Circuit Analysis
EE2001

NANYANG TECHNOLOGICAL UNIVERSITY

Two-Port Networks

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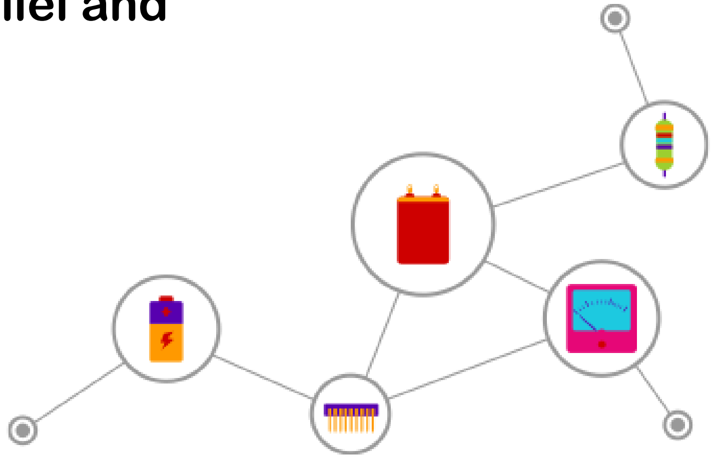
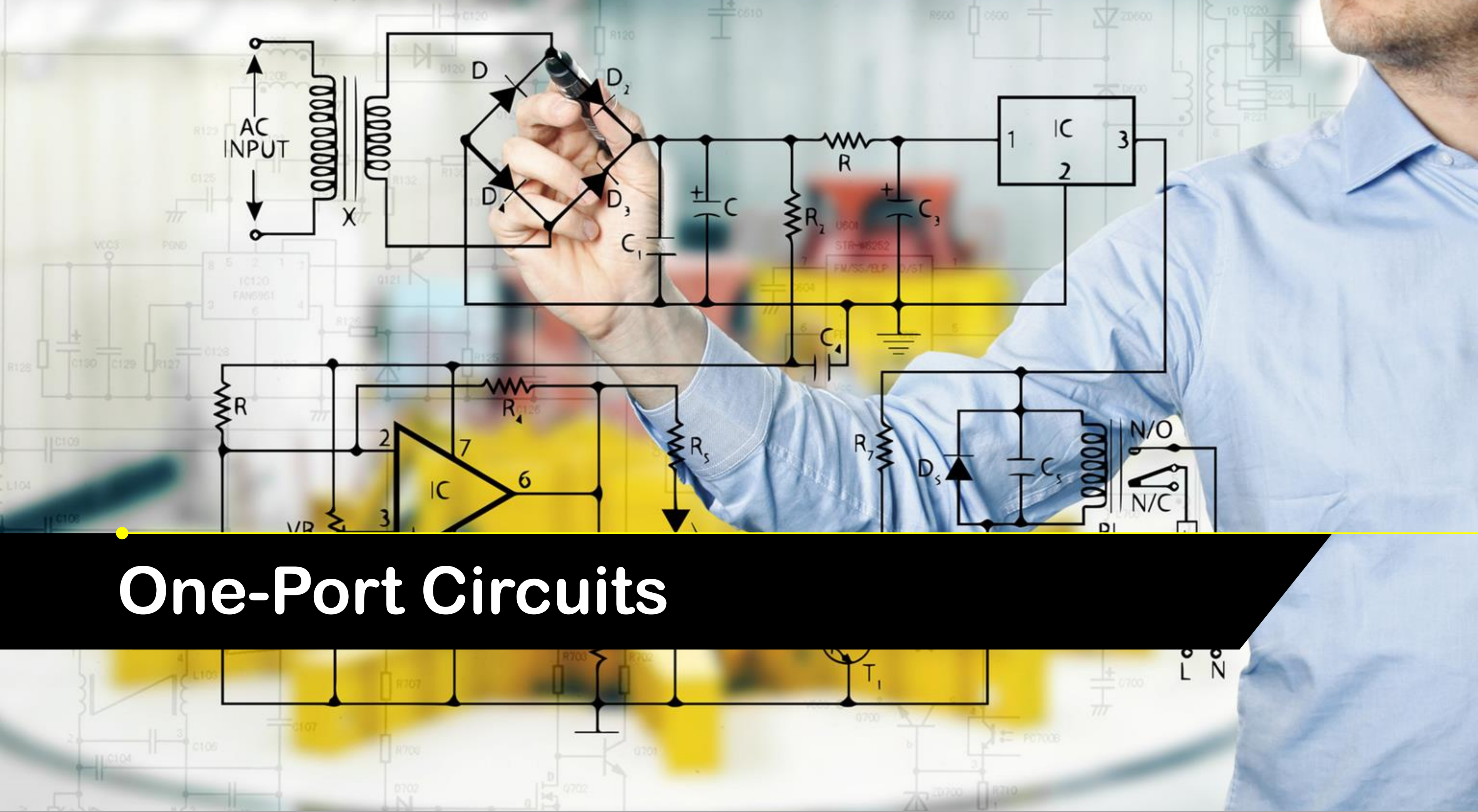
Overview

- One-Port Circuits
- Introduction to Two-Port Network
- Relationship Among Parameters
- Reciprocal and Symmetrical Networks
- Interconnection of Networks

By the end of this lesson, you should be able to…

- Explain the key characteristics of one-port circuits.
- Explain the key characteristics of two-port networks.
- Explain the relationship between parameters.
- Identify reciprocal and symmetrical networks.
- Explain the key characteristics of series, parallel and cascade interconnections.



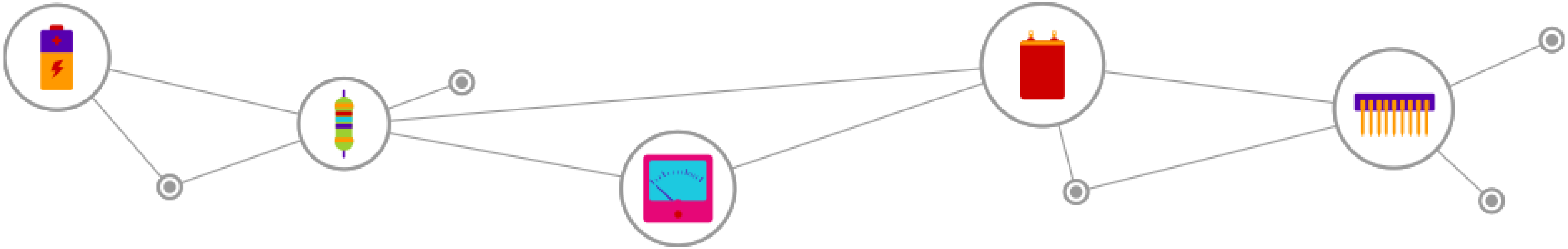


One-Port Circuits

One-Port Circuits

So far in our study of circuits, we have been concerned with determining currents or voltages after being given the sources and the details of the circuit structure.

In many applications, however, we have no interest in the internal parts of the circuits; we want to know only the relationships among voltages and currents at the locations where an exciting source will be connected or an output will be taken.



One-Port Circuits

The figure below shows a general one-port network whose two terminals satisfy the property that for any voltage V across the terminals, the current entering one terminal, say, I, equals to the current leaving the second terminal.

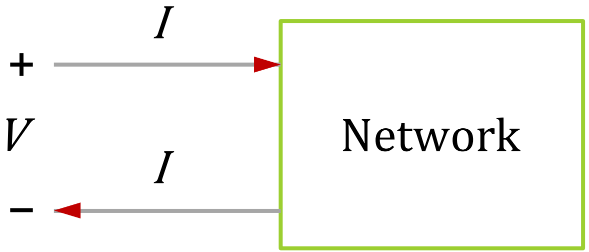
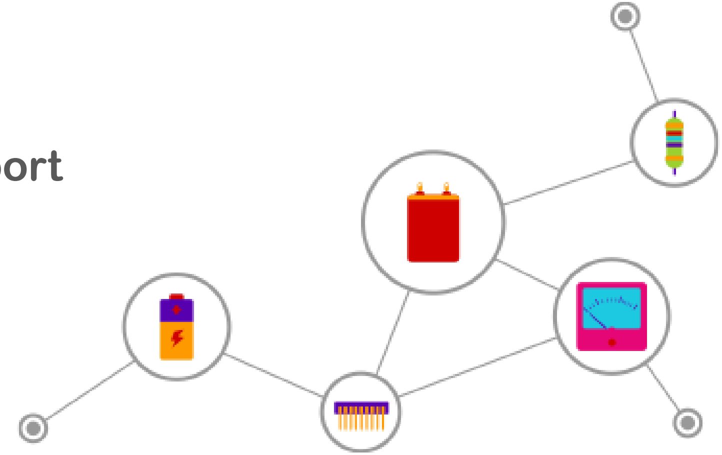


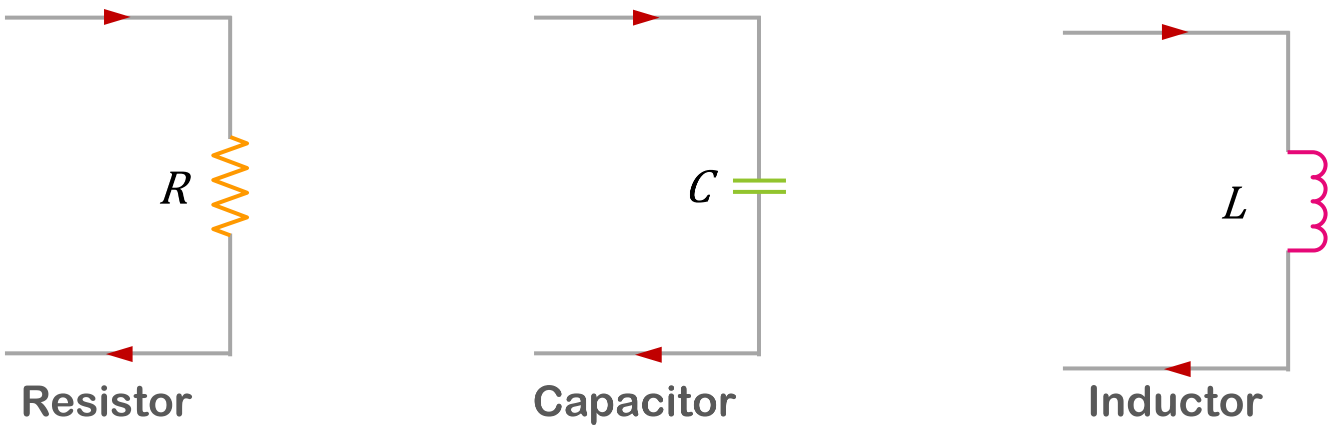
Illustration of a General One-port

This property is called the port condition.



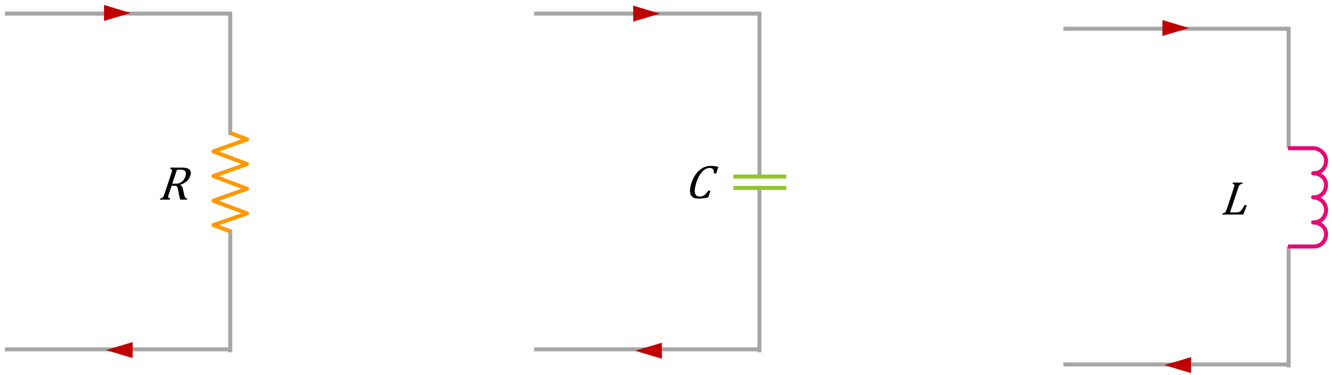
One-Port Circuits

A resistor (or capacitor, or inductor) is a one-port because the current entering one terminal equals the current leaving the other terminal.



A general one-port contains any number of interconnected resistors, capacitors, inductors, and other devices.

In a one-port, only the relationship between the port voltage and current is of interest. For example, the port voltage and current in R, C and L satisfy the relationships as seen below.

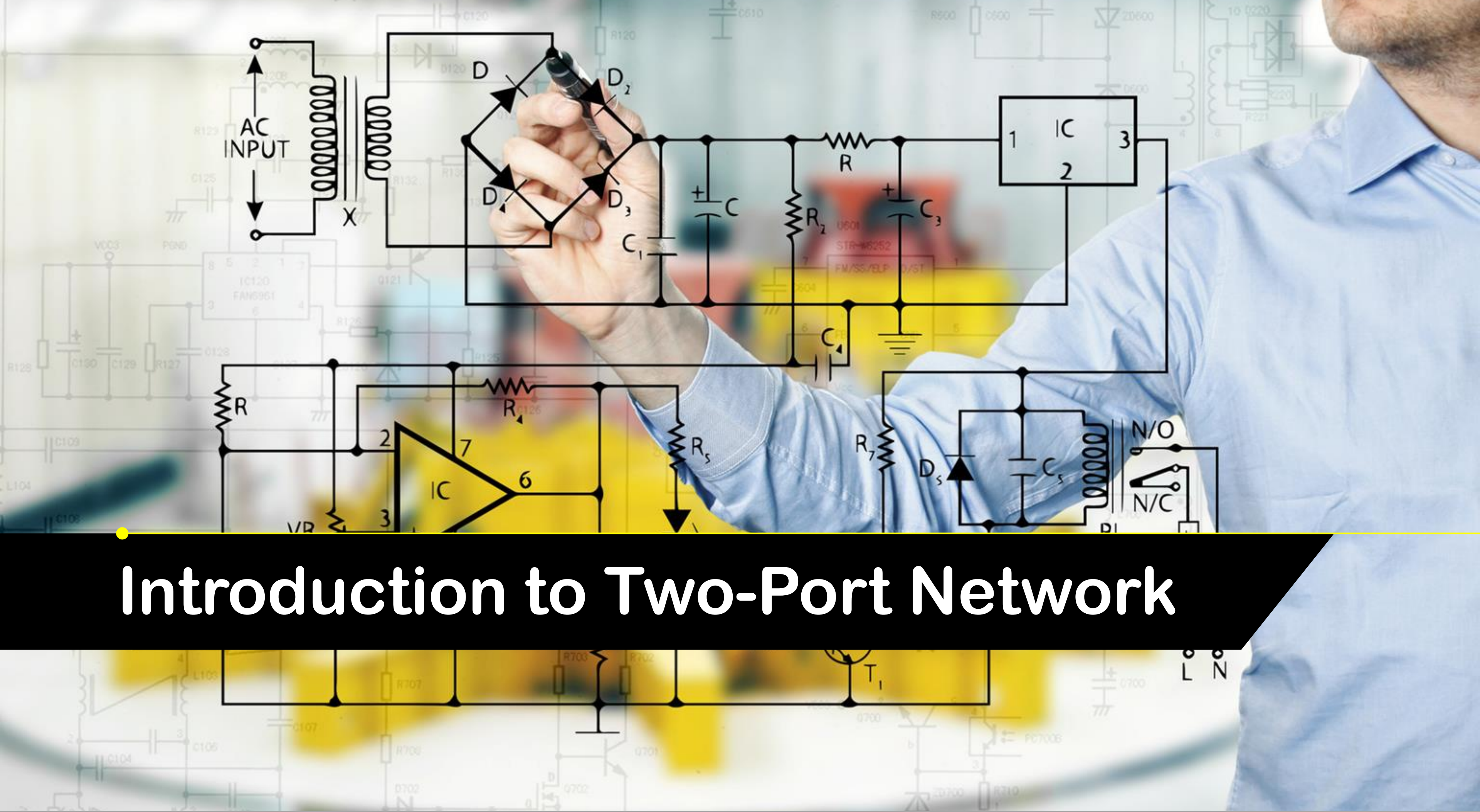


Resistor Capacitor Inductor

$$v_R = i_R R$$

$$i_C = C\frac{dv_C}{dt}$$

$$v_L = L\frac{di_L}{dt}$$



Introduction to Two-Port Network

Introduction to Two-Port Network

Unlike the one-port network, other networks have more pairs of terminals to which external connections can be made. The figure below shows a two-port.

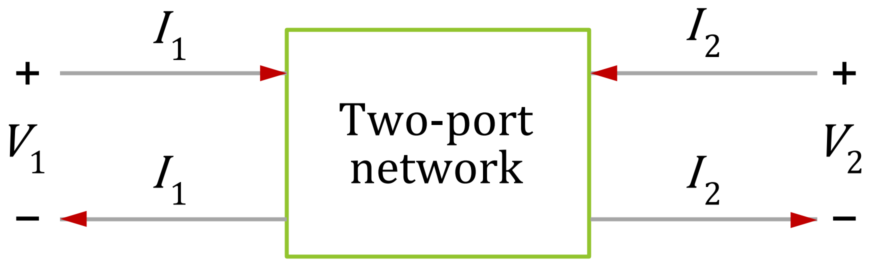


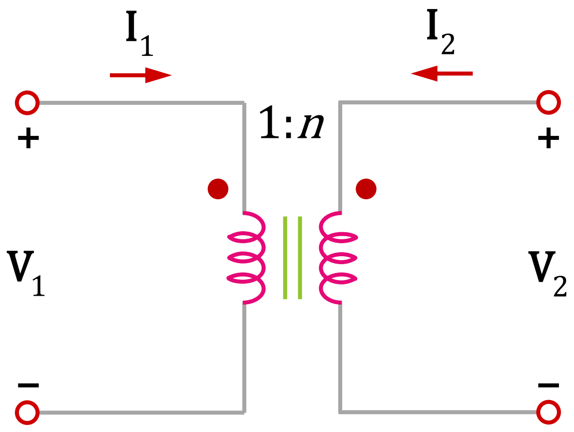
Illustration of a General Two-port

A two-port network is a linear network having two pairs of terminals.

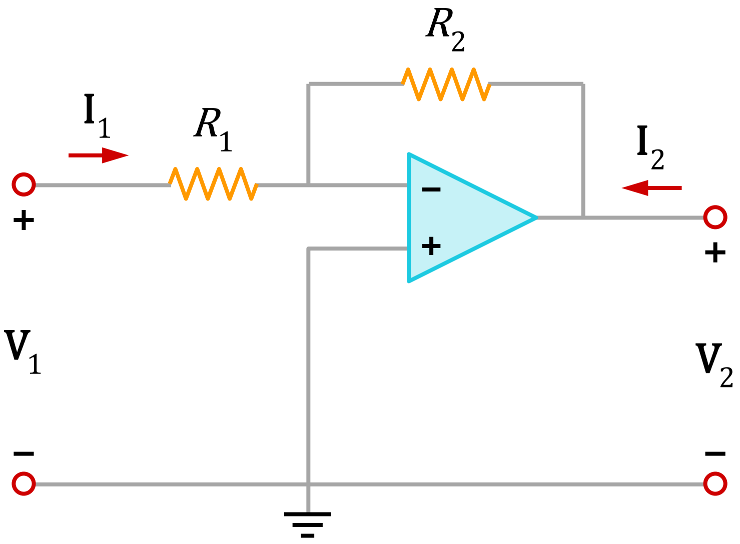
Each pair behaves as a port, i.e., the current entering one terminal of a port equals the current leaving the second terminal of the same port for all voltages across the port.

Introduction to Two-Port Network

Examples



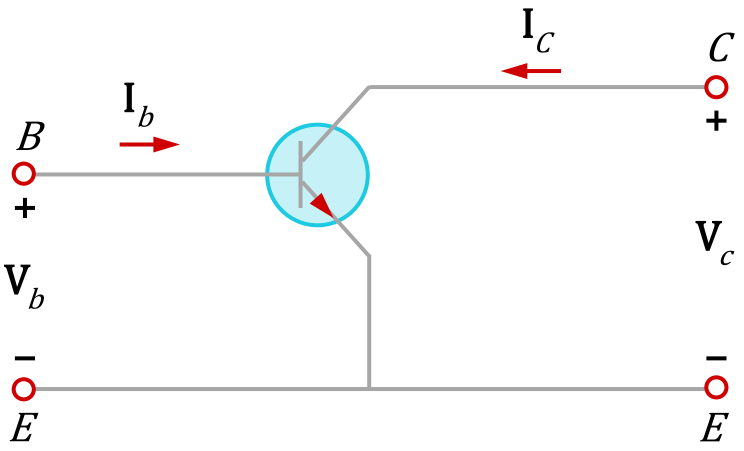
An Ideal Transformer



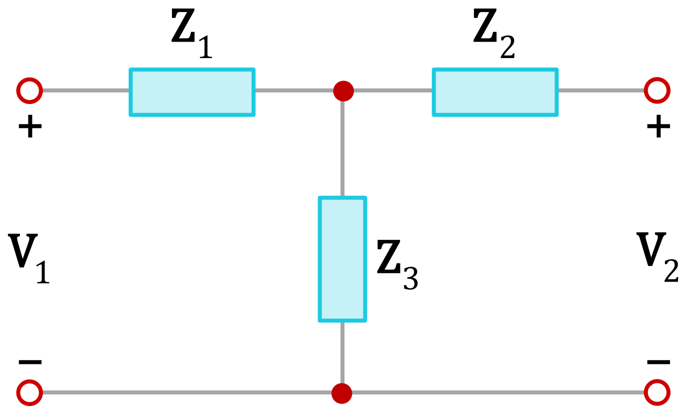
An Op-amp Circuit

Introduction to Two-Port Network

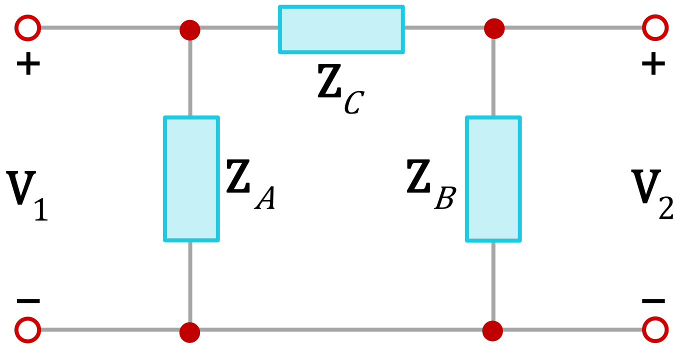
Examples



A Transistor Configuration



A T Network

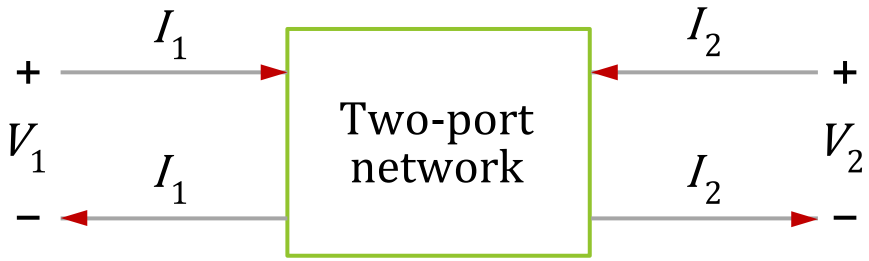


A π Network

Introduction to Two-Port Network

Many practical circuits have just two ports of access: two places where signals may be input or output.

For instance, a coaxial cable between two places has two ports (one at each of those places), an amplifier, a filter, or a transmission line (power or telephone).



