Tutorial 1 (Solutions) (Tutorial 7)

1a) Let $y = \ln(i^{1/2})$

$$e^y = i^{1/2}$$

$$= \left[e^{i(\frac{\pi}{2} + 2n\pi)}\right]^{1/2} \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$= e^{i(\frac{\pi}{4} + n\pi)}$$

$$\therefore \underline{\underline{y = i\left(\frac{\pi}{4} + n\pi\right) \qquad n = 0, \pm 1, \pm 2, \ldots}}$$

b) Let $y = i^i$

$$\ln y = i \ln i$$

$$= i \ln e^{i(\frac{\pi}{2} + 2n\pi)} \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$= i\left[i\left(\frac{\pi}{2} + 2n\pi\right)\right]$$

$$= -\left(\frac{\pi}{2} + 2n\pi\right)$$

$$\therefore \underline{\underline{y = e^{-(\frac{\pi}{2} + 2n\pi)} \qquad n = 0, \pm 1, \pm 2, \ldots}}$$

which is real-valued.

(b) (Cont'd).

Let $y = z^{i}$

$$\ln y = i \ln z$$
$$= i \ln re^{i(\theta + 2n\pi)} \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$= i \ln r - (\theta + 2n\pi)$$

$$y = z^{i} = e^{i \ln r} \cdot e^{-(\theta + 2n\pi)}$$

$$e^{i \ln r} = \cos(\ln r) + i \sin(\ln r).$$

For $y = z^{i}$ to be real,

$$\sin(\ln r) = 0.$$
$$\ln r = \pm k\pi \qquad k = 0, 1, 2, \ldots$$
$$\underline{r = e^{\pm k\pi}}.$$

$$\therefore z^{i} = (re^{i\theta})^{i}$$
$$= (e^{\pm k\pi} \cdot e^{i\theta})^{i} \text{ is real.}$$

The values of z for real z^{i} are

$$\underline{\underline{z = e^{\pm k\pi} \cdot e^{i\theta} \qquad k = 0, 1, 2, \ldots}}$$

2 a) $$f(z) = \frac{x^2 y}{x^3 + y^3} + i x y$$

For the limit to exist, $\lim_{z \to z_0} f(z)$ need to be unique and independent of the directions in which z approaches z_0.

Let the direction be given by $y = kx$, k is a constant.

$$\lim_{z \to 0} f(z) = \lim_{x \to 0,\, y = kx} f(z)$$

$$= \lim_{x \to 0} \frac{k x^3}{x^3 + k^3 x^3} + i k x^2$$

$$= \frac{k}{1 + k^3}$$ which depends on k

(the direction in which x, y approach zero)

$\Rightarrow$ the limit does not exist

2b) let $z = re^{i\theta}$

$$\lim_{z \to 0} f(z) = \lim_{r \to 0} f(z)$$

$$= \lim_{r \to 0} \left[\frac{re^{-i\theta}}{re^{i\theta}} - \frac{re^{i\theta}}{re^{-i\theta}} - \frac{r^2 e^{i2\theta}}{r^2 e^{-i2\theta}} \right].$$

$$= e^{-i2\theta} - e^{i2\theta} - e^{i4\theta}$$

$\Rightarrow$ the limit does not exist

3 a). A function $f(z)$ is continuous at $z = z_0$

if (a) $f(z_0)$ is defined, and

(b) $\lim_{z \to z_0} f(z) = f(z_0)$.

$$f(z) = \begin{cases} \mathrm{Re}\left[\frac{z}{|z|}\right] & z \neq 0 \\ 0 & z = 0 \end{cases}.$$

$$\lim_{z \to 0} f(z) = \lim_{r \to 0} \mathrm{Re}\left[\frac{re^{i\theta}}{|re^{i\theta}|}\right]$$

$$= \lim_{r \to 0} \mathrm{Re}\left[\cos\theta + i\sin\theta\right]$$

$$= \cos\theta.$$

$\Rightarrow$ limit does not exist

$\Rightarrow$ $f(z)$ is not continuous at $z = 0$.

b) $$\lim_{z \to 0} f(z) = \lim_{r \to 0} \mathrm{Im}\left[\frac{re^{i\theta}}{1+|re^{i\theta}|}\right]$$

$$= \lim_{r \to 0} \mathrm{Im}\left[\frac{re^{i\theta}}{1+r}\right].$$

$$= \lim_{r \to 0} \frac{r}{1+r} \sin\theta$$

$$= 0.$$

At $z = 0$, $f(z) = 0$

$\Rightarrow$ $f(z)$ is continuous at $z = 0$.

4) $$f(z) = \begin{cases} \operatorname{Im}\left[\frac{z}{|z|}\right] & z \neq 0 \\ 0 & z = 0 \end{cases}.$$

$f(z)$ is continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

<u>For $z = 0$</u>

$$\lim_{z \to 0} f(z) = \lim_{r \to 0} \operatorname{Im}\left[\frac{\cancel{r}e^{i\theta}}{|\cancel{r}e^{i\theta}|}\right]$$

$$= \sin\theta$$

$\Rightarrow$ limit does not exist

$\Rightarrow$ $f(z)$ is <u>not</u> continuous at <u>$z = 0$</u>.

<u>For $z = 5$</u>

$$\lim_{z \to z_0} f(z) = \lim_{r \to 0} \operatorname{Im}\left[\frac{z_0 + re^{i\theta}}{|z_0 + re^{i\theta}|}\right], \quad z_0 = 5.$$

$$= \lim_{r \to 0} \frac{r\sin\theta}{5}$$

$$= 0$$

$$f(5) = \operatorname{Im}\left[\frac{5}{|5|}\right] = 0.$$

$\Rightarrow$ $$\lim_{z \to 5} f(z) = f(5)$$

$\Rightarrow$ <u>$f(z)$ at $z = 5$ is continuous</u>

4) (Cont'd).

For $z = 5+i$

$$\lim_{z \to z_0} f(z) = \lim_{r \to 0} \operatorname{Im}\left[\frac{z_0 + re^{i\theta}}{|z_0 + re^{i\theta}|}\right] \qquad z_0 = 5+i$$

$$= \lim_{r \to 0} \operatorname{Im} \frac{5+i + re^{i\theta}}{|5+i+\underbrace{re^{i\theta}}_{\to 0}|}.$$

$$= \lim_{r \to 0} \left[\frac{1 + r\sin\theta}{|5+i|}\right]$$

$$= \frac{1}{|5+i|} = \frac{1}{\sqrt{26}}.$$

$$f(5+i) = \operatorname{Im}\left[\frac{5+i}{|5+i|}\right]$$

$$= \frac{1}{\sqrt{26}}.$$

$$\therefore \lim_{z \to z_0} f(z) = f(z_0), \quad z_0 = 5+i.$$

$\Rightarrow$ <u>function is continuous at $z = 5+i$</u>.