NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2017–2018

MH1812 – Discrete Mathematics

May 2018 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT** an **OPEN BOOK** exam.
5. Calculators are allowed.
6. Candidates should clearly explain their reasoning used in each of their answers.

QUESTION 1. **(25 marks)**

(a) Let $S = \{1, 2, 4\}$ and let P be the set of prime numbers. Determine the truth value of the following proposition:

$$\neg\left(\exists x \in S, \forall y \in S,\ x + y \notin P\right).$$

Justify your answer.

(b) Decide whether or not the following argument is valid:

$$\begin{aligned}
&p \vee q;\\
&p \to s;\\
&q \to r;\\
&\neg r \vee p;\\
&\therefore s
\end{aligned}$$

Justify your answer.

Solution:

(a) The truth value of

$$\forall x \in S, \exists y \in S,\ x + y \in P$$

is true. Indeed, for each element $x \in S$ take $y = 1$.

(b) The argument is valid. The premise $p \vee q$ implies that at least one of p and q is true. If p is true then with the proposition $p \to s$, by modus ponens, the conclusion s is true. If q is true then since we have $q \to r$ we must have r is true. Thence, using $\neg r \vee p$ we see that p must be true, and hence the conclusion is true.

QUESTION 2. **(25 marks)**

(a) Let $S = \{1, 2, 3\}$. How many binary relations R on S are there such that

(i) R is reflexive?

(ii) R is symmetric?

(iii) R is an equivalence relation?

Justify your answers.

(b) Define the function $f : \mathbb{Q} \to \mathbb{Q}$ by $f(x) = 2x/3 + 5$.

(i) Prove that the function f is bijective.

(ii) What is the inverse of f?

Solution:

(a) (i) $2^{3^2-3} = 2^6$

(ii) $2^{(3^2-3)/2+3} = 2^6$

(iii) There are five equivalence relations: $\{\{1\}, \{2\}, \{3\}\}$, $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, $\{\{1\}, \{2, 3\}\}$, $\{\{1, 2, 3\}\}$

(b) (i) injective: $f(x) = f(y)$ implies that $2x/3 + 5 = 2y/3 + 5$, which implies that $x = y$.

surjective: let $y \in \mathbb{Q}$. For $x = 3(y - 5)/2$, we have $f(x) = y$.

(ii) $f^{-1} = 3(x - 5)/2$.

QUESTION 3. **(25 marks)**

(a) Solve the recurrence relation

$$a_0 = 2,\ a_1 = 3, \quad a_n = 3a_{n-1} - 2a_{n-2} + 1 \quad \text{for all } n \geqslant 2,$$

that is, write a_n in terms of n. Justify your answer.

(b) Prove that, for all $n \in \mathbb{N}$,

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = n(n+1)(n+2)(n+3)/4.$$

Solution:

(a) Use backtracking to find the formula

$$a_n = (2^{i+1} - 1)a_{n-i} - (2^i - 1)2a_{n-i-1} + 2^{i+1} - i - 2.$$

For $i = n-1$ we have $a_n = (2^n - 1)a_1 - (2^{n-1} - 1)2a_0 + 2^n - n - 1$. Using $a_0 = 2$ and $a_1 = 3$ we obtain the formula $a_n = 2^{n+1} - n$. Then, using induction, we see that this is the correct formula for a_n.

(b) Let $P(k)$ be the hypothesis that

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2) = k(k+1)(k+2)(k+3)/4.$$

Basis case: $n = 1$ we have $1 \cdot 2 \cdot 3 = 1 \cdot 2 \cdot 3 \cdot 4/4$. So $P(1)$ is true. Assume that $P(k)$ is true for some $k \in \mathbb{N}$. Now consider $P(k+1)$. Using the hypothesis $P(k)$ we see that the LHS of $P(k+1)$ is

$$\begin{aligned}&1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3)\\ &= k(k+1)(k+2)(k+3)/4 + (k+1)(k+2)(k+3).\end{aligned}$$

But $k(k+1)(k+2)(k+3)/4 + (k+1)(k+2)(k+3) = (k+1)(k+2)(k+3)(k+4)/4$, as required.

QUESTION 4. **(25 marks)**

(a) Let G be an undirected graph with n vertices. Find the minimum number of edges required such that

   (i) G is connected;

   (ii) G has a Hamiltonian circuit;

   (iii) G has an Euler path.

   Justify your answers.

(b) Does the graph X have

   (i) an Euler path?

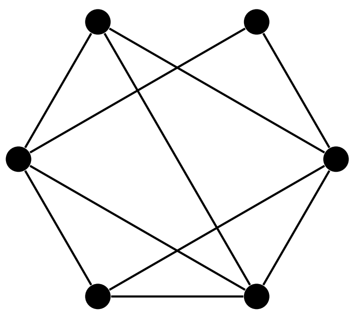
   (ii) a Hamiltonian path?

   (iii) an Euler circuit?

   (iv) a Hamiltonian circuit?

   Justify your answers.

The graph X:



Solution:

(a) (i) $n-1$

   (ii) n

   (iii) 0

(b) (i) yes

   (ii) yes

   (iii) no

   (iv) yes

END OF PAPER