domain to the s-domain



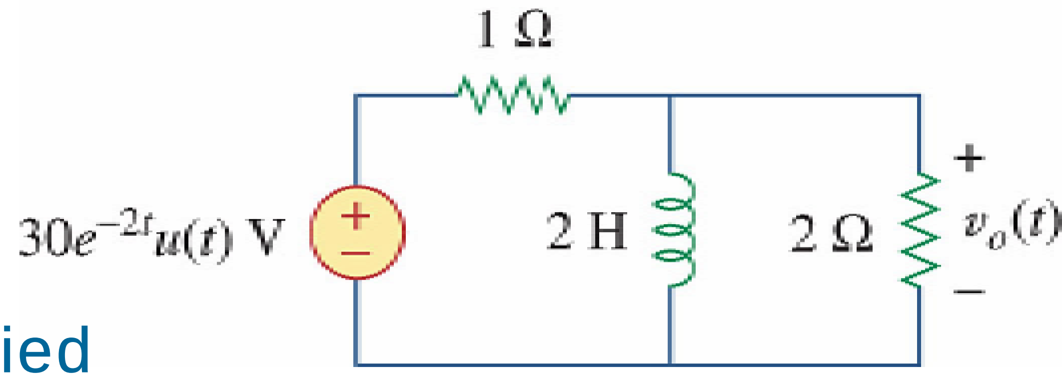
$$I_o = \frac{\frac{4}{s}}{\frac{4}{s} + s + 4} \cdot \frac{5}{s} = \frac{20}{s\left(s^2 + 4s + 4\right)}$$

$$V_o(s) = 4I_o = \frac{80}{s(s+2)^2}$$

$$= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2} = \frac{20}{s} - \frac{20}{s+2} - \frac{40}{(s+2)^2}$$

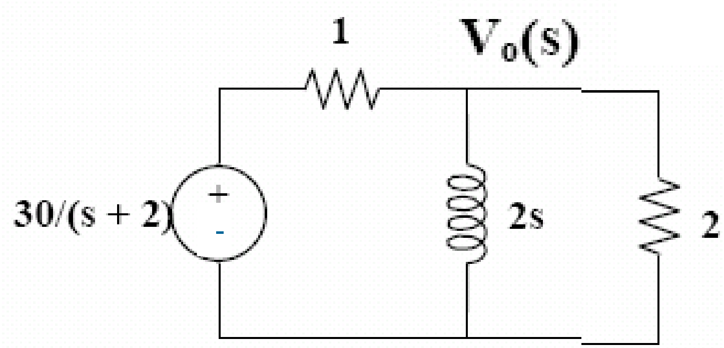
$$v_o(t) = \underline{20\left(1 - e^{-2t} - 2te^{-2t}\right)u(t)\ \mathrm{V}}$$

Example 3:
Determine $v_0(t)$ in the circuit.



Since the input is multiplied by $u(t)$, the voltage source is a short for all $t<0$ and *thus* $i(0)=0$

The s-domain circuit is:



At the top node (KCL):

$$\frac{V_o - \dfrac{30}{s+2}}{1} + \frac{V_o}{2s} + \frac{V_o}{2} = 0$$

$$\left(1 + \frac{1}{2} + \frac{1}{2s}\right)V_o = \frac{30}{s+2} \Rightarrow V_o = \frac{60s}{(s+2)(3s+1)} = \frac{20s}{(s+2)(s+1/3)} = \frac{24}{s+2} - \frac{4}{s+1/3}$$

$$\Rightarrow \quad v_o(t) = \underline{\left(24e^{-2t} - 4e^{-t/3}\right)\mathbf{u(t)V}}$$

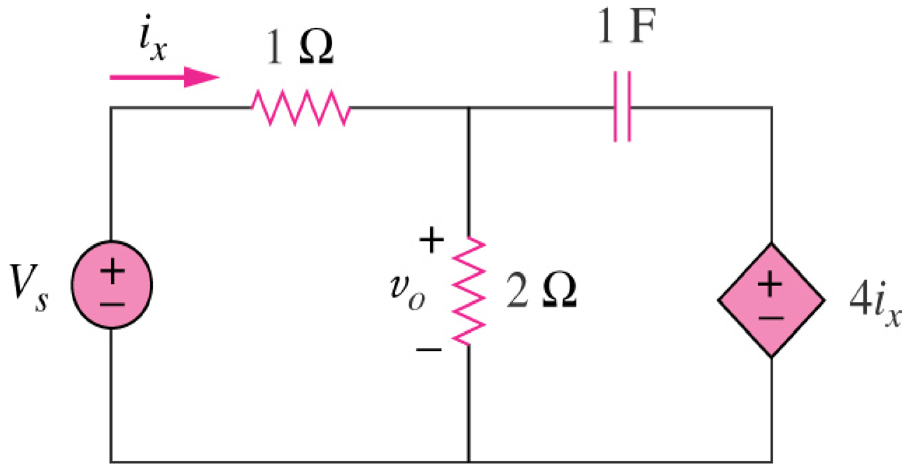
Example 4:

The initial energy in the circuit is zero at $t=0$. Assume that $v_s=15u(t)$ V.

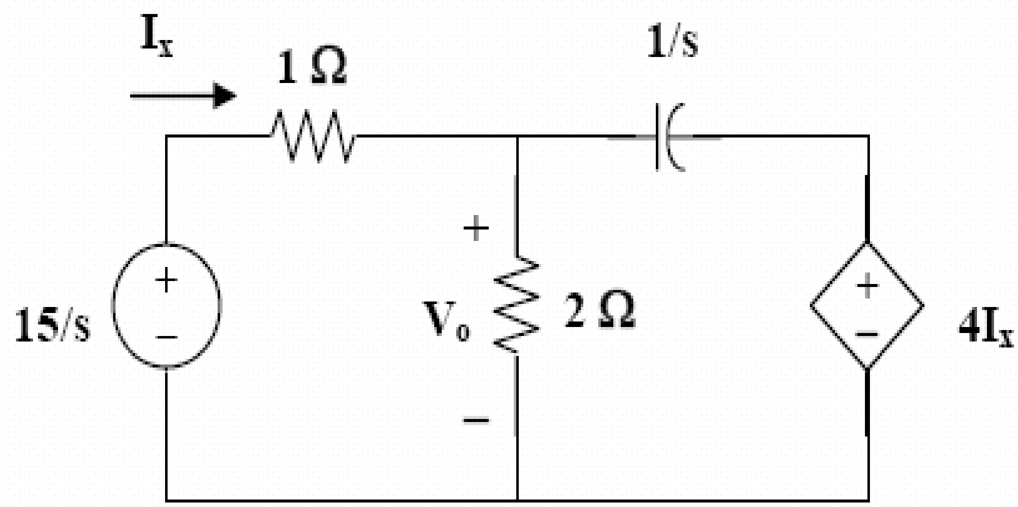
(a) Find $V_0(s)$ using the Thevenin theorem.

(b) Apply the initial- and final-value theorem to find $v_0(0)$ and $v_0(\infty)$.

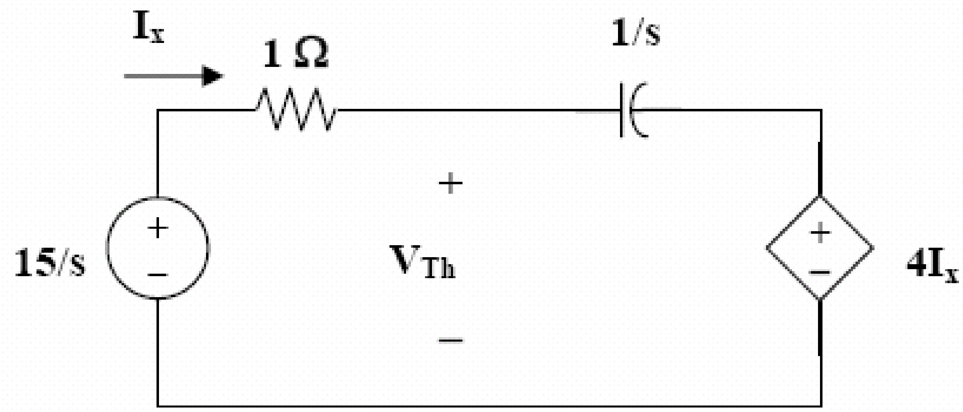
(c) Obtain $v_0(t)$.



The s-domain circuit is:



(a) Take out the 2 Ω and find V_{th} of the Thevenin equivalent circuit



Using mesh analysis we get,

$$-15/s + 1I_x + I_x/s + 4I_x = 0$$

$$(1 + 1/s + 4)I_x = 15/s \qquad I_x = 15/(5s+1)$$

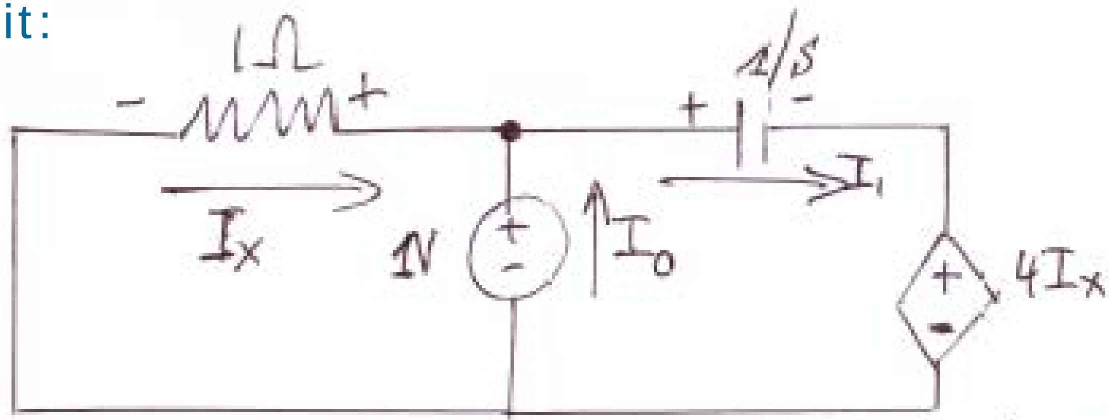
$$V_{Th} = 15/s - 15/(5s+1)$$

$$= (75s+15-15s)/(s(5s+1))$$

$$= 15(4s+1)/(s(5s+1))$$

$$= \mathbf{12(s+0.25)/(s(s+0.2))}$$

To find Z_{th} **of the Thevenin equivalent circuit**, we need to obtain I_0 in the following circuit:



We have inserted a 1V voltage source between the terminals.

