# NANYANG TECHNOLOGICAL UNIVERSITY

## SEMESTER I EXAMINATION 2021-2022

## EE2007 / IM2007 – ENGINEERING MATHEMATICS II

November / December 2021 | Time Allowed: 2.5 hours

**INSTRUCTIONS**

1. This paper contains 4 questions and comprises 5 pages.
2. Answer all 4 questions.
3. All questions carry equal marks.
4. This is a closed book examination.
5. Unless specifically stated, all symbols have their usual meanings.
6. A list of useful formulae is given in the Appendix A.

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1. (a) Matrix $B$ is obtained by performing the following Elementary Row Operations (EROs) on matrix $A$: First, $R_1 \leftrightarrow R_2$, then $R_2 \leftarrow R_2 + \beta R_3$, followed by $R_4 \leftarrow \alpha R_4$, where $\alpha$ and $\beta$ are non-zero constants.

(i) How is the determinant of $B$ related to the determinant of $A$?

(ii) In general, matrix $B$ can be expressed as $B = EA$. If $A$ is a $4 \times 7$ matrix, write down the matrix $E$ as well as its inverse.

(iii) Let $C$ be a matrix of 4 columns, i.e., $C = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}$. If $D = CE^T$ where $E$ is the matrix obtained from part (ii). Express the columns of $D$ in terms of the columns of $C$.

(15 Marks)

(b) Show that

$$\det\left(\begin{bmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{bmatrix}\right) = 0$$

represents the equation of the line passing through the points $(a_1, b_1)$ and $(a_2, b_2)$.

(5 Marks)

(c) If $B = M^{-1}AM$, how is $\det(A)$ related to $\det(B)$? Hence, compute $\det(A^{-1}B)$. Show your working clearly and justify your answer.

(5 Marks)

2. (a) Consider the following system

$$\begin{aligned} x + 4y - 2z &= 1 \\ 2x + 7y - 6z &= 6 \\ 3y + qz &= t \end{aligned}$$

where $q$ and $t$ are unknown real numbers. For what values of $q$ and $t$ will this system has (i) unique solution, (ii) many solutions, and (iii) no solution? Hence, find the solution that has $z = 1$.

(10 Marks)

(b) What is the maximum number of vectors that can be taken from the following to form a set of linearly independent vectors? Give one example of this set of linearly independent vectors.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

Note: Question No. 2 continues on page 3.

$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

(5 Marks)

(c) For what values of $a$ will the following matrix be non-singular?

$$\begin{bmatrix} a & 2 & 3 & 4 \\ a & a & 5 & 6 \\ a & a & a & 7 \\ a & a & a & a \end{bmatrix}$$

(5 Marks)

(d) Consider the matrix $A = \begin{bmatrix} -16 & 2 & 24 \\ 11 & -1 & 12 \\ -16 & 2 & 23 \end{bmatrix}$. Determine which of the following vectors is (are) eigenvector(s) of $A$, and if so, what is (are) the corresponding eigenvalue(s)?

$$\mathbf{v}_1 = \begin{bmatrix} 8 \\ 4 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ -3 \\ -2 \end{bmatrix}.$$

(5 Marks)

3. (a) For the following function, does its limit at the origin exist?

$$f(z) = \frac{5x^3y^2}{2x^5 + 2y^5} + i\,6xy^2$$

(5 Marks)

(b) Using the Cauchy-Riemann equations, determine the analyticity of the following function and find its derivatives at the points where they exist.

$$f(z) = \frac{z^{100} + i}{z^{100}}$$

(8 Marks)

(c) Evaluate the integral $\oint_C \frac{1}{z(z^2+1)}\, dz$, along each of the following counter-clockwise paths: (i) $C : |z| = \frac{1}{2}$, (ii) $C : |z| = \frac{3}{2}$, and (iii) $C : |z - i| = \frac{3}{2}$.

(7 Marks)

(d) Determine the real and imaginary parts of $(1 + \cos\theta + i\sin\theta)^n$.

(5 Marks)

4. (a) Consider the function $f(x, y, z) = xyz$. Find its derivative along the downward normal direction of the surface $2z - xy = 0$ at the point $(2, 3, 3)$.

