NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2019-2020

EE2007 / IM2007 – ENGINEERING MATHEMATICS II

November / December 2019 Time Allowed: $2\frac{1}{2}$ hours

INSTRUCTIONS

1. This paper contains 4 questions and comprises 4 pages.
2. Answer all 4 questions.
3. All questions carry equal marks.
4. This is a closed book examination.
5. Unless specifically stated, all symbols have their usual meanings.
6. A list of useful formulae is given in the Appendix A on page 4.

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1. Consider the block matrix

$$P = \begin{bmatrix} B & 0 \\ C & A \end{bmatrix},$$

where A and B are square matrices, C and 0 are arbitrary and zero matrices of appropriate dimensions.

(a) Let $A = \begin{bmatrix} 3 & 2 \\ 7 & 8 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A. Show your working clearly.

(10 Marks)

(b) Show that the eigenvalues of matrix A are also the eigenvalues of P. Hence, find the eigenvectors of P corresponding to the eigenvalues of A. You may assume that B is an arbitrary n-by-n matrix. Justify your answers.

(5 Marks)

Note: Question 1 continues on page 2.

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(c) Let $B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$. Find the remaining eigenvalues of matrix P.

(5 Marks)

(d) Let $C = \begin{bmatrix} 0 & -2 \\ -2 & -1 \end{bmatrix}$. Find the eigenvector of matrix P corresponding to one of the eigenvalues of matrix B found in part (c).

(5 Marks)

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 & 7 \\ 4 & 11 & 21 & 36 \\ 3 & 21 & 43 & 70 \\ 2 & 16 & 46 & 74 \end{bmatrix}.$$

(a) Use elementary row operations to reduce A to the row echelon form. Hence, find the determinant of A.

(10 Marks)

(b) Based on your working in part (a), or otherwise, find a matrix E that will transform A to the row echelon matrix you obtained in part (a). In other words, find E such that the product EA is the row echelon matrix you obtained in part (a).

(5 Marks)

(c) Consider the system

$$B\mathbf{x} = \mathbf{b},$$

where B is the first three columns of A, i.e.,

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 11 & 21 \\ 3 & 21 & 43 \\ 2 & 16 & 46 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

Determine:

(i) the condition(s) for b_1, b_2, b_3 and b_4 such that the system $B\mathbf{x} = \mathbf{b}$ is consistent.

(ii) the null-space and row space of B.

(10 Marks)

3. (a) Given $f(z) = |z^2| + \left|\frac{1}{z}\right|$, where $z = x + iy$, determine:

   (i) the limit of $f(z)$ as $z \to i$,

   (ii) if $f(z)$ is continuous at $z = i$.

   Using the Cauchy-Riemann equations, comment on the differentiability and analyticity of $f(z)$.

(13 Marks)

(b) Evaluate

$$\oint_C \left[5e^{2z} + z - 1 + \frac{z}{(z-1)^2(z^2-5z+6)}\right] dz$$

along the following paths C (counter-clockwise), where

   (i) C is the circle $|z| = \frac{1}{2}$.

   (ii) C is the circle $|z-1| = \frac{1}{2}$.

   (iii) C is the circle $|z-2i| = 3$.

(12 Marks)

4. (a) For any $f(x, y, z)$,

$$\iint_S \text{curl (grad } f) \cdot d\mathbf{A} = 0.$$

   Justify the truth of this equation with proof(s).

(6 Marks)

(b) A vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + xz\,\mathbf{j} + x\,\mathbf{k}$ cuts a planer surface $S : 3x + 2y + 6z = 6$, $x \geq 0$, $y \geq 0$, $z \geq 0$. Evaluate $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{A}$.

(11 Marks)

(c) Hence, using the result from part 4(b), or otherwise, find the work done in moving a particle along the straight line from $(0, 0, 1)$ to $(2, 0, 0)$.

