NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

EE3001 Engineering Electromagnetics

Session 4-1

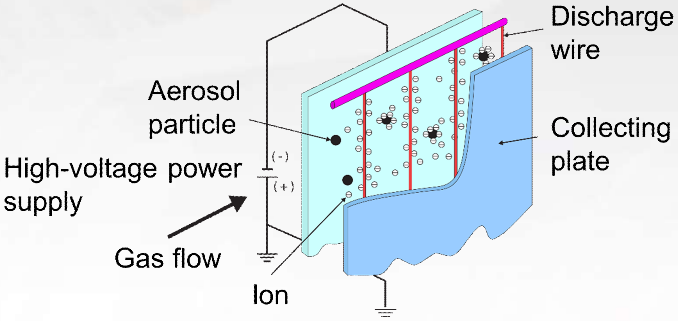
Application of Electric Force

Learning Objectives

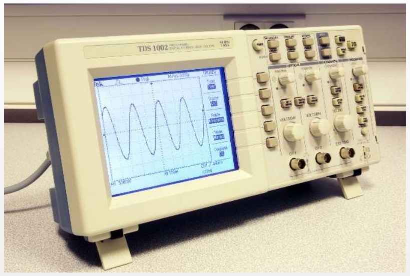
- State another application that uses electric force between electric charges; and
- Describe the operation of a cathode ray tube.

Review of Applications of Electric Force

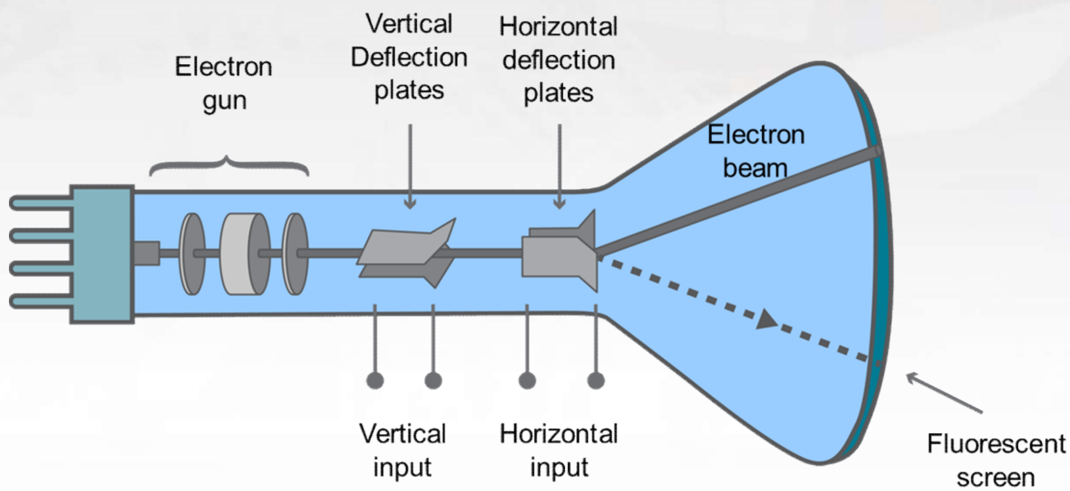
- Electrostatic precipitator



- Oscilloscope and photocopier

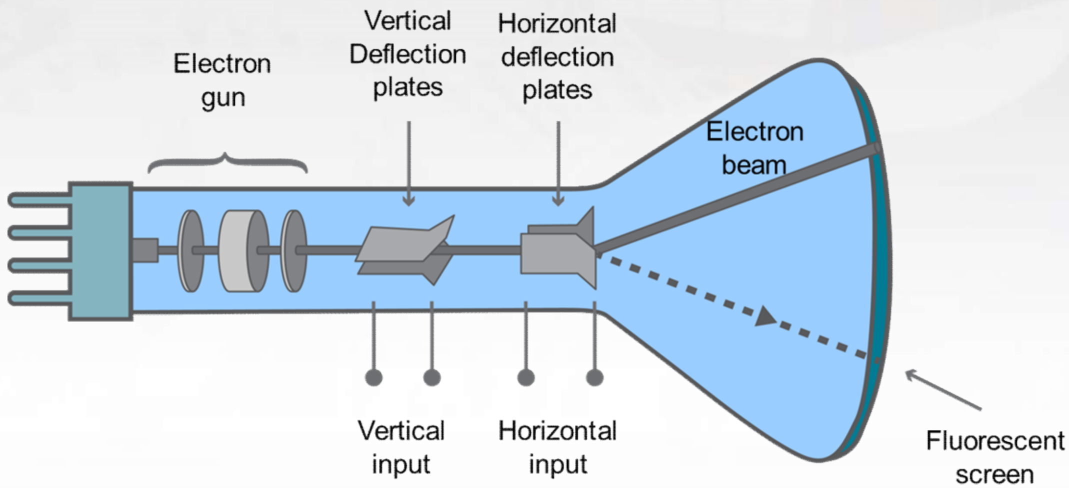


Cathode Ray Tube



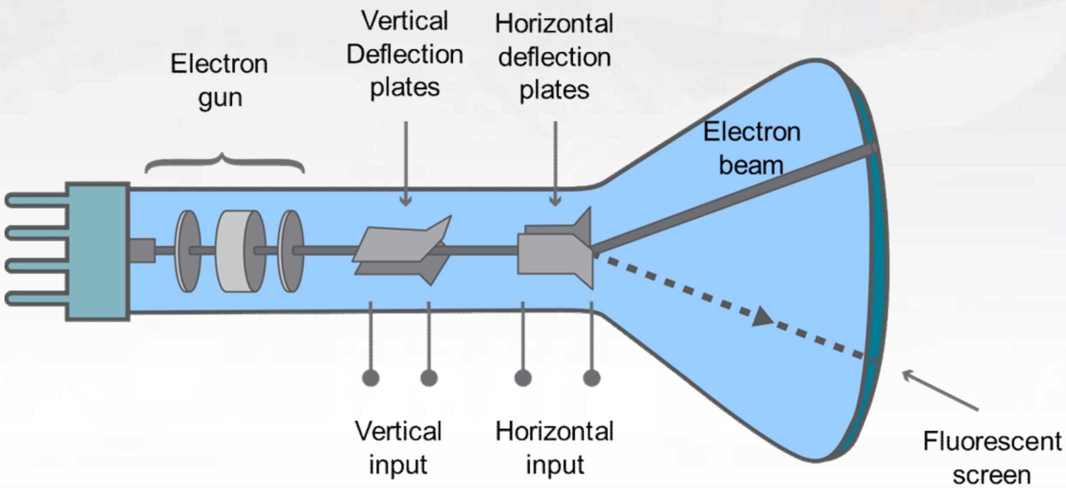
- A CRT is commonly used to obtain a visual display of electronic information in oscilloscopes, radar systems, televisions, etc.

Cathode Ray Tube



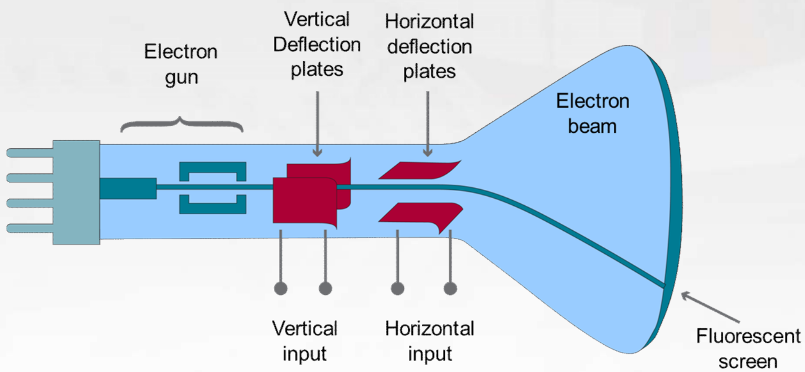
- The CRT is a vacuum tube in which a beam of electrons is accelerated and deflected under the influence of electric or magnetic force.

Cathode Ray Tube



- The electrons are deflected in various directions by two sets of plates.
- The placing of charge on the plates creates the electric force and allows the beam to be steered.

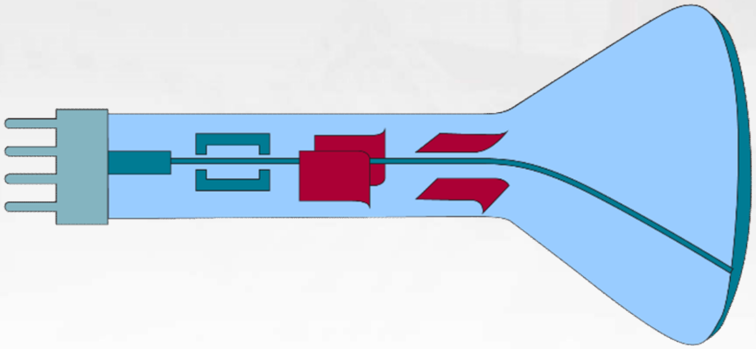
Cathode Ray Tube (CRT)



- Animation on Cathode Ray Tube: http://highered.mheducation.com/olcweb/cgi/pluginpop.cgi?it=swf::100%::100%::/sites/dl/free/0072512644/117354/01_Cathode_Ray_Tube.swf::Cathode%20Ray%20Tube

From Animation Centre – McGraw-Hill

Questions



How much is the charge on bodies of different shapes?