$$Z_{TH} = \frac{V_o}{I_o} = \frac{1V}{I_o}$$

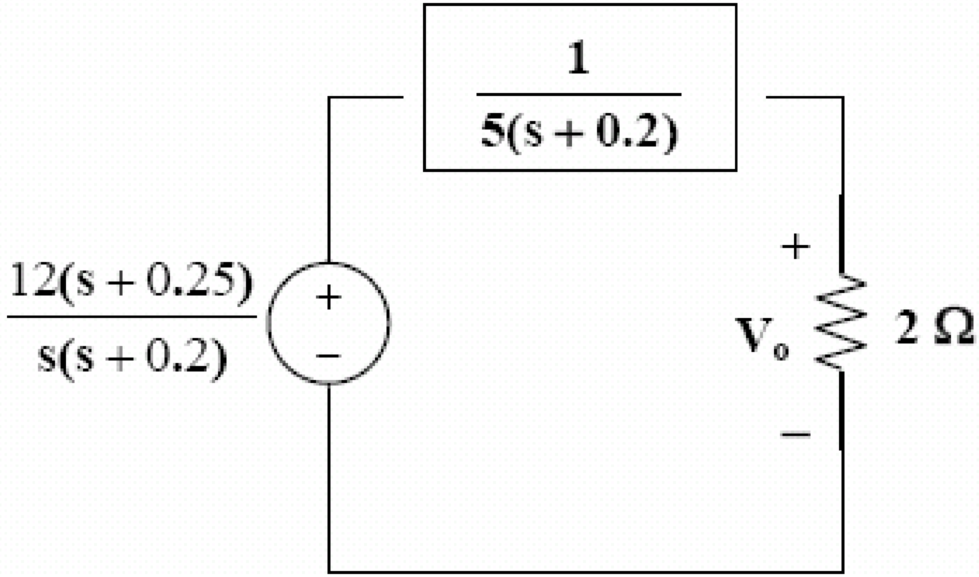
(KCL) $I_o + I_x = I_1 \Rightarrow I_o = I_1 - I_x = I_1 + 1$

$$= (1 - 4I_x)s + 1$$

$$= 5s + 1$$

$$\Rightarrow Z_{TH} = \frac{1}{5s+1} = \frac{1}{5(s+0.2)}$$

The Thevenin equivalent circuit:



$$V_o = \frac{\dfrac{12(s+0.25)}{s(s+0.2)}}{\dfrac{1}{5(s+0.2)}+2}\,2 = \frac{12(s+0.25)}{s(s+0.3)}$$

(b) Initial value:

$$v_o(0^+) = \lim_{s \to \infty} sV_o = \underline{\mathbf{12\ V}}$$

Final value:

$$v_o(\infty) = \lim_{s \to 0} sV_o$$
$$= 12(0+0.25)/(0+0.3) = \underline{\mathbf{10\ V}}$$

(c) Partial fraction expansion leads to

$$V_0 = 10/s + 2/(s+0.3)$$

Taking the inverse Laplace transform we get,
(**also apply time-domain method to verify!**)

$$v_o(t) = (10 + 2e^{-0.3t})u(t) \quad \text{V}$$

Circuit Analysis *With Non-zero Initial Conditions*

Steps:

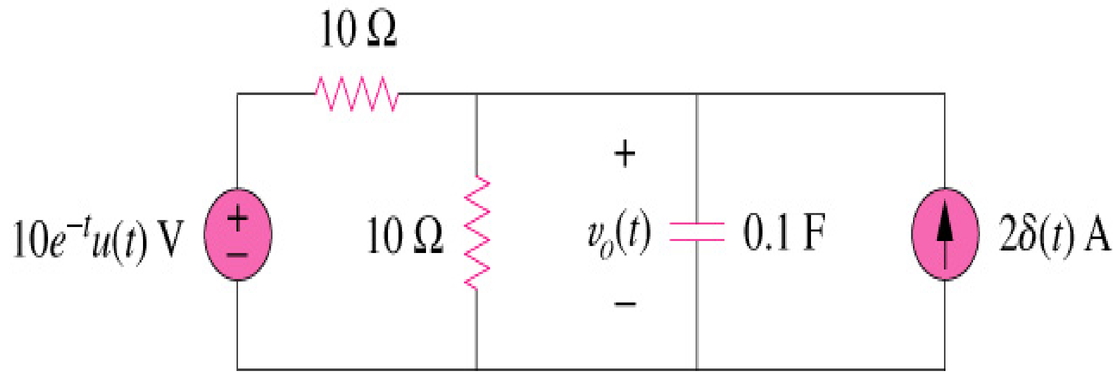
1) If initial conditions are not given, find the values of all capacitor voltages and inductor currents at $t = 0^-$. Under steady-state and using DC input, the capacitor will be open circuit and the inductor will be short circuit.

2) For $t>0$, transform the circuit from the time domain to the s-domain. Draw the s-domain circuit with all initial conditions properly represented.

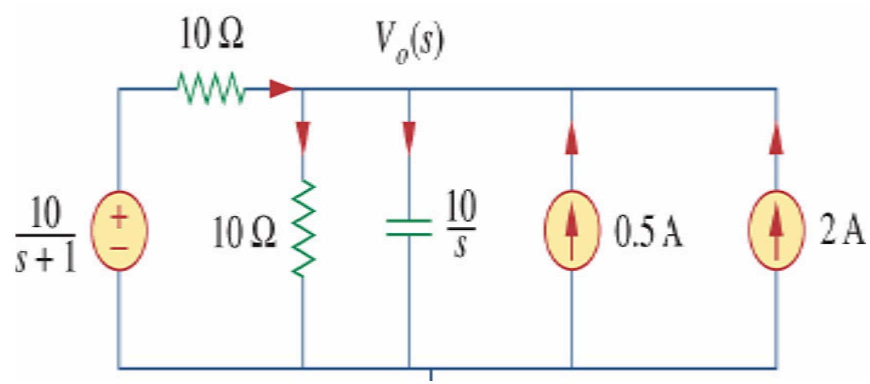
3) Solve the circuit using circuit theorems and relationships developed.

4) Take the inverse transform of the solution and thus obtain the solution in the time domain.

Example 5:



Find $v_0(t)$ in the circuit. Assume $v_0(0)=5V$.



By including the initial condition as a current source $Cv_0(0)=0.1(5)=0.5A$, the s-domain circuit is:

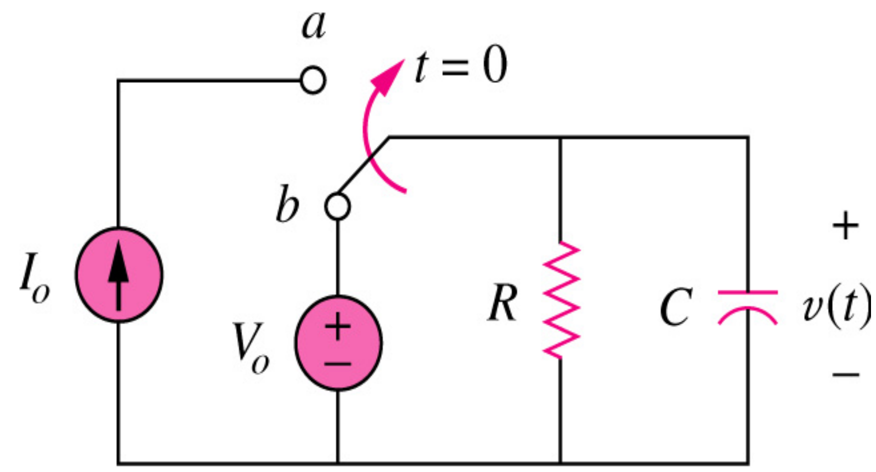
At the top node:
$$\frac{10/(s+1)-V_0}{10}+2+0.5=\frac{V_0}{10}+\frac{V_0}{10/s} \Rightarrow$$

$$\frac{1}{s+1}+2.5=\frac{2V_0}{10}+\frac{sV_0}{10/s}=\frac{1}{10}V_0(s+2) \Rightarrow \frac{10}{s+1}+25=V_0(s+2)$$

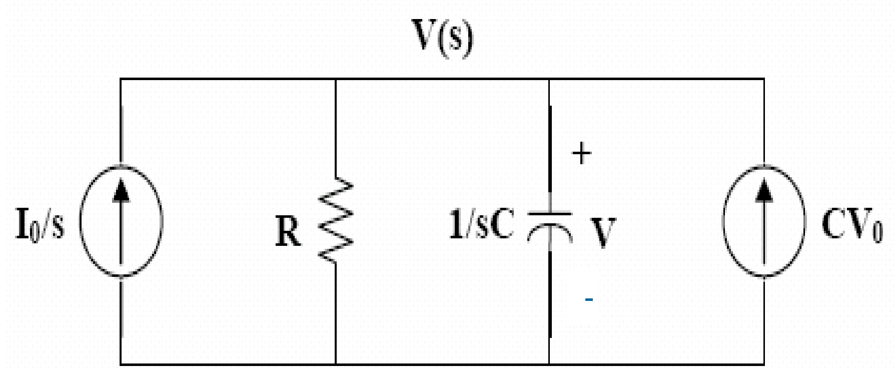
$$\Rightarrow V_0=\frac{25s+35}{(s+1)(s+2)}=\frac{10}{s+1}+\frac{15}{s+2} \Rightarrow \boxed{v_0(t)=(10e^{-t}+15e^{-2t})u(t)\quad \mathrm{V}}$$

Example 6:

The switch in the circuit has been in position *b* for a long time. It is moved to position *a* at *t=0*. Determine *v(t)* for t > 0.



Clearly, $v_0(0)=V_0$. The s-domain circuit is:



At the top node (KCL):

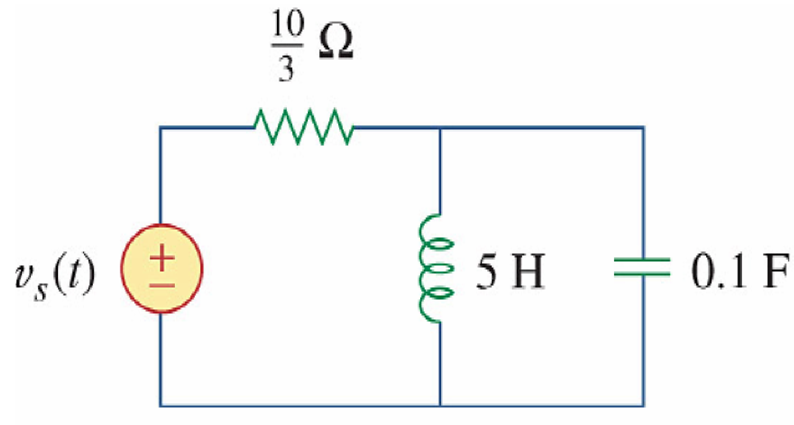
$$\frac{I_0}{s}+CV_0=\frac{V}{R}+sCV=\left(sC+\frac{1}{R}\right)V \Rightarrow V=\frac{I_0}{s(sC+1/R)}+\frac{CV_0}{sC+1/R}$$

$$\Rightarrow V=\frac{V_0}{s+1/RC}+\frac{I_0/C}{s(s+1/RC)}=\frac{V_0}{s+1/RC}+\frac{I_0R}{s}-\frac{I_0R}{s+1/RC}$$

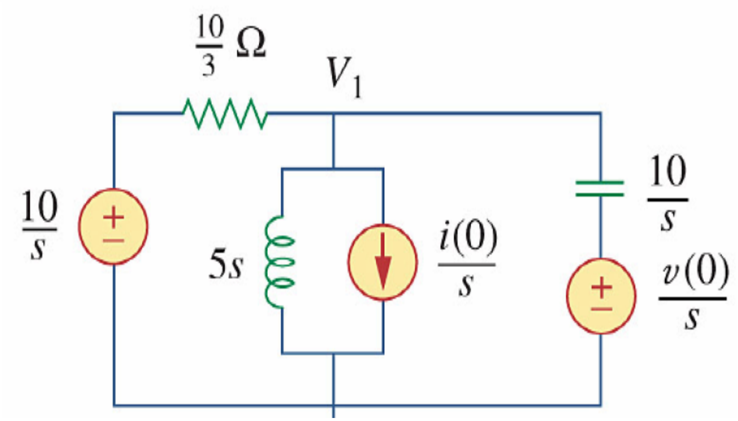
$$\Rightarrow \boxed{v(t)=(V_0-I_0R)e^{-t/\tau}+I_0R,\ \ t>0, \text{ where } \tau=RC}$$

Example 7:

Find the capacitor voltage, if $v_s(t)=10u(t)$ V and at t=0, -1A flows through the inductor and +5 is across the capacitor.



