EE2007: ENGINEERING MATHEMATICS II
Complex Analysis

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EE2007: ENGINEERING MATHEMATICS II
Complex Variables

Text: Kreyszig, E. (2006). Advanced Engineering Mathematics, 9th Ed. John Wiley & Sons, Inc.

Reference:

1. Brown and Churchill (1996). Complex Variables and Applications, 6th Ed., McGraw-Hill. [NTU Library, QA331.7.B878]
2. Matthews and Howell (2001). Complex Analysis for Mathematics and Engineering, 4th Ed., Jones and Bartlett Publishers. [NTU Library, Red spot, QA331.7M439]

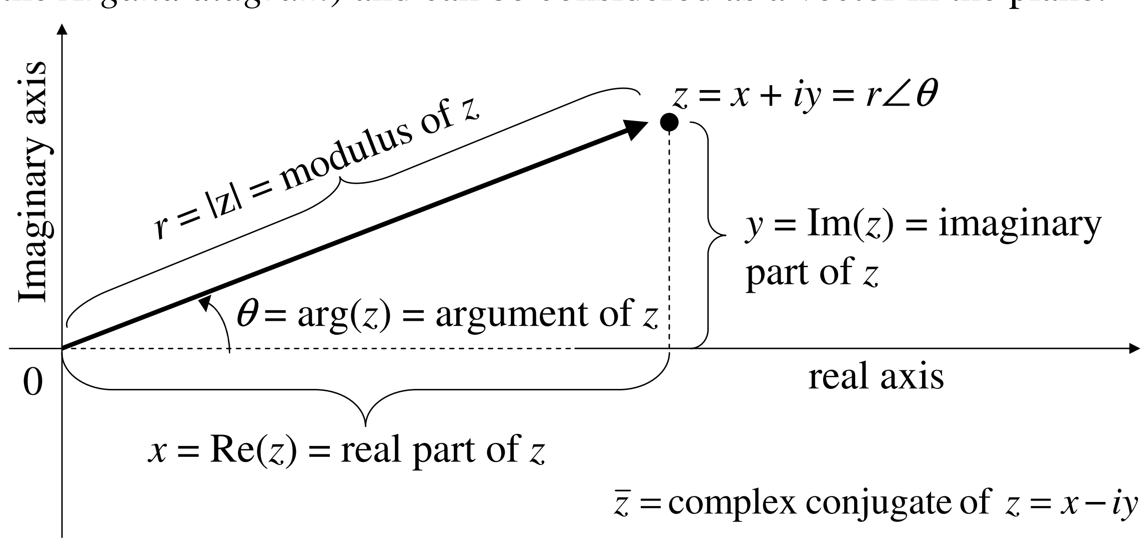
Contents

Revision: Complex Numbers

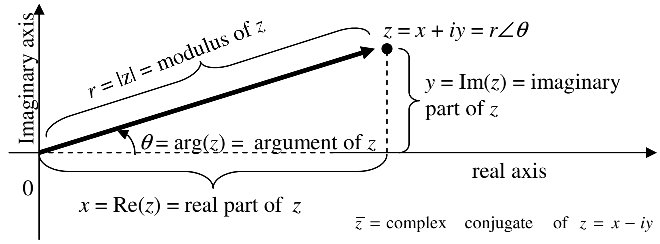
A complex number z is defined as

$$z = x + iy \qquad \text{where} \qquad i = \sqrt{-1}$$

Geometrically, a complex number is a point in the complex plane (or the *Argand diagram*) and can be considered as a vector in the plane.



$\bar{z}$ = complex conjugate of $z = x - iy$



It can be seen from the figure that

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$r = |z| = \sqrt{x^2 + y^2} = |\bar{z}| = \sqrt{z\bar{z}}$$

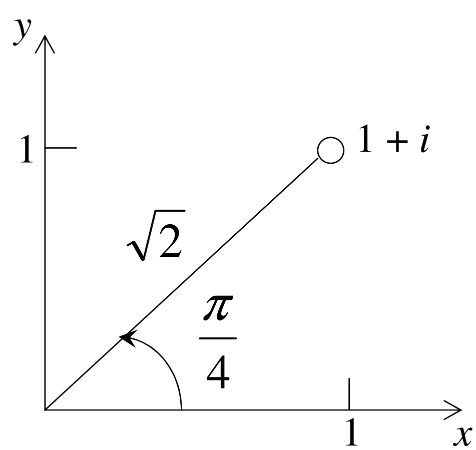
$$\theta = \arg(z) = \arctan\frac{y}{x}\ \textbf{radians}$$

$$= \mathrm{Arg}(z) + 2n\pi, \qquad n = 0, \pm1, \pm2, \ldots$$

where Arg(z) is the **principal values** of arg(z) and satisfies

$$-\pi < Arg(z) \le \pi$$

Example:



Let $z = 1 + i$. Then

$$r = |z| = \sqrt{1+1} = \sqrt{2}$$

$$\arg z = \arctan\frac{1}{1}$$

$$= \frac{\pi}{4} \pm 2n\pi, \quad n = 0, 1, 2, \ldots$$

The **principal value** of the argument is $\frac{\pi}{4}$.

If $z = 1 - i$, then $\arg z = \arctan\frac{-1}{1} = \frac{-\pi}{4} \pm 2n\pi, \quad n = 0, 1, 2, \ldots$

The **principal value** of the argument is $\frac{-\pi}{4}$.

Euler's Formula

From Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \text{and} \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

We have

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

From **Euler's formula** $e^{i\theta} = \cos\theta + i\sin\theta$ for any real value of θ, we can write the **polar form** of a complex number as

$$z = re^{i\theta} = r\angle\theta$$

Algebraic Rules

Let $z_1 = x_1 + iy_1 = r_1\angle\theta_1, \quad z_2 = x_2 + iy_2 = r_2\angle\theta_2$. Then

- Addition and Subtraction

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

- Multiplication

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

- Division

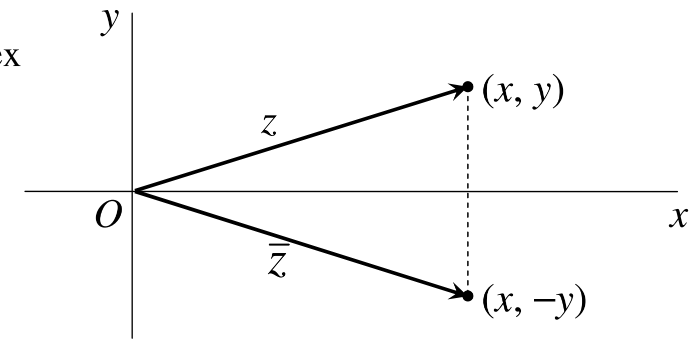
$$\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

