Discrete Mathematics

MH1812

Topic 9.1 - Functions I

Dr. Wang Huaxiong

Topic Overview

What's in store…

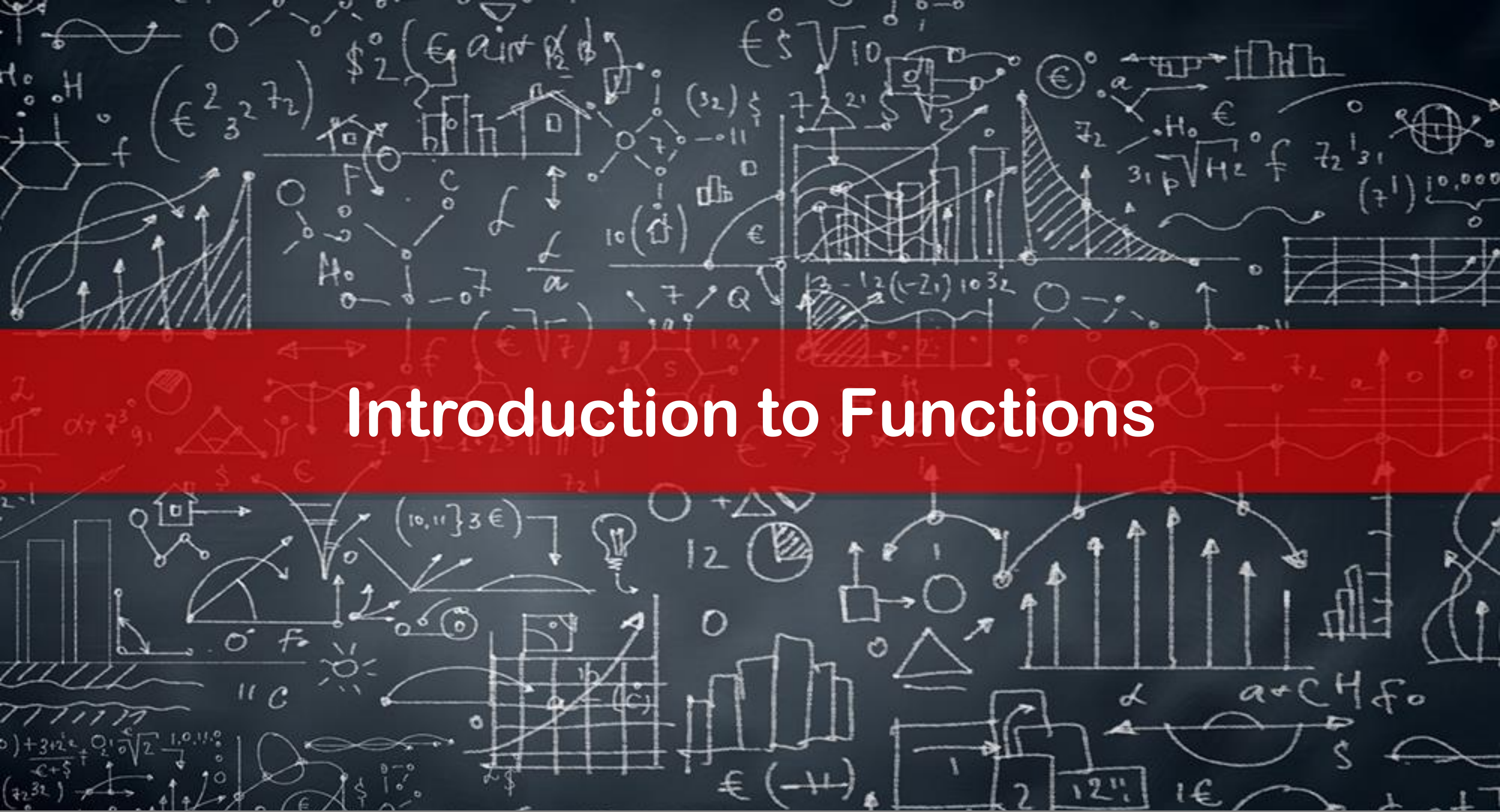
Introduction to Functions

Injectivity

Surjectivity

By the end of this lesson, you should be able to…

- **Explain the concepts of functions.**
- **Explain the concepts of injective functions.**
- **Explain the concepts of surjective functions.**



Introduction to Functions

Introduction to Functions: Definition

Let X and Y be sets. A **function** f from X to Y is a rule that assigns every element x of X to a unique y in Y. We write $f: X \to Y$ and $f(x) = y$.

$$(\forall x \in X\ \exists y \in Y, y = f(x)) \wedge (\forall x_1, x_2 \in X, f(x_1) \neq f(x_2) \to x_1 \neq x_2)$$

X =	Domain
Y =	Codomain
y =	**Image** of x under f
x =	**Preimage** of y under f
Range =	Subset of Y with preimages

Introduction to Functions: Example 1

$$(\forall x \in X\ \exists y \in Y, y = f(x)) \wedge (\forall x_1, x_2 \in X, f(x_1) \neq f(x_2) \rightarrow x_1 \neq x_2)$$

Domain $X = \{a,b,c\}$

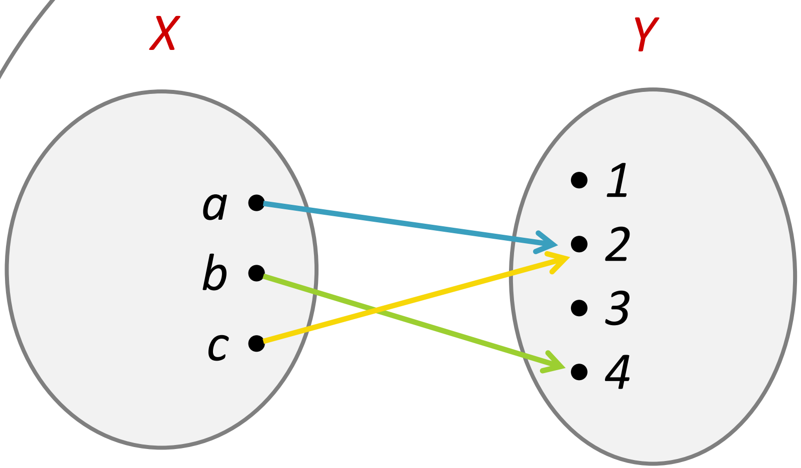
Codomain $Y = \{1,2,3,4\}$

$f = \{(a,2), (b,4), (c,2)\}$

Preimage of 2 is $\{a,c\}$

Range = $\{2,4\}$

Arrow Diagram of f:



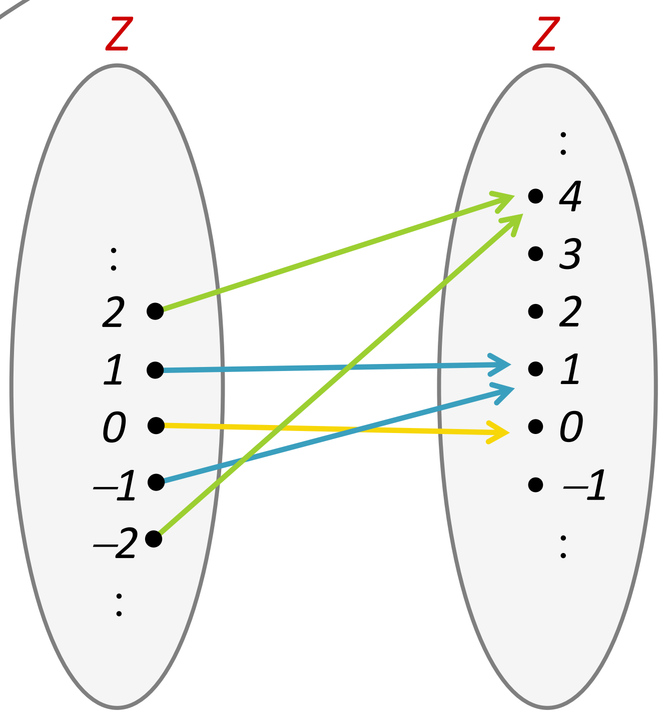
Introduction to Functions: Example 2

Let f be the function from Z to Z that assigns the square of an integer to this integer.

