Circuit Analysis
EE2001

Network Functions
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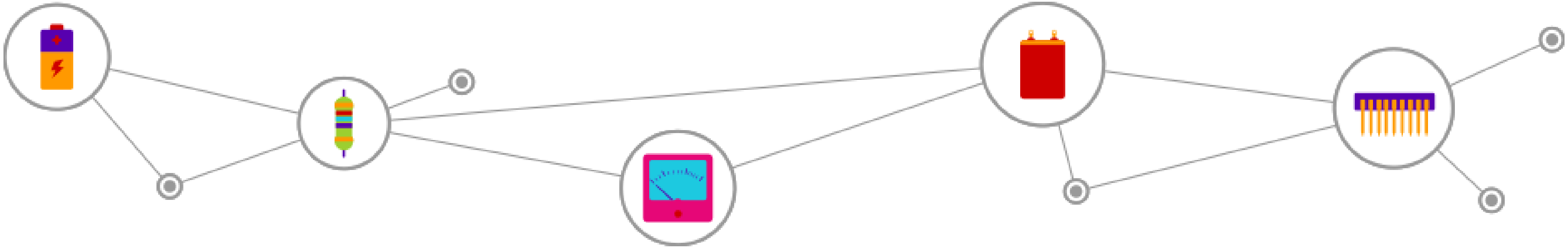
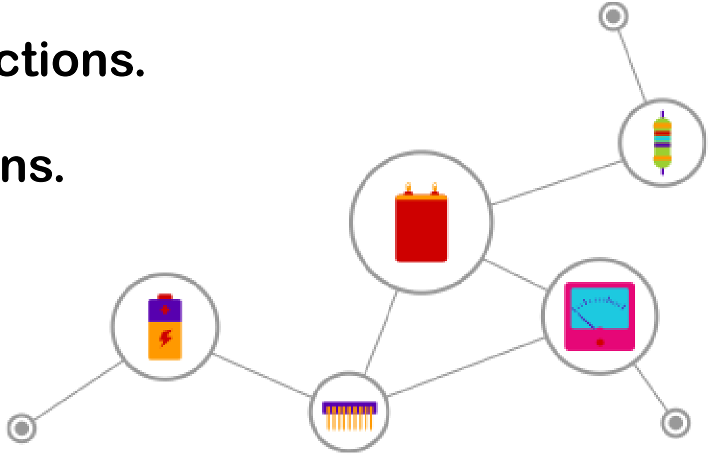
Overview

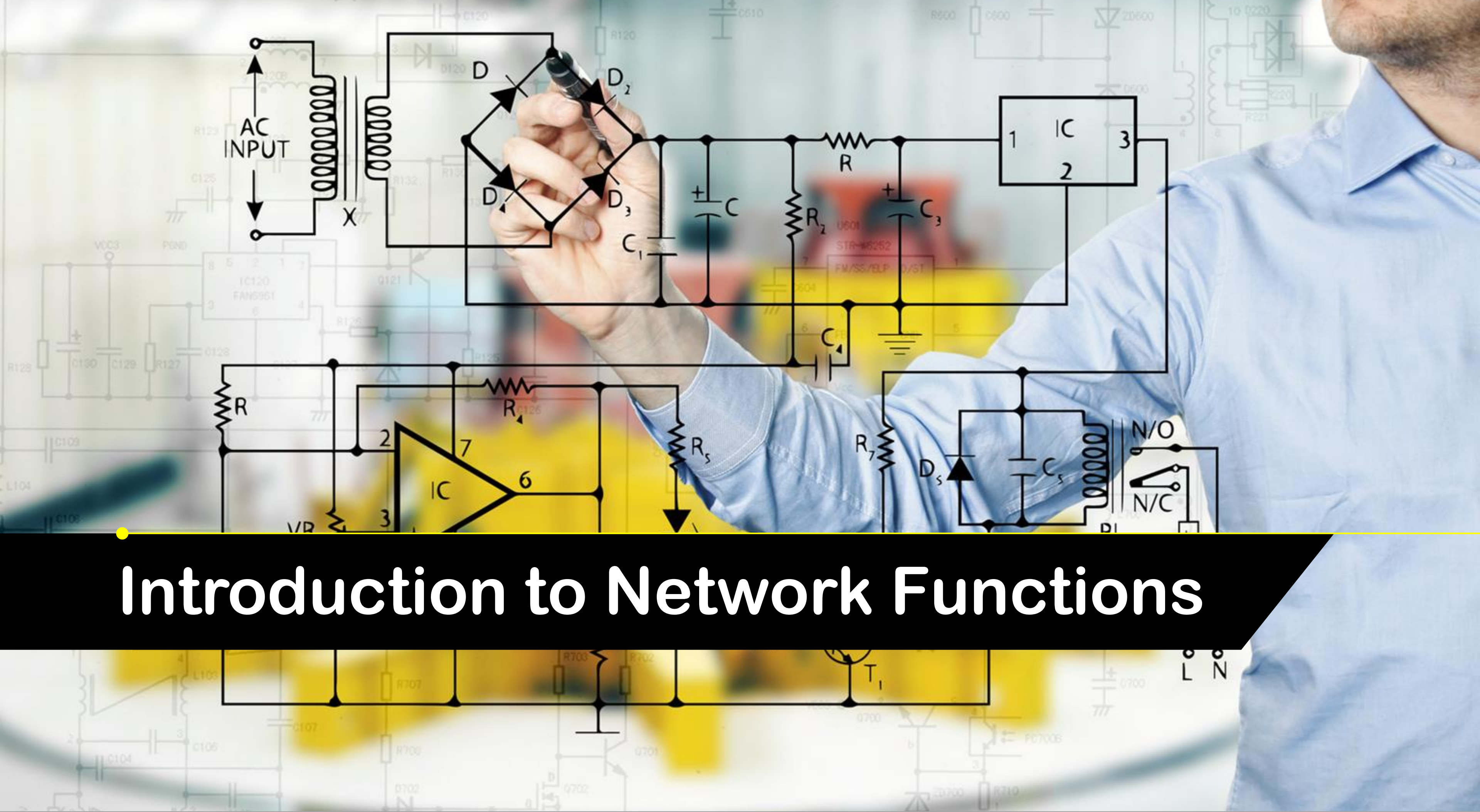
- Introduction to Network Functions
- Poles and Zeros of Transfer Function $H(s)$
- Time Domain Response from Pole-zero Plot
- Poles and Stability
- Transfer Function and Impulse Response
- Transfer Function and Step Response
- DC Steady-state Response (for Step Input)
- AC Steady-state Response (for AC Input)



By the end of this lesson, you should be able to…

- Derive network functions.
- Explain the key characteristics of poles and zeros.
- Determine the time domain response from the pole and zero plot of a given network function.
- Use the locations of the poles to identify stable circuits.
- Explain the key characteristics of impulse functions.
- Explain the key characteristics of step functions.
- Determine the DC steady-state response.
- Determine the AC steady-state response.



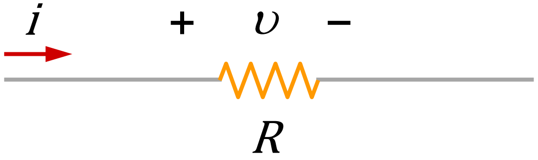


Introduction to Network Functions

Introduction to Network Functions

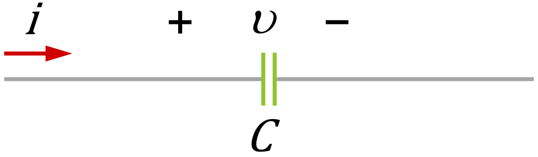
Review: Time-domain equations for R, L, C

Resistor



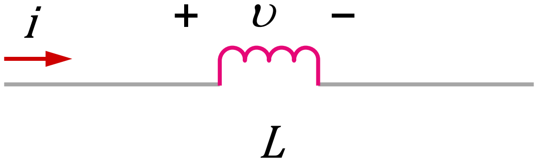
$$v = iR$$

Capacitor



$$i = C\frac{dv}{dt}$$

Inductor

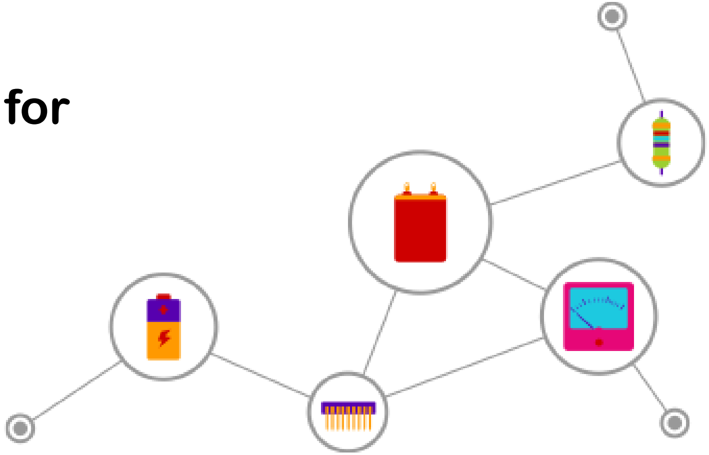


$$v = L\frac{di}{dt}$$

Introduction to Network Functions

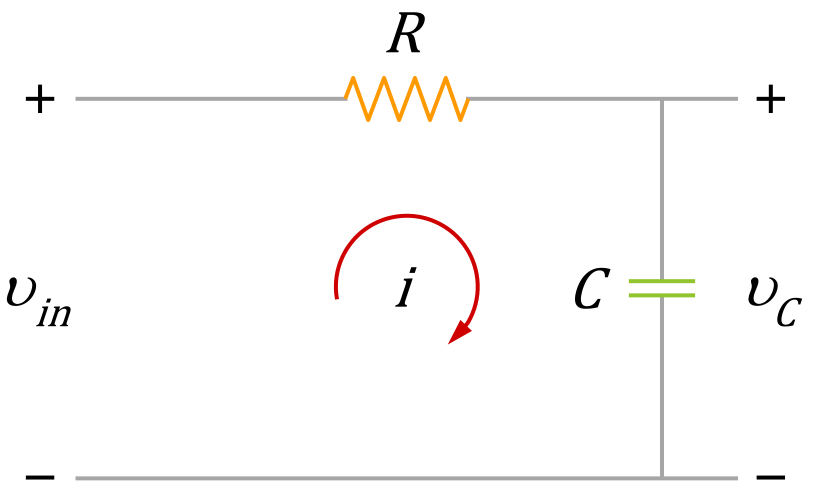
The concept of network function is very important, in terms of circuit analysis and other areas of engineering.

- **Network functions are useful when we are primarily concerned with one input and one resulting output, when we do not want to solve for all the currents and voltages in a circuit.**
- **They give the relation between currents or voltages at different parts of the network.**
- **A network function is a useful analytical tool for finding the response of a circuit.**
- **The form of the network function contains a great deal of information about the behaviour of the circuit.**



Introduction to Network Functions: Example 1

We are interested in the response of the RC circuit below, say, v_C, when a voltage v_{in} is applied at the input of the circuit.

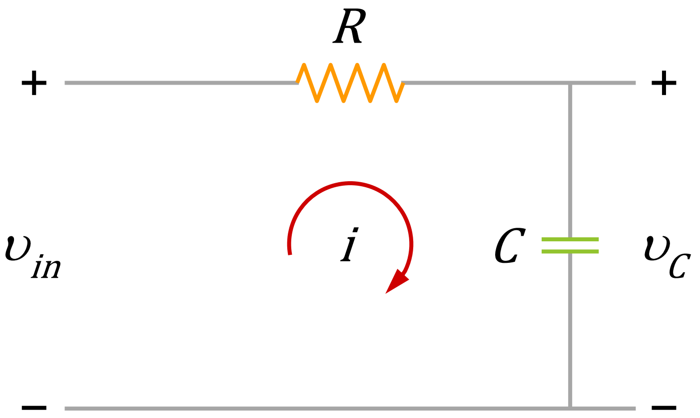


Introduction to Network Functions: Example 1

Applying KVL gives

$$\boxed{v_{in} = iR + v_C}$$

$$\boxed{i = C\frac{dv_C}{dt}}$$



Their Laplace transforms are

$$\boxed{V_{in}(s) = RI(s) + V_C(s)} \quad \textbf{(1)}$$

$$\boxed{I(s) = C(sV_C(s) - v_C(0))} \quad \textbf{(2)}$$

With $v_C(0) = 0$**, i.e., zero initial condition,** $I(s) = sCV_C(s)$**.**

Using this in (1) gives

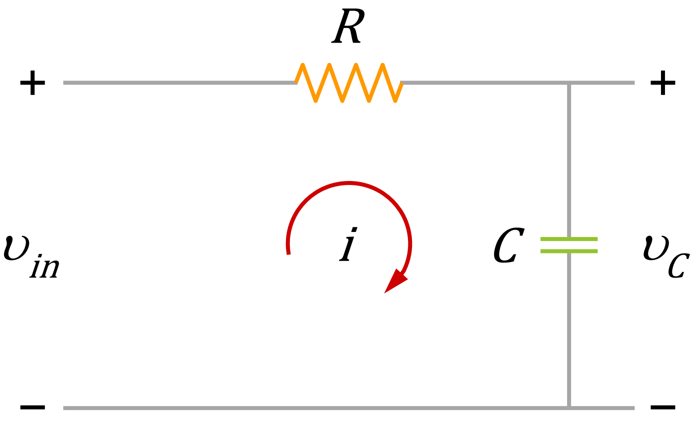
$$\boxed{V_C(s) = \frac{1}{sRC + 1}V_{in}(s)} \quad \textbf{(3)}$$

Introduction to Network Functions: Example 1

If we let $H_1(s) = \frac{1}{sRC+1}$, then

$$V_C(s) = \frac{1}{sRC+1} V_{in}(s) \quad (3)$$

$$V_C(s) = H_1(s) V_{in}(s) \quad (4)$$



The expression in (4) is an input-output relationship between the transforms of v_C and v_{in} .

$H_1(s)$ is known as a **network function** as it gives the relation between voltages at different parts of the network.

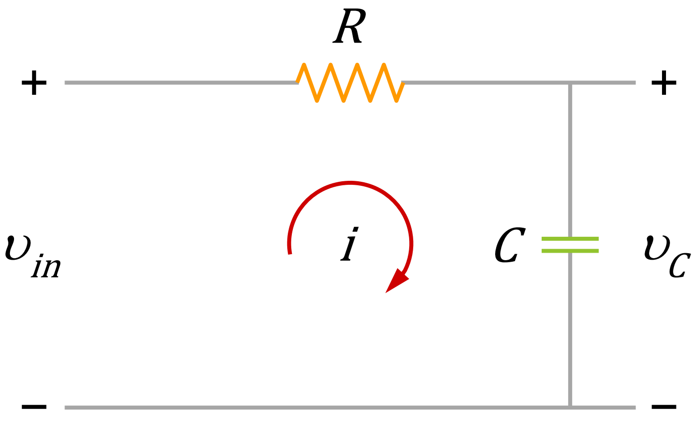
Introduction to Network Functions: Example 1

Suppose the response of interest is now i.

Using $V_C(s) = I(s)/sC$ in (1) gives

$$V_{in}(s) = RI(s) + V_C(s) \quad (1)$$

$$I(s) = \frac{sC}{sRC+1} V_{in}(s) \quad (5)$$

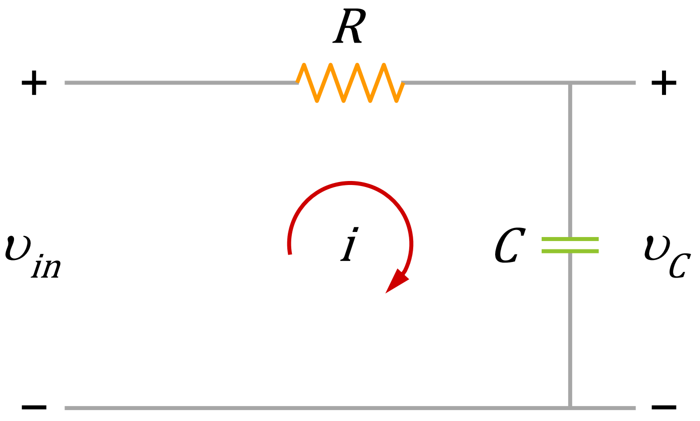


If we let $H_2(s) = \frac{sC}{sRC+1}$, then

$$I(s) = H_2(s) V_{in}(s) \quad (6)$$

Introduction to Network Functions: Example 1

$$\boxed{I(s) = H_2(s)V_{in}(s)} \quad (6)$$



The expression in (6) is an input-output relationship between the transforms of i and v_{in}.

$H_2(s)$ is known as a **network function** as it gives the relation between the current and the voltage at different parts of the network.

It is seen that a given network may be described by different network functions, one for each response of interest.

Introduction to Network Functions: Transfer Function

Transfer function is also called the network function.

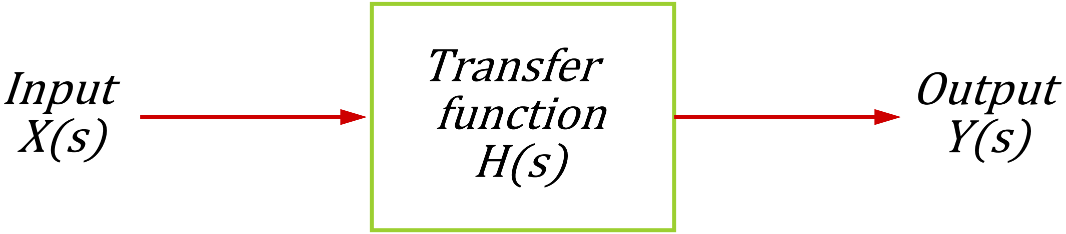
Definition: The transfer function is defined as $H(s) = \dfrac{Y(s)}{X(s)}$

Where

$Y(s)$ = **Laplace transform of the output** $y(t)$

$X(s)$ = **Laplace transform of the input** $x(t)$

This definition assumes that all the initial conditions are zero.



Block diagram for an s-domain input-output relationship

Introduction to Network Functions: Transfer Function

It is customary to call $H(s)$ the **driving-point function**, if the output is measured at the same place where the input has been applied.

The term **driving-point** means that the circuit is driven at one **port** (a pair of terminals) and the response is observed at the same port.

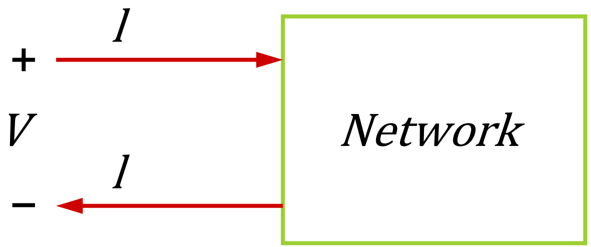


Illustration of a port

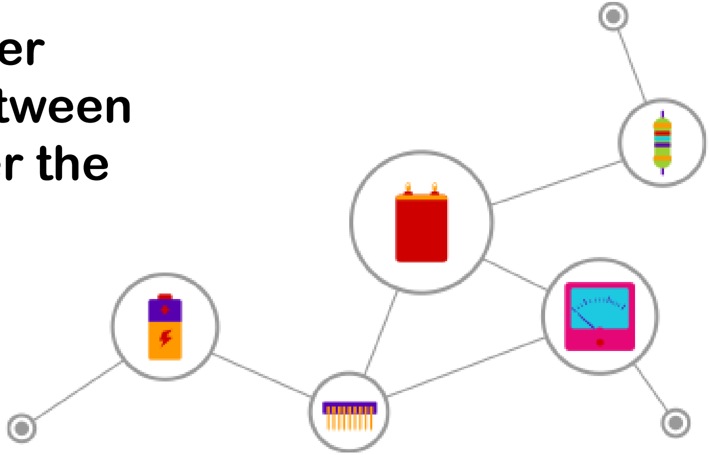
$$Z(s) = \frac{V(s)}{I(s)} \quad \text{or} \quad Y(s) = \frac{I(s)}{V(s)}$$

The **driving-point impedance** $Z(s)$ relates the voltage and current at a port. The **driving-point admittance** is $Y(s) = \frac{1}{Z(s)}$.

Introduction to Network Functions: Transfer Function

A network with two input terminals and two output terminals is called a two-port network.

For this network, there are a few kinds of transfer functions. These functions give the relations between either the voltage or current at one port to either the voltage or current at the other port.



Introduction to Network Functions: Transfer Function

Examples:

Voltage transfer ratio: $H_{21}(s) = \dfrac{V_2(s)}{V_1(s)}$

Current transfer ratio: $\alpha_{12}(s) = \dfrac{I_1(s)}{I_2(s)}$

Transfer impedance: $Z_{21}(s) = \dfrac{V_2(s)}{I_1(s)}$

Transfer admittance: $Y_{21}(s) = \dfrac{I_2(s)}{V_1(s)}$

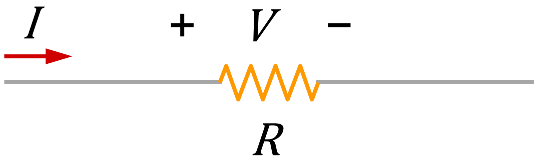
We have earlier seen that the transfer function $H(s)$ can be found if the response $y(t)$ is given by a differential equation.

However, $H(s)$ is usually more easily found by the analysis of the s-domain diagram.

Introduction to Network Functions: Transfer Function

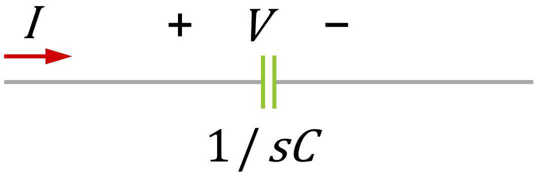
Review: s-domain equations for R, L, C with zero initial conditions

Resistor



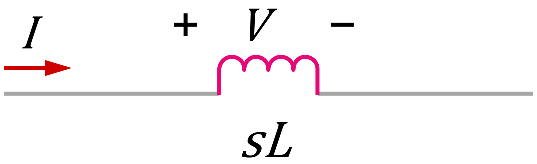
$$V(s) = I(s)R$$

Capacitor



$$V(s) = \frac{1}{sC} I(s)$$

Inductor

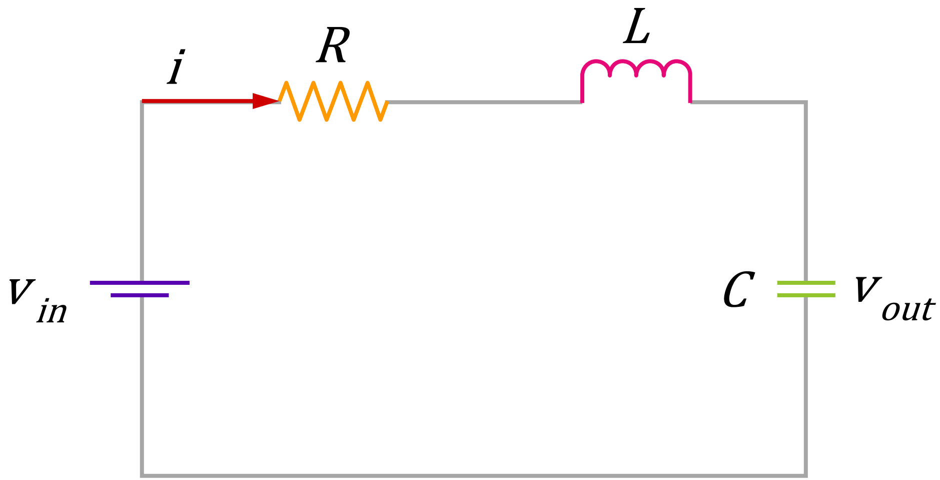


$$V(s) = sLI(s)$$

Introduction to Network Functions: Example 2

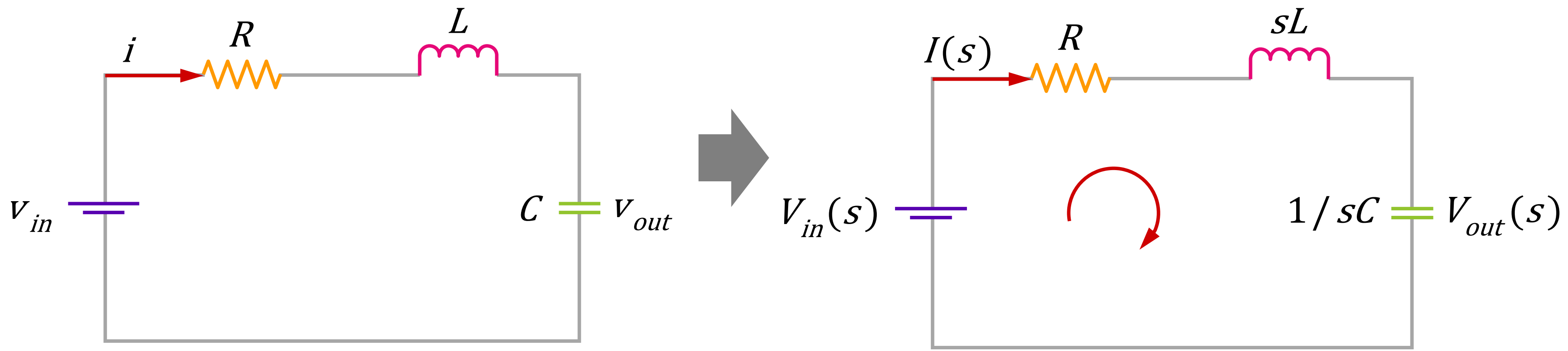
Consider the electrical network as shown.