The s-domain circuit is:



At the top node:

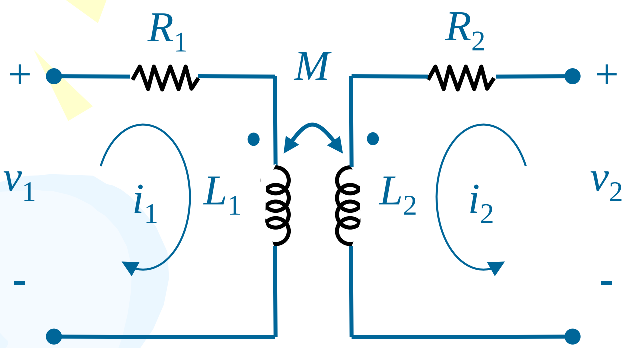
$$\frac{V_1-10/s}{10/3}+\frac{V_1-0}{5s}+\frac{i(0)}{s}+\frac{V_1-[v(0)]/s}{10/s}=0$$

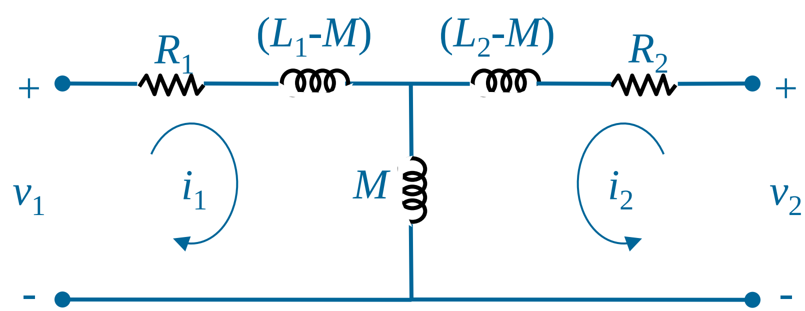
$$\Rightarrow 0.1(s+3+\frac{2}{s})V_1=\frac{3}{s}+\frac{1}{s}+0.5 \Rightarrow (s^2+3s+2)V_1=40+5s$$

$$\Rightarrow V_1=\frac{40+5s}{(s+1)(s+2)}=\frac{35}{s+1}-\frac{30}{s+2} \Rightarrow \boxed{v_1(t)=(35e^{-t}-30e^{-2t})u(t)\ \ \text{V}}$$

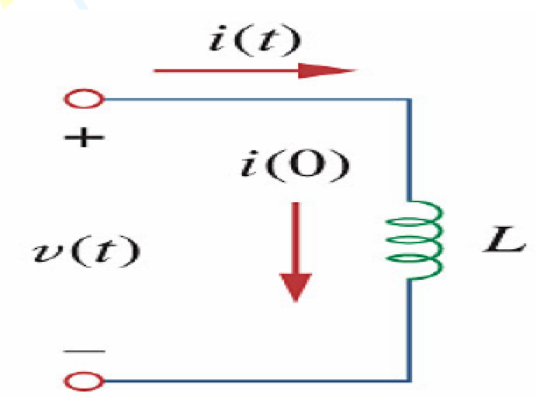
Inclusion of Mutual Inductance

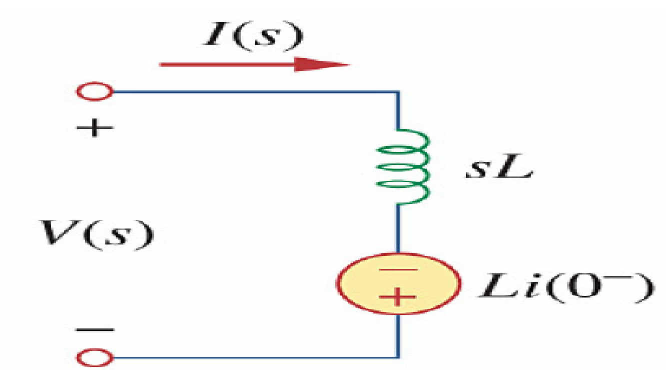
Recall the T-equivalent of magnetically-coupled coils





A t-domain inductor is transformed to the s-domain:





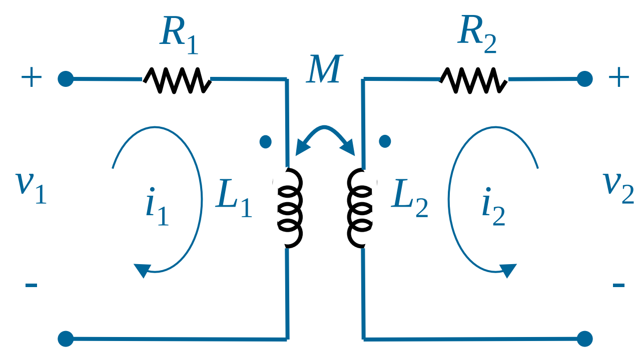
Example 14:

T-equivalent circuit for magnetically coupled coils

For coil 1,

$$v_1 = i_1 R_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

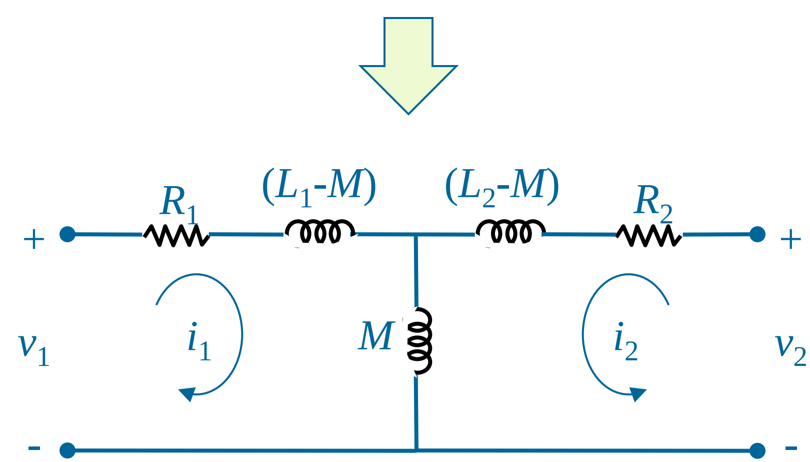
$$v_1 = i_1 R_1 + (L_1 - M)\frac{di_1}{dt} + M \frac{d}{dt}(i_1 + i_2)$$



For coil 2,

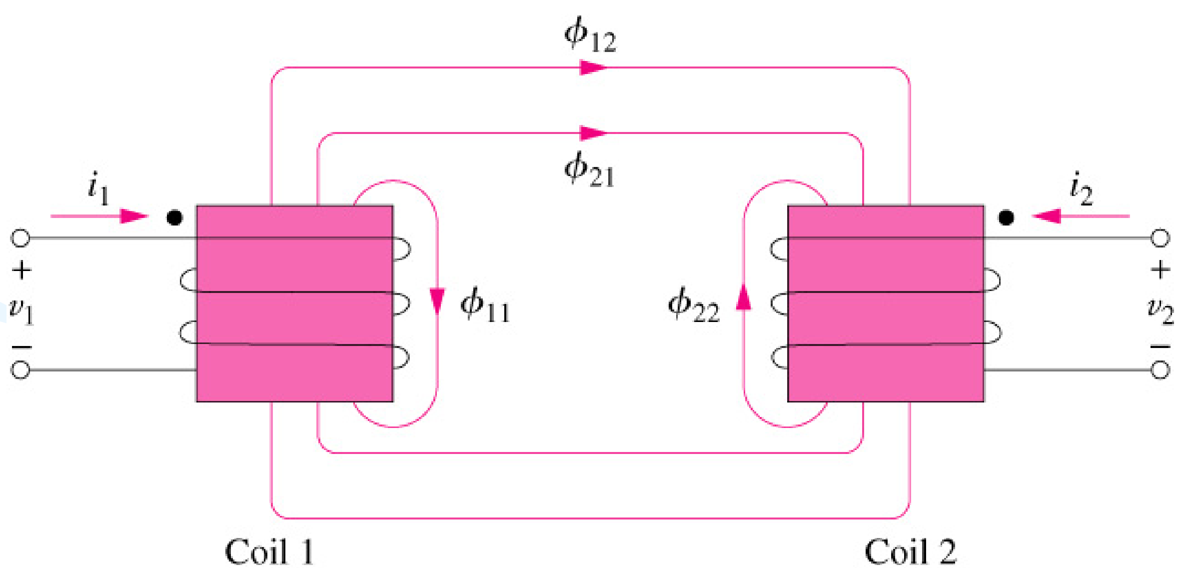
$$v_2 = i_2 R_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

$$v_2 = i_2 R_2 + (L_2 - M)\frac{di_2}{dt} + M \frac{d}{dt}(i_1 + i_2)$$



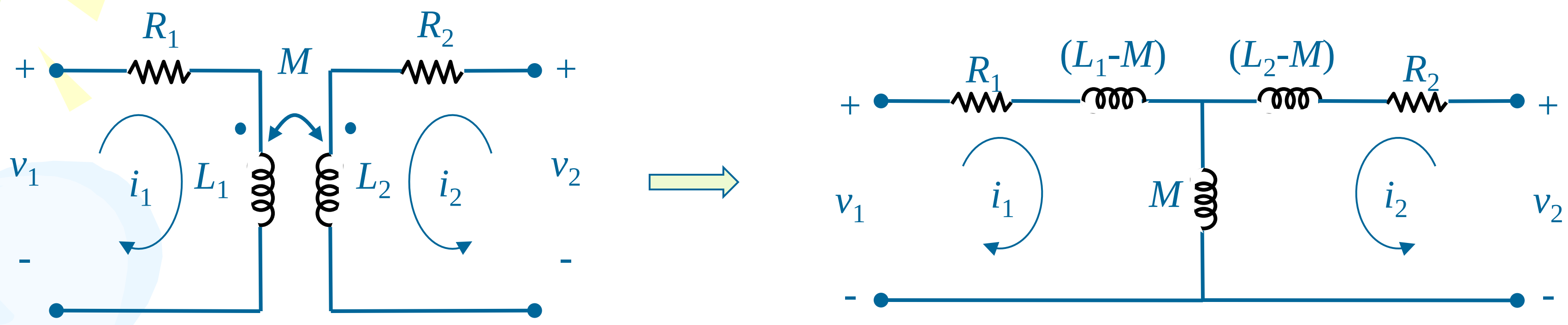
The circuit is equivalent to that on the right

- How to determine the polarity of the induced voltages?
  - Depends on way the coils are wound and reference direction of the coil currents
  - Keep track of polarities by a method known as **dot convention**, where a dot is placed on one terminal of each winding
- If a current enters the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is positive at the dotted terminal of the second coil.

