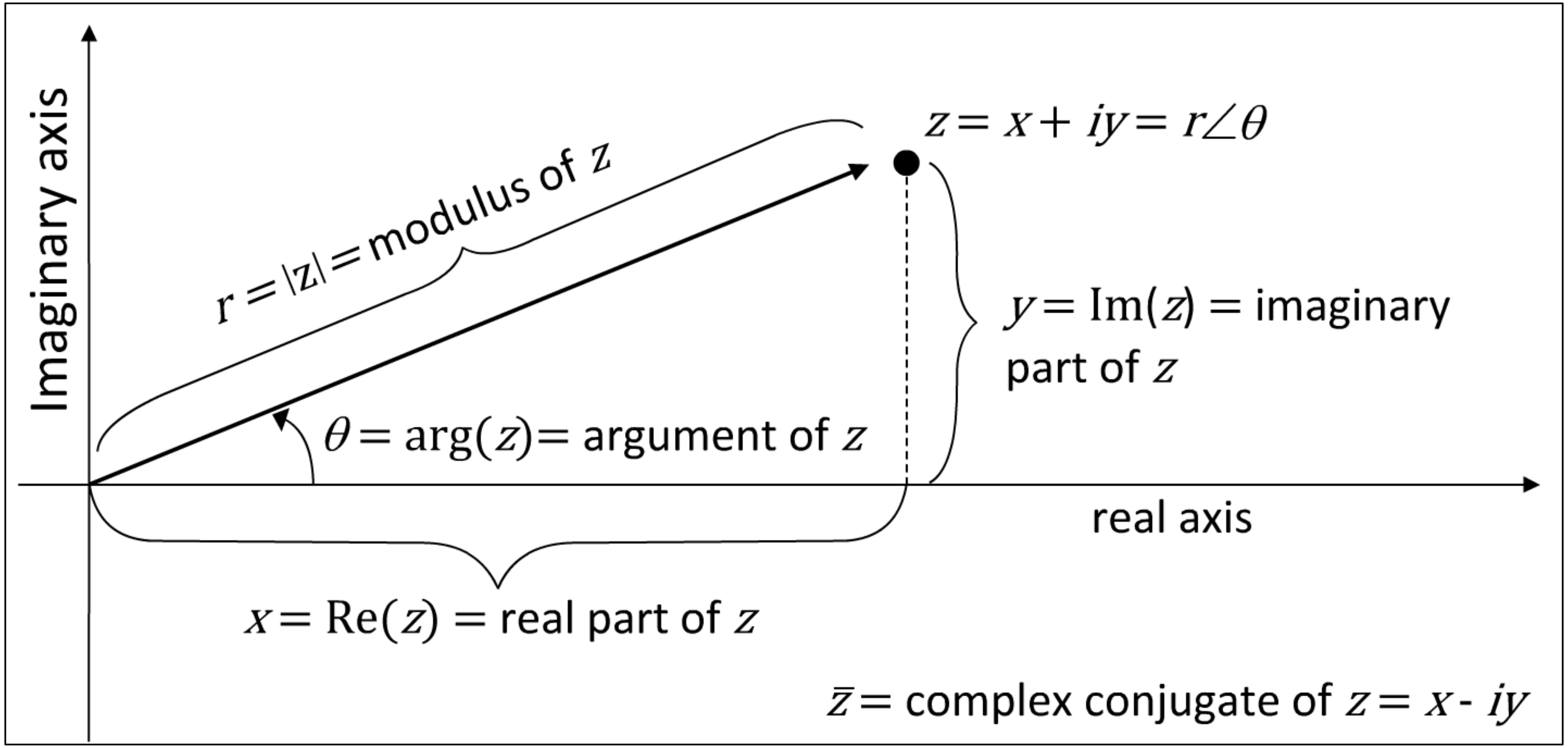
Complex Numbers

EE2007 – Engineering Mathematics II

At the end of this lesson, you should be able to:

- Define the basics of complex numbers.
- Derive Euler’s Formula and De Moivre’s Formula.
- Derive the complex logarithm and its general power.

A complex number z is defined as $z = x + iy$, where $i = \sqrt{-1}$. Geometrically, a complex number is a point in the complex plane (or the Argand diagram) and can be considered as a vector in the plane. A diagrammatic representation of the complex number is shown below.



Here is an explanation of the equation depicted in the diagram.