(8 Marks)

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Appendix A

1. Complex Analysis

   (a) Complex Power: $z^c = e^{c \ln z}$

   (b) De Moivre's Formula: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$

   (c) Cauchy-Riemann equations:

   $$u_x = v_y, v_x = -u_y, \quad \text{or} \quad u_r = \frac{1}{r}v_\theta, v_r = \frac{-1}{r}u_\theta$$

   (d) Cauchy Integral Formula:

   $$\int_C \frac{f(z)}{(z-z_o)^m}\,dz = \frac{2\pi i}{(m-1)!}\frac{d^{(m-1)}}{dz^{(m-1)}}f(z)|_{z=z_o}$$

2. Vector Analysis. Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$.

   (a) Scalar Triple Product: $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$

   (b) Gradient: $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$

   (c) Divergence: $\nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

   (d) Curl: $\nabla\times\mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

   (e) Gauss Theorem: $\iiint_T \nabla\cdot\mathbf{F}\,dv = \iint_S \mathbf{F}\cdot\mathbf{n}\,dA$

   (f) Stokes Theorem: $\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}\,dA = \int_C \mathbf{F}\cdot\mathbf{dr}$

END OF PAPER

1. $P = \begin{bmatrix} B & 0 \\ C & A \end{bmatrix}$

a. $A = \begin{bmatrix} 3 & 2 \\ 7 & 8 \end{bmatrix}$

Eigen values & vector definition:

$A\vec{v} = \lambda\vec{v}$, λ is eigen value, $\vec{v}$ is eigen vector

solving eigen value:

$$A\vec{v} - \lambda\vec{v} = 0$$
$$(A - I\lambda)\vec{v} = 0$$

Hence $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \left| \begin{bmatrix} 3-\lambda & 2 \\ 7 & 8-\lambda \end{bmatrix} \right| = 0$$
$$0 = (3-\lambda)(8-\lambda) - 14$$
$$0 = \lambda^2 - 11\lambda + 24 - 14$$
$$0 = \lambda^2 - 11\lambda + 10$$
$$0 = (\lambda - 10)(\lambda - 1)$$
$$\lambda = 10, 1$$

$\therefore$ the eigen value of A are 1 & 10

to find eigenvector, we need to find the null space of the matrix: $(A - \lambda I)$

for $\lambda = 1$: $\begin{bmatrix} 2 & 2 \\ 7 & 7 \end{bmatrix} \xrightarrow[R_2 = R_2 - 7R_1]{R_1 = \frac{1}{2}R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Hence, $x_2 = c$, $c \in R$

$$x_1 + x_2 = 0$$
$$x_1 = -c$$

Thus the eigen vector is: $c\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Which c can be any real number

for $\lambda = 10$: $\begin{bmatrix} -7 & 2 \\ 7 & -2 \end{bmatrix} \xrightarrow[R_1 = R_1 \times -\frac{1}{7}]{R_2 = R_2 + R_1} \begin{bmatrix} 1 & -\frac{2}{7} \\ 0 & 0 \end{bmatrix}$

Hence, $x_2 = b$, $b \in R$

$$x_1 + -\frac{2}{7}x_2 = 0$$
$$x_1 = \frac{2}{7}b$$

Thus the eigen vector is: $b\begin{bmatrix} \frac{2}{7} \\ 1 \end{bmatrix} = b'\begin{bmatrix} 2 \\ 7 \end{bmatrix}$

for b can be any real number

b. i) $P = \begin{bmatrix} B & 0 \\ C & A \end{bmatrix}$ can be factorice into

$$P = \begin{bmatrix} B & 0 \\ C & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & A \end{bmatrix}$$

Hence, let's define the definition of eigenvalue of P and eigen vector

$$P\vec{v} = \lambda \vec{v}$$

$$\det(P - \lambda I) = 0$$

$$= \left| \begin{bmatrix} B - \lambda I_B & 0 \\ C & A - \lambda I_A \end{bmatrix} \right|$$, I_B = n by n matrix Identity, I_A = 2 by 2 matrix Identity

$$= \left| \begin{bmatrix} B - \lambda I_B & 0 \\ C & I_m \end{bmatrix} \right| \times \left| \begin{bmatrix} I_n & 0 \\ 0 & A - \lambda I_A \end{bmatrix} \right| = 0$$