It is sometimes more convenient to do multiplication and division in the polar form

$$z_1 z_2 = r_1 r_2\angle(\theta_1 + \theta_2), \qquad \frac{z_1}{z_2} = \frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

Complex Conjugate

Given $z = x + iy$, the complex conjugate of z is defined as



$$\bar{z} = x - iy$$

Thus, we can write

- $\mathrm{Re}(z) = \dfrac{1}{2}(z + \bar{z}) \qquad \mathrm{lm}(z) = \dfrac{1}{2i}(z - \bar{z})$

- $z\bar{z} = x^2 + y^2 = |z|^2 \qquad \dfrac{z_1}{z_2} = \dfrac{z_1 \overline{z_2}}{|z_2|^2}$

- $\overline{(z_1 \pm z_2)} = \overline{z_1} \pm \overline{z_2}, \qquad \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}, \qquad \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$

De Moivre's Formula

Let $z = x + iy = r(\cos\theta + i\sin\theta) = r\angle\theta$

Then, for any integer n,

$$z^n = r^n(\cos\theta + i\sin\theta)^n$$

$$z^n = \underbrace{z.z...z}_{n} = \underbrace{r.r...r}_{n}\angle\underbrace{(\theta + \theta + ... + \theta)}_{n} = r^n\angle(n\theta)$$

$$= r^n(\cos n\theta + i\sin n\theta)$$

This gives the **De Moivre's Formula**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

which is useful in deriving certain trigonometric identities.

Example: Find identities for cos 2θ and sin 2θ.

$$\left(\cos\theta + i\sin\theta\right)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$$
$$= \cos 2\theta + i\sin 2\theta$$

Therefore

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \text{and} \quad \sin 2\theta = 2\cos\theta\sin\theta$$

Example: Express $\cos^4\theta$ in terms of multiples of θ.

Since $2\cos\theta = e^{i\theta} + e^{-i\theta}$,

$$2^4\cos^4\theta = \left(e^{i\theta} + e^{-i\theta}\right)^4$$
$$= \left(e^{i4\theta} + e^{-i4\theta}\right) + 4\left(e^{i2\theta} + e^{-i2\theta}\right) + 6$$
$$= 2\cos 4\theta + 8\cos 2\theta + 6$$
$$\Rightarrow \cos^4\theta = \frac{1}{8}\left[\cos 4\theta + 4\cos 2\theta + 3\right]$$

Roots of Complex Numbers

- Consider $z = w^n, \quad n = 1,2,\ldots$

- For a given $z \neq 0$, the solution of w in the above equation is called the n^th root of z and is denoted by $w = \sqrt[n]{z}$

- Given a $z \neq 0$, w can be found as follows:

  First, we write $z = r\angle(\theta + 2k\pi)$. Next, we let $w = R\angle\phi$. Then, $z = w^n$ gives

  $$r\angle\left(\theta + 2k\pi\right) = R^n\angle\left(n\phi\right)$$

  Thus we have,

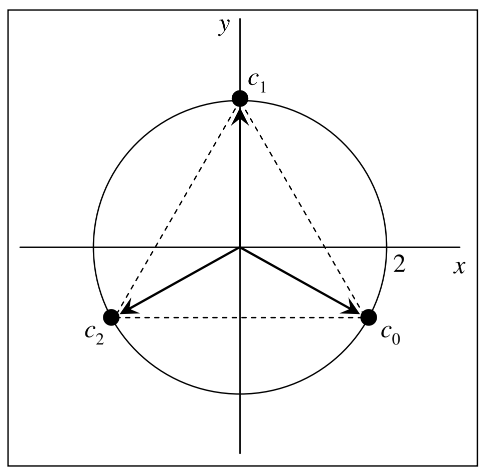
  $$R = \sqrt[n]{r}, \quad \text{and} \quad \phi = \frac{\theta + 2k\pi}{n}, \quad k = 0, 1, \ldots, (n-1)$$

- In summary

$$w_k = \sqrt[n]{z} = \sqrt[n]{r}\angle\left(\frac{\theta + 2k\pi}{n}\right), \qquad k = 0, 1, \ldots, (n-1)$$

Geometrically, the entire set of roots lie at the vertices of a regular polygon of *n* sides inscribed in a circle of radius $\sqrt[n]{r}$.

Example Let us find all values of $(-8i)^{1/3}$, i.e. $\sqrt[3]{-8i}$.



Solution: First, we write

$$-8i = 8\angle\left(\frac{-\pi}{2} + 2k\pi\right), k = 0, \pm 1, \pm 2, \ldots$$

and we see that the desired roots are

$$w_k = 2\angle\left(\frac{-\pi}{6} + \frac{2k\pi}{3}\right), \; k = 0, 1, 2$$

The roots lie at the vertices of an equilateral triangle, inscribed in the circle $|z| = 2$ *and are equally spaced around that circle every* $2\pi/3$ *radians, starting with the principal root*

$$w_0 = 2\angle\left(\frac{-\pi}{6}\right) = \sqrt{3} - i$$

The Exponential Function e^z

- Is defined as

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = e^x (\cos y + i \sin y)$$

- If $x = 0$, have to the so-called **Euler formula:** $e^{iy} = \cos y + i \sin y$

  Hence the **polar form** of a complex number may be written as

$$z = r(\cos\theta + i \sin\theta) = re^{i\theta}$$

- If $z = e^{ix} = \cos x + i \sin x$, then

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) = \frac{1}{2i}(z - \bar{z}), \cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \frac{1}{2}(z + \bar{z})$$

- It is also geometrically obvious that $e^{i\pi} = -1, e^{-i\pi/2} = -i,$ and $e^{-i4\pi} = 1.$

The Complex Logarithm and General Power

- The **natural logarithm** of $z = x + iy$ is denoted by ln$z$ and is defined as the inverse of the exponential function.
- That is, $w = \ln z$ is defined for $z \neq 0$ by the relation $e^w = z$.
- So, if $z = re^{i\theta}$, $r > 0$, then $\ln z = \ln r + i\theta$
- Note that the complex logarithm is **infinitely many-valued**.
- The general power of a complex number, z^c, can be derived as follows:

$$\text{Let } y = z^c, \Rightarrow \ln y = c \ln z, \Rightarrow y = \boxed{z^c = e^{c \ln z}}, z \neq 0$$

Example

- Evaluate ln(3 – 4*i*).

  Solution:

$$\ln(3-4i)=\ln|3-4i|+i\arg(3-4i)$$
$$=1.609-i(0.927\pm 2n\pi) \qquad n=0,1,\ldots$$

  Principal value: *when n* = 0.

- Solve $\ln z=-2-\frac{3}{2}i$.