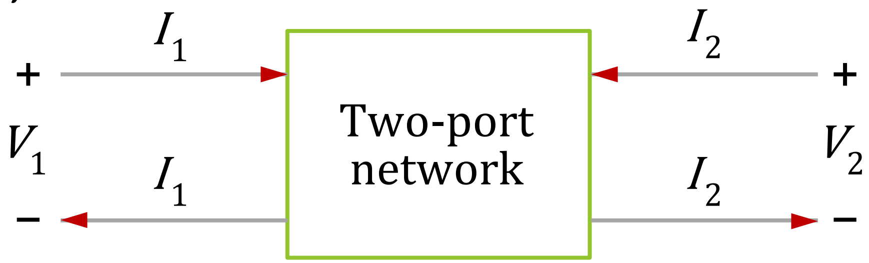
A Two-port Network

Signals are fed into the input port and extracted from the output port.

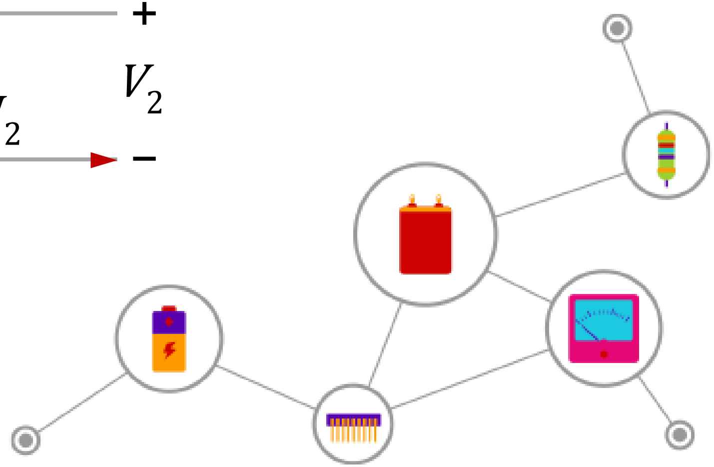
Introduction to Two-Port Network

Suppose if we want to model a circuit that is inside a "black box", where the components, their values, and their interconnections are hidden.

We can perform two simple experiments on such a black box to create a model that consists of just 4 values – the two-port parameter model for the circuit (without stating anything about the internal structure).

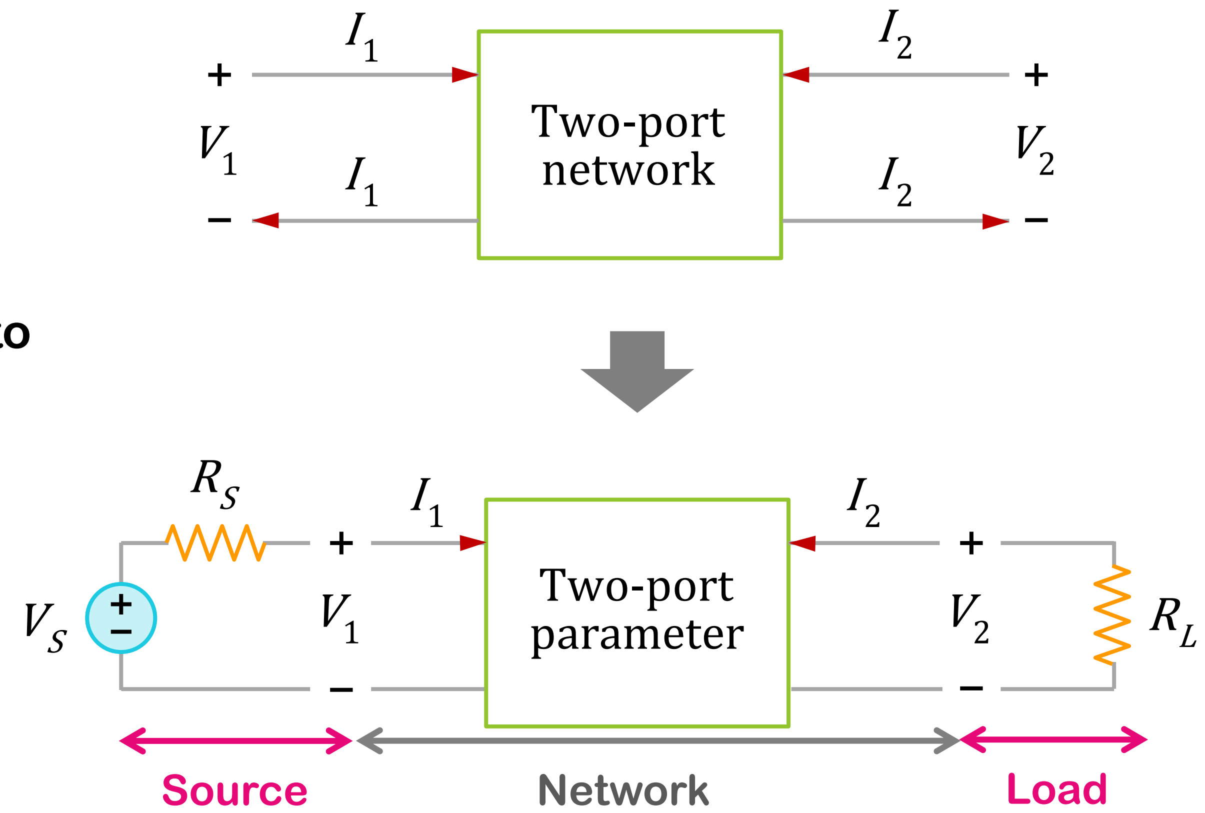


A Two-port Network



Introduction to Two-Port Network

Then we can use the two-port parameter model to predict the behavior of the circuit once we have attached a power source to one of its ports and a load to the other port.



Introduction to Two-Port Network

A two-port description is an efficient way to describe a circuit when the circuit's function is conveyed by the few input-output variables.

The box may contain anything from a single element to a complicated network.

A Two-port Network

However, we are only interested in the voltage and current relationships at the terminals (I_1, V_1, I_2, V_2). It is thus important to find parameters to express their behavior conveniently.

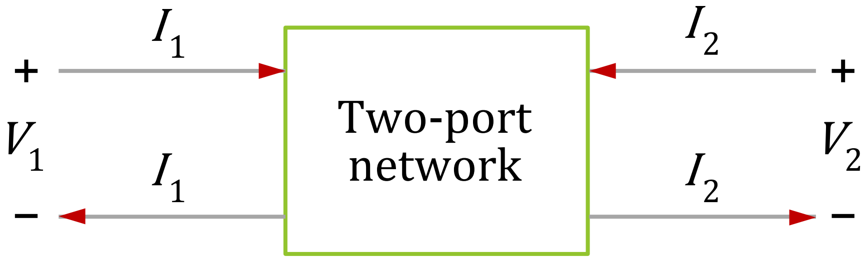
Introduction to Two-Port Network

A two-port network has four variables – two currents and two voltages.

Different groupings of current and voltage variables lead to different kinds of characteristic parameters.

Four such two-port parameters are commonly studied, namely,

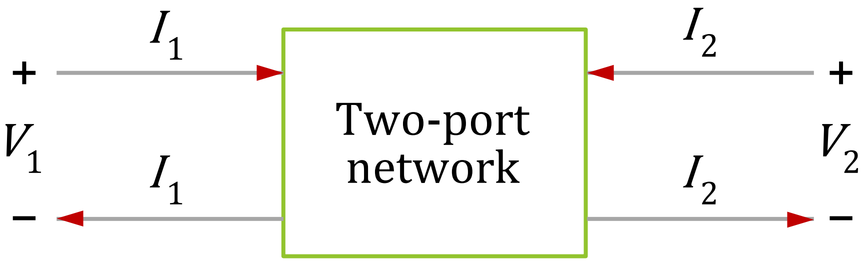
1. Admittance (or y),
2. Impedance (or z),
3. Hybrid (or h) and
4. Transmission (or $ABCD$) parameters.



A Two-port Network

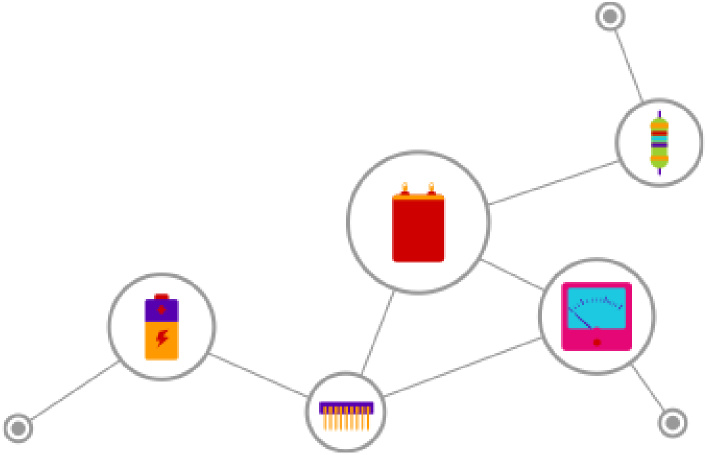
Introduction to Two-Port Network

In a linear two-port network, a variable pair is related to the other variable pair by a pair of linear equations.



A Two-port Network

All the equivalent circuits considered can be used to simplify the analysis of circuits used in communication and power transmission systems.



Introduction to Two-Port Network: Admittance

1. Admittance Parameters

Under the assumptions that there are no internal independent sources and zero initial conditions, the admittance parameters of a two-port relate the port voltages V_1 and V_2 to the terminal currents I_1 and I_2, i.e.,

$$\boxed{I_1 = y_{11}V_1 + y_{12}V_2} \quad \textbf{(1)}$$

$$\boxed{I_2 = y_{21}V_1 + y_{22}V_2} \quad \textbf{(2)}$$

A Two-port Network

Where the $y_{ij}{}'s$ are known as the admittance parameters expressed in S, Siemens (Ω^{-1}).

Introduction to Two-Port Network: Admittance

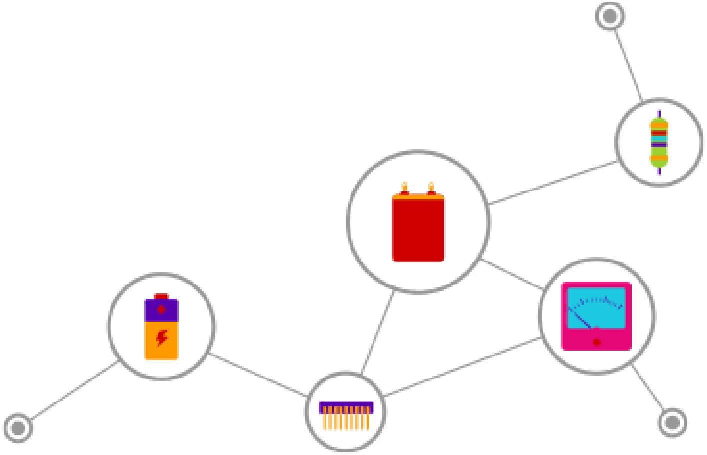
In matrix form, we have

$$I_1 = y_{11}V_1 + y_{12}V_2 \quad \textbf{(1)}$$

$$I_2 = y_{21}V_1 + y_{22}V_2 \quad \textbf{(2)}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

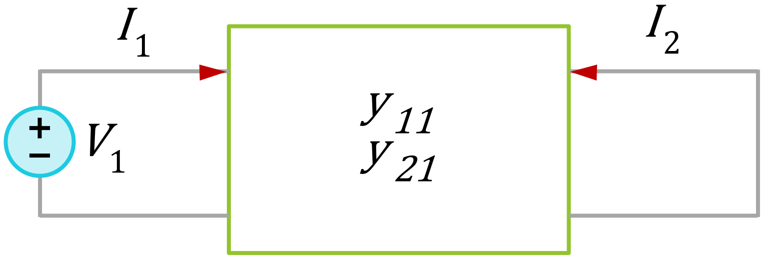
$$[y] = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$$



Introduction to Two-Port Network: Admittance

One can calculate the y parameters using the superposition from (1) – (2) as follows:

Connect an independent voltage source V_1 to port 1 and **short-circuit port 2** $(V_2 = 0)$. Measure I_1 and I_2.



From (1) – (2)

$$I_1 = y_{11}V_1 + \cancel{y_{12}V_2} \quad (1)$$

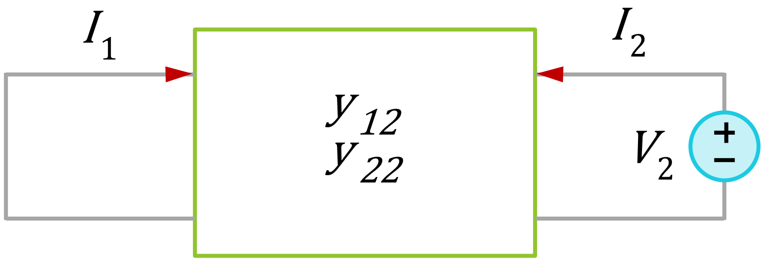
$$I_2 = y_{21}V_1 + \cancel{y_{22}V_2} \quad (2)$$

$$y_{11} = \frac{I_1}{V_1}\Big|_{V_2=0}$$

$$y_{21} = \frac{I_2}{V_1}\Big|_{V_2=0}$$

Introduction to Two-Port Network: Admittance

Connect an independent voltage source V_2 to port 2 and short-circuit port 1 $(V_1 = 0)$. Measure I_1 and I_2.



From (1) – (2)

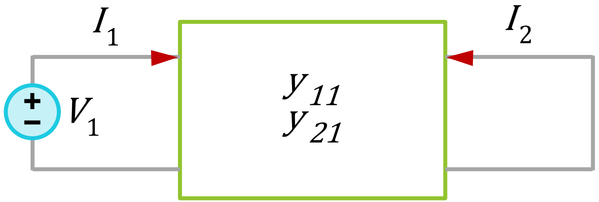
$$I_1 = \cancel{y_{11}V_1} + y_{12}V_2 \quad \textbf{(1)}$$

$$y_{12} = \frac{I_1}{V_2}\Big|_{V_1=0}$$

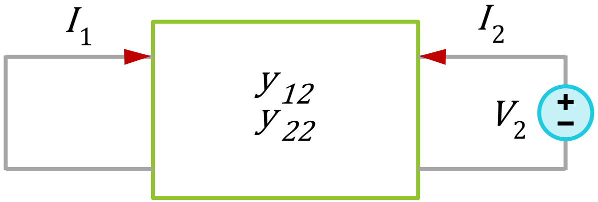
$$I_2 = \cancel{y_{21}V_1} + y_{22}V_2 \quad \textbf{(2)}$$

$$y_{22} = \frac{I_2}{V_2}\Big|_{V_1=0}$$

Introduction to Two-Port Network: Admittance



Short-circuit Port 2



Short-circuit Port 1

Definitions:

y_{11} = short-circuit input admittance

y_{21} = short-circuit forward transfer admittance

y_{12} = short-circuit reverse transfer admittance

y_{22} = short-circuit output admittance

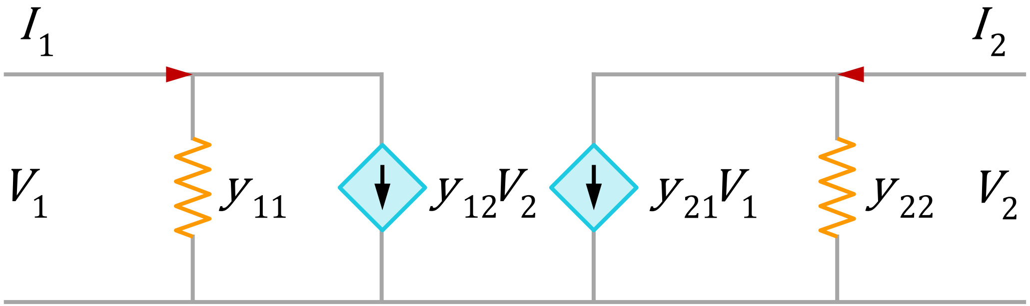
Introduction to Two-Port Network: Admittance

The y parameters are also referred to as the **short-circuit admittance parameters** since one of the ports is always short-circuited while making the measurements.

A circuit representation of (1) – (2) is shown.

$$\boxed{I_1 = y_{11}V_1 + y_{12}V_2}$$

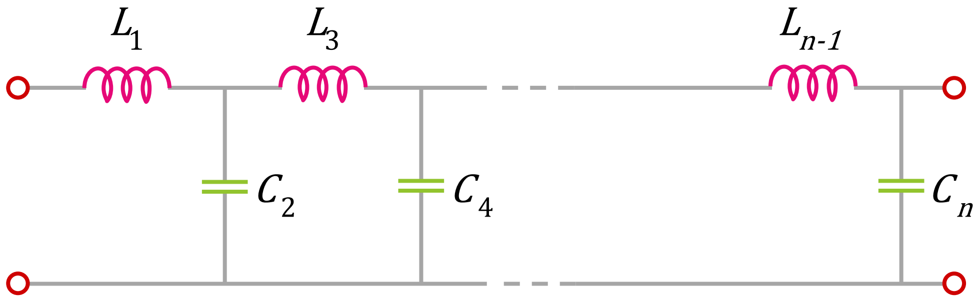
$$\boxed{I_2 = y_{21}V_1 + y_{22}V_2}$$

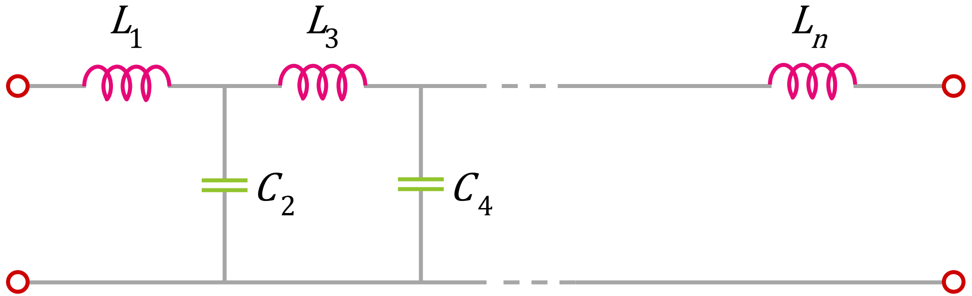


Two-dependent Source Equivalent Circuit

Introduction to Two-Port Network: Admittance

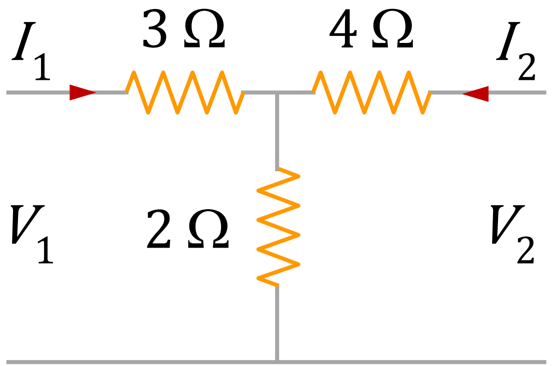
Remark: An application of the y parameters is the synthesis of ladder networks used in designing passive low-pass filters.





Introduction to Two-Port Network: Example 1

Find the y parameters for the network shown. The resistances are in Ω.



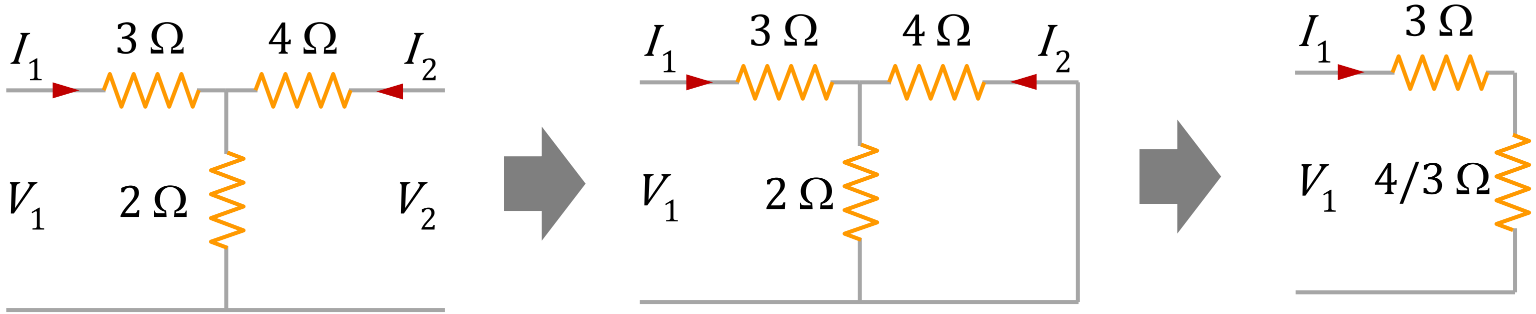
Introduction to Two-Port Network: Example 1

Short-circuit Port 2

$$I_1 = y_{11}V_1 + \cancel{y_{12}V_2} \Rightarrow y_{11} = \frac{I_1}{V_1}$$

$$I_2 = y_{21}V_1 + \cancel{y_{22}V_2} \Rightarrow y_{21} = \frac{I_2}{V_1}$$

With $V_2 = 0$ and applying a voltage source to port 1, we have

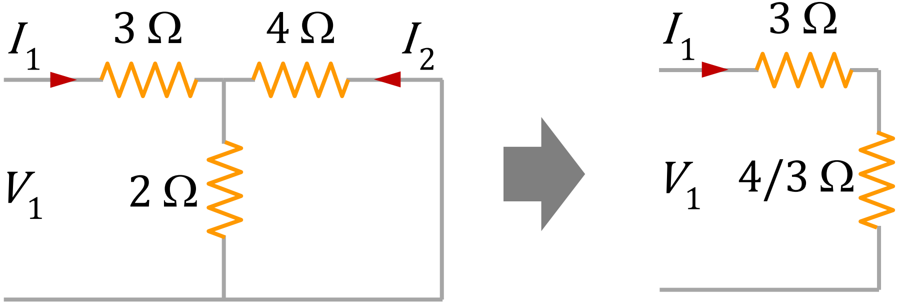


Introduction to Two-Port Network: Example 1

$$y_{11} = \frac{I_1}{V_1} = \frac{1}{3 + \frac{4}{3}} = \frac{3}{13}\,\Omega^{-1}$$

$$I_2 = -\frac{2}{2+4} I_1 = -\frac{1}{3} I_1$$ **(current division)**

$$y_{21} = \frac{I_2}{V_1} = \frac{-I_1}{3V_1} = -\frac{1}{13}\,\Omega^{-1}$$



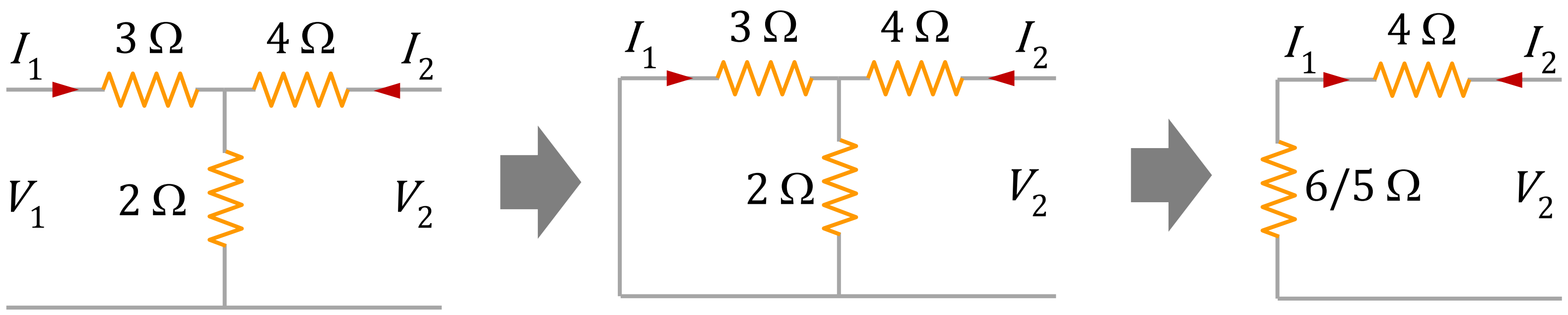
Introduction to Two-Port Network: Example 1

Short-circuit Port 1

$$I_1 = \cancel{y_{11}V_1} + y_{12}V_2 \quad \Rightarrow \quad \boxed{y_{12} = \frac{I_1}{V_2}}$$

$$I_2 = \cancel{y_{21}V_1} + y_{22}V_2 \quad \Rightarrow \quad \boxed{y_{22} = \frac{I_2}{V_2}}$$

With $V_1 = 0$ and applying a voltage source to port 2, we have

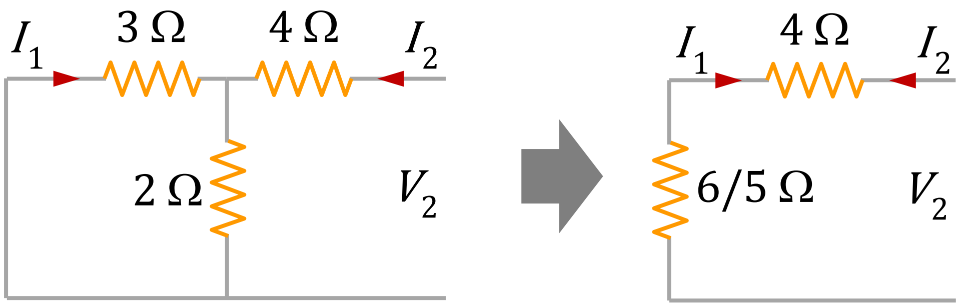


Introduction to Two-Port Network: Example 1

$$y_{22} = \frac{I_2}{V_2} = \frac{1}{4+\frac{6}{5}} = \frac{5}{26}\,\Omega^{-1}$$

$$I_1 = -\frac{2}{3+2} I_2 = -\frac{2}{5} I_2$$ (current division)

$$y_{12} = \frac{I_1}{V_2} = -\frac{2I_2}{5V_2} = -\frac{1}{13}\,\Omega^{-1}$$



Introduction to Two-Port Network: Example 1

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} \frac{3}{13} & -\frac{1}{13} \\ -\frac{1}{13} & \frac{5}{26} \end{bmatrix}$$

Remark: The method used here is tedious in that we must set up two separate experiments.