How much is the force between / due to charged bodies of different shapes?

Summary

- Electrostatic precipitator, oscilloscope and photocopier are examples that use electric force between electric charges.

- The CRT is a vacuum tube in which a beam of electrons is accelerated and deflected under the influence of electric or magnetic force.
  - The electrons are deflected in various directions by two sets of plates.

  - The placing of charge on the plates creates the electric force and allows the beam to be steered.

EE3001 Engineering Electromagnetics

Session 4-2

Electric Field Intensity

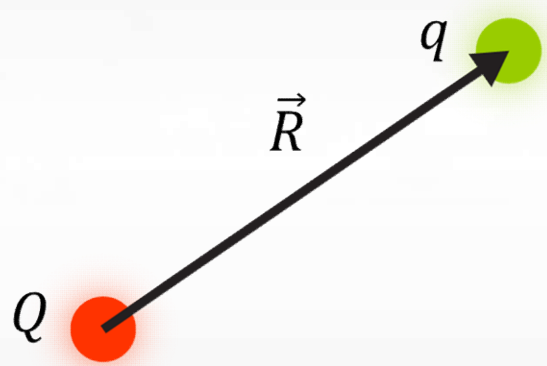
Learning Objectives

- Define electric field intensity;
- State the formula for the electric field intensity of a point charge;
- Calculate the electric field intensity of a given point charge; and
- Obtain the electric field intensity of an electric dipole.

Concept of Electric Field

- The force between two stationary point charges Q and q,
  $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{R^2}\vec{a}_R.$

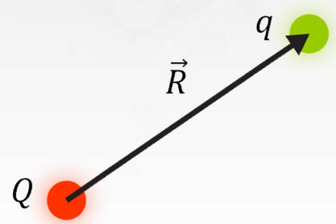
Action at a distance!



Electric field around the charge

Definition of Electric Field Intensity

- The **electric field intensity** $\vec{E}$ is defined as the force exerted on a unit positive charge brought near a charge or a charge distribution:

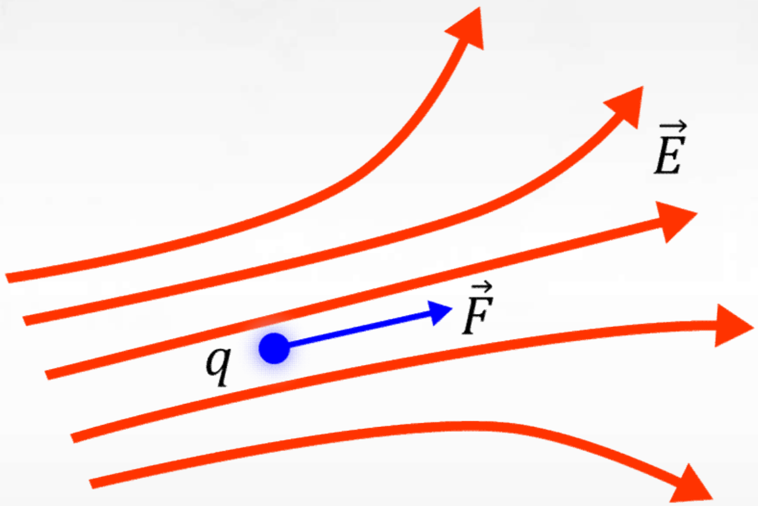


$$\vec{E} = \lim_{q\to 0}\frac{\vec{F}}{q}\ (Newtons/Coulomb)$$

- Limiting operation in the above equation:
  - So that the test charge q does not disturb the original charge distribution and its field distribution.
  - Units of electric field intensity = N/C, **which is equivalent to V/m.**

Electric Force in an Electric Field

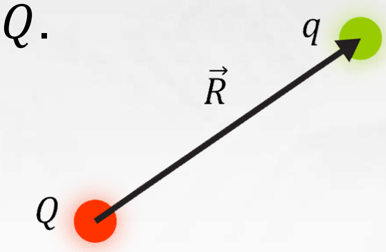
The force acting on a point charge q placed in an electric field $\vec{E}$ is,



$$\vec{F} = q\vec{E}$$

Electric Field Due to Point Charge: Expression

Consider a point charge Q in free space. We want to determine electric field at a point at a distance R away from Q.



We introduce a test (or a probe) point charge q at the point where the field is to be determined.
Using Coulomb's law, we get:

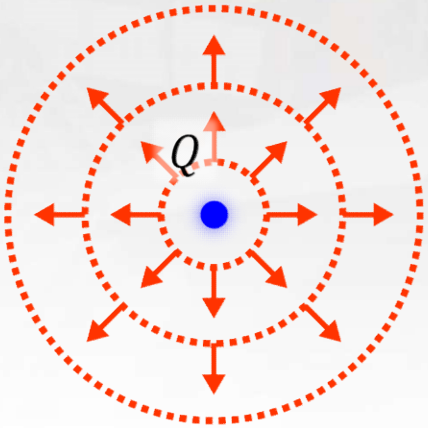
$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{R^2}\vec{a}_R$$

$$\vec{E} = \lim_{q\to 0}\frac{\vec{F}}{q}$$

$$\boxed{\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\vec{a}_R = \frac{Q}{4\pi\varepsilon_0 R^3}\vec{R}}$$

Electric Field Due to Point Charge: Nature

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\vec{a}_R$$



The electric field intensity of a positive point charge:

- ❑ is in the radially outward direction;
- ❑ has a magnitude proportional to the charge; and
- ❑ is inversely proportional to the square of the distance from the point charge.

Quiz

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Hayt and Buck, Ch 2, Q2:

The electric field intensity due to a point charge Q_1 at a distance $R_1 = 1\ \mathrm{cm}$ away from it is $E_1 = 1\ \mathrm{V/m}$. What is the intensity E_2 of the field of a charge $Q_2 = 4Q_1$ at a distance $R_2 = 2\ \mathrm{cm}$ from it?

A: $E_2 = 1\ \mathrm{V/m}$

B: $E_2 = 2\ \mathrm{V/m}$

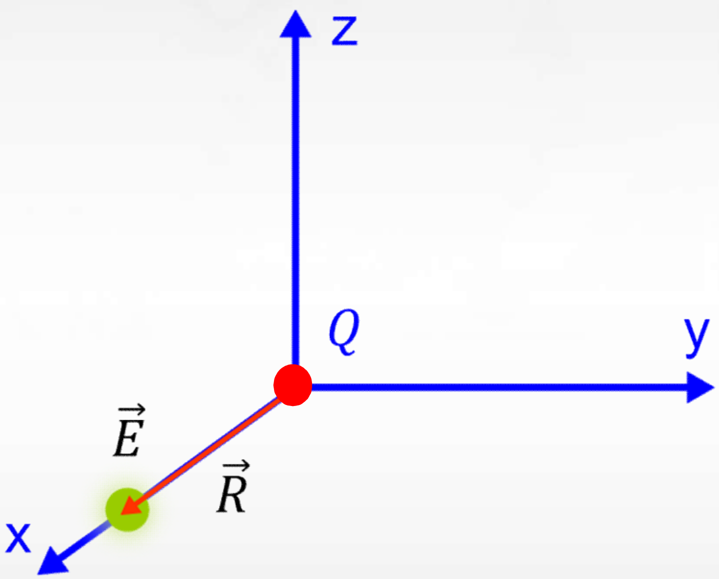
C: $E_2 = 4\ \mathrm{V/m}$

D: $E_2 = \frac{1}{2}\ \mathrm{V/m}$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\vec{a}_R$$

Example: Electric Field Due to Point Charge

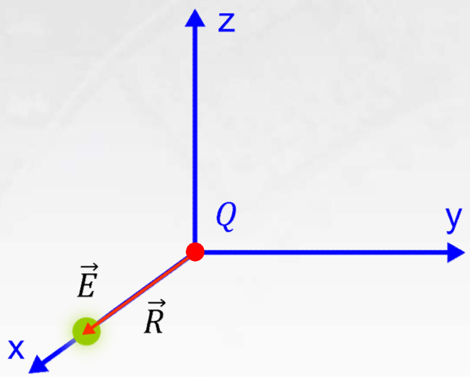
A charge Q is located at the origin and has a charge of $3\ \mu\text{C}$. Find the electric field intensity at point $(3,0,0)\ \text{m}$.



What is the direction of the electric field at point $(3,0,0)$?