  Solution:

$$z=e^{-2-\frac{3}{2}i}=e^{-2}e^{-i\frac{3}{2}}=e^{-2}\left(\cos\frac{3}{2}-i\sin\frac{3}{2}\right)$$
$$=0.010-i0.135$$

Example: Find the principal value of $(1+i)^i$.

Solution: *Let* $y=(1+i)^i$. *Then*

$$\ln y=i\ln(1+i), \quad \text{or} \quad y=e^{i\ln(1+i)}$$

Hence, $(1+i)^i=e^{i\ln(1+i)}$. *But,*

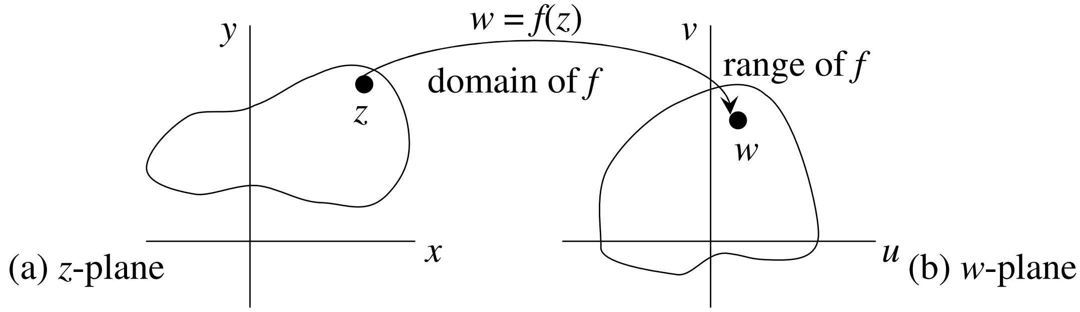
$$\ln(1+i)=\ln\left(\sqrt{2}e^{i(\pi/4+2k\pi)}\right)$$
$$=\ln\sqrt{2}+i(\pi/4+2k\pi), \qquad k=0,\pm 1,\ldots$$

and the principal value is when k = 0. Therefore,

$$e^{i\ln(1+i)}=e^{i\left(\ln\sqrt{2}+i\pi/4\right)}=e^{-\frac{\pi}{4}+i\left(\ln\sqrt{2}\right)}$$

Complex Functions

Complex analysis is concerned with complex functions that are differentiable in some domain. Hence we shall first say what we mean by a complex function and then define the concepts of limits and derivative in complex analogous to calculus.



- A complex function f is a rule (or mapping) that assigns to every complex number z in S a complex number w in T.
- Mathematically, we write $w = f(z)$
- The set S is called **domain** of f and the set T is called the range of f.
- If $z = x + iy$ and $w = u + iv$, then we may write

$$w = f(z) = u(x, y) + iv(x, y)$$

Example Let $w = f(z) = z^2 + 3z$. Find u and v and calculate the value of f at $z = 1 + 3i$.

Solution: *Let $z = x + iy$. Then*

$$w = z^2 + 3z$$
$$= (x + iy)^2 + 3(x + iy)$$
$$= x^2 - y^2 + i2xy + 3x + i3y$$

Hence

$$u = \mathrm{Re}(w) = x^2 - y^2 + 3x$$
$$v = \mathrm{Im}(w) = 2xy + 3y$$

and

$$f(1 + i3) = u(1,3) + v(1,3) = -5 + i15$$

Try using polar form, $z = r\angle\theta$, and check if you get the same answer.

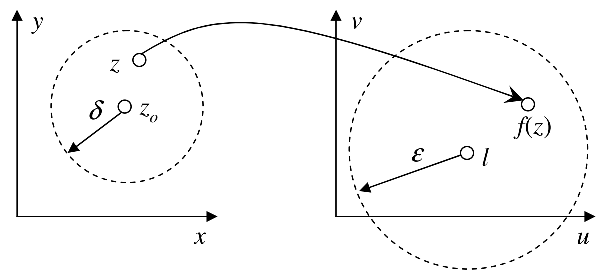
Limit

- A function $f(z)$ is said to have the **limit** L as z approaches a point z_o if
  1. $f(z)$ is defined in the neighbourhood of z_o (except perhaps at z_o itself), and
  2. $f(z)$ approaches the **same** complex number L as $z \to z_o$ from **all** directions within its neighbourhood.
- Mathematically, we write

$$\lim_{z \to z_o} f(z) = L$$

if given $\in > 0$, there exists $\delta > 0$, such that

$$\left| f(z) - L \right| < \in, \forall 0 < \left| z - z_o \right| < \delta$$



- In words, the above means that the point $f(z)$ can be made arbitrarily close to the point L if we choose the point z sufficiently close to, but not equal to, the point z_o.

Examples

- $\displaystyle \lim_{z \to \infty} \frac{2z + i}{z + 1} = \lim_{z \to \infty} \frac{2 + (i/z)}{1 + (1/z)} = 2$

- $\displaystyle \lim_{z \to \infty} \frac{2z^3 - 1}{z^2 + 1} = \lim_{z \to \infty} \frac{2 - (1/z^3)}{(1/z) + (1/z^3)} = \lim_{z \to \infty} \frac{2}{0} = \infty$

- $\displaystyle \lim_{z \to 0} \frac{z}{\overline{z}}$ does not exist because:

  Let $y \to 0$ first and then let $x \to 0$. In this case, we have

$$\lim_{x \to 0, y = 0} \frac{x + i0}{x - i0} = 1.$$

  Now let $x \to 0$ first and then let $y \to 0$. In this case, we get

$$\lim_{x = 0, y \to 0} \frac{0 + iy}{0 - iy} = -1.$$

  This means condition '2' in page 21 is not satisfied.

Continuity

A function $f(z)$ is said to be **continuous** at $z = z_o$ if

1. $f(z_o)$ exists, 2. $\lim_{z \to z_o} f(z)$ exists, and 3. $\lim_{z \to z_o} f(z) = f(z_o)$

Note: Just writing statement (3) implies the truth of (1) and (2).

We say that f is a continuous function if f is continuous for all z in the domain S.

Example: Let $f(0) = 0$, and for $z \neq 0$, $f(z) = \mathrm{Re}(z^2)/|z^2|$.
Determine whether $f(z)$ is continuous at the origin.

Solution:

$$\lim_{z \to 0} \mathrm{Re}(z^2)/|z^2| = \lim_{z \to 0} \frac{x^2 - y^2}{x^2 + y^2} = \begin{cases} 1 & if \quad y \to 0 \ first \\ -1 & if \quad x \to 0 \ first \end{cases}$$

Hence, f is not continuous at the origin.

Alternatively,

$$\lim_{z \to 0} \mathrm{Re}(z^2)/|z^2| = \lim_{r \to 0} \frac{r^2 \cos 2\theta}{r^2} = \cos 2\theta$$

The limit does not exists because it depends on the direction of approach to the origin.