Then $f: Z \to Z, f(x) = x^2$

Domain and codomain of f: Z

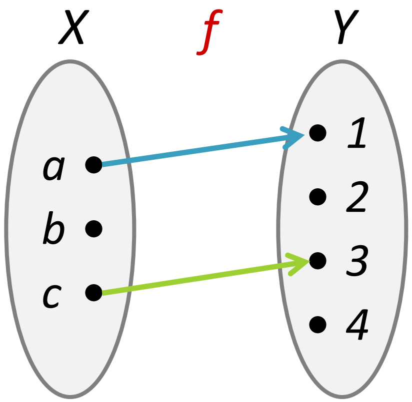
Range $(f) = \{0, 1, 4, 9, 16, 25, ...\}$



Introduction to Functions: Functions vs. Non-functions

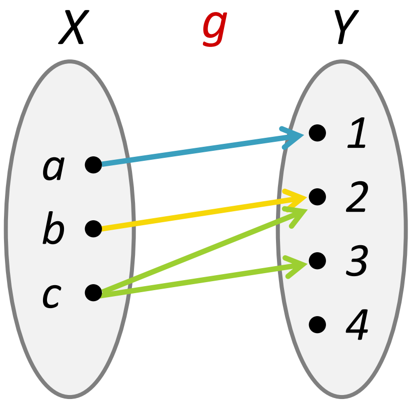
$$(\forall x \in X\ \exists y \in Y, y = f(x)) \land (\forall x_1, x_2 \in X, f(x_1) \neq f(x_2) \rightarrow x_1 \neq x_2)$$

X = {a,b,c} **to** Y = {1,2,3,4}



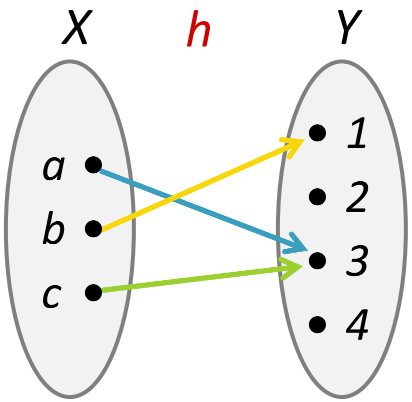
No!

(*b* has no image)



No!

(*c* has two images)

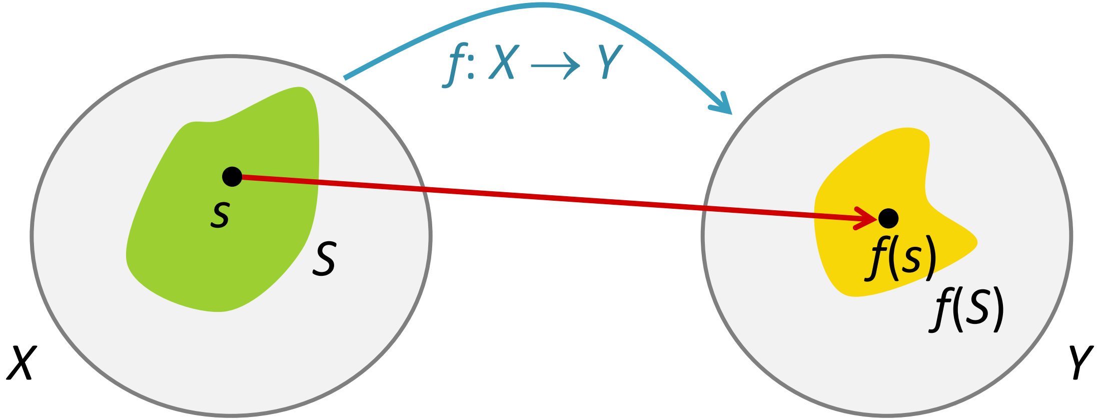


Yes!

(Each element of *X* has exactly one image)

Introduction to Functions: Image of a Set

Let f be a function from X to Y and $S \subseteq X$. The **image of S** is the subset of Y that consists of the images of the elements of S: $f(S) = \{f(s) \mid s \in S\}$.



Injectivity

Injectivity: One-to-one Function

A function f is one-to-one (or injective), if and only if $f(x) = f(y)$ implies $x = y$ for all x and y in the domain of f.

In words…

"All elements in the domain of f have different images".

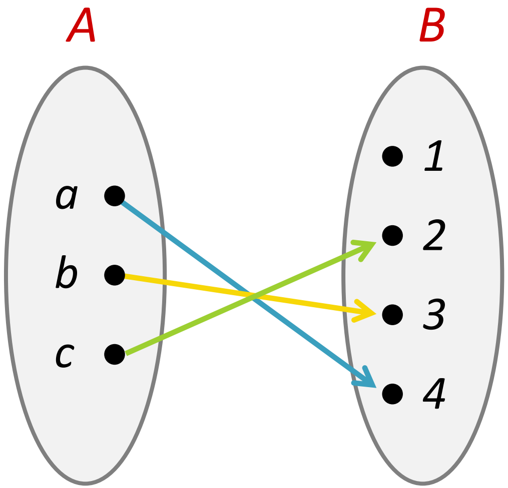
Mathematical Description

$$f: A \rightarrow B \text{ is one-to-one} \Leftrightarrow \forall x_1, x_2 \in A \ (f(x_1) = f(x_2) \Longrightarrow x_1 = x_2)$$

or

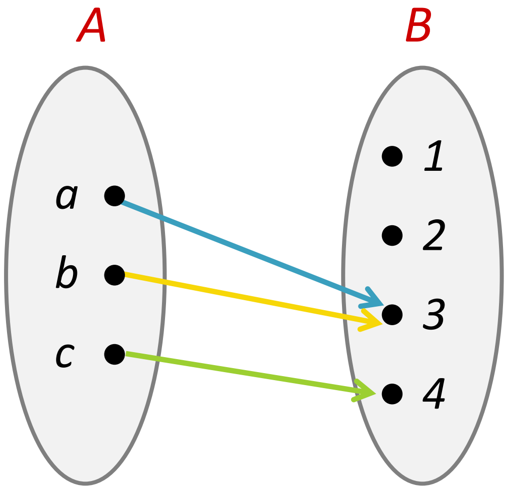
$$f: A \rightarrow B \text{ is one-to-one} \Leftrightarrow \forall x_1, x_2 \in A \ (x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2))$$

Injectivity: One-to-one Example



One-to-one

(All elements in A have a different image)

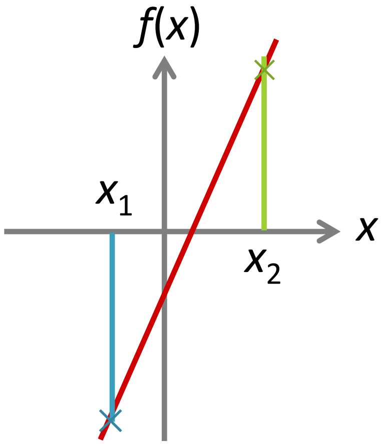


Not one-to-one

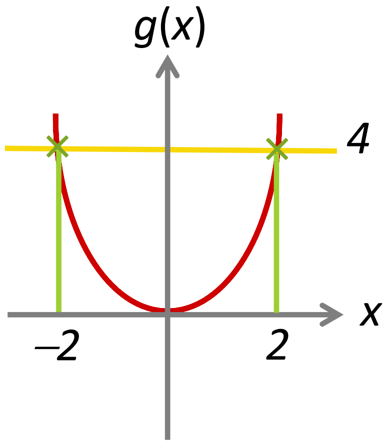
(a and b have the same image)

Injectivity: One-to-one Example

$f: R \to R, f(x) = 4x - 1$



$g: R \to R, g(x) = x^2$



Does each element in R have a different image?

Yes!

No!

To show $\forall x_1, x_2 \in R\ (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$,

take some $x_1, x_2 \in R$ **with** $f(x_1) = f(x_2)$.

Then $4x_1 - 1 = 4x_2 - 1 \Rightarrow 4x_1 = 4x_2 \Rightarrow x_1 = x_2$.

Take $x_1 = 2$ **and** $x_2 = -2$.

Then $g(x_1) = 2^2 = 4 = g(x_2)$ **and** $x_1 \neq x_2$.

Surjectivity

Surjectivity: Onto Function

$f(x)$

A function f from X to Y is **onto** (or **surjective**), if and only if for every element $y \in Y$ there is an element $x \in X$ with $f(x) = y$.

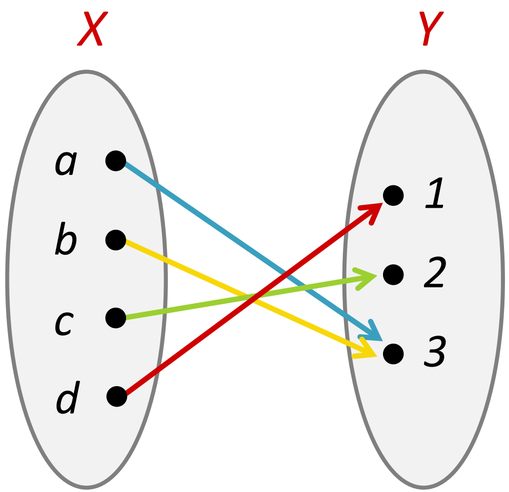
In words…

"Each element in the codomain of f has a preimage".