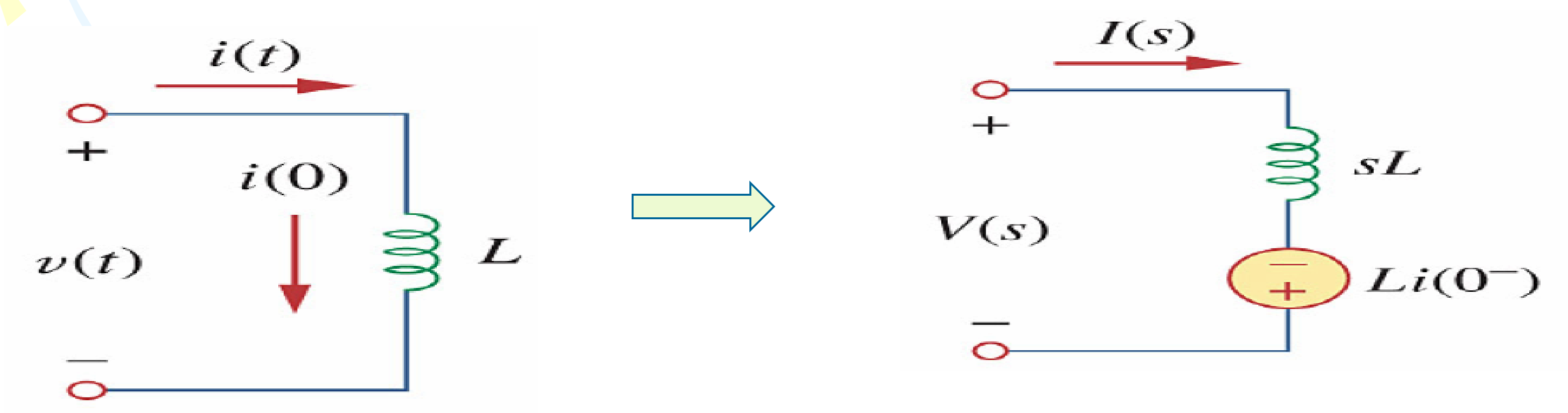


Inclusion of Mutual Inductance

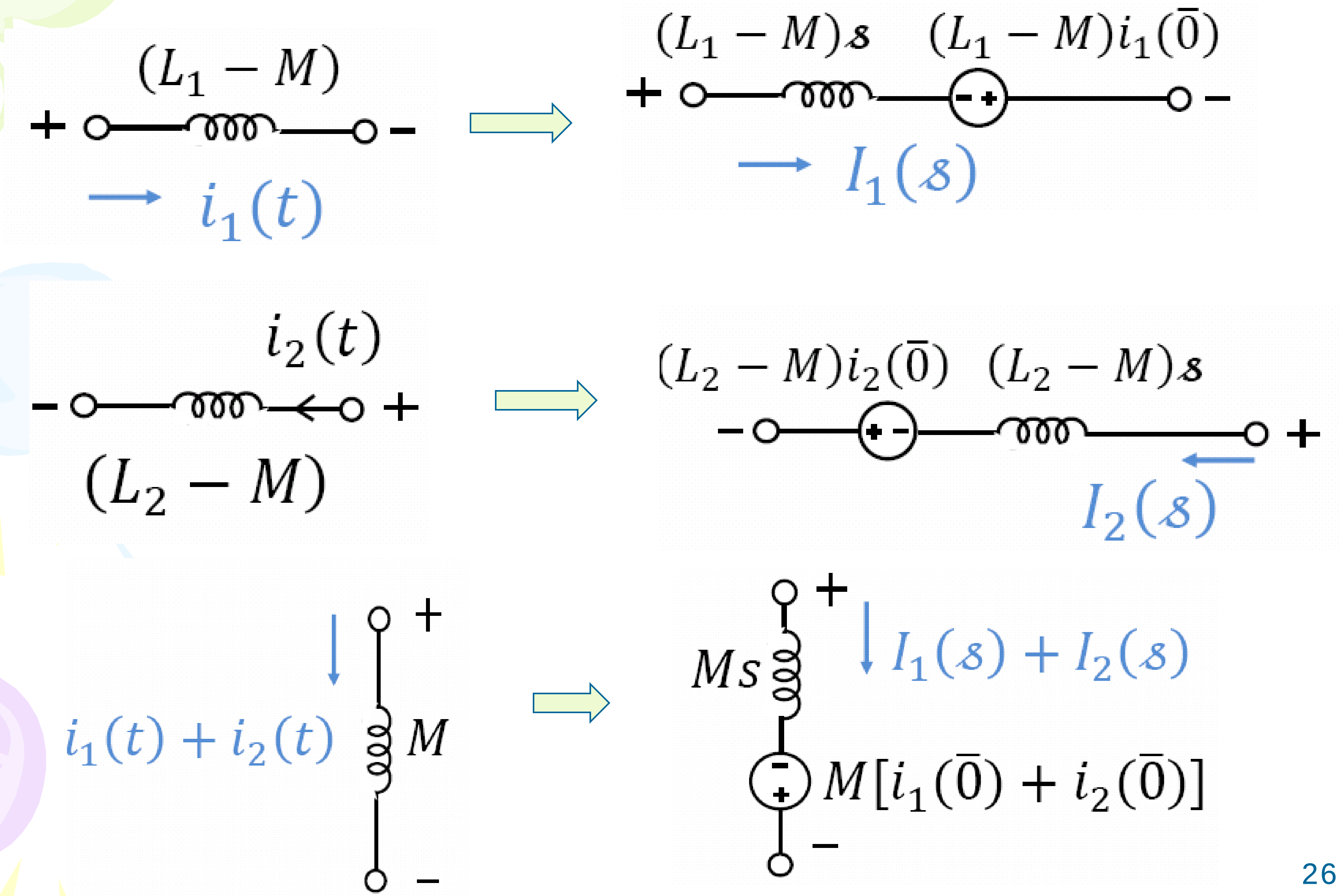
Recall the T-equivalent of magnetically-coupled coils



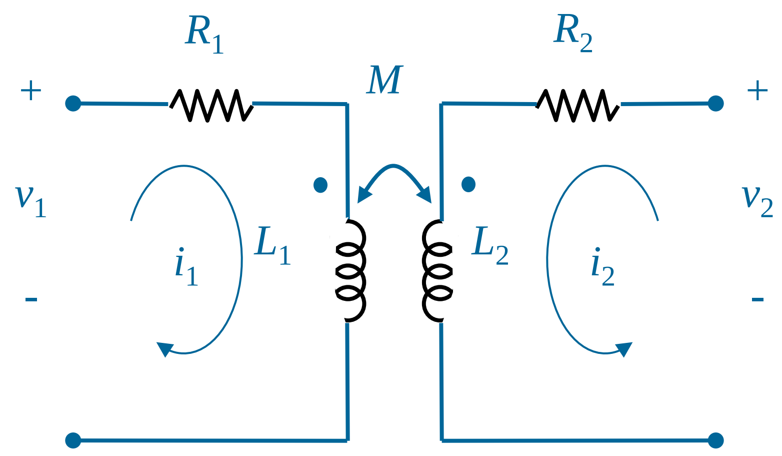
A t-domain inductor is transformed to the s-domain:

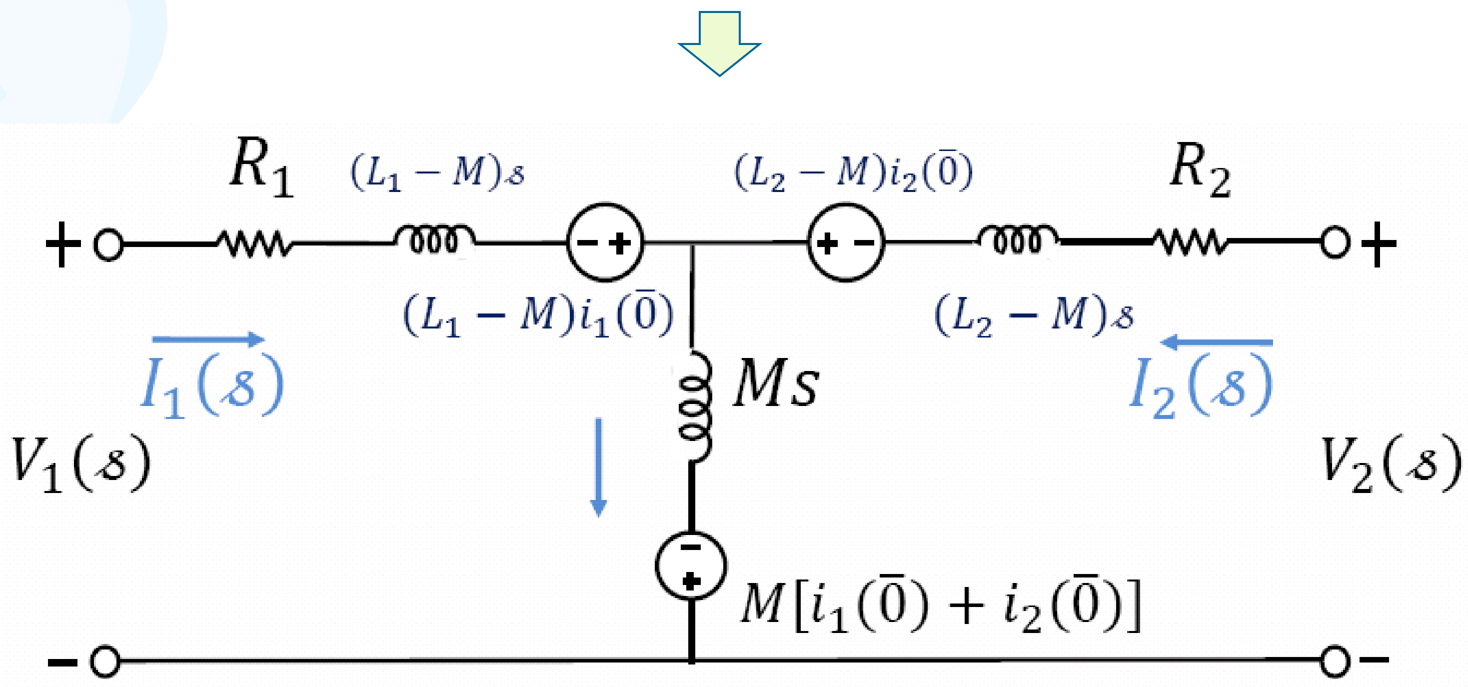


We can transform inductors L_1-M, L_1-M, and M in a similar way as:

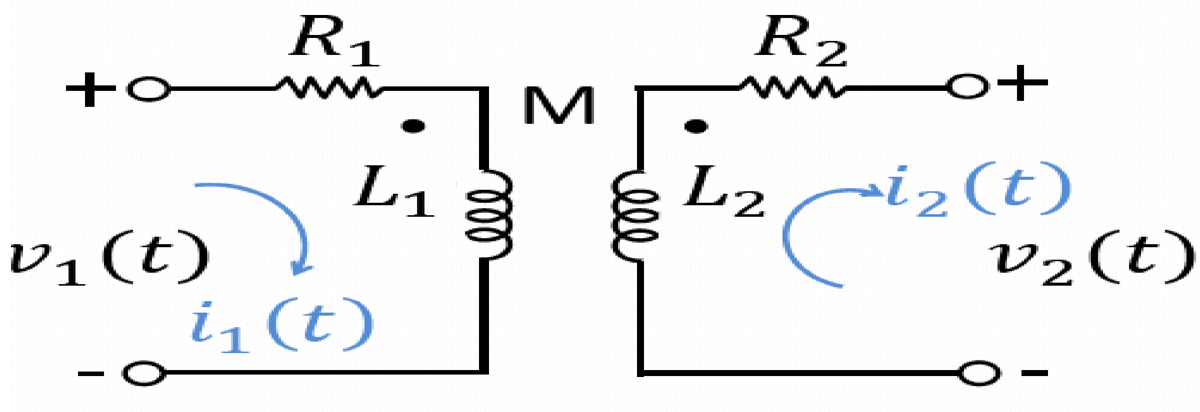


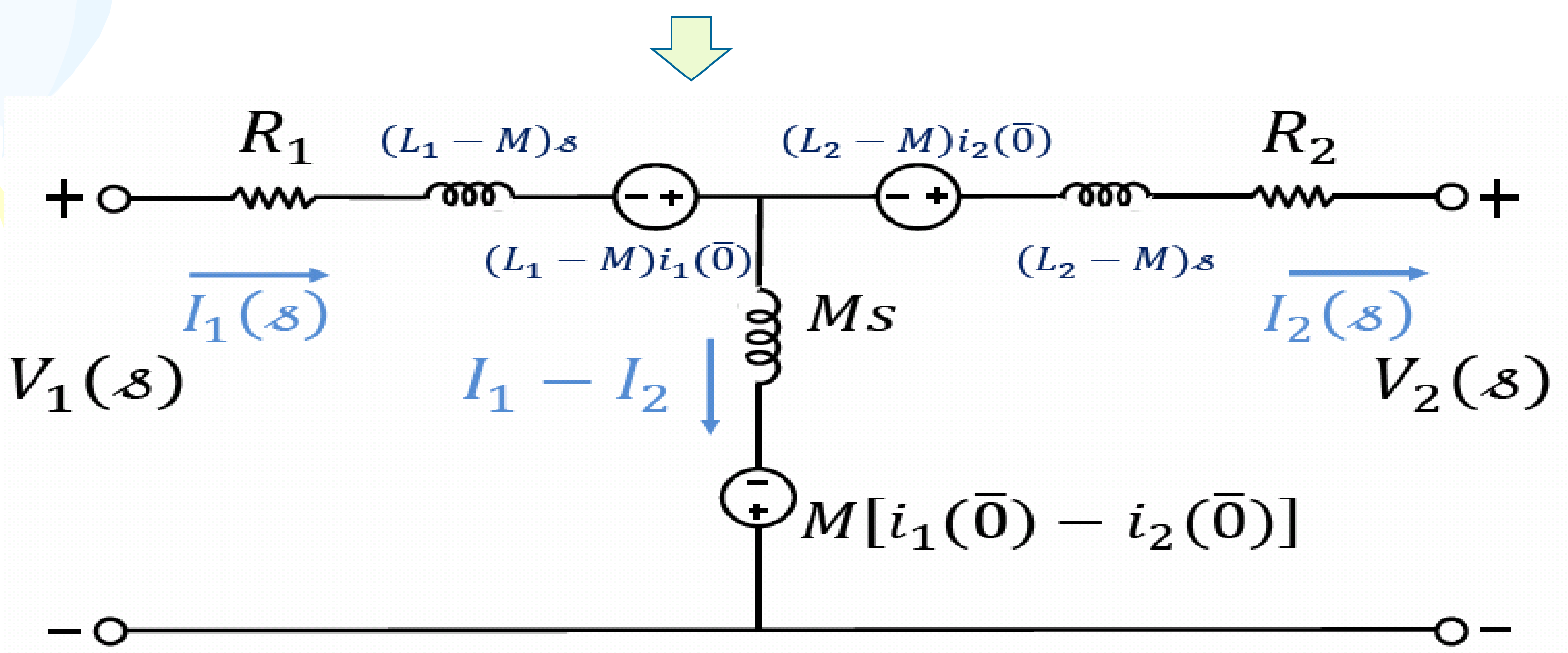
T-equivalent s-model

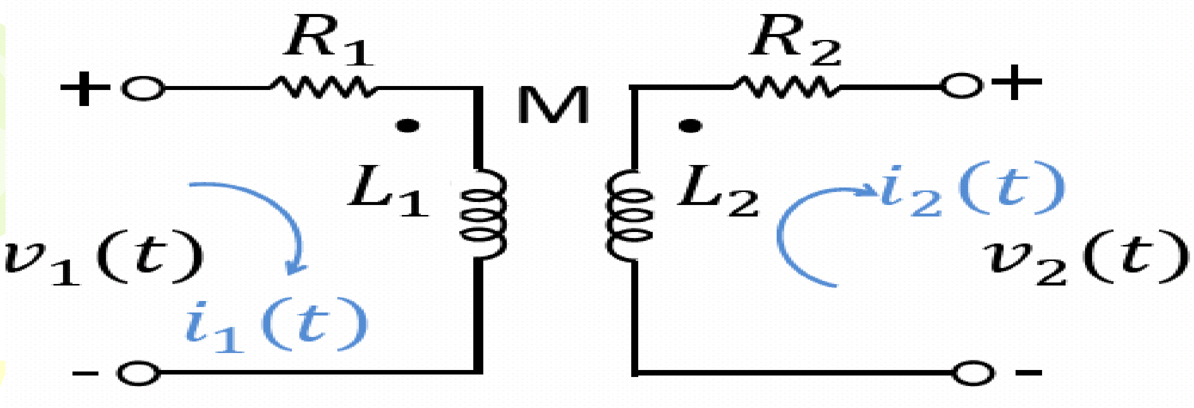




T-model depends on dot convention and current directions







COIL 1

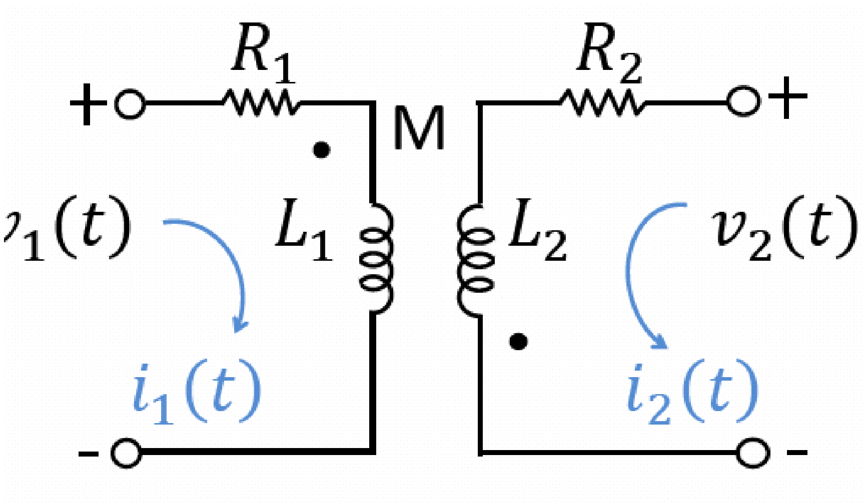
$$V_1 = i_1R_1 + L_1\frac{di_1}{dt} - M\frac{di_2}{dt} + M\frac{di_1}{dt} - M\frac{di_1}{dt}$$

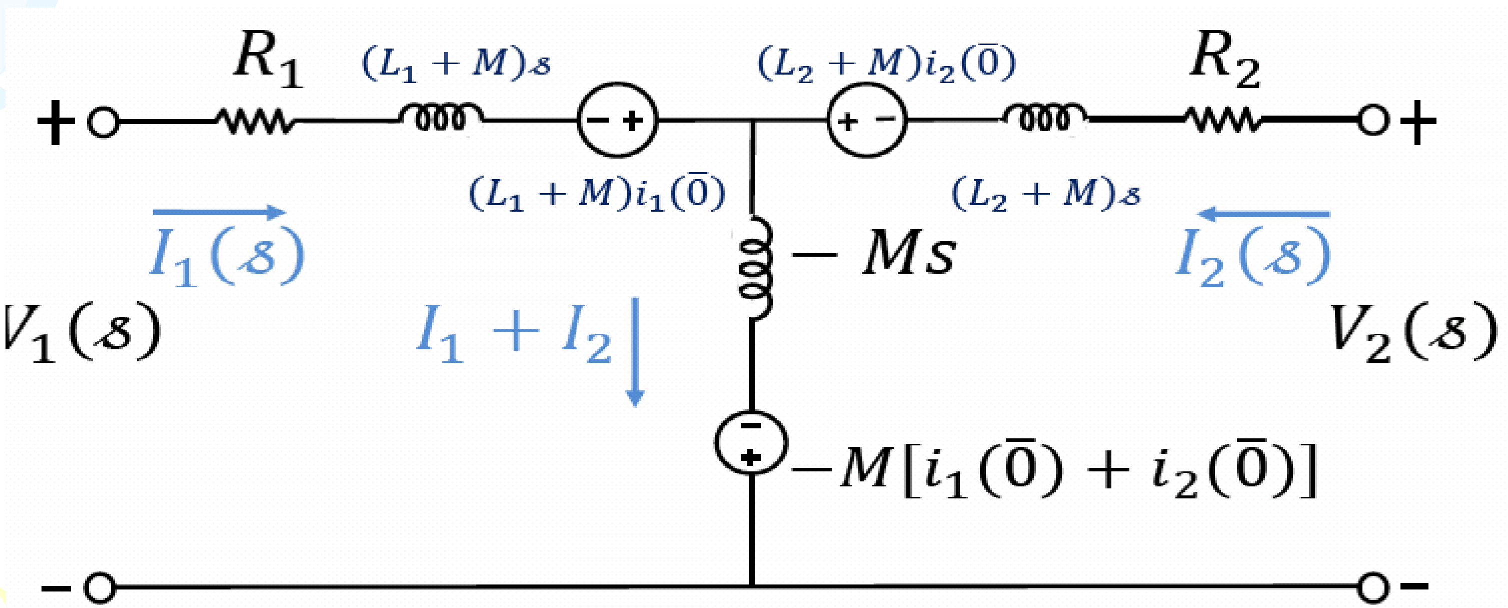
$$V_1 = i_1R_1 + (L_1 - M)\frac{di_1}{dt} + M\frac{d}{dt}(i_1 - i_2)$$