Another way of calculating the y parameters without the need for experiments is to write **KVL** or **KCL** equations for the two-port, and then arrange them in the form as (1) – (2).

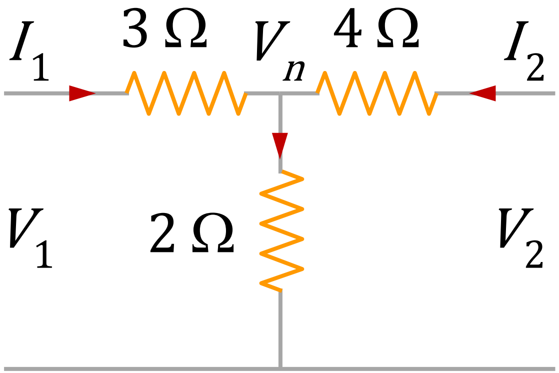
Introduction to Two-Port Network: Example 1

The y parameters could be obtained using nodal analysis as follows:

At the upper node

$$I_1 + \frac{V_2 - V_n}{4} - \frac{V_n}{2} = 0 \quad \text{(i)}$$

$$I_2 + \frac{V_1 - V_n}{3} - \frac{V_n}{2} = 0 \quad \text{(ii)}$$



Express I_1 in terms of V_n and V_1 gives

$$\frac{V_1 - V_n}{3} + \frac{V_2 - V_n}{4} - \frac{V_n}{2} = 0 \Rightarrow V_n = \frac{4}{13}V_1 + \frac{3}{13}V_2 \quad \text{(iii)}$$

Introduction to Two-Port Network: Example 1

Using (iii) in (i) and (ii) gives

$$I_1 + \frac{V_2 - V_n}{4} - \frac{V_n}{2} = 0 \quad \text{(i)}$$

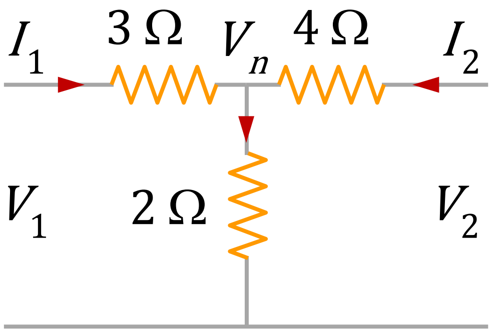
$$I_2 + \frac{V_1 - V_n}{3} - \frac{V_n}{2} = 0 \quad \text{(ii)}$$

$$V_n = \frac{4}{13}V_1 + \frac{3}{13}V_2 \quad \text{(iii)}$$

$$I_1 = \frac{3}{13}V_1 + (-\frac{1}{13})V_2$$

$$I_2 = (-\frac{1}{13})V_1 + \frac{5}{26}V_2$$

And the answer is the same as before.



Introduction to Two-Port Network: Example 2

Now, if a $39\ \mathrm{V}$ source is connected at port 1, and a $26\ \mathrm{V}$ source is connected at port 2. Find the currents I_1 and I_2.

The y parameter equations are given by

$$I_1 = y_{11}V_1 + y_{12}V_2 \quad \textbf{(1)}$$

$$I_2 = y_{21}V_1 + y_{22}V_2 \quad \textbf{(2)}$$

With $V_1 = 39\ \mathrm{V}$ and $V_2 = 26\ \mathrm{V}$, using (1) and (2), we have

$$I_1 = \frac{3}{13}(39) + (-\frac{1}{13})(26) = 7\ \mathrm{A}$$

$$I_2 = (-\frac{1}{13})(39) + \frac{5}{26}(26) = 2\ \mathrm{A}$$

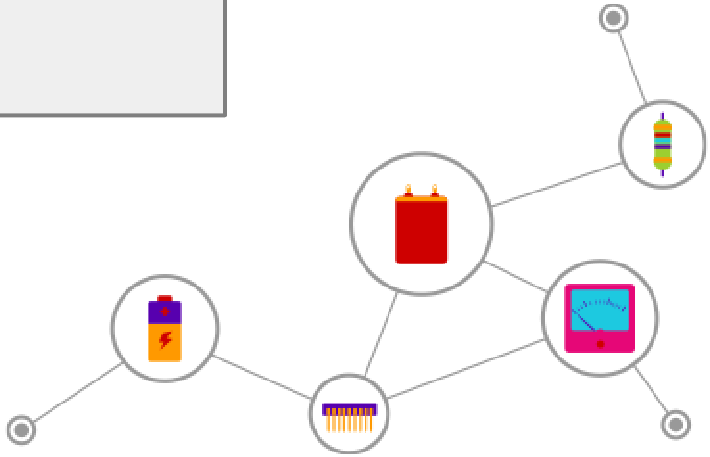
Introduction to Two-Port Network: Example 2

Note: The example uses DC voltage sources and currents. But the equations (1) and (2) apply equally well with alternating voltages and currents.

V and I are then the transforms of voltage and current, and the admittances are complex quantities, functions of $j\omega$ or s.

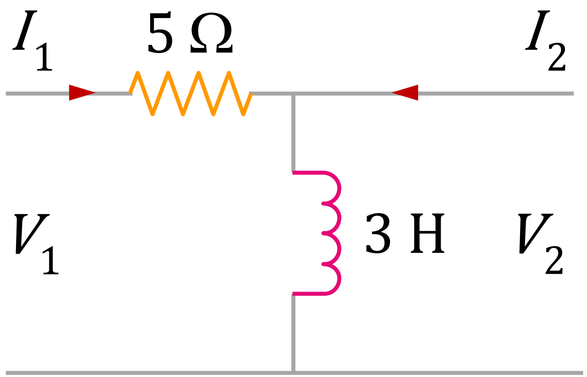
$$\boxed{I_1 = y_{11}V_1 + y_{12}V_2} \quad (1)$$

$$\boxed{I_2 = y_{21}V_1 + y_{22}V_2} \quad (2)$$



Introduction to Two-Port Network: Example 3

In the network shown, compute the y parameters.



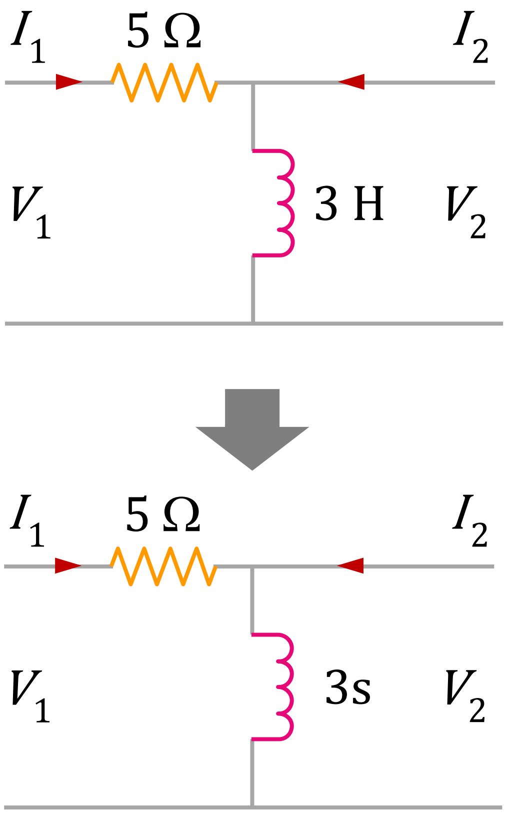
Introduction to Two-Port Network: Example 3

Due to the presence of an energy storage element, we will work in the s-domain.

At the upper node, we have

$$I_1 + I_2 = \frac{V_2}{3s} \quad \textbf{(i)}$$

$$I_2 = \frac{V_2}{3s} - \frac{V_1 - V_2}{5} \quad \textbf{(ii)}$$



Using (ii) in (i) and rearranging (ii) gives

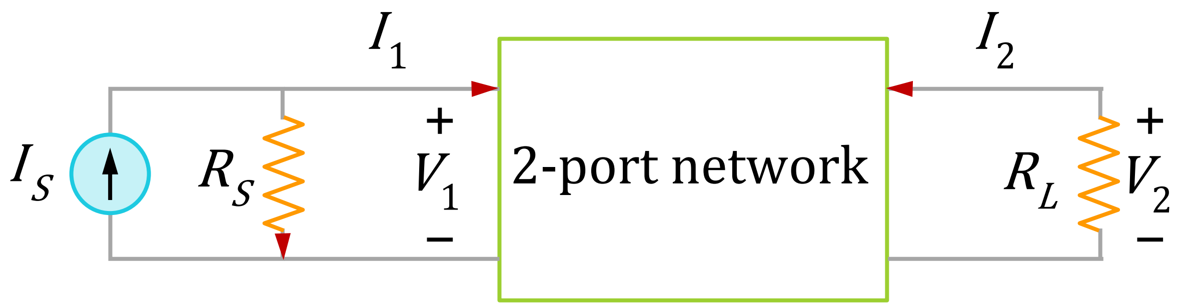
$$I_1 = \frac{1}{5}V_1 - \frac{1}{5}V_2$$

$$I_2 = -\frac{1}{5}V_1 + \frac{3s+5}{15s}V_2$$

Introduction to Two-Port Network: Example 4

The two-port network shown is driven by a current source with $R_s = 200\ \Omega$ and is loaded with $R_L = 20\ \Omega$.

The network parameters are: $y_{11} = 15\ mS$, $y_{12} = -5\ mS$, $y_{21} = 100\ mS$, $y_{22} = 10\ mS$. Find (a) the input resistance V_1/I_1, (b) the current gain I_2/I_1, and (c) the voltage gain V_2/V_1.

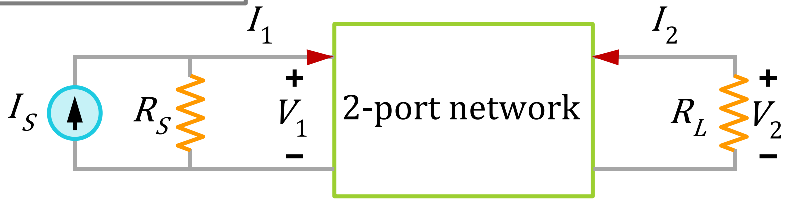


Introduction to Two-Port Network: Example 4

The y parameter equations are

$$I_1 = y_{11}V_1 + y_{12}V_2 \quad \textbf{(1)} \Rightarrow \quad I_1 = 15V_1 - 5V_2 \text{ mA}$$

$$I_2 = y_{21}V_1 + y_{22}V_2 \quad \textbf{(2)} \Rightarrow \quad I_2 = 100V_1 + 10V_2 \text{ mA}$$



But $V_2 = -20I_2 \text{ mV},\ V_1 = 200(I_s - I_1) \text{ mV}.$

Using them in I_1 and I_2 gives

$$I_1 = 3(I_s - I_1) + 0.1I_2$$

$$I_2 = 20(I_s - I_1) - 0.2I_2$$

or

$$4I_1 - 0.1I_2 = 3I_s$$

$$20I_1 + 1.2I_2 = 20I_s$$

Introduction to Two-Port Network: Example 4

Solving for I_1, I_2, V_1 and V_2 gives

$$I_1 = 15V_1 - 5V_2 \quad \text{mA}$$

$$I_2 = 100V_1 + 10V_2 \quad \text{mA}$$

$$4I_1 - 0.1I_2 = 3I_s$$

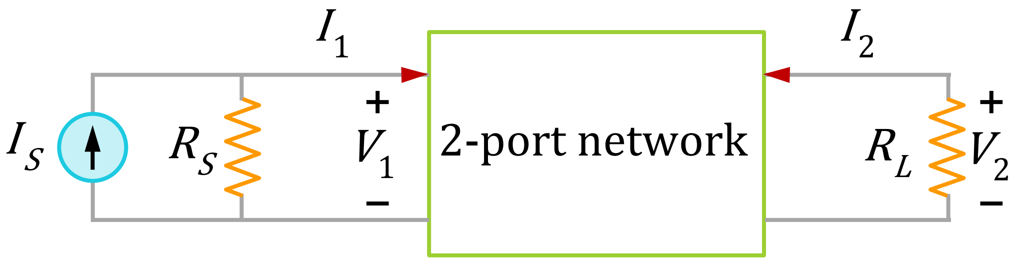
$$20I_1 + 1.2I_2 = 20I_s$$

$$I_1 = 0.00082I_s \text{ A}$$

$$I_2 = 0.002941I_s \text{ A}$$

$$V_1 = 0.2(I_s - 0.82I_s) = 0.0352I_s \text{ V}$$

$$V_2 = -20I_2 = -0.02(2.941)I_s = -0.0588I_s \text{ V}$$



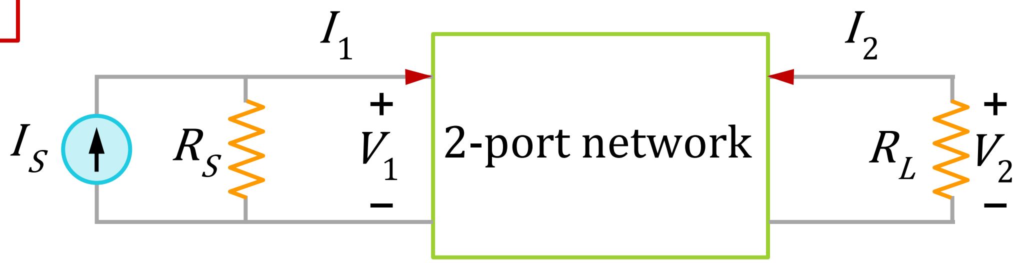
Introduction to Two-Port Network: Example 4

Thus,

$$Z_{in} = \frac{V_1}{I_1} = \frac{0.0352 I_s}{0.00082 I_s} = 42.75\ \Omega$$

$$\frac{I_2}{I_1} = \frac{0.002941 I_s}{0.00082 I_s} = 3.57$$

$$\frac{V_2}{V_1} = -\frac{0.0588 I_s}{0.0352 I_s} = -1.67$$



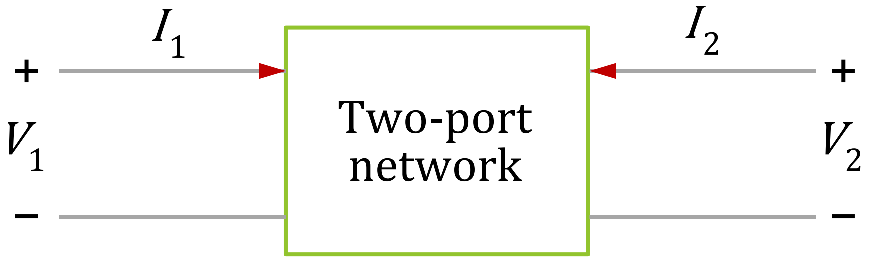
Introduction to Two-Port Network: Impedance

2. Impedance Parameters

Under the assumptions that there are no internal independent sources and zero initial conditions, the impedance parameters of a two-port relate the terminal currents I_1 and I_2 to the port voltages V_1 and V_2, i.e.,

$$\boxed{V_1 = z_{11}I_1 + z_{12}I_2} \quad (3)$$

$$\boxed{V_2 = z_{21}I_1 + z_{22}I_2} \quad (4)$$



A Two-port Network

Where the $z_{ij}'s$ are known as the impedance parameters expressed Ω.

Introduction to Two-Port Network: Impedance

In matrix form, we have

$$V_1 = z_{11} I_1 + z_{12} I_2 \qquad \textbf{(3)}$$

$$V_2 = z_{21} I_1 + z_{22} I_2 \qquad \textbf{(4)}$$

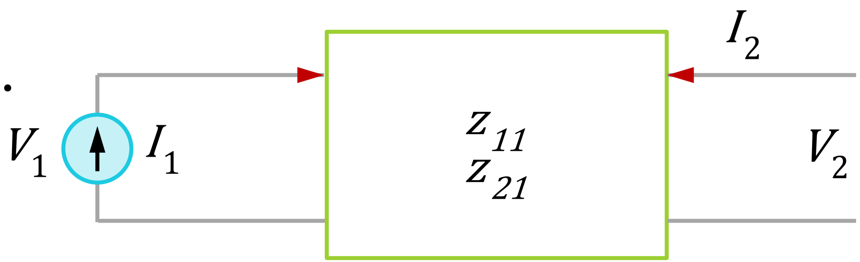
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$[z] = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

Introduction to Two-Port Network: Impedance

One can calculate the z parameters using the superposition from (3) – (4) as follows:

Connect an independent current source I_1 at port 1 and leave **port 2 open-circuited** $(I_2 = 0)$. Measure V_1 and V_2 .



From (3) – (4)

$$V_1 = z_{11}I_1 + \cancel{z_{12}I_2} \quad \textbf{(3)}$$

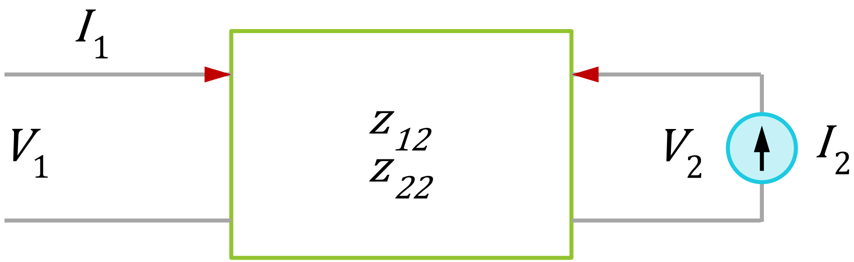
$$z_{11} = \frac{V_1}{I_1}\Big|_{I_2=0}$$

$$V_2 = z_{21}I_1 + \cancel{z_{22}I_2} \quad \textbf{(4)}$$

$$z_{21} = \frac{V_2}{I_1}\Big|_{I_2=0}$$

Introduction to Two-Port Network: Impedance

Connect an independent current source I_2 at port 2 and leave port 1 open-circuited $(I_1 = 0)$. Measure V_1 and V_2 .



From (3) – (4)

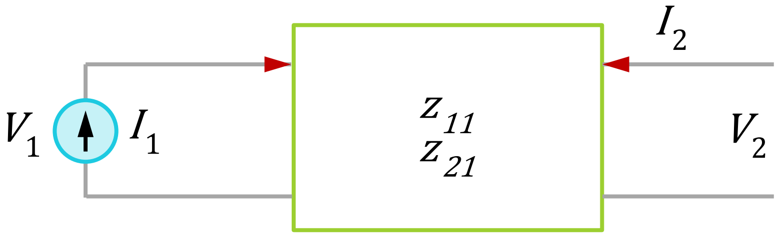
$$V_1 = \cancel{z_{11}I_1} + z_{12}I_2 \quad \textbf{(3)}$$

$$z_{12} = \frac{V_1}{I_2}\Big|_{I_1=0}$$

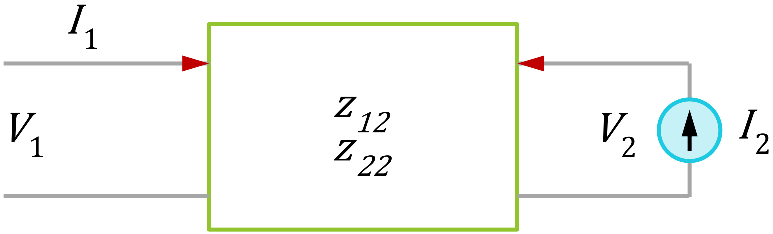
$$V_2 = \cancel{z_{21}I_1} + z_{22}I_2 \quad \textbf{(4)}$$

$$z_{22} = \frac{V_2}{I_2}\Big|_{I_1=0}$$

Introduction to Two-Port Network: Impedance



Open-circuit Port 2



Open-circuit Port 1

Definitions:

z_{11} = open-circuit input impedance

z_{21} = open-circuit transfer impedance

z_{12} = open-circuit reverse transfer impedance

z_{22} = open-circuit output impedance

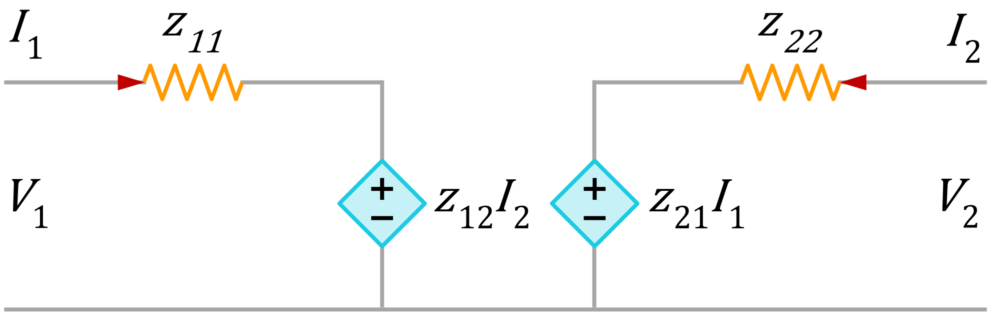
Introduction to Two-Port Network: Impedance

These parameters are referred to as **open-circuit impedance parameters** since one of the ports is always open circuit while making the measurements at the other port.

A circuit representation of (3) – (4) is shown.

$$V_1 = z_{11}I_1 + z_{12}I_2$$

$$V_2 = z_{21}I_1 + z_{22}I_2$$



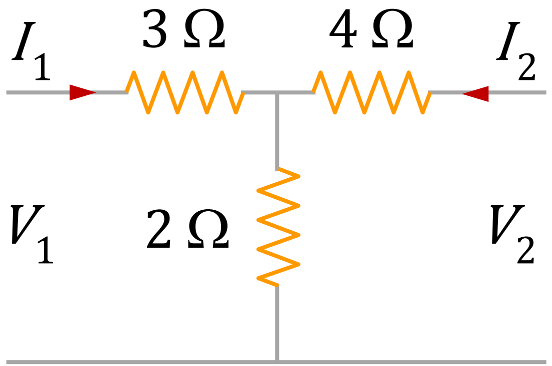
Two-dependent Source Equivalent Circuit

Introduction to Two-Port Network: Impedance

Remark: An application of the z parameters is the synthesis of ladder networks used in designing passive low-pass filters.

Introduction to Two-Port Network: Example 5

Find the z parameters for the network shown. The resistances are in Ω.