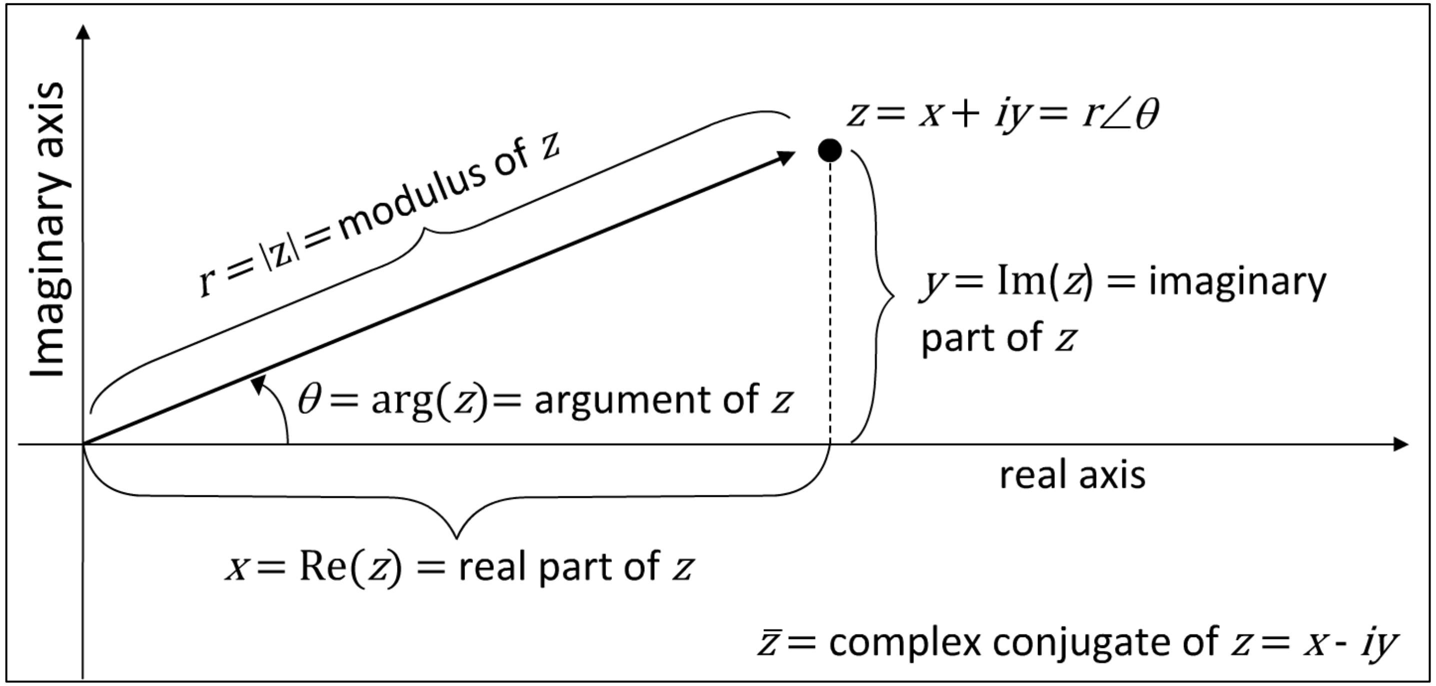
$x = r\cos\theta$, and $y = r\sin\theta$

$$r = |z| = \sqrt{x^2 + y^2} = |\bar{z}| = \sqrt{z\bar{z}}$$

$$\theta = \arg(z) = \arctan\frac{y}{x}\,\text{radians}$$

$$= \mathrm{Arg}(z) + 2n\pi, \qquad n = 0, \pm 1, \pm 2, \ldots$$

Where, $\mathrm{Arg}(z)$ is the principal value of $\arg(z)$ and satisfies $-\pi < \mathrm{Arg}(z) \leq \pi$



Let us look at an example to understand the concept of complex numbers.

Example 1

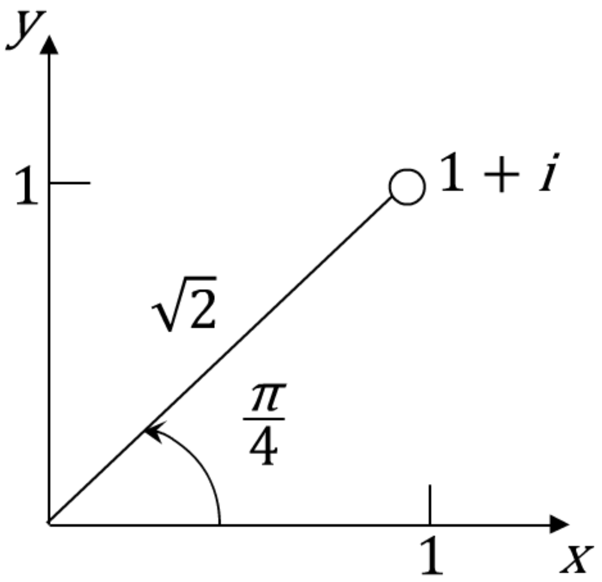
i. Let $z = 1 + i$

Then, $r = |z| = \sqrt{1+1} = \sqrt{2}$

$$\arg z = \arctan\frac{1}{1}$$

$$= \frac{\pi}{4} \pm 2n\pi, n = 0,1,2, \ldots$$

The principal value of the argument is $\frac{\pi}{4}$.



ii. If $z = 1 - i$, then $\arg z = \arctan\frac{-1}{1} = \frac{-\pi}{4} \pm 2n\pi, n = 0,1,2, \ldots$

The principal value of the argument is $\frac{-\pi}{4}$.