(8 Marks)

(b) Find the work done in moving a particle in the force field given by $\mathbf{F} = 2xyz^2\mathbf{i} + (x^2z^2 + \cos y)\mathbf{j} + 2x^2yz\mathbf{k}$ from $(1, \pi/2, 0)$ to $(7, \pi/6, 6)$ along an arbitrary path.

(11 Marks)

(c) Given that $u = zx^2y$ and $v = x^2 + y^2 - z^2$, determine $\nabla\cdot(\nabla u \times \nabla v)$ and $\nabla\times(\nabla u \times \nabla v)$.

(6 Marks)

END OF PAPER

# Appendix A

## Some Useful Formulae for Complex Analysis

1. Complex Power: $z^c = e^{c \ln z}$

2. Euler's Formula: $e^{ix} = \cos x + i \sin x$

3. De Moivre's Formula: $(\cos x + i \sin x)^n = \cos nx + i \sin nx$

4. Cauchy-Riemann equations:

$$u_x = v_y, \quad v_x = -u_y, \quad \text{or} \quad u_r = \frac{1}{r} v_\theta, \quad v_r = \frac{-1}{r} u_\theta$$

5. Derivative, if exists: $f'(z) = u_x + iv_x = e^{-i\theta}(u_r + iv_r)$

6. Cauchy Integral Formula:

$$\int_C \frac{f(z)}{(z - z_o)^m} \, dz = \frac{2\pi i}{(m-1)!} \frac{d^{(m-1)}}{dz^{(m-1)}} f(z)|_{z=z_o}$$

## Some Useful Formulae for Vector Calculus

Let $\mathbf{F} = F_1 \, \mathbf{i} + F_2 \, \mathbf{j} + F_3 \, \mathbf{k}$.

1. Scalar Triple Product: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$

2. Gradient: $\nabla f = \frac{\partial f}{\partial x} \, \mathbf{i} + \frac{\partial f}{\partial y} \, \mathbf{j} + \frac{\partial f}{\partial z} \, \mathbf{k}$

3. Divergence: $\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

4. Curl: $\nabla \times \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

5. Gauss Theorem: $\iiint_T \nabla \cdot \mathbf{F} \, dv = \iint_S \mathbf{F} \cdot \mathbf{n} \, dA$

6. Stokes Theorem: $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dA = \int_C \mathbf{F} \cdot \mathbf{dr}$

**EE2007 ENGINEERING MATHEMATICS II**
**IM2007 ENGINEERING MATHEMATICS II**

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.

**Solution for EE2007/IM2007**

**SEMESTER I EXAMINATION 2021-2022**

1. (a) (i) Let $E_i$ for $i = 1, 2, 3$ denotes:

$E_1: R_1 \leftrightarrow R_2$

$E_2: R_2 \leftarrow R_2 + \beta R_3$

$E_3: R_4 \leftarrow \alpha R_4$

Then, $B$ is defined by

$$B = E_3 E_2 E_1 A$$

Therefore,

$$\det(B) = \det(E_3) \cdot \det(E_2) \cdot \det(E_1) \cdot \det(A)$$

$$\det(B) = (\alpha) \cdot (1) \cdot (-1) \cdot \det(A)$$

$$\boldsymbol{\det(B) = -\alpha \cdot \det(A)}$$

(ii) Since $E$ must be a square matrix, $E$ must be $4 \times 4$ matrix, where $E = E_3 \cdot E_2 \cdot E_1$. For simplicity, we can just take $E_1$ and perform ERO of $E_2$ and $E_3$. Therefore,

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \beta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}$$

By using Augmented Matrix, we can find the inverse easily.

$$E^{-1} = \begin{bmatrix} 0 & 1 & -\beta & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} \end{bmatrix}$$

(iii) We just need to apply one of the transpose identity.

$$D = CE^T = (EC^T)^T$$

$$D = \left( E \cdot \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{bmatrix} \right)^T$$

$$D = [\mathbf{v}_2 \quad \mathbf{v}_1 + \beta \mathbf{v}_3 \quad \mathbf{v}_3 \quad \alpha \mathbf{v}_4]$$