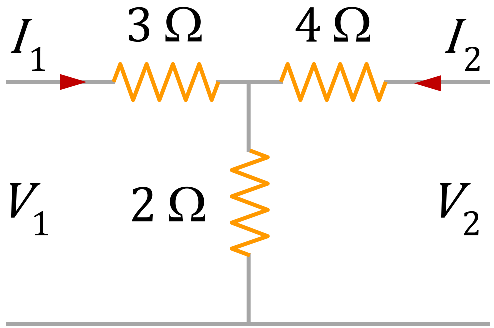


Introduction to Two-Port Network: Example 5

The z parameter equations are given by

$$V_1 = z_{11}I_1 + z_{12}I_2 \quad \textbf{(3)}$$

$$V_2 = z_{21}I_1 + z_{22}I_2 \quad \textbf{(4)}$$



Using Mesh analysis, we get

$$V_1 = 3I_1 + 2(I_1 + I_2) = 5I_1 + 2I_2$$

$$V_2 = 4I_2 + 2(I_1 + I_2) = 2I_1 + 6I_2$$

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 6 \end{bmatrix}$$

Introduction to Two-Port Network: Example 6

Suppose an independent current source forces $4\ \mathrm{A}$ to enter port 1, while another source forces $1\ \mathrm{A}$ to enter port 2 (both in the arrow directions). What are the voltages at the two ports?

The z parameter equations are given by

$$V_1 = 5I_1 + 2I_2$$

$$V_2 = 2I_1 + 6I_2$$

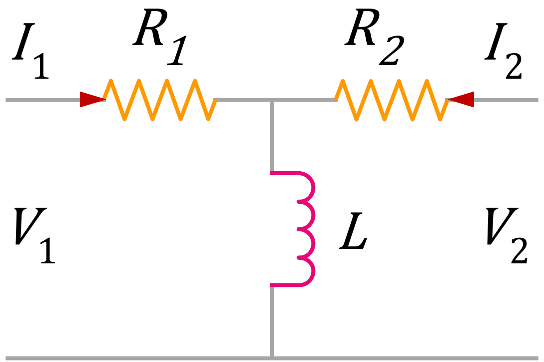
With $I_1 = 4\ \mathrm{A}$ and $\mathrm{I}_2 = 1\ \mathrm{A}$, we have

$$V_1 = 5(4) + 2(1) = 22\ \mathrm{V}$$

$$V_2 = 2(4) + 6(1) = 14\ \mathrm{V}$$

Introduction to Two-Port Network: Example 7

Consider the following two-port configuration.
Compute the z parameters.



Introduction to Two-Port Network: Example 7

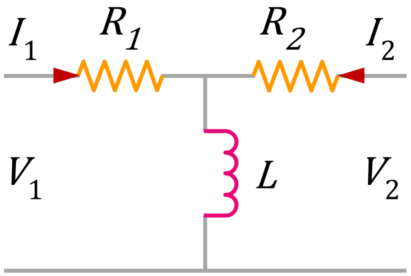
Using mesh analysis, we have

$$V_1 = R_1 I_1 + sL(I_1 + I_2) \quad \text{(i)}$$

$$V_1 = z_{11} I_1 + z_{12} I_2 \quad \text{(3)}$$

$$V_2 = R_2 I_2 + sL(I_1 + I_2) \quad \text{(ii)}$$

$$V_2 = z_{21} I_1 + z_{22} I_2 \quad \text{(4)}$$



Rearranging (i) and (ii) in the form of (3) – (4) gives

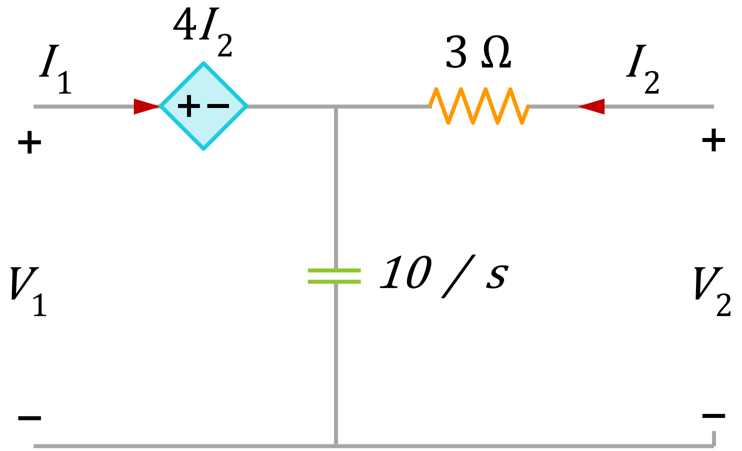
$$V_1 = (R_1 + sL) I_1 + sL I_2$$

$$V_2 = sL I_1 + (R_2 + sL) I_2$$

$$\boxed{\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} R_1 + sL & sL \\ sL & R_2 + sL \end{bmatrix}}$$

Introduction to Two-Port Network: Example 8

Find the z parameters for the circuit as shown.



Introduction to Two-Port Network: Example 8

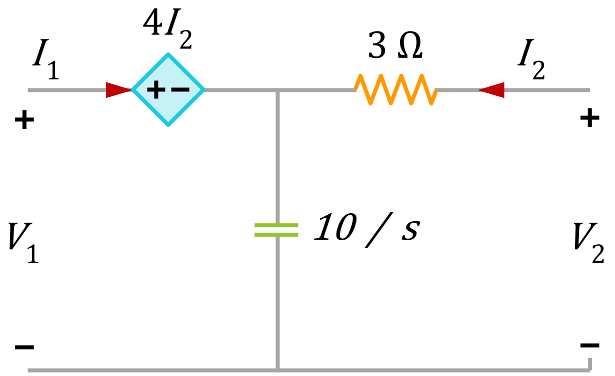
The loop equations are

$$V_1 = 4I_2 + \frac{10}{s}(I_1 + I_2) \quad \Rightarrow \quad V_1 = \frac{10}{s}I_1 + (4 + \frac{10}{s})I_2$$

$$V_2 = 3I_2 + \frac{10}{s}(I_1 + I_2) \quad \Rightarrow \quad V_2 = \frac{10}{s}I_1 + (3 + \frac{10}{s})I_2$$

Rearranging gives

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} \frac{10}{s} & 4 + \frac{10}{s} \\ \frac{10}{s} & 3 + \frac{10}{s} \end{bmatrix}$$



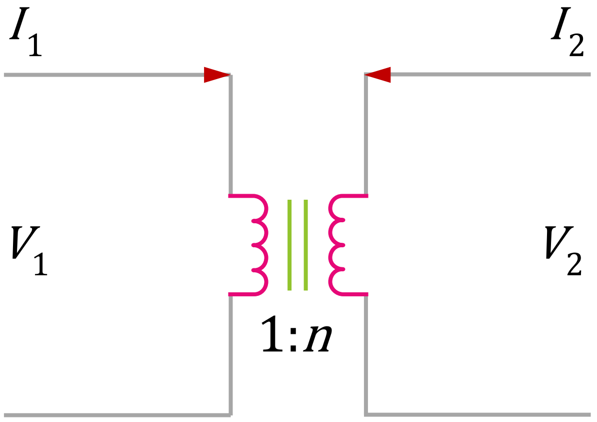
Introduction to Two-Port Network: Hybrid

3. Hybrid Parameters

There are some circuits that have z parameters, but not y parameters, and conversely. A circuit element that has neither is the ideal transformer as shown.

For the ideal transformer, we have $V_2 = nV_1$ and $I_1 = -nI_2$.

Clearly, V_1 and V_2 cannot be expressed as functions of I_1 and I_2; nor can I_1 and I_2 be expressed as functions of V_1 and V_2.



An Ideal Transformer

Thus, we need an alternative modeling technique for two-port analysis. The hybrid parameters provide one of several alternatives.

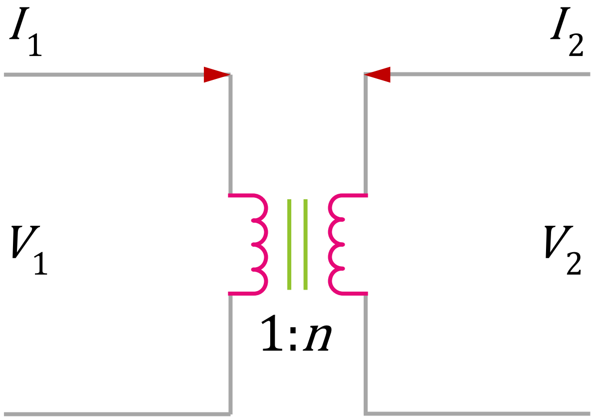
Introduction to Two-Port Network: Hybrid

3. Hybrid Parameters

If the two-port contains no internal independent sources and has no initial stored energy, then the hybrid parameters are defined by the equations:

$$\boxed{V_1 = h_{11}I_1 + h_{12}V_2} \quad \textbf{(5)}$$

$$\boxed{I_2 = h_{21}I_1 + h_{22}V_2} \quad \textbf{(6)}$$



An Ideal Transformer

Where h_{11} has units of Ohms (Ω), h_{12} and h_{21} are dimensionless, and h_{22} has units of Siemens (Ω^{-1}).

Introduction to Two-Port Network: Hybrid

In matrix form, we have

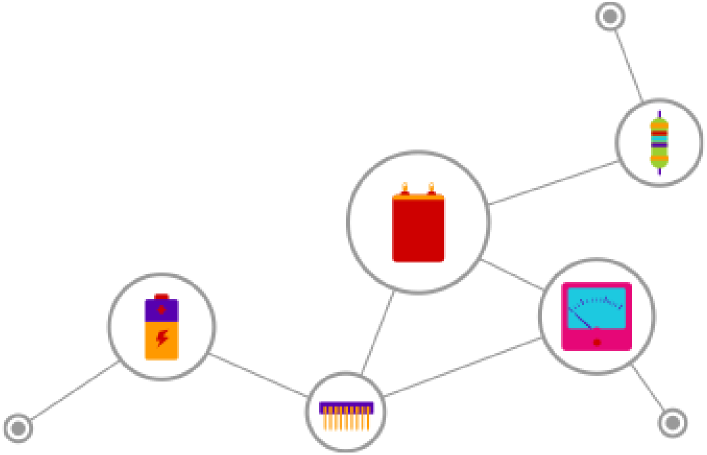
$$V_1 = h_{11} I_1 + h_{12} V_2 \quad \textbf{(5)}$$

$$I_2 = h_{21} I_1 + h_{22} V_2 \quad \textbf{(6)}$$

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

$$[h] = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

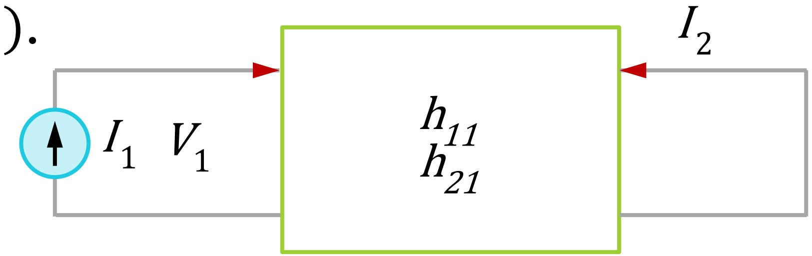
These parameters are called hybrid because we have chosen the independent quantities I_1 and V_2 from both ports.



Introduction to Two-Port Network: Hybrid

The h parameters may be computed as follows:

Connect an independent current source I_1 at port 1 and short-circuit port 2 $(V_2 = 0)$. Measure V_1 and I_2 .



From (5) – (6)

$$V_1 = h_{11} I_1 + \cancel{h_{12} V_2} \quad (5)$$

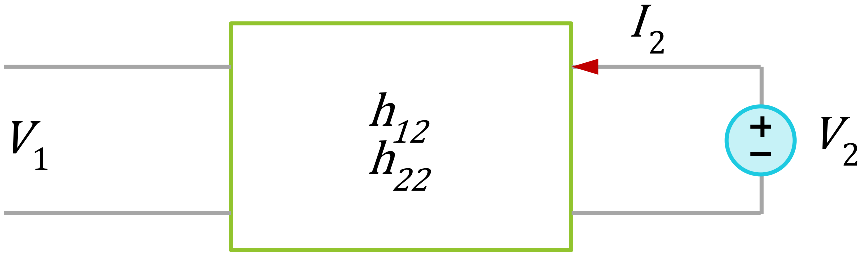
$$h_{11} = \frac{V_1}{I_1}\Big|_{V_2=0}$$

$$I_2 = h_{21} I_1 + \cancel{h_{22} V_2} \quad (6)$$

$$h_{21} = \frac{I_2}{I_1}\Big|_{V_2=0}$$

Introduction to Two-Port Network: Hybrid

Connect an independent voltage source V_2 at port 2 and leave port 1 open-circuited $(I_1 = 0)$. Measure V_1 and I_2 .



From (5) – (6)

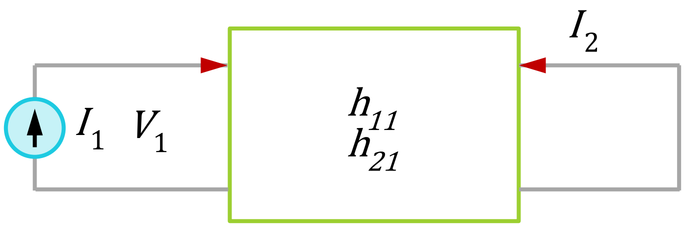
$$V_1 = \cancel{h_{11}I_1} + h_{12}V_2 \quad \textbf{(5)}$$

$$h_{12} = \frac{V_1}{V_2}\Big|_{I_1=0}$$

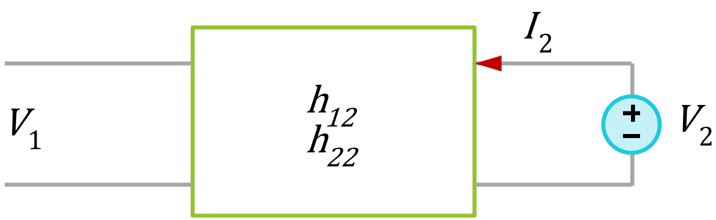
$$I_2 = \cancel{h_{21}I_1} + h_{22}V_2 \quad \textbf{(6)}$$

$$h_{22} = \frac{I_2}{V_2}\Big|_{I_1=0}$$

Introduction to Two-Port Network: Hybrid



Short-circuit Port 2



Open-circuit Port 1

Definitions:

h_{11} = short-circuit input impedance

h_{21} = short-circuit forward current gain

h_{12} = open-circuit reverse voltage gain

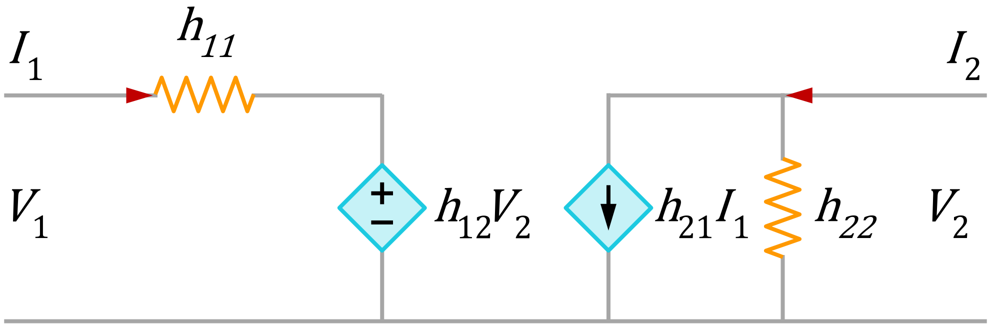
h_{22} = open-circuit output admittance

Introduction to Two-Port Network: Hybrid

A circuit representation of (5) – (6) is shown.

$$\boxed{V_1 = h_{11}I_1 + h_{12}V_2}$$

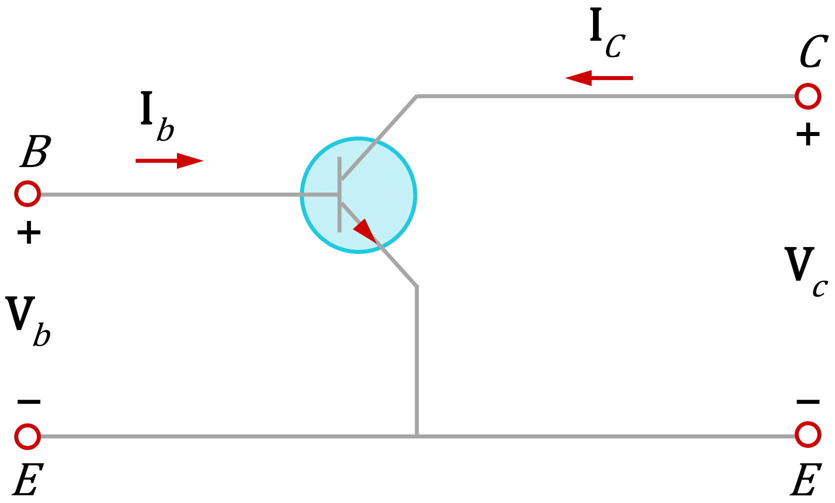
$$\boxed{I_2 = h_{21}I_1 + h_{22}V_2}$$



Two-dependent Source Equivalent Circuit

Introduction to Two-Port Network: Hybrid

Remark: The use of h parameters is well suited to transistor circuits.



Common-emitter Configuration

Introduction to Two-Port Network: Example 9

Determine the h parameters with the following data:

i. **With the output terminals short-circuited,** $V_1 = 25\ V$, $I_1 = 1\ A$, $I_2 = 2\ A$.

ii. **With the input terminals open-circuited,** $V_1 = 10\ V$, $V_2 = 50\ V$, $I_2 = 2\ A$.

The h parameter equations are given by

$$\boxed{V_1 = h_{11}I_1 + h_{12}V_2} \quad \textbf{(5)}$$

$$\boxed{I_2 = h_{21}I_1 + h_{22}V_2} \quad \textbf{(6)}$$

Introduction to Two-Port Network: Example 9

i. **With the output terminals short-circuited,**
$V_1 = 25\ V,\ I_1 = 1\ A,\ I_2 = 2\ A.$

$$\boxed{V_1 = h_{11}I_1 + h_{12}V_2} \quad \textbf{(5)}$$

$$\boxed{I_2 = h_{21}I_1 + h_{22}V_2} \quad \textbf{(6)}$$

With output short-circuited $(V_2 = 0)$

$$\boxed{h_{11} = \frac{V_1}{I_1} = \frac{25}{1} = 25\ \Omega}$$

$$\boxed{h_{21} = \frac{I_2}{I_1} = \frac{2}{1} = 2}$$

Introduction to Two-Port Network: Example 9

ii. With the input terminals open-circuited,

$V_1 = 10\ V, V_2 = 50\ V,\ \ I_2 = 2\ A.$

$$\boxed{V_1 = h_{11}I_1 + h_{12}V_2} \quad \textbf{(5)}$$

$$\boxed{I_2 = h_{21}I_1 + h_{22}V_2} \quad \textbf{(6)}$$

With input open-circuited $(I_1 = 0)$

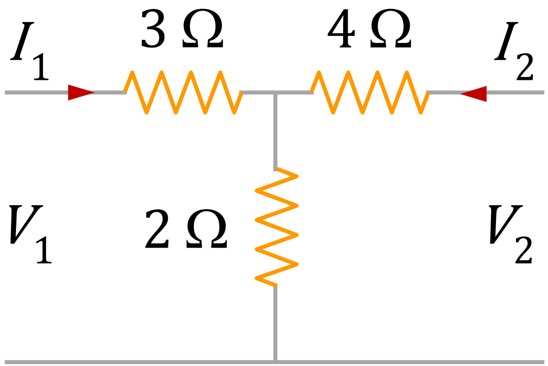
$$\boxed{h_{12} = \frac{V_1}{V_2} = \frac{10}{50} = 0.2}$$

$$\boxed{h_{22} = \frac{I_2}{V_2} = \frac{2}{50} = 0.04\ \ \Omega^{-1}}$$

$$\Rightarrow \quad \boxed{\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 25 & 0.2 \\ 2 & 0.04 \end{bmatrix}}$$

Introduction to Two-Port Network: Example 10

Find the h parameters for the network shown. The resistances are in Ω.

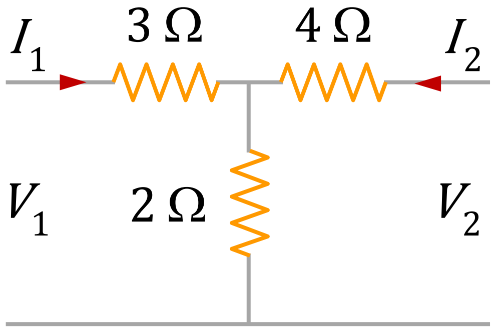


Introduction to Two-Port Network: Example 10

The h parameter equations are given by

$$V_1 = h_{11}I_1 + h_{12}V_2 \quad \textbf{(5)}$$

$$I_2 = h_{21}I_1 + h_{22}V_2 \quad \textbf{(6)}$$



Using Mesh analysis, we get

$$V_1 = 3I_1 + 2(I_1 + I_2) = 5I_1 + 2I_2 \quad \textbf{(i)}$$

$$V_2 = 4I_2 + 2(I_1 + I_2) = 2I_1 + 6I_2 \Rightarrow I_2 = \frac{1}{6}(V_2 - 2I_1)$$

Introduction to Two-Port Network: Example 10

Using I_2 in (i) gives

$$V_1 = 3I_1 + 2(I_1 + I_2) = 5I_1 + 2I_2 \quad \text{(i)}$$

$$I_2 = \frac{1}{6}(V_2 - 2I_1)$$

$$V_1 = 5I_1 + \frac{2}{6}(-2I_1 + V_2) = \frac{13}{3}I_1 + \frac{1}{3}V_2$$

Thus,

$$V_1 = \frac{13}{3}I_1 + \frac{1}{3}V_2$$

$$I_2 = -\frac{1}{3}I_1 + \frac{1}{6}V_2$$

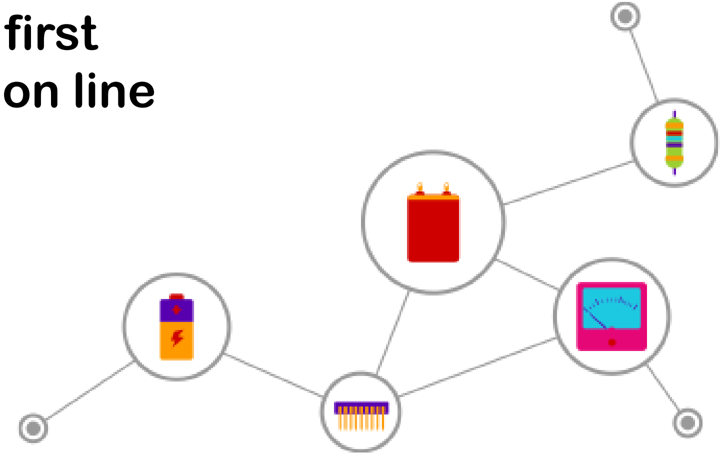
$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \frac{13}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

Introduction to Two-Port Network: Transmission (*ABCD*)

4. Transmission (*ABCD*)

Two-port problems can be solved with the aid of either the y, z or h parameters. Nevertheless, we shall find it convenient to introduce another set, the $ABCD$ parameters. They are convenient because they give voltage and current at one port in terms of voltage and current at the other port.

Historically, the transmission parameters were first used by power system engineers for transmission line analysis and are still used today.