Mathematical Description

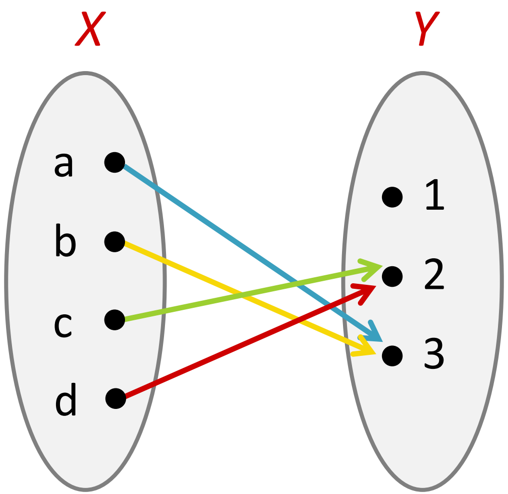
$$f: X \rightarrow Y \text{ is onto} \Leftrightarrow \forall y \in Y \; \exists x \in X, \; f(x) = y$$

Surjectivity: Onto Example



Onto

(All elements in *Y* have a preimage)



Not onto

(*1* has no preimage)

Surjectivity: Onto Example

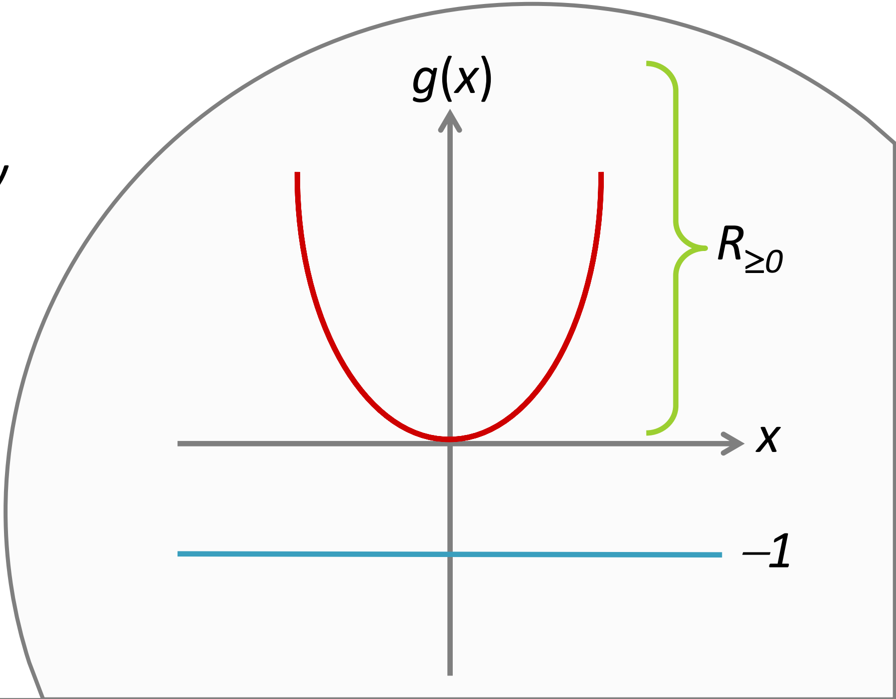
$g: R \to R,\ g(x) = x^2$

Does each element in R have a preimage?

No!

- **To show** $\exists y \in R$ **such that** $\forall x \in R\ g(x) \neq y$
- **Take** $y = -1$
- **Then any** $x \in R$ **holds** $g(x) = x^2 \neq -1 = y$

But $g: R \to R_{\geq 0}$, $g(x) = x^2$ **(where** $R_{\geq 0}$ **denotes the set of non-negative real numbers) is onto!**



Topic Summary

Let's recap…

- Functions:
  - Domain
  - Codomain
  - Image
  - Preimage
  - Range
- Injective functions (one-to-one)
- Surjective functions (onto)

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Topic 9.2 - Functions II
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Topic Overview

What's in store…

B ijectivity

I dentity and Inverse

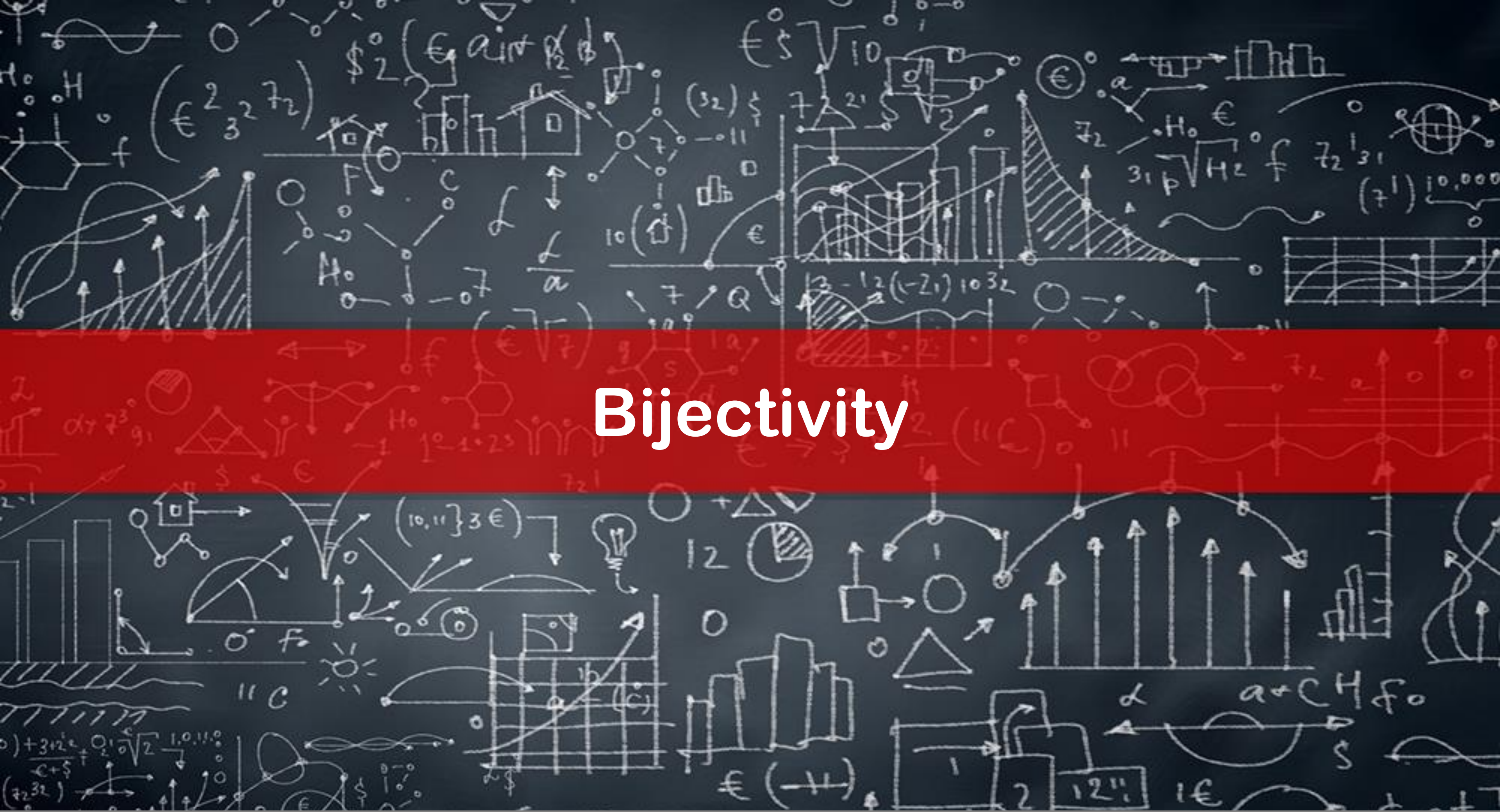
C omposition and Properties

By the end of this lesson, you should be able to…

- Explain the concepts of bijective functions.

- Explain the concepts of identity and inverse functions.

- Explain the composition of functions.



Bijectivity

Bijectivity: One-to-one Correspondence

A function f is a one-to-one correspondence (or bijection), if and only if it is both one-to-one and onto.

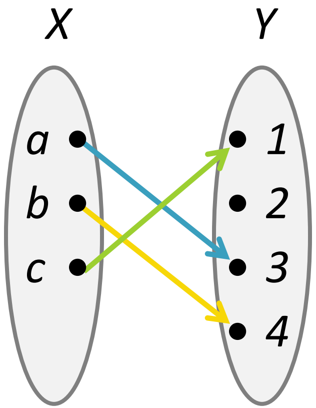
In words…

"No element in the codomain of f has two (or more) preimages" (one-to-one)

and

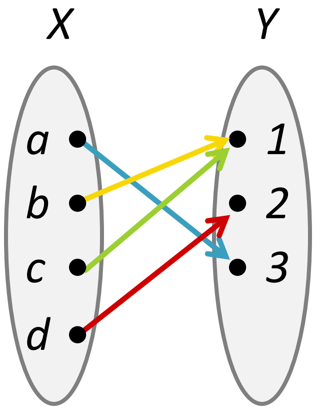
"Each element in the codomain of f has a preimage" (onto)

Bijectivity: Example (Bijection)



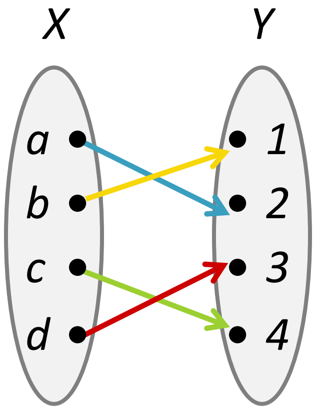
No!

(Not onto as *2* has no preimage)



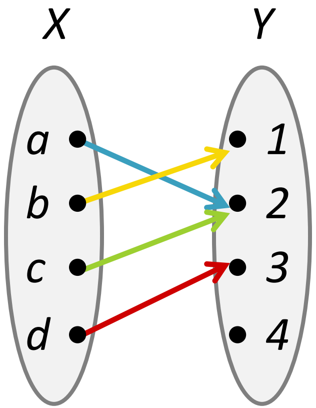
No!