Derivatives of Complex Functions

- The derivative of a complex function f at a point z_o is written as $f'(z_o)$ and is defined as

$$f'(z_o) = \lim_{z \to z_o} \frac{f(z) - f(z_o)}{z - z_o} \quad \text{if the limit exists.}$$

- or, by substituting $z = z_o + \Delta z$

$$f'(z_o) = \lim_{\Delta z \to 0} \frac{f(z_o + \Delta z) - f(z_o)}{\Delta z}$$

Example

$$\frac{d}{dz}\left(z^2\right) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0}(2z + \Delta z) = 2z.$$

Thus $f(z) = z^2$ is differentiable for all z.

The usual differentiation formulae hold as for real variables. For example

$$\frac{d}{dz}(c) = 0, \quad \frac{d}{dz}(z) = 1, \quad \frac{d}{dz}\left(z^n\right) = nz^{n-1}$$

and

$$\frac{d}{dz}\left(2z^2 + i\right)^5 = 5\left(2z^2 + i\right)^4 \cdot 4z = 20z\left(z^2 + i\right)^4$$

However, care is required for more unusual functions.

Example Discuss the differentiability of $\bar{z}$.

Solution: We begin from the definition. Let $f(z)=\bar{z}$ *, then*

$$f'(z)=\lim_{\Delta z\to 0}\frac{\overline{z+\Delta z}-\bar{z}}{\Delta z}=\lim_{\Delta z\to 0}\frac{\overline{\Delta z}}{\Delta z} \quad ; \quad (\text{Note}: \overline{z+\Delta z}=\bar{z}+\overline{\Delta z})$$

Now, consider $\Delta z = \Delta r e^{i\theta}$*. Then* $\Delta z \to 0$ *from all directions when* $\Delta r \to 0$.

Thus, we could determine the limit as follows:

$$f'(z)=\lim_{\Delta z\to 0}\frac{\overline{\Delta z}}{\Delta z}=\lim_{\Delta z\to 0}\frac{\Delta r e^{-i\theta}}{\Delta r e^{i\theta}}=e^{-i2\theta}$$

The limit depends on θ *so it doesn't exist. Hence,* $f(z)=\bar{z}$ *is not differentiable anywhere.*

Analytic Functions

- A function $f(z)$ is said to be **analytic** at a point z_o if its derivative exists not only at z_o but also in some neighbourhood of z_o. A function $f(z)$ is said to be analytic in a domain D if it is analytic at each point in D.
- Hence analyticity implies differentiability and continuity.
- The point $z = z_o$ where $f(z)$ ceases to be analytic is called the **singular point** or **singularity**[1] of $f(z)$.

Examples (see pages 25 and 27)

- $f(z) = z^2$ is analytic everywhere in the complex plane.
- $f(z)=\bar{z}$ is NOT analytic at any point.

[1]We shall restrict ourselves to only singular point called "pole of order *m*". The general classification of singularity which requires background in Laurent's Series will be omitted due to time constraint.

A Test for Analyticity

Theorem 1. [Cauchy-Riemann Equations] *The complex function*

$f(z) = u(x, y) + iv(x, y)$

is analytic at a point z_o *if for every point in the neighbourhood of* z_o

1. *u, v and their partial derivatives exist and are continuous, and*
2. *The Cauchy-Riemann equations*

$$u_x = v_y \qquad \text{and} \qquad v_x = -u_y \qquad \text{are satisfied.}$$

If the above two conditions are satisfied in some domain D, then the function is analytic in D.

Derivation of the C-R Equations

Recall that the derivatives of a complex function f at a point z_o is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta x, \Delta y \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y}$$

Consider along the x-axis, i.e. $\Delta y = 0$, we have

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x, y) - iv(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x, y) + i\left(v(x+\Delta x, y) - v(x, y)\right)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Similarly, along y-axis, i.e. $\Delta x = 0$, we have

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - u(x, y) - iv(x, y)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x, y) + i\big(v(x, y+\Delta y) - v(x, y)\big)}{i\Delta y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

for the derivative to exist, the two limits must agree, i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Thus the C-R equations are $\boxed{u_x = v_y \text{ and } v_x = -u_y.}$ It can be shown that, when $z \neq 0$, the C-R equations in polar coordinates are[3]

$$\boxed{u_r = \frac{1}{r}v_\theta \qquad \text{and} \qquad v_r = -\frac{1}{r}u_\theta}$$

[3] see, for example, Brown and Churchill (1996), pp. 52-53.

Derivatives of Complex Functions

If $f(z) = u(x, y) + iv(x, y)$ and $f'(z)$ exists, then

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= v_y - iu_y \\ &= u_x - iu_y \\ &= v_y + iv_x \end{aligned}$$

In polar form, if $f(z) = u(r, \theta) + iv(r, \theta)$, and $f'(z)$ exists, then

$$\begin{aligned} f'(z) &= e^{-i\theta}\left(u_r + iv_r\right) \\ &= \frac{1}{r}e^{-i\theta}\left(v_\theta - iu_\theta\right) \end{aligned}$$

Example: [C-R Equations] Verify that $f(z) = \bar{z}$ is not analytic.

Solution: *We have seen earlier that* $\bar{z}$ *is not differentiable, hence* $\bar{z}$ *is not analytic.*

Alternatively, using C-R equations with

$$u(x, y) = x, \quad v(x, y) = -y$$

we see that

$$u_y = -v_x = 0, \quad but \quad u_x = 1, \quad but \quad v_y = -1. \quad Hence\ u_x \neq v_y$$

C-R equations are not satisfied, hence the function is not analytic.

Example: Is $f(z) = z^3$ analytic?

Solution: *In general,* **polynomials of complex variables are analytic**. *Here we shown using C-R equations.*

Given $f(z) = z^3$,

$$u(r, \theta) = r^3\cos 3\theta, \quad v(r, \theta) = r^3\sin 3\theta$$

Therefore

$$u_r = 3r^2\cos 3\theta, \quad u_\theta = -3r^3\sin 3\theta$$

$$v_r = 3r^2\sin 3\theta, \quad and \quad v_\theta = 3r^3\cos 3\theta$$

Since C-R equations

$$u_r = \frac{1}{r}v_\theta \qquad and \qquad v_r = -\frac{1}{r}u_\theta$$

are satisfied and the functions u, v and their partial derivatives are continuous, z^3 *is analytic.*

Example: Discuss the analyticity of the function

$$f(z) = x^2 + iy^2.$$

Solution: With $u = x^2$, $v = y^2$, we have

$$u_x = 2x, \quad v_y = 2y$$
$$v_x = 0, \quad u_y = 0$$

Thus, from C-R equations, $f(z)$ is differentiable only for those values of z that lie along the straight line $x = y$.