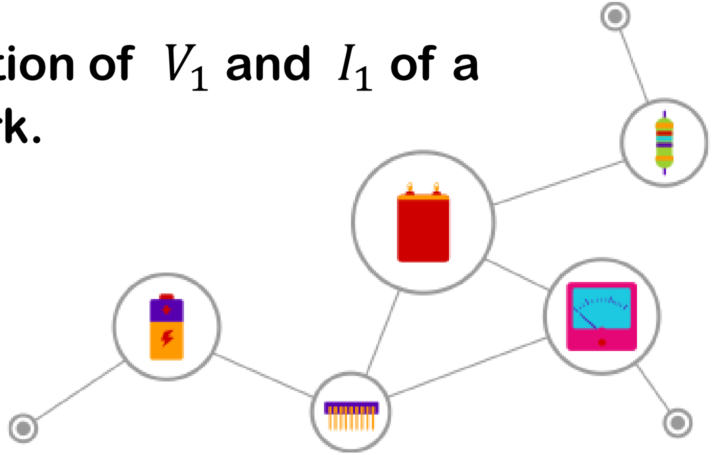


Introduction to Two-Port Network: Transmission (*ABCD*)

Power system engineers often have to solve for the magnitude of sending-end voltage to be maintained in the line in order to ensure a pre-specified voltage magnitude at the receiving-end, when the receiving-end is delivering a pre-specified power at a certain power factor.

In addition, they would like to solve for the sending-end line current, sending-end active and reactive power, etc., under this condition.

The problem essentially involves the determination of V_1 and I_1 of a two-port network given V_2 and I_2 of the network.



Introduction to Two-Port Network: Transmission (*ABCD*)

A similar problem arises in filtering context.

The filter engineer wants to study the attenuation and phase shift suffered by a signal when it goes through a linear time-invariant two-port network.

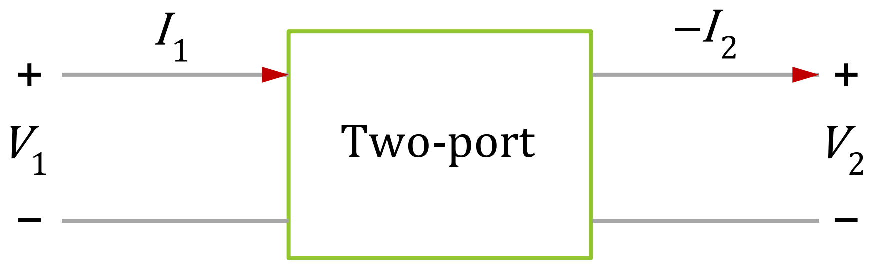
Introduction to Two-Port Network: Transmission (*ABCD*)

Consider the two-port network where at port 1, current I_1 is going into the network and at port 2, current I_2 is going out of the network.

Suppose we take V_2 and I_2 of port 2 as independent quantities and express V_1 and I_1 of port 1 as follows:

$$\boxed{V_1 = AV_2 - BI_2} \quad (7)$$

$$\boxed{I_1 = CV_2 - DI_2} \quad (8)$$



A Two-port Network

Note that A and D have no units, B has the units of Ohm and C has the units of Siemens.

Introduction to Two-Port Network: Transmission (*ABCD*)

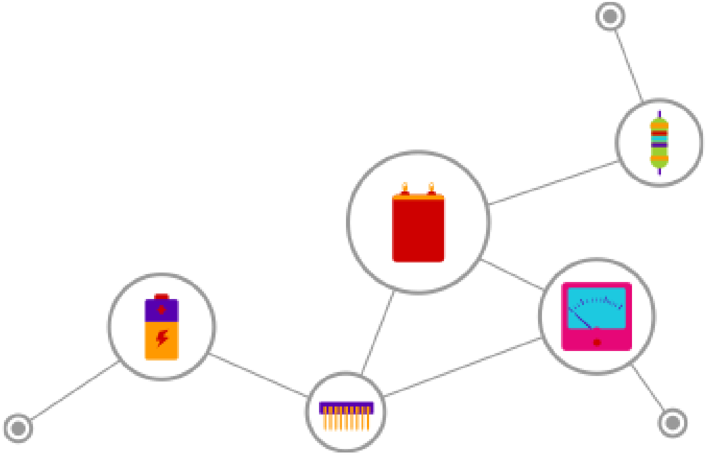
In matrix form, we have

$$V_1 = AV_2 - BI_2 \quad \textbf{(7)}$$

$$I_1 = CV_2 - DI_2 \quad \textbf{(8)}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

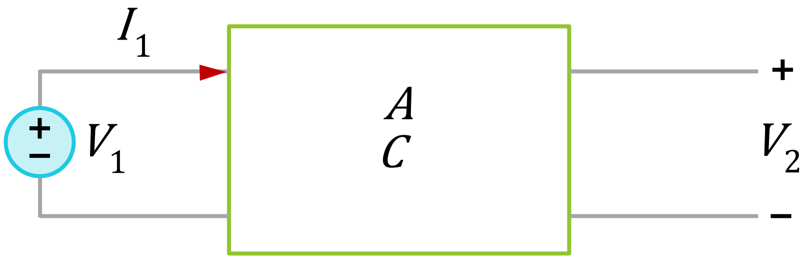
$$[T] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$



Introduction to Two-Port Network: Transmission (*ABCD*)

The $ABCD$ parameters may be computed as follows:

Connect an independent voltage source V_1 at port 1and leave port 2 open-circuited $(I_2 = 0)$. Measure V_2 and I_1.



From (7) – (8)

$$V_1 = AV_2 - \cancel{BI_2} \quad (7)$$

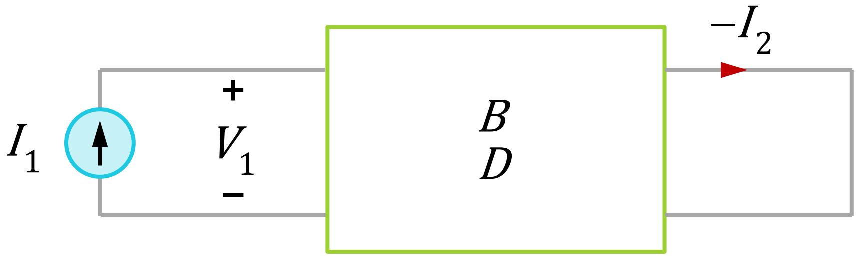
$$A = \frac{V_1}{V_2}\Big|_{I_2=0}$$

$$I_1 = CV_2 - \cancel{DI_2} \quad (8)$$

$$C = \frac{I_1}{V_2}\Big|_{I_2=0}$$

Introduction to Two-Port Network: Transmission (*ABCD*)

Connect an independent current source I_1 at port 1 and short-circuit port 2 $(V_2 = 0)$. Measure V_1 and I_2.



From (7) – (8)

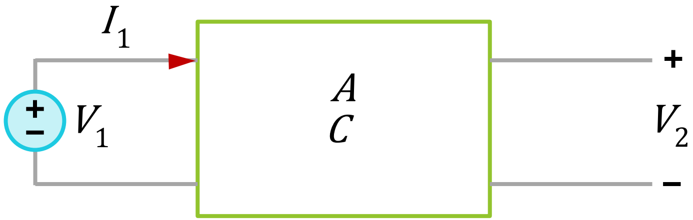
$$V_1 = \cancel{AV_2} - BI_2 \quad \textbf{(7)}$$

$$B = \frac{V_1}{-I_2}\Big|_{V_2=0}$$

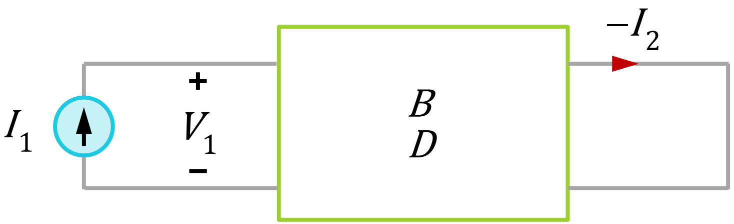
$$I_1 = \cancel{CV_2} - DI_2 \quad \textbf{(8)}$$

$$D = \frac{I_1}{-I_2}\Big|_{V_2=0}$$

Introduction to Two-Port Network: Transmission (*ABCD*)



Open-circuit Port 2



Short-circuit Port 2

Definitions:

A = open-circuit reverse voltage transfer ratio

B = short circuit reverse transfer impedance

C = open-circuit reverse transfer admittance

D = short circuit reverse current transfer ratio

Introduction to Two-Port Network: Example 11

The following measurements were made on a two-port resistive circuit:

i. Port 2 open-circuited: $V_1 = 10\,mV,\; I_1 = 50\,\mu A,\; V_2 = 20\,V.$

ii. Port 2 short-circuited: $V_1 = 40\,mV,\; I_1 = 100\,\mu A,\; I_2 = -1\,mA.$

Determine the $ABCD$ parameters of the circuit.

The $ABCD$ parameter equations are given by

$$\boxed{V_1 = AV_2 - BI_2} \quad \textbf{(7)}$$

$$\boxed{I_1 = CV_2 - DI_2} \quad \textbf{(8)}$$

Introduction to Two-Port Network: Example 11

i. **Port 2 open-circuited:** $V_1 = 10\ mV,\ I_1 = 50\ \mu A,\ V_2 = 20\ V.$

$$\boxed{V_1 = AV_2 - BI_2} \quad \textbf{(7)}$$

$$\boxed{I_1 = CV_2 - DI_2} \quad \textbf{(8)}$$

With port 2 open-circuited $(I_2 = 0)$

$$\boxed{A = \frac{V_1}{V_2} = \frac{10mV}{20V} = 5\ x\ 10^{-4}}$$

$$\boxed{C = \frac{I_1}{V_2} = \frac{50\mu A}{20V} = 2.5\ x\ 10^{-6}\ \ \Omega^{-1}}$$

Introduction to Two-Port Network: Example 11

ii. Port 2 short-circuited: $V_1 = 40\ mV,\ I_1 = 100\ \mu A,\ I_2 = -1\ mA.$

$$\boxed{V_1 = AV_2 - BI_2} \quad \textbf{(7)}$$

$$\boxed{I_1 = CV_2 - DI_2} \quad \textbf{(8)}$$

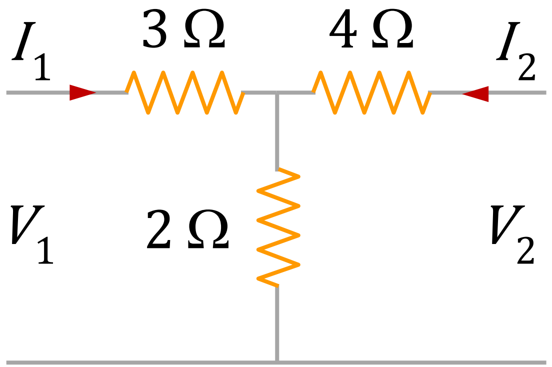
With port 2 short-circuited $(V_2 = 0)$

$$\boxed{B = \frac{V_1}{-I_2} = \frac{40mV}{-(-1mA)} = 40\ \Omega}$$

$$\boxed{D = \frac{I_1}{-I_2} = \frac{100\mu A}{-(-1mA)} = 0.1}$$

Introduction to Two-Port Network: Example 12

Find the $ABCD$ parameters for the network shown. The resistances are in Ω.

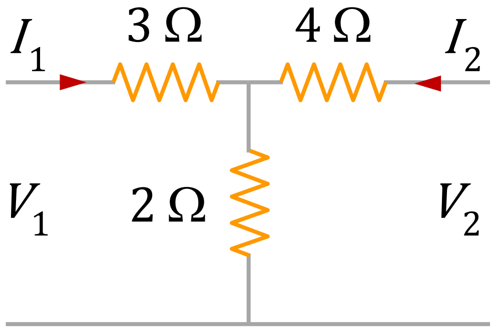


Introduction to Two-Port Network: Example 12

The $ABCD$ parameter equations are given by

$$\boxed{V_1 = AV_2 - BI_2} \quad \textbf{(7)}$$

$$\boxed{I_1 = CV_2 - DI_2} \quad \textbf{(8)}$$



Using Mesh analysis, we get

$$\boxed{V_1 = 3I_1 + 2(I_1 + I_2) = 5I_1 + 2I_2} \quad \textbf{(i)}$$

$$\boxed{V_2 = 4I_2 + 2(I_1 + I_2) = 2I_1 + 6I_2} \Rightarrow \boxed{I_1 = \frac{1}{2}(V_2 - 6I_2) = \frac{1}{2}V_2 - 3I_2} \quad \textbf{(ii)}$$

Introduction to Two-Port Network: Example 12

Using (ii) in (i) gives

$$V_1 = 3I_1 + 2(I_1 + I_2) = 5I_1 + 2I_2 \quad \text{(i)}$$

$$I_1 = \frac{1}{2}(V_2 - 6I_2) = \frac{1}{2}V_2 - 3I_2 \quad \text{(ii)}$$

$$V_1 = \frac{5}{2}V_2 - 13I_2$$

Thus,

$$V_1 = \frac{5}{2}V_2 - 13I_2$$

$$I_1 = \frac{1}{2}V_2 - 3I_2$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 2.5 & 13 \\ 0.5 & 3 \end{bmatrix}$$

Relationship Among Parameters

Relationship Among Parameters

The 4 parameters (y, z, h and $ABCD$) we have seen so far all describe the same two-port network.

Therefore, if one set of parameters is known, it should be possible to convert from one set of parameter to another via some algebraic manipulations.

Relationship Among Parameters

Suppose we want to find the relationship between z and y parameters. The idea here is to put the z equations in the same form as the y equations and then equate coefficients.

The z parameters are defined by

$$V_1 = z_{11}I_1 + z_{12}I_2$$

$$V_2 = z_{21}I_1 + z_{22}I_2$$

Solving for I_1 and I_2 gives

$$I_1 = \frac{z_{22}}{\Delta z}V_1 + \frac{-z_{12}}{\Delta z}V_2$$

$$I_2 = \frac{-z_{21}}{\Delta z}V_1 + \frac{z_{11}}{\Delta z}V_2$$

$$\Delta z = z_{11}z_{22} - z_{12}z_{21} \neq 0$$

Comparing them with y parameters

$$I_1 = y_{11}V_1 + y_{12}V_2$$

$$I_2 = y_{21}V_1 + y_{22}V_2$$

The y parameters in terms of the z parameters are given by

$$y_{11} = \frac{z_{22}}{\Delta z} \qquad y_{12} = \frac{-z_{12}}{\Delta z}$$

$$y_{21} = \frac{-z_{21}}{\Delta z} \qquad y_{22} = \frac{z_{11}}{\Delta z}$$

Relationship Among Parameters

Remark:

If the z parameter matrix is known, then the y parameter matrix is given by

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}^{-1} = \frac{\mathbf{1}}{\Delta z}\begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}$$

$$\Delta z = z_{11}z_{22} - z_{12}z_{21}$$

If the y parameter matrix is known, then the z parameter matrix is given by

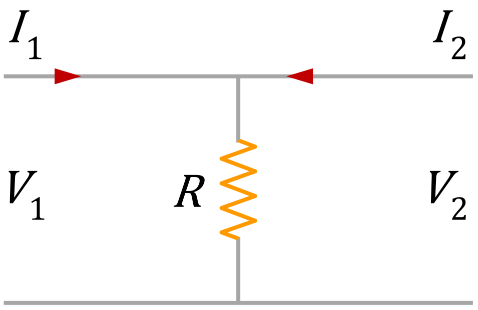
$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}^{-1} = \frac{\mathbf{1}}{\Delta y}\begin{bmatrix} y_{22} & -y_{12} \\ -y_{21} & y_{11} \end{bmatrix}$$

$$\Delta y = y_{11}y_{22} - y_{12}y_{21}$$

Relationship Among Parameters

Note: Some circuits have z parameters but not y parameters. This happens when $\Delta z = 0$.

Consider the following network.



The z parameters two-port equations are

$$V_1 = RI_1 + RI_2$$

$$V_2 = RI_1 + RI_2$$

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} R & R \\ R & R \end{bmatrix}$$

The circuit does not have y parameters as
$\Delta z = z_{11}z_{22} - z_{12}z_{21} = R^2 - R^2 = 0.$

Relationship Among Parameters: Table 1

The table specifies the relationships among the parameters studied thus far. (There is no need to memorise this table.)

$$\Delta z = z_{11}z_{22} - z_{12}z_{21}$$

$$\Delta y = y_{11}y_{22} - y_{12}y_{21}$$

$$\Delta h = h_{11}h_{22} - h_{12}h_{21}$$

$$\Delta t = AD - BC$$

	z	y	h	$ABCD$
z	$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$	$\begin{bmatrix} \frac{y_{22}}{\Delta y} & \frac{-y_{12}}{\Delta y} \\ \frac{-y_{21}}{\Delta y} & \frac{y_{11}}{\Delta y} \end{bmatrix}$	$\begin{bmatrix} \frac{\Delta h}{h_{22}} & \frac{h_{12}}{h_{22}} \\ \frac{-h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{bmatrix}$	$\begin{bmatrix} \frac{A}{C} & \frac{\Delta t}{C} \\ \frac{1}{C} & \frac{D}{C} \end{bmatrix}$
y	$\begin{bmatrix} \frac{z_{22}}{\Delta z} & \frac{-z_{12}}{\Delta z} \\ \frac{-z_{21}}{\Delta z} & \frac{z_{11}}{\Delta z} \end{bmatrix}$	$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{h_{11}} & \frac{-h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\Delta h}{h_{11}} \end{bmatrix}$	$\begin{bmatrix} \frac{D}{B} & \frac{-\Delta t}{B} \\ \frac{-1}{B} & \frac{A}{B} \end{bmatrix}$
h	$\begin{bmatrix} \frac{\Delta z}{z_{22}} & \frac{z_{12}}{z_{22}} \\ \frac{-z_{21}}{z_{22}} & \frac{1}{z_{22}} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{y_{11}} & \frac{-y_{12}}{y_{11}} \\ \frac{y_{21}}{y_{11}} & \frac{\Delta y}{y_{11}} \end{bmatrix}$	$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$	$\begin{bmatrix} \frac{B}{D} & \frac{\Delta t}{D} \\ \frac{-1}{D} & \frac{C}{D} \end{bmatrix}$
$ABCD$	$\begin{bmatrix} \frac{z_{11}}{z_{21}} & \frac{\Delta z}{z_{21}} \\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}} \end{bmatrix}$	$\begin{bmatrix} \frac{-y_{22}}{y_{21}} & \frac{-1}{y_{21}} \\ \frac{-\Delta y}{y_{21}} & \frac{-y_{11}}{y_{21}} \end{bmatrix}$	$\begin{bmatrix} \frac{-\Delta h}{h_{21}} & \frac{-h_{11}}{h_{21}} \\ \frac{-h_{22}}{h_{21}} & \frac{-1}{h_{21}} \end{bmatrix}$	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

Relationship Among Parameters: Example 1

The following measurements were made on a two-port resistive circuit:

i. Port 2 open-circuited: $V_1 = 10\,mV,\ I_1 = 50\,\mu A,\ V_2 = 20\,V.$

ii. Port 2 short-circuited: $V_1 = 40\,mV,\ I_1 = 100\,\mu A,\ I_2 = -1\,mA.$

Determine the h parameters of the circuit

The h parameter equations are given by

$$\boxed{V_1 = h_{11}I_1 + h_{12}V_2}$$

$$\boxed{I_2 = h_{21}I_1 + h_{22}V_2}$$

Relationship Among Parameters: Example 1

ii. Port 2 short-circuited: $V_1 = 40\ mV,\ I_1 = 100\ \mu A,\ I_2 = -1\ mA.$

$$\boxed{V_1 = h_{11} I_1 + h_{12} V_2}$$

$$\boxed{I_2 = h_{21} I_1 + h_{22} V_2}$$

With port 2 short-circuited $(V_2 = 0)$

$$\boxed{h_{11} = \frac{V_1}{I_1} = \frac{40mV}{100\mu A} = 400\ \Omega}$$

$$\boxed{h_{21} = \frac{I_2}{I_1} = \frac{-1mA}{100\mu A} = -10}$$

But $h_{12} = \frac{V_1}{V_2}|_{I_1=0}$ **and** $h_{22} = \frac{I_2}{V_2}|_{I_1=0}$ **cannot be obtained from the given measurements because these do not include the case of port 1 open-circuited.**

Relationship Among Parameters: Example 1

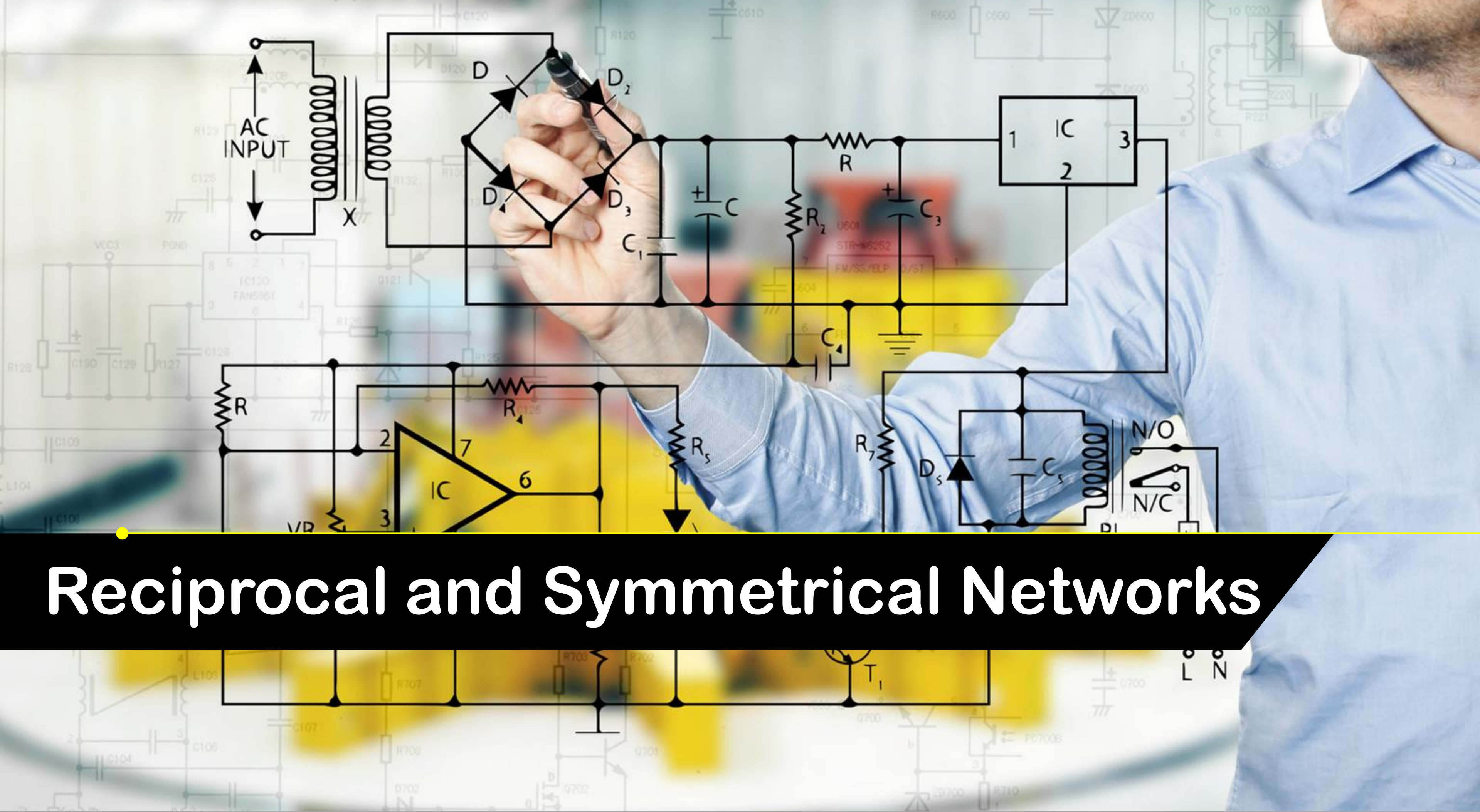
However, we have obtained the $ABCD$ parameters previously as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 5\ x\ 10^{-4} & 40 \\ 2.5\ x\ 10^{-6} & 0.1 \end{bmatrix}$$

Using Table 1, we have the relationship between the h and $ABCD$ parameters as

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{B}}{\mathrm{D}} & \frac{\Delta\mathrm{t}}{\mathrm{D}} \\ \frac{-1}{\mathrm{D}} & \frac{C}{\mathrm{D}} \end{bmatrix} \Rightarrow \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 400 & -5\ x\ 10^{-4} \\ -10 & 2.5\ x\ 10^{-5} \end{bmatrix}$$

Where $\Delta\mathrm{t} = AD - BC = (5x10^{-4})(0.1) - (40)(2.5\ x\ 10^{-6}) = -5\ x\ 10^{-5}$

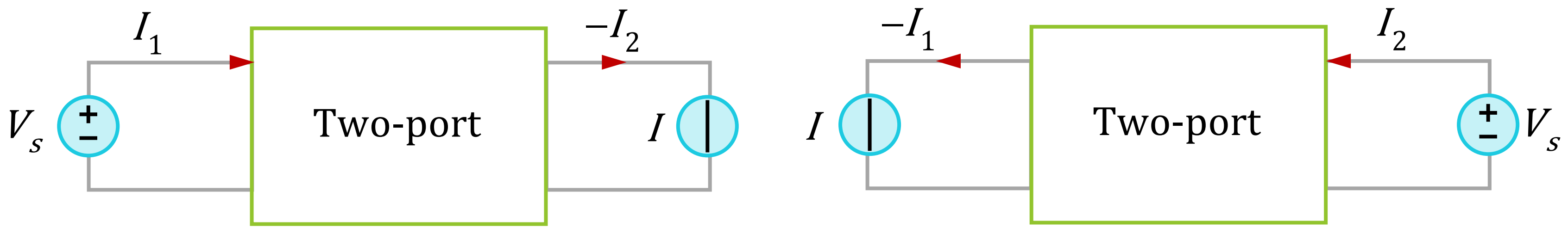


Reciprocal and Symmetrical Networks

Reciprocal Network

We look at two special cases of networks.