Solution

Solution



The vector $\vec{R}$ from the Charge Q at point $(0,0,0)$ to the field point $(3,0,0)$ m is

$$\vec{R} = 3\vec{a}_x$$

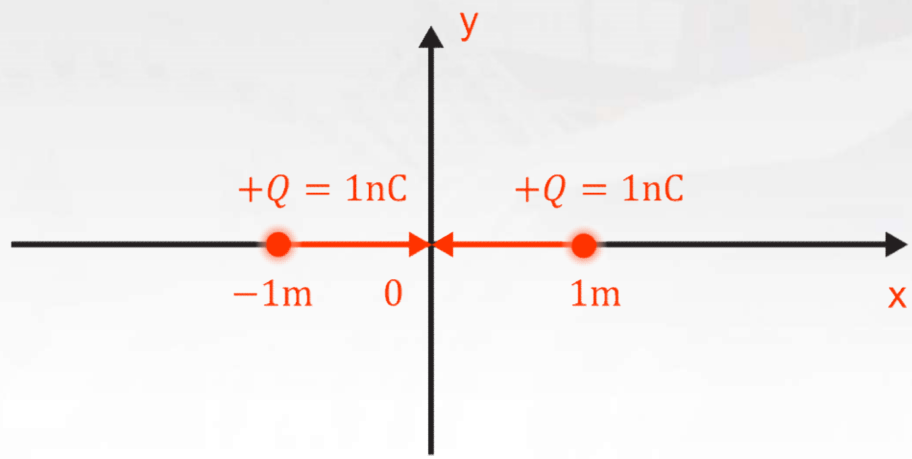
The electric field at point $(3,0,0)$ m is then

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\vec{a}_R = \frac{Q}{4\pi\varepsilon_0 R^3}\vec{R} = \frac{9\times 10^9 \times 3\times 10^{-6} \times 3\vec{a}_x}{3^3}$$

$$= 3000\vec{a}_x \ (V/m)$$

Next

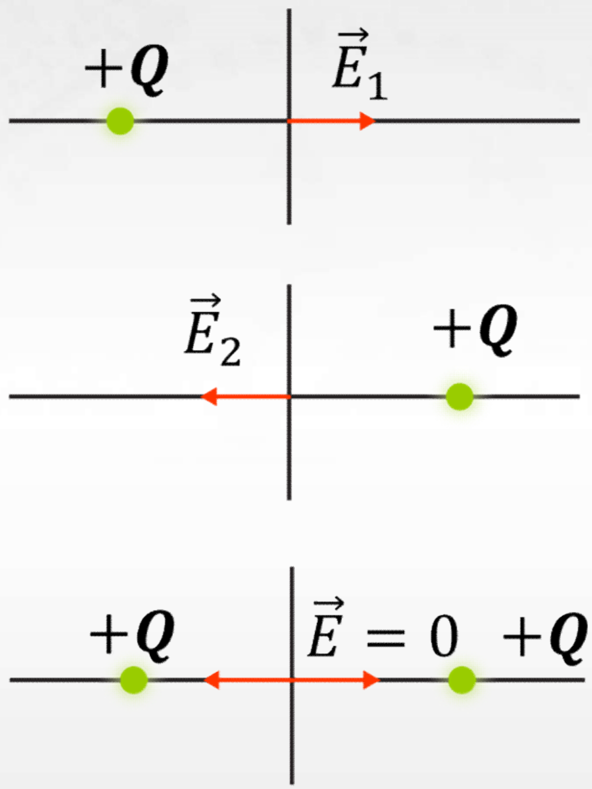
Electric Field Due to Two Positive Point Charges

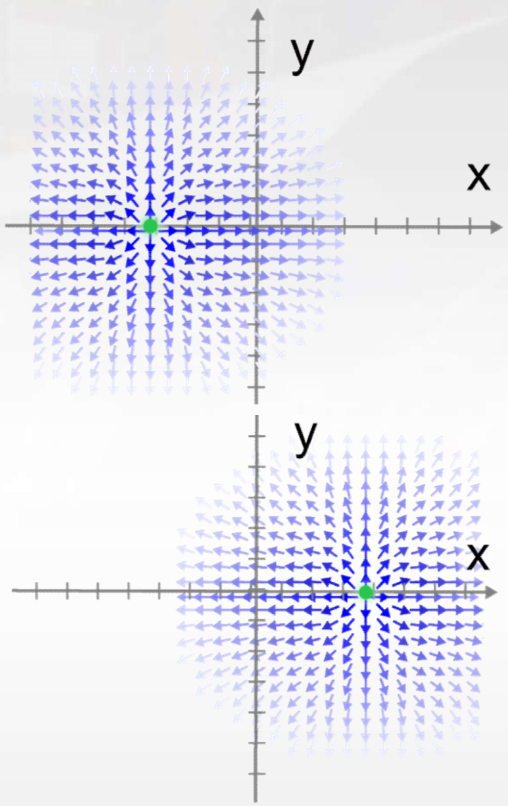


Question:

What is the electric field intensity $\vec{E}$ at the origin?

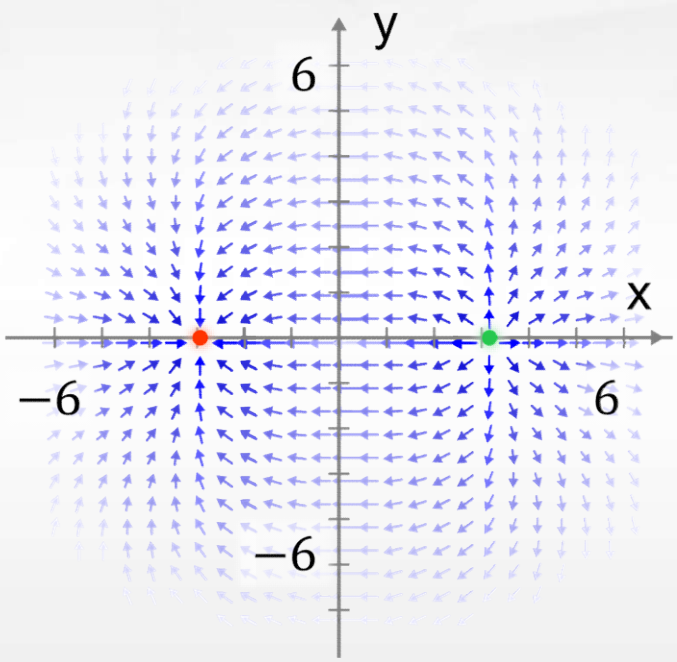
Superposition





Electric Dipole

An electric dipole consists of a pair of point charges: one positive and the other negative; both having the same amount of charge.



Summary

- The electric field intensity $\vec{E}$ is defined as the force exerted on a unit positive charge brought near a charge or a charge distribution:

$$\vec{E} = \lim_{q \to 0} \frac{\vec{F}}{q} \; (Newtons/Coulomb)$$

- The electric field intensity of a positive point charge is in the radially outward direction and is defined as:

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2} \vec{a}_R$$

- An electric dipole consists of a pair of point charges: one positive and the other negative; both having the same amount of charge.

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Session 4-3

Electric Field Due To Different Types of Charge Distributions

Learning Objectives

- Obtain the electric field intensity due to a line charge distribution;
- Obtain the electric field intensity due to a surface charge distribution;
- Obtain the electric field intensity due to a volume charge distribution;
- Explain the concept of source and field points; and
- State the points to note when choosing a source point.

Electric Field Due to a Line Charge Distribution

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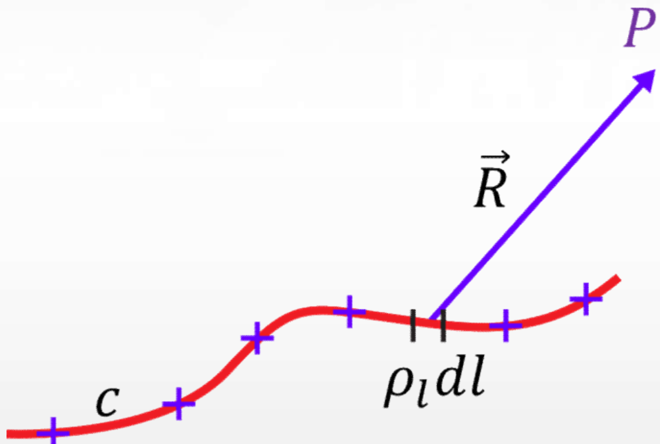
We consider a differential element dl on the line.

- The charge contained in dl is $\rho_l dl$.

According to the formula for a point charge, the differential element $\rho_l dl$ produces a differential electric field at a distance R:



Line charge

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\rho_l dl}{R^2}\vec{a}_R$$

Electric Field Due to a Line Charge Distribution

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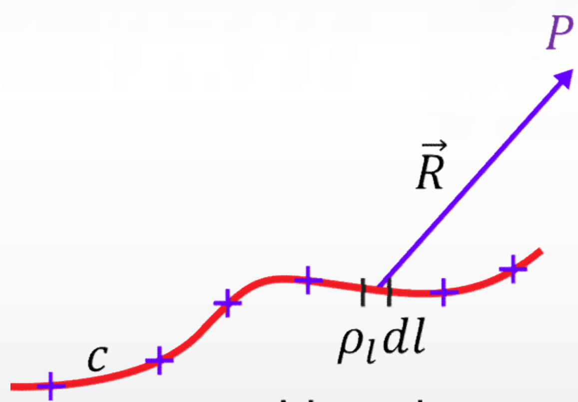
Finally, the electric field at a point P due to a line charge distribution ρ_l is

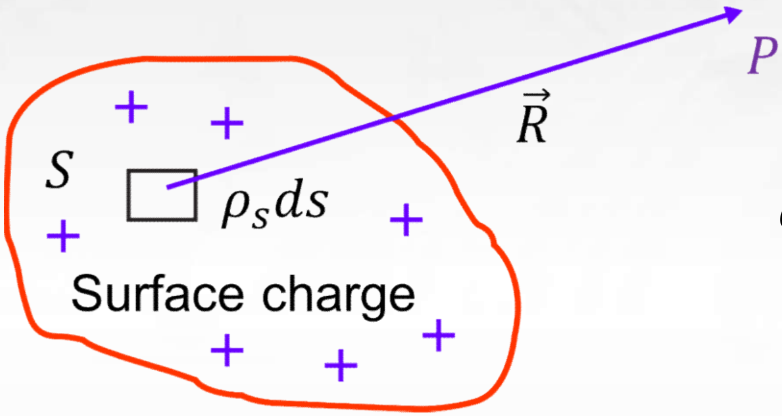
$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_C \frac{\rho_l dl}{R^2}\vec{a}_R = \frac{1}{4\pi\varepsilon_0}\int_C \frac{\rho_l dl}{R^3}\vec{R}}$$