From Euler's formula, it can be found that:

$$e^{i\theta} = \cos\theta + i\sin\theta \text{ and } e^{-i\theta} = \cos\theta - i\sin\theta$$

Thus, $\boxed{\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}}$ and $\boxed{\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}}$

From Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, for any real value of θ, the polar form of a

complex number can be written as $z = re^{i\theta} = r\angle\theta$.

Let us now look at some Algebraic Rules.

Let $z_1 = x_1 + iy_1 = r_1\angle\theta_1$ and $z_2 = x_2 + iy_2 = r_2\angle\theta_2$

Addition and subtraction $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$

Multiplication $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

Division $\dfrac{z_1}{z_2} = \dfrac{x_1 x_2 + y_1 y_2}{x^2 + y^2} + i\,\dfrac{x_2 y_1 - x_1 y_2}{x^2 + y^2}$

Complex Numbers > Euler's Formula

Let us now look at some Algebraic Rules.

Let $z_1 = x_1 + iy_1 = r_1\angle\theta_1$ and $z_2 = x_2 + iy_2 = r_2\angle\theta_2$

Add

It is sometimes more convenient to do multiplication and division in the polar form.

$$z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2),$$

$$\frac{z_1}{z_2} = \frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

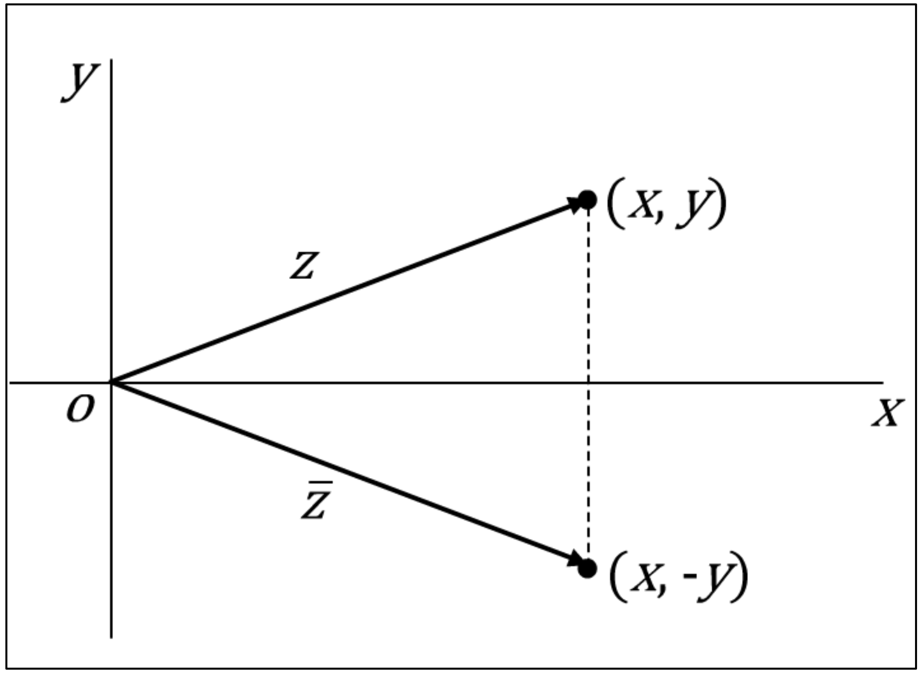
Division

$$\frac{\quad}{z_2} = \frac{\quad}{x^2+y^2} + i\,\frac{\quad}{x^2+y^2}$$

Let us now understand the complex conjugate of z and its algebraic rules.

In the given equation $z = x + iy$, the complex conjugate of z is defined as $\bar{z} = x - iy$.

Thus, it can be written as:



$$Re(z) = \frac{1}{2}(z + \bar{z}),\ Im(z) = \frac{1}{2i}(z - \bar{z})$$

$$z\bar{z} = x^2 + y^2 = |z|^2,\ \frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{|z_2|^2}$$

$$(\overline{z_1 \pm z_2}) = \bar{z}_1 \pm \bar{z}_2,\ \overline{z_1 z_2} = \bar{z}_1\bar{z}_2,\ \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

Here is the derivation of the De Moivre's formula.

- Let $z = x + iy = r(\cos\theta + i\sin\theta) = r\angle\theta$

- Then, for any integer n,

$$z^n = r^n(\cos\theta + i\sin\theta)^n$$

$$z^n = \underbrace{z.z \ldots .z}_{n} = \underbrace{r.r \ldots .r}_{n}\angle\underbrace{(\theta + \theta + \cdots + \theta)}_{n} = r^n\angle(n\theta)$$

$$= r^n(\cos n\theta + i\sin n\theta)$$

- From the above equation, the De Moivre's formula can be expressed as:
$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ which is useful in deriving certain trigonometric identities.

Let us look at a sample problem to understand the concept of complex numbers.

Sample Problem 1

Find identities for $\cos 2\theta$ and $\sin 2\theta$.