1. (b) Recall the equation of the line passing through the points $(a_1, b_1)$ and $(a_2, b_2)$

$$y - b_2 = \left(\frac{b_2 - b_1}{a_2 - a_1}\right)(x - a_2)$$

By expanding along the third column, we get

$$\det\left(\begin{bmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{bmatrix}\right) = (1) \cdot \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + (-1) \cdot \begin{vmatrix} x & y \\ a_2 & b_2 \end{vmatrix} + (1) \cdot \begin{vmatrix} x & y \\ a_1 & b_1 \end{vmatrix}$$

$$0 = a_1 b_2 - a_2 b_1 - x b_2 + a_2 y + x b_1 - a_1 y$$

We somehow need to get the term $(y + b_2)(a_2 - a_1)$. Hence,

$$0 = y(a_2 - a_1) + b_2(a_1 - x \boldsymbol{+ a_2 - a_2}) - a_2 b_1 + x b_1$$

$$0 = y(a_2 - a_1) + b_2(a_1 \boldsymbol{- a_2}) + b_2(\boldsymbol{a_2} - x) - a_2 b_1 + x b_1$$

$$a_2 b_1 - x b_1 + b_2(x \boldsymbol{- a_2}) = y(a_2 - a_1) - b_2(\boldsymbol{a_2} - a_1)$$

$$-b_1(x - a_2) + b_2(x \boldsymbol{- a_2}) = y(a_2 - a_1) - b_2(\boldsymbol{a_2} - a_1)$$

Therefore, by dividing each side with $(a_2 - a_1)$

$$\boldsymbol{\left(\frac{b_2 - b_1}{a_2 - a_1}\right)(x - a_2) = y - b_2}$$

(c) By determinant identity, we get

$$\det(B) = \frac{1}{\det(M)} \cdot \det(A) \cdot \det(M)$$

$$\mathbf{det}(\boldsymbol{B}) = \mathbf{det}\,(\boldsymbol{A})$$

Therefore,

$$\det(A^{-1}B) = \frac{1}{\det\,(A)} \cdot \det(B) = \mathbf{1}$$

2. (a) By reducing it to RRE form, we get

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & q-6 & t+12 \end{array}\right]$$

This system will have:

i. Unique solution

$$(q-6)z = t+12$$

$$z = \frac{t+12}{q-6}$$

Therefore,

$$\boldsymbol{(q, t)} \in \mathbb{R} \textbf{ where } \boldsymbol{q \neq 6}$$

Which means any $(q, t)$ solutions are valid as long as $q \neq 6$

ii. Many solutions

$$(q, t) = (6, -12)$$

iii. No Solution

$$\boldsymbol{q = 6} \textbf{ and } \boldsymbol{z} \in \mathbb{R}$$

For $z = 1$, we get

$$q - 6 = t + 12$$

$$q - t = 18$$

We just pick $(\boldsymbol{q}, \boldsymbol{t}) = (\mathbf{19}, \mathbf{1})$ which makes $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (\mathbf{27}, \mathbf{-6}, \mathbf{1})$

(b) Let $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5 \quad \mathbf{v}_6]$. By reducing it from the ERO form,

$$A = \begin{bmatrix} \mathbf{1} & 0 & 0 & -1 & -1 & 0 \\ 0 & \mathbf{1} & 0 & 1 & 0 & -1 \\ 0 & 0 & \mathbf{1} & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The linear independent vectors are those non-zero pivots in the corresponding row $(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$. Hence, the maximum number of vectors to form a set of linearly independent vectors is **3**.

(c) A matrix is non-singular if the determinant **is not** 0. By reducing it to RREF,

$$A = \begin{bmatrix} a & 2 & 3 & 4 \\ 0 & a-2 & 2 & 2 \\ 0 & 0 & a-5 & 1 \\ 0 & 0 & 0 & a-7 \end{bmatrix}$$

Where the determinant, $\det(A) = a \cdot (a-2) \cdot (a-5) \cdot (a-7)$. Hence, the matrix is non-singular for

$$a \in \mathbb{R}, a \neq 0, 2, 5, 7$$

(d) By multiplying $A$ with each of the vectors, we will get

$$\left\{ \begin{bmatrix} 24 \\ 156 \\ 18 \end{bmatrix}, \begin{bmatrix} 60 \\ 45 \\ 57 \end{bmatrix}, \begin{bmatrix} 90 \\ -6 \\ 88 \end{bmatrix} \right\}$$

We can clearly see that none of these vectors is eigenvector of $A$. Hence, we cannot find any eigenvalue from these vectors.