(Not one-to-one as *1* has two preimages)



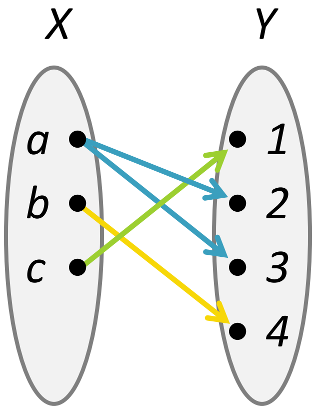
Yes!

(Each element has exactly one preimage)



No!

(Neither one-to-one nor onto)



No!

(Not a function as *a* has two images)

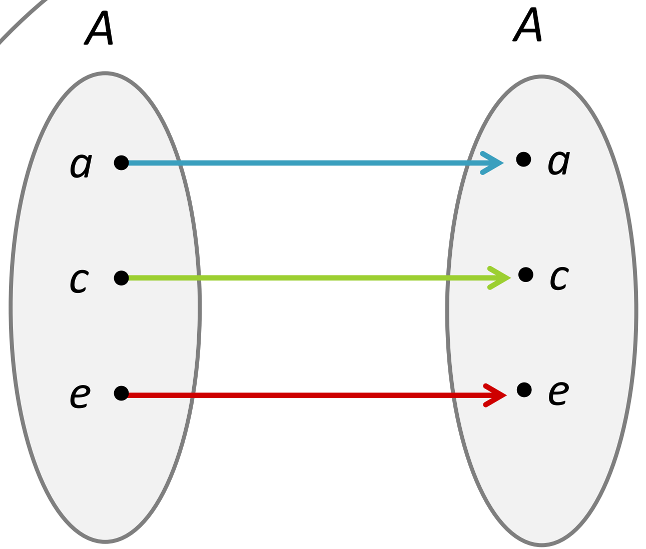
Identity and Inverse

Identity and Inverse: Identity Function

The **identity function** on a set A is defined as:
$i_A: A \to A$, $i_A(x) = x$.

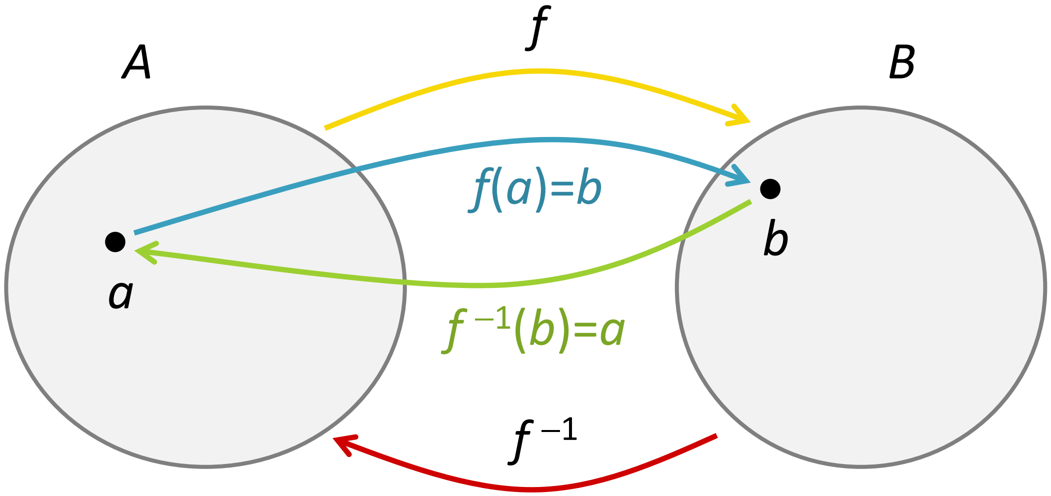
Example

All identity functions are bijections (e.g., for $A = \{a, c, e\}$).



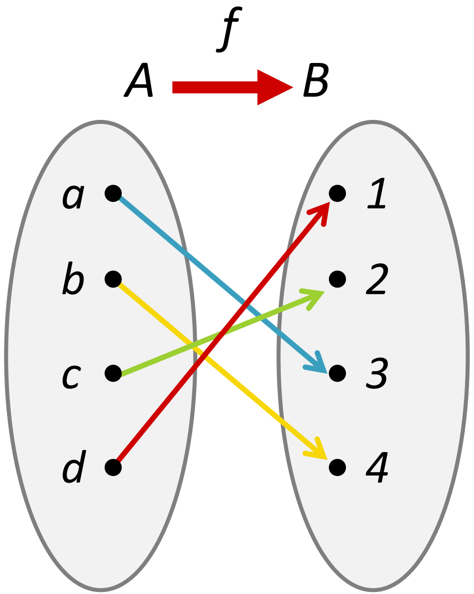
Identity and Inverse: Inverse Function

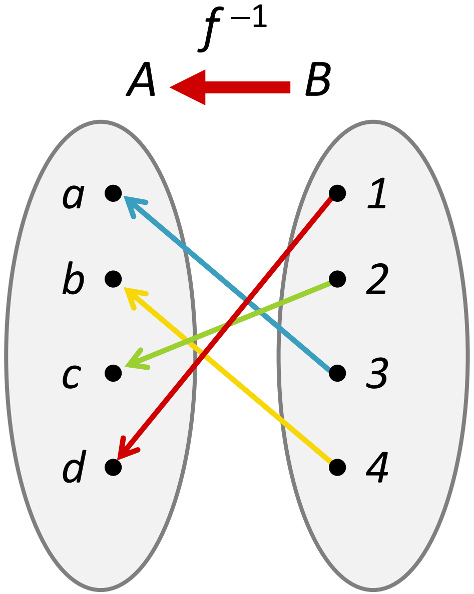
Let $f: A \to B$ be a one-to-one correspondence (bijection). Then the inverse function of f, $f^{-1}: B \to A$, is defined by: $f^{-1}(b)$ = that unique element $a \in A$ such that $f(a) = b$. We say that f is invertible.



Identity and Inverse: Example 1

Find the inverse function of the following function:

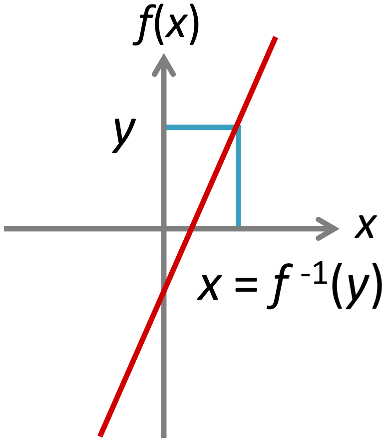




Let f: $A \to B$ be a one-to-one correspondence and f^{-1}: $B \to A$ its inverse.
Then $\forall b \in B\ \forall a \in A\ (f^{-1}(b) = a \Leftrightarrow b = f(a))$.

Identity and Inverse: Example 2

What is the inverse of
$f{:}R \to R$, $f(x) = 4x{-}1$**?**

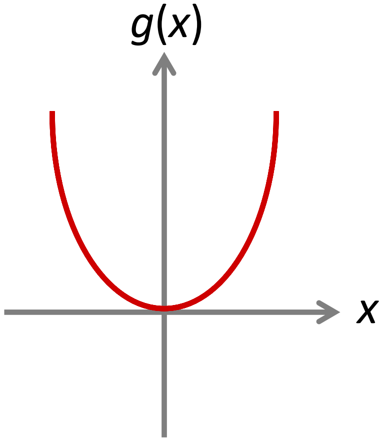


Let $y \in R$.

Calculate x **with** $f(x) = y$: $y = 4x{-}1 \Leftrightarrow (y{+}1)/4 = x$.

Hence $f^{-1}(y) = (y{+}1)/4$.

What is the inverse of
$g{:}R \to R$, $g(x) = x^2$ **?**



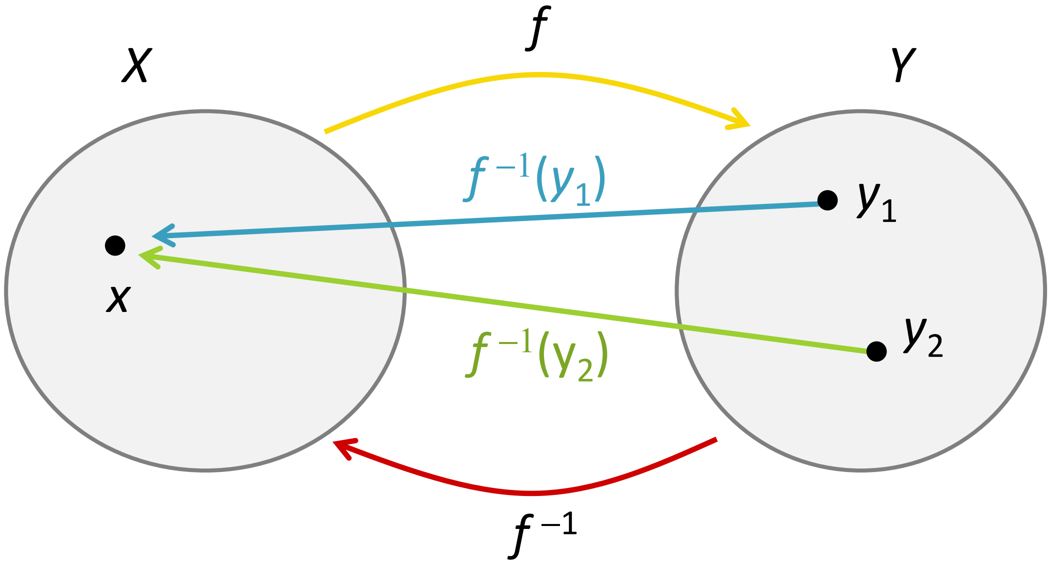
Identity and Inverse: One-to-one Correspondence

Theorem 1: If $f: X \to Y$ is a one-to-one correspondence, then $f^{-1}: Y \to X$ is a one-to-one correspondence.