If z_o lies on this line, any circle centered at z_o will contain points for which $f'(z)$ does not exist. Thus $f(z)$ is nowhere analytic.

Some Common (and Important) Functions

- **Polynomials**, i.e., functions of the form

  $$f(z) = c_o + c_1 z + c_2 z^2 + \ldots + c_n z^n$$

  where c_o, …, c_n are complex constants, are analytic in the entire complex plane.
- **Rational functions**, i.e. quotient of two polynomails

  $f(z) = \dfrac{g(z)}{h(z)}$ are analytic except at points where $h(z) = 0$.
- **Partial fractions** of the form

  $$f(z) = \frac{c}{(z - z_o)^m}$$

  where c and z_o complex, m a positive integer are analytic except at z_o.

Complex Integration

- In the case of a **real definite** integral
  $\int_a^b f(x)dx$
  means we integrate along the x-axis from a to b, and the integrand $f(x)$ is defined for each point between a and b.

- In the case of a complex definite integral, or **line integral**, $\int_C f(z)dz$, we integrate along curve C, in a given direction, in the complex plane and the integrand is defined for each point on C and C is called the **contour** or **path of integration**.

- If C is closed contour, we sometimes denote the complex line integral by $\oint_C f(z)dz$

- If C is on the real axis, then $z = x$ and the complex integral becomes a real definite integral.

Parametric Representation of a Path or Contour

A **contour** or **path of integration** on the complex plane can be represented in the form of

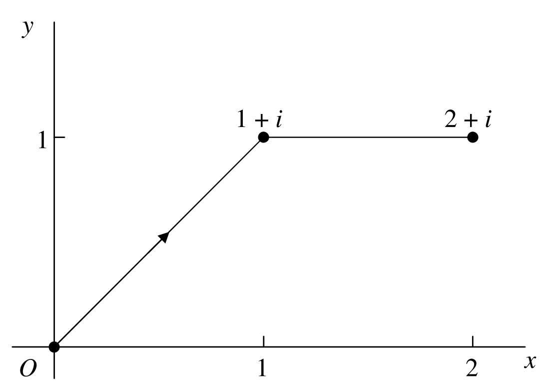
$$z(t) = x(t) + iy(t), \quad a \le t \le b$$

where t is the real parameter. This establishes a continuous mapping of the interval $a \le t \le b$ into the xy- or z-plane, and the direction of the path is according to the increasing values of t.

Example:

The path in the figure on the right can be represented by

$$z = \begin{cases} x + ix, & 0 \le x \le 1 \\ x + i, & 1 \le x \le 2 \end{cases}$$



Complex Line Integration

Example: Evaluate $\int_C \bar{z}dz$ where C is given by

$$x = 3t, \quad y = t^2, \quad -1 \le t \le 4.$$

Solution: *Since $z = x + iy$, we write*

$z(t) = 3t + it^2$ *, which gives* $dz(t) = (3 + i2t)dt$

and hence

$$\int_C \bar{z}dz = \int_{-1}^{4}\left(3t - it^2\right)(3 + i2t)dt$$
$$= \int_{-1}^{4}\left(2t^3 + 9t\right)dt + i\int_{-1}^{4} 3t^2 dt = 195 + i65$$

Example: Evaluate $\oint_C \frac{1}{z}dz$ where C is the unit circle in the complex plane, counter-clockwise.

Solution:

The path C can be represented by

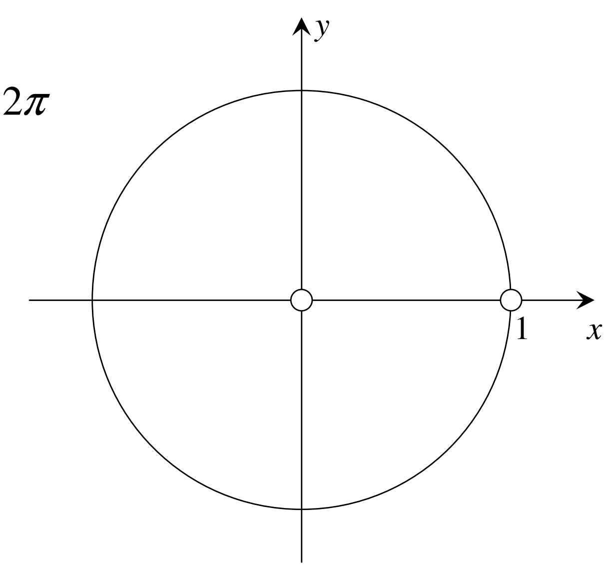
$$z(t) = \cos t + i\sin t = e^{it}, \; 0 \le t \le 2\pi$$

and

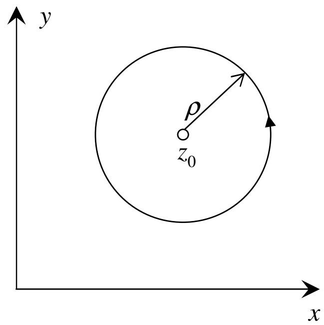
$$dz(t) = ie^{it}dt = izdt$$

Hence

$$\oint_C \frac{1}{z}dz = i\int_0^{2\pi} dt = 2\pi i$$



Example: Evaluate $\int_C (z-z_o)^m\,dz$ where C is a CCW circle of radius ρ with centre at z_o.



The path is represented as

$$z(\theta) = z_o + \rho e^{i\theta}, \qquad 0 \le \theta \le 2\pi$$

Then

$$(z-z_o)^m = \rho^m e^{im\theta}, \qquad dz = i\rho e^{i\theta}\,d\theta$$

Hence,

$$\int_C (z-z_o)^m\,dz = \int_0^{2\pi} \rho^m e^{im\theta} i\rho e^{i\theta}\,d\theta = i\rho^{m+1}\int_0^{2\pi} e^{i(m+1)\theta}\,d\theta$$

$$= \begin{cases} 2\pi i & (m=-1) \\ 0 & (m \ne -1, m \text{ integer}) \end{cases}$$

Integration by the use of the path

From the previous examples, we arrive at a practical means of evaluating a complex line integral:

Theorem 2. [Integration by the use of the path]

Let C be a piecewise smooth path, represented by $z = z(t)$, *where* $a \le t \le b$. *Let* $f(z)$ *be a continuous function on C. Then*

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\frac{dz}{dt}\,dt$$

Proof: See Text

Basic Properties of Complex Line Integrals

1. Linearity

$$\int_C [k_1 f_1(z) + k_2 f_2(z)]dz = k_1 \int_C f_1(z)dz + k_2 \int_C f_2(z)dz$$

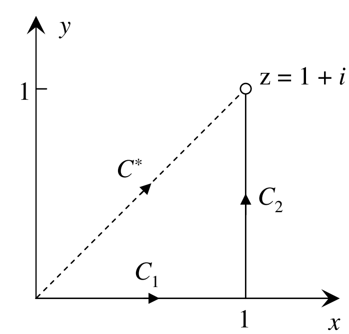
2. Subdivision of path

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

3. Sense of integration

$$\int_{z_1}^{z_2} f(z)dz = -\int_{z_2}^{z_1} f(z)dz$$

Example: Evaluate $\int_0^{1+i} \mathrm{Re}\, z\, dz$ along (a) C^*, (b) C_1 and C_2.