Solution:

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$$

$$= \cos 2\theta + i\sin 2\theta$$

Therefore,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \text{ and } \sin 2\theta = 2\cos\theta\sin\theta$$

Let us look at another sample problem explaining the concept of complex numbers.

Sample Problem 2

Express $\cos^4\theta$ in terms of multiples of θ.

Solution:

Since $2\cos\theta = e^{i\theta} + e^{-i\theta}$

$$2^4\cos^4\theta = (e^{i\theta} + e^{-i\theta})^4$$

$$= \left(e^{i4\theta} + e^{-i4\theta}\right) + 4\left(e^{i2\theta} + e^{-i2\theta}\right) + 6$$

$$= 2\cos 4\theta + 8\cos 2\theta + 6$$

$$\Rightarrow \cos^4\theta = \frac{1}{8}[\cos 4\theta + 4\cos 2\theta + 3]$$

Consider $z = w^n, n = 1,2, \ldots$

For a given $z \neq 0$, the solution of w in the above equation is called the n^{th} root of z and is denoted by $w = \sqrt[n]{z}$.

First, $z = r\angle(\theta + 2k\pi)$.

Next, let $w = R\angle\phi$.

Then, $z = w^n$ gives

$r\angle(\theta + 2k\pi) = R^n\angle(n\phi)$.

Thus, $R = \sqrt[n]{r}$, and

$$\phi = \frac{\theta + 2k\pi}{n}, \mathrm{k} = 0,1, \ldots, (n-1).$$

Consider $z = w^n, n = 1,2, \ldots$

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To summarise,

$$w_k = \sqrt[n]{z} = \sqrt[n]{r}\angle\left(\frac{\theta + 2k\pi}{n}\right),$$
$$k = 0,1, \ldots., (n-1)$$

Geometrically, the entire set of roots lies at the vertices of a regular polygon of n sides inscribed in a circle of radius $\sqrt[n]{r}$.

Thus, $R = \sqrt[n]{r}$, and

$$\phi = \frac{\theta + 2k\pi}{n}, \mathrm{k} = 0,1, \ldots, (n-1).$$

Let us look at an example to understand the concept of roots of complex numbers.

Example 2

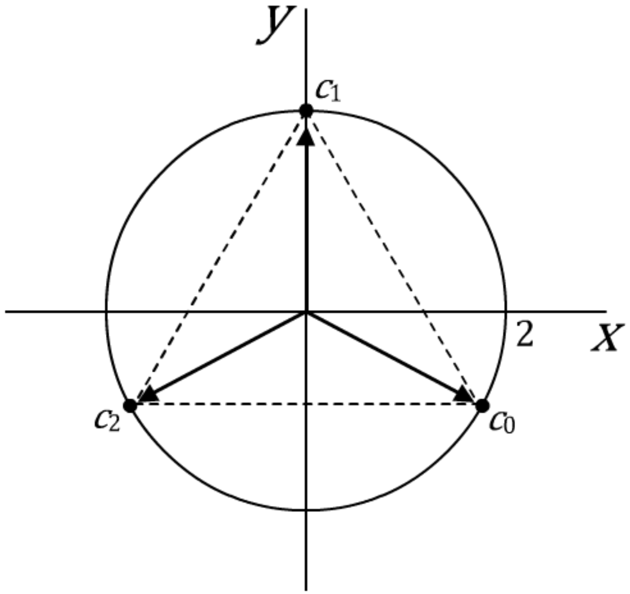
Let us find all values of $(-8i)^{1/3}$, that is, $\sqrt[3]{-8i}$.

First,

$$-8i = 8\angle\left(\frac{-\pi}{2} + 2k\pi\right), k = 0, \pm 1, \pm 2, \ldots$$

The desired roots are:

$$w_k = 2\angle\left(\frac{-\pi}{6} + \frac{2k\pi}{3}\right), k = 0,1,2$$

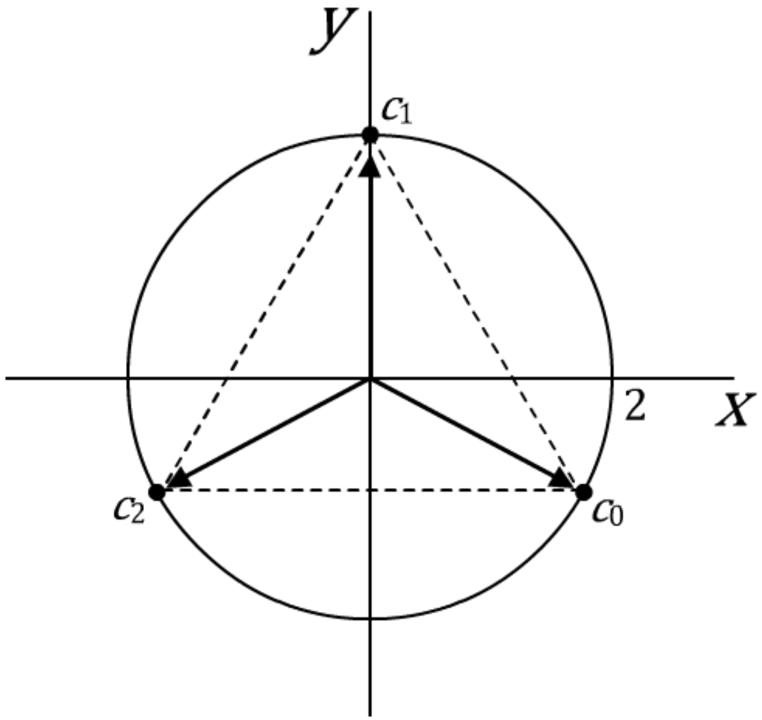


Let us look at an example to understand the concept of roots of complex numbers.

Example 2 (contd.)

The roots lie at the vertices of an equilateral triangle, inscribed in the circle $|z| = 2$ and are equally spaced around that circle every $^{2\pi}/_{3}$ radians, starting with the principal root

$$w_0 = 2\angle\left(\frac{-\pi}{6}\right) = \sqrt{3} - i.$$



Let us now define the exponential function.

The exponential function e^z is defined as:

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = e^x \ (\cos y + i \sin y).$$

If $x = 0$, then the Euler formula becomes: $e^{iy} = \cos y + i \sin y$.

Hence, the polar form of a complex number may be written as

$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

If $z = e^{ix} = \cos x + i \sin x$, then

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) = \frac{1}{2i}(z - \bar{z}),$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \frac{1}{2}(z + \bar{z}).$$

It is also geometrically obvious that $e^{i\pi} = -1, e^{-i\pi/2} = -i$ and $e^{-i4\pi} = 1$.

The natural logarithm of $z = x + iy$ is denoted by $\ln z$ and is defined as the inverse of the exponential function.

Since, $w = \ln z$ is defined for $z \neq 0$ by the relation $e^w = z$.

So, if $z = re^{i\theta}, r > 0$, then $\ln z = \ln r + i\theta$.

Note that the complex logarithm is infinitely many-valued.

The general power of a complex number, z^c, can be derived as follows:

Let $y = z^c, \Rightarrow \ln y = c\ln z, \Rightarrow y = z^c = e^{c\ln z}, z \neq 0$.

Let us look at a sample problem to understand the concept of complex logarithm.

Sample Problem 3

i) Evaluate $\ln(3-4i)$.

ii) Solve $\ln z = -2 - \frac{3}{2}i$.

Solution:

i) $\ln(3-4i) = \ln|3-4i| + i\arg(3-4i)$

$$= 1.609 - i(0.927 \pm 2n\pi), n = 0,1,\ldots$$

Principal value: When $n = 0$

ii) $z = e^{-2-\frac{3}{2}i} = e^{-2}e^{-i\frac{3}{2}} = e^{-2}\left(\cos\frac{3}{2} - i\sin\frac{3}{2}\right)$

$$= 0.010 - i\,0.135$$

Here is another sample problem explaining the concept of complex logarithm.

Sample Problem 4

Find the principal value of $(1+i)^i$.

Solution:

Let $y = (1+i)^i$. Then, $\ln y = i\ln(1+i)$, or $y = e^{i\ln(1+i)}$

Hence, $(1+i)^i = e^{i\ln(1+i)}$

But, $\ln(1+i) = \ln\left(\sqrt{2}e^{i(\pi/4+2k\pi)}\right)$

$$= \ln\sqrt{2} + i\left(\pi/4 + 2k\pi\right), k = 0, \pm 1, \ldots$$

and the principal value is when $k = 0$.

Therefore, $e^{i\ln(1+i)} = e^{i(\ln\sqrt{2}+i\pi/4)} = e^{-\frac{\pi}{4}+i(\ln\sqrt{2})}$

Summary

Key points discussed in this lesson:

- A complex number z is defined as $z = x + iy$, where $i = \sqrt{-1}$. Geometrically, a complex number is a point in the complex plane (or the Argand diagram) and can be considered as a vector in the plane.

- In the given complex number $z = x + iy$, the complex conjugate of z is defined as $\bar{z} = x - iy$.

- From Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$, and $e^{-i\theta} = \cos\theta - i\sin\theta$. Then,
$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.

Key points discussed in this lesson:

- For complex number $z = x + iy = r(\cos\theta + i\sin\theta) = r\angle\theta$. The De Moivre's formula is given as: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$.

- The exponential function e^z is defined as: $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = e^x\ (\cos y + i \sin y)$.

- The natural logarithm of $z = x + iy$ is denoted by $\ln z$ and is defined as the inverse of the exponential function.