Line charge

Electric Field Due to Surface Charge Distribution

If there is a surface charge distribution, the electric field at a point P due to a small element of surface ds is:

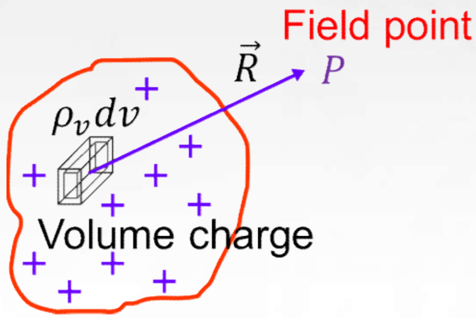


$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\rho_s ds}{R^2}\vec{a}_R$$

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\iint_S \frac{\rho_s ds}{R^2}\vec{a}_R = \frac{1}{4\pi\varepsilon_0}\iint_S \frac{\rho_s ds}{R^3}\vec{R}}$$

Electric Field Due to Volume Charge Distribution

For a volume charge distribution, the electric field at a point P due to a small element of volume dv is:

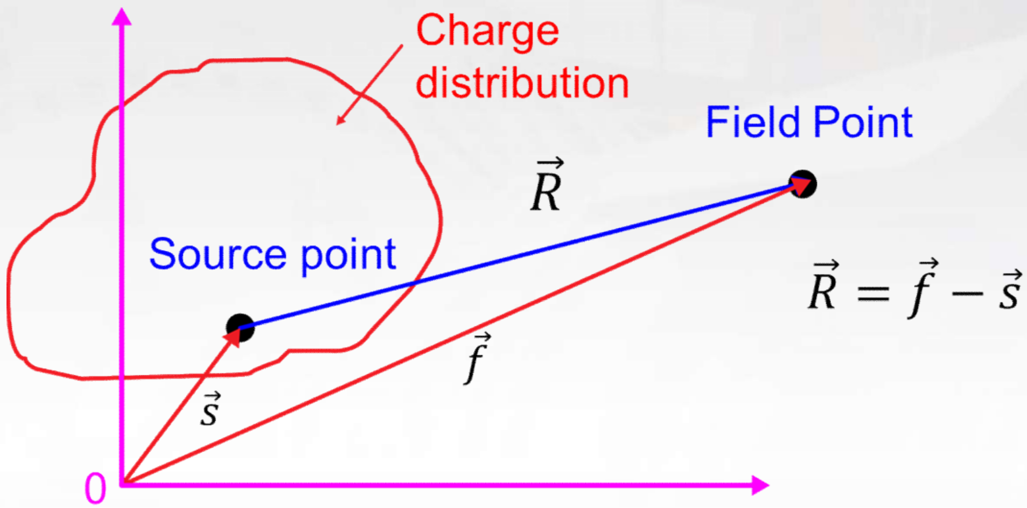


$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\rho_v dv}{R^2}\vec{a}_R$$

The electric field intensity at P due to the entire volume charge distribution then becomes:

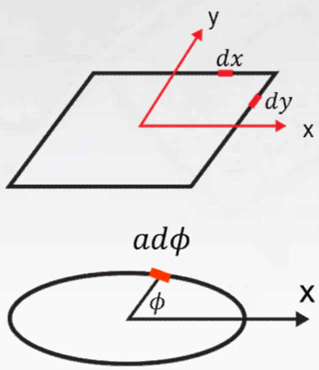
$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho_v dv}{R^2}\vec{a}_R = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho_v dv}{R^3}\vec{R}}$$

Source Point and Field Point

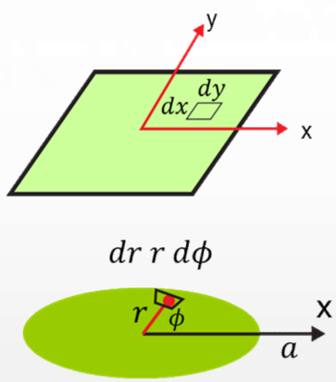


- **Source point** ($\vec{s}$) is one of the points where the electric charge (source) is located.
- **Field point** ($\vec{f}$) is the point where the field is to be determined.

How to Choose the Source Point



Charged loops



Charged plates

- **Source point** is where the differential element of charge (source) is located.
- **Source point** should always be on or within the charge distribution.
- **Source element** must include coordinates that are varying so that it can represent any point on the entire charge distribution.

Summary

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- The electric field due to a line charge distribution with charge density ρ_l:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_C \frac{\rho_l dl}{R^2}\vec{a}_R = \frac{1}{4\pi\varepsilon_0}\int_C \frac{\rho_l dl}{R^3}\vec{R}$$

- The electric field due to a surface charge distribution with charge density ρ_s:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\iint_S \frac{\rho_s ds}{R^2}\vec{a}_R = \frac{1}{4\pi\varepsilon_0}\iint_S \frac{\rho_s ds}{R^3}\vec{R}$$

Summary

- The electric field intensity due to a volume charge distribution is:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \iiint\limits_V \frac{\rho_v dv}{R^2} \vec{a}_R = \frac{1}{4\pi\varepsilon_0} \iiint\limits_V \frac{\rho_v dv}{R^3} \vec{R}$$

- Source point ($\vec{s}$) is one of the points where the electric charge (source) is located. Field point ($\vec{f}$) is the point where the field is to be determined.

Summary

- When we choose a source point, we have to take note of the following:
  - Source point is where the differential element of charge (source) is located.
  - Source point should always be on or within the charge distribution.
  - Source element must include coordinates that are varying so that it can represent any point on the entire charge distribution.

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Session 4-4

Example of Calculation of Electric Field Intensity

Learning Objectives

- Apply the formula learnt to find the electric field intensity due to a line charge distribution; and
- Use the result so obtained to determine the electric field intensity of a line charge.

Example: An Infinitely Long Line Charge

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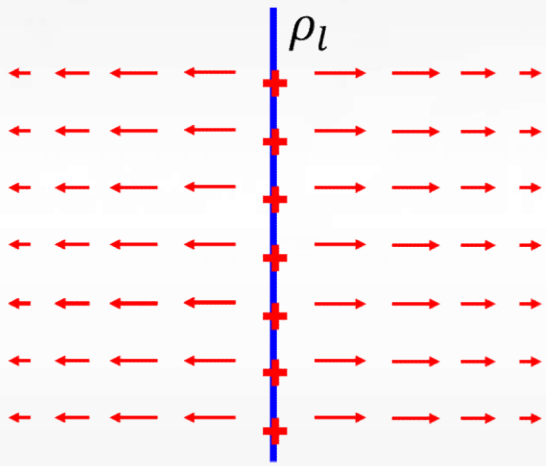
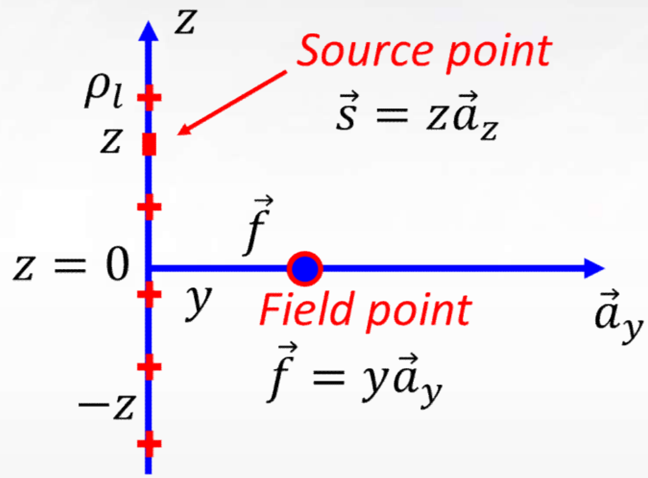
An infinitely long straight line charge, with a uniform line charge density ρ_l, is situated in air, along the z −axis, as shown. Determine the electric field intensity at a point at a distance y away from the line charge along the y −axis.





Solution

An Infinitely Long Line Charge Solution

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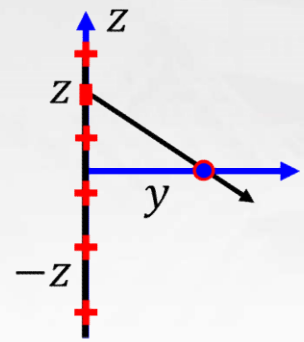
Field point $\vec{f} = y\vec{a}_y$

Source point $\vec{s} = z\vec{a}_z$

Source element $dl = dz$



We obtain

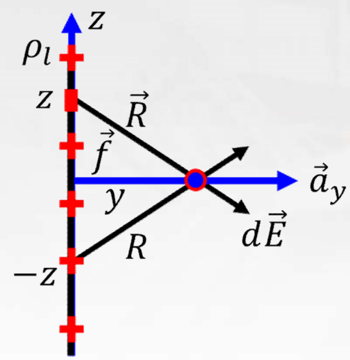
$$\vec{R} = \vec{f} - \vec{s} = y\vec{a}_y - z\vec{a}_z \qquad R = \sqrt{y^2 + z^2}$$

Then the electric field intensity is

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_l \vec{R}}{R^3} dl = \frac{\rho_l}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty} \frac{y\vec{a}_y - z\vec{a}_z}{(y^2 + z^2)^{3/2}} dz$$

Next

An Infinitely Long Line Charge Solution (cont.)