A two-port network is said to be **reciprocal** if interchanging an ideal independent voltage source at one port, with an ideal ammeter at the other port, does not change the ammeter reading. This is shown below.



Condition for reciprocity

$$\boxed{\frac{I_2}{V_1}\Big|_{V_2=0} = \frac{I_1}{V_2}\Big|_{V_1=0}}$$

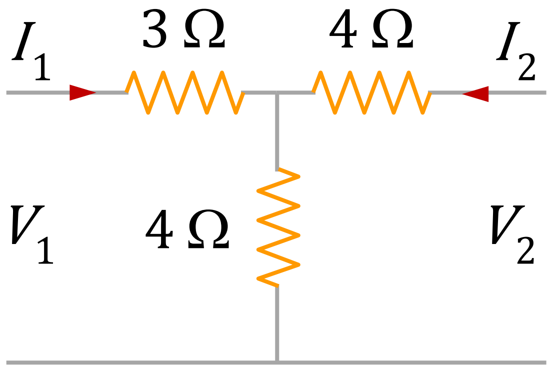
Reciprocal Network

Parameters	Condition of Reciprocity
$[z]$	$z_{12} = z_{21}$
$[y]$	$y_{12} = y_{21}$
$[h]$	$h_{12} = -h_{21}$
$[ABCD]$	$AD - BC = 1$

Note: A circuit containing R's, L's, C's and transformers, but no dependent sources or independent sources, is a reciprocal network.

Reciprocal Network: Example 1

Consider the network as shown. Find out if this is a reciprocal network.



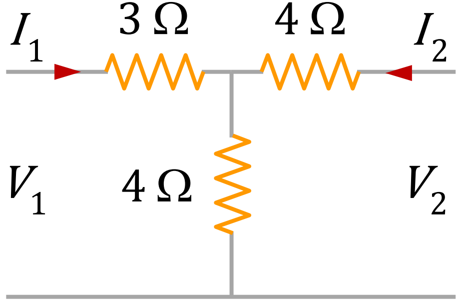
Reciprocal Network: Example 1

When a voltage source of $10\ \mathrm{V}$ is applied to port 1, the ammeter at port 2 reads $1\ \mathrm{A}$. The mesh equations are:

$$3I_1 + 4(I_1 + I_2) = 10$$

$$4(I_1 + I_2) + 4I_2 = 0$$

Solving gives $I_2 = -1$.



Also, when a voltage source of $10\ \mathrm{V}$ is applied to port 2, an ammeter at port 1 reads $1\ \mathrm{A}$. The mesh equations are:

$$4I_2 + 4(I_1 + I_2) = 10$$

$$3I_1 + 4(I_1 + I_2) = 0$$

Solving gives $I_1 = -1$.

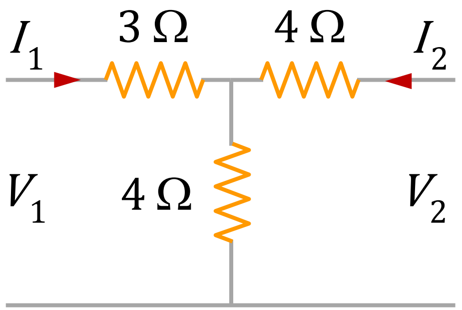
Reciprocal Network: Example 1

The z parameters of the network are given by

$$V_1 = 3I_1 + 4(I_1 + I_2) = 7I_1 + 4I_2$$

$$V_2 = 4I_2 + 4(I_1 + I_2) = 4I_1 + 8I_2$$

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 4 & 8 \end{bmatrix}$$



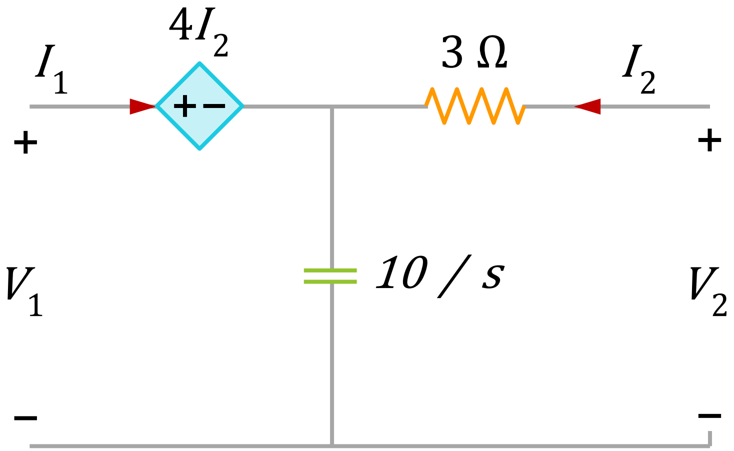
Thus, the network is reciprocal as $z_{12} = z_{21}$.

This circuit only contains resistors and no dependent sources or independent sources. Thus, it is a reciprocal network.

Now, the number of parameters required to characterise a two-port network is reduced from 4 to 3 since $z_{12} = z_{21}$.

Reciprocal Network: Example 2

Consider the following circuit with a dependent source. Determine if this is a reciprocal network.



Reciprocal Network: Example 2

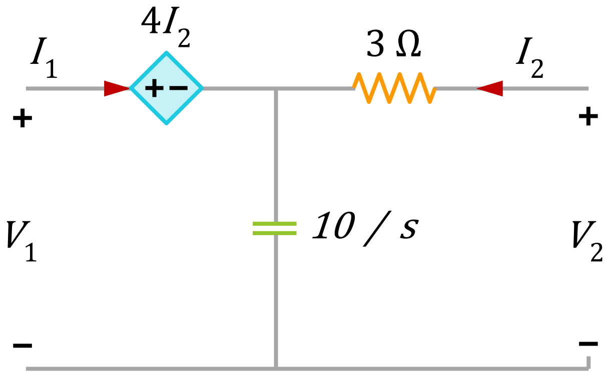
The loop equations are

$$V_1 = 4I_2 + \frac{10}{s}(I_1 + I_2) \quad \Rightarrow \quad V_1 = \frac{10}{s}I_1 + (4 + \frac{10}{s})I_2$$

$$V_2 = 3I_2 + \frac{10}{s}(I_1 + I_2) \quad \Rightarrow \quad V_2 = \frac{10}{s}I_1 + (3 + \frac{10}{s})I_2$$

Rearranging gives

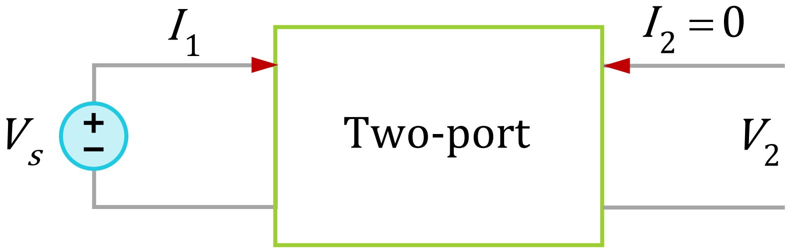
$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} \frac{10}{s} & \frac{4s+10}{s} \\ \frac{10}{s} & \frac{3s+10}{s} \end{bmatrix}$$



Since $z_{12} \neq z_{21}$, the given circuit is **not a reciprocal circuit**.

Symmetrical Network

A network is said to be symmetrical if its ports can be interchanged without altering its terminal currents and voltages, i.e., it is electrically the same when viewed from port 1 or port 2. This is as shown (where $I_1 = I_2$).



Condition for symmetry

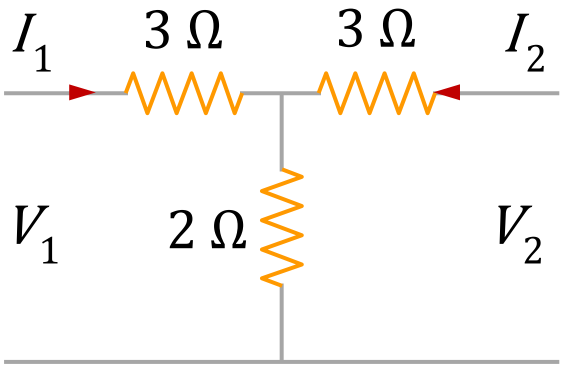
$$\frac{V_1}{I_1}\Big|_{I_2=0} = \frac{V_2}{I_2}\Big|_{I_1=0}$$

Symmetrical Network

Parameters	Condition of Symmetry
$[Z]$	$z_{11} = z_{22}$
$[Y]$	$y_{11} = y_{22}$
$[h]$	$h_{11}h_{22} - h_{12}h_{21} = 1$
$[ABCD]$	$A = D$

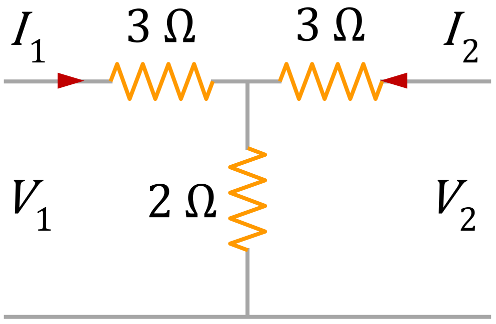
Symmetrical Network: Example 3

Consider the network as shown. Determine if this is a reciprocal and symmetrical network.



Symmetrical Network: Example 3

When a voltage source of $10\ \mathrm{V}$ is applied to port 1 with port 2 open-circuited $(I_2 = 0)$, I_1 is given by $2\ \mathrm{A}$.



Also, when a voltage source of $10\ \mathrm{V}$ is applied to port 2 with port 1 open-circuited $(I_1 = 0)$, I_2 is given by $2\ \mathrm{A}$.

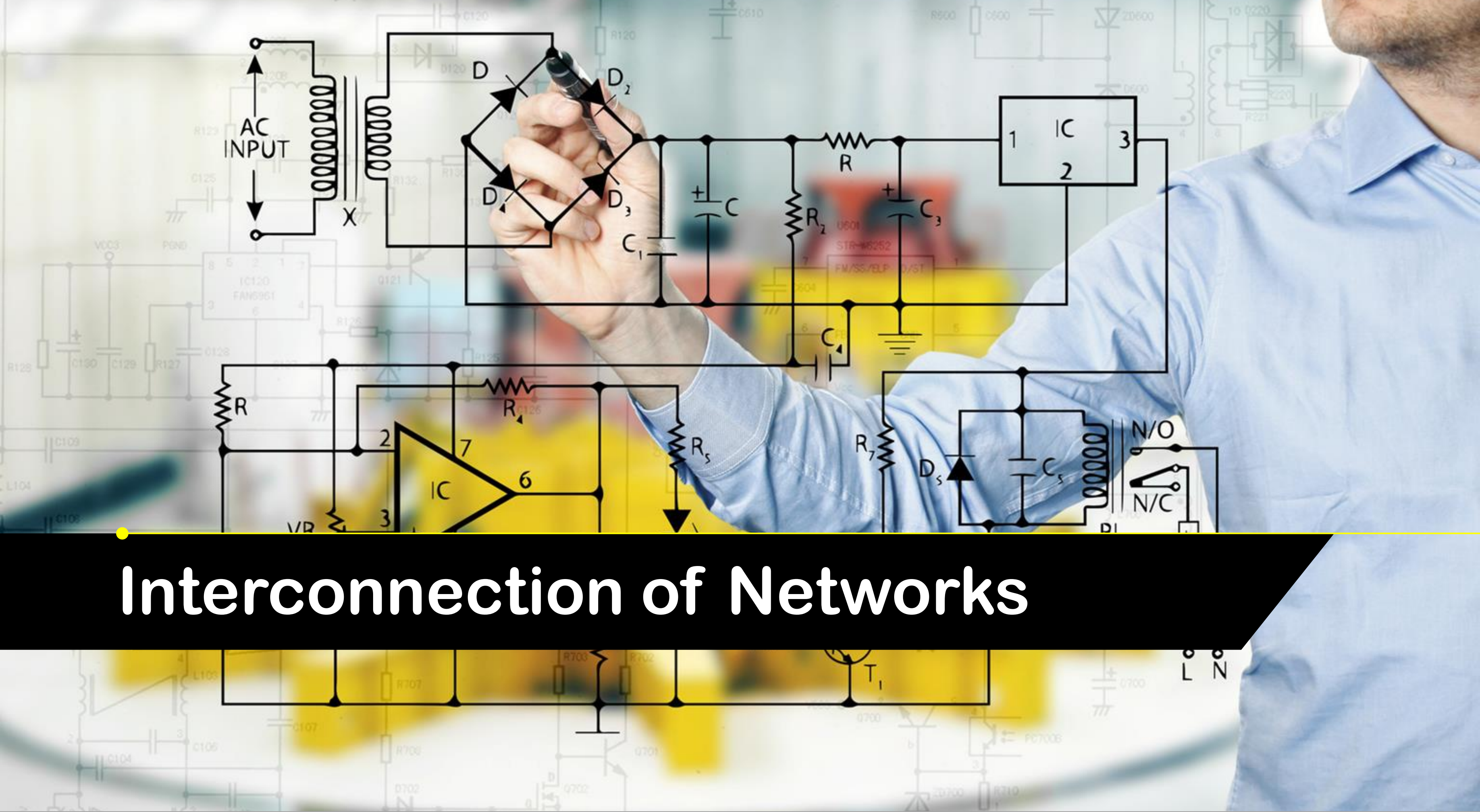
The z parameters of the network are given by

$$\boxed{V_1 = 5I_1 + 2I_2}$$

$$\boxed{V_2 = 2I_1 + 5I_2}$$

$$\Rightarrow \boxed{\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}}$$

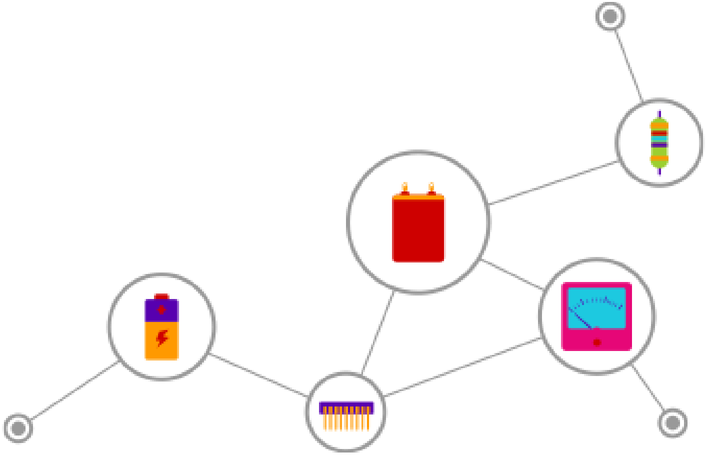
Thus, the network is reciprocal as $z_{12} = z_{21}$. It is also symmetrical as $z_{11} = z_{22}$. Networks in communications are often symmetrical.



Interconnection of Networks

Interconnection of Networks

- A complex system can be viewed as an interconnection of several simpler two-port networks.
- Conversely, a two-port network which is to be built up can be designed by combining simple two-port structures as building blocks.
- From the designer’s point of view, it is easier to design simple blocks and then interconnecting them than to design a single complex network.
- Thus, interconnection of networks allows us to analyse several general systems.

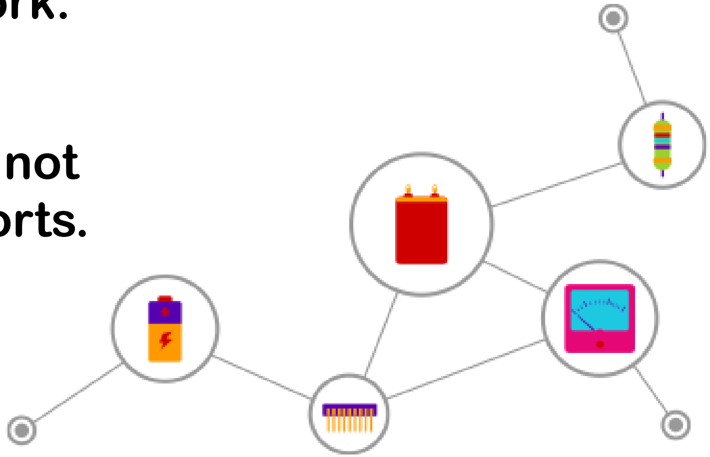


Interconnection of Networks

Two-port networks can be interconnected in various configurations. We only consider three of them as follows:

Series **Parallel** **Cascade**

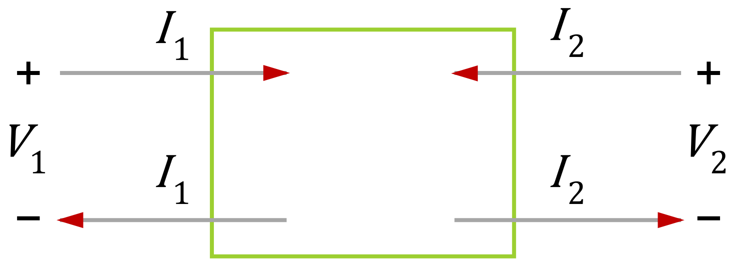
- For each configuration, a set of parameters may be more useful than others to describe the network.
- But, we must ensure that when two-ports are connected together, the inter-connections do not change the properties of the individual two-ports.
- Otherwise, the two-port analysis method would be invalid.



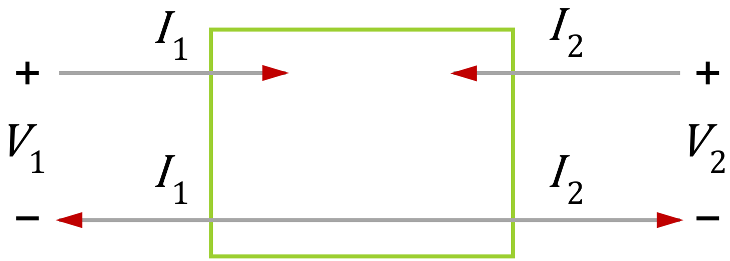
Interconnection of Networks: Port Condition

A general linear two-port has four external terminals for connection to other networks. In the case when the input and output ports share a common terminal, there are only three external terminals.

Such a two-port is often referred to as a **common ground two-port**.



A General Two-port



A Common-ground Two-port

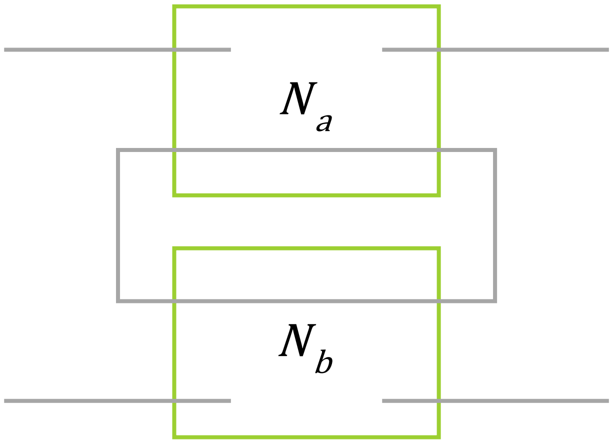
Interconnection of Networks: Port Condition

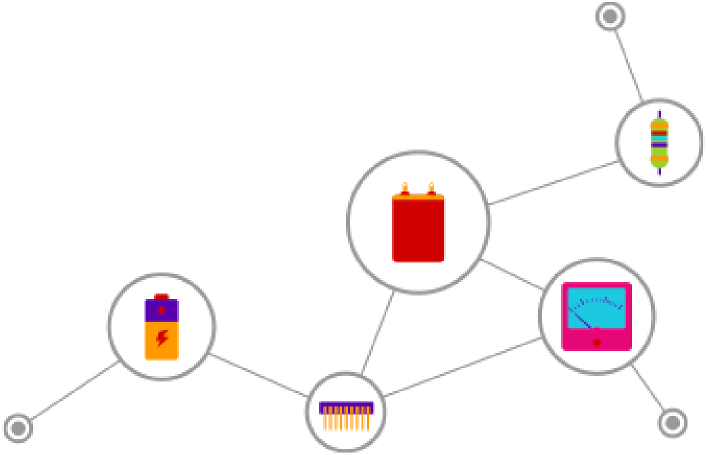
The new interconnected two-ports have new two-port parameters computed from the two two-ports, provided the port condition is met, i.e., the current entering and leaving a port must be the same (refer to the next slide).

Interconnection of Networks: Port Condition

Series

The series connection equation holds only for the case as shown (common ground terminals tied together).

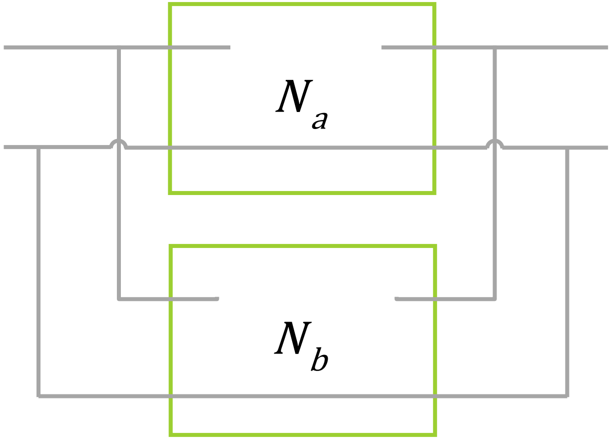


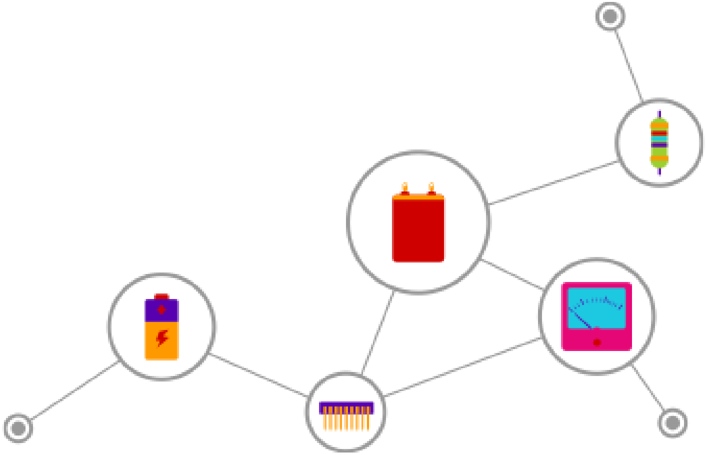


Interconnection of Networks: Port Condition

Parallel

The parallel connection equation holds for common ground two-port connections as shown (common ground terminals tied together).