Find the transfer function $\dfrac{V_{out}(s)}{V_{in}(s)}$.



Introduction to Network Functions: Example 2

The circuit is first transformed to the s-domain form as shown (with zero initial conditions).

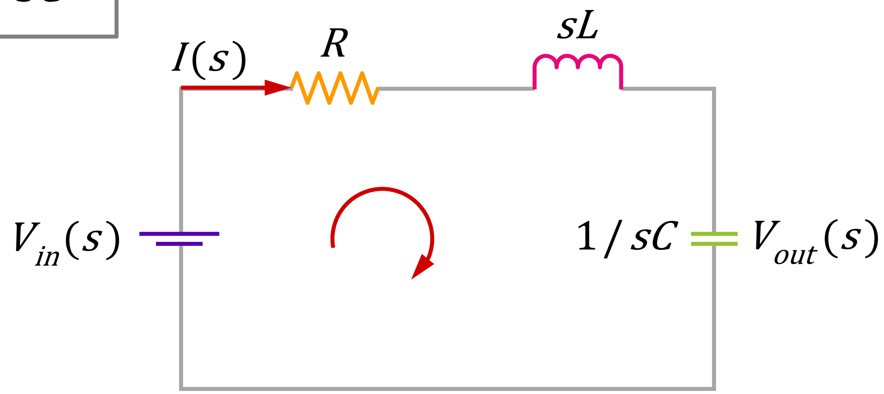


Introduction to Network Functions: Example 2

Applying KVL, we have

$$\left(R + sL + \frac{1}{sC}\right) I(s) = V_{in}(s)$$

$$V_{out}(s) = \frac{I(s)}{sC}$$



This gives

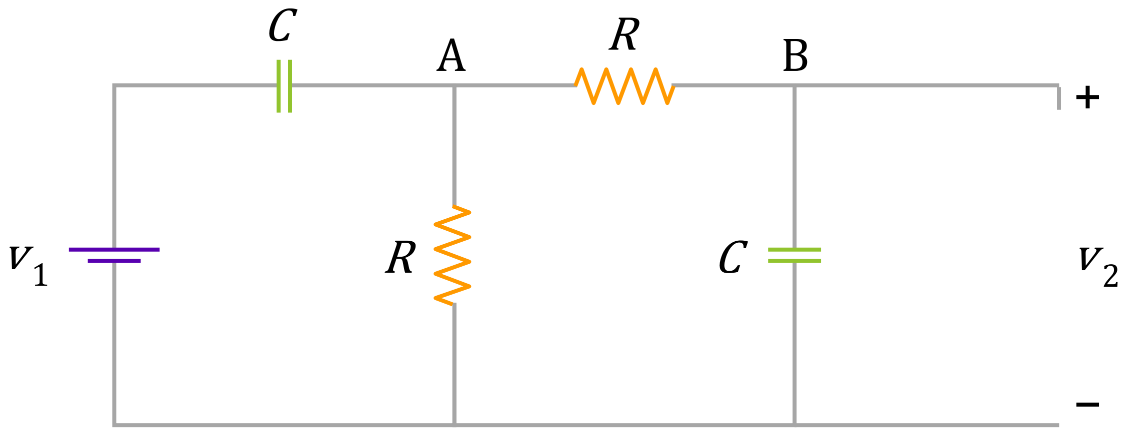
$$\left(R + sL + \frac{1}{sC}\right) sCV_{out}(s) = V_{in}(s)$$

The transfer function of the RLC network is

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{sC(R + sL + \frac{1}{sC})} = \frac{1/(LC)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

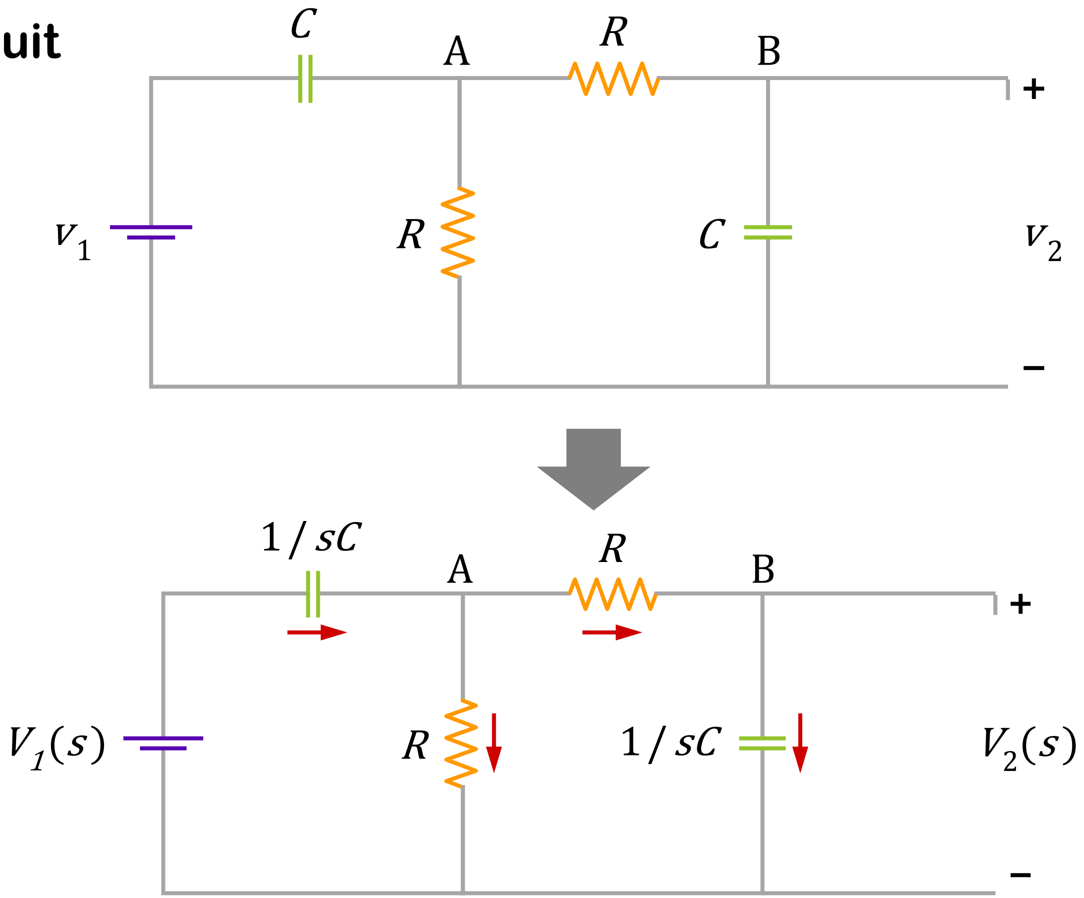
Introduction to Network Functions: Example 3

For the circuit shown, find the voltage transfer function $\dfrac{V_2(s)}{V_1(s)}$.



Introduction to Network Functions: Example 3

The transformed circuit is as shown.



Introduction to Network Functions: Example 3

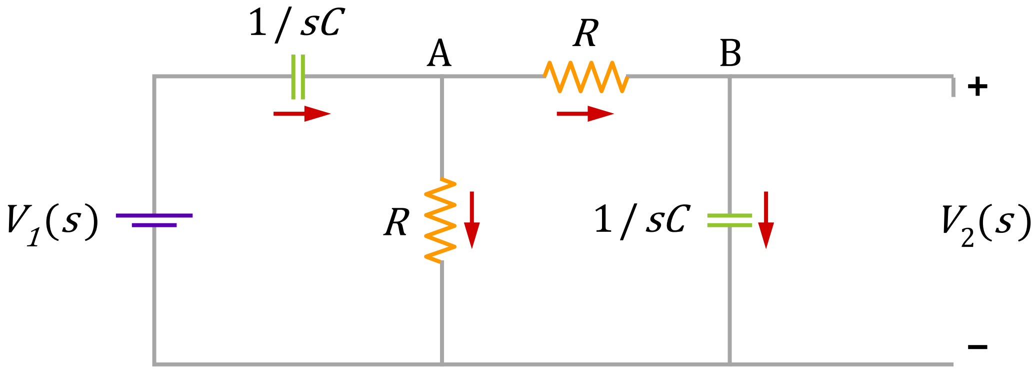
Using nodal currents, we have

At node A

$$sC(V_1(s) - V_A(s)) = \frac{V_A(s)}{R} + \frac{V_A(s) - V_2(s)}{R} \quad (1)$$

At node B

$$\frac{V_A(s) - V_2(s)}{R} = sCV_2(s) \quad (2)$$

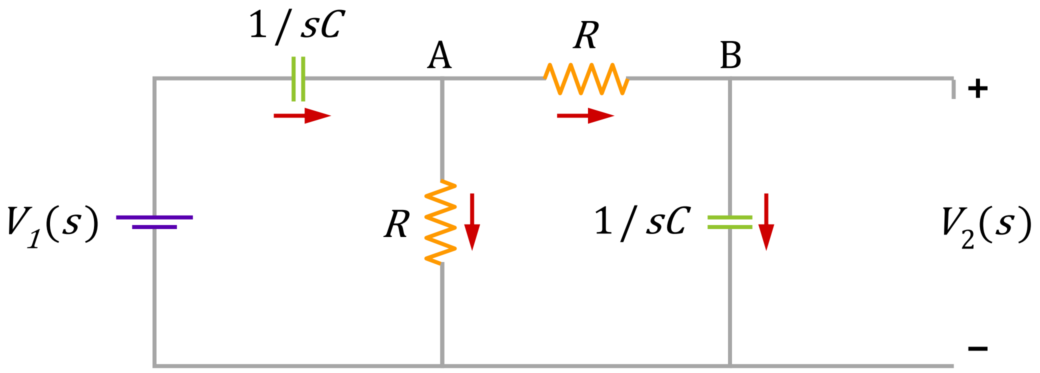


Introduction to Network Functions: Example 3

From (2), we get (3)

$$\frac{V_A(s) - V_2(s)}{R} = sCV_2(s) \quad (2)$$

$$V_A(s) = (1 + sRC)V_2(s) \quad (3)$$



Using (3) in (1) gives

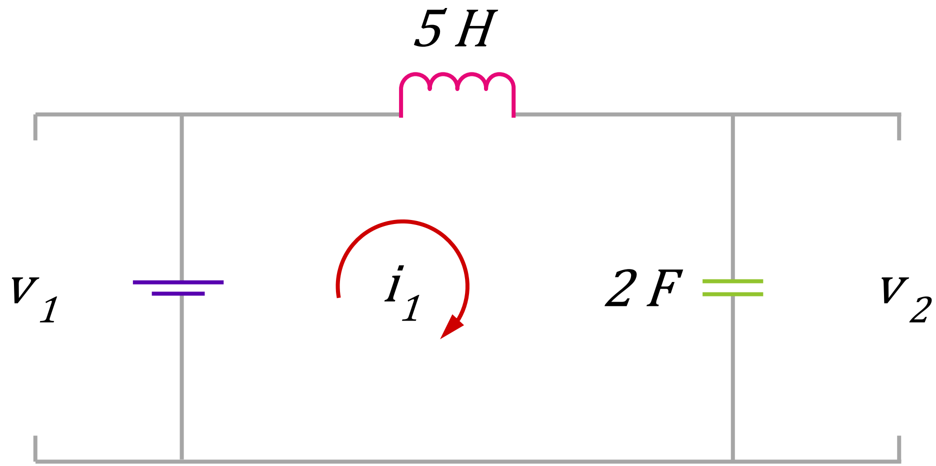
$$V_A(s) = (1 + sRC)V_2(s) \quad (3)$$

$$sC(V_1(s) - V_A(s)) = \frac{V_A(s)}{R} + \frac{V_A(s) - V_2(s)}{R} \quad (1)$$

$$\boxed{\frac{V_2(s)}{V_1(s)} = \frac{sRC}{(sRC)^2 + 3RCs + 1}}$$

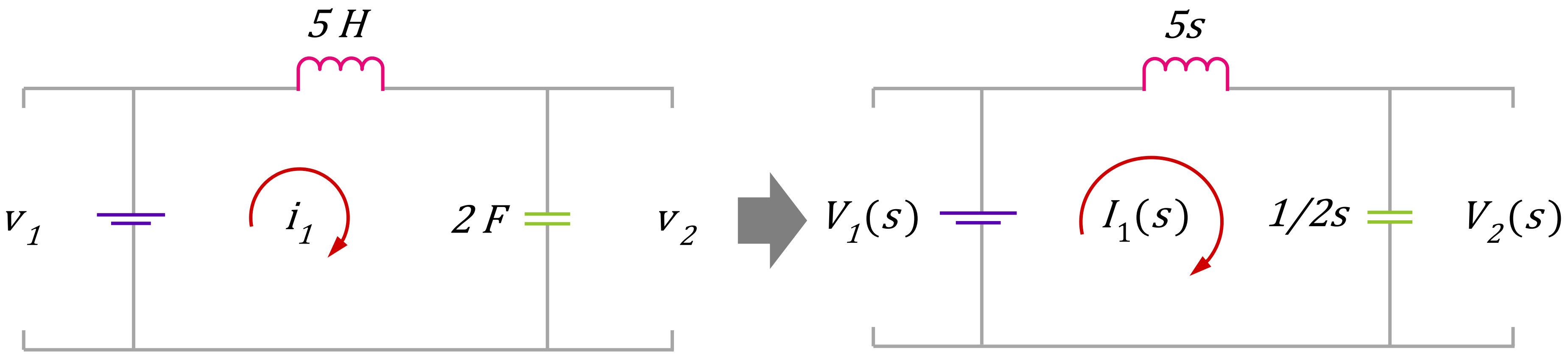
Introduction to Network Functions: Example 4

Determine the transfer functions $H_{21}(s)$, $Z_{21}(s)$ **and the driving-point impedance function** $Z_{11}(s)$ **for the network below.**



Introduction to Network Functions: Example 4

The transformed network is as shown.

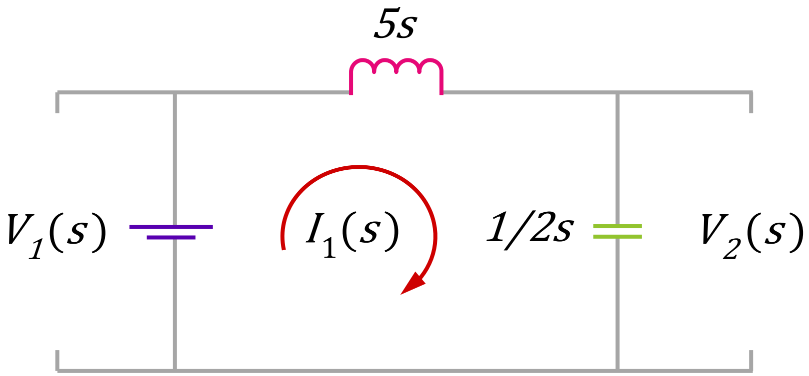


Introduction to Network Functions: Example 4

Applying KVL, we get

$$V_1(s) = (5s + \frac{1}{2s})I_1(s)$$

$$V_2(s) = \frac{1}{2s}I_1(s)$$



Therefore, the voltage transfer ratio is

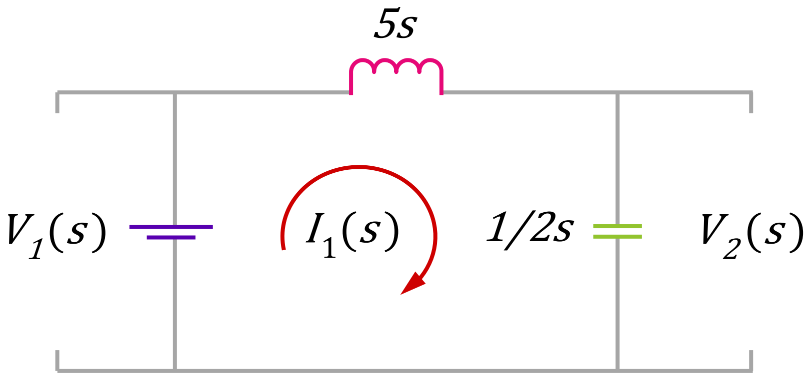
$$H_{21}(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{2s(5s + \frac{1}{2s})} = \frac{1}{10s^2 + 1}$$

The transfer impedance is

$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)} = \frac{1}{2s}$$

Introduction to Network Functions: Example 4

The driving-point impedance $Z_{11}(s)$ and the driving-point admittance $Y_{11}(s)$ are

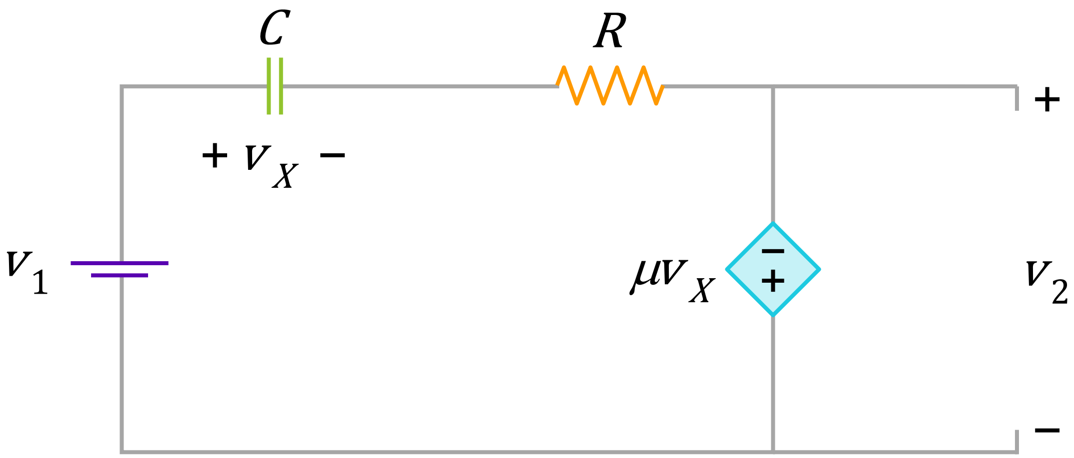


$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)} = 5s + \frac{1}{2s} = \frac{10s^2 + 1}{2s}$$

$$Y_{11}(s) = \frac{1}{Z_{11}(s)} = \frac{2s}{10s^2 + 1}$$

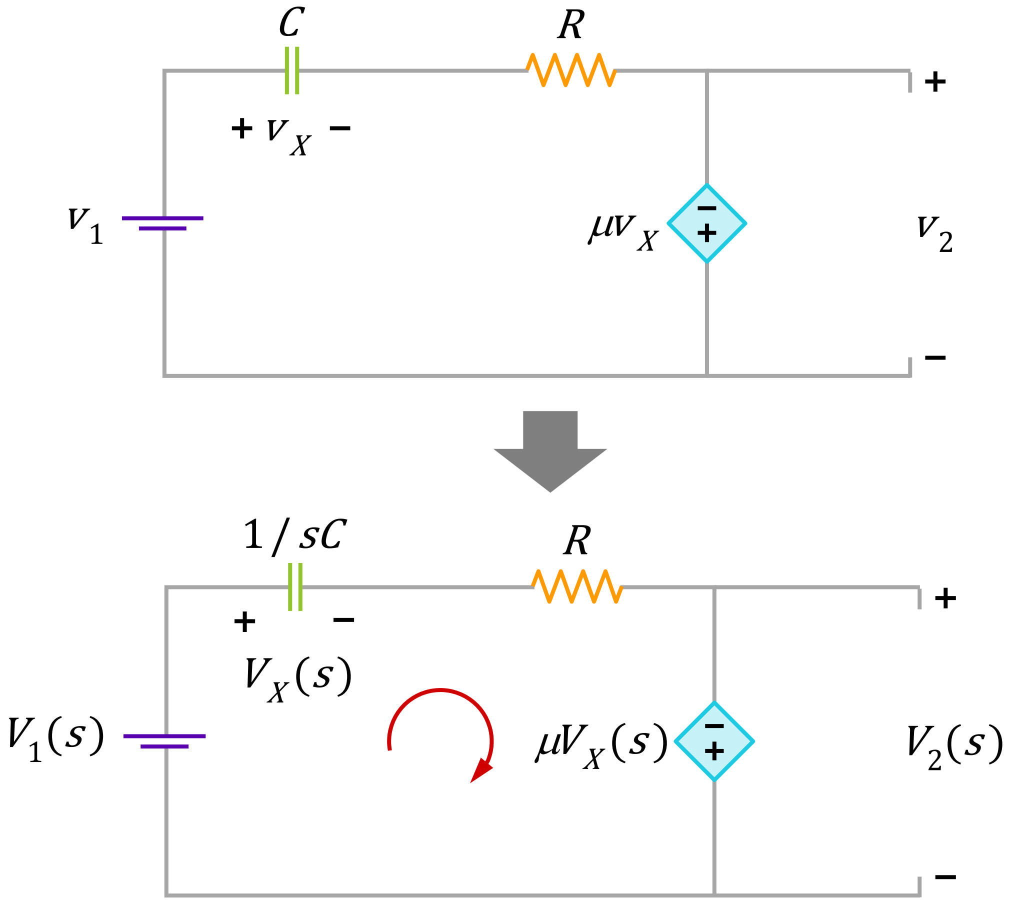
Introduction to Network Functions: Example 5

Find the voltage transfer function $\dfrac{V_2(s)}{V_1(s)}$ of the following circuit which contains a voltage-controlled voltage source.



Introduction to Network Functions: Example 5

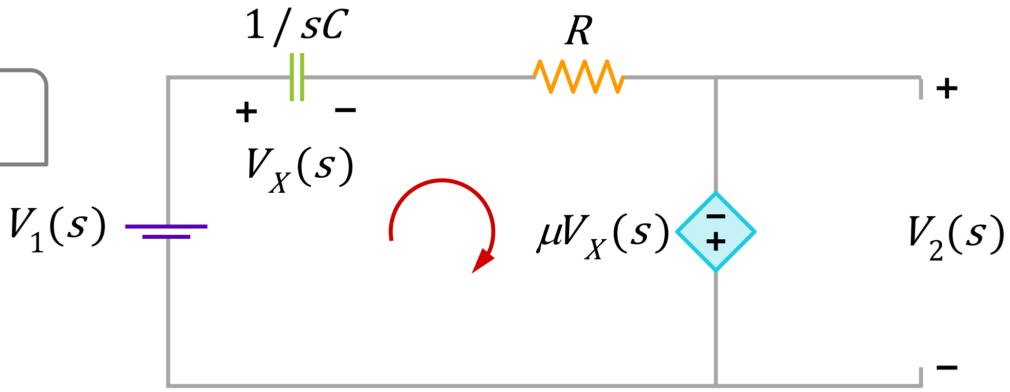
The transformed network is as shown.



Introduction to Network Functions: Example 5

Using KVL, we have

$$V_1(s) = I(s)R + V_X(s) - \mu V_X(s)$$



Where

$$I(s) = sCV_X(s)$$

This gives

$$V_1(s) = (sRC + 1 - \mu)V_X(s)$$

But,

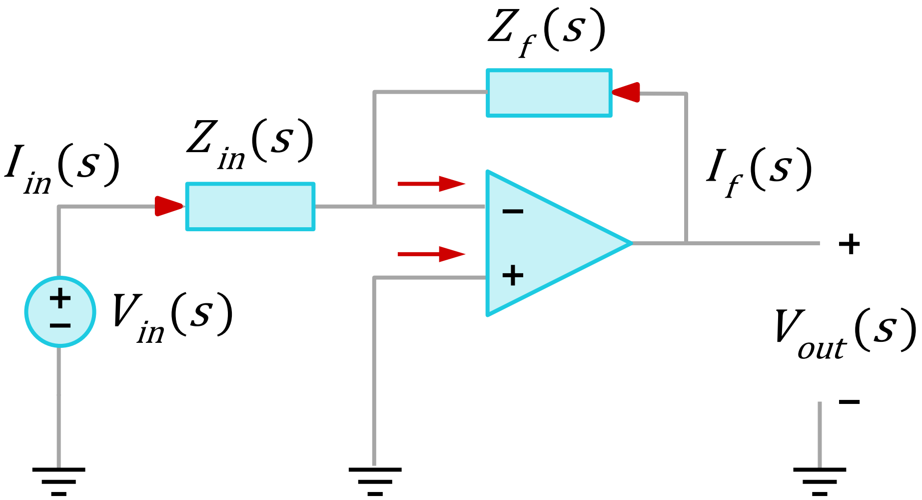
$$V_2(s) = -\mu V_X(s)$$

Thus,

$$\boxed{\frac{V_2(s)}{V_1(s)} = \frac{-\mu}{sRC + 1 - \mu}}$$

Introduction to Network Functions: Example 6

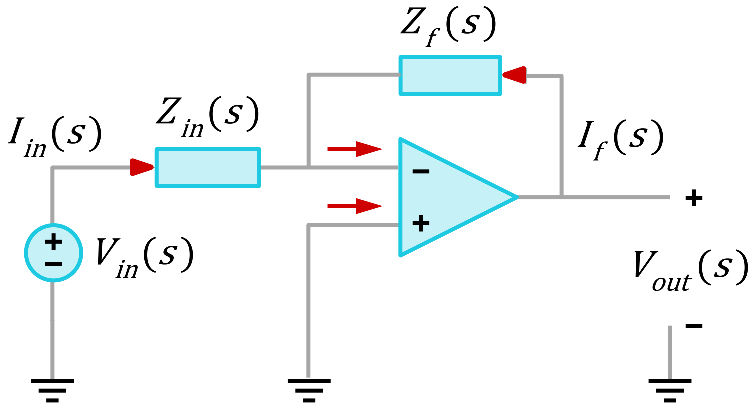
Determine the transfer function of the ideal operational-amplifier circuit as shown.