* $\det \begin{vmatrix} I_n & 0 \\ 0 & A - \lambda I_A \end{vmatrix} = 1 \det \begin{vmatrix} I_{n-1} & 0 \\ 0 & A - \lambda I_A \end{vmatrix}$

$$= \underbrace{1 \times 1 \times \dots 1}_{n \text{ times}} \det(A - \lambda I_A) = 0$$

Which is the definition of eigenvalue of A

* $\det \begin{vmatrix} B - \lambda I_B & 0 \\ C & I_m \end{vmatrix} = 1 \det \begin{vmatrix} B - \lambda I_B & 0 \\ C_{m-1} & I_m \end{vmatrix}$

$$= \det(B - \lambda I_B) = 0$$

ii) Hence, eigenvalues of P are eigenvalues of A & B.

1st condition:

* Since eigenvalues of A is also eigenvalues of P, which we define as λ_1 with eigen vector V_1; (size 2x1), then it's also an eigenvalue of the matrix P, with the same eigen vector with augmented zeros. ($AV_1 = \lambda V_1$)

So: $PV = \lambda_1 V$, V is eigenvector of P

$$\begin{pmatrix} B & 0 \\ C & A \end{pmatrix} \begin{pmatrix} 0 \\ V_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 \\ V_1 \end{pmatrix}$$

Hence, eigen vector of P: $\lambda = 1, 7$ respectively $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 2 \\ 7 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ AV_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_1 V_1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ \lambda_1 V_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_1 V_1 \end{pmatrix}$$, which is true, Hence if λ is the eigenvalue of A it is also eigenvalue of P which we had shown above.

$V = \begin{pmatrix} 0 \\ V_1 \end{pmatrix}$ → n times rows, → 2 rows

2nd condition:

* let λ_2 be the eigen value of B with eigenvector V_2, if λ_2 is also eigenvalue of A we have proven that it is also eigenvalue of A, so let's assume it is not the eigenvalue of A - hence $|A - \lambda_2 I| \neq 0$)

Thus; $$\begin{pmatrix} B & 0 \\ C & A \end{pmatrix} \begin{pmatrix} V_2 \\ x \end{pmatrix} = \begin{pmatrix} BV_2 \\ CV_2 + Ax \end{pmatrix} = \begin{pmatrix} \lambda_2 V_2 \\ Ax + CV_2 \end{pmatrix}$$

we can make $Ax + CV_2 = \lambda_2 x$ by choosing $x = -(A - \lambda_2 I)^{-1} C V_2$

we found eigenvector for P with λ_2 as eigenvalue. Here, if λ is eigenvalue of A & B it must be eigenvalue of P (which was proven above)

3rd Condition: if λ is eigenvalue of P, the it must be eigenvalue of A or B.

$$\begin{pmatrix} B & 0 \\ C & A \end{pmatrix} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} Bx_2 \\ Cx_2 + Ax_1 \end{pmatrix} = \begin{pmatrix} \lambda x_2 \\ \lambda x_1 \end{pmatrix}$$, if $x_2 = 0$, then λ is eigen value of B; if $x_2 \neq 0$, it is eigenvalue of A

c. $B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$

$\det(B-\lambda I_B) = \left| \begin{bmatrix} 4-\lambda & 3 \\ 2 & 5-\lambda \end{bmatrix} \right| = (\lambda-4)(\lambda-5) - 6$

$= \lambda^2 - 9\lambda + 14 = 0$

$= (\lambda-7)(\lambda-2) = 0$

$\lambda = 7, 2$

d. $C = \begin{bmatrix} 0 & -2 \\ -2 & -1 \end{bmatrix}$

$P = \begin{bmatrix} 4 & 3 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 0 & -2 & 3 & 2 \\ -2 & -1 & 7 & 0 \end{bmatrix}$

eigenvector:

$(P-\lambda I) =$ for $\lambda = 2$, the nullspace

$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -2 & 1 & 2 \\ -2 & -1 & 7 & 6 \end{bmatrix} \xrightarrow[R_4 = R_4 + R_1]{R_2 = R_2 - R_1} \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 2 \\ 0 & 2 & 7 & 6 \end{bmatrix} \xrightarrow[R_1 = \frac{1}{2}R_1,\ R_3 = -\frac{1}{2}R_3]{R_4 = R_4 + R_3} \begin{bmatrix} 1 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -1 \\ 0 & 0 & 8 & 8 \end{bmatrix}$

let $x_4 = d \in R$

$x_3 + x_4 = 0$

$x_3 = -d$

$x_2 - \frac{1}{2}x_3 - x_4 = 0$

$x_2 = \frac{1}{2}d$

$x_1 + \frac{3}{2}x_2 = 0$

$x_1 = -\frac{3}{4}d$

Hence the eigenvector is $d\begin{bmatrix} -\frac{3}{4} \\ \frac{1}{2} \\ -1 \\ 1 \end{bmatrix}$

Optional: eigenvector for $\lambda = 7$

$P - \lambda I = \begin{bmatrix} -3 & 3 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & -2 & -4 & 2 \\ -2 & -1 & 7 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -\frac{9}{13} \\ 0 & 1 & 0 & -\frac{9}{13} \\ 0 & 0 & 1 & -\frac{2}{13} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_4 = e \in R$ $\quad x_3 = \frac{2}{13}e$ $\quad x_2 = \frac{9}{13}e$ $\quad x_1 = \frac{9}{13}e$ $\rightarrow$ eigenvector is $e\begin{bmatrix} \frac{9}{13} \\ \frac{9}{13} \\ \frac{2}{13} \\ 1 \end{bmatrix}, e \in R$

*: * we can also use the formula $x = -(A-\lambda_2 I)^{-1} C v_2$ from previous page

for $\lambda = 2$, $(B-\lambda I_B) = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \rightarrow \vec{v_2} = k\begin{bmatrix} -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix}, k \in R$ $\quad x = -\begin{bmatrix} 1 & 2 \\ 7 & 6 \end{bmatrix}^{-1} \times \begin{bmatrix} 0 & -2 \\ -2 & -1 \end{bmatrix} k \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Hence, $v_2 = k\begin{bmatrix} -\frac{3}{4} \\ \frac{1}{2} \\ -1 \\ 1 \end{bmatrix}, k \in R$ for $\lambda = 2$

Similarly, $\lambda = 7$ $\quad (B-\lambda I_B) = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \rightarrow \vec{v_2} = k\begin{bmatrix} 9 \\ 9 \end{bmatrix}, k \in R$ $\quad x = -\begin{bmatrix} -4 & 2 \\ 7 & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} 0 & -2 \\ -2 & -1 \end{bmatrix} k \begin{bmatrix} 9 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \end{bmatrix}$

Hence, $v_2 = k\begin{bmatrix} 9 \\ 9 \\ \frac{2}{13} \\ 13 \end{bmatrix}, k \in R$ for $\lambda = 7$

2 a. $A = \begin{bmatrix} 1 & 2 & 4 & 7 \\ 4 & 11 & 21 & 36 \\ 3 & 21 & 43 & 70 \\ 2 & 16 & 46 & 74 \end{bmatrix} \xrightarrow[R_4 = R_4 - 2R_1]{R_2 = R_2 - 4R_1,\ R_3 = R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 3 & 5 & 8 \\ 0 & 15 & 31 & 49 \\ 0 & 12 & 30 & 60 \end{bmatrix} \xrightarrow[R_4 = R_4 - 4R_2]{R_3 = R_3 - 5R_2} \begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 6 & 9 \\ 0 & 0 & 18 & 28 \end{bmatrix} \xrightarrow{R_4 = R_4 - 3R_3} \dots$

$\begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 6 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ define as X

$\det(A) = \det(X)$
$\det(A) = 1 \times 3 \times 6 \times 1$
$\det(A) = 18$

b. Translating Row operation to E, Start by $I_{4 \times 4}$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_4 = R_4 - 2R_1]{R_2 = R_2 - 4R_1,\ R_3 = R_3 - 3R_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_4 = R_4 - 4R_2]{R_3 = R_3 - 5R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 17 & -5 & 1 & 0 \\ 14 & -4 & 0 & 1 \end{bmatrix} \xrightarrow{R_4 = R_4 - 3R_3} E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 17 & -5 & 1 & 0 \\ -37 & 11 & -3 & 1 \end{bmatrix}$

c. $B = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 11 & 21 \\ 3 & 21 & 43 \\ 2 & 16 & 46 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$