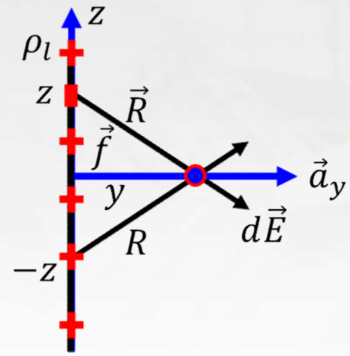


- When **symmetry is used** considering two symmetrical elements at z and $-z$, respectively, the addition of the two differential electric fields is in the y (or radial) direction.

- Due to the symmetry of the charge distribution, the z −directed component vanishes.

Next

An Infinitely Long Line Charge Solution (cont.)



- Use the integral formula, $\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}}$

$$\vec{E} = \frac{\rho_l}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty} \frac{y\vec{a}_y}{(y^2+z^2)^{3/2}}dz = \frac{y\rho_l\vec{a}_y}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{dz}{(y^2+z^2)^{3/2}}$$

$$= \frac{y\rho_l\vec{a}_y}{4\pi\varepsilon_0}\left[\frac{z}{y^2\sqrt{y^2+z^2}}\right]_{-\infty}^{+\infty} = \vec{a}_y\frac{\rho_l}{2\pi\varepsilon_0 y}\ V/m$$

$$\rightarrow \vec{a}_r\frac{\rho_l}{2\pi\varepsilon_0 r}\ V/m$$

Next

An Infinitely Long Line Charge Solution (cont.)

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$$\left[\frac{z}{y^2\sqrt{y^2+z^2}}\right]_{-\infty}^{+\infty}$$

$$= \left.\frac{z}{y^2\sqrt{y^2+z^2}}\right|_{z\to+\infty} - \left.\frac{z}{y^2\sqrt{y^2+z^2}}\right|_{z\to-\infty}$$

$$= \left.\frac{2z}{y^2 z\sqrt{\left(\frac{y}{z}\right)^2+1}}\right|_{z\to+\infty}$$

$$= \frac{2}{y^2}$$

Next

Quiz

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Hayt and Buck, Ch 2, Q3:

The intensity of the field due to a line charge ρ_{l1} a distance $r_1 = 1\ \text{cm}$ away from it is $E_1 = 1\ \text{V/m}$. What is the intensity E_2 of the field of the line charge $\rho_{l2} = 4\rho_{l1}$ at a distance $r_2 = 2\ \text{cm}$ from it?

A: $E_2 = 1\ \text{V/m}$

B: $E_2 = 4\ \text{V/m}$

C: $E_2 = 2\ \text{V/m}$

D: $E_2 = \frac{1}{2}\ \text{V/m}$

$$\vec{E} = \vec{a}_y \frac{\rho_l}{2\pi\varepsilon_0 y}\ \text{V/m} \rightarrow \vec{a}_r \frac{\rho_l}{2\pi\varepsilon_0 r}$$

Next

Summary

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- The following is the formula used to find the electric field intensity due to a line charge distribution:
  - $\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_L \frac{\rho_l \vec{R}}{R^3} dl$

- The electric field intensity of a line charge:

$$\vec{E} = \vec{a}_y \frac{\rho_l}{2\pi\varepsilon_0 y} \ \mathrm{V/m} \rightarrow \vec{a}_r \frac{\rho_l}{2\pi\varepsilon_0 r}$$

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EE3001 Engineering Electromagnetics

Session 5-1

2nd Example of Calculation of Electric Field Intensity

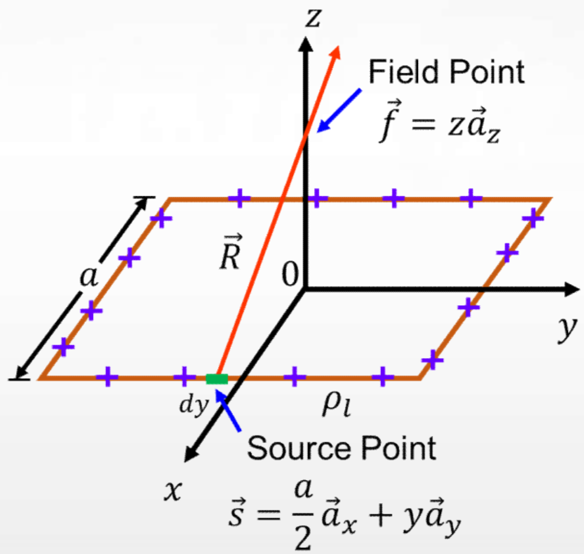
Learning Objectives

- ❑ Apply the formula to find the electric field intensity due to a square loop; and
- ❑ Explain the uses of electric charges in a laser printer.

Example: A Charged Square Loop

A Charged Square Loop:
A square loop of side length a, positioned in the xy −plane and centered at the origin, is uniformly charged with a line charge density ρ_l.

Determine the electric field intensity at a point $(0,0,z)$.



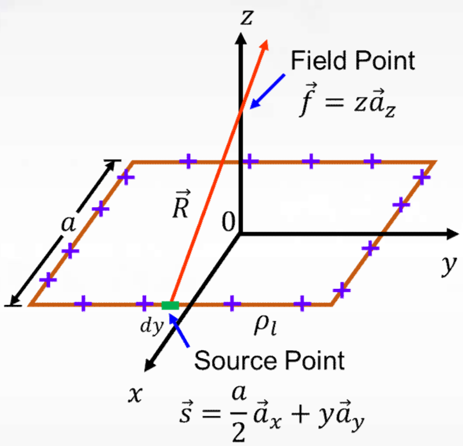
Solution to Example

Choosing line section at $x = \frac{a}{2}$:

Field point is at $(0,0,z)$: $\vec{f} = z\vec{a}_z$

Source point is at $(\frac{a}{2}, y, 0)$: $\vec{s} = \frac{a}{2}\vec{a}_x + y\vec{a}_y$

Source element $dl = dy$



Then

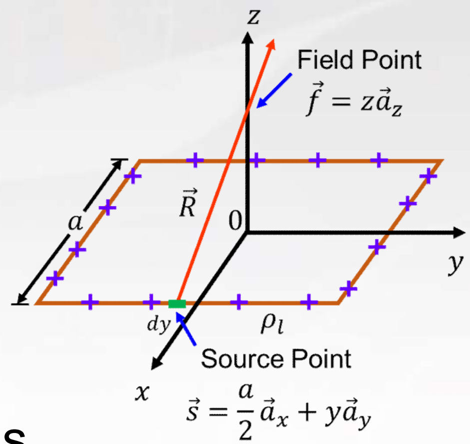
$$\vec{R} = \vec{f} - \vec{s} = z\vec{a}_z - \left(\frac{a}{2}\vec{a}_x + y\vec{a}_y\right)$$

$$R = \sqrt{\left(\frac{a}{2}\right)^2 + y^2 + z^2}$$

Solution to Example (cont.)

We know: $\vec{R} = \vec{f} - \vec{s} = z\vec{a}_z - \left(\frac{a}{2}\vec{a}_x + y\vec{a}_y\right)$

$$R = \sqrt{\left(\frac{a}{2}\right)^2 + y^2 + z^2}$$

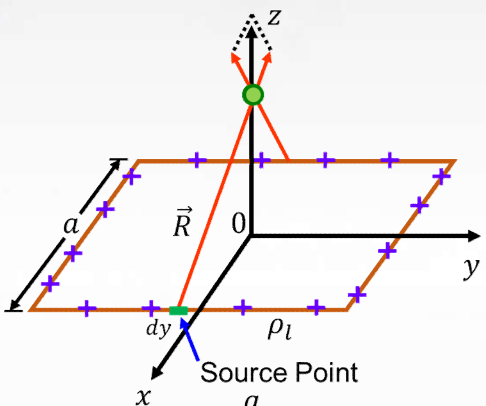


So, the electric field intensity due to this line section is

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\int_{one\ side}\frac{\rho_l\vec{R}}{R^3}dl = \frac{\rho_l}{4\pi\varepsilon_0}\int_{-\frac{a}{2}}^{\frac{a}{2}}\frac{z\vec{a}_z - \left(\frac{a}{2}\vec{a}_x + y\vec{a}_y\right)}{\left(\left(\frac{a}{2}\right)^2 + y^2 + z^2\right)^{3/2}}dy$$

Solution to Example (cont.)