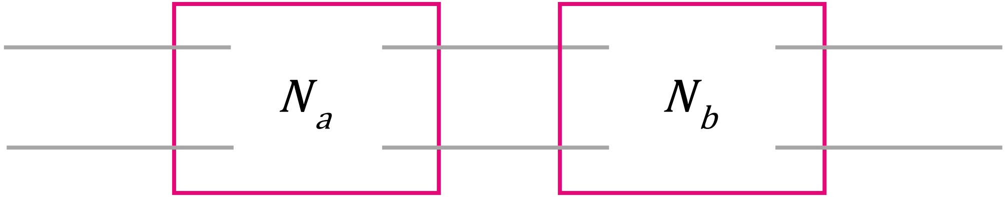


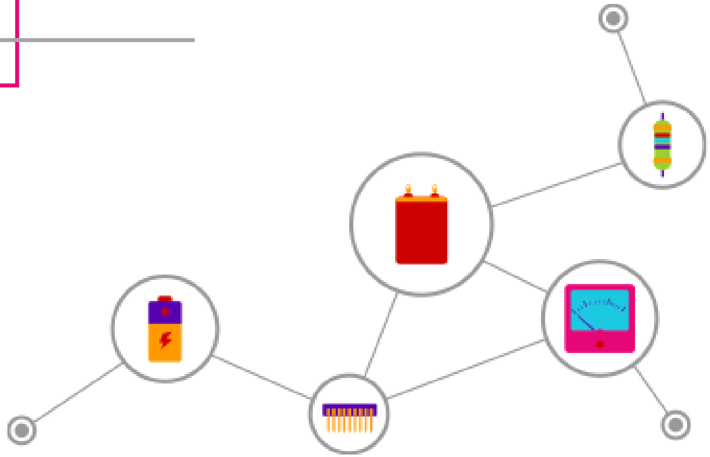


Interconnection of Networks: Port Condition

Cascade

The cascade connection equation holds whether or not the component two-ports are of the common-ground type.

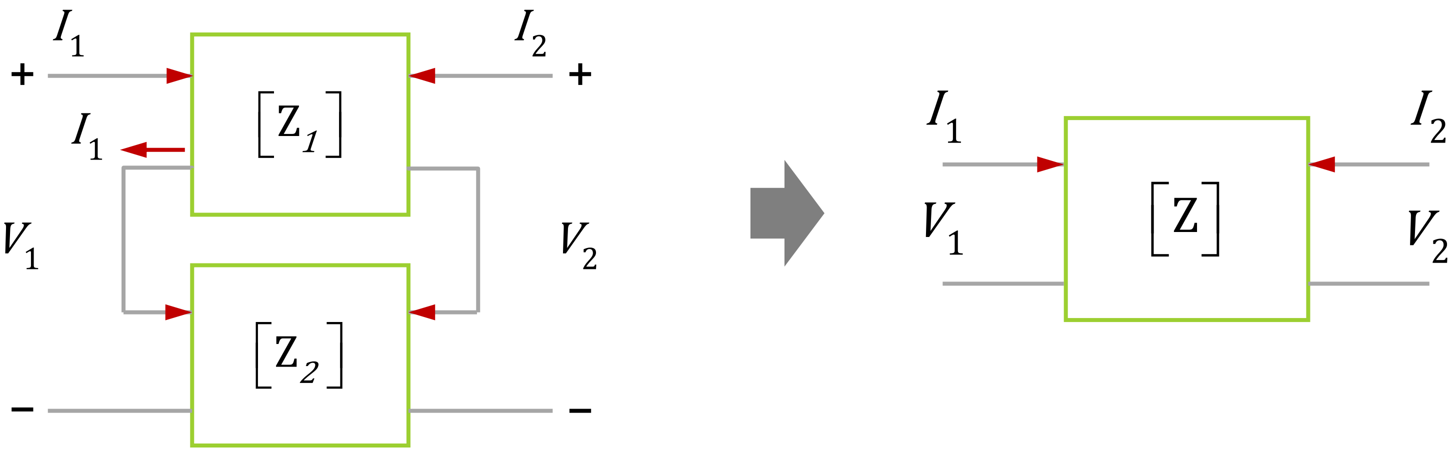




Interconnection of Networks: Series Connection

The figure below shows a series connection of two two-port networks. The two input ports carry the same current I_1 and the two output ports carry the same current I_2.

Series

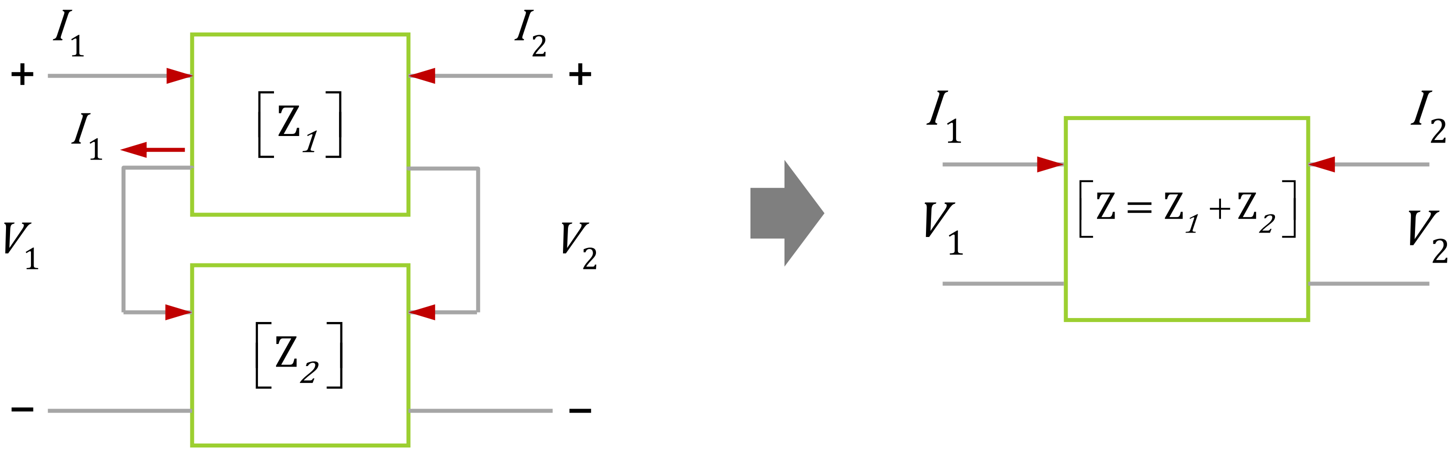


Series Connection of Two Two-Ports

Interconnection of Networks: Series Connection

The z parameters can be easily used to find the overall parameters of the series-connected two-ports. If $[z_1]$ and $[z_2]$ are the z parameters of the network N_1 and N_2, respectively, then for the overall network

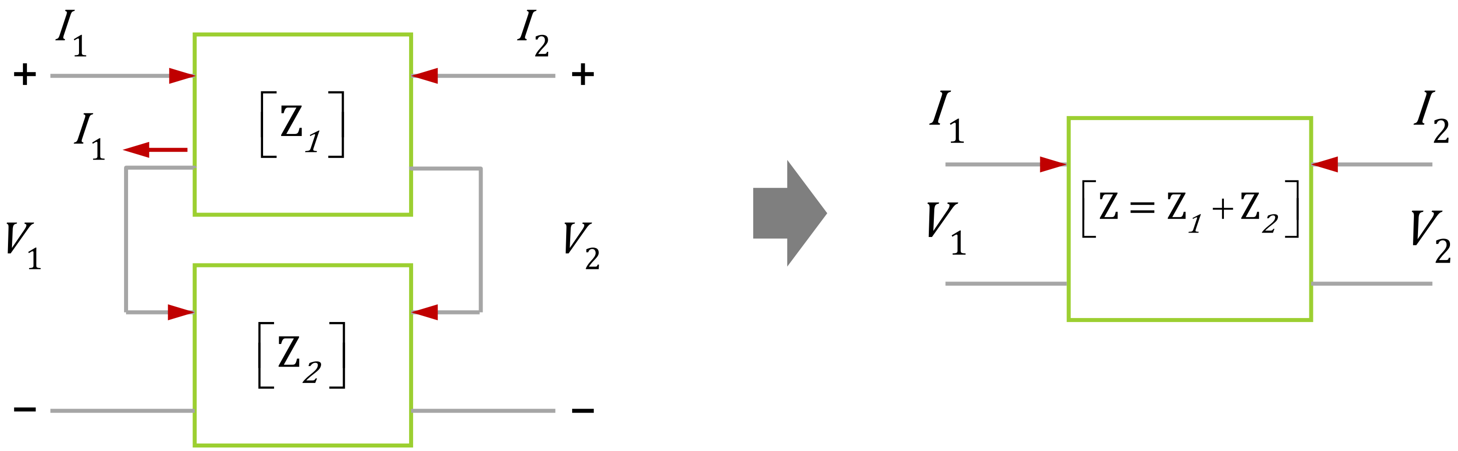
$$[z] = [z_1] + [z_2]$$



Series Connection of Two Two-Ports

Interconnection of Networks: Series Connection

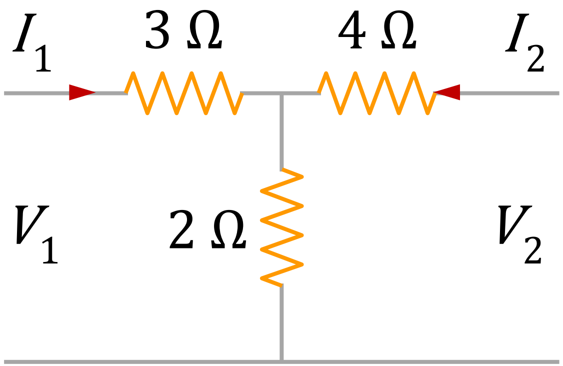
The overall z parameter matrix for a series connected two-port networks is the sum of z parameter matrices of each individual two-port connected in series.



Series Connection of Two Two-Ports

Interconnection of Networks: Example 1

Two identical networks as shown are connected in series. Determine the overall z parameters of the combined network.



Interconnection of Networks: Example 1

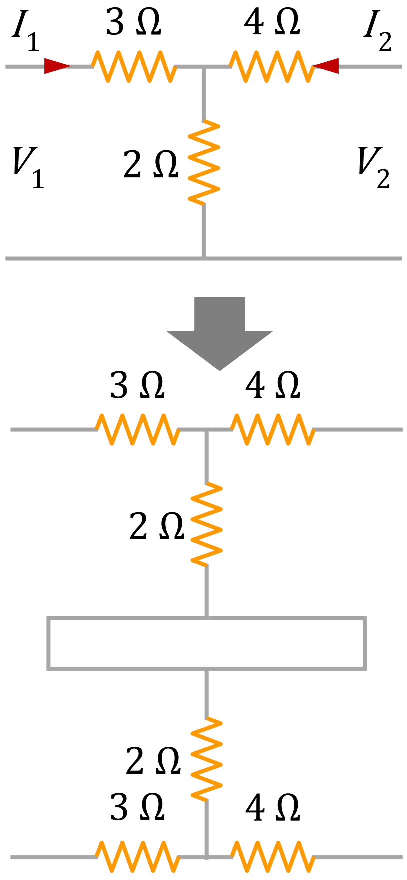
The combined network is as shown. The port condition is met.

The z parameters of the single two-port network are defined by

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad \Rightarrow \quad [z_1] = \begin{bmatrix} 5 & 2 \\ 2 & 6 \end{bmatrix}$$

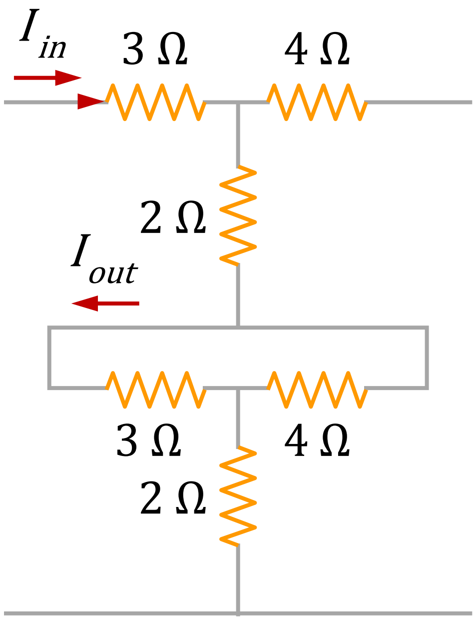
The overall z parameters are

$$[z] = [z_1] + [z_2] = 2[z_1] = \begin{bmatrix} 10 & 4 \\ 4 & 12 \end{bmatrix}$$



Interconnection of Networks: Example 2

The combined network below violates the port condition and the result $[z] = [z_1] + [z_2]$ does not hold.



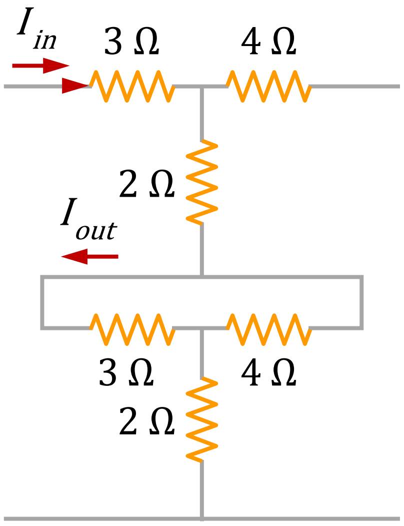
Interconnection of Networks: Example 2

The mesh equations in this case are

$$V_1 = 3I_1 + \frac{40}{7}(I_1 + I_2) = \frac{61}{7}I_1 + \frac{40}{7}I_2$$

$$V_2 = 4I_2 + \frac{40}{7}(I_1 + I_2) = \frac{40}{7}I_1 + \frac{68}{7}I_2$$

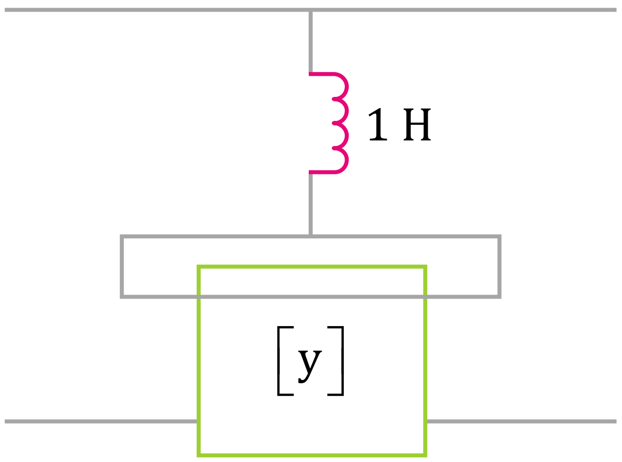
$$\begin{bmatrix} \frac{61}{7} & \frac{40}{7} \\ \frac{40}{7} & \frac{68}{7} \end{bmatrix} \neq \begin{bmatrix} 10 & 4 \\ 4 & 12 \end{bmatrix}$$



Note that the current entering the top terminal of port 1 (I_{in}) is not equal to the current leaving the bottom terminal of port 1 (I_{out}), i.e., it is no longer a two-port network.

Interconnection of Networks: Example 3

Find the z parameters of the two two-ports connected in series as shown with $[y] = \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}$.



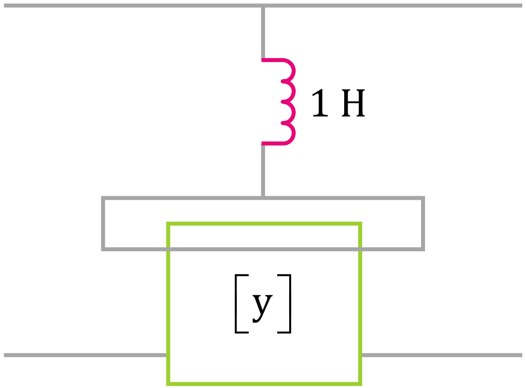
Interconnection of Networks: Example 3

The z parameters of the 1st network are given by

$$V_1 = s(I_1 + I_2) = sI_1 + sI_2$$

$$V_2 = s(I_1 + I_2) = sI_1 + sI_2$$

$$[Z_1] = \begin{bmatrix} s & s \\ s & s \end{bmatrix}$$



Now, $[z]$ for the 2nd network is given by

$$[z_2] = [y]^{-1} = \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}^{-1}$$

$$[z_2] = \frac{1}{(4)(1) - (-1)(-2)} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 2 \end{bmatrix}$$

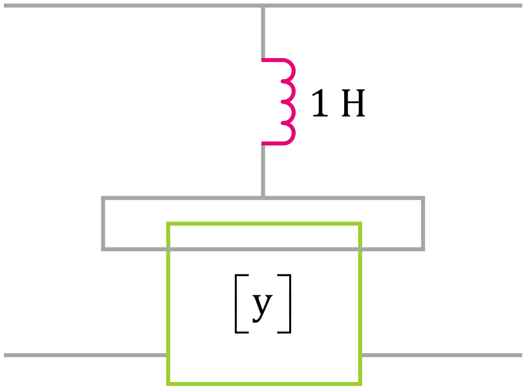
Interconnection of Networks: Example 3

$$[Z_1] = \begin{bmatrix} s & s \\ s & s \end{bmatrix}$$

$$[z_2] = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 2 \end{bmatrix}$$

Therefore,

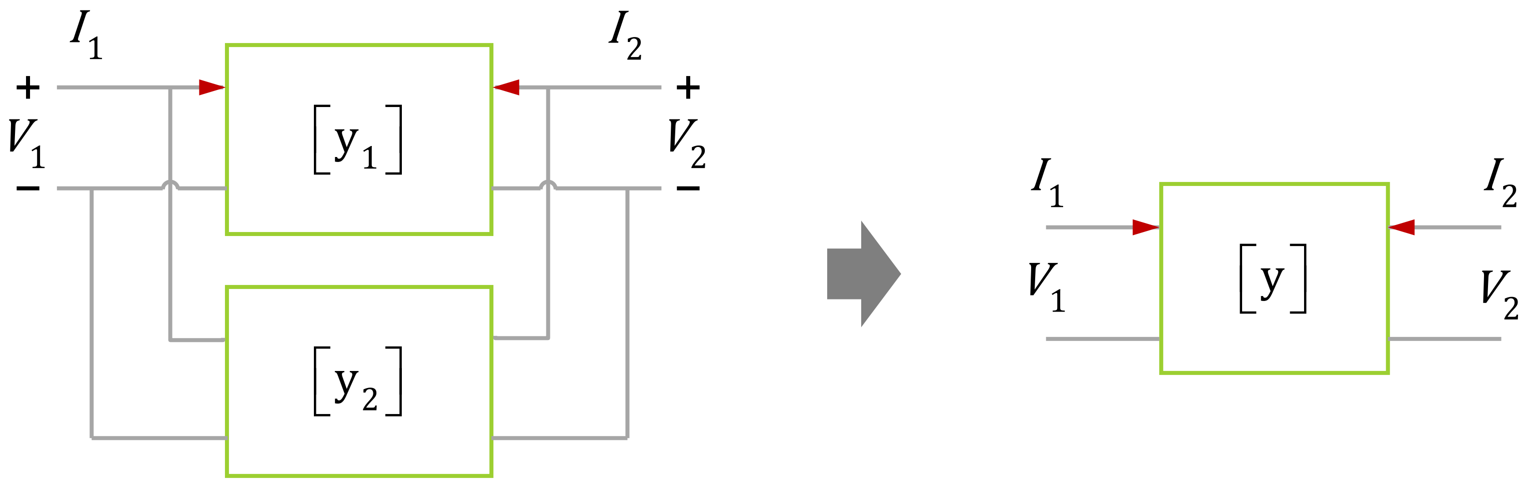
$$[z] = [z_{1]}+[z_2] = \begin{bmatrix} s+0.5 & s+0.5 \\ s+1 & s+2 \end{bmatrix}$$



Interconnection of Networks: Parallel Connection

The figure below shows a parallel connection of two two-port networks.

Parallel

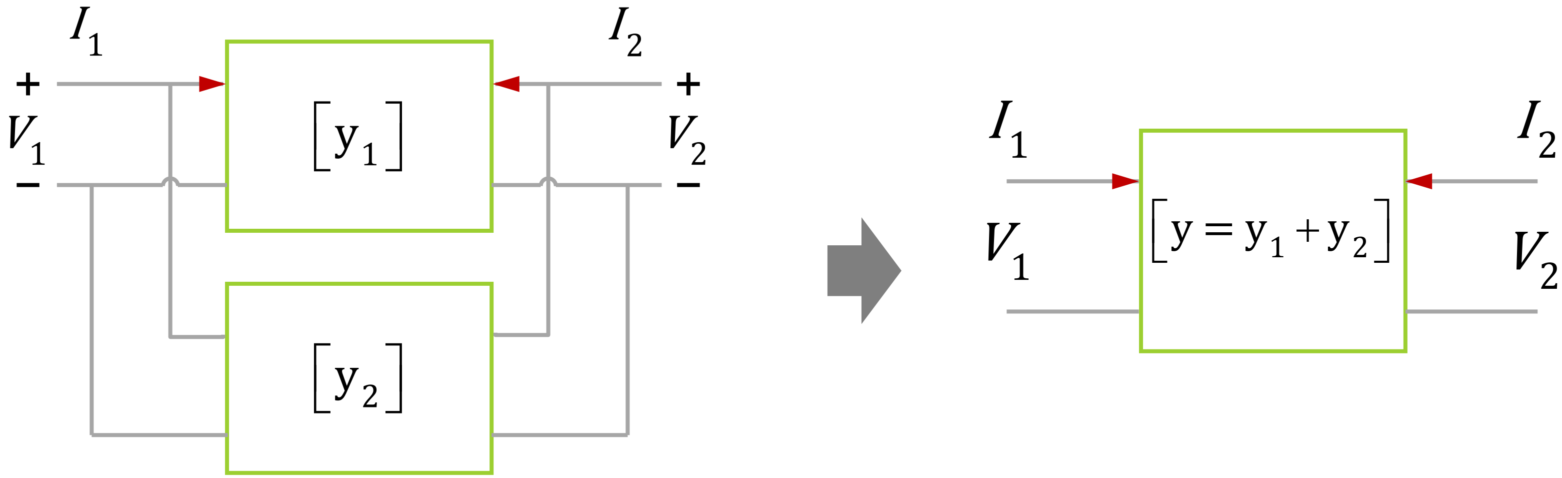


Parallel Connection of Two Two-Ports

Interconnection of Networks: Parallel Connection

The y parameters can be easily used to find the overall parameters of the parallel-connected two-ports. If $[y_1]$ and $[y_2]$ are the y parameters of the network N_1 and N_2, respectively, then for the overall network

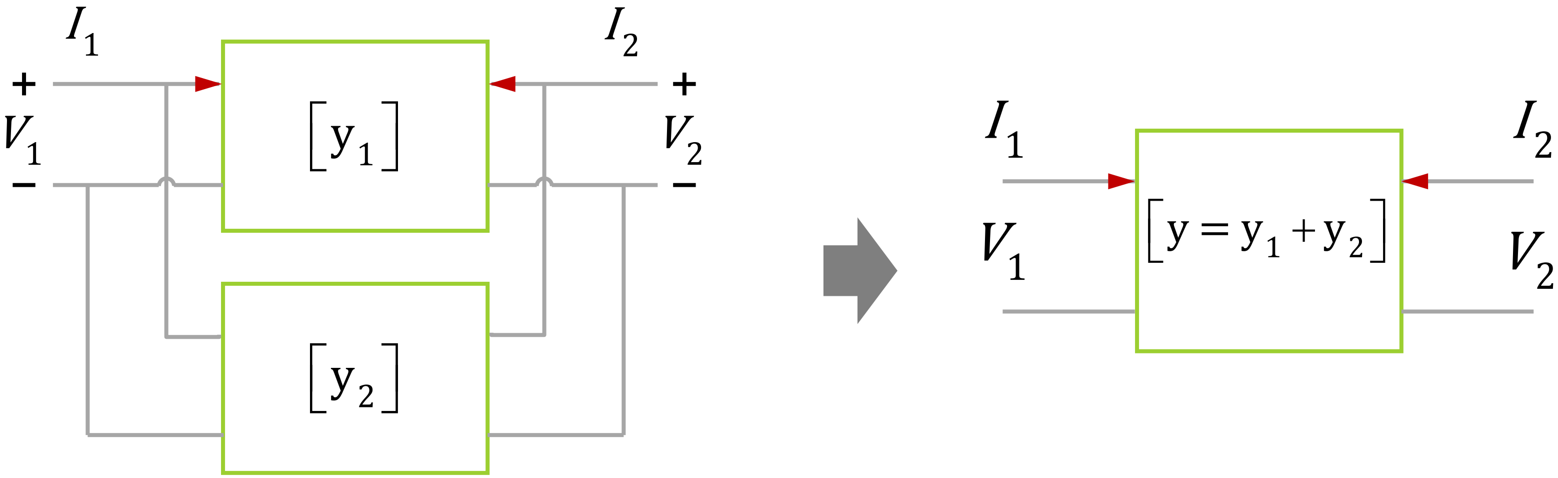
$$\boxed{[y] = [y]_1 + [y_2]}$$



Parallel Connection of Two Two-Ports

Interconnection of Networks: Parallel Connection

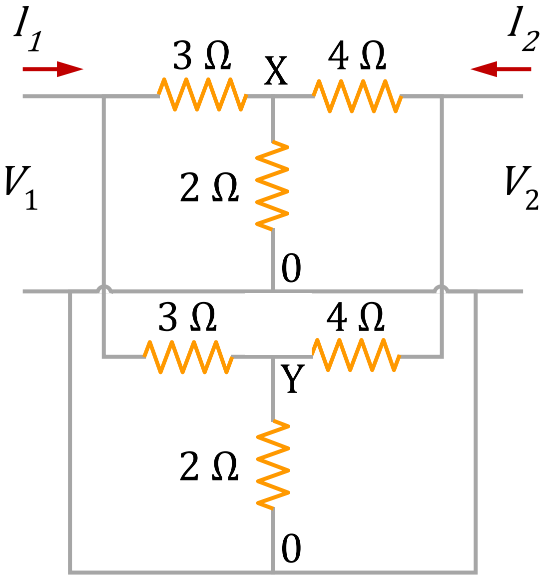
The overall y parameter matrix for a parallel-connected two-port networks is the sum of y parameter matrices of each individual two-port connected in parallel.