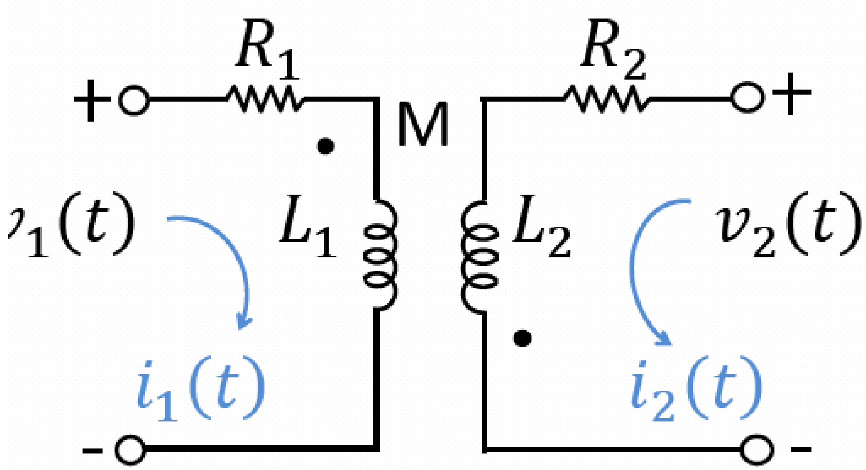
COIL 2

$$V_2 = -i_2R_2 - L_2\frac{di_2}{dt} + M\frac{di_1}{dt} - M\frac{di_2}{dt} + M\frac{di_2}{dt}$$

$$= -i_2R_2 - (L_2 - M)\frac{di_2}{dt} + M\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right)$$


R_1
R_2
M
$v_1(t)$
L_1
L_2
$v_2(t)$
$i_1(t)$
$i_2(t)$


R_1
$(L_1+M)s$
$(L_2+M)i_2(\bar{0})$
R_2
$(L_1+M)i_1(\bar{0})$
$(L_2+M)s$
$I_1(s)$
$-Ms$
$I_2(s)$
$V_1(s)$
I_1+I_2
$V_2(s)$
$-M[i_1(\bar{0})+i_2(\bar{0})]$



COIL 1

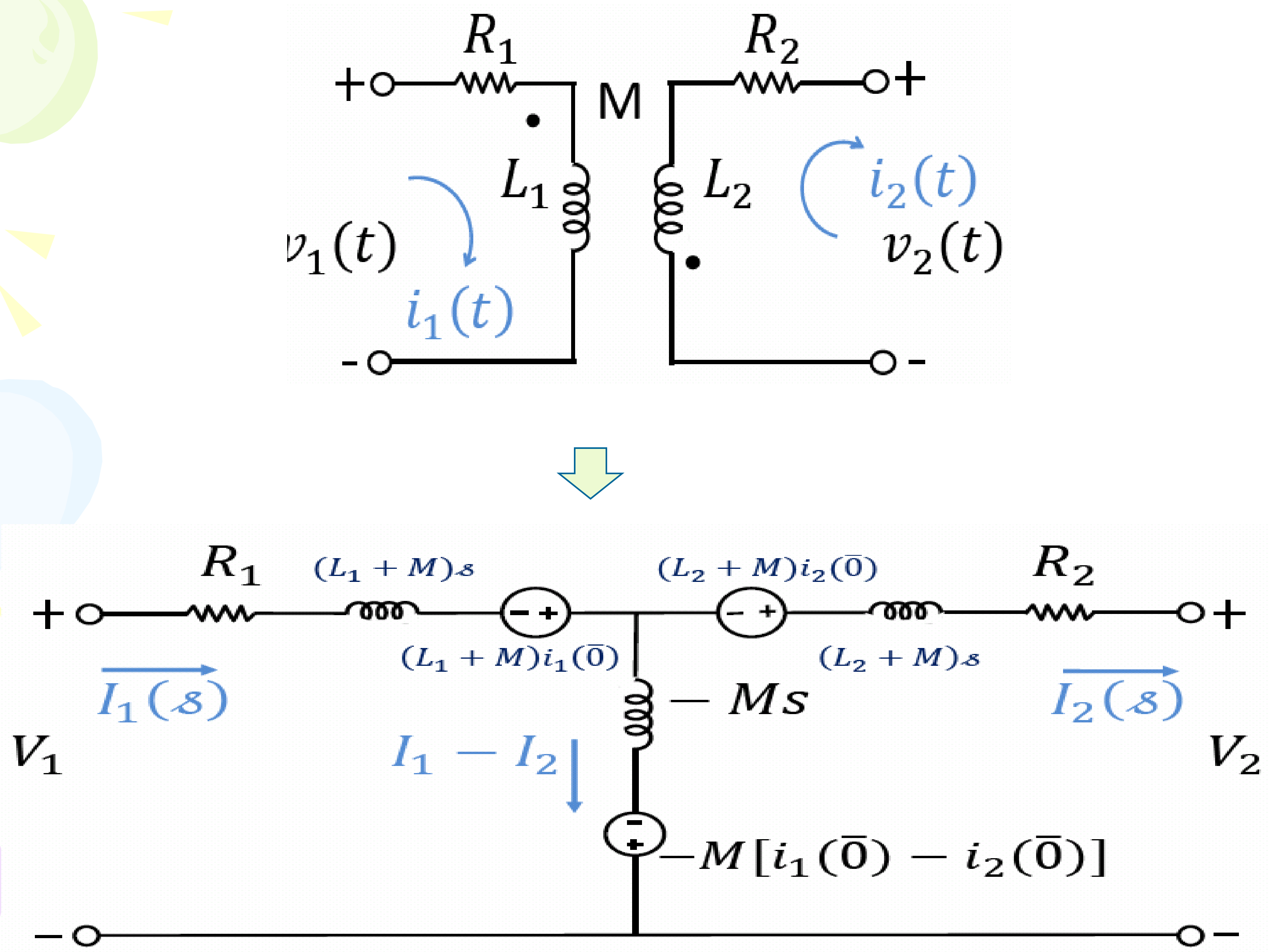
$$V_1 = i_1 R_1 + L_1 \frac{di_1}{dt} - M\frac{di_2}{dt} - M\frac{di_1}{dt} + M\frac{di_1}{dt}$$

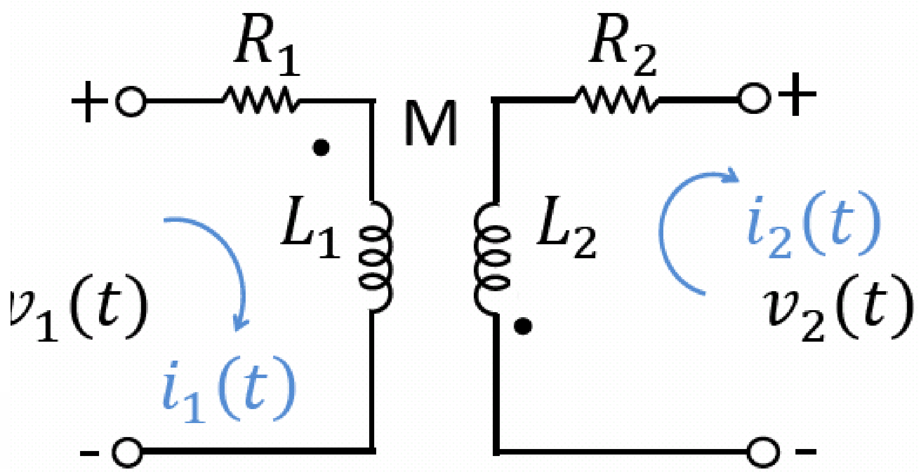
$$= i_1 R_1 + (L_1 + M)\frac{di_1}{dt} - M\frac{d(i_1+i_2)}{dt}$$

COIL 2

$$V_2 = i_2 R_2 + L_2 \frac{di_2}{dt} - M\frac{di_1}{dt} - M\frac{di_2}{dt} + M\frac{di_2}{dt}$$

$$= i_2 R_2 + (L_2 + M)\frac{di_2}{dt} - M\frac{d(i_1+i_2)}{dt}$$


R1
R2
M
L1
L2
i2(t)
v1(t)
v2(t)
i1(t)
R1
(L1 + M)s
(L2 + M)i2(0̄)
R2
(L1 + M)i1(0̄)
(L2 + M)s
I1(s)
− Ms
I2(s)
V1
I1 − I2
V2
−M[i1(0̄) − i2(0̄)]



COIL 1

$$V_1 = i_1 R_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} - M \frac{di_1}{dt} + M \frac{di_1}{dt}$$

$$= i_1 R_1 + (L_1 + M)\frac{di_1}{dt} - M\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right)$$

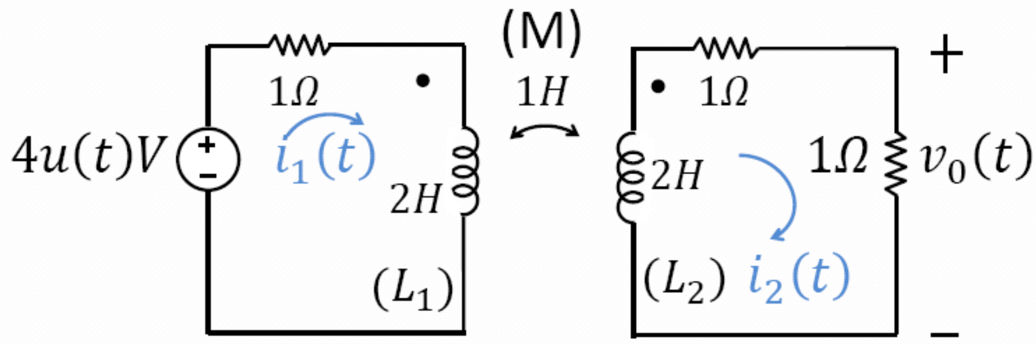
COIL 2

$$V_2 = -i_2 R_2 - L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} + M \frac{di_2}{dt} - M \frac{di_2}{dt}$$