Due to the symmetry of the charge distribution about the z −axis, the x and y components vanish, as seen in the figure.



$$\vec{s} = \frac{a}{2}\vec{a}_x + y\vec{a}_y$$

$$\int \frac{dx}{(x^2+b)^{3/2}} = \frac{x}{b\sqrt{x^2+b}}$$

$$\vec{E}_1 = \frac{\rho_l}{4\pi\varepsilon_0}\int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{z\vec{a}_z}{\left(\left(\frac{a}{2}\right)^2 + y^2 + z^2\right)^{3/2}} dy$$

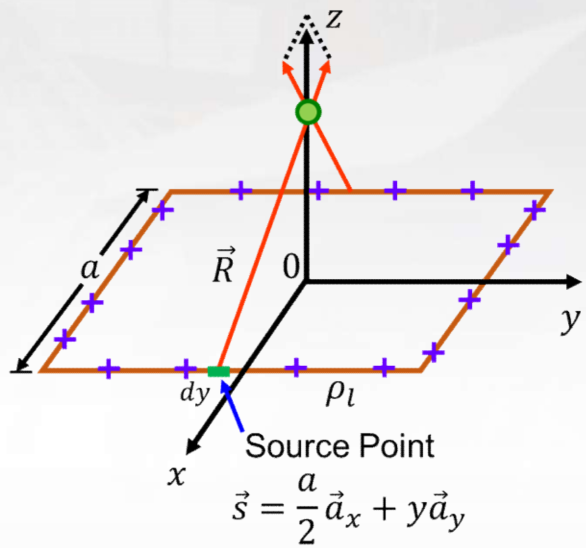
$$= \frac{\rho_l \vec{a}_z}{4\pi\varepsilon_0} \frac{z}{\left(\frac{a}{2}\right)^2 + z^2} \frac{y}{\left(y^2 + \left(\frac{a}{2}\right)^2 + z^2\right)^{1/2}}\Bigg|_{-\frac{a}{2}}^{\frac{a}{2}}$$

$$= \frac{\rho_l \vec{a}_z}{4\pi\varepsilon_0} \frac{z}{\left(\frac{a}{2}\right)^2 + z^2} \frac{a}{\left(\frac{a^2}{2} + z^2\right)^{1/2}} \; V/m$$

Solution to Example (cont.)

For the entire loop,

$$\vec{E} = \frac{\rho_l \vec{a}_z}{\pi \varepsilon_0} \frac{z}{\left(\frac{a}{2}\right)^2 + z^2} \frac{a}{\left(\frac{a^2}{2} + z^2\right)^{1/2}} \; V/m$$



Application of Electric Charges

Laser printer

https://www.youtube.com/watch?v=yMmupCmo-70

Uploaded by Eco Device Enterprise

Appendix

Summary

- ❑ Apply the formula to find the electric field intensity due to a square loop; and
- ❑ Explain the uses of electric charges in a laser printer.

Laser Printer - 1

- Below is the list of the use of electric charges in a laser printer:
  - The drum brushes against an electrically charged roller that coats the drum's surface with negative charges of electricity.
  - As the laser zaps the drum, it creates neutral spaces on the surface of the drum until it forms an image of the document on the drum.
  - The toner is stirred inside the hopper such that the friction from the stirring generates static electricity that causes the toner to be negatively charged.

Laser Printer - 2

- Below is the list of the use of electric charges in a laser printer:
  - When the drum turns and passes a roller that is covered with toner, the toner is attracted to the neutral spaces on the drum. At the same time, the toner is repelled by the negative charges that covered the rest of the drum.
  - While all these are happening, the printer is charging the piece of paper with positive electric charges.
  - When the paper passes through the rolling drum that carries its toner image, the force of attraction pulls the toner off the drum and onto the page.

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EE3001 Engineering Electromagnetics

Session 5-2

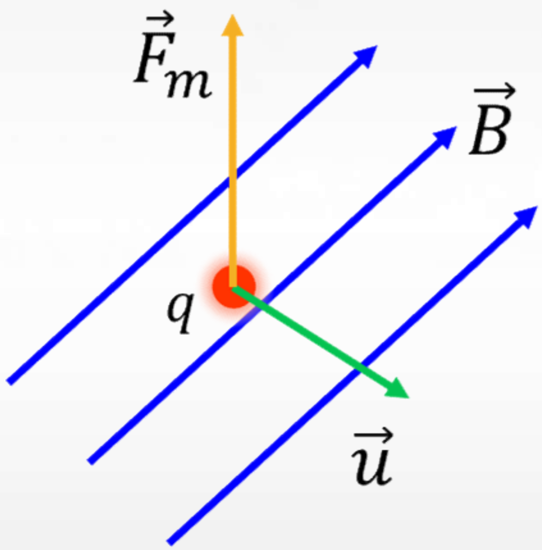
Lorentz Force

Learning Objectives

- Express the force on a charged particle in a magnetic field;
- Explain Lorentz force; and
- Apply the concept of Lorentz force.

Force on a Charged Particle in a Magnetic Field

Based on experiments conducted to determine the motion of charged particles in magnetic fields, it was established that the magnetic force $\vec{F}_m$ acting on a particle of charge q can be written in the form:

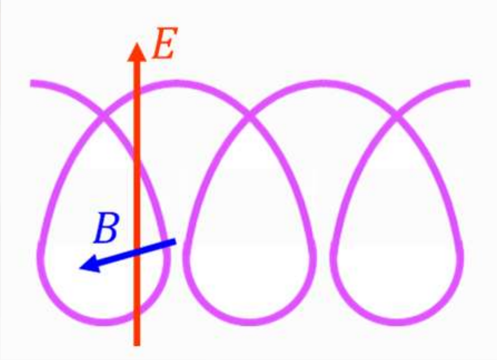


$$\vec{F}_m = q\vec{u} \times \vec{B} \ (N)$$

where $\vec{u}$ is the velocity at which the charged particle is moving.

Lorentz Force

If a charged particle is under the influence of both an electric field $\vec{E}$ and a magnetic field $\vec{B}$, then the total electromagnetic force acting on it is:



$$\vec{F} = \vec{F}_e + \vec{F}_m = q\vec{E} + q\vec{u} \times \vec{B} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

Notes:

- ❑ The electric force is always in the direction of the electric field, while the magnetic force is always perpendicular to the magnetic field.

- ❑ The electric force acts on a charged particle whether or not it is moving, the magnetic force acts on it only when it is in motion.

Quiz

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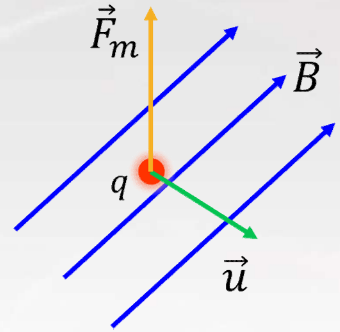
Hayt and Buck, Ch 9, Q1:

$$\boxed{\vec{F} = \vec{F}_e + \vec{F}_m = q\vec{E} + q\vec{u} \times \vec{B} = q(\vec{E} + \vec{u} \times \vec{B})}$$



The net force on a charged particle subject to both electric and magnetic fields may be zero under the following condition:

A: Never

B: If the particle is at rest

C: If the magnetic field is in the direction of the particle's velocity

D: If the magnetic field is perpendicular to the direction of the particle's velocity

Summary

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- The force on a charged particle in a magnetic field is:

$$\vec{F}_m = q\vec{u} \times \vec{B} \ (N)$$

- Lorentz force arises if a charged particle is under the influence of both an electric field $\vec{E}$ and a magnetic field $\vec{B}$; then the total force acting on the charged particle is:

$$\vec{F} = \vec{F}_e + \vec{F}_m = q\vec{E} + q\vec{u} \times \vec{B} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

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Session 6-1

Operations Using Del Operator: Gradient

Learning Objectives

- ❑ Explain the importance of del operator;
- ❑ Explain the physical meaning of gradient;
- ❑ State the formula for calculation of gradient in Cartesian coordinates; and
- ❑ Apply the formula for calculation of gradient in Cartesian coordinates.

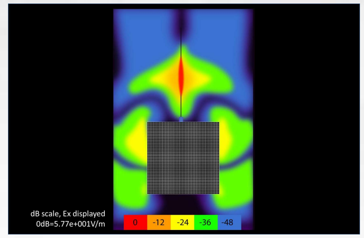
Vector Analysis Information

- This lecture reviews the operations using del operator.

- You are strongly encouraged to refer to:
  - The **Preparatory Notes** in NTULearn for a quick introduction;
    and
  - Chapter 1 of the first text book (Hayt & Buck)
    or
  - Part 1 of the second text book (Sadiku)
    or
  - Notes for EE2007 – Engineering Mathematics II - for detailed explanation and derivation.

∇ Operator

- ∇ **(del or nabla) operator** provides a means of describing the spatial derivatives of a scalar or a vector field.