We can use Gauss Jordan form of system:

$B' = \left[\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 4 & 11 & 21 & b_2 \\ 3 & 21 & 43 & b_3 \\ 2 & 16 & 46 & b_4 \end{array}\right]$, which A is a special case of the matrix with $b_1 = 7$, $b_2 = 36$, $b_3 = 70$, $b_4 = 74$

To do the ERO, we can use matrix E instead of Row operation.

$EB' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 17 & -5 & 1 & 0 \\ -37 & 11 & -3 & 1 \end{bmatrix} \times \left[\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 4 & 11 & 21 & b_2 \\ 3 & 21 & 43 & b_3 \\ 2 & 16 & 46 & b_4 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 0 & 3 & 5 & b_2 - 4b_1 \\ 0 & 0 & 6 & b_3 - 5b_2 + 17b_1 \\ 0 & 0 & 0 & b_4 - 3b_3 + 11b_2 - 37b_1 \end{array}\right]$

Hence, $b_4 - 3b_3 + 11b_2 - 37b = 0$, for $Bx = b$ to be consistent

ii) for nullspace, set $b_1, b_2, b_3, b_4 = 0$, then RREF form, we get:

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow x_3 = c = 0$, $N(B) = c \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $c \in R$

for the row space, The non zero rows in the reduce row echelon form (RREF) are a basis for the row space:

$R(B) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

3 a. $f(z) = |z^2| + \left|\frac{1}{z}\right|$, $z = x+iy$

i) $\lim_{z \to i} f(z) = \lim_{a \to 0, b \to 1} |(a+bi)^2| + \frac{1}{|a+bi|} = \lim_{a \to 0, b \to 1} |a^2+2abi-b^2| + \frac{1}{|a+bi|} = 1 + \frac{1}{1} = 2$

ii) $f(i) = |i^2| + \left|\frac{1}{i}\right| = 2$, Since $f(i) = \lim_{z \to i} f(z)$, it is continuous at i

$f(z) = |z^2| + \left|\frac{1}{z}\right|$, $z = re^{i\theta}$ (Polar form)

$= |z|^2 + \left|\frac{1}{z}\right|$ $|z^2| = |z|^2$, $|z| = r$, $\left|\frac{1}{z}\right| = \frac{1}{|z|}$

$f(z) = \underbrace{\left(r^2 + \frac{1}{r}\right)}_{u} + i\underbrace{0}_{v}$

apply cauchy-Riemann eq:

$u_r = 2r - \frac{1}{r^2}$, $v_r = 0$, $u_\theta = 0$, $v_\theta = 0$

$u_r = \frac{1}{r} v_\theta$, $2r - \frac{1}{r^2} = 0$, which is not true, hence it is not analytic!

$v_r = -\frac{1}{r} u_\theta$, $0 = 0$ ✓

for differentiability: $2r - \frac{1}{r^2} = 0$ must be satisfy!

$2r = \frac{1}{r^2}$, $r \neq 0$

$r^3 = \frac{1}{2} \rightarrow r = \frac{1}{\sqrt[3]{2}}$ must be satisfy!

b. $\oint \left[5e^{2z} + z - 1 + \frac{z}{(z-1)^2(z^2-5z+6)}\right] dz$

1st: analytic function properties: $\oint f(z)\,dz = 0$, for all analytic function

Hence, $5e^{2z}$, z, -1, are all 0 and the integral can be simplified!