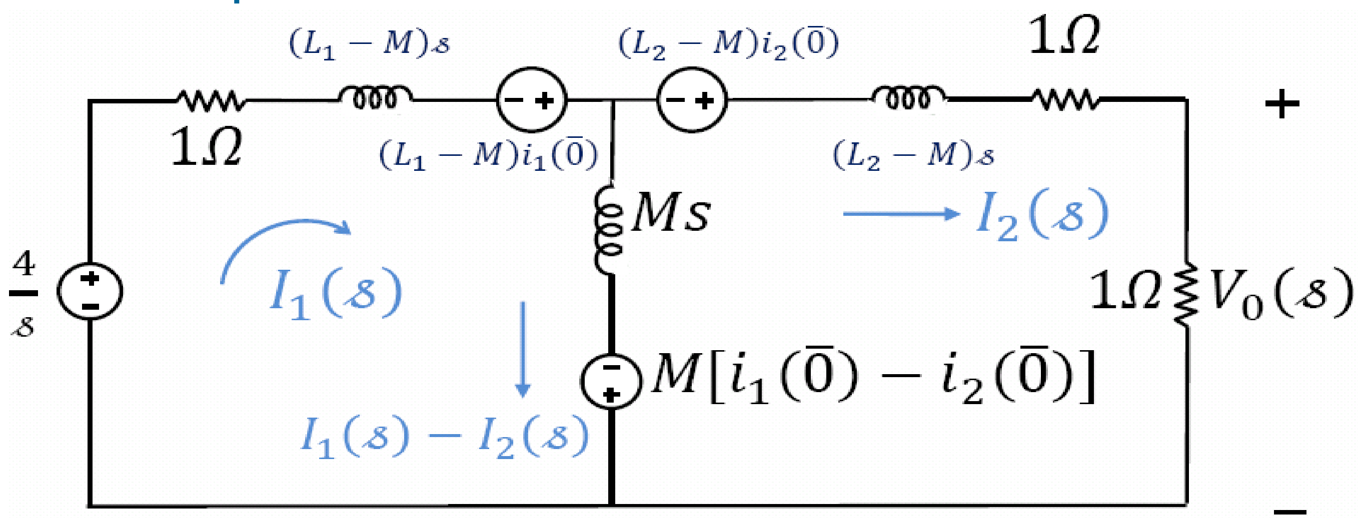
$$= -i_2 R_2 - (L_2 + M)\frac{di_2}{dt} - M \frac{d(i_1 - i_2)}{dt}$$

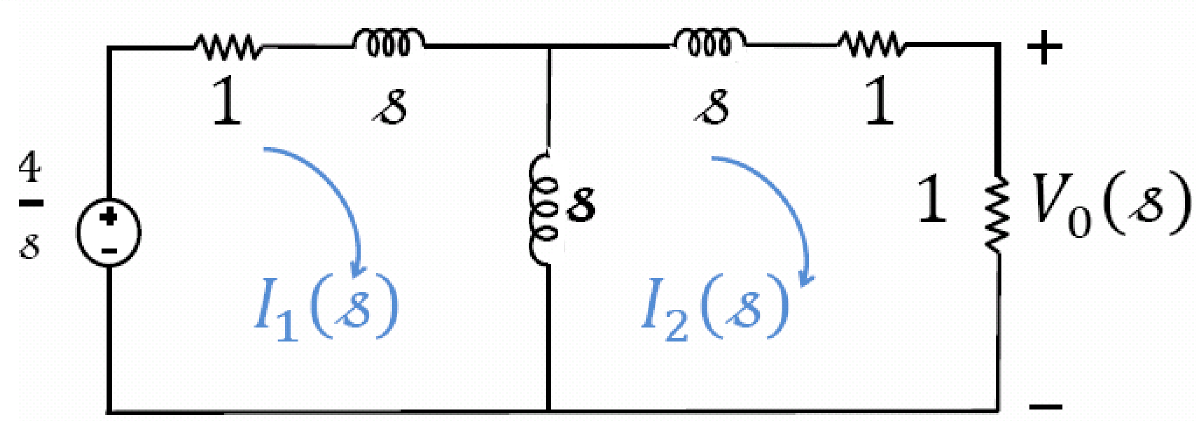
Example 8

Find $v_o(t)$ for $t>0$, given that coupled coils had no initial charge.



The s-domain T-equivalent is





Mesh Equations:

$$\frac{4}{s} = (s + 1 + s)I_1(s) - sI_2(s)$$

$$(2 + 2s)I_2(s) - sI_1(s) = 0$$

$$\Rightarrow I_1(s) = \frac{2(s+1)}{s} I_2(s)$$

Substitution gives

$$\frac{4}{s} = (2s + 1)I_1(s) - sI_2(s)$$

$$= (2s+1)\left[\frac{2(s+1)}{s} I_2(s)\right] - sI_2(s)$$

$$\Rightarrow 4 = [2(s+1)(2s+1) - s^2]I_2(s)$$

$$\Rightarrow I_2(s) = \frac{4}{2[2s^2+3s+1]-s^2} = \frac{4}{3s^2+6s+2} = \frac{4/3}{s^2+2s+0.667}$$

Poles are

$$\frac{-2 \pm \sqrt{4-4 \text{ x } 0.667}}{2} = -0.423, -1.5773$$

$$\therefore I_2(s) = \frac{K_1}{s+0.423} + \frac{K_2}{s+1.5773}$$

$$\Rightarrow K_1 = (s+0.423).I_2(s)|_{s=-0.423} = 1.155$$

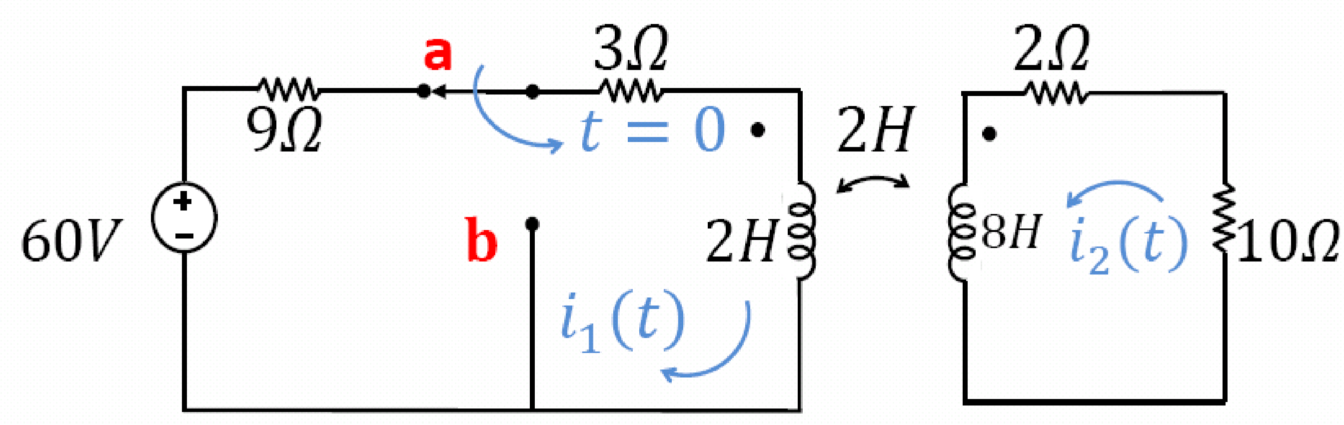
$$K_2 = (s+1.5773).I_2(s)|_{s=-1.5773} = -1.155$$

$$\Rightarrow i_2(t) = [1.155e^{-0.423t} - 1.155e^{-1.5773t}]u(t)A$$

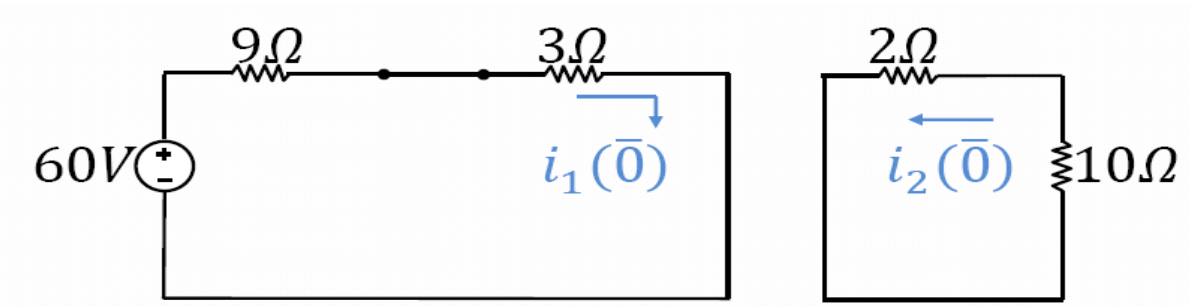
$$\boxed{v_o(t) = i_2(t) \times 1 = (1.155e^{-0.423t} - 1.155e^{-1.5773t})u(t) \quad V}$$

Example 9

The switch has been in position “a” for a long time before it is moved to “b” at t = 0. Find for $i_2(t)$ for $t>0$.

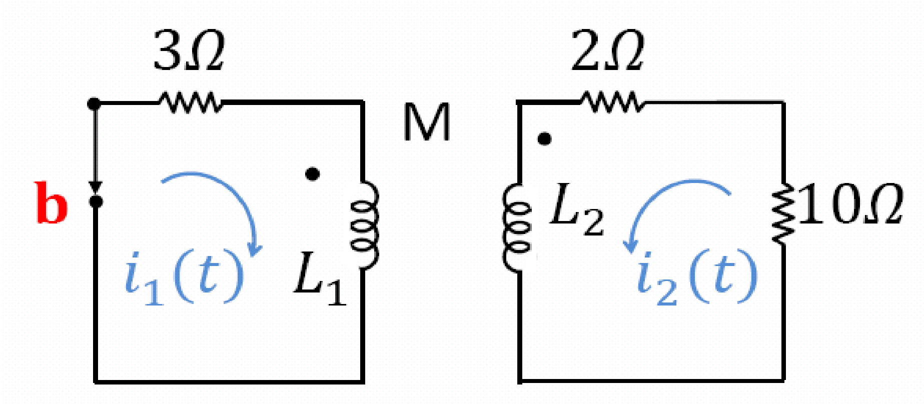


For t<0, (switch in position “**a”**)