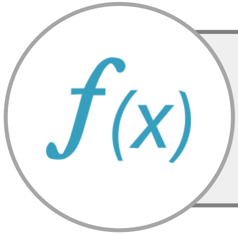
Proof: f^{-1} is one-to-one

Take $y_1, y_2 \in Y$ such that $f^{-1}(y_1) = f^{-1}(y_2) = x$.

Then $f(x) = y_1$ and $f(x) = y_2$, thus $y_1 = y_2$.



Identity and Inverse: One-to-one Correspondence

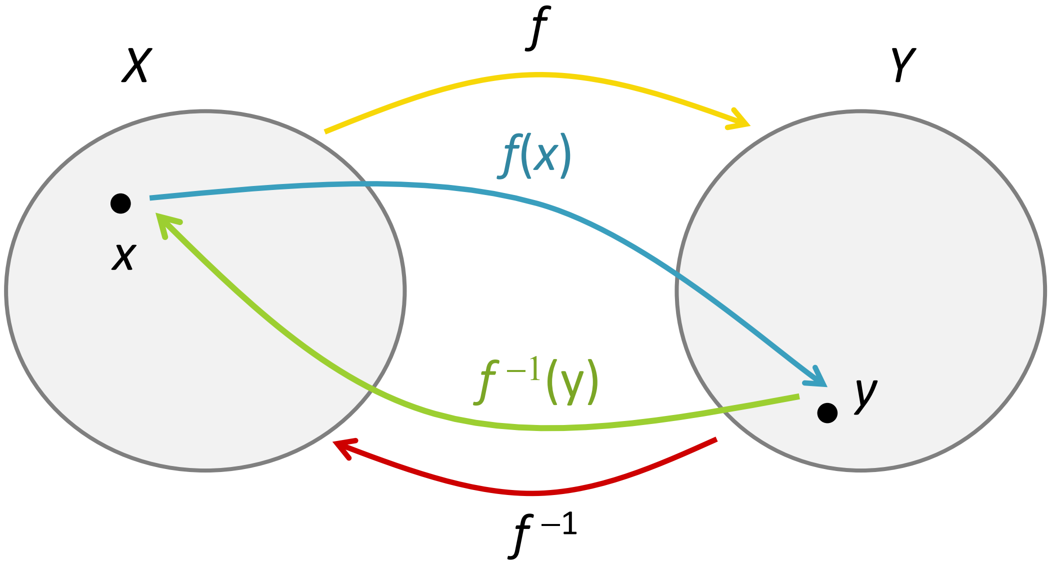


Theorem 1: If $f: X \to Y$ is a one-to-one correspondence, then $f^{-1}: Y \to X$ is a one-to-one correspondence.

Proof: f^{-1} is onto

Take some $x \in X$, and let $y = f(x)$.

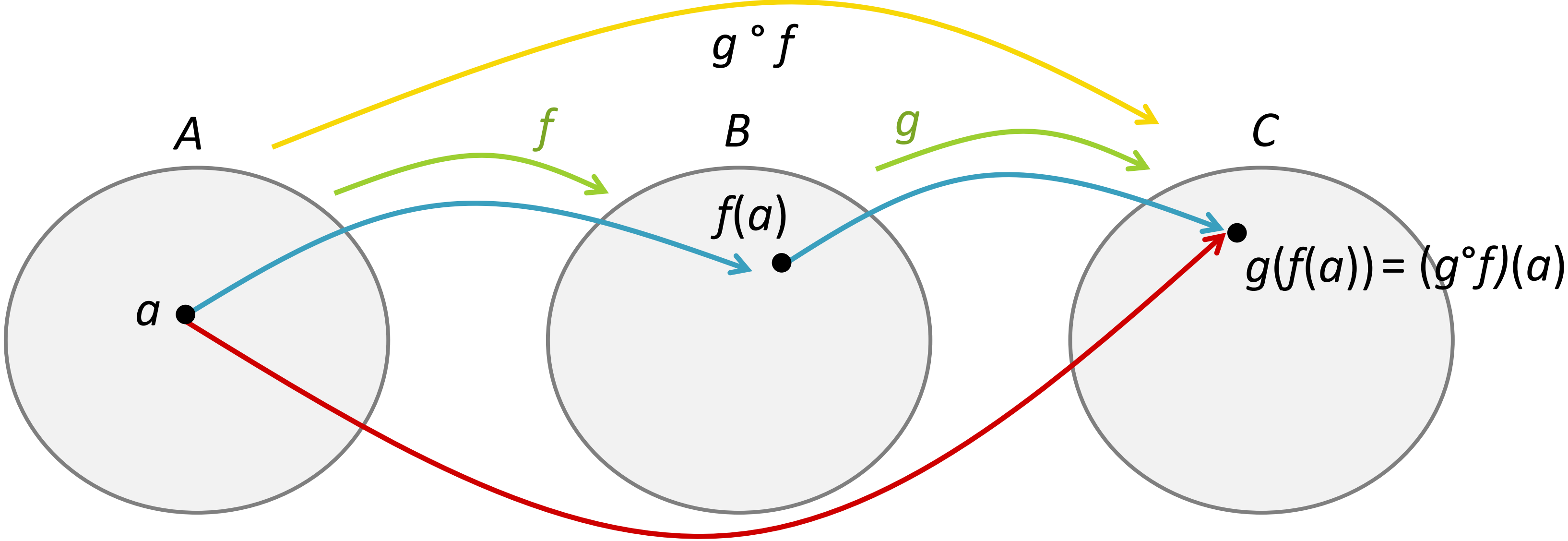
Then $f^{-1}(y) = x$.



Composition and Properties

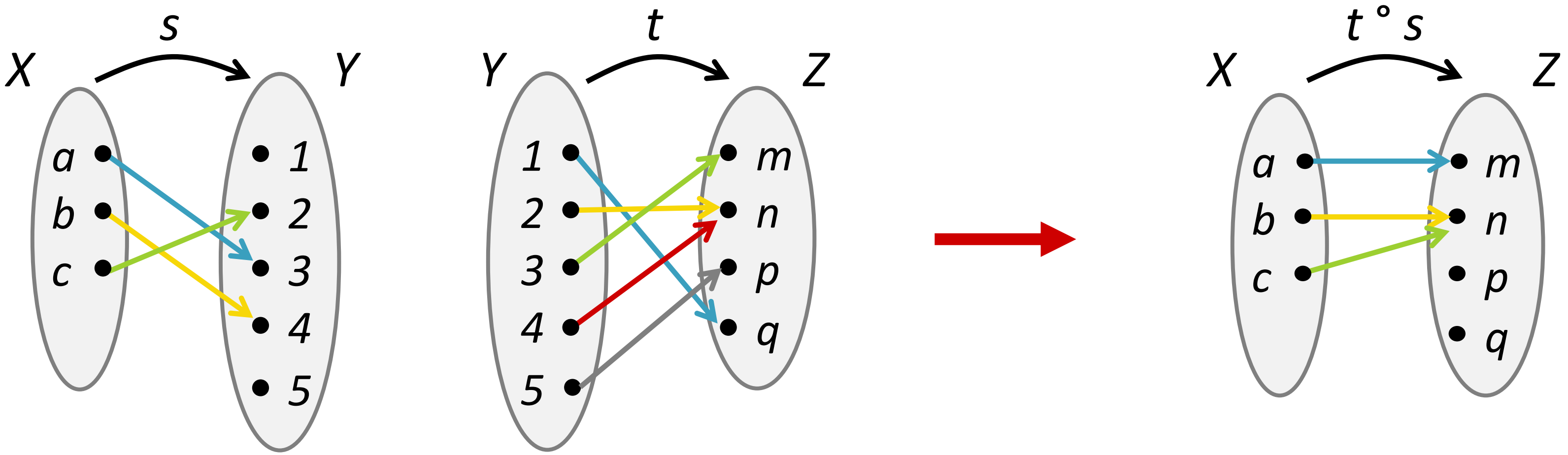
Composition and Properties: Composition of Functions

Let f: $A \to B$ and g: $B \to C$ be functions. The composition of the functions f and g, denoted as $g \circ f$, is defined by: $g \circ f$: $A \to C$, $(g \circ f)(a) = g(f(a))$.