3. (a) Let $x = y$. Therefore,

$$\lim_{y\to 0} \left( \frac{5y^5}{4y^5} + i\ 6y^3 \right) = \frac{5}{4}$$

Let $x = 1$. Therefore,

$$\lim_{y\to 0} \left( \frac{5y^2}{2 + 2y^5} + i\ 6y^2 \right) = 0$$

Since we get different results from different approach, the limit **does not exist at the origin.**

(b) Notice that for $z \neq 0$,

$$f(z) = \frac{z^{100} + i}{z^{100}} = 1 + i\left(\frac{1}{z^{100}}\right)$$

Let $z = re^{i\theta}$. Therefore,

$$f(z) = 1 + \left( \frac{e^{i\frac{\pi}{2}}}{r^{100}e^{100i\theta}} \right)$$

$$f(z) = 1 + r^{-100}\left(e^{i\left(\frac{\pi}{2} - 100\theta\right)}\right)$$

$$f(z) = 1 + r^{-100}(\sin 100\theta + i \cos 100\theta)$$

Hence,

$$u(r,\theta) = 1 + r^{-100} \sin 100\theta$$

$$v(r,\theta) = r^{-100} \cos 100\theta$$

Therefore,

$$u_r = -100r^{-101} \sin 100\theta$$

$$u_\theta = 100r^{-100} \cos 100\theta$$

$$v_r = -100r^{-101} \cos 100\theta$$

$$v_\theta = -100r^{-100} \sin 100\theta$$

Now, we need to check if the Cauchy-Riemann equation holds.

$$\frac{1}{r}v_\theta = -100r^{-101}\sin 100\theta = u_r$$

$$-\frac{1}{r}u_\theta = -100r^{-101}\cos 100\theta = v_r$$

Therefore, $\boldsymbol{f(z)}$ **is analytic for** $\boldsymbol{z \neq 0}$, where the derivative is

$$f'(z) = e^{-i\theta}\left(-100r^{-101}\sin 100\theta + i(-100r^{-101}\cos 100\theta)\right)$$

$$\boldsymbol{f'(z) = -100r^{-101} \cdot e^{i\left(\frac{\pi}{2}-101\theta\right)}}$$

(c) Notice that the integrand is not analytic at $z = 0, i, -i$.

i. $C: |z| = \frac{1}{2}$

Since only $z = 0$ is enclosed by C, then

$$\oint_C \frac{1}{z(z^2+1)}dz = 2\pi i\left(\frac{1}{z^2+1}\right)\Bigg|_{z=0} = 2\pi i$$

ii. $C: |z| = \frac{3}{2}$

All points are enclosed by C. Let's introduce $C_1, C_2,$ and $C_3$ to simplify our calculation.

$$\oint_C \frac{1}{z(z^2+1)}dz = \oint_{C_1} \frac{\frac{1}{z^2+1}}{z}dz + \oint_{C_2} \frac{\frac{1}{z(z-i)}}{z+i}dz + \oint_{C_3} \frac{\frac{1}{z(z+i)}}{z-i}dz$$

Therefore,

$$\oint_C \frac{1}{z(z^2+1)}dz = 2\pi i\left(\frac{1}{z^2+1}\Bigg|_{z=0} + \frac{1}{z(z-i)}\Bigg|_{z=-i} + \frac{1}{z(z+i)}\Bigg|_{z=i}\right)$$

$$\oint_C \frac{1}{z(z^2+1)}dz = 2\pi i\left(1 - \frac{1}{2} - \frac{1}{2}\right) = 0$$

iii. $C: |z-i| = \frac{3}{2}$

Only $z = 0$ and $z = i$ are enclosed by C. Hence,

$$\oint_C \frac{1}{z(z^2+1)}dz = \oint_{C_1} \frac{\frac{1}{z^2+1}}{z}dz + \oint_{C_2} \frac{\frac{1}{z(z+i)}}{z-i}dz$$

$$\oint_C \frac{1}{z(z^2+1)}dz = 2\pi i\left(\frac{1}{z^2+1}\Bigg|_{z=0} + \frac{1}{z(z+i)}\Bigg|_{z=i}\right) = \pi i$$