Introduction to Network Functions: Example 6

Since no current enters at the inputs of an ideal op-amp, $I_{in}(s) = -I_f(s)$.

Furthermore, the voltage at the negative op-amp terminal is driven to virtual ground.



Hence, $\boxed{V_{in}(s) = Z_{in}(s)I_{in}(s)}$

$\boxed{V_{out}(s) = Z_f(s)I_f(s)}$

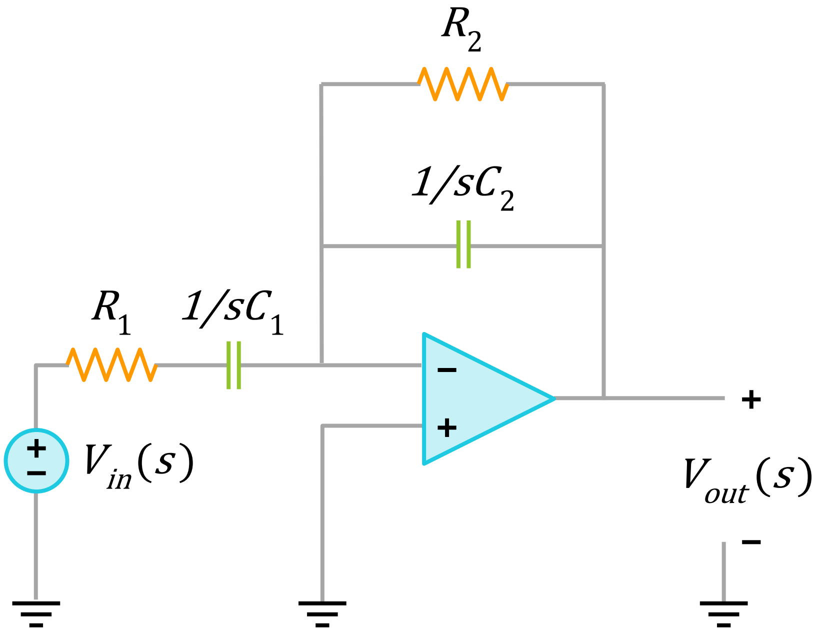
Combining these relationships with $I_{in}(s) = -I_f(s)$ yields

$$\boxed{H(s) = \frac{V_{out}(s)}{V_{in}(s)} = -\frac{Z_f(s)}{Z_{in}(s)}}$$

This is a very handy formula for computing the transfer functions and responses of many op-amp circuits.

Introduction to Network Functions: Example 7

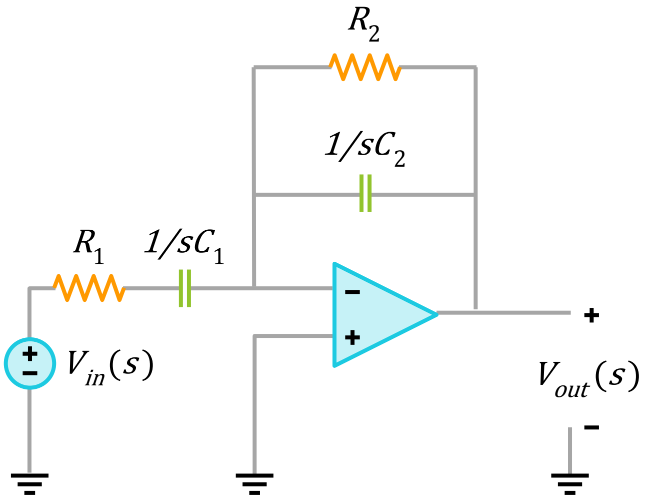
Find the voltage transfer function $H(s) = \dfrac{V_{out}(s)}{V_{in}(s)}$ of the circuit as shown.



Introduction to Network Functions: Example 7

The input impedance $Z_{in}(s)$ of this circuit is

$$Z_{in}(s) = R_1 + \frac{1}{sC_1} = \frac{sR_1C_1 + 1}{sC_1}$$

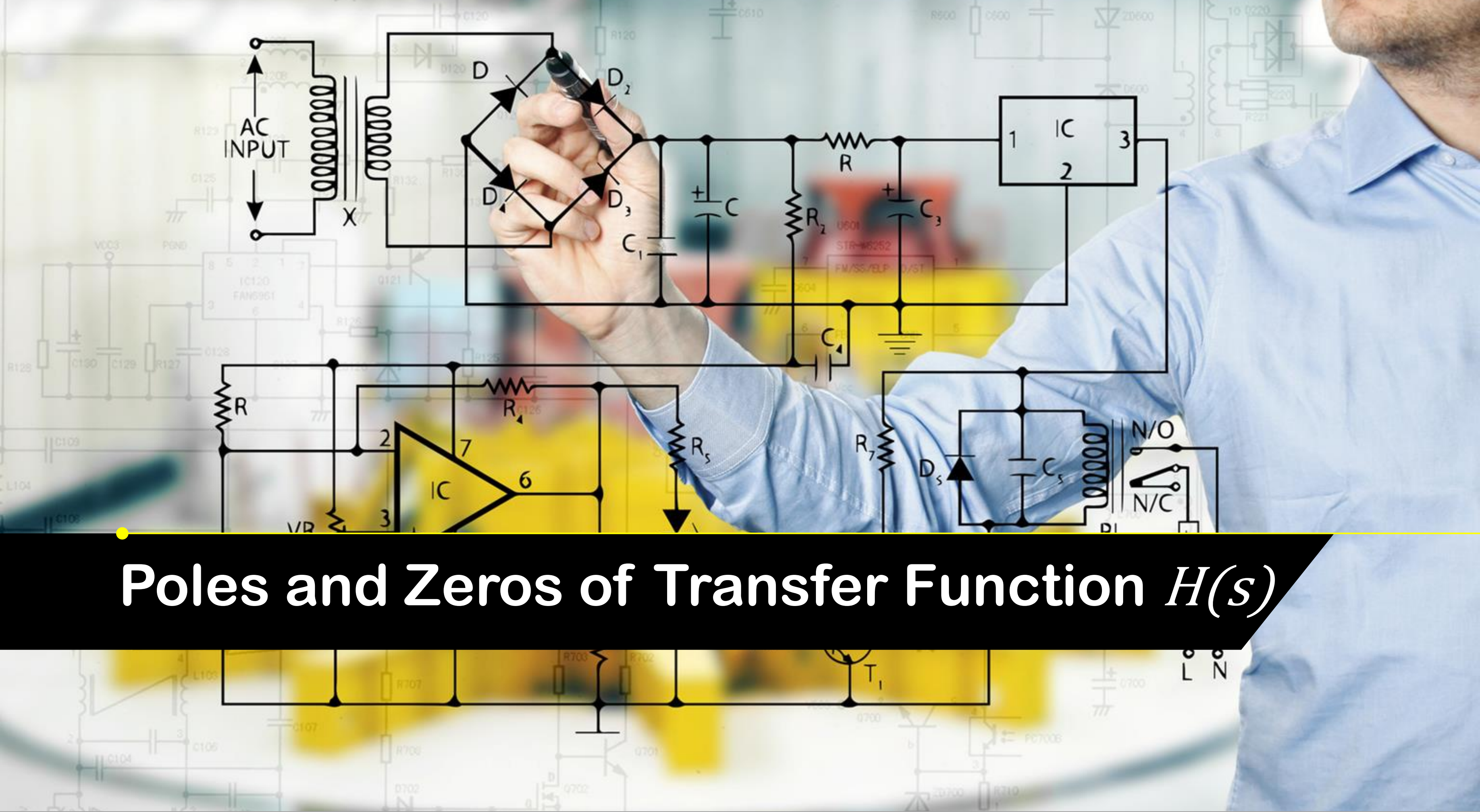


The impedance $Z_f(s)$ in the feedback path is

$$Z_f(s) = \frac{R_2\left(\frac{1}{sC_2}\right)}{R_2 + \frac{1}{sC_2}} = \frac{R_2}{sR_2C_2 + 1}$$

And the voltage transfer function is

$$\frac{V_{out}(s)}{V_{in}(s)} = -\frac{Z_f(s)}{Z_{in}(s)} = -\frac{sR_2C_1}{(sR_1C_1 + 1)(sR_2C_2 + 1)}$$



Poles and Zeros of Transfer Function $H(s)$

Poles and Zeros of Transfer Function $H(s)$

The transfer function of a circuit can be characterised in terms of its poles and zeros, and gain constant.

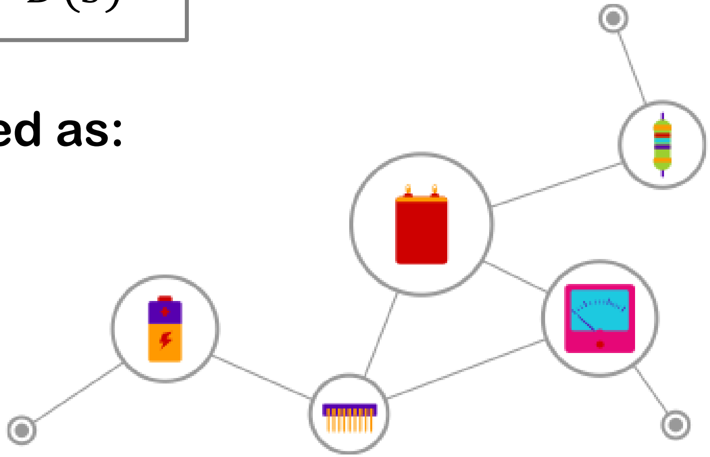
In general, the transfer function $H(s)$ is a ratio of polynomials in the variable s. We designate these two polynomials as $N(s)$ and $D(s)$ so that

$$H(s) = \frac{N(s)}{D(s)}$$

The polynomials $N(s)$ and $D(s)$ can be expressed as:

$$N(s) = a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0$$

$$D(s) = b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0$$



Poles and Zeros of Transfer Function $H(s)$

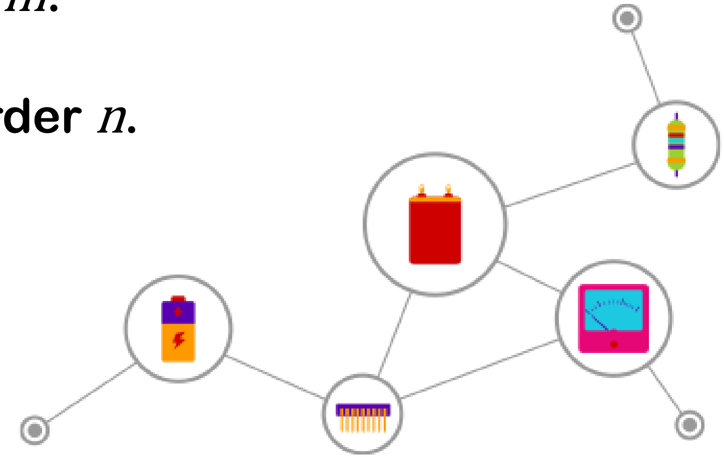
Polynomial ratios like this equation are referred to in general as rational functions, i.e., ratio of two polynomials.

$$H(s) = \frac{N(s)}{D(s)}$$

The order of a transfer function is the value of the larger of m and n.

Thus, if $m > n$, the transfer function is of order m.

Whereas, if $n > m$, the transfer function is of order n.



Poles and Zeros of Transfer Function $H(s)$: Example 1

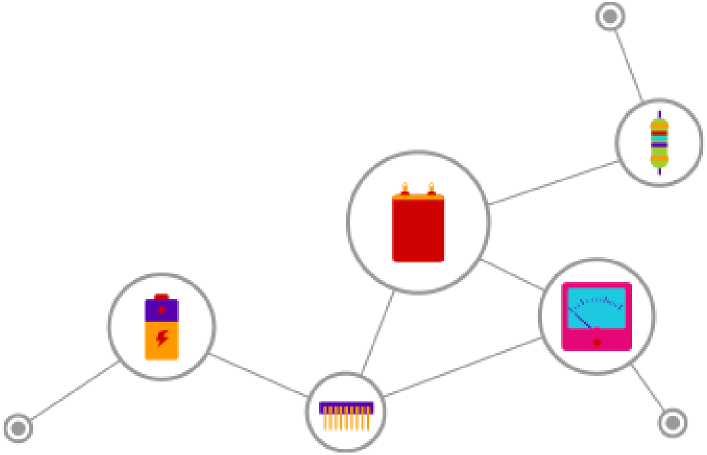
Consider $H(s) = \frac{N(s)}{D(s)} = \frac{2s+16}{s^2+16s+8}$

Since $n = 2 > m = 1$, the order of $H(s)$ is 2.

Each of the polynomial may be factored as

$$N(s) = (s + z_1)(s + z_2) \ldots (s + z_m)$$

$$D(s) = (s + p_1)(s + p_2) \ldots (s + p_n)$$



Poles and Zeros of Transfer Function $H(s)$: Example 1

$$N(s) = (s + z_1)(s + z_2) \dots (s + z_m)$$

$$D(s) = (s + p_1)(s + p_2) \dots (s + p_n)$$

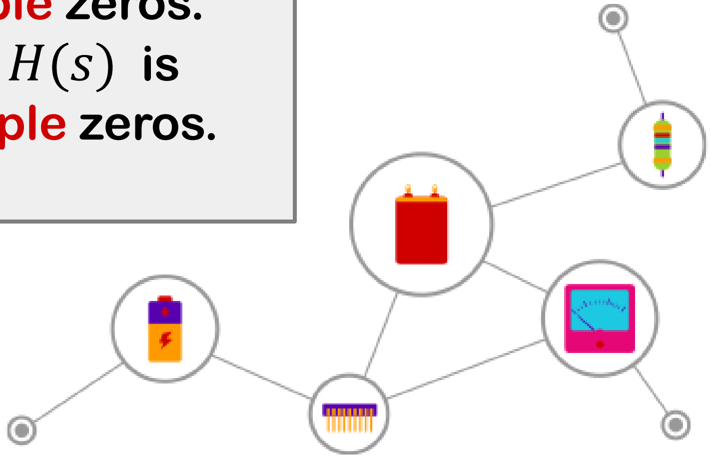
$-z_1, -z_2, \dots,$ are called the finite zeros of $H(s)$; they are the roots of $N(s)$, i.e., values of s at which $N(s)$ and $H(s)$ become zero.

$-p_1, -p_2, \dots,$ are called the finite poles of $H(s)$; they are the roots of $D(s)$, i.e., values of s at which $D(s)$ becomes zero and $H(s)$ becomes infinitely large.

Poles and Zeros of Transfer Function $H(s)$: Example 1

Remark: The denominator polynomial $D(s)$ is called characteristic polynomial. $D(s) = 0$ is called the characteristic equation. The roots of $D(s) = 0$ are the poles of $H(s)$.

If the poles or zeros are not repeated, then $H(s)$ is said to be having simple poles or simple zeros. If the poles or zeros are repeated, then $H(s)$ is said to be having multiple poles or multiple zeros.



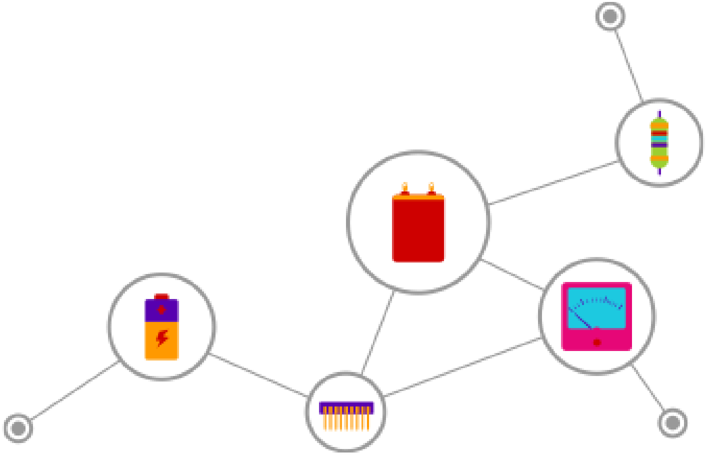
Poles and Zeros of Transfer Function $H(s)$: Example 2

Consider the transfer function

$$\boxed{H(s) = \frac{10(s+2)}{s(s+1)(s+3)^2}}$$

It has two poles at $s = -3$ and simple poles at $s = 0, -1$.

There is a finite simple zero at $s = -2$.



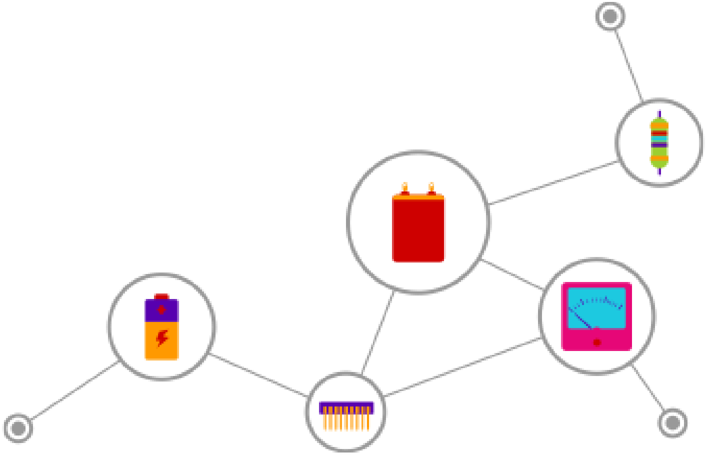
Poles and Zeros of Transfer Function $H(s)$: Example 3

Consider the transfer function

$$H(s) = \frac{2(s^2 + 6s + 25)}{s^3 + 7s^2 + 10s} = \frac{2(s + 3 - j4)(s + 3 + j4)}{s(s + 2)(s + 5)}$$

There are three finite poles at $s = 0, -2, -5$.

There are two finite zeros at $-3 \pm j4$.

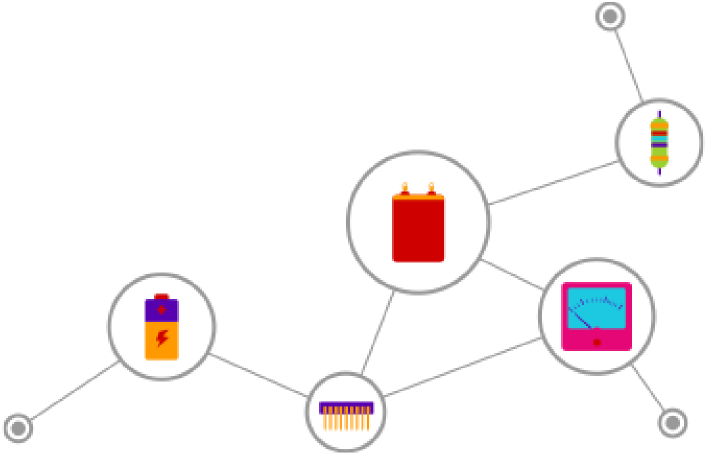


Poles and Zeros of Transfer Function $H(s)$: Example 4

Consider the transfer function

$$H(s) = \frac{10}{s + 10}$$

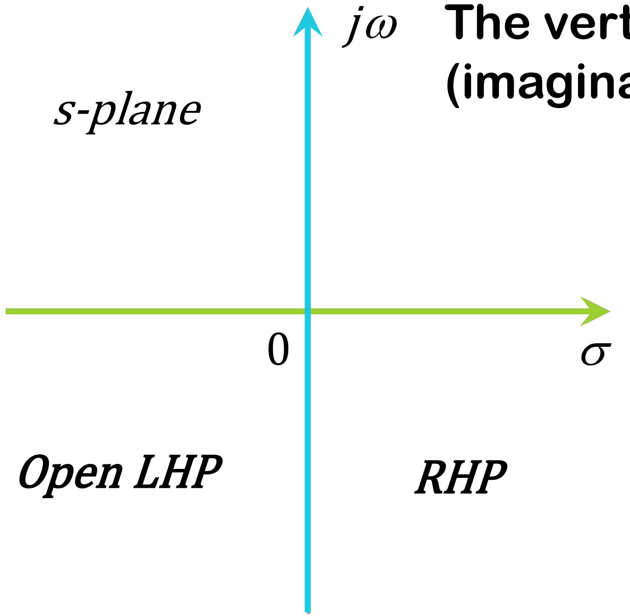
It has a simple pole at $s = -10$ **and there are no finite zeros.**



Poles and Zeros of Transfer Function $H(s)$: Pole-zero Plot

Since poles and zeros are, in general, complex numbers, they may be plotted on the **complex s-plane**, where s is represented as a complex frequency of the form $s = \sigma + j\omega$.

In these **pole-zero** plots, a zero is denoted by ‘o’ and a pole is denoted by ‘x’.



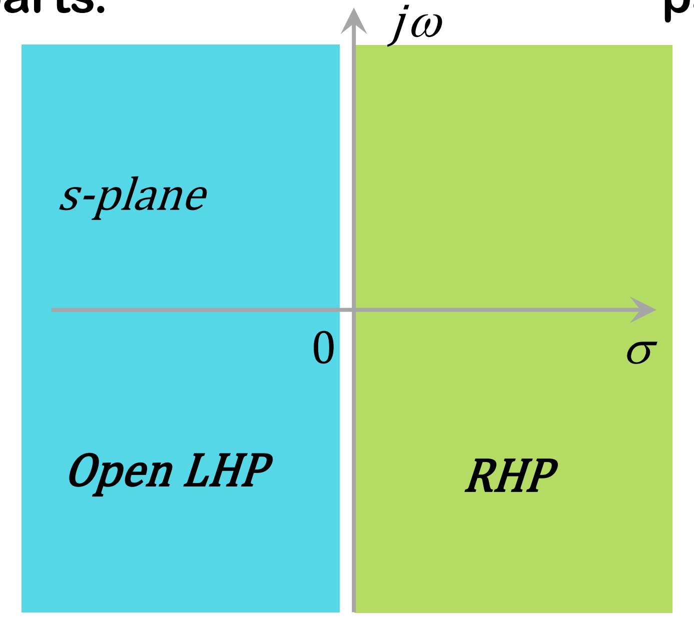
The vertical axis is $j\omega$-axis (imaginary axis)

The horizontal axis is the σ-axis (real axis)

Poles and Zeros of Transfer Function $H(s)$: Pole-zero Plot

The **open left-half plane** (excluding the $j\omega$ axis) contains poles and zeros with strictly negative real parts.

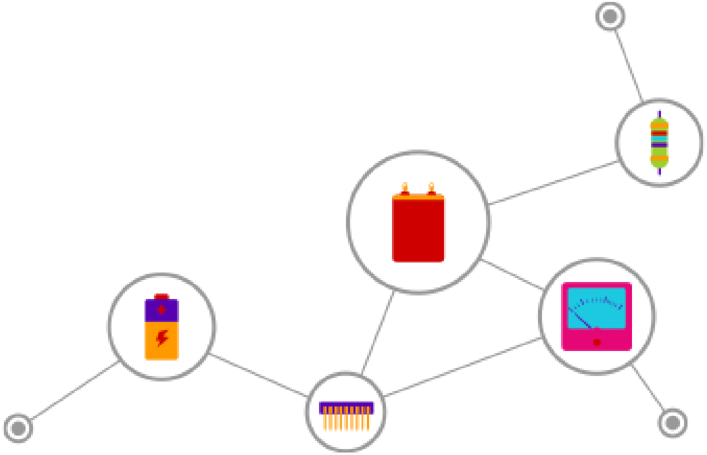
The **right-half plane** (including the $j\omega$ axis) contain poles and zeros with non-negative real parts.