1. C^* can be represented by

$$z(t) = t + it, \quad 0 \le t \le 1$$

which gives $dz = (1 + i)dt$. Hence

$$\int_0^{1+i} \mathrm{Re}\, z\, dz = \int_0^1 t(1+i)dt = \frac{1}{2}(1+i)$$

2. C_1 and C_2 are represented by

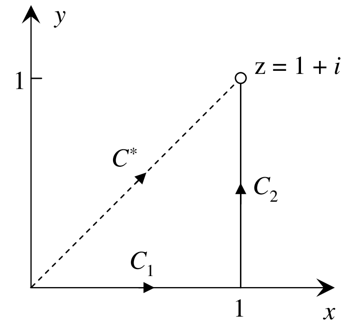
C_1 : $z(t) = t$, $0 \le t \le 1$ giving $dz = dt$
C_2 : $z(t) = 1 + it$, $0 \le t \le 1$ giving $dz = idt$

Along C_1, $\mathrm{Re}(z) = t$ and along C_2, $\mathrm{Re}(z) = 1$.

Hence, $\int_0^{1+i} \mathrm{Re}\, z\, dz = \int_{C_1} + \int_{C_2} = \int_0^1 t\, dt + \int_0^1 1 \cdot i\, dt = \frac{1}{2} + i$

Example: Evaluate $\int_0^{1+i} z\,dz$ along (a) C^*, (b) C_1 and C_2.

1. Along C^*, z is represented by



$$z(t) = t + it, \quad 0 \le t \le 1$$

which gives $dz = (1 + i)dt$. Hence

$$\int_0^{1+i} z\,dz = \int_0^1 (t + it)(1+i)dt$$
$$= \int_0^1 (t - t + i2t)dt = it^2\Big|_0^1 = i$$

2.
Along C_1 : $z(t) = t$, $0 \le t \le 1$ and $dz = dt$
Along C_2 : $z(t) = 1 + it$, $0 \le t \le 1$ and $dz = idt$

Hence, $\int_0^{1+i} z\,dz = \int_{C_1} + \int_{C_2} = \int_0^1 t\,dt + \int_0^1 (1+it)\cdot i\,dt = i$

Example: Evaluate $\int_C \bar{z}dz$ from $z = 0$ to $z = 4 + 2i$ along the curve given by the line $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$

Along $z = 0$ to $z = 2i$: $z(t) = 0 + it$; $0 \le t \le 2$; $dz = i\,dt$
Along $z = 2i$ to $z = 4 + 2i$: $z(t) = t + 2i$; $0 \le t \le 4$; $dz = dt$

$$\int_C \bar{z}dz = \int_0^2 tdt + \int_0^4 (t - 2i)dt$$
$$= 2 + \int_0^4 tdt - 2i\int_0^4 dt$$
$$= 2 + \left[\frac{t^2}{2}\right]_0^4 - 8i$$
$$= 2 + 8 - 8i = 10 - 8i$$

Example: $\int_C \bar{z}dz$ from $z = 0$ to $z = 4 + 2i$ where C is a parabolar $x = y^2$

$$z(t) = t^2 + it\ ;\quad 0 \le t \le 2, dz = (2t+i)dt$$

$$\int_C \bar{z}dz = \int_0^2 \left(t^2 - it\right)(2t+i)dt$$

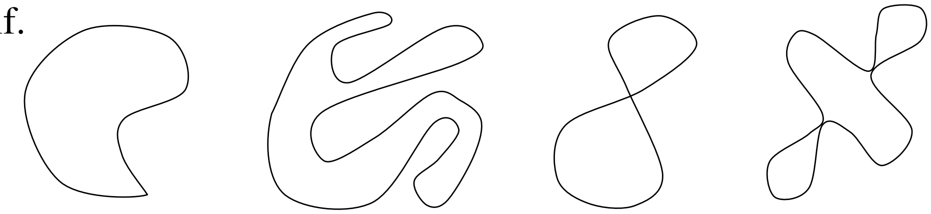
$$= \int_0^2 \left(2t^3 - it^2 + t\right)dt$$

$$= 10 - \frac{8}{3}i$$

- We have seen from the examples that a line integral of $f(z)$ in the complex plane may or may not depend on the choice of the path itself. Sometimes, the integrals evaluated turn out to be zero or $2\pi i$.
- Under what condition will the integral be **independent** of the path?
- Under what condition will the integral be zero?
- Is there something special about the value $2\pi i$?
- to answer these questions, we need additional background:
  - Concept of Simple Closed Path and Simply Connected Domain
  - Cauchy's Integral Theorem

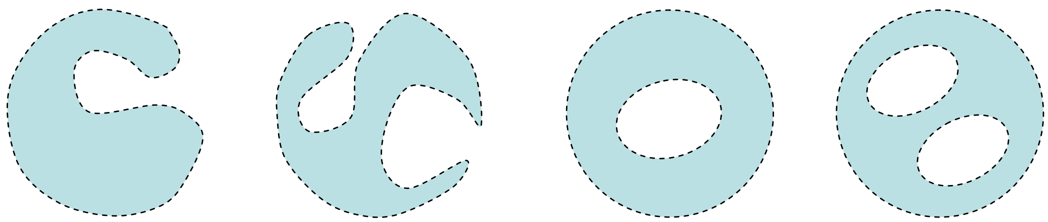
Simple Closed Path and Simply Connected Domain

- A simple closed path is a closed path that does not intersect or touch itself.



Simple   Simple   Not Simple   Not Simple

- A simply connected domain D in the complex plane is a domain such that every simple closed path in D encloses only points of D.



Simply connected   Simply connected   Doubly connected   Triply connected

- A domain that is not simply connected is called **multiply connected**.
- Intuitively, a simply connected domain is one which does not have any “holes” in it, while a multiply connected domain is one which does.

Cauchy’s Integral Theorem

Theorem 3. [Cauchy’s Integral Theorem] *If $f(z)$ is analytic in a simply connected domain D, then for every* **simple closed path** *C in D,*

$$\int_C f(z)dz = 0$$

Example:

$$\int_C e^z dz = 0, \quad \int_C \cos z \, dz = 0 \quad \text{and} \quad \int_C z^n dz = 0, \quad n = 0,1,\ldots$$

for any closed path since these functions are **entire** (analytic for all z).