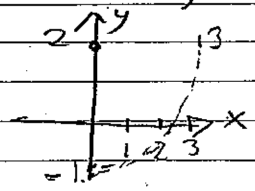
$\oint \frac{z}{(z-1)^2(z^2-5z+6)} dz = \oint \frac{z\,dz}{(z-1)^2(z-3)(z-2)}$

i) $|z| = \frac{1}{2}$, Since there are no poles inside the region, the integral is $\boxed{0}$

ii) $|z-1| = \frac{1}{2}$, circle with radius $\frac{1}{2}$, center at $(1,0)$, there are 2 poles at $z=1$
Hence, we need to use cauchy integral

$\int_C \frac{z/(z^2-5z+6)}{(z-1)^2} = 2\pi i \frac{d}{dz}\left(\frac{z}{z^2-5z+6}\right)\Big|_{z=1} = 2\pi i \times \frac{5}{4} = \frac{5\pi i}{2}$

iii) $|z-2i|=3$, Circle with radius 3, center at $(0,2)$



there are 3 poles inside the region, which are 1, 1, 2
Since, we have calculated 1, 1,
We just need to find at $(2,0)$

$$\int_C \frac{z}{(z-1)^2(z-2)(z-3)}\,dz = \int \frac{z/(z-1)^2(z-3)}{z-2}\,dz = 2\pi i \left.\frac{z}{(z-1)^2(z-3)}\right|_{z=2} = \frac{4\pi i}{-1} = -4\pi i$$

for $z=2$

Hence $\displaystyle\int_C \frac{z}{(z-1)^2(z^2-5z+6)} = \frac{5}{2}\pi i - 4\pi i = -1.5\pi i$

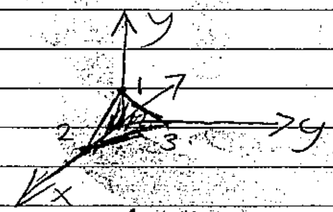
4 a. $f(x,y,z) \to$ scalar field

$$\iint \nabla\times\nabla f \cdot dA = 0$$

$$\nabla\times\nabla f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \hat{i}\left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial y\partial z}\right) - \hat{j}\left(\frac{\partial^2 f}{\partial x\partial z} - \frac{\partial^2 f}{\partial x\partial z}\right) + \hat{k}\left(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial x\partial y}\right) = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Hence, a dot product with 0 vector is 0, which is true for all f!

b. $S: 3x+2y+6z=6$



$$F = \begin{pmatrix} z \\ xz \\ x \end{pmatrix} \qquad \nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & xz & x \end{vmatrix} = (-x)\hat{i} + 0\hat{j} + z\hat{k} = -x\hat{i} + z\hat{k}$$

$$\iint_S \nabla\times F \cdot d\vec{A} = \iint_S \begin{pmatrix} -x \\ 0 \\ z \end{pmatrix} \hat{n}\cdot dA,$$

$3x+2y+6z=6$
$\to z = 1 - \frac{1}{2}x - \frac{1}{3}y$
$g(x,y) = 1 - \frac{1}{2}x - \frac{1}{3}y$
$\nabla g = (-\frac{1}{2}, -\frac{1}{3})$

$$= \iint_S \begin{pmatrix} -x \\ 0 \\ z \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ 1 \end{pmatrix} \cdot dA$$

$dA = dx\,dy$, $x\geq 0, y\geq 0, z\geq 0$

$3x+2y=6$
$x = \frac{6-2y}{3} = 2-\frac{2}{3}y$
$y \leq 3$

$\nabla f = -g_x\hat{i} - g_y\hat{j} + \hat{k}$
$\vec{n} = \nabla f = \frac{1}{2}\hat{i} + \frac{1}{3}\hat{j} + 1\hat{k}$

$$= \int_0^3\int_0^{2-\frac{2}{3}y} -\frac{1}{2}x + 1 - \frac{1}{2}x - \frac{1}{3}y \; dx\,dy$$

$$= \int_0^3 \left[-\frac{x^2}{2} + x - \frac{1}{3}xy\right]_0^{2-\frac{2}{3}y} dy$$

$$= \int_0^3 -\frac{1}{2}\left(\frac{2}{3}y-2\right)^2 + \left(2-\frac{2}{3}y\right) - \frac{1}{3}\left(2-\frac{2}{3}y\right)y$$