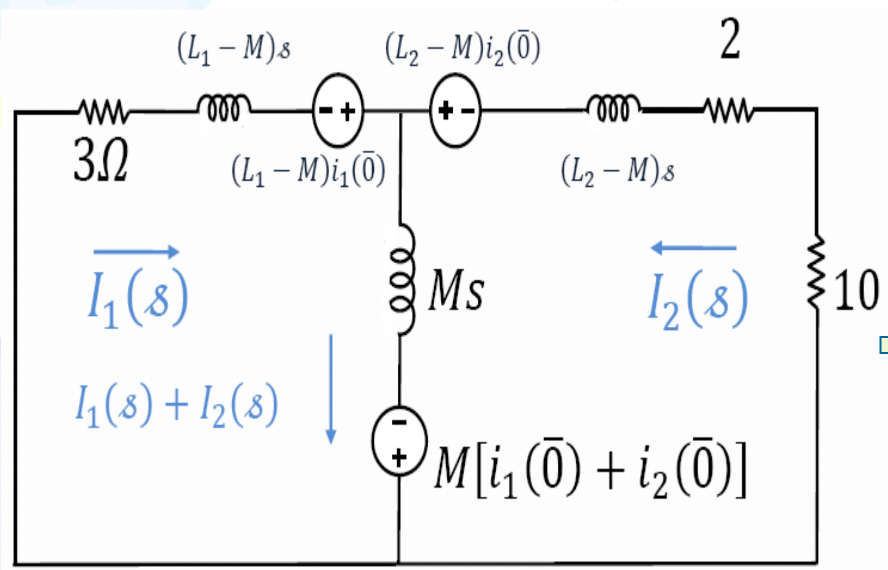


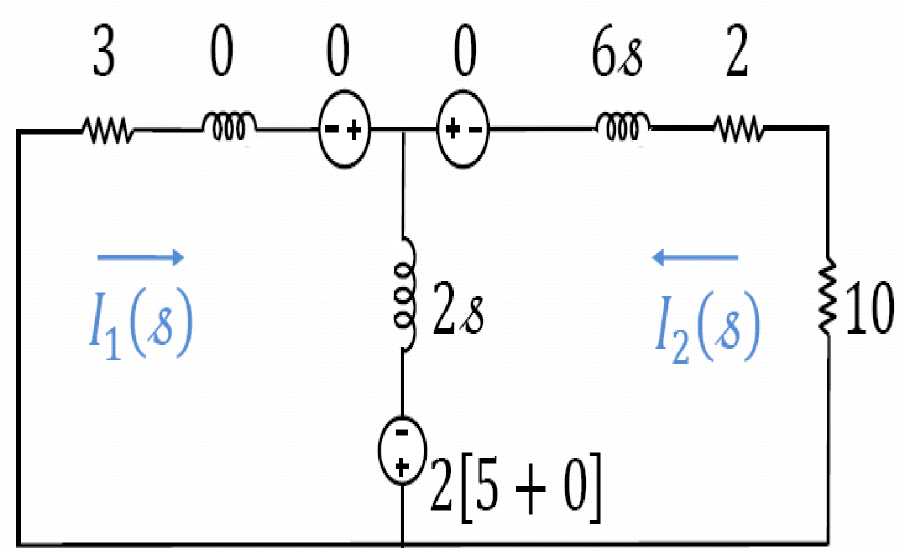
$$\Rightarrow i_1(\bar{0}) = \frac{60V}{(9+3)\Omega} = 5A \qquad i_2(\bar{0}) = 0$$

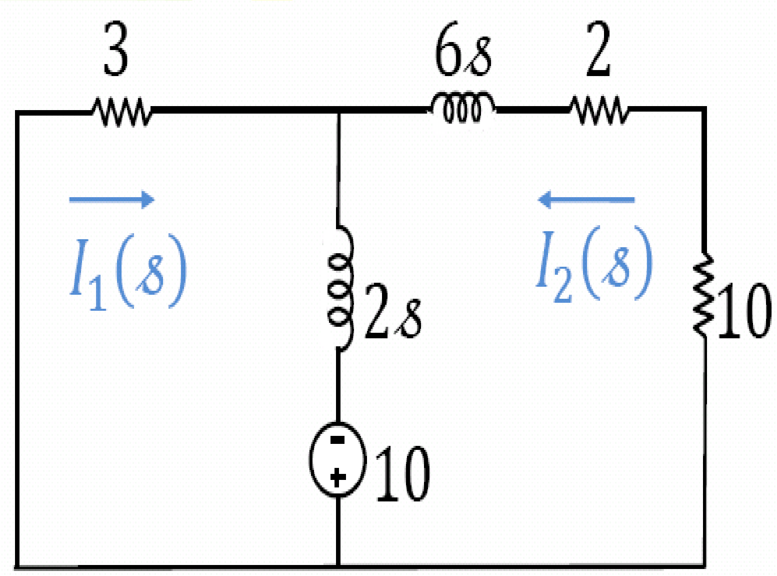
For t≥0, (switch in position "**b**")



The s-domain T-equivalent is







Mesh Equations:

$$(3+2s)I_1(s)+2sI_2(s)=10$$

$$(12+8s)I_2(s)+2sI_1(s)=10$$

$$\Rightarrow I_1(s)=\frac{10-(12+8s)I_2(s)}{2s}$$

$$\therefore (3+2s)\left[\frac{10-(12+8s)I_2}{2s}\right]+2sI_2=10$$

$$\Rightarrow I_2=\frac{30}{12s^2+48s+36}=\frac{30/12}{s^2+4s+3}=\frac{1.25}{s+1}-\frac{1.25}{s+3}$$

$$\Rightarrow i_2(t)=1.25\{e^{-t}-e^{-3t}\}.u(t)A$$

Transfer Functions

- The transfer function of a circuit, denoted as H(s), describes how an output behaves with respect to an input source. It specifies the transfer from the input to the output in the s-domain, assuming no initial energy.

- It is defined as the ratio of the output response Y(s) to the input response X(s), with zero initial conditions

$$\boxed{H(s) = \frac{Y(s)}{X(s)}}$$

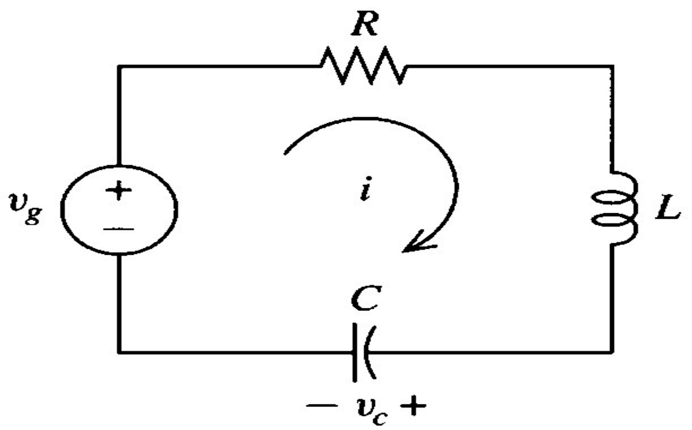
- The inverse Laplace transform is h(t) which is the unit impulse response of the circuit.

- The transfer function depends on what we define as input and output. There are four types of transfer functions:

  1. $H(s)$ = voltage gain = $V_0(s)/V_i(s)$
  2. $H(s)$ = Current gain = $I_0(s)/I_i(s)$
  3. $H(s)$ = Impedance = $V(s)/I(s)$
  4. $H(s)$ = Admittance = $I(s)/V(s)$

Example 10

Consider the RLC circuit with source voltage as the input and the current as the output Find the transfer function.



Mesh Equation:

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt = v_g(t)$$

Taking Laplace transform on both sides gives

$$RI(s) + LsI(s) + \frac{1}{Cs}I(s) = V_g(s)$$

So the transfer function is

$$H(s) = \frac{I(s)}{V_g(s)} = \frac{1}{R + Ls + \frac{1}{Cs}} = \frac{Cs}{LCs^2 + RCs + 1}$$

Example 11:

The output of a circuit is $y(t)=10e^{-t}\cos 4t$ when the input is $x(t)=e^{-t}u(t)$. Find the transfer function of the system and its impulse response.

Transform y(t) and x(t) into s-domain

$$X(s)=\frac{1}{s+1}, \qquad Y(s)=\frac{10(s+1)}{(s+1)^2+16}$$

Apply H(s)=Y(s)/X(s), we get

$$H(s)=\frac{Y(s)}{X(s)} = \frac{10(s+1)^2}{(s+1)^2+16} = 10-40\frac{4}{(s+1)^2+16}$$

Apply inverse transform to H(s), we get

$$h(t)=10\delta(t)-40e^{-t}\sin(4t)u(t)$$

Example 12

The transfer function of a circuit is

$$H(s) = \frac{2s}{s+6}$$

Find the output y(t) due to the input $5e^{-3t}u(t)$ and its impulse response.

$$\text{If } x(t) = 5e^{-3t}\, u(t) \qquad \text{then } X(s) = \frac{5}{s+3}.$$

$$Y(s) = H(s)X(s) = \frac{10s}{(s+3)(s+6)} = \frac{-10}{s+3} + \frac{20}{s+6}$$

$$y(t) = (-10e^{-3t} + 20e^{-6t})u(t)$$

$$H(s) = \frac{2s}{(s+6)} = \frac{2(s+6-6)}{s+6} = 2 - \frac{12}{s+6}$$

$$h(t) = [2\delta(t)\text{-}12e^{-6t}]u(t)$$

Summary:

Laplace transform method allows us to consider circuits described by sets of linear integro-differential equations, and more importantly to ***solve these equations as algebraic equations in s-domain.***

Steps:

1) If initial conditions are not given, find the values of all capacitor voltages and inductor currents at $t = 0^-$.

2) For $t>0$, transform the circuit from the time domain to the s-domain. Draw the s-domain circuit with all initial conditions properly represented.

3) Solve the circuit using any circuit analysis technique with which we are familiar.

4) Take the inverse transform of the solution and thus obtain the solution in the time domain.

Mutual Inductance: equivalent circuit.

Transfer Functions: describes how an output behaves with respect to an input source

Exercises (Problems in Chapter 16 of the textbook)

Try the following exercises:

16.3, **ans:** $i(t) = [16 + 104e^{-15t}]u(t)$ mA

16.5, **ans** $i_o(t) = \left(e^{-2t} - \frac{2}{\sqrt{7}}e^{-0.5t}\sin\left(\frac{\sqrt{7}}{2}t\right)\right)u(t)A$

16.16, **ans:** $i_o(t) = [2.1473e^{-2t} - 5.144e^{-.5t}\cos(1.25t) + 9.582e^{-0.5t}\sin(1.25t)]u(t)$ A

16.21, **ans:** $v_o(t) = \frac{20}{3}\left[1 - e^{-t}\cos 0.7071t - 1.414e^{-t}\sin 0.7071t\right]u(t)V$

16.27, **ans:** $I_1(s) = \frac{20(s+1)}{(s+3)(3s^2+4s+1)}$ $I_2(s) = \frac{10(s+1)}{(s+3)(3s^2+4s+1)}$

16.37 **ans:** $\frac{I_1}{V_s} = \frac{s^2-3}{3s^2+2s-9}$ $\frac{I_2}{V_x} = \frac{-3}{2s}$