Example:

$$\int_C \frac{1}{z^2+4} dz = 0, \qquad C: \ \text{unit circle}$$

although the integrand is not analytic at $z = \pm 2i$, these points are not enclosed by C.

Independence of Path

Now we are ready to answer the question set out on path independence.

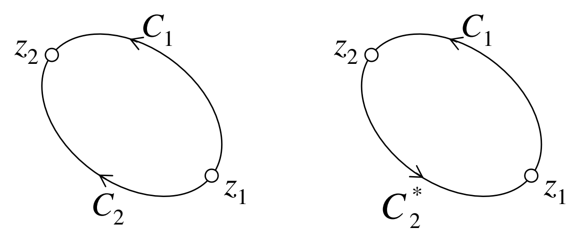
Theorem 4. [Independence of Path] *If f(z) is analytic in a* **simply connected domain** *D, then the integral of f(z) is independent of the path in D.*

Proof: *Let z_1 and z_2 be any point in D. Consider two paths C_1 and C_2 in D from z_1 to z_2 as shown in the figure below. Denote by C_2^* the path C_2 with the direction reversed. From Cauchy's theorem, we have*

$$\int_{C_1} f\ dz + \int_{C_2^*} f\ dz = 0$$

Thus

$$\int_{C_1} f\ dz = -\int_{C_2^*} f\ dz = \int_{C_2} f\ dz$$



Cauchy's Integral Formula

The most important consequence of Cauchy's integral theorem is Cauchy's integral formula. This formula is useful for evaluating integrals of the form

$$\int_C \frac{f(z)}{(z-z_o)^m} dz \quad m = 1,2,3...$$

Theorem 6. [Cauchy's Integral Formula] *Let f(z) be analytic in a simply connected domain D. Then for any point z_o in D and any simple closed path C in D that encloses z_o*

$$\int_C \frac{f(z)}{(z-z_o)} dz = 2\pi i f(z_o)$$

In general,

$$\int_C \frac{f(z)}{(z-z_o)^m} dz = \frac{2\pi i}{(m-1)!} f^{(m-1)}(z_o) \qquad m = 1,2,3...$$

The integration being taken CCW. (See text for proof.)

Examples (Cauchy's Integral Formula)

- $\int_C \frac{e^z}{z-2}dz = 2\pi i e^z\Big|_{z=2} = 2\pi i e^2$

  for any simple closed path enclosing $z_o = 2$.

- Evaluate $\int_C \frac{z^3-6}{2z-i}dz$, C : unit circle, CCW

  Solution: Since C encloses $z = \frac{i}{2}$

$$\int_C = \int_C \frac{z^3-6}{2(z-i/2)}dz = \pi i\left(z^3-6\right)\Big|_{z=i/2} = \pi i\left(\frac{-i}{8}-6\right)$$

- Evaluate $\int_C \frac{\cos z}{(z-\pi i)^2}dz$ where C is any contour enclosing $z = \pi i$, CCW.

  Solution:

$$\int_C = 2\pi i \frac{d}{dz}\cos z\Big|_{z=\pi i} = -2\pi i \sin(\pi i)$$

- Evaluate $\int_C \frac{z^4-3z^2+6}{(z+i)^3}dz$ where C is any contour enclosing $z = -i$, CCW.

  Solution:

$$\int_C = \frac{2\pi i}{2!}\frac{d^2}{dz^2}\left(z^4-3z^2+6\right)\Big|_{z=-i} = \pi i\left(12z^2-6\right)\Big|_{z=-i} = -18\pi i$$

• Evaluate $\int_C \frac{1}{z^2+1} dz$, C : $|z| = 3$, CCW.

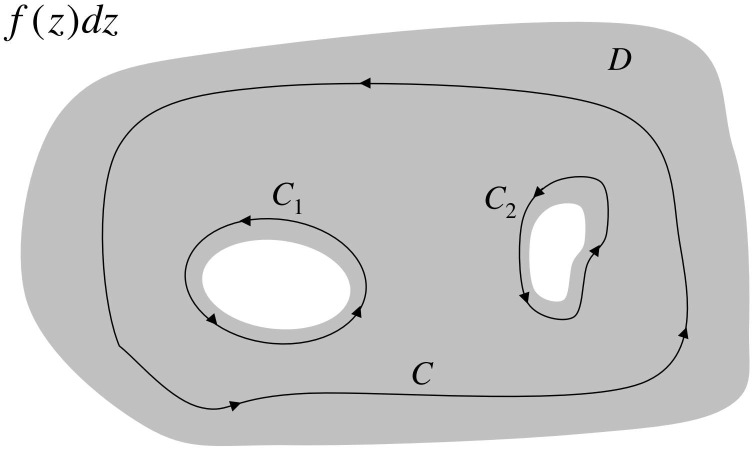
Solution: The integrand is not analytic at $z = \pm i$ which are inside C. Cauchy's formula only applies to one singular point inside C. Use partial fraction decomposition and apply Cauchy's formula

$$\oint_C \frac{dz}{z^2+1} = \oint_C \frac{dz}{(z+i)(z-i)} = \frac{1}{2i}\oint_C \left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz$$
$$= \frac{1}{2i}\left[2\pi i - 2\pi i\right] = 0$$

Cauchy's Theorem for Multiply Connected Domains

Suppose C, C_1, …, C_n are simple closed curves with a positive orientation such that C_1, C_2, …, C_n are interior to C but regions interior to each C_k, $k = 1, 2, \ldots, n$, have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the C_k, $k = 1, 2, \ldots, n$, then

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz$$



Example[5]**:** Evaluate $\oint_C \frac{dz}{z^2+1}$ where C is the circle $|z| = 3$.

