

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2016–2017

MH1812 – Discrete Mathematics

April 2017

TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT** an **OPEN BOOK** exam.
5. Calculators are allowed.
6. Candidates should clearly explain their reasoning used in each of their answers.

QUESTION 1.**(15 marks)**

Decide whether or not the following argument is valid:

$$\begin{aligned}
 &p \wedge q; \\
 &r \rightarrow s; \\
 &\neg r \rightarrow q; \\
 &p \vee r; \\
 &\therefore (p \vee q) \wedge r
 \end{aligned}$$

Briefly justify your answer.

Solution: The argument is invalid. Indeed, set $p = T, q = T, r = F$. Then the assumptions are all true and the conclusion is false.

QUESTION 2.**(15 marks)**

Consider three sets S , T , and U where S is defined to be the set of all even integers, $T = \{n \in \mathbb{Z} : 3 \mid n\}$, and $U = \{n \in \mathbb{Z} : n \equiv 0 \pmod{6}\}$.

(a) Prove the set equality $S \cap T = U$. **(8 marks)**

(b) Determine the truth value of the following proposition **(7 marks)**

$$\neg (\forall x \in U, \exists y \in T, x \cdot y \notin S),$$

where \cdot denotes multiplication. Justify your answer.

Solution:

(a) $S \cap T \subseteq U$: Let $x \in S \cap T$. Then x is both a multiple of 2 and 3. Hence x is a multiple of 6 and therefore $x \in U$.

$U \subseteq S \cap T$: Let $x \in U$. Then $x = 6k$ for some $k \in \mathbb{Z}$. Hence $x = 2(3k)$ and $x = 3(2k)$. Therefore x is both a multiple of 2 and 3 as required.

(b) First note that

$$\neg (\forall x \in U, \exists y \in T, x \cdot y \notin S) \equiv \exists x \in U, \forall y \in T, x \cdot y \in S$$

This statement is true. Indeed, $6 \in U$ and $6 \cdot y$ is even for all $y \in T$.

QUESTION 3.**(30 marks)**

- (a) Using the characteristic equation, solve the recurrence relation
- (10 marks)**

$$a_0 = 2, a_1 = 3, \quad a_n = 7a_{n-1} - 12a_{n-2}.$$

- (b) Consider the recurrence relation given by the initial conditions
- $D_0 = 1, D_1 = 0$
- , and
- $D_n = (n-1)(D_{n-1} + D_{n-2})$
- for all
- $n \geq 2$
- . Prove the equality

$$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}.$$

(Hint: use induction.)

(20 marks)**Solution:**

- (a) We have the characteristic equation

$$x^2 - 7x + 12 = (x-3)(x-4).$$

Hence $a_n = u3^n + v4^n$. Since $a_0 = u + v = 2$ and $a_1 = 3u + 4v = 3$. Therefore $u = 5$ and $v = -3$.

- (b) Basis cases are OK. Let
- $P(n)$
- be our inductive hypothesis
- $D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}$
- . Suppose that
- $P(k)$
- is true for all
- $k \leq n$
- . Consider
- $P(n+1)$
- . We have

$$\begin{aligned} D_{n+1} &= n(D_n + D_{n-1}) \\ &= n \left(n! \sum_{k=0}^n \frac{(-1)^k}{k!} + (n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \right) \\ &= n \left(n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} + (-1)^n + (n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \right) \\ &= n(-1)^n + n \left((n+1)(n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \right) \\ &= n(-1)^n + (n+1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}. \end{aligned}$$

It suffices to show that $n(-1)^n = (n+1)! \left(\frac{(-1)^n}{n!} + \frac{(-1)^{n+1}}{(n+1)!} \right)$. This follows since

$$(n+1)! \left(\frac{(-1)^n}{n!} + \frac{(-1)^{n+1}}{(n+1)!} \right) = (n+1)(-1)^n + (-1)^{n+1}$$

and $(-1)^n + (-1)^{n+1} = 0$ for all $n \in \mathbb{Z}$.

QUESTION 4.**(25 marks)**

- (a) Consider the relation R on the set of integers \mathbb{Z} given by

$$aRb \iff b \equiv a^3 - a \pmod{3}.$$

- (i) Is R reflexive? **(5 marks)**
- (ii) Is R symmetric? **(5 marks)**
- (iii) Is R transitive? **(5 marks)**

Justify your answers.

- (b) Let $S = \{1, 2, \dots, n\}$. Determine

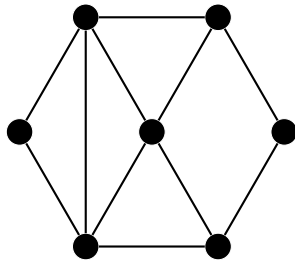
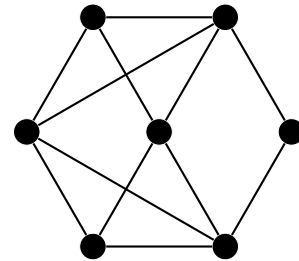
- (i) the cardinality of the set T of all functions $f : S \rightarrow S$? **(5 marks)**
- (ii) the cardinality of the set $U = \{f \in T \mid f \text{ is invertible}\}$? **(5 marks)**

Solution:

- (a)
 - (i) No. $(1, 1) \notin R$.
 - (ii) No. $(1, 0) \in R$ but $(0, 1) \notin R$.
 - (iii) Yes. Suppose $(x, y) \in R$ and $(y, z) \in R$. Then z is a multiple of 3. Now $(x, w) \in R$ for all w that is divisible by 3. So (x, z) is also in R .
- (b)
 - (i) Each function $f : S \rightarrow S$ must map each element of S to precisely one element in S . So each element can be mapped to n elements. Thus the cardinality is n^n .
 - (ii) Each invertible function $f : S \rightarrow S$ must map each element of S to a unique element in S . So the first element can be mapped to n elements; the second to $(n - 1)$ elements, and so on. Thus the cardinality is $n!$.

QUESTION 5.

Consider the two graphs in Figure 1.

 X  Y Figure 1: The graphs X and Y .

- (a) For each of the graphs X and Y
- (i) determine whether or not it has an Euler path; **(4 marks)**
 - (ii) determine whether or not it has an Euler circuit; **(4 marks)**
 - (iii) determine whether or not it has an Hamilton circuit. **(4 marks)**
- (b) Are the graphs X and Y are isomorphic? Justify your answer. **(3 marks)**

Solution:

- (a) (i) Both X and Y have precisely two vertices of odd degree. Hence both have Euler paths.
- (ii) Both X and Y have precisely two vertices of odd degree. Hence both do not have Euler circuits.
- (iii) Both X and Y have Hamilton circuits.
- (b) The graphs are not isomorphic. They have different degree sequences.

END OF PAPER