Composition and Properties: Example

Given functions $s: X \to Y$ **and** $t: Y \to Z$**. Find** $t \circ s$ **and** $s \circ t$**.**



Composition and Properties: Example

f: $Z \to Z$, $f(n) = 2n + 3$, g: $Z \to Z$, $g(n) = 3n + 2$

What is $g \circ f$ and $f \circ g$?

$(f \circ g)(n) = f(g(n)) = f(3n + 2) = 2(3n + 2) + 3 = 6n + 7$

$(g \circ f)(n) = g(f(n)) = g(2n + 3) = 3(2n + 3) + 2 = 6n + 11$

$f \circ g \neq g \circ f$ **(No commutativity for the composition of functions!)**

Composition and Properties: One-to-one Propagation

Theorem 2: Let $f: X \to Y$ and $g: Y \to Z$ be both one-to-one functions. Then $g \circ f$ is also one-to-one.

Proof: $\forall x_1, x_2 \in X\ ((g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow x_1 = x_2)$

Suppose $x_1, x_2 \in X$ with $(g \circ f)(x_1) = (g \circ f)(x_2)$.

Then $g(f(x_1)) = g(f(x_2))$.

Since g is one-to-one, it follows $f(x_1) = f(x_2)$.

Since f is one-to-one, it follows $x_1 = x_2$.

Composition and Properties: Onto Propagation

Theorem 3: Let $f: X \to Y$ and $g: Y \to Z$ be both onto functions. Then $g \circ f$ is also onto.

Proof: $\forall z \in Z\ \exists x \in X$ such that $(g \circ f)(x) = z$

Let $z \in Z$.

Since g is onto, $\exists y \in Y$ with $g(y) = z$.

Since f is onto, $\exists x \in X$ with $f(x) = y$.

Hence, with $(g \circ f)(x) = g(f(x)) = g(y) = z$.

Topic Summary

Let’s recap…

- Bijective functions

- Identify and inverse functions

- Composition of functions and their properties

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Topic 9.3 - Functions III

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Topic Overview

What's in store…

C eiling and Floor

P igeonhole Principle

C ountable Sets

$f(x)$

By the end of this lesson, you should be able to...

- Explain what is a ceiling function and floor function.

- Use the pigeonhole principle.

- Explain the difference between a countable set and an uncountable set.

Ceiling and Floor

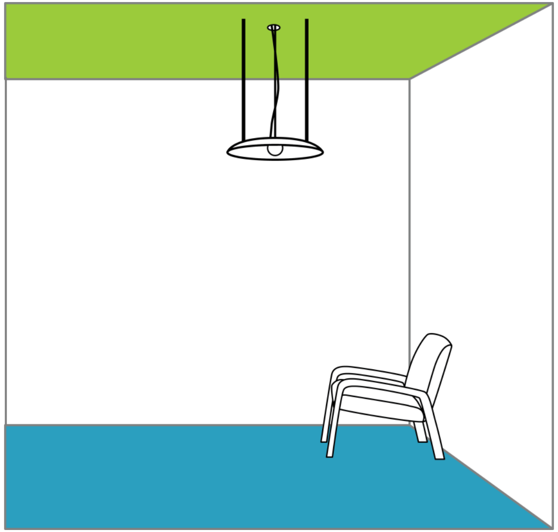
Ceiling and Floor: Definition

The **floor function** assigns to the real number x, the largest integer $\lfloor x \rfloor$ that is less than or equal to x. The **ceiling function** assigns to the real number x, the smallest integer $\lceil x \rceil$ that is greater than or equal to x.

Example

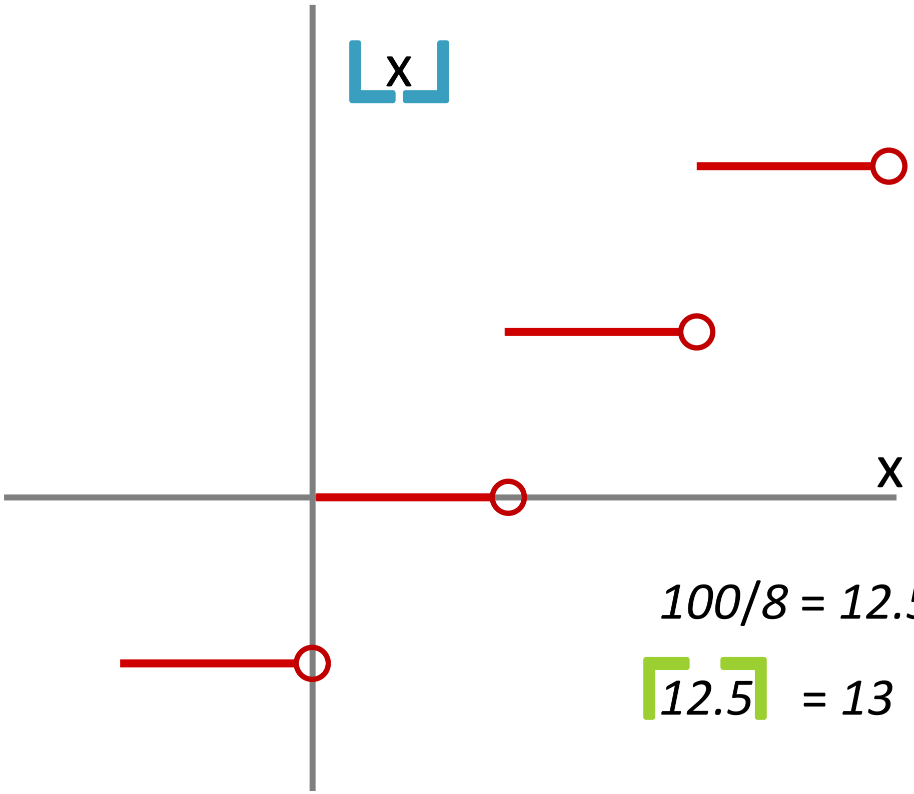
$\lfloor 1/2 \rfloor = 0$ $\qquad$ $\lceil 1/2 \rceil = 1$

$\lfloor -1/2 \rfloor = -1$ $\qquad$ $\lceil -1/2 \rceil = 0$



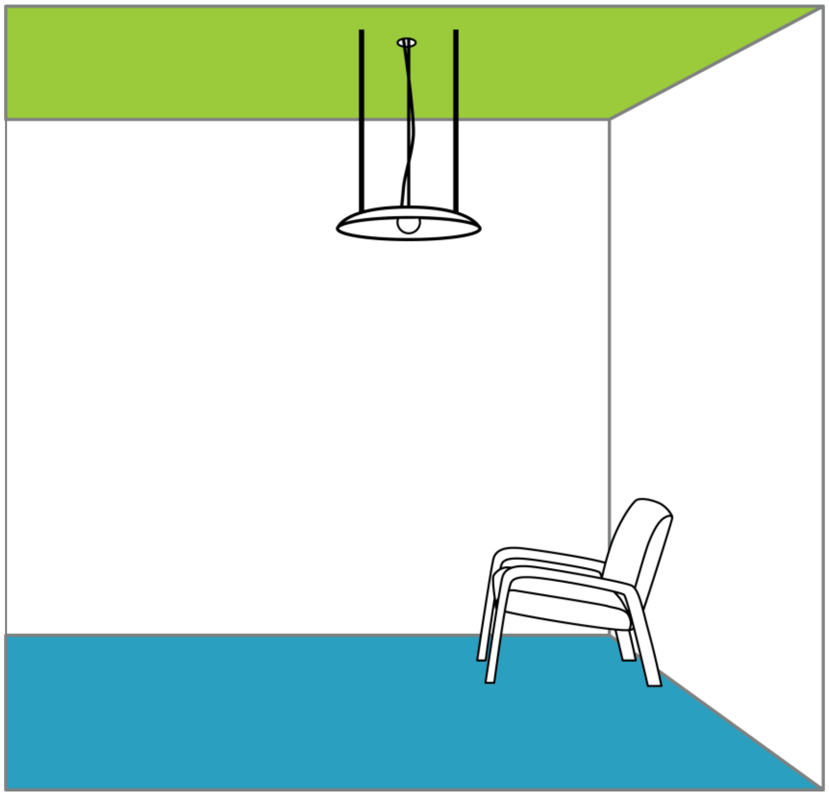
Ceiling and Floor: Example

How many bytes are required to encode *100* bits of data?