Differentiation of Complex Functions

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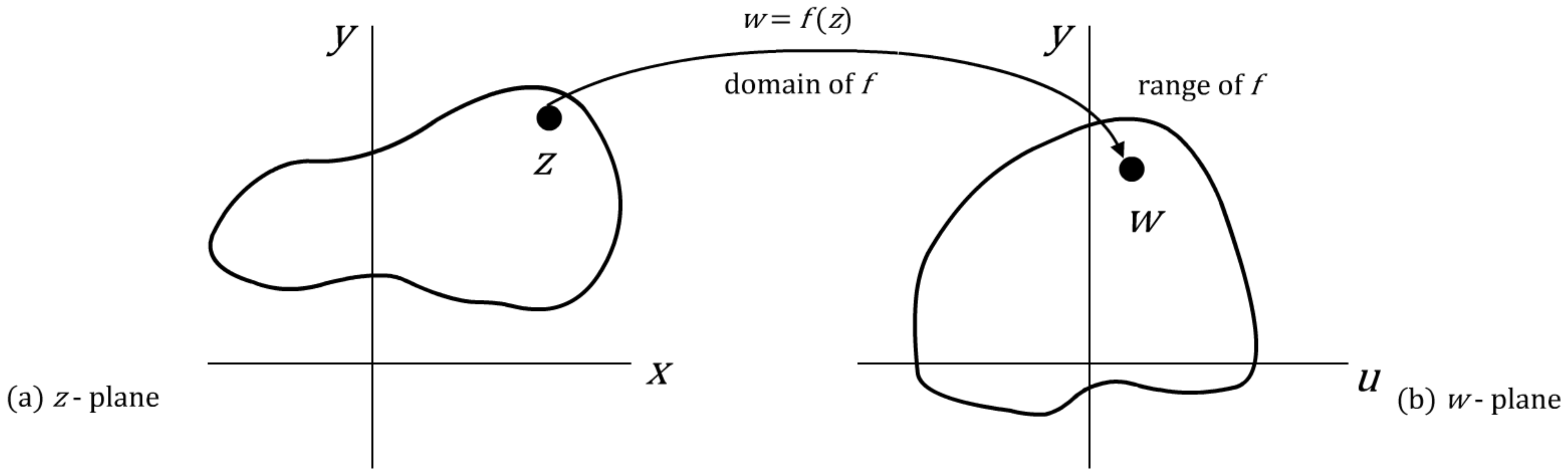
At the end of this lesson, you should be able to:

- Describe the concept of limit and continuity of complex functions.
- Explain the differentiability and analyticity of complex functions.

A complex function f is concerned with complex functions that are differentiable in some domain.



A complex function f is a rule (or mapping) that assigns to every complex number z in a set S, and a complex number w in a set T.

Mathematically, it can be expressed as $w = f(z)$.

The set S is called the domain of f and the set T is called the range of f.

If $z = x + iy$ and $w = u + iv$, then,

$$w = f(z) = u(x, y) + iv(x, y)$$

Let us take a look at a sample problem to understand the concept of complex functions.

Sample Problem 1

Let $w = f(z) = z^2 + 3z$. Find u and v and calculate the value of f at $z = 1 + 3i$.

Solution:

Let $z = x + iy$.

Then, $w = z^2 + 3z$

$$= (x + iy)^2 + 3(x + iy)$$

$$= x^2 - y^2 + i2xy + 3x + i3y$$

Let us take a look at a sample problem to understand the concept of complex functions.

Solution (contd.):

Hence,

$$u = \mathrm{Re}(w) = x^2 - y^2 + 3x$$

$$v = \mathrm{Im}(w) = 2xy + 3y$$

If, $z = x + iy = 1 + i3$

then, $f(z) = u(1,3) + v(1,3) = -5 + i15$

Try using the polar form, $z = r\angle\theta$, and check if you get the same answer.

A function $f(z)$ is said to have the limit L as z approaches a point z_0 if the following conditions are satisfied.

01 $f(z)$ is defined in the neighbourhood of z_0 (except perhaps at z_0 itself).

02 $f(z)$ approaches the same complex number L as $z \to z_0$ from all directions within its neighbourhood.

Mathematically, the limit of a function $f(z)$ can be expressed as:

$$\lim_{z \to z_0} f(z) = L$$

If given $\in$, there exists $\delta > 0$, such that,

$$|f(z) - L| < \in, \forall\, 0 < z - z_0 < \delta$$

The given equation means that the point $f(z)$ can be made arbitrarily close to the point L if the point z is chosen in such a way that it is sufficiently close to, but not equal to the point z_0.

Examples

1

$$\lim_{z\to\infty}\frac{2z+i}{z+1}=\lim_{z\to\infty}\frac{2+(i/z)}{1+(1/z)}=2$$

2

$$\lim_{z\to\infty}\frac{2z^3-1}{z^2+1}=\lim_{z\to\infty}\frac{2-(1/z^3)}{(1/z)+(1/z^3)}=\lim_{z\to\infty}\frac{2}{0}=\infty$$

Examples (contd.)

3 $\lim\limits_{z\to\infty}\frac{z}{\bar{z}}$ does not exist.