Solution: The integrand $\frac{1}{z^2+1}$ *is not analytic at* $z = \pm i$*. Both of these points lie within the contour* C*. Introduce* C_1 *and* C_2 *as shown in the figure to exclude these points and using the Cauchy's Theorem to this multiply connected domain, we have*

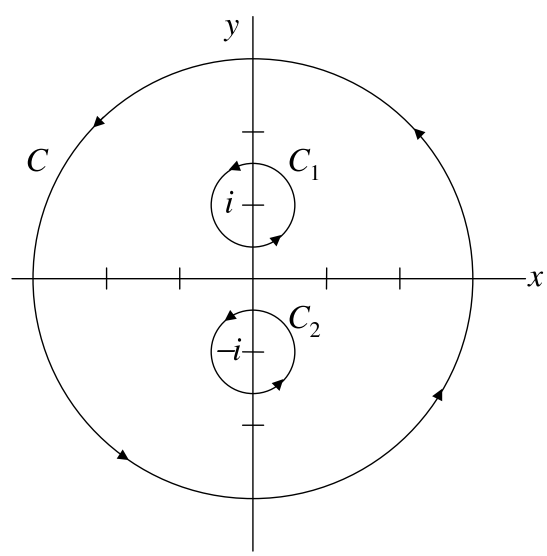
[5] This is the same example in page 55 where we solved using partial fraction. Here we shall apply Cauchy's theorem to Multiply Connected Domains.

$$\oint_C \frac{dz}{z^2+1} = \oint_C \frac{dz}{(z+i)(z-i)}$$

$$= \oint_{C_1} \frac{1/(z+i)}{z-i}dz + \oint_{C_2} \frac{1/(z-i)}{z+i}dz$$

$$= 2\pi i\left[\frac{1}{z+i}\right]\Bigg|_{z=i} + 2\pi i\left[\frac{1}{z-i}\right]\Bigg|_{z=-i}$$

$$= 2\pi i\frac{1}{2i} + 2\pi i\frac{1}{-2i} = 0$$



Evaluation of Real Integrals

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta$$

Where $F(\cos\theta, \sin\theta)$ is a real function of $\cos\theta$ and $\sin\theta$ and is finite on the interval of integration.

Basic Idea:

- Let $z = e^{i\theta}$, we have

$$\cos\theta = \frac{z+\bar{z}}{2}, \sin\theta = \frac{z-\bar{z}}{2i} \text{ and } dz = ie^{i\theta}d\theta \;\Rightarrow d\theta = \frac{1}{iz}dz$$

- This allows us to convert $F(\cos\theta, \sin\theta)$ into $f(z)$ and the integration interval of $0 \le \theta \le 2\pi$ is changed to the unit circle.
- Thus

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta = \oint_C f(z)\frac{1}{iz}dz, \quad C: \text{ unit circle, CCW}$$

Example: Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta}$

Solution: *Let $z = e^{i\theta}$. Substitute*

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) \qquad \textit{and} \qquad d\theta = \frac{dz}{iz}$$

The real integral becomes

$$\oint_C \frac{dz/iz}{\sqrt{2}-\frac{1}{2}\left(z+\frac{1}{z}\right)} = -\frac{2}{i}\oint_C \frac{dz}{\left(z-\left(\sqrt{2}+1\right)\right)\left(z-\left(\sqrt{2}-1\right)\right)}$$

where C is the CCW unit circle.

The integrand has simple pole at $z = \sqrt{2} - 1$ inside C and $z = \sqrt{2} + 1$ outside C. Hence, using Cauchy's Integral Formula, the integral is

$$
\begin{aligned}
&-\frac{2}{i}\oint_C \frac{dz}{\left(z-\left(\sqrt{2}+1\right)\right)\left(z-\left(\sqrt{2}-1\right)\right)} \\
&= -\frac{2}{i}\oint_C \frac{\overline{\dfrac{1}{z-\left(\sqrt{2}+1\right)}}}{z-\left(\sqrt{2}-1\right)}dz \\
&= -\frac{2}{i}(2\pi i)\left.\frac{1}{z-\left(\sqrt{2}+1\right)}\right|_{z=\sqrt{2}-1} = 2\pi
\end{aligned}
$$

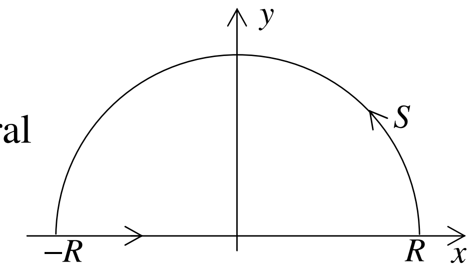
Improper Integrals of Rational Functions

$$\int_{-\infty}^{\infty} f(x)dx = \oint_{\text{UHP}} f(z)dz = \sum_{k=1}^{n} \oint_{C_k \text{ in UHP}} f(z)dz, \qquad \textbf{if}$$

1. $f(x) = \dfrac{p(x)}{q(x)}$ is a real function with $p(x)$ and $q(x)$ no common factors and $q(x) \neq 0$ for all real x.
2. $\deg\left(q(x)\right) \geq \deg\left(p(x)\right) + 2.$

E.g. $f(x) = \dfrac{1}{1+x^4}$ satisfies the above conditions but $f(x) = \dfrac{x^3}{1+x^4}$ does not.

Proof: Basic Idea



- Consider the corresponding contour integral
$$\int_C f(z)dz$$
- where C is the Upper Half Plane (UHP) contour shown on the right
- Then $\oint_C f(z)dz = \int_S f(z)dz + \int_{-R}^{R} f(x)dx$
- Since $f(x)$ is rational, $f(z)$ has finitely many poles in the UHP. If we choose R large enough, the C contour will enclose all these poles.
- In addition, due to assumption 2, it can be shown that $\lim_{R\to\infty} \int_S f(z)dz = 0.$ Hence,
$$\int_{-\infty}^{\infty} f(x)dx = \oint_{\text{UHP}} f(z)dz$$

Example: Show that $\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}.$

Solution: First, check that $f(x)$ satisfies our assumptions. Now, consider $f(z) = \frac{1}{1+z^4}$ which has four simple poles at $z = e^{\pi i/4}, e^{3\pi i/4}, e^{-3\pi i/4}, e^{-\pi i/4}$.

Only the first two poles, $e^{\pi i/4}$ and $e^{3\pi i/4}$, lie inside the UHP. The corresponding complex integral is

$$\oint_{UHP} \frac{1}{1+z^4} dz = \oint_{UHP} \frac{1}{\left(z-e^{\pi i/4}\right)\left(z-e^{3\pi i/4}\right)\left(z-e^{-3\pi i/4}\right)\left(z-e^{-\pi i/4}\right)} dz$$

$$= \oint_{c_1} \frac{\frac{1}{\left(z-e^{3\pi i/4}\right)\left(z-e^{-3\pi i/4}\right)\left(z-e^{-\pi i/4}\right)}}{z-e^{\pi i/4}} dz + \oint_{c_2} \frac{\frac{1}{\left(z-e^{\pi i/4}\right)\left(z-e^{-3\pi i/4}\right)\left(z-e^{-\pi i/4}\right)}}{z-e^{3\pi i/4}} dz$$

$$= 2\pi i\left(-\frac{1}{4}e^{\pi i/4} + \frac{1}{4}e^{-\pi i/4}\right)$$

Since $\dfrac{1}{1+x^4}$ *is even,*

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^4}$$

$$= -\frac{\pi i}{4}\left(e^{i\pi/4} - e^{-i\pi/4}\right) = \frac{\pi}{2}\sin\frac{\pi}{4}$$

$$= \frac{\pi}{2\sqrt{2}}$$