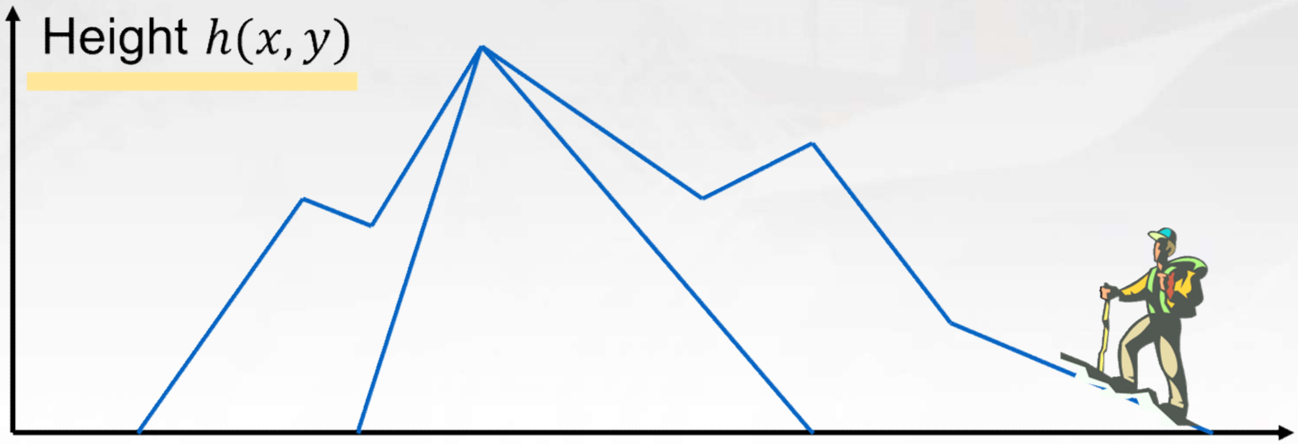


- The **gradient** of a **scalar** function (or field):
  Relates electric field and electric potential

- The **divergence** of a **vector** field:
  Used in Maxwell's equations (Gauss Law)

- The **curl** of a **vector** field:
  Used in Maxwell's equations (Ampere's Law, Faraday's Law)

Slope of a Mountain



What is the maximum slope of the mountain at any point?

Given $h(x, y)$

Gradient of a Scalar Field

The **gradient** of a scalar field is defined as a vector that represents

- The maximum space rate of increase of that scalar field in both magnitude and direction.

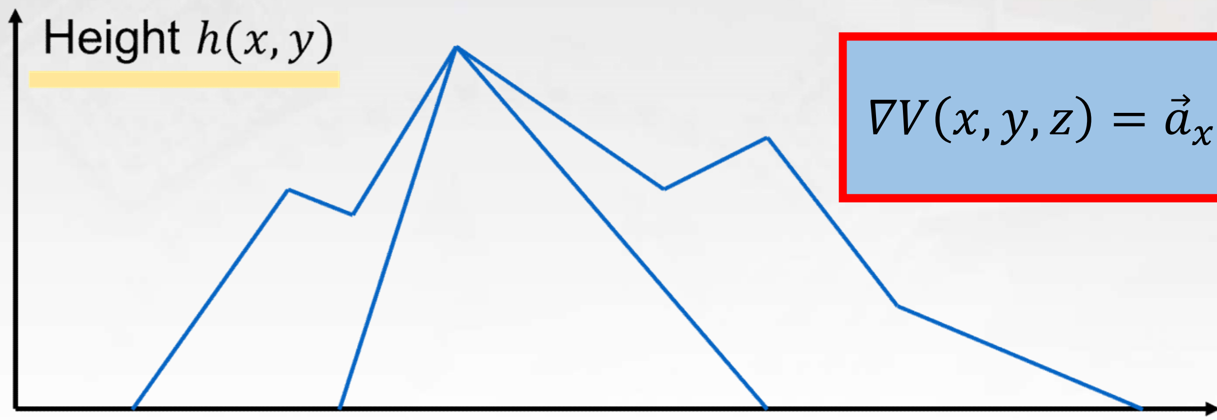
$$grad V = \nabla V = \vec{a}_n \frac{dV}{dn}$$

In Cartesian coordinates,

$$\boxed{\nabla V(x,y,z) = \vec{a}_x \frac{\partial V}{\partial x} + \vec{a}_y \frac{\partial V}{\partial y} + \vec{a}_z \frac{\partial V}{\partial z}}$$

How do we get this formula?

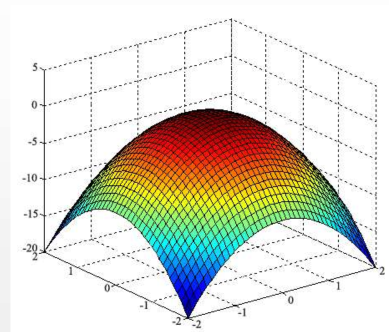
Maximum Slope of a Mountain: Example



$$\nabla V(x, y, z) = \vec{a}_x \frac{\partial V}{\partial x} + \vec{a}_y \frac{\partial V}{\partial y} + \vec{a}_z \frac{\partial V}{\partial z}$$

The maximum slope of the mountain at a point is the gradient of the height function at that point.

Example: Given $h(x, y) = 4 - x^2 - 2y^4$



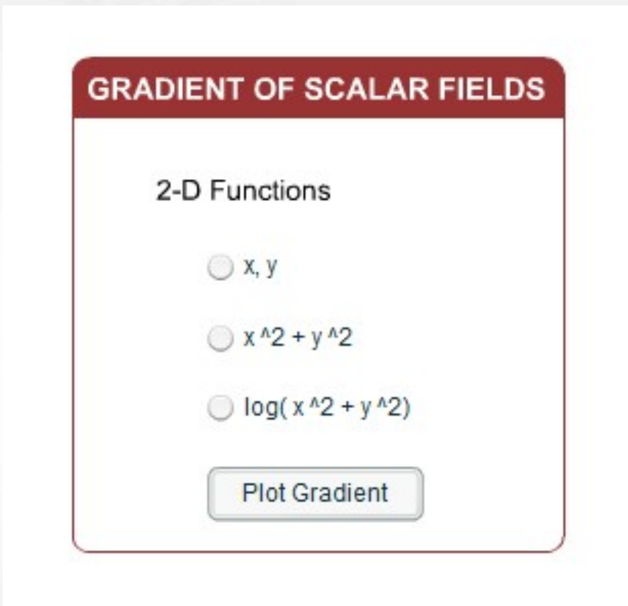
$$\nabla h(x, y) = \vec{a}_x \frac{\partial h(x, y)}{\partial x} + \vec{a}_y \frac{\partial h(x, y)}{\partial y} + 0$$

$$= \vec{a}_x(-2x) + \vec{a}_y(-8y^3)$$

Interactive Display 1

Gradient of scalar fields

- Hayt and Buck, Ch.4, Interactive 3



Appendix

Summary

- ∇ (del or nabla) operator provides a means of describing the spatial derivatives of a scalar or a vector field which is a function of multiple variables.

- The gradient of a scalar field is defined as a vector that represents
  - The maximum space rate of increase of that scalar field,
  - In both magnitude and direction.

- In Cartesian coordinate system, the gradient of a scalar function is

$$\nabla V(x, y, z) = \vec{a}_x \frac{\partial V}{\partial x} + \vec{a}_y \frac{\partial V}{\partial y} + \vec{a}_z \frac{\partial V}{\partial z}$$

Additional Information on Gradient

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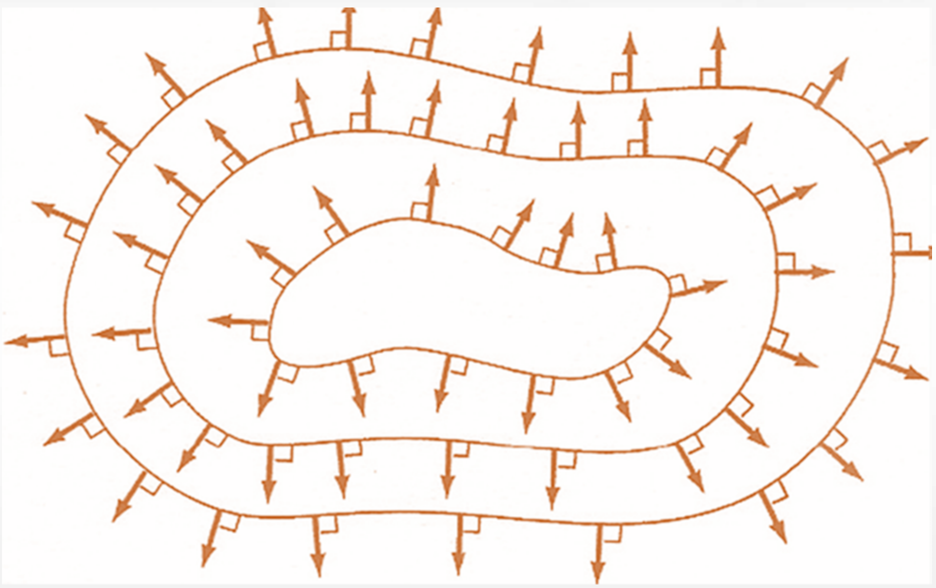
From the definition of gradient operation, it is obvious that ∇V is a vector normal to the surface $V(x, y, z) = Const.$, since on the constant V surface the tangential variation of the scalar function $V(x, y, z)$ is zero.