(d) Let $1+\cos\theta+i\sin\theta=re^{i\alpha}$. Therefore,

$$r=\sqrt{(1+\cos\theta)^2+\sin^2\theta}$$

$$r=\sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta}=\sqrt{2(1+\cos\theta)}=2\cos\frac{\theta}{2}$$

Also,

$$\tan\alpha=\frac{\sin\theta}{1+\cos\theta}=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}=\tan\frac{\theta}{2}$$

$$\alpha=\frac{\theta}{2}$$

Therefore,

$$1+\cos\theta+i\sin\theta=2\cos\frac{\theta}{2}e^{\frac{i\theta}{2}}$$

Hence,

$$Re((1+\cos\theta+i\sin\theta)^n)=2^n\cos^n\frac{\theta}{2}\cos\frac{n\theta}{2}$$

$$Im((1+\cos\theta+i\sin\theta)^n)=2^n\cos^n\frac{\theta}{2}\sin\frac{n\theta}{2}$$

4. (a) First, let's find the unit normal of $2z-xy=0$ at the point $(2,3,3)$

$$\mathbf{n}=\frac{\nabla(2z-xy)}{\|\nabla(2z-xy)\|}=\frac{-y\mathbf{i}-x\mathbf{j}+2\mathbf{k}}{\|-y\mathbf{i}-x\mathbf{j}+2\mathbf{k}\|}=\frac{1}{\sqrt{17}}(-3\mathbf{i}-2\mathbf{j}+2\mathbf{k})$$

But since the direction is downward (negative $k$ component), we use

$$\mathbf{n}=\frac{1}{\sqrt{17}}(3\mathbf{i}+2\mathbf{j}-2\mathbf{k})$$

Also, at the point $(2,3,3)$

$$\nabla f(x,y,z)=9\mathbf{i}+6\mathbf{j}+6\mathbf{k}$$

Therefore, the directional derivative is

$$\nabla f\cdot\mathbf{n}=\frac{1}{\sqrt{17}}(3\mathbf{i}+2\mathbf{j}-2\mathbf{k})\cdot(9\mathbf{i}+6\mathbf{j}+6\mathbf{k})=\frac{27}{\sqrt{17}}$$

(b) First, we should check whether **F** is conservative or not.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + \cos y & 2x^2yz \end{vmatrix} = 0$$

Therefore, **F** is conservative. Now let's deduce $V$ such that,

$$\mathbf{F} = \nabla V$$

$$\frac{\partial V}{\partial x} = F_1 = 2xyz^2 \Longrightarrow V = x^2yz^2 + f(y,z)$$

$$\frac{\partial V}{\partial y} = F_2 = x^2z^2 + \cos y \Longrightarrow V = x^2yz^2 + \sin y + f(x,z)$$

$$\frac{\partial V}{\partial z} = F_3 = 2x^2yz \Longrightarrow V = x^2yz^2 + f(x,y)$$

Therefore,

$$V = x^2yz^2 + \sin y + C$$

Hence, the work done

$$\int_C \mathbf{F} \cdot dr = \int_A^B dV = V(B) - V(A)$$

$$\int_C \mathbf{F} \cdot dr = V\left(7, \frac{\pi}{6}, 6\right) - V\left(1, \frac{\pi}{2}, 0\right) = 294\pi - \frac{1}{2}$$

(c) Let's find $\nabla u \times \nabla v$

$$\nabla u = (2xyz)\mathbf{i} + (x^2z)\mathbf{j} + (x^2y)\mathbf{k}$$

$$\nabla v = (2x)\mathbf{i} + (2y)\mathbf{j} + (-2z)\mathbf{k}$$

$$\nabla u \times \nabla v = (-2x^2z^2 - 2x^2y^2)\mathbf{i} + (2x^3y + 4xyz^2)\mathbf{j} + (4xy^2z - 2x^3z)\mathbf{k}$$

Therefore,

$$\nabla \times (\nabla u \times \nabla v) = 0\mathbf{i} + \left(2z(x^2 - 2y^2)\right)\mathbf{j} + \left(2y(5x^2 + 2z^2)\right)\mathbf{k}$$

$$\nabla \cdot (\nabla u \times \nabla v) = 0$$