$$100/8 = 12.5$$

$$\lceil 12.5 \rceil = 13$$



Pigeonhole Principle

Pigeonhole Principle: Definition

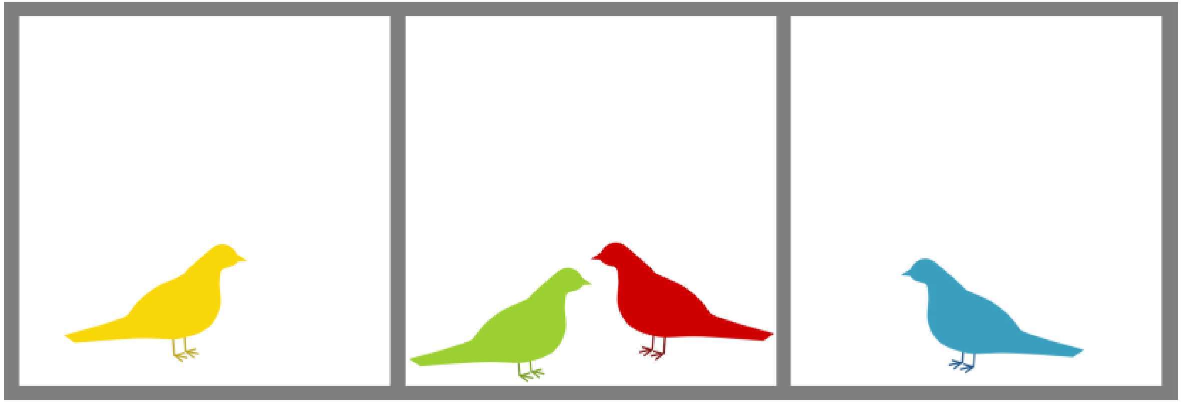
- k pigeonholes, n pigeons, $n > k$
- At least one pigeonhole contains at least two pigeons

Peter Gustav Lejeune Dirichlet (1805 - 1859)

Peter Gustav Lejeune Dirichlet under WikiCommons (PD-US)

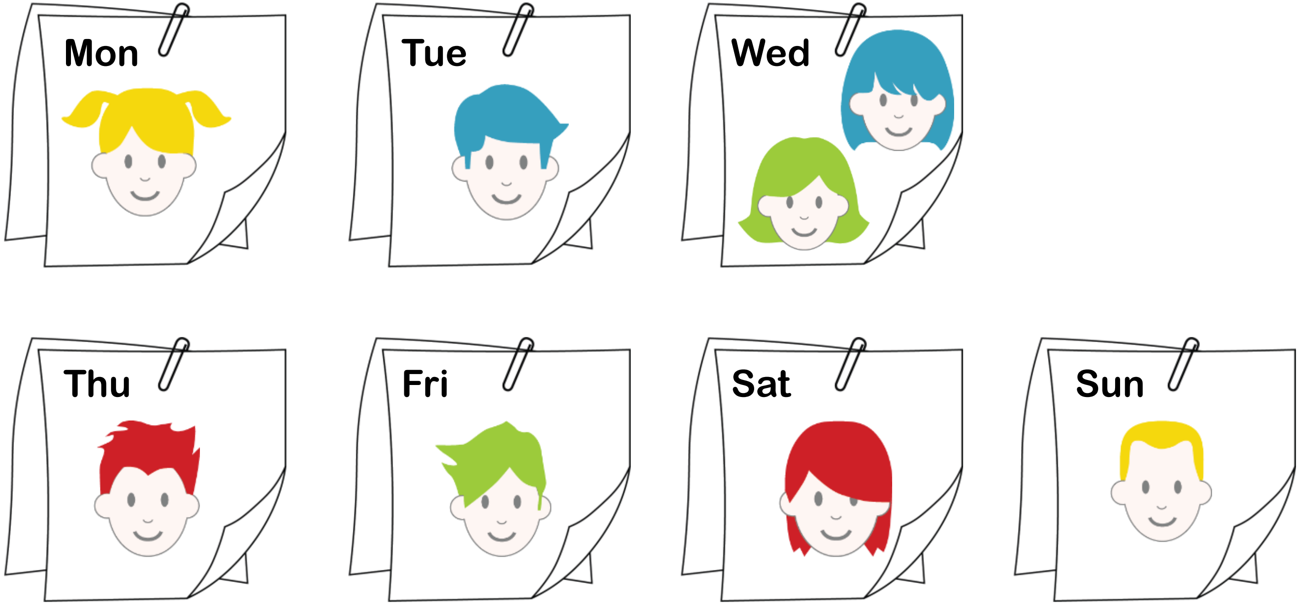
Pigeonhole Principle

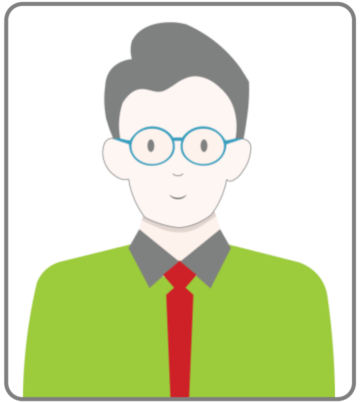
A function from one finite set to a smaller finite set cannot be one-to-one: there must be at least two elements in the domain that have the same image in the codomain.



Pigeonhole Principle: Scenario 1

Consider Bob and his 8 children. At least two of his children were born on the same day of the week.