Let $y \to 0$ first and then, let $x \to 0$. In this case,

$$\lim_{x\to 0, y=0}\frac{x+i0}{x-i0}=1$$

Now, let $x \to 0$ first and then, let $y \to 0$. In this case,

$$\lim_{x=0, y\to 0}\frac{0+iy}{0-iy}=-1$$

As the function does not approach the same value from all directions within its neighbourhood, the limit does not exist.

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A function $f(z)$ is said to be continuous at $z = z_0$ if it satisfies the following three conditions.

01 $f(z_0)$ exists

02 $\lim_{z \to z_0} f(z)$ exists

03 $\lim_{z \to z_0} f(z) = f(z_0)$

Note that if condition (3) is true, it implies that conditions (1) and (2) are true as well.

f is said to be a continuous function, if f is continuous for all z in the domain S.

Let us see how to test the continuity of a function with the help of the following sample problem.

Sample Problem 2

Let $f(0) = 0$, and for $z \neq 0$, $f(z) = \mathrm{Re}(z^2)/|z^2|$. Determine whether $f(z)$ is continuous at the origin.

Solution:

$$\lim_{z\to 0} \mathrm{Re}(z^2)/|z^2| = \lim_{z\to 0} \frac{x^2 - y^2}{x^2 + y^2} = \begin{cases} 1 \text{ if } y \to 0 \text{ first} \\ -1 \text{ if } x \to 0 \text{ first} \end{cases}$$

Hence, f is not continuous at the origin.

Let us see how to test the continuity of a function with the help of the following sample problem.

Solution (contd.):

Alternatively, using polar representation,

$$\begin{aligned} z &= re^{i\theta} \\ &= r\cos\theta + ir\sin\theta \end{aligned}$$

$$\lim_{z\to 0} \mathrm{Re}(z^2)/|z^2| = \lim_{r\to 0} \frac{r^2\cos 2\theta}{r^2} = \cos 2\theta$$

The limit does not exist because it depends on the direction of approach to the origin.

The derivative of a complex function f at a point z_0 is written as $f'(z_0)$ and is defined as:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$, provided that the limit exists.

Or, by substituting $z = z_0 + \Delta z$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

For example,

$$\frac{d}{dz}(z^2) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

Thus, $f(z) = z^2$ is differentiable for all z.

The usual differentiation formulae (as in the case of real variables) hold for complex functions. Let us refer to an example.

01	$\frac{d}{dz}(c) = 0$
02	$\frac{d}{dz}(z) = 1$
03	$\frac{d}{dz}(z^n) = nz^{n-1}$
04	$\frac{d}{dz}(2z^2 + i)^5 = 5(2z^2 + i)^4.4z = 20z(2z^2 + i)^4$

However, care is required for more unusual functions.

Let us take a look at a sample problem to understand the concept of differentiability of complex functions.

Sample Problem 3

Discuss the differentiability of $\bar{z}$.

Solution:

Let $f(z) = \bar{z}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z}$$

Using the property $\overline{z + \Delta z} = \bar{z} + \overline{\Delta z}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

Let us take a look at a sample problem to understand the concept of differentiability of complex functions.

Solution (contd.):

Now, consider $\Delta z = \Delta r e^{i\theta}$. Then, $\Delta z \to 0$ from all directions when $\Delta r \to 0$.

Thus, the limit can be determined as follows:

$$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta r e^{-i\theta}}{\Delta r e^{i\theta}} = e^{-i2\theta}$$

The limit depends on θ, and therefore, it does not exist. Hence, $f(z) = \bar{z}$ is not differentiable anywhere.

A function $f(z)$ is said to be analytic at a point z_0 if its derivative exists not only at z_0, but also in some neighbourhood of z_0.

A function $f(z)$ is said to be analytic in the domain D if it is analytic at each point in D.

Hence, analyticity implies differentiability and continuity.

The point $z = z_0$, where $f(z)$ ceases to be analytic. It is called the singular point or singularity of $f(z)$.

For example,

- $f(z) = z^2$ is analytic everywhere in the complex plane
- $f(z) = \bar{z}$ is not analytic at any point

Cauchy-Riemann (C-R) Equations can be used to test the analyticity of a complex function.

Theorem 1: The complex function $f(z) = u(x, y) + iv(x, y)$ is analytic at a point z_0 if for every point in the neighbourhood of z_0.

1. u, v, and their partial derivatives exist and are continuous.

2. Cauchy-Riemann equations, $u_x = v_y$ and $v_x = -u_y$ are satisfied.

If these two conditions are satisfied in some domain D, then the function is analytic in D.