Poles and Zeros of Transfer Function $H(s)$: Example 5

Show the pole-zero plot of

$$H(s) = \frac{s + 0.5}{(s + 2)(s + 2.5)^2(s^2 + 2s + 2)}$$



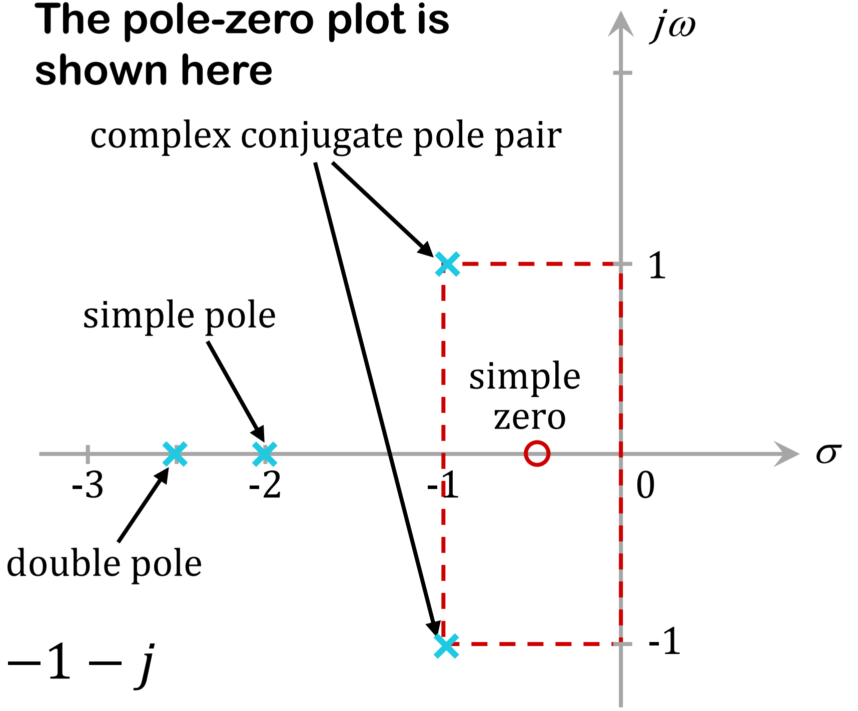
Poles and Zeros of Transfer Function $H(s)$: Example 5

$H(s)$ **can be written as**

$$H(s) = \frac{s+0.5}{(s+2)(s+2.5)^2(s^2+2s+2)}$$

$$H(s) = \frac{s+0.5}{(s+2)(s+2.5)^2(s+1+j)(s+1-j)}$$

The pole-zero plot is shown here



There are simple poles at $s = -2, -1+j, -1-j$ **and a double pole at** $s = -2.5$.

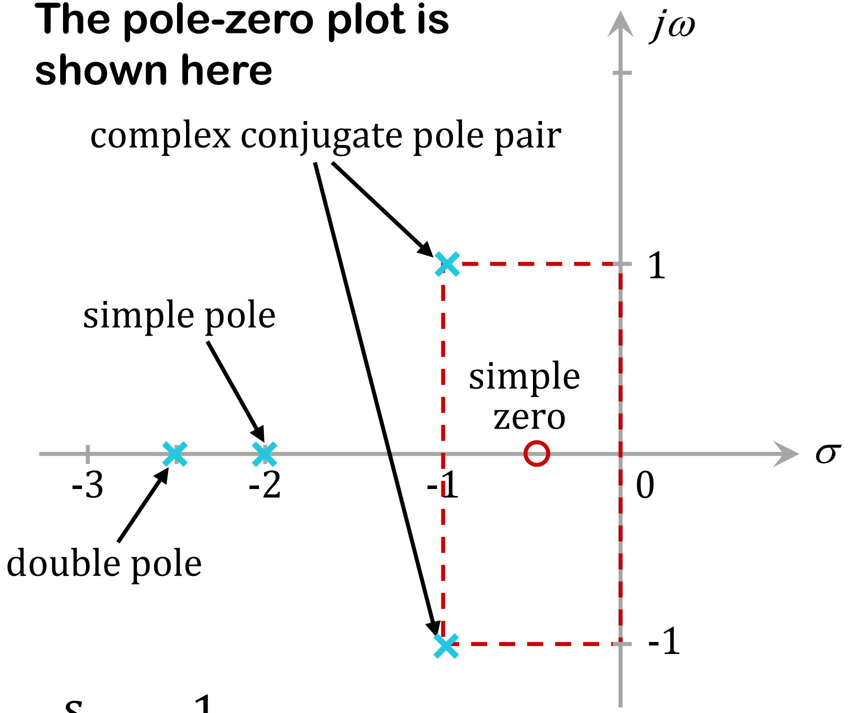
There is a finite simple zero at $s = -0.5$.

Poles and Zeros of Transfer Function $H(s)$: Example 5

Again, consider

$$H(s) = \frac{s + 0.5}{(s + 2)(s + 2.5)^2(s^2 + 2s + 2)}$$

The pole-zero plot is shown here



Note that the poles and zeros of $H(s)$ are located in the finite s plane. $H(s)$ can also have zeros at infinity.

For very large values of s, $H(s) \approx \frac{s}{s.s^2s^2} = \frac{s}{s^5} = \frac{1}{s^4}$.

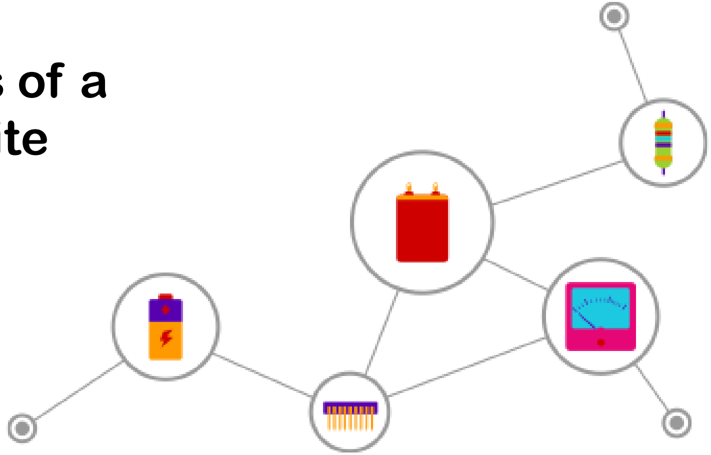
Poles and Zeros of Transfer Function $H(s)$: Example 5

Now, $H(s) = 0$ when $s = \infty$.

This means that $H(s)$ has four zeros at infinity (as zeros are the values of s at which $H(s)$ becomes zero).

However, in this course, we are interested in the poles and zeros located in the finite s plane.

Therefore, when we refer to the poles and zeros of a rational function of s, we are referring to the finite poles and zeros.



Poles and Zeros of Transfer Function $H(s)$: Partial Fraction Expansion

We want to find the inverse transform of a function that has the form (where $n > m$)

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}$$

1. Simple Poles of $F(s)$: Real and Distinct

$$F(s) = \frac{K(s+z_1)(s+z_2)}{s(s+p_1)(s+p_2)} = \frac{K_1}{s} + \frac{K_2}{s+p_1} + \frac{K_3}{s+p_2}$$

Using the transform pairs

$$1 \Leftrightarrow \frac{1}{s}$$

$$e^{-at} \Leftrightarrow \frac{1}{s+a}$$

$$\mathcal{L}^{-1}\{F(s)\} = (K_1 + K_2 e^{-p_1 t} + K_3 e^{-p_2 t}) u(t)$$

Poles and Zeros of Transfer Function $H(s)$: Partial Fraction Expansion

We want to find the inverse transform of a function that has the form (where $n > m$)

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}$$

2. Repeated Real Roots

$$F(s) = \frac{(a_2 s^2 + a_2 s + a_1)}{s(s+p)^2} = \frac{K_1}{s} + \frac{K_2}{(s+p)^2} + \frac{K_3}{s+p}$$

Using

$$1 \Leftrightarrow \frac{1}{s} \qquad e^{-at} \Leftrightarrow \frac{1}{s+a} \qquad te^{-at} \Leftrightarrow \frac{1}{(s+a)^2}$$

$$\mathcal{L}^{-1}\{F(s)\} = (K_1 + K_2 t e^{-pt} + K_3 e^{-pt}) u(t)$$

Poles and Zeros of Transfer Function $H(s)$: Partial Fraction Expansion

3. Distinct Complex Roots

$$F(s) = \frac{K(s+z)}{(s+p)(s+\sigma-j\omega)(s+\sigma+j\omega)}$$

Rewrite

$$F(s) = \frac{K_1}{s+p} + \frac{K_2 s + K_3}{(s+\sigma)^2+\omega^2} = \frac{K_1}{s+p} + \frac{K_2(s+\sigma)}{(s+\sigma)^2+\omega^2} + \frac{K_3-\sigma K_2}{(s+\sigma)^2+\omega^2}$$

Poles and Zeros of Transfer Function $H(s)$: Partial Fraction Expansion

Using

$$e^{-at} cos\omega t \Leftrightarrow \frac{s+a}{(s+a)^2+\omega^2}$$

$$e^{-at} sin\omega t \Leftrightarrow \frac{\omega}{(s+a)^2+\omega^2}$$

$$\mathcal{L}^{-1}\{F(s)\} = K_1 e^{-pt} + K_2\, e^{-\sigma t}\cos\omega t + K_4\, e^{-\sigma t}\sin\omega t$$

$$= K_1 e^{-pt} + e^{-\sigma t}(K_2 \cos\omega t + K_4 \sin\omega t)$$

$$= K_1 e^{-pt} + K_5 e^{-\sigma t}\sin(\omega t + \theta)$$

Note

$$A_1 \sin\omega t + A_2 \cos\omega t = A sin(\omega t + \theta)$$

$$A = \sqrt{A_1^2 + A_2^2}$$

$$\theta = \tan^{-1}\frac{A_2}{A_1}$$

Time Domain Response from Pole-zero Plot

Time Response from Pole-zero Plot

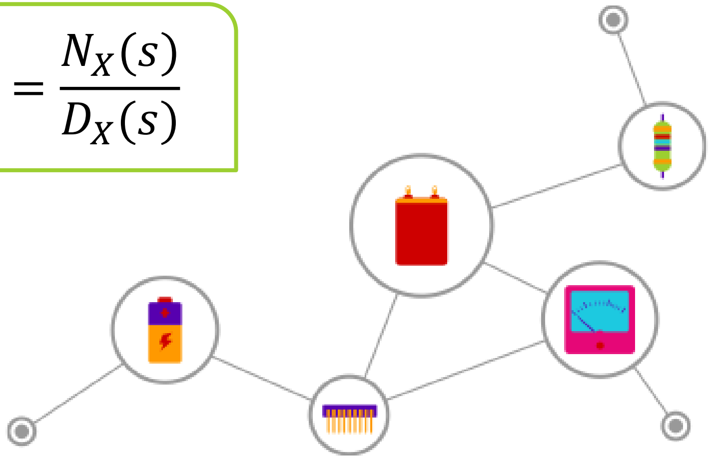
Using the knowledge of the pole locations of a transfer function, we can categorise the kinds of responses that are due to different kinds of terms in the partial fraction expansion of the transfer function.

Consider the input-output relationship expressed in terms of the transfer function

$$Y(s) = H(s)X(s)$$

$$H(s) = \frac{N_H(s)}{D_H(s)} \qquad X(s) = \frac{N_X(s)}{D_X(s)}$$

Thus, the poles of $Y(s)$ **come from either the poles of** $H(s)$ **or the poles of** $X(s)$.



Time Response from Pole-zero Plot

If $Y(s)$ has only simple poles, then

$$Y(s) = \sum_{i=1}^{i=n} \frac{K_i}{s+p_i} + \sum_{j=1}^{j=l} \frac{K_j}{s+p_j}$$

where the $-p_i$'s are the poles of $H(s)$ and the $-p_j$'s are the poles of $X(s)$.

Thus,

$$y(t) = \Big(\underbrace{\sum_{i=1}^{i=n} K_i e^{-p_i t}}_{\text{Natural Response}} + \underbrace{\sum_{j=1}^{j=l} K_j e^{-p_j t}}_{\text{Forced Response}}\Big) u(t)$$

Time Response from Pole-zero Plot

The response $y(t)$ is the sum of the forced response and the natural response.

$$y(t) = \Big(\underbrace{\sum_{i=1}^{i=n} K_i e^{-p_i t}}_{\text{Natural Response}} + \underbrace{\sum_{j=1}^{j=l} K_j e^{-p_j t}}_{\text{Forced Response}}\Big) u(t)$$

The **natural response** is produced by the **poles** of $H(s)$ **(natural poles)**.

The **forced response** is produced by the **poles** of $X(s)$ **(forced poles)**.

Thus, the time domain response of a given network may be assessed from the pole and zero plot of a given network function and from the knowledge of the network sources.

Time Response from Pole-zero Plot: Example 1

Consider the transfer function $H(s) = \dfrac{V_2(s)}{V_1(s)} = \dfrac{10}{(s+1)(s+10)}$.

This gives

$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{10}{(s+1)(s+10)}$$

$$V_2(s) = \frac{10}{(s+1)(s+10)} V_1(s)$$

Suppose the input is $v_1(t) = u(t)$, **so that** $V_1(s) = 1/s$.

Then, the output response is given by

$$V_2(s) = \frac{H(s)}{s} = \frac{10}{s(s+1)(s+10)}$$

$$= \frac{1}{s} - \frac{10}{9}\frac{1}{(s+1)} + \frac{1}{9}\frac{1}{(s+10)}$$

Time Response from Pole-zero Plot: Example 1

$$V_2(s) = \frac{1}{s} - \frac{10}{9}\frac{1}{(s+1)} + \frac{1}{9}\frac{1}{(s+10)}$$

Forced Pole

Natural Poles

$$v_2(t) = u(t) - (\frac{10}{9}e^{-t} - \frac{1}{9}e^{-10t})u(t)$$

Forced Response

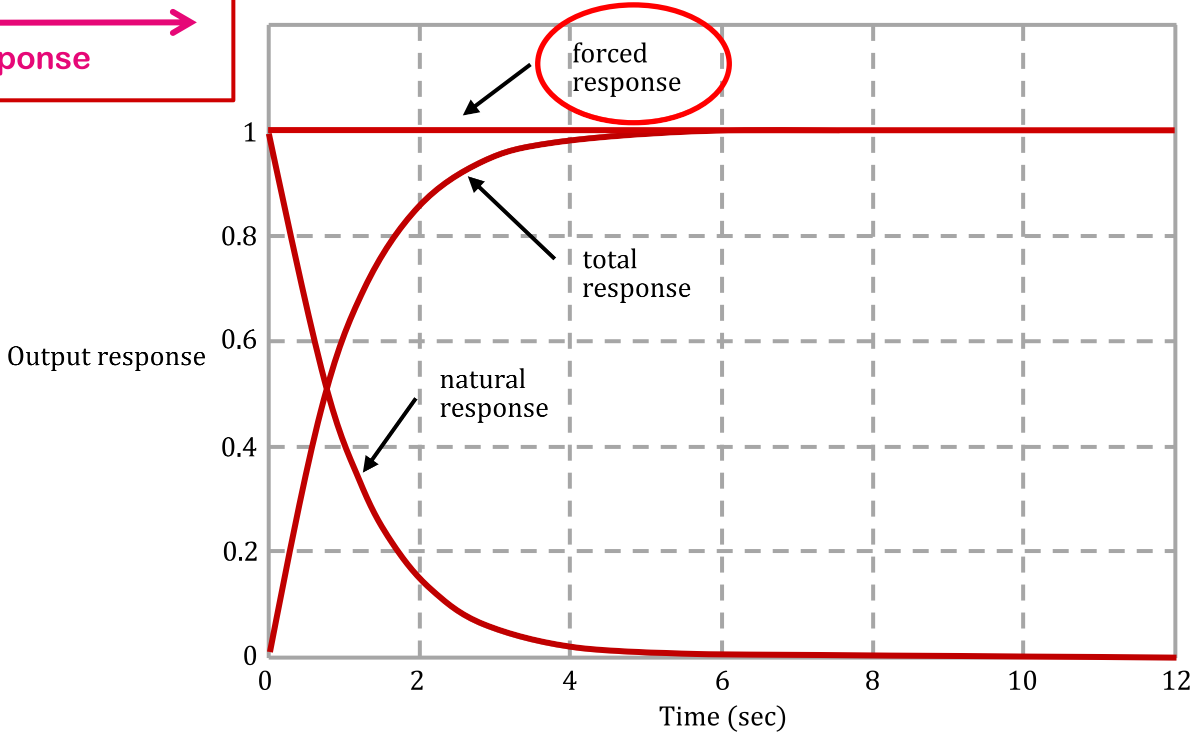
Natural Response

Time Response from Pole-zero Plot: Example 1

The total response is shown below.

$$v_2(t) = u(t) - \left(\frac{10}{9}e^{-t} - \frac{1}{9}e^{-10t}\right)u(t)$$

Forced Response — Natural Response



Time Response from Pole-zero Plot: Example 2

Consider the transfer function $H(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{s+1}$.

Suppose the input is $v_1(t) = 10\cos 2t$, so that $V_1(s) = \frac{10s}{s^2+4}$ $\left(\cos\omega t \Leftrightarrow \frac{s}{s^2+\omega^2}\right)$.

Then, the output response is given by

$$V_2(s) = \frac{10s}{(s+1)(s^2+4)}$$

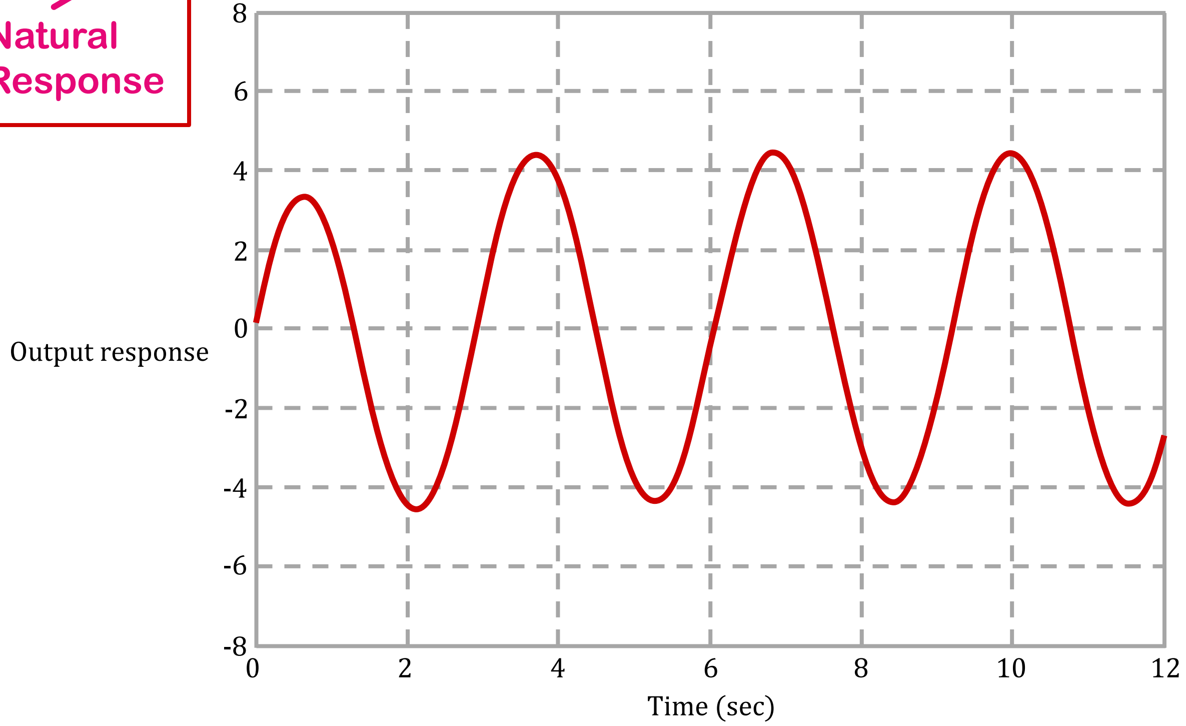
$$V_2(s) = \underbrace{\frac{2s+8}{s^2+4}}_{\text{Forced Poles}} - \underbrace{\frac{2}{(s+1)}}_{\text{Natural Pole}}$$

$$v_2(t) = (\underbrace{2\cos 2t + 4sin2t}_{\text{Forced Response}} \underbrace{- 2e^{-t}}_{\text{Natural Response}})u(t)$$

Time Response from Pole-zero Plot: Example 2

The total response is shown below.

$$v_2(t) = (\underbrace{2\cos 2t + 4\sin 2t}_{\text{Forced Response}} \underbrace{- 2e^{-t}}_{\text{Natural Response}})u(t)$$



Time Response from Pole-zero Plot: Example 3

Consider the transfer function

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{(s+1)^2+1}{(s+1)(s+2)(s+3)}.$$

Time Response from Pole-zero Plot: Example 3

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{(s+1)^2+1}{(s+1)(s+2)(s+3)}$$

Let the input be $v_{in}(t) = e^{-t}\text{sint}.u(t)$,
so that $V_{in}(s) = \frac{1}{(s+1)^2+1}$
(since $e^{-at}\sin\omega t \Leftrightarrow \frac{\omega}{(s+a)^2+\omega^2}$).

The output is given by

$$V_{out}(s) = H(s)V_{in}(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

$$V_{out}(s) = \underbrace{\frac{0.5}{s+1} - \frac{1}{s+2} + \frac{0.5}{s+3}}_{\text{Natural Poles}}$$

$$v_{out}(t) = \underbrace{(0.5e^{-t} - e^{-2t} + 0.5e^{-3t})}_{\text{Natural Response}}u(t)$$

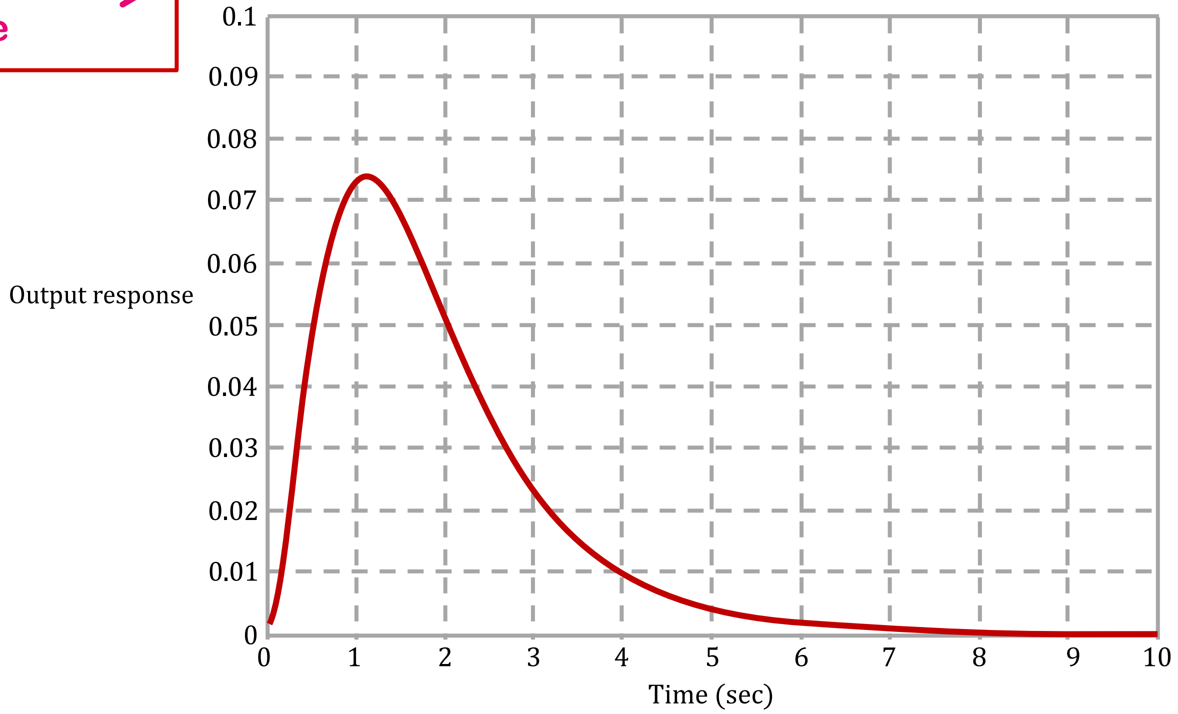
All the poles are natural poles and response is only the natural response. Poles from input are cancelled by zeros of $H(s)$.

Time Response from Pole-zero Plot: Example 3

The response is shown below.

$$v_{out}(t) = (0.5e^{-t} - e^{-2t} + 0.5e^{-3t})u(t)$$

Natural Response

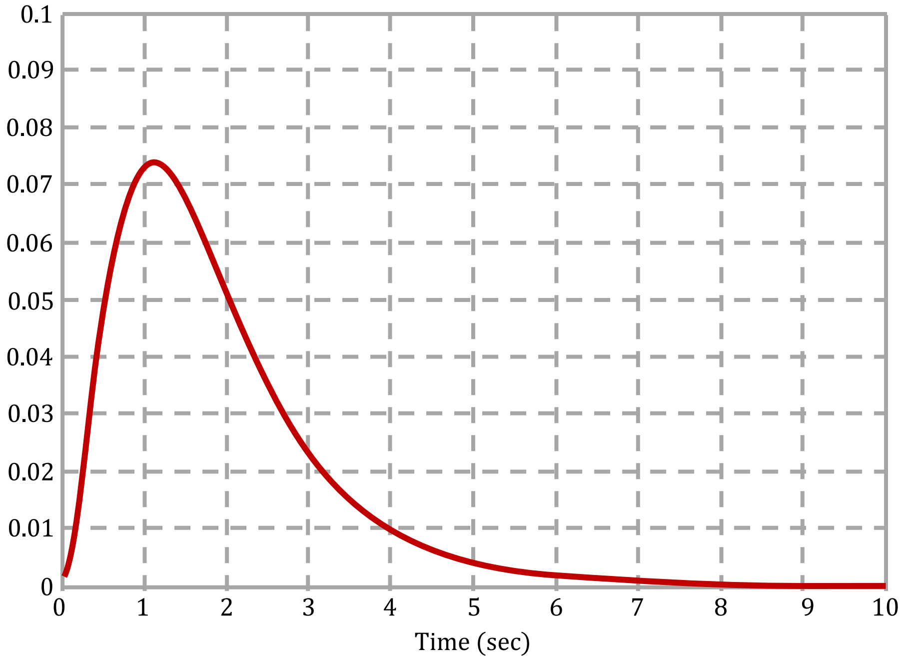


Time Response from Pole-zero Plot: Example 3

Note that the response does not have any term similar to the input signal. This follows because the input poles ($s = -1 \pm j$) coincide with the zeros of $H(s)$.

