- Let S be the set of irrational numbers, and Δ be the addition. Is S closed under Δ? Justify your answer.

Solution.

- Take two rational numbers $\frac{m}{n}$ and $\frac{m'}{n'}$. Then

$$\frac{m}{n}\frac{m'}{n'} = \frac{mm'}{nn'}$$

which is a rational number. Thus the answer is yes, S is closed under multiplication.

- The subtraction of two natural numbers does not always give a number natural, for example,

$$5 - 10 = -5.$$

Thus S is not closed under subtraction.

- The addition of two irrational numbers does not always give an irrational number, for example,

$$(2+\sqrt{2}) + (2-\sqrt{2})) = 4$$

and 4 is not an irrational number. Thus S is not closed under addition. Note that we are using here the claim that $2+\sqrt{2}$ is irrational. Indeed, suppose that $2+\sqrt{2}$ were rational, that is $2+\sqrt{2} = \frac{m}{n}$ for m, n some integers. Then

$$\sqrt{2} = \frac{m}{n} - 2 = \frac{m-2n}{n}$$

which is a contradiction to the fact that $\sqrt{2}$ is irrational.

Exercises for Chapter 2

Exercise 11. Decide whether the following statements are propositions. Justify your answer.

1. $2+2=5$.
2. $2+2=4$.
3. $x=3$.
4. Every week has a Sunday.

5. Have you read “Catch 22”?

Solution.

1. $2+2=5$: this is a proposition, because it is a statement that always takes the truth value ”false”.
2. $2+2=4$: this is a proposition, because it is a statement that always takes the truth value ”true”.
3. $x=3$: the statement depends on the value of x. Maybe it is true (if x was assigned the value 3), or maybe it is false (if x was assigned a different value). Thus this is not a proposition.
4. Every week has a Sunday: this is a proposition, because it is a statement that always takes the truth value ”true”.
5. Have you read “Catch 22”?: this is a question, thus it is not a proposition.

Exercise 12. Show that

$$\neg(p \vee q) \equiv \neg p \wedge \neg q.$$

This is the second law of De Morgan.

Solution. We show the equivalence using truth tables:

p	q	$p \vee q$	$\neg(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Since both truth tables are the same, the two logical expressions are equivalent.

Exercise 13. Show that the second absorption law $p \wedge (p \vee q) \equiv p$ holds.

Solution. We show the equivalence using a truth table:

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

Since the column of p is the same as that of $p \wedge (p \vee q)$, both logical expressions are equivalent.

Exercise 14. These two laws are called distributivity laws. Show that they hold:

1. Show that $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$.
2. Show that $(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$.

Solution. We use truth tables:

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$	$p \vee r$	$q \vee r$	$(p \vee r) \wedge (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	T	F	F
F	T	T	F	T	T	T	T
F	T	F	F	F	F	T	F
F	F	T	F	T	T	T	T
F	F	F	F	F	F	F	F

p	q	r	$p \vee q$	$(p \vee q) \wedge r$	$p \wedge r$	$q \wedge r$	$(p \wedge r) \vee (q \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	F	T
T	F	F	T	F	F	F	F
F	T	T	T	T	F	T	T
F	T	F	T	F	F	F	F
F	F	T	F	F	F	F	F
F	F	F	F	F	F	F	F

Exercise 15. Verify $\neg(p \vee \neg q) \vee (\neg p \wedge \neg q) \equiv \neg p$ by

- constructing a truth table,
- developing a series of logical equivalences.

Solution. We start with a truth table:

p	q	$\neg p$	$\neg q$	$p \vee \neg q$	$\neg(p \vee \neg q)$	$\neg p \wedge \neg q$	$\neg(p \vee \neg q) \vee (\neg p \wedge \neg q)$
T	T	F	F	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	F	T	F	T
F	F	T	T	T	F	T	T

Next we want to prove this result without using the truth table, but by developing logical equivalences:

$$\begin{aligned}\neg(p \vee \neg q) \vee (\neg p \wedge \neg q) &\equiv (\neg p \wedge q) \vee (\neg p \wedge \neg q) \text{ De Morgan}\\ &\equiv \neg p \wedge (q \vee \neg q) \text{ Distributivity}\\ &\equiv \neg p \wedge T \text{ since}(q \vee \neg q) \equiv T\\ &\equiv \neg p.\end{aligned}$$

Exercise 16. Using a truth table, show that:

$$\neg q \to \neg p \equiv p \to q.$$

Solution. We compute the truth table:

p	q	$\neg p$	$\neg q$	$\neg q \to \neg p$	$p \to q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Exercise 17. Show that $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$.