Derivation of the C-R Equations

The derivative of a complex function f at a point z_0 is given by:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta x, \Delta y \to 0} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y}$$

Along the x-axis, that is, $\Delta y = 0$,

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y) + i(v(x + \Delta x, y) - v(x, y))}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Derivation of the C-R Equations

Similarly, along the y-axis, that is, $\Delta x = 0$,

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - u(x, y) - iv(x, y)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y) + i(v(x, y + \Delta y) - v(x, y))}{i\Delta y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

For the derivative to exist, the two limits must agree, that is:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Derivation of the C-R Equations

Thus, the C-R equations are:

$$u_x = v_y \text{ and } v_x = -u_y$$

When $z \neq 0$, the C-R equations in polar coordinates are:

$$u_r = \frac{1}{r} v_\theta \text{ and } v_r = -\frac{1}{r} u_\theta$$

Derivatives of Complex functions

If $f(z) = u(x,y) + iv(x,y)$ and $f'(z)$ exists, then,

$$f'(z) = u_x + iv_x$$
$$= v_y - iu_y$$
$$= u_x - iu_y$$
$$= v_y + iv_x$$

In polar form, if $f(z) = u(r,\theta) + iv\,(r,\theta)$ and $f'(z)$ exists, then,

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$
$$= \frac{1}{r}e^{-i\theta}(v_\theta - iu_\theta)$$

The following sample problem demonstrates how Cauchy-Riemann equations are used to test the analyticity of a complex function.

Sample Problem 4

Verify that $f(z) = \bar{z}$ is not analytic.

Solution:

Using C-R equations,

$$u(x, y) = x \text{ and } v(x, y) = -y$$

Now, $u_y = -v_x = 0$

However, $u_x = 1$ and $v_y = -1$

As the C-R equations are not satisfied, the given function is not analytic.

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The following sample problem demonstrates how Cauchy-Riemann equations are used to test the analyticity of a complex function.

Sample Problem 4

Verify that $f(z) = \bar{z}$ is not analytic.

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Now, u_x

However, $u_x = 1$ and $v_y = -1$

As the C-R equations are not satisfied, the given function is not analytic.

As the function $f(z) = z$ is not differentiable, it can be simply stated that the function is not analytic, without even using the C-R equations.

The following sample problem demonstrates how Cauchy-Riemann equations are used to test the analyticity of a complex function.

Sample Problem 5

Is $f(z) = z^3$ analytic?

Solution:

In general, polynomials of complex variables are analytic. Let's solve the given problem using C-R equations.

$$f(z) = z^3$$

$u(r, \theta) = r^3 \cos 3\theta$ and $v(r, \theta) = r^3 \sin 3\theta$

The following sample problem demonstrates how Cauchy-Riemann equations are used to test the analyticity of a complex function.

Solution (contd.):

Therefore, $u_r = 3r^2\cos 3\theta$ and $u_\theta = -3r^3\sin 3\theta$

$$v_r = 3r^2\sin 3\theta \text{ and } v_\theta = 3r^3\cos 3\theta$$

As the C-R equations $u_r = \frac{1}{r} v_\theta$ and $v_r = -\frac{1}{r} u_\theta$ are satisfied, and the functions u, v, and their partial derivatives are continuous, the function $f(z) = z^3$ is analytic.

Here is another sample problem that helps us understand how these equations are used to test the analyticity of a complex function.

Sample Problem 6

Discuss the analyticity of the function $f(z) = x^2 + iy^2$.

Solution:

With $u = x^2$ and $v = y^2$: $u_x = 2x$ and $v_y = 2y$

$$v_x = 0 \text{ and } u_y = 0$$

Thus, from C-R equations, $f(z)$ is differentiable only for those values of z that lie along the straight line $x = y$. If z_0 lies on this line, any circle centered at z_0 will contain points for which $f'(z)$ does not exist. Therefore, the given function is not analytic at any point.

NANYANG TECHNOLOGICAL UNIVERSITY

Some Common (and Important) Functions

Polynomials, that is, functions of the form, $f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$ (where c_0, c_1, ….., c_n are complex constants) are analytic in the entire complex plane.

Rational functions, that is, quotient of two polynomials, $f(z) = \dfrac{g(z)}{h(z)}$ are analytic except at points where $h(z) = 0$.

Partial fractions of the form $f(z) = \dfrac{c}{(z - z_0)^m}$, where c and z_0 are complex, and m is a positive integer, are analytic except at z_0.

Summary

Key points discussed in this lesson:

- A complex function f is a rule (or mapping) that assigns to every complex number z in a set S, and a complex number w in a set T.

- A function $f(z)$ is said to have the limit L as z approaches a point z_0 if:
  - $f(z)$ is defined in the neighbourhood of z_0 (except perhaps at z_0 itself)
  - $f(z)$ approaches the same complex number L as $z \to z_0$ from all directions within its neighbourhood

- A function $f(z)$ is said to be continuous at $z = z_0$ if:
  - $f(z_0)$ exists
  - $\lim_{z \to z_0} f(z)$ exists
  - $\lim_{z \to z_0} f(z) = f(z_0)$ exists

Key points discussed in this lesson:

- The derivative of a complex function f at a point z_0 is written as $f'(z_0)$ and is defined as:

  $f'(z_0) = \lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$, provided that the limit exists.

- A function $f(z)$ is said to be analytic at a point z_0 if its derivative exists not only at z_0, but also in some neighbourhood of z_0.