The effect of the input frequencies (poles) is cancelled out by the zeros of $H(s)$, and such frequencies are absent from the circuit response.

$$v_{out}(t) = \underbrace{(0.5e^{-t} - e^{-2t} + 0.5e^{-3t})u(t)}_{\text{Natural Response}}$$



Time Response from Pole-zero Plot: Time Domain Output Responses

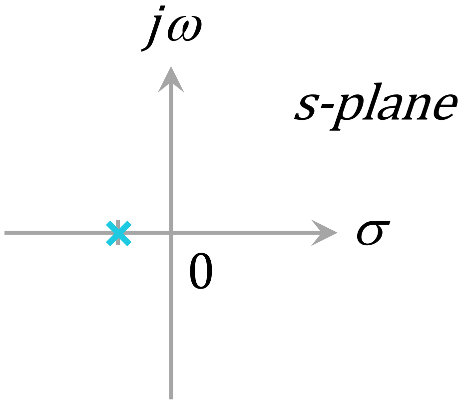
In what follows, we describe the **output response** of a network given its **pole locations**.

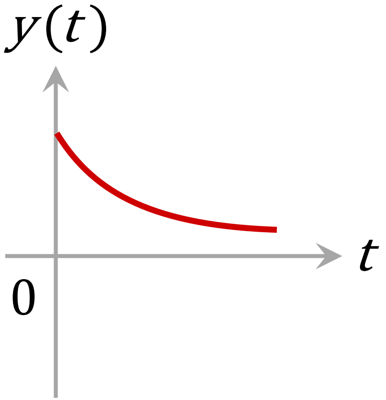
Case One

$$Y(s) = \frac{1}{s+p}$$

The pole is real and simple. We have

$$y(t) = \mathcal{L}^{-1}\{\frac{1}{s+p}\} = e^{-pt}u(t) .$$

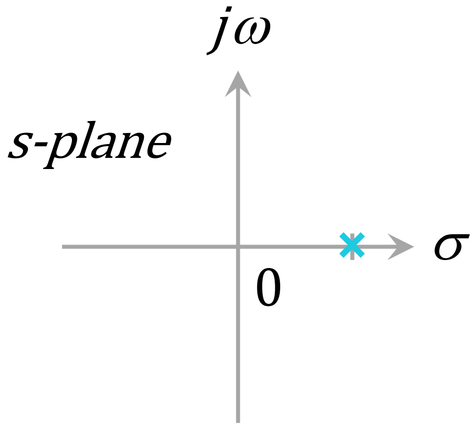


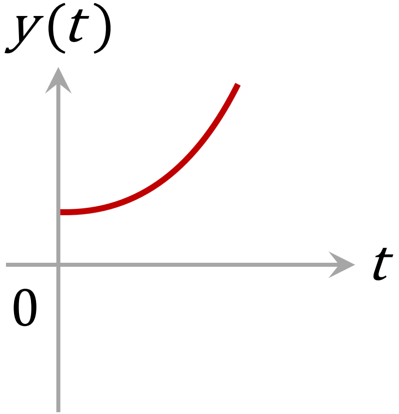


Pole is in the open LHP ($p > 0$). The response $y(t)$ is an **exponential decay**.

Time Response from Pole-zero Plot: Time Domain Output Responses

Case One

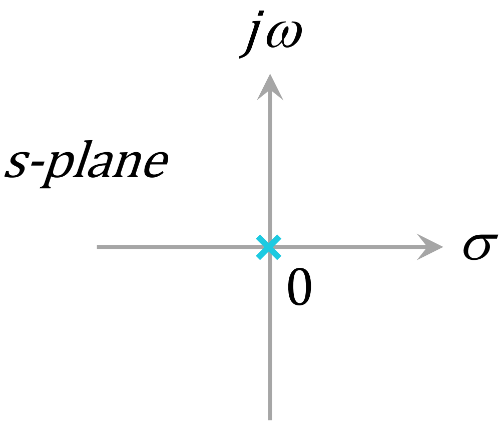


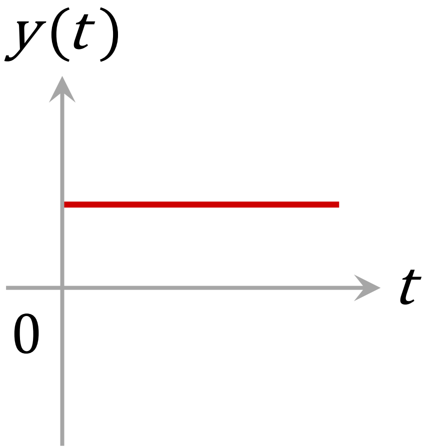


Pole in the RHP ($p < 0$). The response $y(t)$ is an exponential rise.

Time Response from Pole-zero Plot: Time Domain Output Responses

Case One





Pole at the origin $s = 0$ $(p = 0)$. The response $y(t)$ is a constant.

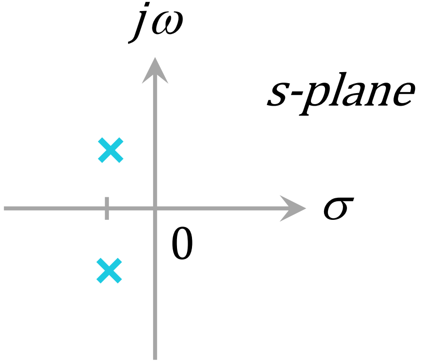
Time Response from Pole-zero Plot: Time Domain Output Responses

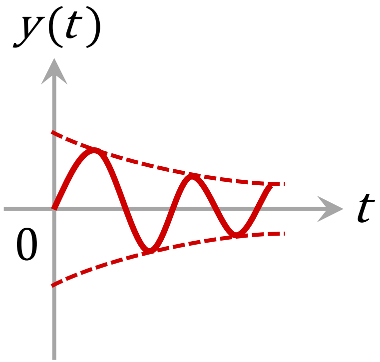
Case Two

$$Y(s) = \frac{\omega}{(s+\sigma+j\omega)(s+\sigma-j\omega)}$$

There is a pair of simple complex-conjugate poles at $s_{1,2} = -\sigma \pm j\omega$.

Thus, $y(t) = \mathcal{L}^{-1}\{\frac{\omega}{(s+\sigma)^2+\omega^2}\} = e^{-\sigma t}\sin\omega t . u(t)$

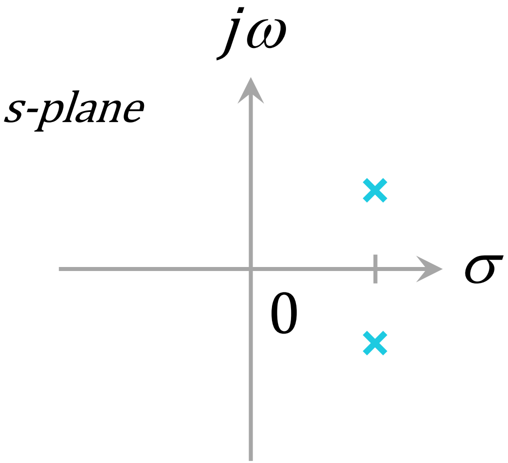


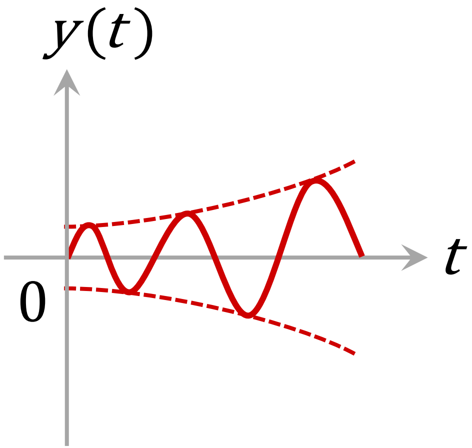


Poles are in the open LHP $(\sigma > 0)$. $y(t)$ **is an exponentially decaying sinusoid.**

Time Response from Pole-zero Plot: Time Domain Output Responses

Case Two

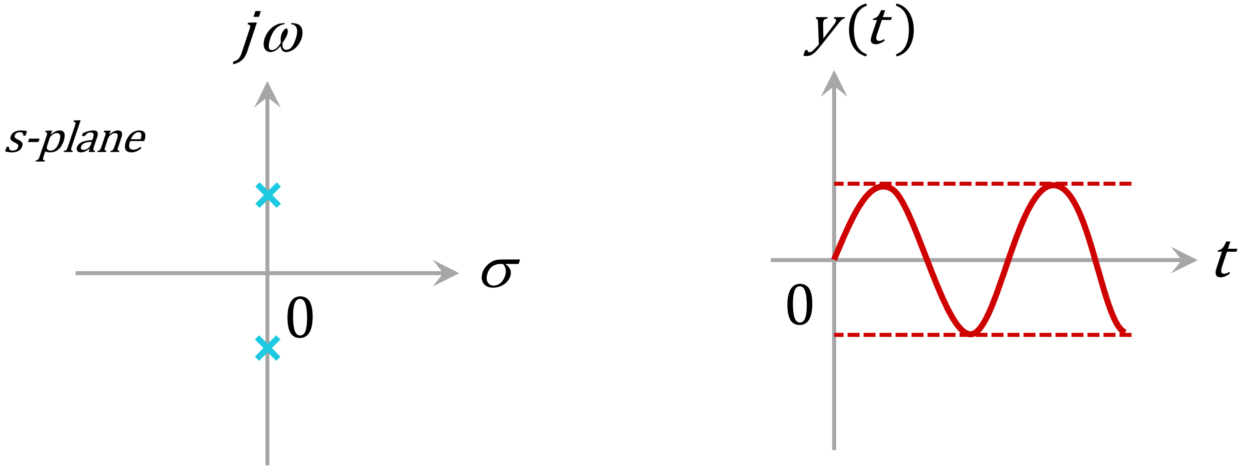




Poles are in the RHP $(\sigma < 0)$. $y(t)$ is an exponentially growing sinusoid.

Time Response from Pole-zero Plot: Time Domain Output Responses

Case Two



Poles are on the imaginary axis $(\sigma = 0)$. The response is oscillatory.

Time Response from Pole-zero Plot: Time Domain Output Responses

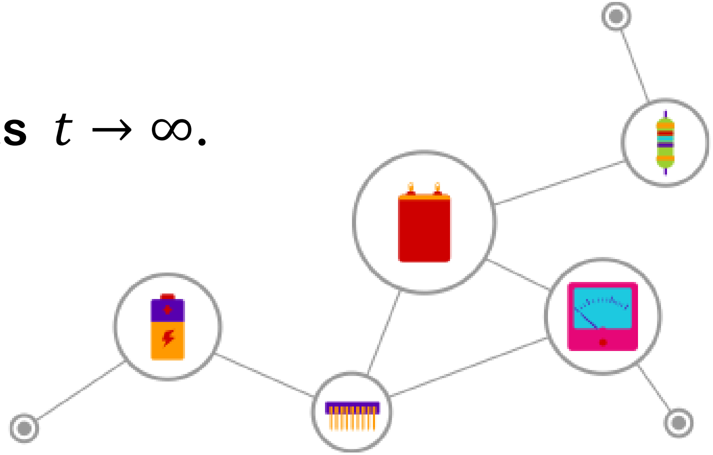
Case Three

If the transfer function has repeated poles, then the response grows exponentially with time if the poles are in the RHP.

For $(s) = \frac{1}{(s+p)^2}$ **, the response is** $y(t) = te^{-pt}u(t)$.

If $p < 0$**, then** $y(t) = te^{|p|t}u(t) \rightarrow \infty$ **as** $t \rightarrow \infty$.

However, if $p > 0$**, then** $y(t) = te^{-pt}u(t) \rightarrow 0$ **as** $t \rightarrow \infty$.



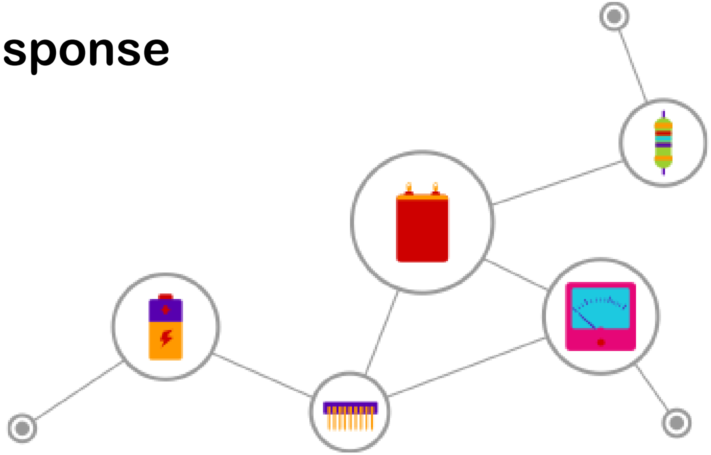
Time Response from Pole-zero Plot: Time Domain Output Responses

Case Four

For any arbitrary pattern of poles, we may classify them according to the three cases discussed.

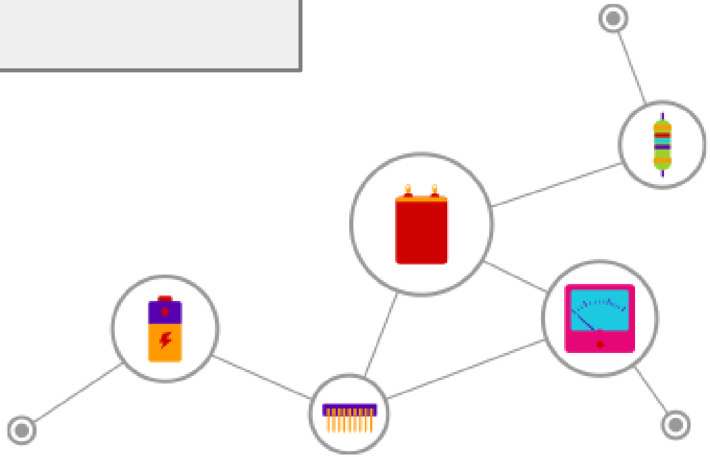
The total response corresponding to these poles may be simply found by adding each of the individual factors.

For $(s) = \frac{K_1}{s} + \frac{K_2}{s+p} + \frac{K_3}{(s+\sigma)^2+\omega^2}$ **, then the total response will be** $y(t) = (K_1 + K_2 e^{-pt} + K_4 e^{-\sigma t} \sin\omega t) u(t)$**.**



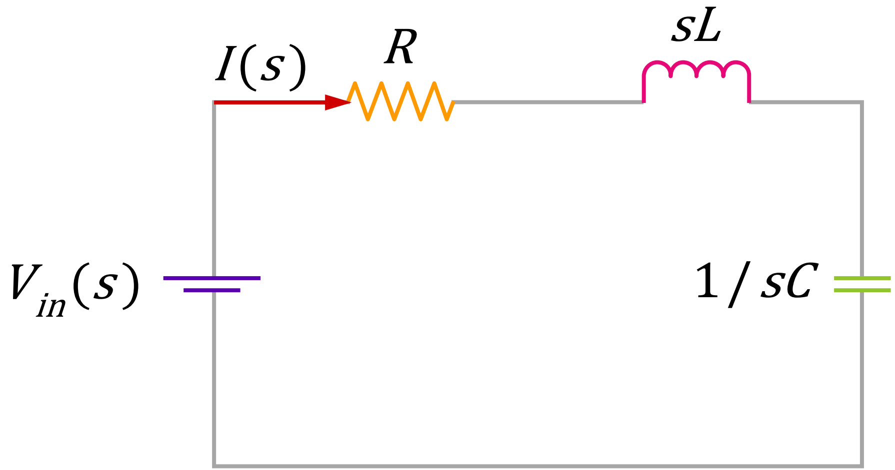
Time Response from Pole-zero Plot: Time Domain Output Responses

Looking at the four cases in the previous slides, it is clear that for $y(t)$ not to grow exponentially with time, the poles of $Y(s)$ must lie in open LHP or (simple) on the $j\omega$ axis.



Time Response from Pole-zero Plot: Example 4

Consider the current response of the series RLC circuit excited by a voltage source.



Time Response from Pole-zero Plot: Example 4

The current response is given by

$$I(s) = \frac{1}{L}\frac{s}{(s^2+\frac{R}{L}s+\frac{1}{LC})}V_{in}(s)$$

If $V_{in}(s) = \frac{V}{s}$

Then

$$I(s) = \frac{V}{L}\frac{1}{(s^2+a_1 s + a_0)}$$

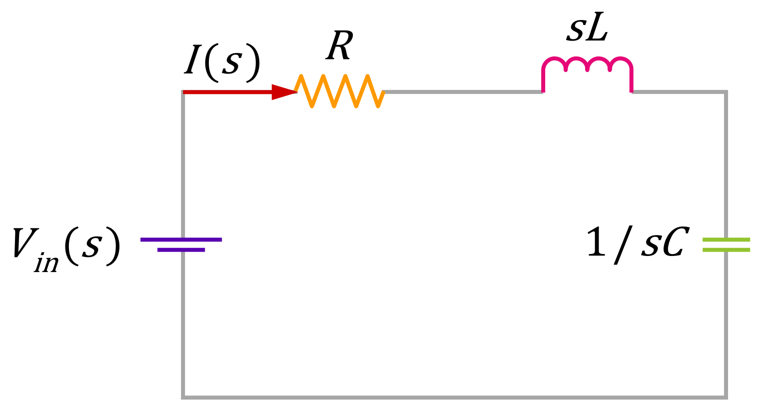
Where

$$a_1 = \frac{R}{L}$$

$$a_0 = \frac{1}{LC}$$

The poles of $I(s)$ are

$$s_{1,2} = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - a_0}$$



Time Response from Pole-zero Plot: Example 4

There are three possible forms for the roots:

1. $$\boxed{\frac{a_1^2}{4} - a_0 = 0}$$ $$\boxed{s_{1,2} = -\alpha}$$

2. $$\boxed{\frac{a_1^2}{4} - a_0 > 0}$$ $$\boxed{s_{1,2} = -\alpha \pm \beta}$$

Note that the roots all lie in the open LHP. The corresponding responses are:

1. **Critically damped** – roots are real and equal

$$\boxed{i(t) = \frac{V}{L} t e^{-\alpha t} u(t)}$$

2. **Overdamped** – roots are real and unequal

$$\boxed{i(t) = \frac{V}{\sqrt{R^2 - \frac{4L}{C}}} \left(e^{s_1 t} - e^{s_2 t}\right) u(t)}$$

Time Response from Pole-zero Plot: Example 4

There are three possible forms for the roots:

3. $$\frac{a_1^2}{4} - a_0 < 0$$ $$s_{1,2} = -\alpha \pm j\beta$$

Note that the roots all lie in the open LHP. The corresponding responses are:

3. **Underdamped** – roots are complex conjugate

$$i(t) = \frac{V}{\sqrt{\frac{L}{C} - \frac{R^2}{4}}} e^{-\alpha t} \sin\beta t . u(t)$$

Time Response from Pole-zero Plot: Example 4

1. **Critically damped**

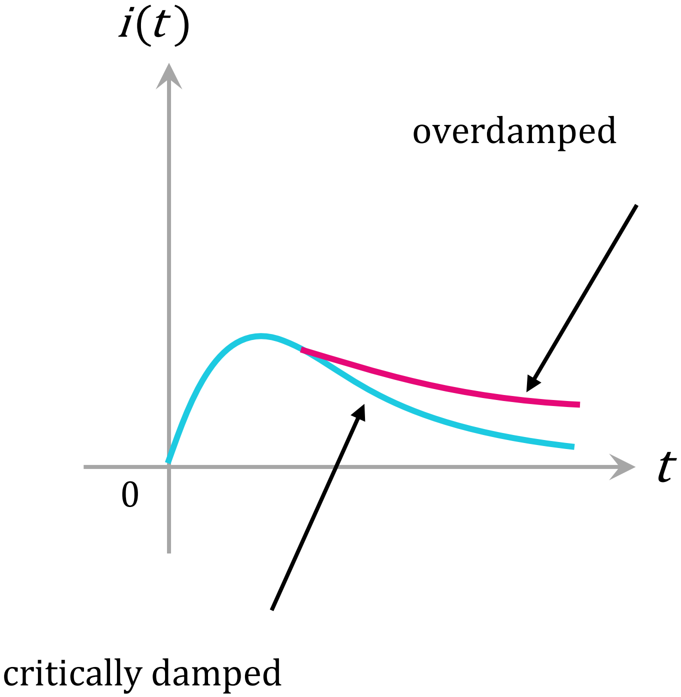
$$i(t) = \frac{V}{L} t e^{-\alpha t} u(t)$$

2. **Overdamped**

$$i(t) = \frac{V}{\sqrt{R^2 - \frac{4L}{C}}} \left(e^{s_1 t} - e^{s_2 t}\right) u(t)$$

3. **Underdamped**

$$i(t) = \frac{V}{\sqrt{\frac{L}{C} - \frac{R^2}{4}}} e^{-\alpha t} \sin\beta t . u(t)$$



Time Response from Pole-zero Plot: Example 4

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$$i(t) = \frac{V}{L} t e^{-\alpha t} u(t)$$

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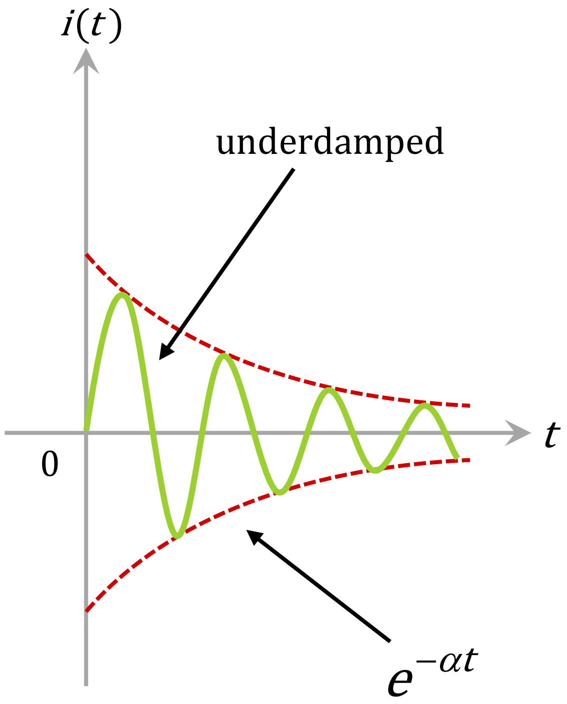


Poles and Stability

Poles and Stability

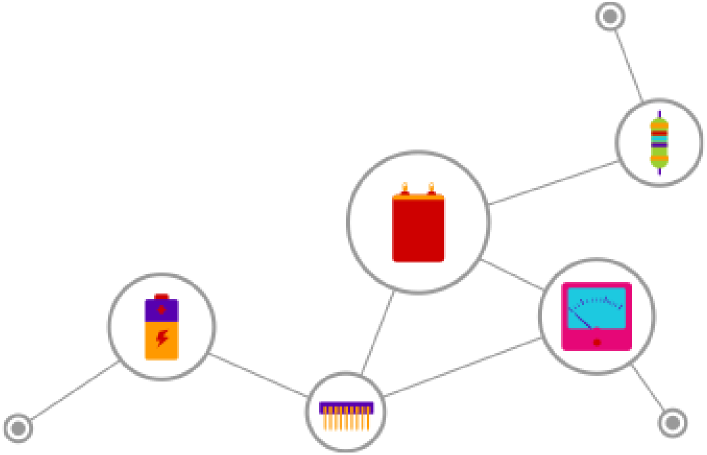
Knowledge of the locations of the poles of the response $Y(s) = H(s)X(s)$ identifies the behavior of the circuit with time.

Thus, the very important circuit behavior called **stability** is of interest to us.

Consider the transfer function $H(s) = \frac{Y(s)}{X(s)}$.

Then the response is $Y(s) = H(s)X(s)$.

The response will consist of two parts: A wave-form similar to the input $x(t)$ plus a natural waveform characteristic of the network itself. The natural response of the network appears in every type of input.



Poles and Stability

The terms in a partial fraction expansion of the response establish the types of behavior present in the response.