Solution. We use logical equivalences:

$$\begin{aligned}p \vee q \to r &\equiv (p \vee q) \to r \text{ precedence}\\ &\equiv \neg(p \vee q) \vee r \text{ conversion theorem}\\ &\equiv (\neg p \wedge \neg q) \vee r \text{ De Morgan}\\ &\equiv (\neg p \vee r) \wedge (\neg q \vee r) \text{ Distributivity}\\ &\equiv (p \to r) \wedge (q \to r) \text{ conversion theorem}\end{aligned}$$

Exercise 18. Are $(p \to q) \vee (q \to r)$ and $p \to r$ equivalent statements?

Solution. They are not equivalent. Here is a proof using truth tables:

p	q	r	$p \to q$	$q \to r$	$(p \to q) \vee (q \to r)$	$p \to r$
T	T	T	T	T	T	T
T	T	F	T	F	T	F
T	F	T	F	T	T	T
T	F	F	F	T	T	F
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

We see that the first rows for example are giving different truth values. This can be done using equivalences as well:

$$\begin{aligned}(p \to q) \vee (q \to r) &\equiv (\neg p \vee q) \vee (\neg q \vee r) \text{ conversion theorem}\\ &\equiv \neg p \vee r \vee T \text{ since } \neg q \vee q \equiv T\\ &\equiv T.\end{aligned}$$

Since $p \to q$ is not equivalent to T, both statements cannot be equivalent.

Exercise 19. Prove or disprove the following statement:

$$p \wedge (\neg(q \to r)) \equiv (p \to r).$$

Solution. One should disprove the statement. There are several ways to do so. For example, note that

$$p \wedge (\neg(q \to r)) \equiv p \wedge (\neg(\neg q \vee r)) \equiv p \wedge (q \wedge \neg r) \equiv p \wedge q \wedge \neg r$$

using the conversion theorem and De Morgan law. Now we see, for example, that if p is true and r is false, then $p \wedge q \wedge \neg r$ can take the value true when q is true, but $p \to r$ is then false, no matter the value of q. It is also possible to find the same conclusion by doing a truth table.

p	q	r	$q \to r$	$p \wedge \neg(q \to r)$	$p \to r$
T	T	T	T	F	T
T	T	F	F	T	F
T	F	T	T	F	T
T	F	F	T	F	F
F	T	T	T	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T

Exercise 20. Show that this argument is valid:

$$\boxed{\neg p \to F; \therefore p.}$$

Solution. The premise is $\neg p \to F$, which is true when $\neg p$ is false, which is exactly when p is true.

Exercise 21. Show that this argument is valid, where C denotes a contradiction.

$$\boxed{\neg p \to C; \therefore p.}$$

Solution. The premise is $\neg p \to C$, which is true when $\neg p$ is false, which is exactly when p is true.

Exercise 22.
1. Prove or disprove the following statement:

$$(p \wedge q) \to p \equiv T.$$

2. Decide whether the following argument is valid.

   $\neg d \to h$;
   $\neg h \to d$;
   $\therefore \neg d \vee \neg h$

Solution.
1. One should prove the statement. Using the conversion theorem, and De Morgan's Law

$$\neg(p \wedge q) \vee p \equiv (\neg p \vee \neg q) \vee p \equiv T$$

   since $\neg p \vee p$ is always true. Alternatively, $r \to p$ is always true, but for r true and p false. Here this means that we would need p false and $p \wedge q$ true, which is not possible, therefore it is always true. A 3rd way is to use a truth table.