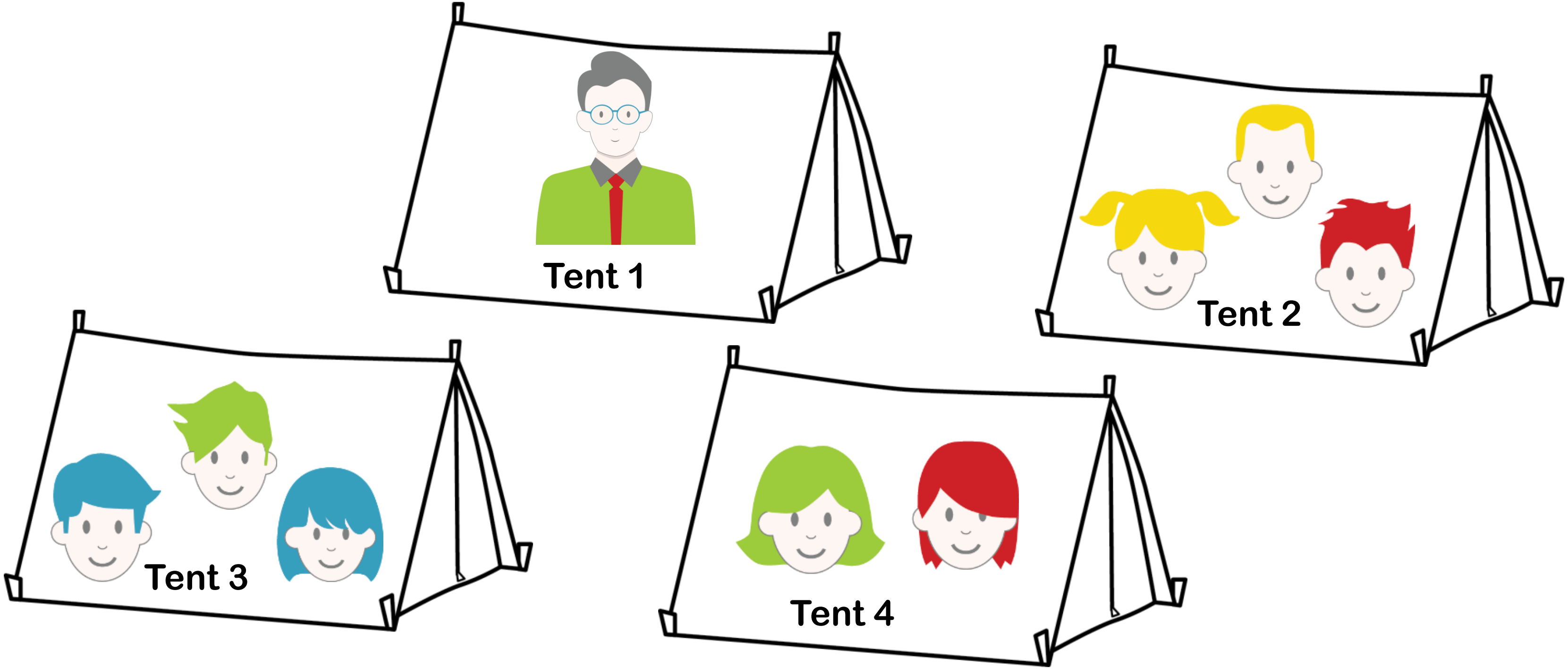




Bob

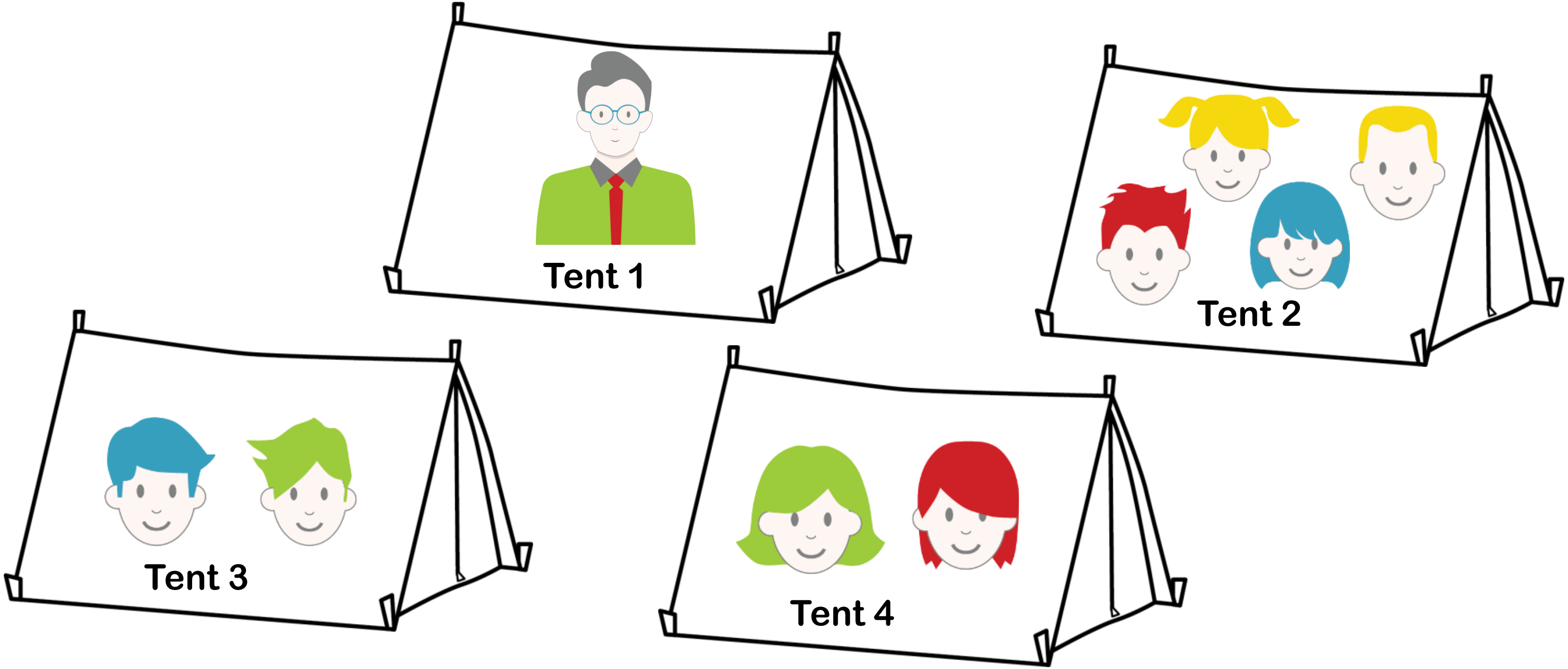
Pigeonhole Principle: Scenario 2

They go camping at the lake. Bob gets a tent of his own, but the others get to share 3 tents. Then, there are at least 3 children sleeping in at least one of them.



Pigeonhole Principle: Scenario 3

They go camping at the lake. Bob gets a tent of his own, but the others get to share 3 tents. Then, there are at least 3 children sleeping in at least one of them.



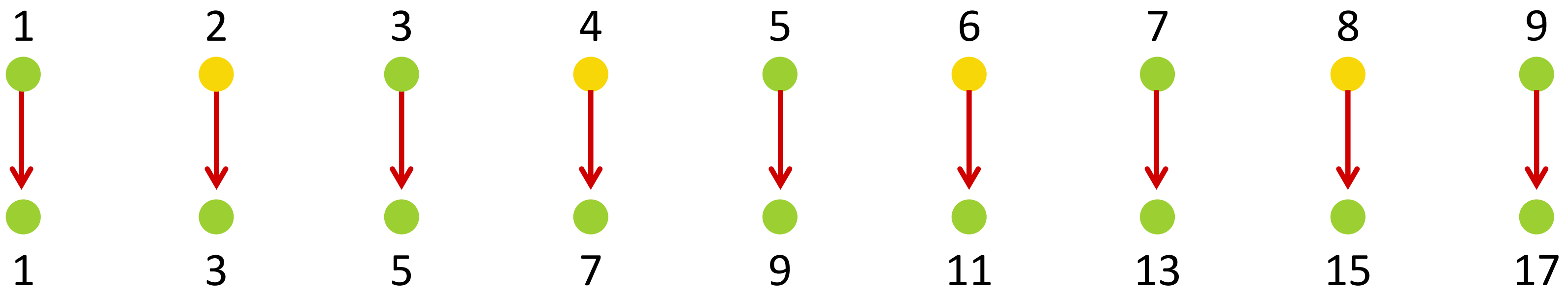
Countable Sets

Countable Sets: Definition

A set that is either finite, or has the same cardinality as the set of positive integers is called countable.
A set that is not countable is called uncountable.

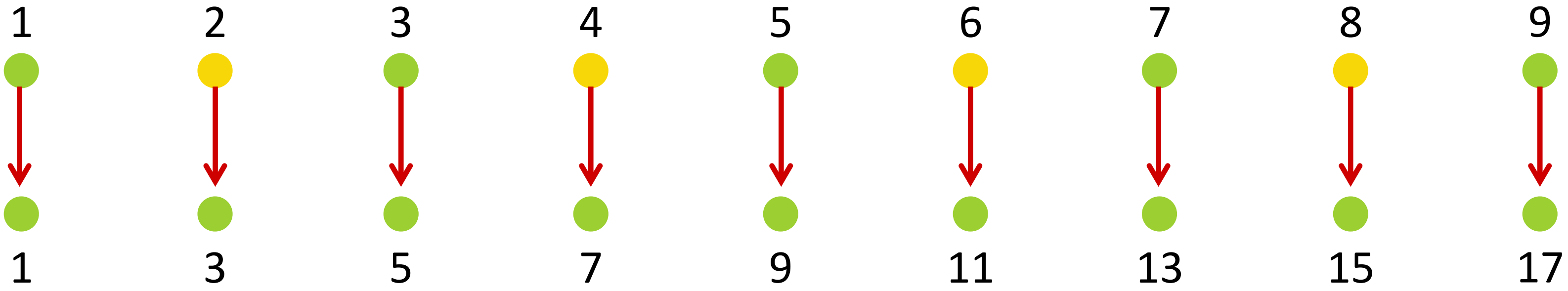
Countable Sets: Example

The set of odd positive integers is a countable set.



- To show that the set of positive odd integers is countable, find a one-to-one correspondence between this set and the set of positive integers.
- Consider the function $f(n) = 2n - 1$.
- $f(n)$ goes from the set of positive integers to the set of odd positive integers.

Countable Sets: Example

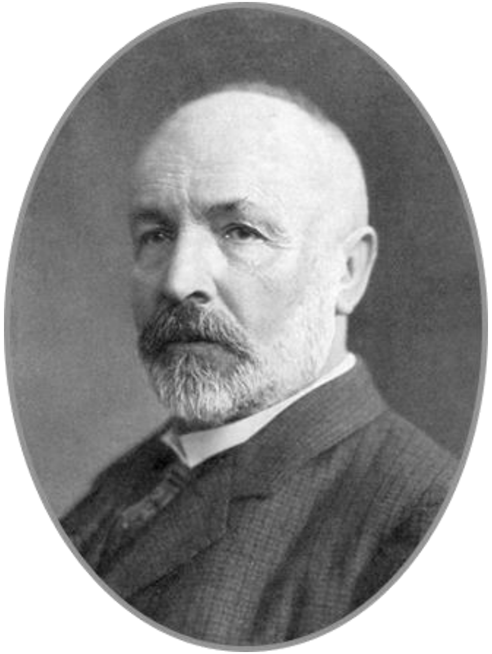


- $f(n)$ is one-to-one: suppose $f(n) = f(m)$, then $2n - 1 = 2m - 1$. Hence, $n = m$.
- $f(n)$ is onto: take m as an odd positive integer. Then m is less than an even integer $2k$ (k a natural number). Thus $m = 2k - 1 = f(k)$.

Countable Sets: An Uncountable Set?

What would be an example of an uncountable set?

- **Real numbers**
- **Proven in 1879 by Cantor**
- **Proof is called "Cantor diagonalisation argument"**
- **Proof method is widely used in the theory of computation**



Georg Ferdinand Ludwig Philipp Cantor 1845 - 1918

Georg Cantor under WikiCommons (PD-US)

Countable Sets: Cantor Diagonalisation

- Suppose that the set of real numbers is countable.
- Then, we will get a contradiction.
- If the set of real numbers is countable, then the set of real numbers that falls between *0* and *1* is also countable.
- Since there is a one-to-one correspondence with positive integers, we can label all of them:

$$r_1, r_2, r_3, \ldots$$

Countable Sets: Cantor Diagonalisation

- **Write these numbers in decimal representation:**

$$r_1 = 0.\, d_{11}\ d_{12}\ d_{13} \ldots$$
$$r_2 = 0.\, d_{21}\ d_{22}\ d_{23} \ldots$$
$$r_3 = 0.\, d_{31}\ d_{32}\ d_{33} \ldots$$

- **Note that all d_{ij} belong to $\{0,1,2,\ldots 9\}$**

- **Form a new real number r with decimal expansion**

$$r = 0.\, d_1\ d_2\ d_3 \ldots$$

where d_i is 5 if $d_{ii} = 4$ and 4 otherwise

Countable Sets: Cantor Diagonalisation

- **The number r is different from all other real numbers listed in the interval $[0,1]$.**

- **This is because r differs from the decimal expansion of r_i in the *ith* place by construction.**

- **We thus found a contradiction to the fact that we are able to list all the real numbers in $[0,1]$, since r does not belong!**

Topic Summary

Let's recap…

- Ceiling and floor functions
- Pigeonhole principle
- Countable and uncountable sets