Each term has only one of several possible forms. The very common terms are $(j > 1)$, where p_i is real.

$$\boxed{\frac{K}{s}} \quad \boxed{\frac{K}{s^j}} \quad \boxed{\frac{K}{(s + p_i)}} \quad \boxed{\frac{K}{(s + p_i)^j}} \quad \boxed{\frac{As + B}{(s + \sigma)^2 + \omega^2}}$$

The corresponding time response of each term is given by (for $t > 0$):

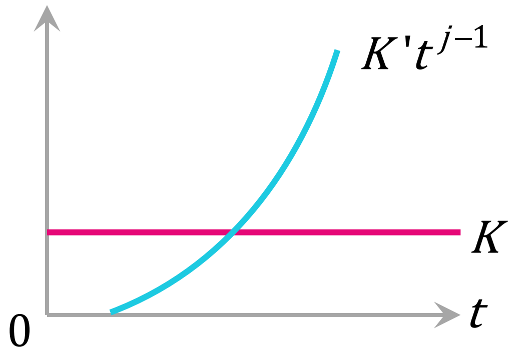
$$\boxed{K} \quad \boxed{K' t^{j-1}} \quad \boxed{K e^{-p_i t}} \quad \boxed{K' t^{j-1} e^{-p_i t}} \quad \boxed{K e^{-\sigma t} \sin(\omega t + \theta)}$$

Poles and Stability

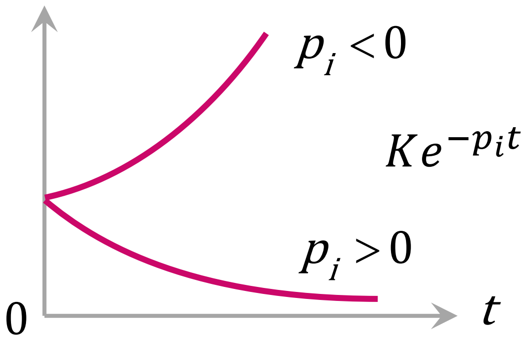
The response types are shown as follows:

$\boxed{K}$

$\boxed{K' t^{j-1}}$



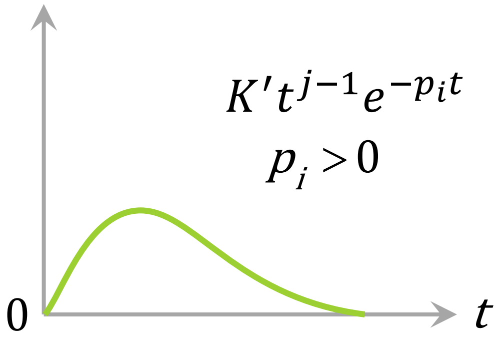
$\boxed{K e^{-p_i t}}$



Poles and Stability

The response types are shown as follows:

$$\boxed{K' t^{j-1} e^{-p_i t}}$$



Poles and Stability

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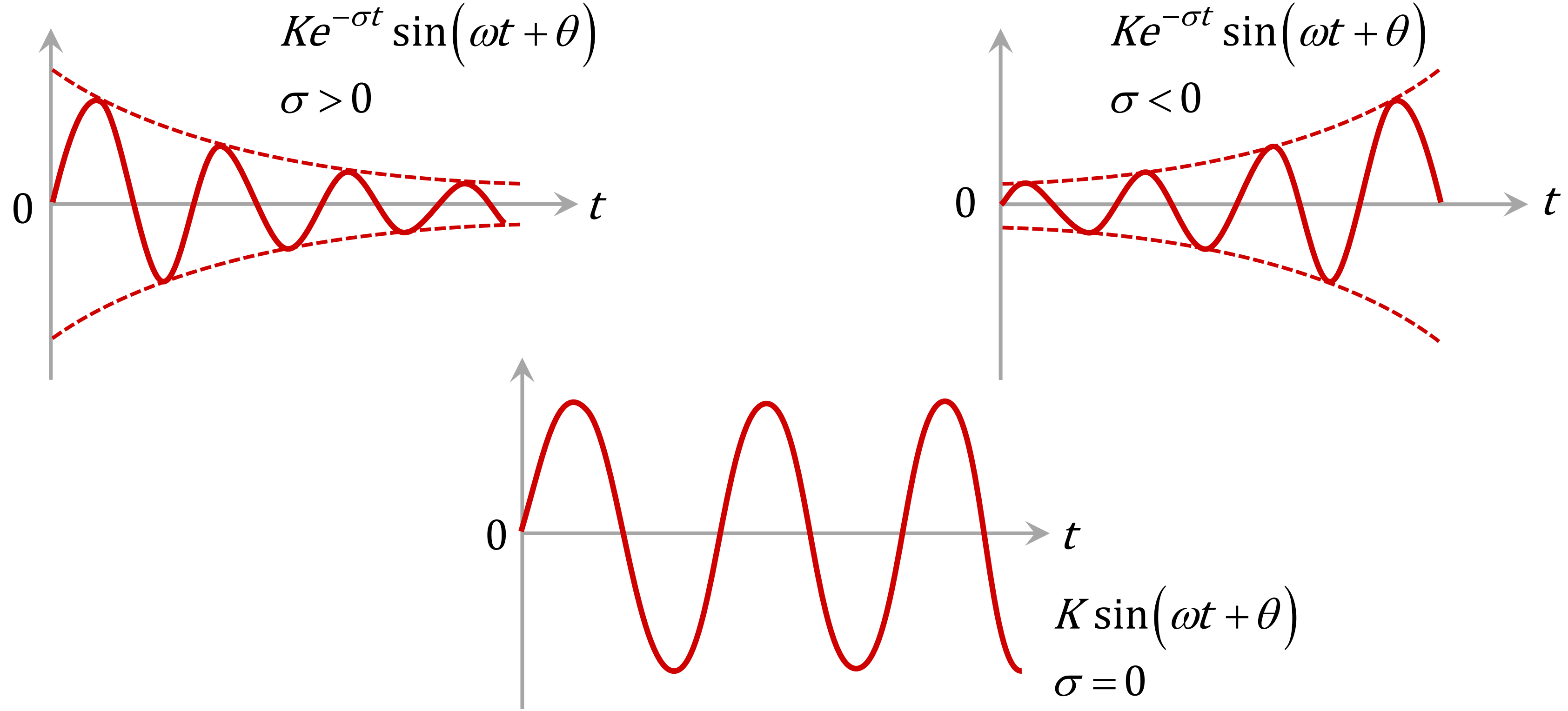
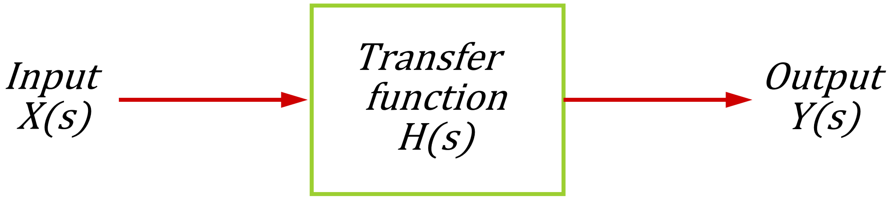


Poles and Stability

Earlier on, we have seen that for $y(t)$ not to grow exponentially with time, the poles of $Y(s)$ must lie in the open LHP or (simple) on the $j\omega$ axis.

Thus, knowledge of the locations of the poles of the response $Y(s) = H(s)X(s)$ identifies the behavior of the circuit with time. We are interested in the very important circuit behavior called **stability**.

A **circuit** represented by a transfer function $H(s)$ is called **stable** if every bounded input yields a bounded output response.

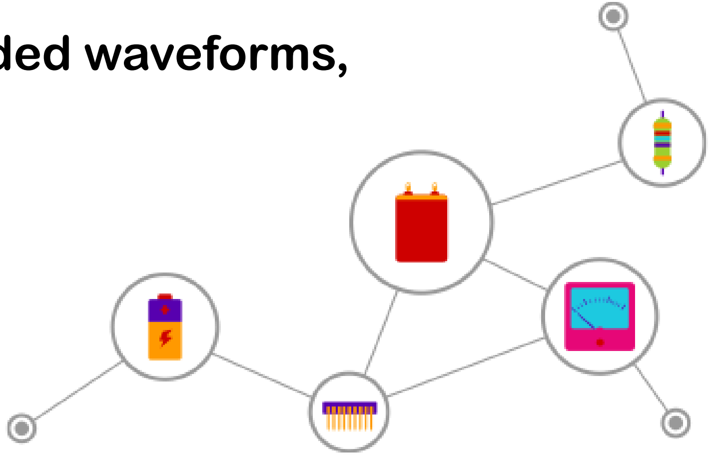


Poles and Stability

A waveform $f(t)$ is bounded if $|f(t)| < K < \infty$, for all t and for some constant K.

A waveform is bounded if its magnitude has a finite height.

$f(t) = 1, f(t) = 3e^{-2t}, f(t) = 5\sin 2t$ **are bounded waveforms, whereas $f(t) = 2t$ is unbounded.**



Poles and Stability

Suppose $H(s) = \frac{N_H(s)}{D_H(s)}$ and $X(s) = \frac{N_X(s)}{D_X(s)}$.

Then the poles of $Y(s) = H(s)X(s)$ come from the poles of $H(s)$ and the poles of $X(s)$. By assumption, $x(t)$ is bounded, and so the poles of $X(s)$ are in the open LHP or (simple) on the $j\omega$ axis.

We now have the following result:

- A circuit is stable if and only if all the poles of the transfer function $H(s)$ lie in the open LHP.
- For stability, $H(s)$ should have no poles in the RHP (which includes the $j\omega$ axis).
- A circuit that is not stable is said to be unstable.

Poles and Stability

A transfer function $H(s)$ with first-order poles on the $j\omega$ axis is sometimes called marginally stable.

But, it can lead to instability for a certain type of inputs (see following example).

Poles and Stability: Example 1

Consider the circuit with transfer function $H(s) = \dfrac{s(s+2)}{(s+6)(s^2+8s+25)}$.

The poles of $H(s)$ are $s = -6$, $s_{1,2} = -4 \pm j3$.

The real part of $s_{1,2}$ (-4) is negative.

So the circuit is stable because all poles lie in the open LHP.

The presence of the $\mathrm{j}\omega$-axis zero (or even a RHP zero) does not affect the stability of a transfer function.

Poles and Stability: Example 2

Consider the transfer function $\frac{Y(s)}{X(s)} = H(s) = \frac{2(s^2+6s+25)}{s(s^2+7s+10)}$.

The poles of $H(s)$ are $s = 0, -2, -5$. A simple pole lies on the $j\omega$-axis (at the origin), so the circuit is marginally stable.

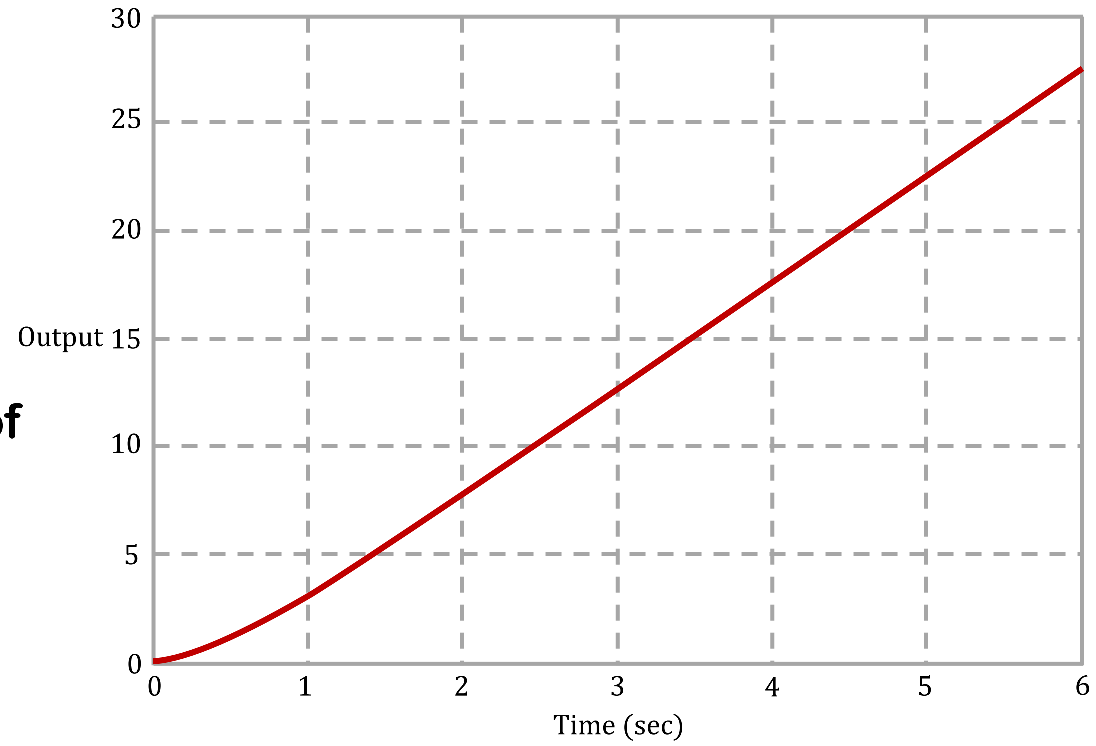
But if a unit step $u(t)$ is applied to this system giving $X(s) = 1/s$, then it gives rise to the term $1/s^2$ (time response is $tu(t)$) .

Poles and Stability: Example 2

This represents an unstable output response.

Although neither $H(s)$ nor $X(s)$ was unstable, the combination produced an unstable response.

This represents the concept of exciting a circuit at a natural frequency (a pole of $H(s)$), which can lead to an unstable response.



Poles and Stability: Example 3

Consider $H(s) = \frac{1}{s+1}$

Let the input $x(t)$ **be** $e^{-\alpha t}$ **with** $\alpha > 0.$

Then $\mathcal{L}\{e^{-\alpha t}\} = \frac{1}{s+\alpha}$.

The output response is

$$\boxed{Y(s) = H(s)X(s) = \frac{1}{(s+1)(s+\alpha)} = \frac{K_1}{s+1} + \frac{K_2}{s+\alpha}}$$

$\alpha \neq 1$

This means

$$\boxed{y(t) = (K_1 e^{-t} + K_2 e^{-\alpha t})u(t)}$$

Poles and Stability: Example 3

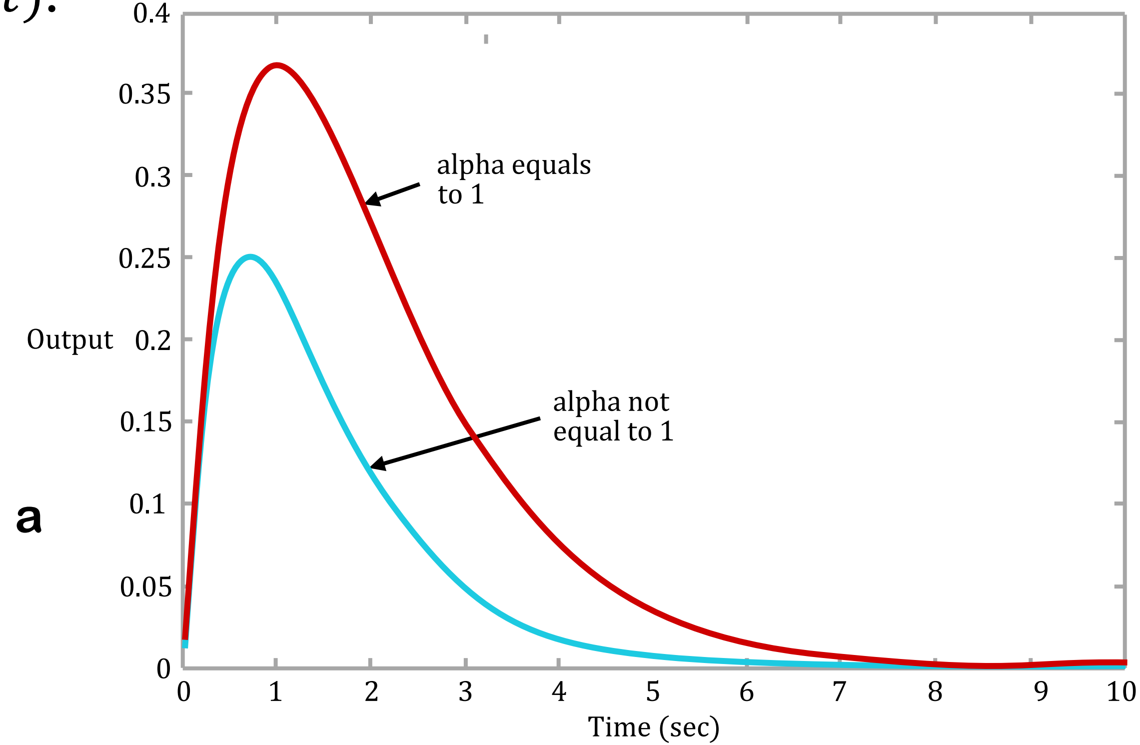
If $\alpha = 1$, then $Y(s) = \frac{1}{(s+1)^2}$.

This leads to $y(t) = te^{-t}u(t)$.

Here, $\lim_{t\to\infty} y(t) = 0$.

We excited a natural frequency of the network $(p_1 = -1)$ with the source (e^{-t}). This is resonance, which is the condition when a natural frequency of the network is excited.

The response is as shown:



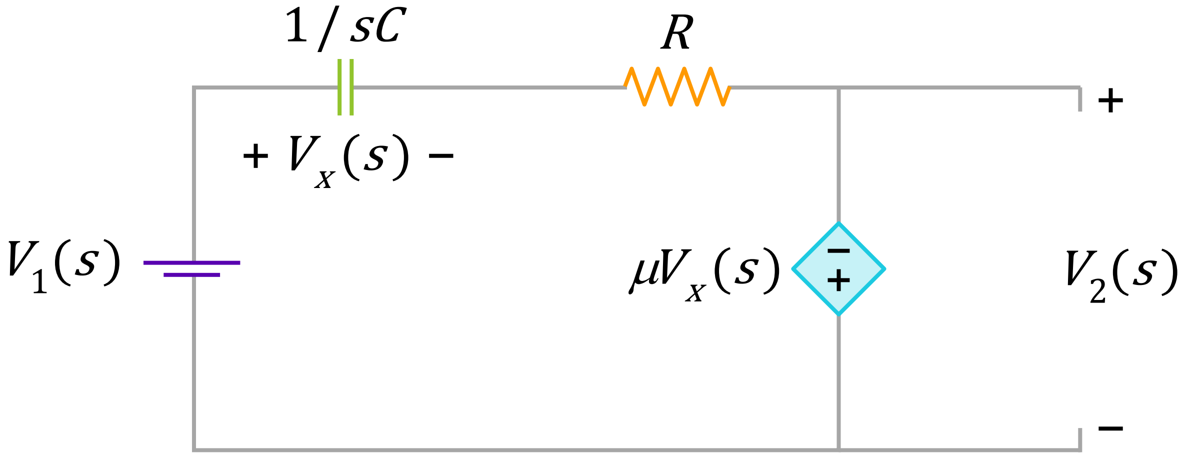
Poles and Stability: Stability Criterion for Active Networks

Passive networks (circuits with only R, L, and C) will always be stable.

However, active networks (containing controlled sources) are not always stable.

Poles and Stability: Example 4

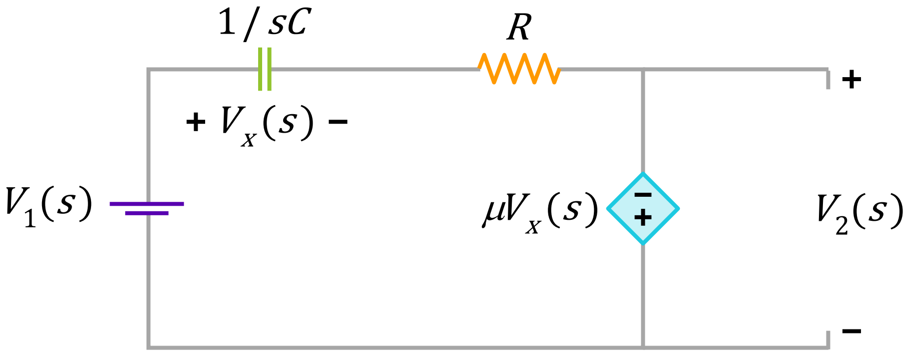
Consider the active network as shown ($\mu > 0$)**.**



Poles and Stability: Example 4

We derived the transfer function earlier as

$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{-\mu}{RCs + 1 - \mu}$$



The pole of $H(s)$ is $= -\frac{1-\mu}{RC}$.

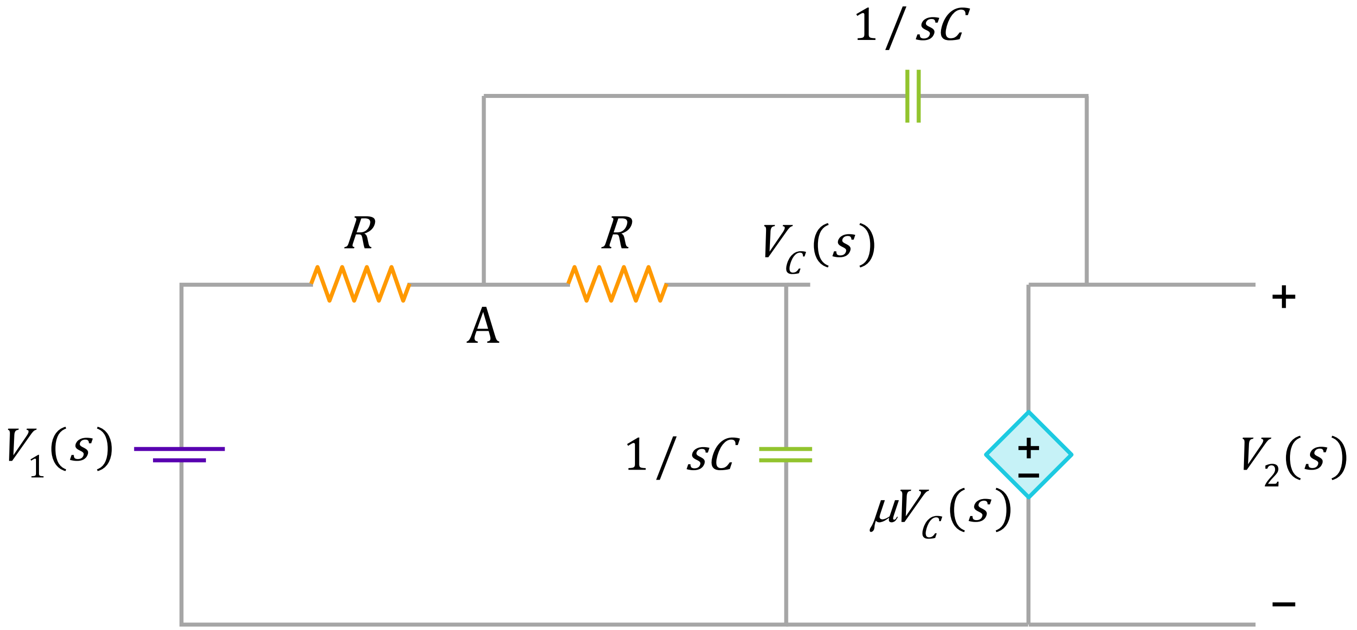
The active circuit will be stable if the pole lies in the open LHP.

Thus, $s = -\frac{1-\mu}{RC}$ has to be negative to ensure the stability of the circuit, i.e., $\mu < 1$.

If $\mu > 1$, the circuit will be unstable as the pole lies in the RHP.

Poles and Stability: Example 5

Consider the active network as shown.