2. The argument is not valid. There are several ways to see it. One is a truth table, which shows that when $d = T$ and $h = T$, then the conclusion is false. Yet, the premises are true, since $\neg d$ and $\neg h$ are false. Another way is to notice that one premise is the contrapositive of the other, therefore both of them are equivalent, from which one finds the same counterexample.

Exercise 23. Determine whether the following argument is valid:

$$\begin{array}{l} \neg p \to r \wedge \neg s \\ t \to s \\ u \to \neg p \\ \neg w \\ u \vee w \\ \therefore t \to w. \end{array}$$

Solution. We start by noticing that we have

$$u \vee w; \neg w; \therefore u.$$

Indeed, if $u \vee w$ and $\neg w$ are both true, then w is false, and u must be true. Next

$$u \to \neg p; u; \therefore \neg p.$$

Indeed, if $u \to \neg p$ is true, either u is true and $\neg p$ is true, or u is false. But u is true, thus $\neg p$ is true (Modus Ponens). Then

$$\neg p \to r \wedge \neg s; \neg p; \therefore r \wedge \neg s,$$

this is again Modus Ponens. Then

$$r \wedge \neg s; \therefore \neg s.$$

Indeed, for $r \wedge \neg s$ to be true, it must be that $\neg s$ is true. Finally

$$t \to s; \neg s; \therefore \neg t$$

since for $t \to s$ to be true, we need either t to false, or t and s to be true, but since s is false, t must be false (Modus Tollens), and

$$\neg t \therefore \neg t \vee w$$

or equivalently

$$\neg t \vee w \equiv t \to w$$

using the Conversion theorem, which shows that the argument is valid.

Exercise 24. Determine whether the following argument is valid:

$$
\begin{array}{l}
p \\
p \vee q \\
q \to (r \to s) \\
t \to r \\
\therefore \neg s \to \neg t.
\end{array}
$$

Solution. For this question, there is no obvious way to combine the known statements with inference rules. The only 2 related statements are p and $p \vee q$, and assuming that both are true, all can be deduced is that q is either true or false. Now if q is false, $q \to (r \to s)$ is always true, while if q is true, $q \to (r \to s)$ is true only if $(r \to s)$ is true, which excludes the possibility $r = T$ and $s = F$.

q	r	s	t
F	T	T	
F	T	F	
F	F	T	
F	F	F	
F	T	T	
T	F	T	
T	F	F	

Now we look at the last premise $t \to r$. For it to be true, we need t false, or t true and r true.

q	r	s	t
F	T	T	T/F
F	T	F	T/F
F	F	T	F
F	F	F	F
T	T	T	T/F
T	F	T	F
T	F	F	F

Now if s is true, then $\neg s$ is always false, and the conclusion is always true. We thus focus on s is false, and $\neg t$ is false, that is t is true. The second row gives a counter-example:

$$q = F,\ r = T,\ s = F,\ t = T.$$

Exercise 25. Decide whether the following argument is valid:

$(p \vee q) \to \neg r$;
$\neg r \to s$;
p;
$\therefore s$

Solution. We start by noticing that

p;
$\therefore (p \vee q)$

Then

$(p \vee q) \to \neg r$;
$p \vee q$;
$\therefore \neg r$

Finally

$\neg r \to s$;
$\neg r$;
$\therefore s$

and we conclude that the argument is valid.

We can come to the same conclusion using a truth table. Note that we care only about the critical rows, those for which the premises are true. Thus in the table below, we assume that p is always true.

s	q	r	$p \vee q$	$p \vee q \to \neg r$	$\neg r \to s$	
T	T	T	T	F		
T	T	F	T	T	T	critical
T	F	T	T	F		
T	F	F	T	T	T	critical
F	T	T	T	F		
F	T	F	T	T	F	
F	F	T	T	F		
F	F	F	T	T	F	

We see that there are only 2 critical rows, for which s is true, therefore the argument is valid.

Exercises for Chapter 3

Exercise 26. Consider the predicates $M(x, y) =$ "x has sent an email to y", and $T(x, y) =$ "x has called y". The predicate variables x, y take values in