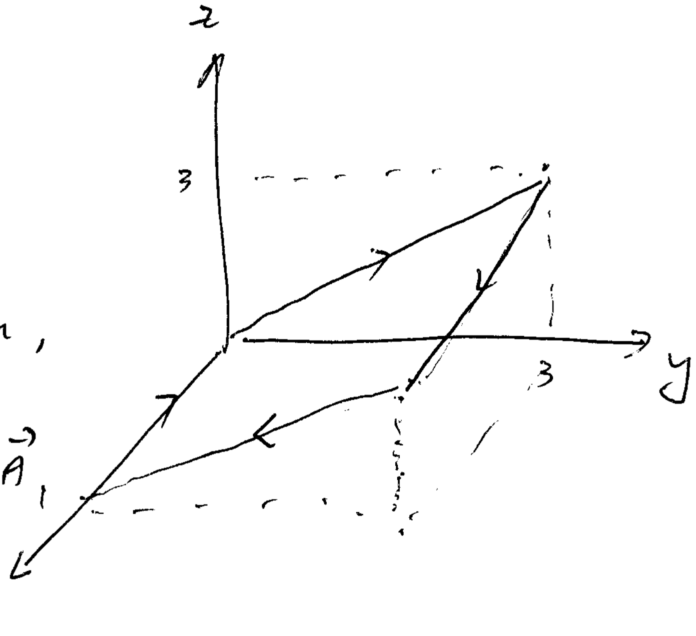
Tutorial 6 (Tutorial 12) Solutions

1) W.D. = $\oint_C \vec{F} \cdot d\underset{\sim}{r}$.



Using Stokes' Theorem,

$$\oint_C \vec{F} \cdot d\underset{\sim}{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{A},$$

For planar surface S, let

$$\left.\begin{aligned} x &= u \\ y &= v \\ z &= y = v \end{aligned}\right\} \quad \begin{aligned} 0 &\le u \le 1 \\ 0 &\le v \le 3. \end{aligned}$$

$$\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ v \end{pmatrix}$$

$$\underset{\sim}{r_u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \underset{\sim}{r_v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \quad \vec{N} = \underset{\sim}{r_u} \times \underset{\sim}{r_v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

For the given path, the normal of the plane should be orientated downwards.

Redefine $\vec{N} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

$$\vec{F} = \begin{pmatrix} x^2 \\ 4xy^3 \\ y^2 x \end{pmatrix}.$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 4xy^3 & xy^2 \end{vmatrix}.$$

$$= \begin{pmatrix} 2xy \\ -y^2 \\ 4y^3 \end{pmatrix}.$$

$$\therefore \iint \operatorname{curl} \vec{F} \cdot d\vec{A} = \iint \operatorname{curl} \vec{F} \cdot \vec{N} \, du \, dv$$

$$= \int_{v=0}^{v=3} \int_{u=0}^{u=1} \begin{pmatrix} 2xy \\ -y^2 \\ 4y^3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} du \, dv.$$

$$= \int_{v=0}^{v=3} \int_{u=0}^{u=1} (-y^2 - 4y^3) \, du \, dv$$

$$= \int_{v=0}^{v=3} (-v^2 - 4v^3) dv \cdot \int_{u=0}^{u=1} 1 \, du.$$

$$= \left[-\frac{v^3}{3} - v^4 \right]_0^3 = \underline{\underline{-90 \text{ units}}}$$

2) Using Stokes Theorem,

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{A} = \oint_C \vec{F} \cdot d\underset{\sim}{r}$$

where path c is the boundary of S on the xy plane.

$S: \quad z = 4 - x^2 - y^2, \quad z \geq 0.$

$C: \quad z = 0 \Rightarrow x^2 + y^2 = 4$ (circle of radius = 2).

$$\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2\cos t \\ 2\sin t \\ 0 \end{pmatrix} \quad 0 \leq t \leq 2\pi.$$

$$\vec{F} = \begin{pmatrix} 2z \\ 3x \\ 5y \end{pmatrix} = \begin{pmatrix} 0 \\ 6\cos t \\ 10\sin t \end{pmatrix}.$$

$$d\underset{\sim}{r} = \begin{pmatrix} -2\sin t \\ 2\cos t \\ 0 \end{pmatrix} dt.$$

$$\therefore \oint \vec{F} \cdot d\underset{\sim}{r} = \int_{t=0}^{2\pi} \begin{pmatrix} 0 \\ 6\cos t \\ 10\sin t \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ 2\cos t \\ 0 \end{pmatrix} dt.$$

$$= \int_0^{2\pi} 12\cos^2 t \, dt.$$

$$= \int_0^{2\pi} 6(\cos 2t + 1)\, dt.$$

$$= 6\left[\frac{\sin 2t}{2} + t\right]_0^{2\pi}$$

$$= \underline{\underline{12\pi}}$$

3) Using Divergence Th.

$$\oiint_S \vec{F} \cdot d\vec{A} = \iiint_V \operatorname{div} \vec{F} \, dV$$

$$\operatorname{div} \vec{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = 1$$

$$\therefore \oiint_S \vec{F} \cdot d\vec{A} = \iiint_V 1 \, dv \quad \text{(for sphere).}$$

$$= \frac{4}{3}\pi a^3 \quad \text{(volume of sphere)}$$

4). $$\oiint_S \vec{F} \cdot d\vec{A} = \iiint_V \operatorname{div} \vec{F} \, dv.$$

$$\operatorname{div} \vec{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix} = 3x^2 + 3y^2 + 3z^2.$$

$$\therefore \iiint_V \operatorname{div} \vec{F} \, dv = 3 \iiint \underbrace{(x^2+y^2+z^2)}\, \underbrace{dv}$$

$$= 3 \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\rho=0}^{\rho=a} \underbrace{\rho^2} \; \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}.$$

$$= 3 \left[\int_{\rho=0}^{\rho=a} \rho^4 \, d\rho \right] \left[\int_{\phi=0}^{\phi=\frac{\pi}{2}} \sin\phi \, d\phi \right] \left[\int_{\theta=0}^{\theta=2\pi} 1 \, d\theta \right].$$

$$= 3 \left[\frac{\rho^5}{5} \right]_0^a \left[-\cos\phi \right]_0^{\pi/2} \left[2\pi \right].$$

$$= \frac{6}{5}\pi a^5$$

5).
$$\oiint_S \vec{F} \cdot d\vec{A} = \iiint_V \operatorname{div} \vec{F} \, dV$$

$$\operatorname{div} \vec{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} 2x \\ 3y \\ z^2 \end{pmatrix} = 2 + 3 + 2z$$
$$= 5 + 2z .$$

$$\iiint_V \operatorname{div} \vec{F} \, dV = \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} (5 + 2z) \, dx \, dy \, dz .$$

$$= \left[\int_{x=0}^{1} 1 \, dx \right] \cdot \left[\int_{y=0}^{1} 1 \, dy \right] \cdot \left[\int_{z=0}^{1} (5 + 2z) \, dz \right]$$

$$= (1)(1) \left[5z + z^2 \right]_0^1$$

$$= \underline{\underline{6}} .$$