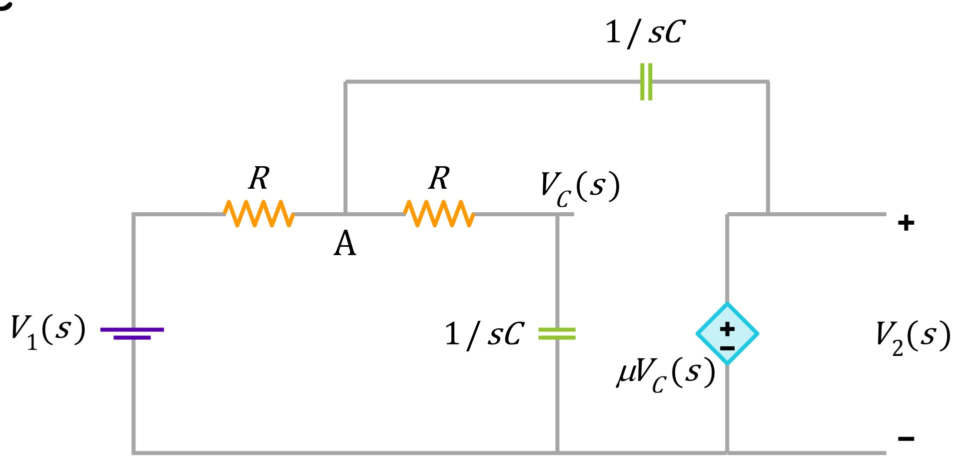
Poles and Stability: Example 5

The transfer function was found to be

$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{\mu}{(RCs)^2 + (3-\mu)RCs + 1}$$

The poles of $H(s)$ are given by

$$s_{1,2} = -\frac{3-\mu}{2RC} \pm \frac{1}{2RC}\sqrt{(3-\mu)^2 - 4}$$

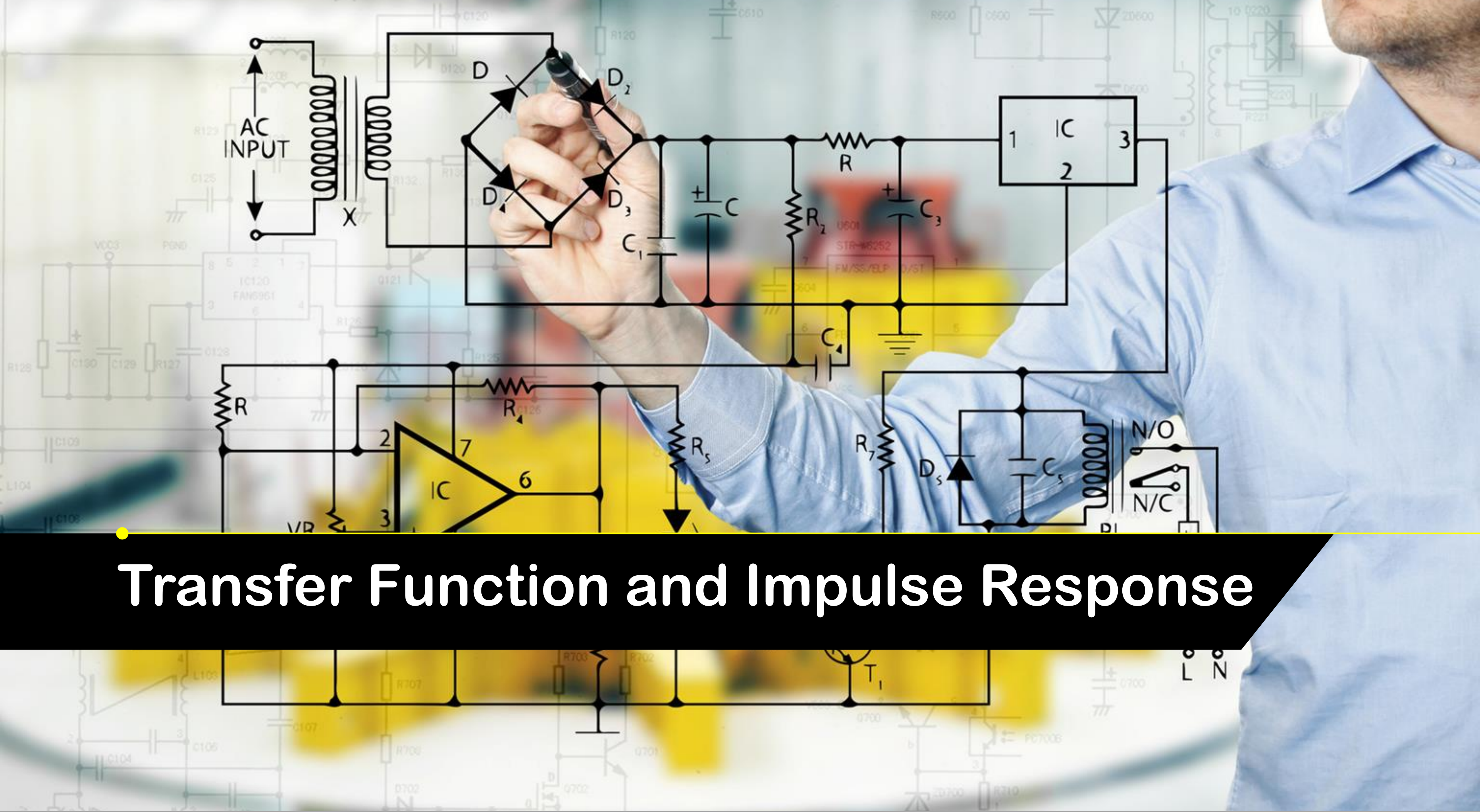


Thus, to ensure that poles are in the open LHP, i.e., stability, $0 < \mu < 3$ (using the following result).

Result: Consider the transfer function of a circuit given by

$$\frac{V_o(s)}{V_i(s)} = \frac{K}{bs^2 + as + K}$$

The circuit is always stable for $a, b, K > 0$.



Transfer Function and Impulse Response

Transfer Function and Impulse Response

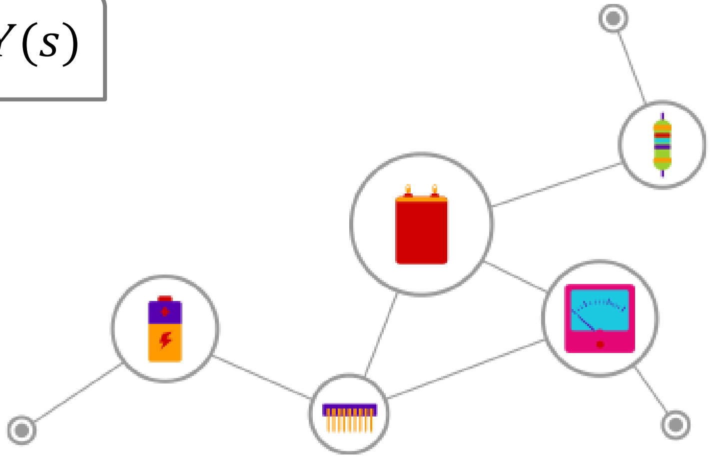
The impulse response $h(t)$ is the response of a circuit having a single input excitation of a unit impulse, assuming zero initial conditions.

Consider the Input/Output (I/O) relationship given by $Y(s) = H(s)X(s)$.

When the input is $x(t) = \delta(t)$ (i.e., a unit impulse), then $X(s) = \mathcal{L}\{\delta(t)\} = 1$, and the input-output relationship is now

$$\boxed{Y(s) = H(s).1 = H(s)} \qquad \boxed{\mathcal{L}\{h(t)\} = Y(s)}$$

The transform of the impulse response, i.e., $\mathcal{L}\{h(t)\}$ equals to the transfer function $H(s)$.



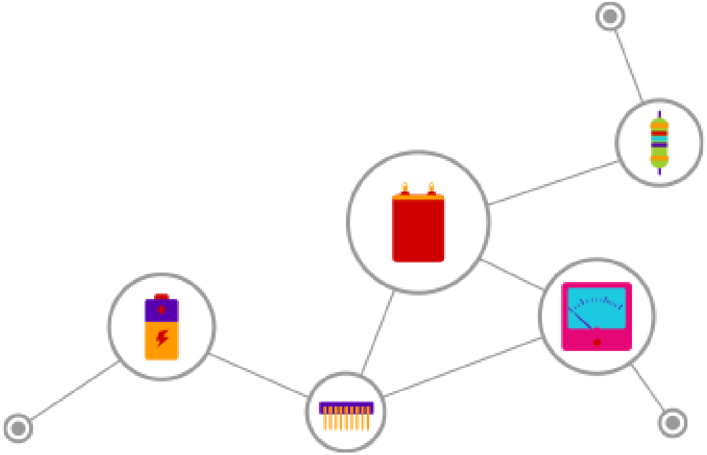
Transfer Function and Impulse Response

Thus, the impulse function contributes no poles to $Y(s)$. If there are no repeated poles, the partial fraction expansion of $H(s)$ is

$$H(s) = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \cdots + \frac{K_n}{s - p_n}$$

The impulse response $h(t) = \mathcal{L}^{-1}\{H(s)\}$ is

$$h(t) = (\underbrace{K_1 e^{p_1 t} + K_2 e^{p_2 t} + \cdots K_n e^{p_n t}}_{\text{Natural Response}})u(t)$$



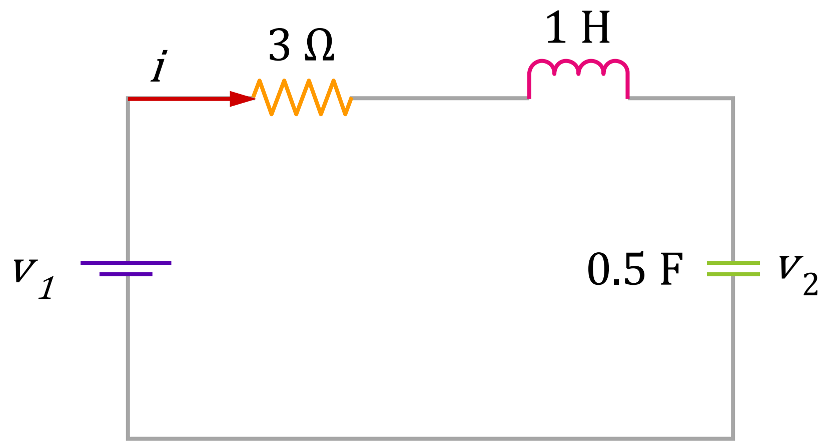
Transfer Function and Impulse Response

When the circuit is stable, all the natural poles are in the open LHP and $\lim_{t\to\infty} h(t) = 0$.

Every linear circuit having constant parameter values for its elements can be represented in the time domain by its impulse response.

Transfer Function and Impulse Response: Example 1

Figure below shows a RLC circuit. The impulse response of the RLC circuit is $h(t) = (2e^{-t} - 2e^{-2t})u(t)$. Find the transfer function of the circuit.



The transform of the circuit impulse response $h(t)$ is the transfer function $H(s)$, given by

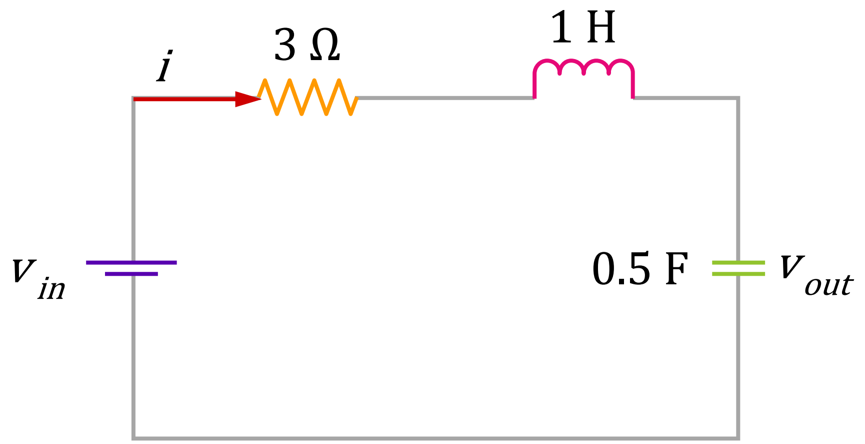
$$\boxed{H(s) = \mathcal{L}\{h(t)\}}$$

$$\boxed{= \mathcal{L}\{(2e^{-t} - 2e^{-2t})u(t)\}}$$

$$\boxed{= \frac{2}{s+1} - \frac{2}{s+2}}$$

Transfer Function and Impulse Response: Example 2

The following RLC circuit has the transfer function given by $H(s) = \frac{2}{(s+1)(s+2)}$.



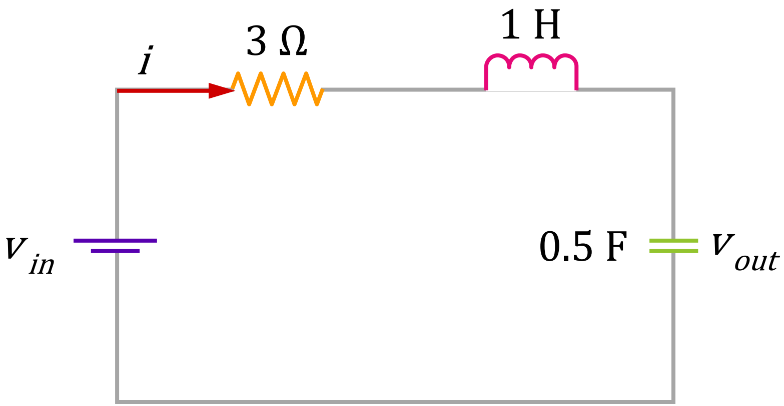
Transfer Function and Impulse Response: Example 2

The impulse response given by

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{(s+1)(s+2)}\right\}$$

$$= \mathcal{L}^{-1}\{\frac{2}{s+1} - \frac{2}{s+2}\}$$

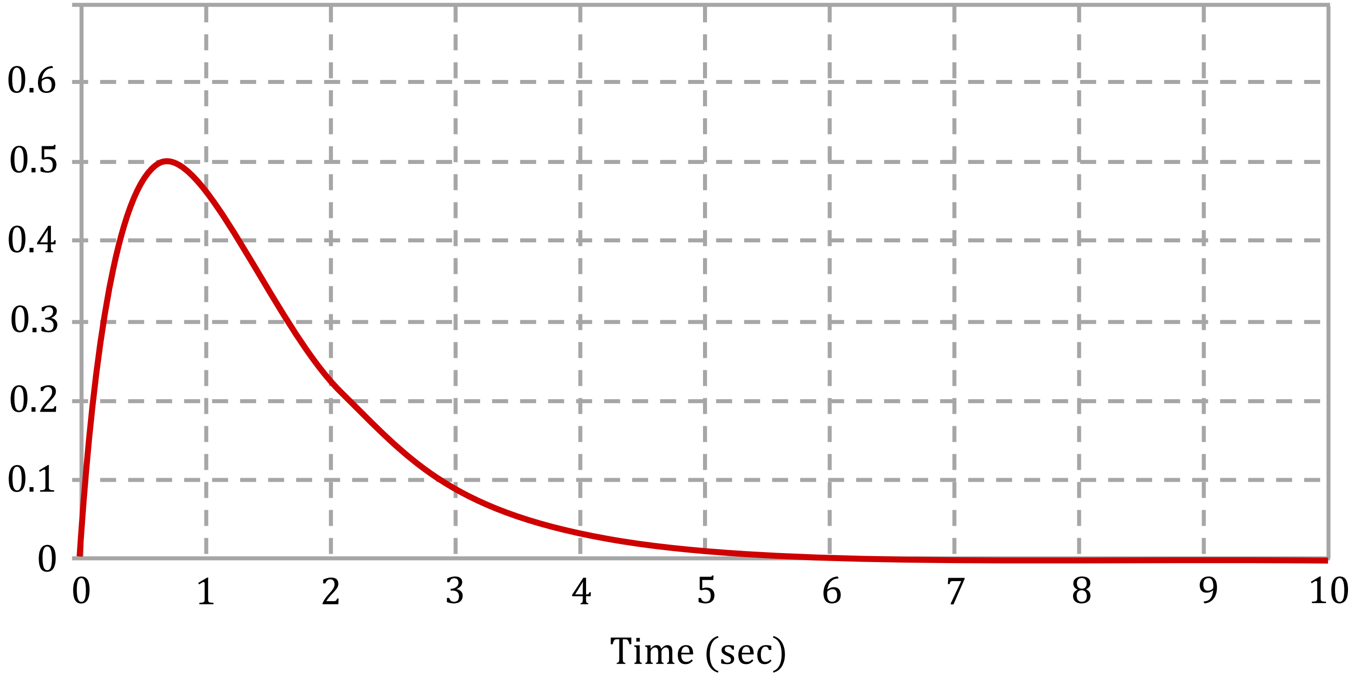
$$\boxed{= (2e^{-t} - 2e^{-2t})u(t)}$$

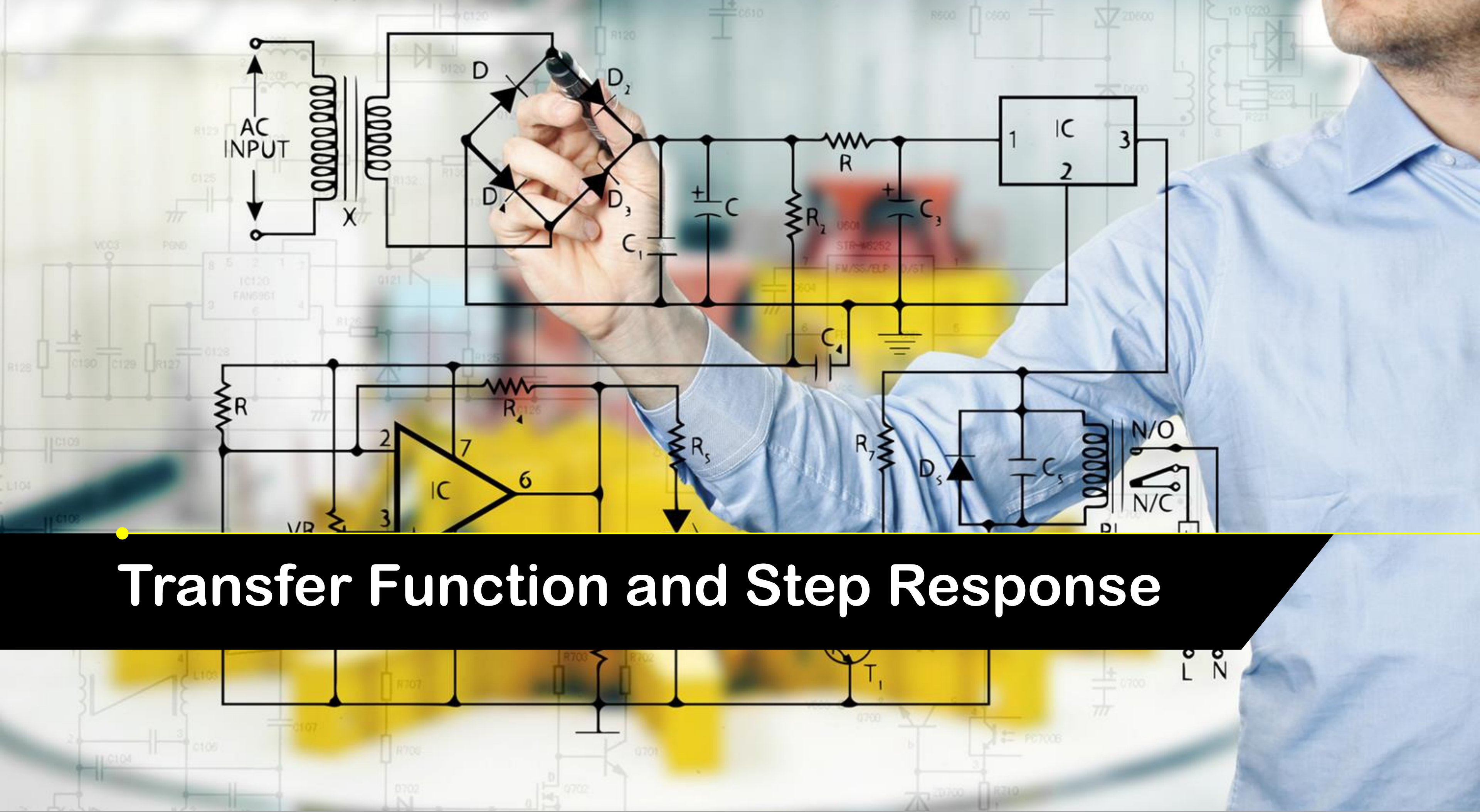


Transfer Function and Impulse Response: Example 2

The impulse response is shown here:

$$\boxed{h(t) = (2e^{-t} - 2e^{-2t})u(t)}$$





Transfer Function and Step Response

Transfer Function and Step Response

The step response is the response of a circuit to a step function, assuming all initial conditions are zero.

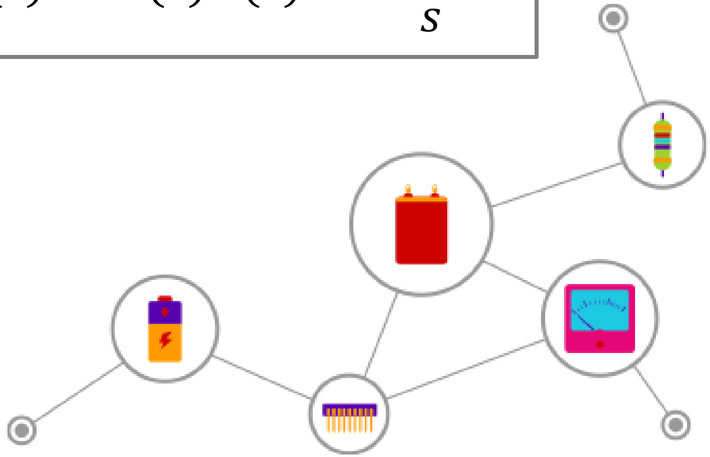
When the input $x(t)$ is a step function $Au(t)$, with $A > 0$ a constant, $X(s) = \mathcal{L}\{Au(t)\} = \frac{A}{s}$.

The input-output relationship now becomes

$$\boxed{Y(s) = H(s)X(s) = \frac{AH(s)}{s}}$$

The waveform of the step response is

$$\boxed{y(t) = \mathcal{L}^{-1}\{\frac{AH(s)}{s}\}}$$



Transfer Function and Step Response

If $Y(s) = \frac{AH(s)}{s}$ has only **simple poles**, then

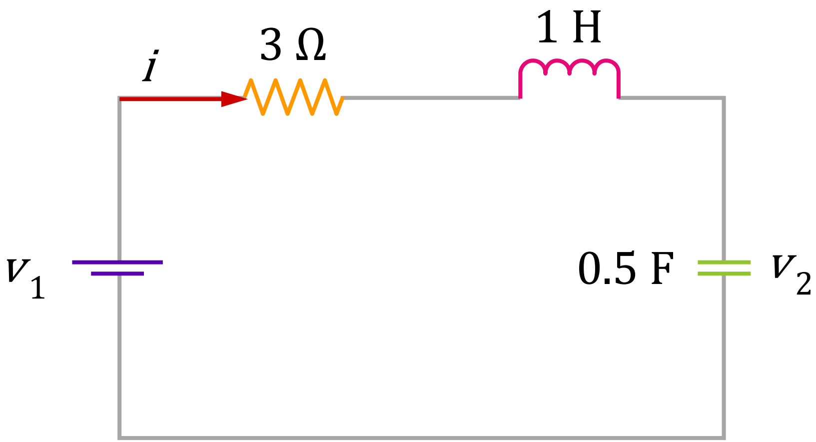
$$Y(s) = \frac{K}{s} + \sum_{i=1}^{i=n} \frac{K_i}{s + p_i}$$

$$y(t) = \underbrace{Ku(t)}_{\text{Forced Response}} + \underbrace{\left(\sum_{i=1}^{i=n} K_i \, e^{-p_i t} \right) u(t)}_{\text{Natural Response}}$$

If p_i 's are positive, then the natural response will decay to zero, and $\lim_{t \to \infty} y(t) = K$.

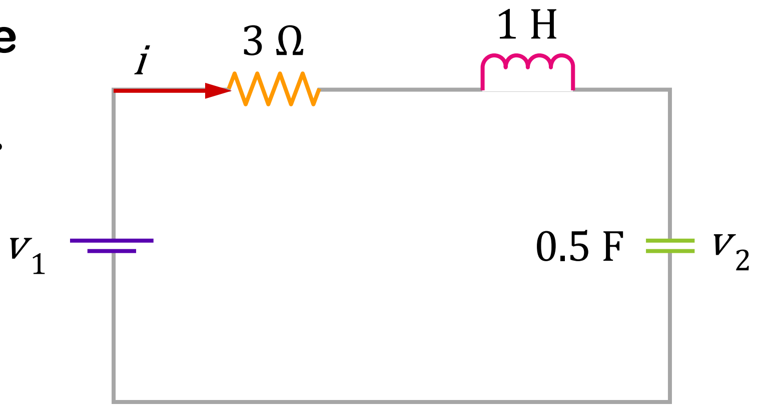
Transfer Function and Step Response: Example 1

The following RLC circuit has the transfer function given by $H(s) = \frac{2}{(s+1)(s+2)}$.



Transfer Function and Step Response: Example 1

This gives $V_2(s) = \frac{2}{(s+1)(s+2)} V_1(s)$. Suppose the input is $v_1(t) = u(t)$, so that $V_1(s) = \frac{1}{s}$.