Parallel Connection of Two Two-Ports

Interconnection of Networks: Example 4

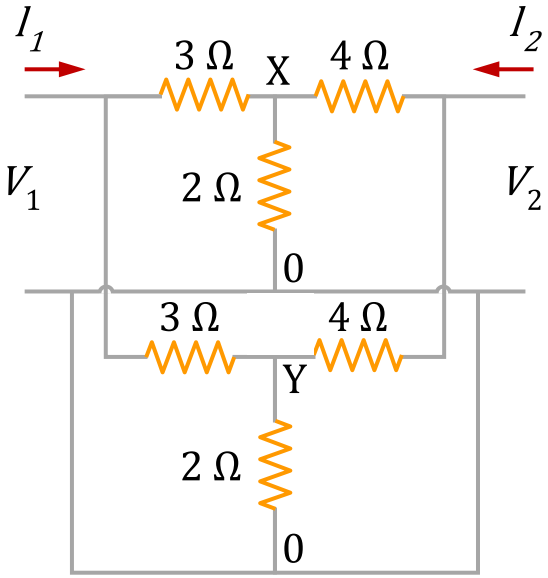
Two identical networks as shown are connected in parallel. Determine the overall y parameters of the combined network.



Interconnection of Networks: Example 4

The y parameters of the single two-port network are defined by

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{13} & -\frac{1}{13} \\ -\frac{1}{13} & \frac{5}{26} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$



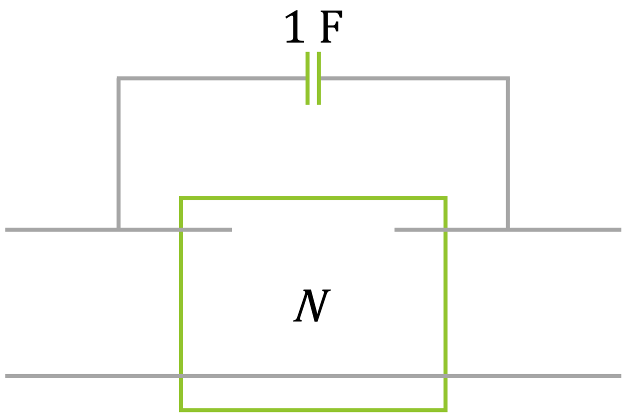
The overall y parameters are

$$[y] = [y_1] + [y_2] = 2[y_1]$$

$$[y] = 2\begin{bmatrix} \frac{3}{13} & -\frac{1}{13} \\ -\frac{1}{13} & \frac{5}{26} \end{bmatrix} = \begin{bmatrix} \frac{6}{13} & -\frac{2}{13} \\ -\frac{2}{13} & \frac{5}{13} \end{bmatrix}$$

Interconnection of Networks: Example 5

Find the y parameters of the two two-ports connected in parallel, where for the two-port *N*, the z parameter matrix is $[z] = \begin{bmatrix} 7 & 2 \\ 10 & 3 \end{bmatrix}$.



Interconnection of Networks: Example 5

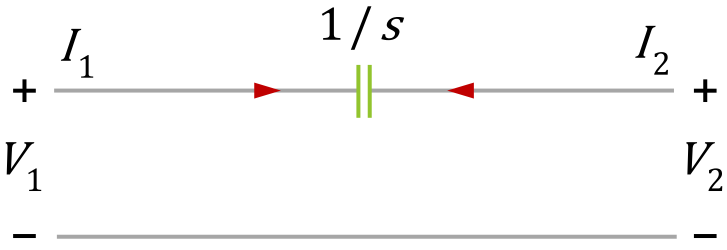
Consider the first network as shown.

Applying KVL gives

$$V_1 = \frac{1}{s} I_1 + V_2$$ **or** $$I_1 = sV_1 - sV_2$$

$$V_2 = \frac{1}{s} I_2 + V_1$$ **or** $$I_2 = -sV_1 + sV_2$$

$$[y_1] = \begin{bmatrix} s & -s \\ -s & s \end{bmatrix}$$



Interconnection of Networks: Example 5

And

$$[y_2] = [z]^{-1} = \frac{1}{7x3 - 2x10}\begin{bmatrix} 3 & -2 \\ -10 & 7 \end{bmatrix}$$

$$[y_2] = \begin{bmatrix} 3 & -2 \\ -10 & 7 \end{bmatrix}$$

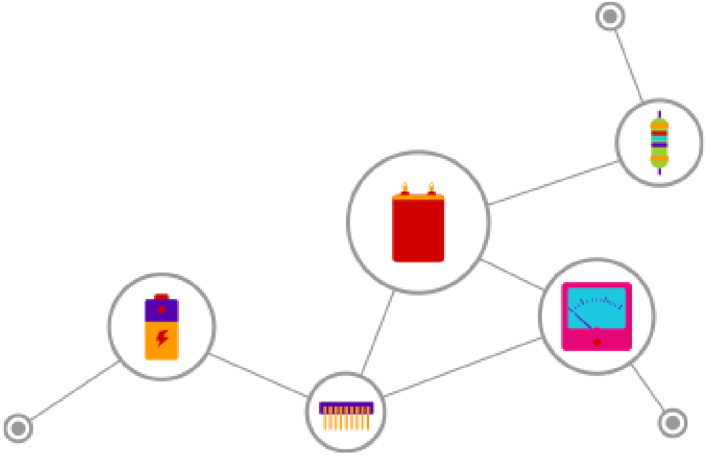
Interconnection of Networks: Example 5

Thus,

$$[y] = [y_1] + [y_2]$$

$$[y] = \begin{bmatrix} s & -s \\ -s & s \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ -10 & 7 \end{bmatrix}$$

$$\boxed{[y] = \begin{bmatrix} s+3 & -(s+2) \\ -(s+10) & s+7 \end{bmatrix}}$$

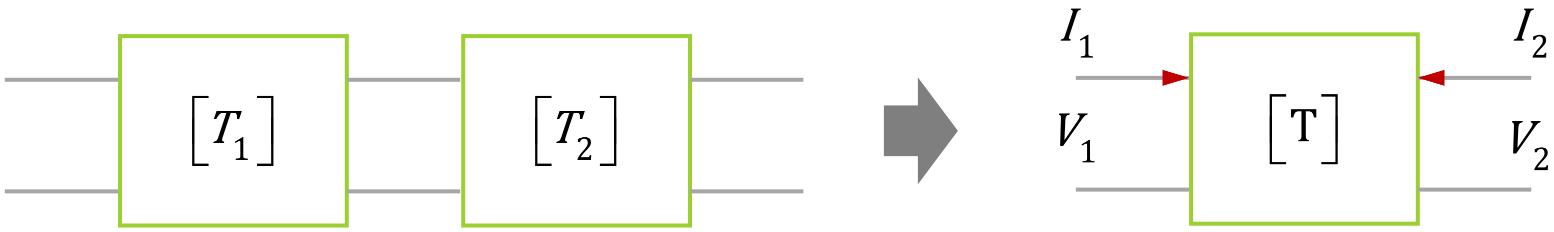


Interconnection of Networks: Cascade Connection

Two two-port networks are said to be connected in cascade if the output port of the first network becomes the input of the second network (refer to figure below). Two transmission lines are connected in this manner.

So transmission is suitable for cascade connection.

Cascade



Cascade Connection of Two Two-Ports

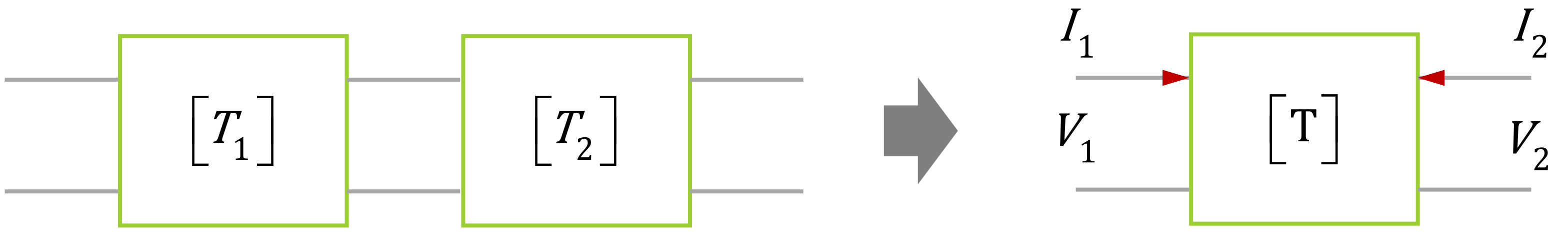
Interconnection of Networks: Cascade Connection

The $ABCD$ parameters can be easily used to find the overall parameters of the cascaded two-ports. If $[T_1]$ and $[T_2]$ are the $ABCD$ parameters of the networks N_1 and N_2, respectively, then for the overall network

$$\boxed{[T] = [T_1][T_2]}$$

or

$$\boxed{\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}}$$



Cascade Connection of Two Two-Ports

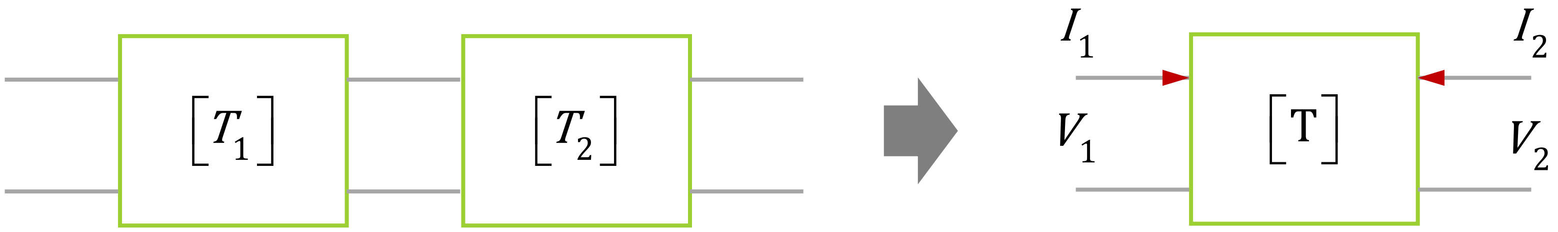
Interconnection of Networks: Cascade Connection

The overall $ABCD$ matrix for the two cascaded networks is equal to the product of $ABCD$ matrices of the individual networks taken in order.

Note:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{bmatrix}$$

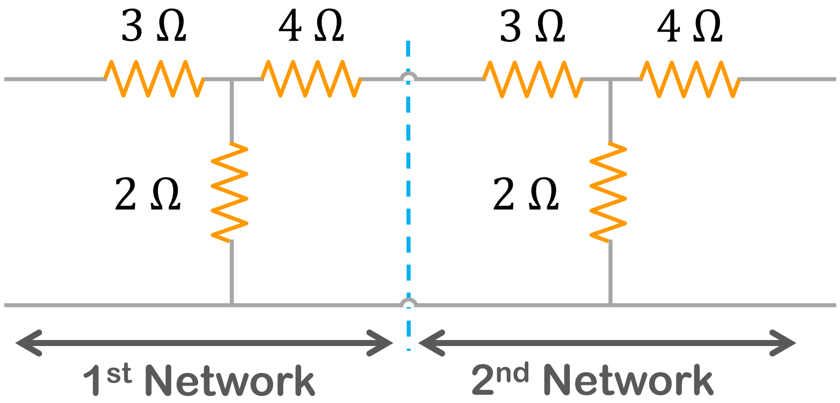
$$[T_1][T_2] \neq [T_2][T_1]$$



Cascade Connection of Two Two-Ports

Interconnection of Networks: Example 6

Two identical networks as shown are connected in cascade. Determine the overall $ABCD$ parameters of the combined network.



Interconnection of Networks: Example 6

The parameters of the single two-port network are defined by

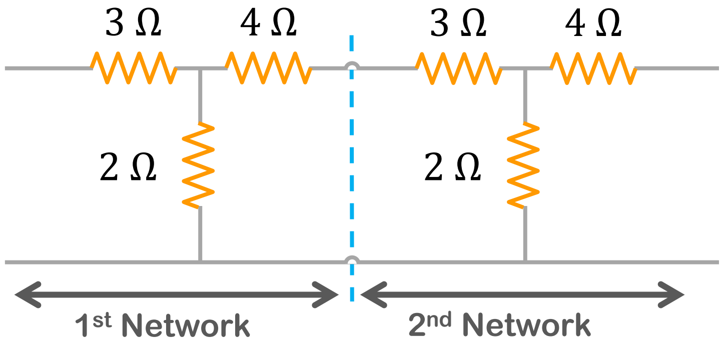
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 2.5 & 13 \\ 0.5 & 3 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

$$[T_1] = \begin{bmatrix} 2.5 & 13 \\ 0.5 & 3 \end{bmatrix}$$

The overall $ABCD$ parameters are given by

$$[T] = [T_1][T_2] = \begin{bmatrix} 2.5 & 13 \\ 0.5 & 3 \end{bmatrix} \begin{bmatrix} 2.5 & 13 \\ 0.5 & 3 \end{bmatrix}$$

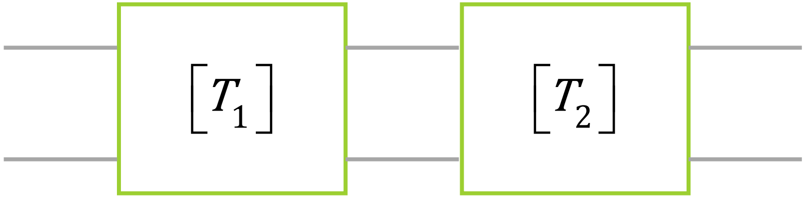
$$[T] = \begin{bmatrix} 12.75 & 71.5 \\ 2.75 & 15.5 \end{bmatrix}$$



Interconnection of Networks: Example 7

Find the $ABCD$ parameters of the cascade connection as shown, where

$$[T_1] = \begin{bmatrix} 0.1 & 0 \\ 0 & 10 \end{bmatrix}, \quad [T_2] = \begin{bmatrix} 2 & 10 \\ 0.1 & 1 \end{bmatrix}$$

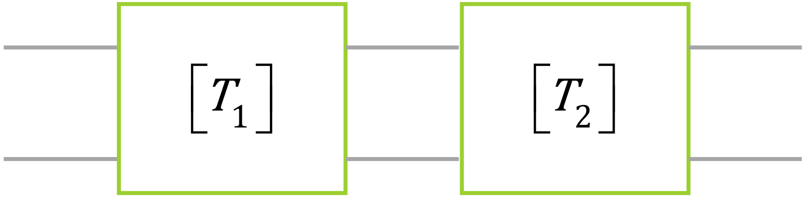


Interconnection of Networks: Example 7

Cascade connection

$$[T] = [T_1][T_2] = \begin{bmatrix} 0.1 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 2 & 10 \\ 0.1 & 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0.2 & 1 \\ 1 & 10 \end{bmatrix}$$



Summary

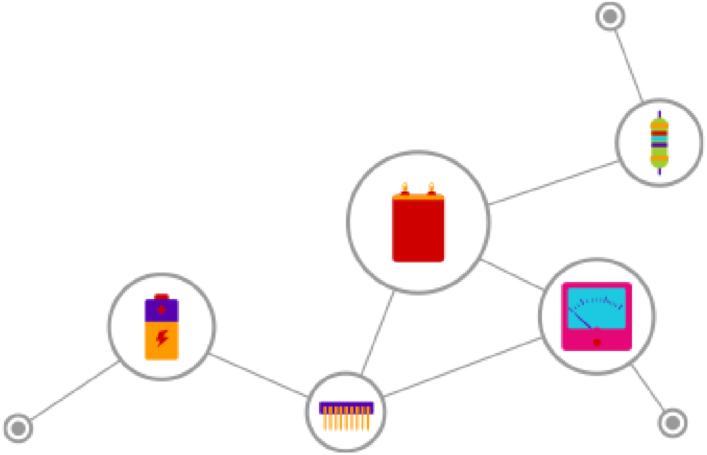
Summary

- Admittance parameters of a two-port network are defined by

$$\boxed{I_1 = y_{11}V_1 + y_{12}V_2} \quad (1)$$

$$\boxed{I_2 = y_{21}V_1 + y_{22}V_2} \quad (2)$$

  – Where the parameters are determined by short-circuit tests on port 1 and port 2 separately.



Summary

- Impedance parameters of a two-port network are defined by

$$\boxed{V_1 = z_{11}I_1 + z_{12}I_2} \quad \textbf{(3)}$$

$$\boxed{V_2 = z_{21}I_1 + z_{22}I_2} \quad \textbf{(4)}$$

  – Where the parameters are determined by open-circuit tests on port 1 and port 2 separately.

Summary

- Hybrid parameters of a two-port network are defined by

$$\boxed{V_1 = h_{11}I_1 + h_{12}V_2} \quad (5)$$

$$\boxed{I_2 = h_{21}I_1 + h_{22}V_2} \quad (6)$$

  – Where h_{11} and h_{21} are determined by short-circuit test on port 2. The parameters h_{12} and h_{22} are obtained by open-circuit test on port 1.

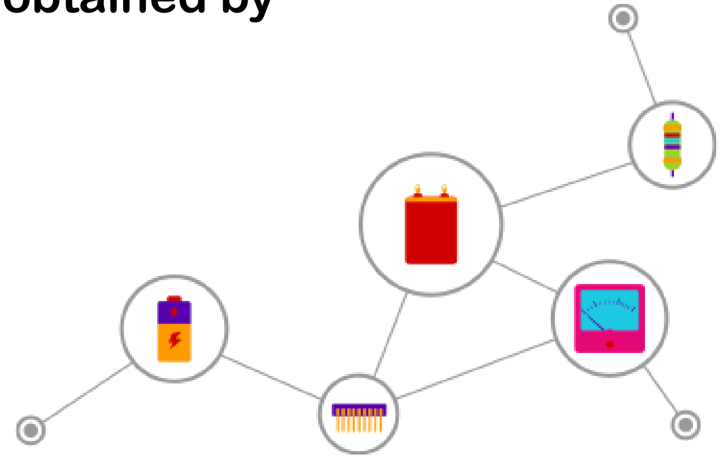
Summary

- Transmission parameters of a two-port network are defined by

$$\boxed{V_1 = AV_2 - BI_2} \quad \textbf{(7)}$$

$$\boxed{I_1 = CV_2 - DI_2} \quad \textbf{(8)}$$

  - Where A and C are determined by open-circuit test on port 2. The parameters B and D are obtained by short-circuit test on port 2.



Summary

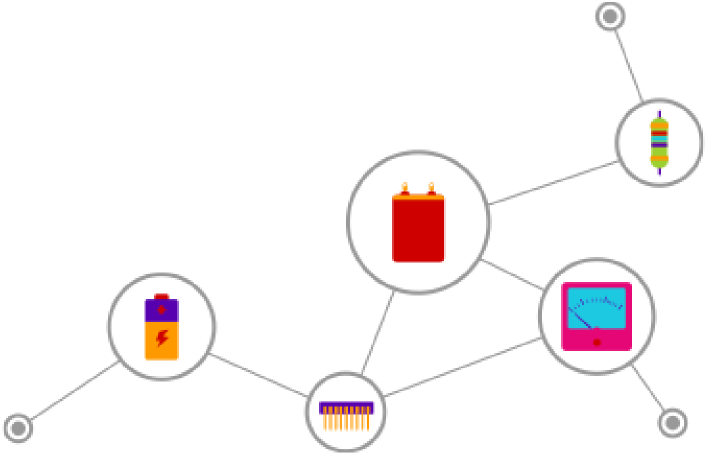
- A two-port circuit is reciprocal when

$$\frac{I_2}{V_1}\Big|_{V_2=0} = \frac{I_1}{V_2}\Big|_{V_1=0}$$

  - Or

Condition of Reciprocity
$z_{12} = z_{21}$
$y_{12} = y_{21}$
$h_{12} = -h_{21}$
$AD - BC = 1$

  - A circuit with only passive circuit elements (R, L, C) and transformers is a reciprocal network.



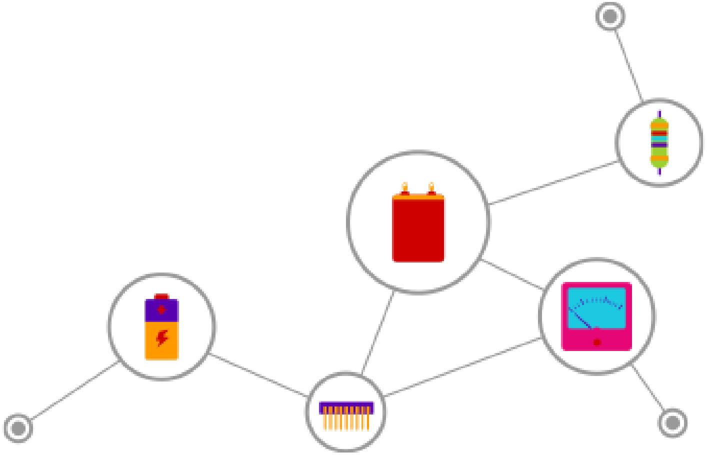
Summary

- A two-port circuit is symmetrical when

$$\frac{V_1}{I_1}\Big|_{I_2=0} = \frac{V_2}{I_2}\Big|_{I_1=0}$$

  – Or

Condition of Symmetry
$z_{11} = z_{22}$
$y_{11} = y_{22}$
$h_{11}h_{22} - h_{12}h_{21} = 1$
$A = D$



Summary

- **Let $[T_1]$ and $[T_2]$ be the ABCD parameters of the two-port networks N_1 and N_2 respectively. Then the $ABCD$ parameters of the cascade connection of N_1 and N_2 is given by** $\boxed{[T_1][T_2]}$