$$= \int_0^3 -\frac{2}{9}y^2 + \frac{2}{9}y^2 + \frac{4}{3}y - \frac{2}{3}y - \frac{2}{3}y + 2 - 2\,dy$$

$$= \Big[0y + 0y + 0y\Big]_0^3$$

$$= [0] = 0$$

We can also use $r_u \times r_v$ to find the n

$S: r(u,v) = u\hat{i} + v\hat{j} + (1 - \frac{1}{2}u - \frac{1}{3}v)\hat{k}$

$r_u = \hat{i} + -\frac{1}{2}\hat{k}$

$r_v = \hat{j} + -\frac{1}{3}\hat{k}$

$$[r_u \times r_v] = \begin{vmatrix} i & j & k \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{3} \end{vmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ 1 \end{pmatrix}$$

With $u=x$, $v=y$

Hence we arrive at the same results $\vec{n} = (\frac{1}{2}, \frac{1}{3}, 1)$

C. $\iint \nabla \times F \, dA = \oint F \cdot d\vec{l}$ (Stokes theorem)

from $(0,0,1)$ to $(2,0,0)$ let $x = 2t$ $\rightarrow r(t) = 2t\hat{i} + 0\hat{j} + (1-t)\hat{k}$

define as path A $y = 0$ $dr = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} dt$

$z = 1-t$, $0 \le t \le 1$

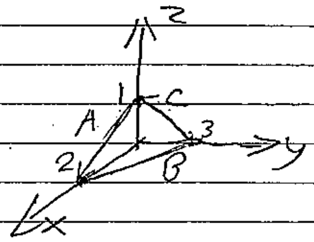
$$\int_A \vec{F} \cdot d\vec{r} = \int_0^1 \begin{pmatrix} 1-t \\ 2t(1-t) \\ 2t \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} dt$$

$$= \int_0^1 2 - 2t - 2t$$

$$= 2t - 2t^2 \Big]_0^1$$

$$= 0$$

Hence, the work done is 0



additional: * to show part B is correct, we can define 2 more path
path B: $(2,0,0)$ to $(0,3,0)$, path C: $(0,3,0)$ to $(0,0,1)$

for B: let $x = 2-2t$ $r(t) = (2-2t)\hat{i} + 3t\hat{j} + 0\hat{k}$

$y = 3t$ $dr = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} dt$, $0 \le t \le 1$

$z = 0$

$$\int_B \vec{F} \cdot d\vec{r} = \int_0^1 \begin{pmatrix} 0 \\ 0 \\ 2-2t \end{pmatrix} \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} dt = 0$$

for C: let $x = 0$ $r(t) = 0\hat{i} + (3-3t)\hat{j} + t\hat{k}$

$y = 3-3t$ $dr = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} dt$, $0 \le t \le 1$

$z = t$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} dt = 0$$

Hence, $\iint_S \nabla \times \vec{F} \, d\vec{A} = \oint \vec{F} \, d\vec{l}$

$$\iint \nabla \times \vec{F} \cdot d\vec{A} = \int_A \vec{F} \, d\vec{r} + \int_B \vec{F} \cdot d\vec{r} + \int_C \vec{F} \cdot d\vec{r}$$

$$= 0 + 0 + 0$$

$\iint \vec{\nabla} \times \vec{F} \cdot d\vec{A} = 0$, which is true in part B

Tips and Tricks:

LA:

* Understand how to do ERO properly (Elementary Row Operation) as it is a basic for Linear algebra.
* Understand the condition for One solution, many solution, no solutions (see previous semester solution for examples)
* Learn Block Matrix configuration and operation as it is different from normal matrix.
* Know the meaning of basis, subspace (Column or row), rank, dimension, Nullspace.

Complex Analysis

* Understand the condition for analyticity and differentiability using Cauchy Riemann
* Integration in Complex Analysis:
  i) given the boundary, evaluate the integral.
  ii) given improper integral know how to change to Complex Analysis.

Vector Calculus

* Understand conservative field with the concept of gradient of scalar field (∇V) and Curl of vector field ($\nabla \times \vec{F}$) $= 0$ for conservative field.
* Know how to apply theorem (Null identity, Stokes theorem, Gauss Theorem)