Then, the output response is given by

$$V_2(s) = H(s)X(s) = \frac{H(s)}{s} = \frac{2}{s(s+1)(s+2)}$$

$$V_2(s) = \frac{1}{s} - \frac{2}{(s+1)} + \frac{1}{(s+2)}$$

$$v_2(t) = u(t) - (2e^{-t} - e^{-2t})u(t)$$

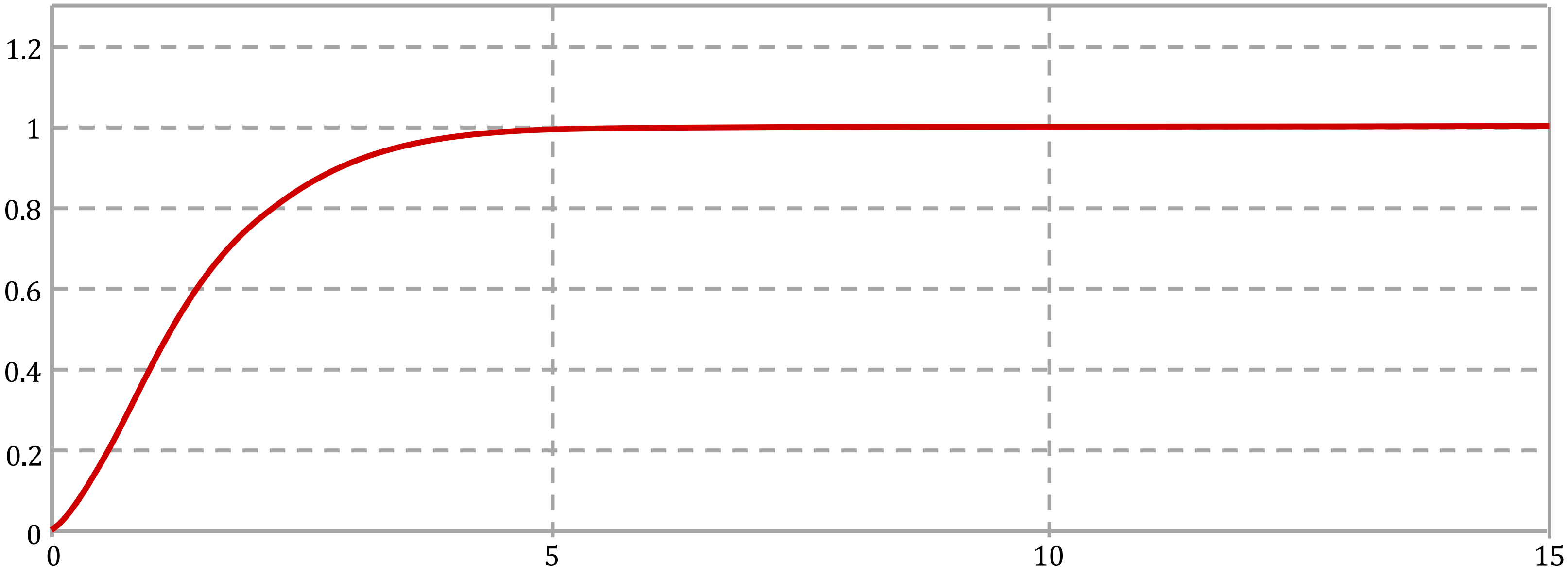
Forced Response

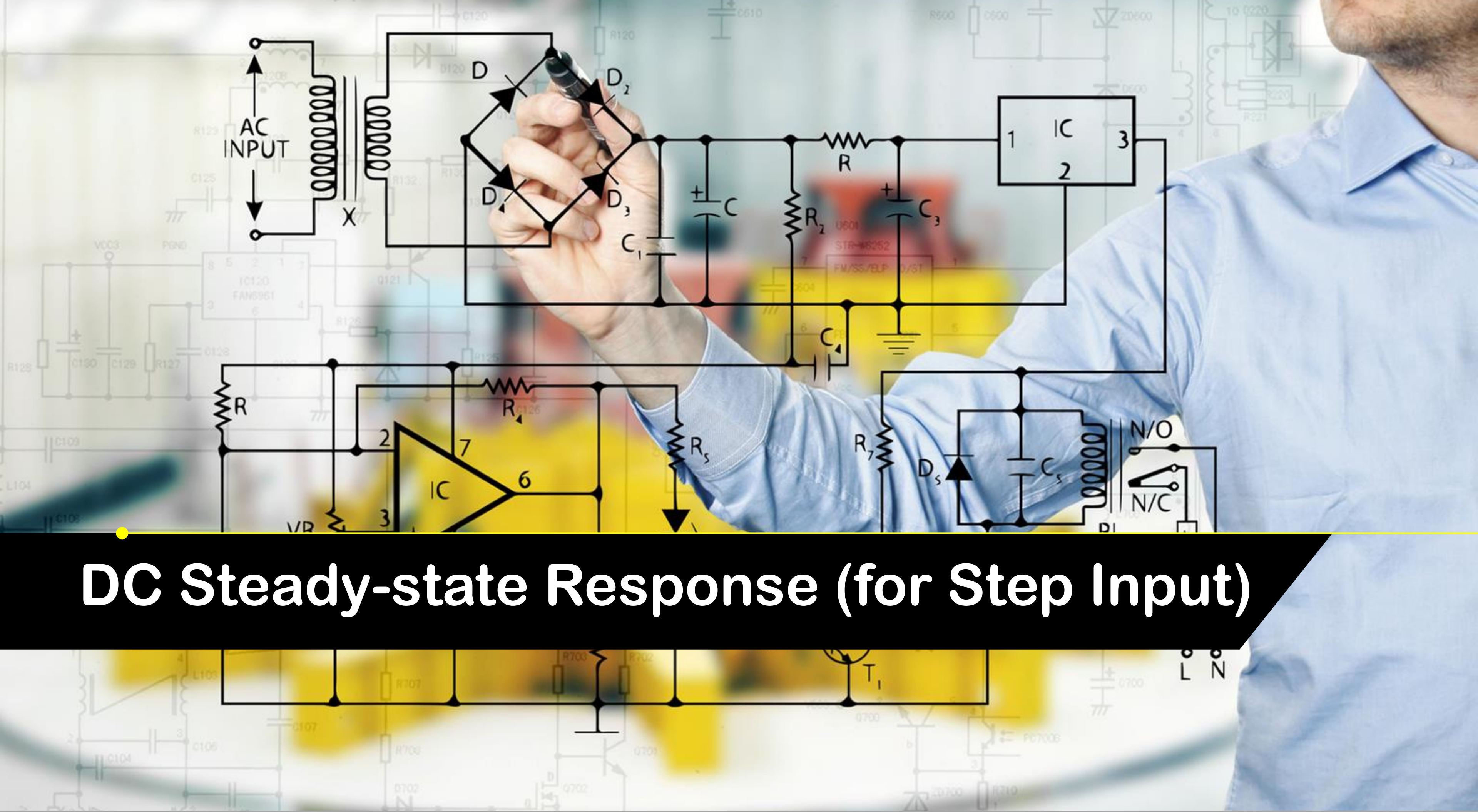
Natural Response

Transfer Function and Step Response: Example 1

The step response is shown here:

$$v_2(t) = \underbrace{u(t)}_{\text{Forced Response}} - \underbrace{(2e^{-t} - e^{-2t})u(t)}_{\text{Natural Response}}$$





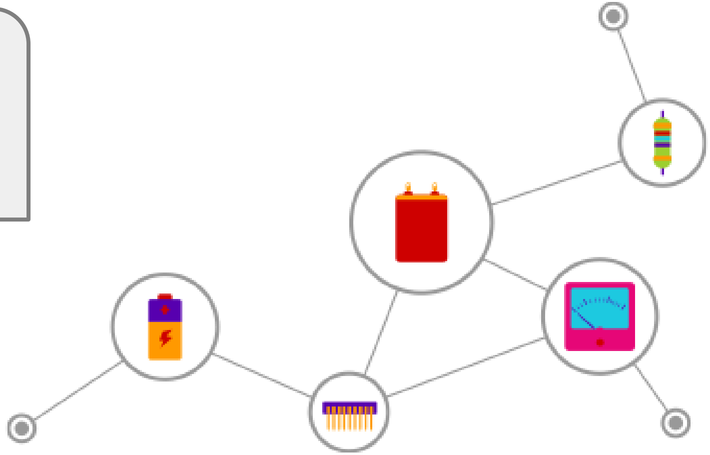
DC Steady-state Response (for Step Input)

DC Steady-state Response (for Step Input)

When the circuit is stable, the natural response decays to zero, leaving a forced component called the DC steady-state response, i.e., $\lim_{t\to\infty} y(t) = K$.

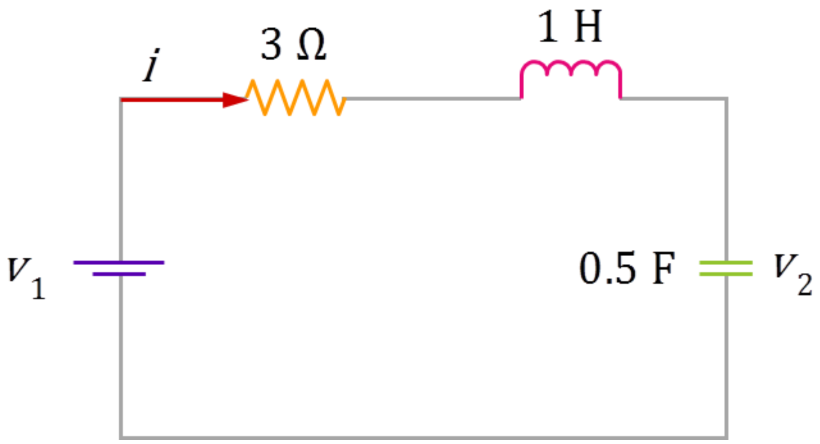
For a step input $Au(t)$, the amplitude of the DC steady-state response equals A times $H(s)$ at $s = 0$, i.e., $AH(0)$.

If the circuit is unstable, the steady-state response is never reached, i.e., it does not exist.



DC Steady-state Response (for Step Input): Example 1

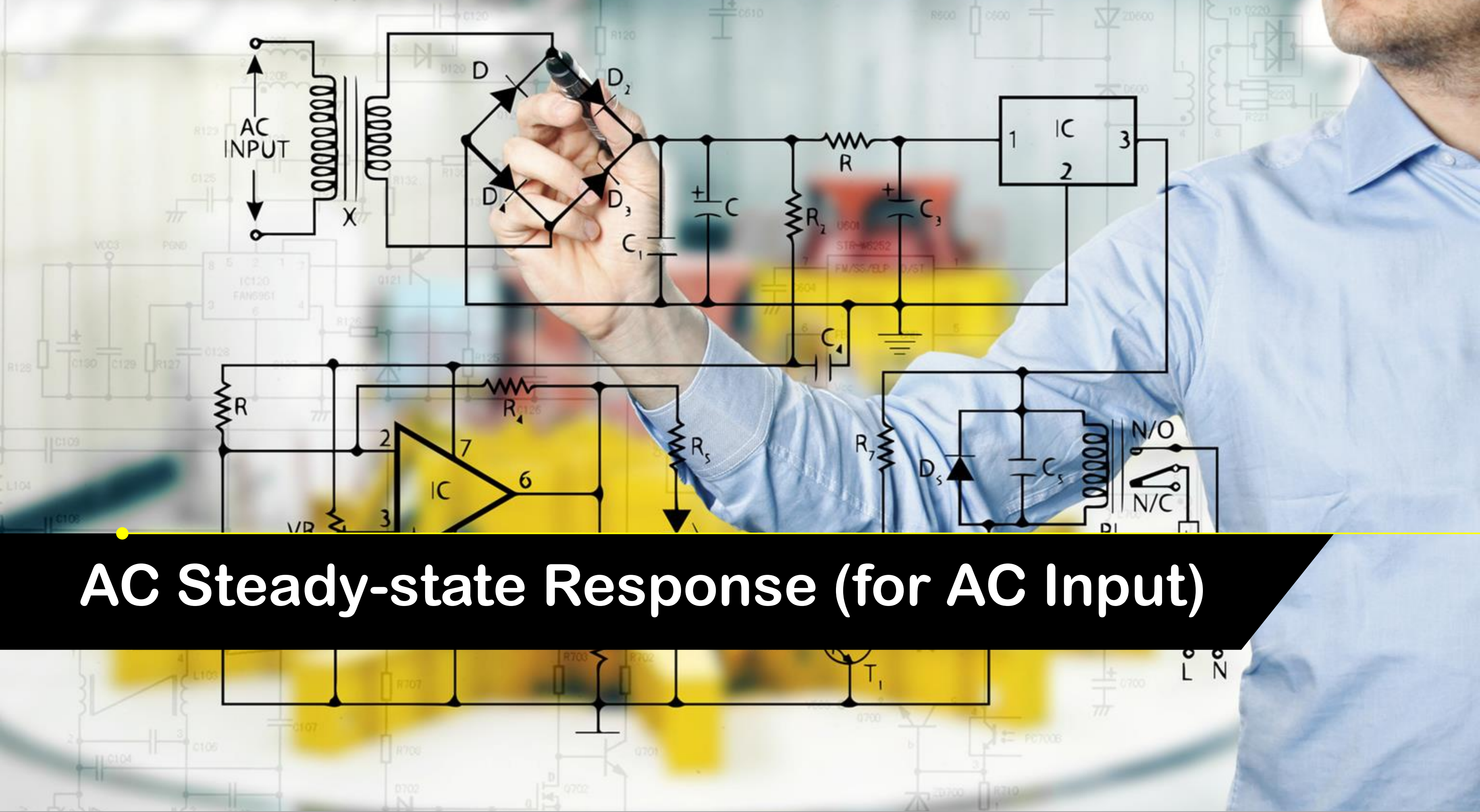
The RLC circuit has a transfer function $H(s) = \frac{2}{(s+1)(s+2)}$.
Find the DC steady-state response for $x(t) = Au(t)$.



The DC steady-state response is given by

$$H(0) = \frac{2}{(0+1)(0+2)}A = A.$$

The response is as shown for $A = 1$.



AC Steady-state Response (for AC Input)

AC Steady-state Response (for AC Input)

We can use a circuit's transfer function $H(s)$ to determine the steady-state response of the circuit to a sinusoidal input of the form given by $A\cos(\omega t+\phi).u(t)$. We omit the tedious derivation.

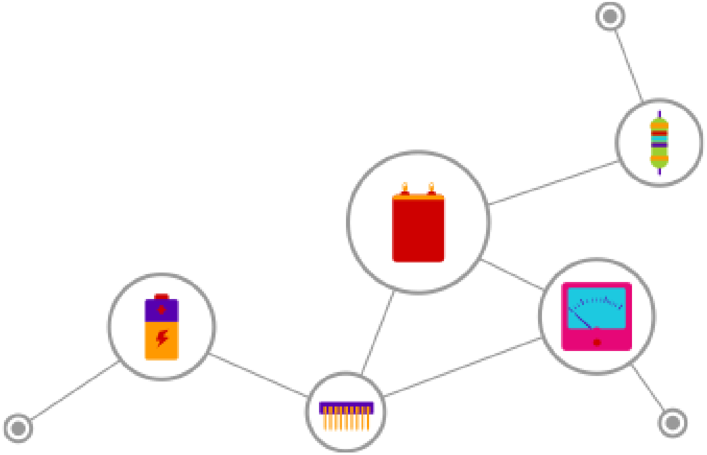
The steady-state output $y_{ss}(t)$ is given by

$$\boxed{y_{ss}(t) = A|H(j\omega)|\cos(\omega t+\phi+\angle H(j\omega))}$$

Where $\boxed{H(j\omega) = H(s)|_{s=j\omega}}$ a complex quantity

$$H(j\omega) = |H(j\omega)|\angle H(j\omega)$$

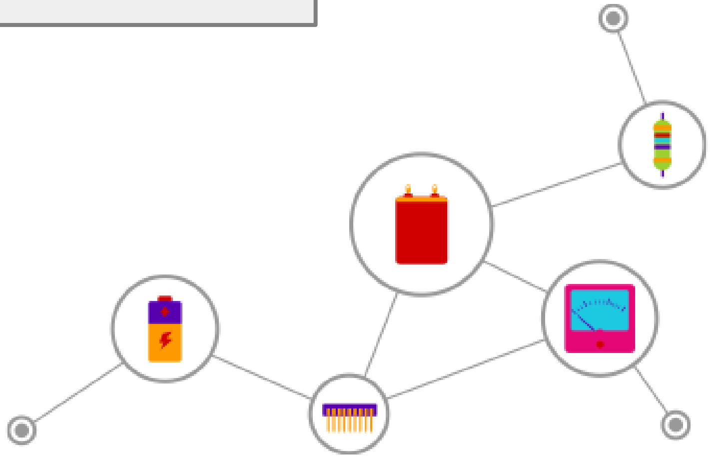
Magnitude of Transfer Phase Angle Function



AC Steady-state Response (for AC Input)

So, we can say three things about the steady-state response:

1. **Output frequency = input frequency** ω
2. **Output amplitude = Input amplitude x** $|H(j\omega)|$
3. **Output phase = Input phase +** $\angle H(j\omega)$



AC Steady-state Response (for AC Input): Example 1

Consider a circuit having the transfer function $H(s) = \frac{s^2 - 0.5s + 5}{s^2 + 0.5s + 5.7321}$ driven by a sinusoidal input $v_{in}(t) = \sqrt{2}\cos(2t + 45^{\circ})\,u(t)$.

AC Steady-state Response (for AC Input): Example 1

At $s = j\omega$

$(j^2 = -1)$

$$H(j\omega) = \frac{(j\omega)^2 - j0.5\omega + 5}{(j\omega)^2 + j0.5\omega + 5.7321}$$

$$H(j\omega) = \frac{5 - \omega^2 - j0.5\omega}{5.7321 - \omega^2 + j0.5\omega}$$

For $\omega = 2$

$$H(j2) = \frac{5 - 2^2 - j0.5(2)}{5.7321 - 2^2 + j0.5(2)} = \frac{1 - j}{1.7321 + j}$$

$$H(j2) = \frac{\sqrt{2}\angle - 45^{\mathrm{o}}}{\sqrt{4}\,\angle 30^{\mathrm{o}}} = \frac{1\angle - 75^{\mathrm{o}}}{\sqrt{2}}$$

$$|H(j2)| = \frac{1}{\sqrt{2}}$$

$$\angle H(j2) = -75^{\mathrm{o}}$$

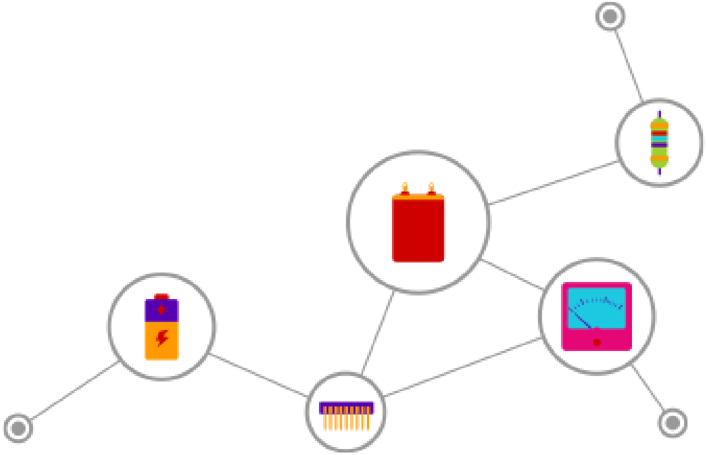
AC Steady-state Response (for AC Input): Example 1

The sinusoidal steady-state output is

$$v_{outss}(t) = |H(j\omega)| \cdot A\cos\big(\omega t + \phi + \angle H(j\omega)\big)$$

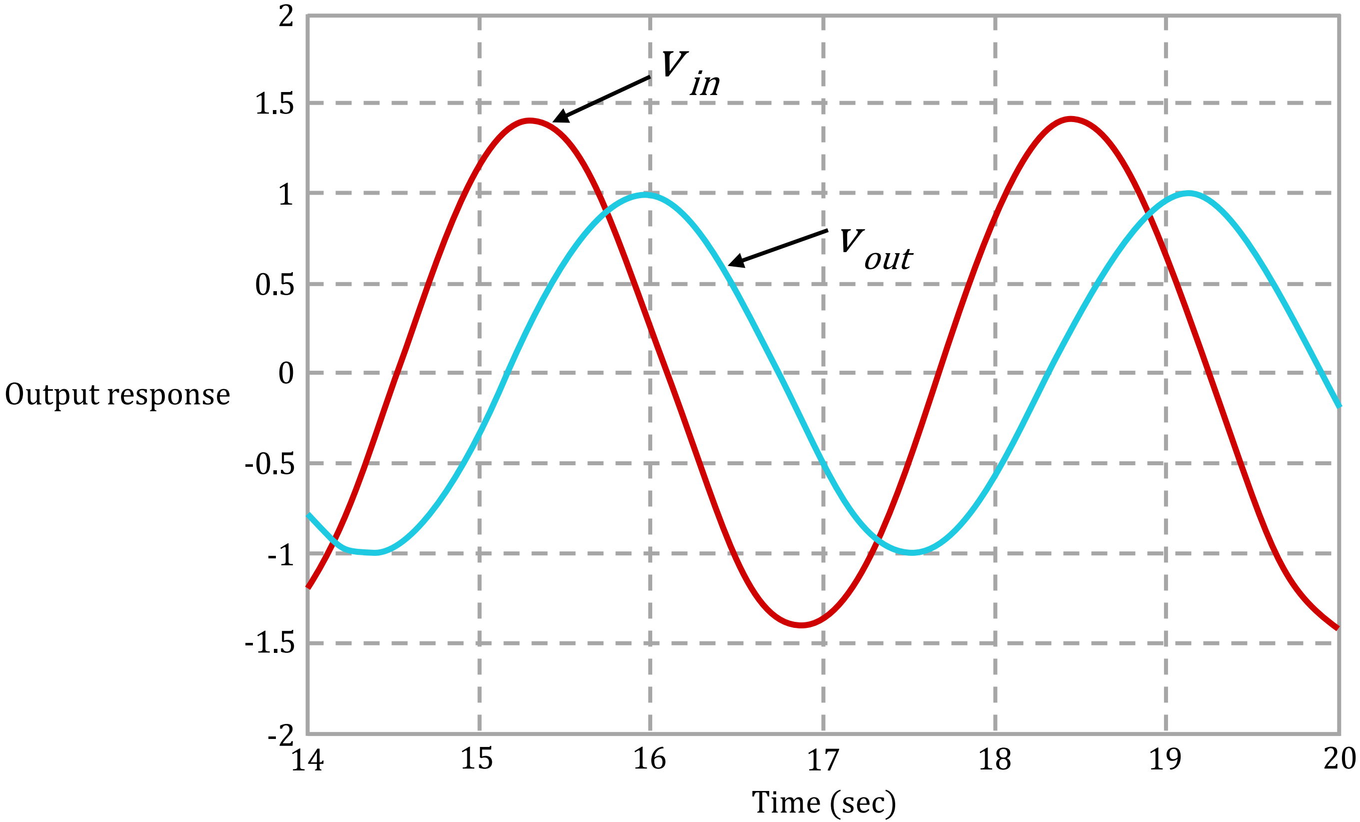
$$= \frac{1}{\sqrt{2}}\sqrt{2}\cos(2t + 45^{o} - 75^{o})$$

$$= \cos(2t - 30^{o})$$



AC Steady-state Response (for AC Input): Example 1

The steady-state response is as shown.



Summary

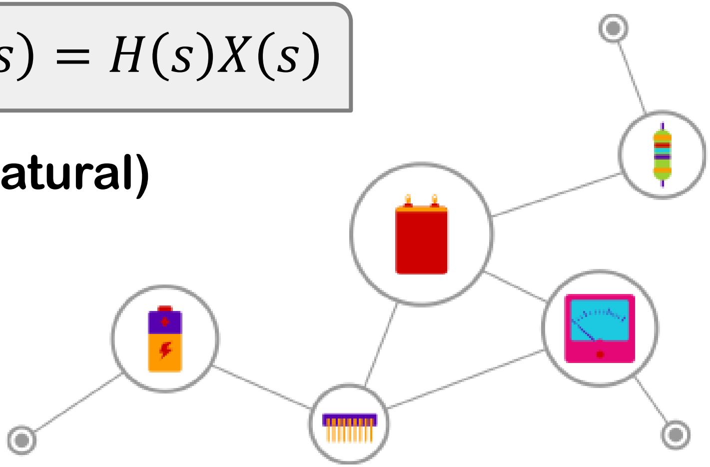
Summary

- The transfer function is defined as $$H(s) = \frac{Y(s)}{X(s)}$$
  - Where all the initial conditions are zero and

    $Y(s)$ = Laplace transform of the output $y(t)$

    $X(s)$ = Laplace transform of the input $x(t)$

- The input-output relationship is given by $$Y(s) = H(s)X(s)$$
  - Hence, poles of $Y(s)$ come from poles (natural) of $H(s)$ and poles (forced) of $X(s)$.

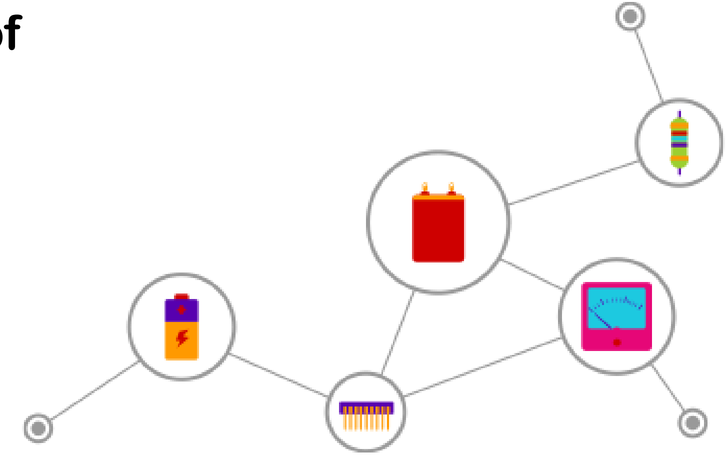


Summary

- The output response $y(t)$ is the sum of the natural response (produced by the natural poles of $H(s)$) and the forced response (produced by the forced poles of $X(s)$.

- The output response $y(t)$ has resonance when one of the poles of $H(s)$ is the same as one of the poles of $X(s)$.

- A system is stable if and only if all the poles of $H(s)$ lie in the open LHP.



Summary

- There are three types of stable output responses $y(t)$:
  1. Critically damped - All the poles of $Y(s)$ are real, negative with at least two same poles.
  2. Overdamped - All the poles of $Y(s)$ are real, negative and distinct.
  3. Underdamped - All the poles of $Y(s)$ lie in the open LHP with at least one pair of complex conjugate poles.

- The Laplace transform of the impulse response $h(t)$ is the transfer function $H(s)$.