$1, 2, 3, 4, 5, \ldots$. Now when $n = 1$, we get $f(1) = 0$ and we are left to check that the negative integers are obtained. But $3, 5, 7, 9, 11, \ldots$ are mapped to $-1, -2, -3, -4, \ldots$. So this function seems to do what we want! So let us give a proof. We have to show that $f(n)$ is injective. So suppose $f(n) = f(m)$ for some positive integers n, m. So either $f(n) = f(m)$ is positive, then we have

$$n/2 = m/2 \Rightarrow n = m$$

or $f(n) = f(m)$ is negative and

$$-(n-1)/2 = -(m-1)/2 \Rightarrow n = m.$$

So this shows the function is injective. Now let us proof that it is surjective (or onto). We need to show that for any arbitrary m which is an integer, there exists a positive integer n such that $f(n) = m$. If $m > 0$, pick n to be $n = 2m$ (note that n defined like that is indeed a positive integer), and $f(n) = 2m/2 = m$. If $m = 0$, pick $n = 1$ and $f(1) = 0$. If $m < 0$, then pick $n = -2m + 1$. We have $f(n) = -[(-2m+1) - 1]/2 = m$ and because m is negative, $-2m + 1$ is indeed a positive integer.

Exercises for Chapter 10

Exercise 97. Prove that if a connected graph G has exactly two vertices which have odd degree, then it contains an Euler path.

Solution. Suppose that v and w are the vertices of G which have odd degrees, while all the other vertices have an even degree. Create a new graph G', formed by G, with one more edge e, which connects v and w. Every vertex in G' has even degree, so by the theorem on Euler cycles, there is an Euler cycle. Say this Euler cycle is

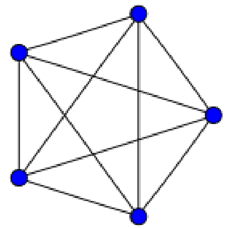
$$v, e_1, v_2, e_2, \ldots, w, e, v$$

then

$$v, e_1, v_2, e_2, \ldots, w$$

is an Euler path.

Exercise 98. Draw a complete graph with 5 vertices.



Solution.

Exercise 99. Show that in every graph G, the number of vertices of odd degree is even.

Solution. Let E denote the set of edges, and write the set V of vertices as $V' \cup V''$ where V' is the set of nodes with odd degrees, and V'' is the set of nodes with even degrees. Suppose that the number of vertices of odd degree is odd, then

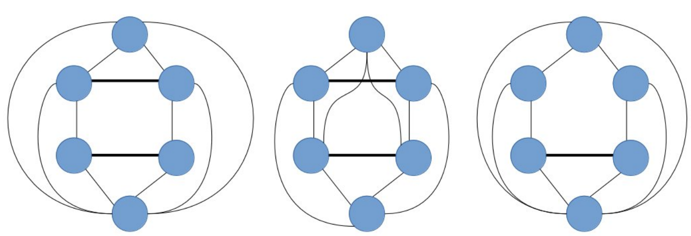
$$2|E| = \sum_{v \in V} \deg(v) = \sum_{v \in V'} + \sum_{v \in V''}$$

where the first sum (over V') is odd and the second sum is even, a contradiction.

Exercise 100. Show that in very simple graph (with at least two vertices), there must be two vertices that have the same degree.

Solution. Suppose there are n nodes. If all degrees are different, they must be exactly $0, 1, \ldots, n-1$, which is impossible: one cannot have one node of degree 0, yet another one with degree $n-1$!

Exercise 101. Decide whether the following graphs contain a Euler path/cycle.



Solution. The first graph (left hand side) contains a Euler path and no Euler circuit, the middle graph contains a Euler circuit, the third one contains none!