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EE3001 Engineering Electromagnetics

Session 6-2

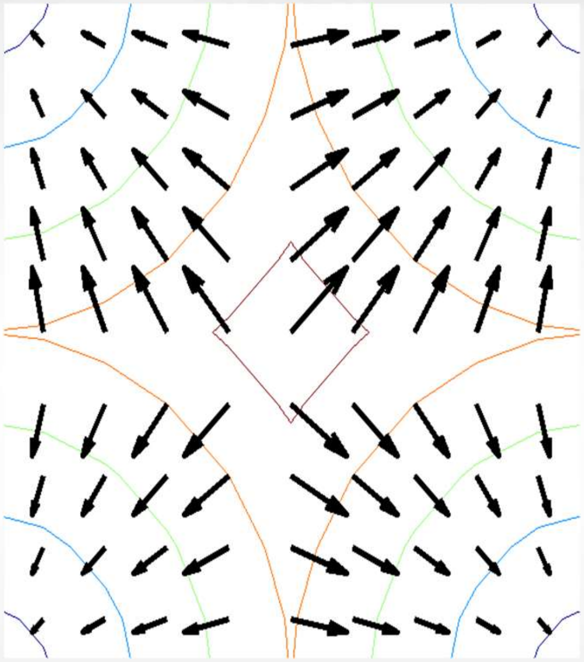
Divergence and Surface Integral

Learning Objectives

- Explain the physical meaning of divergence;
- State the formula for calculation of divergence in Cartesian coordinates;
- Apply the formula for calculation of divergence in Cartesian coordinates;
- Explain the concept of vector element of area;
- Explain the concept of surface integral; and
- State the formula for calculating surface integral.

Divergence

Consider air coming out of a punctured balloon.



What is the net outward 'flux'?

Divergence of a Vector Field

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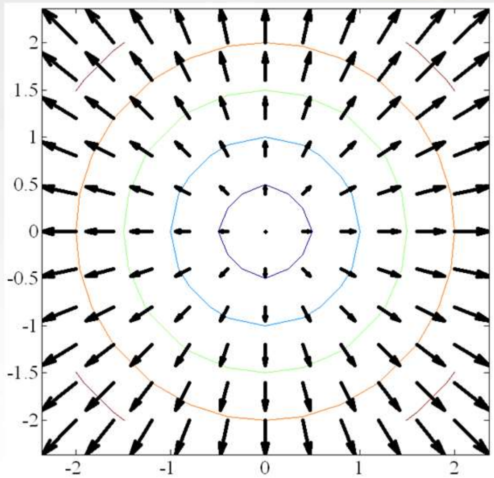
- ❑ The **divergence** of a vector field is a measure of:
  - the net outward flux from an elemental volume dV at a point in the field.

- ❑ In Cartesian coordinates, we obtain:

$$\boxed{\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}}$$

Appendix

Divergence: Example 1

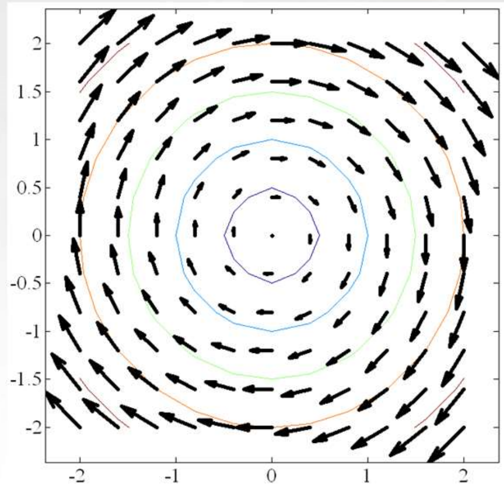


- Given $\vec{F} = x\vec{a}_x + y\vec{a}_y$

$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$= \frac{\partial (x)}{\partial x} + \frac{\partial (y)}{\partial y} + \frac{\partial (0)}{\partial z} \quad = 2$$

Divergence: Example 2



- Given $\vec{F} = y\vec{a}_x - x\vec{a}_y$

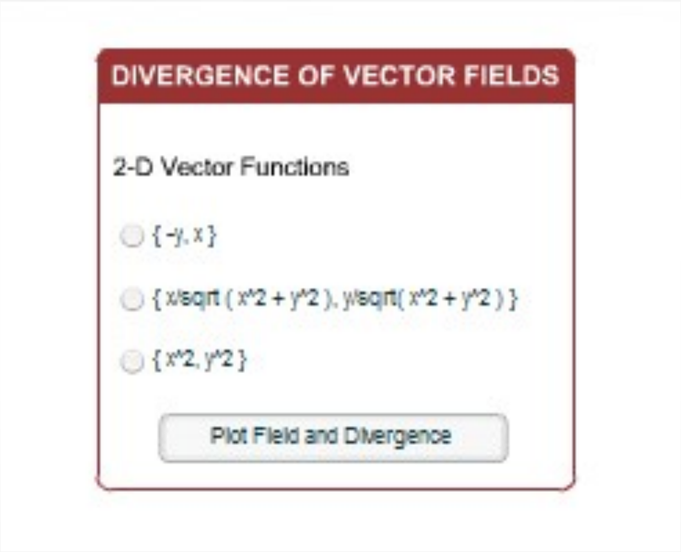
$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$= \frac{\partial (y)}{\partial x} + \frac{\partial (-x)}{\partial y} + \frac{\partial (0)}{\partial z} = 0$$

Interactive Display 2

Divergence of vector fields

- Hayt and Buck, Ch.3, Interactive 1

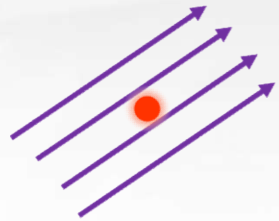


Physical Meaning of Divergence

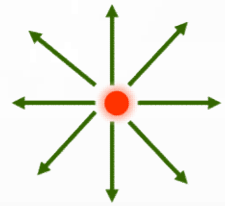
Divergence is a measure of the net outward flow (or source strength) at a point.

Examples: Air from a balloon; car exhaust; kitchen sink; etc

$\nabla \cdot \vec{A} = 0$: The flux which is flowing into a point is equal to the flux which is flowing out



$\nabla \cdot \vec{A} > 0$: More flux flowing out than flowing in (existence of a "source")



$\nabla \cdot \vec{A} < 0$: More flux is flowing in than flowing out (existence of a "sink")

Question

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How do we get?

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

Through the concept of flux!

Flux is related to surface integral

(one needs to calculate surface integral to determine electric flux, magnetic flux and current)

Vector $\vec{ds}$ and Scalar ds

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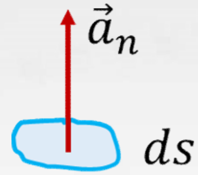
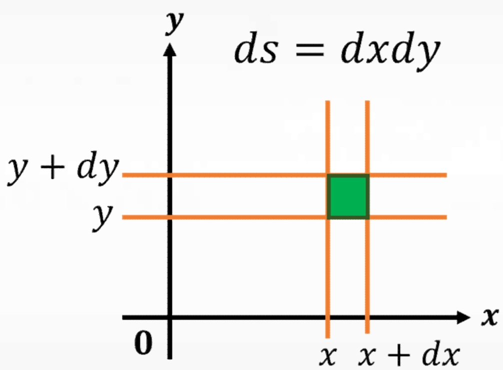
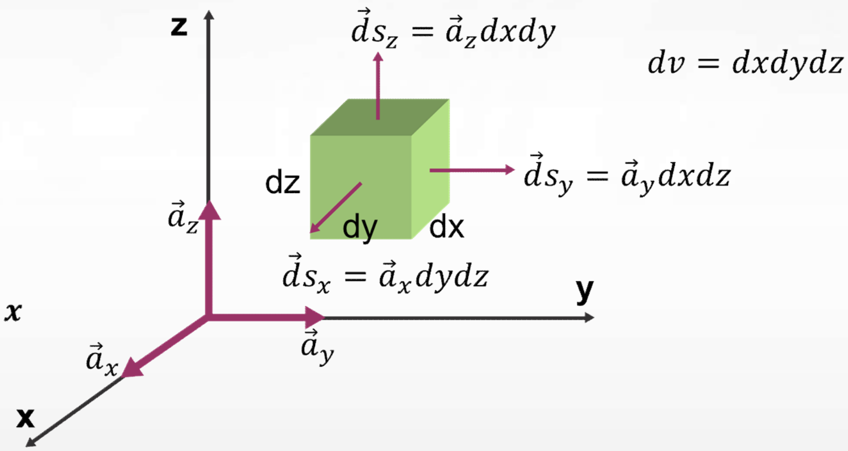
$$\vec{ds} = \vec{a}_n ds$$







$$\vec{ds} = \vec{a}_z dxdy$$

Surface Integral (or Flux)

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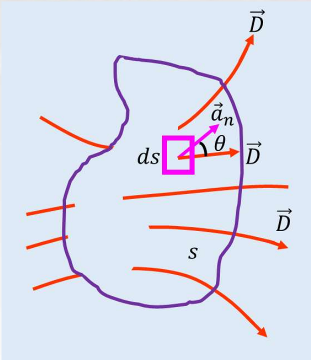
- The **surface integral** or flux of any vector field $\vec{D}$ through a surface S is defined as



$$\iint_s \vec{D} \cdot \vec{a}_n ds = \iint_s |\vec{D}| \cos\theta \, ds$$

- ds is a differential element of surface area at a point on the surface.

$$\iint_s \vec{D} \cdot \vec{a}_n ds = \iint_s \vec{D} \cdot \vec{d}s$$

- θ is the angle between the vector field $\vec{D}$ and the unit vector $\vec{a}_n$ normal to the surface at the point.

$$\vec{d}s = \vec{a}_n ds$$

Quiz

Hayt and Buck, Ch 3, Q3:

The flux of a vector quantity crossing a surface:

A: Is always zero

B: Is related to the quantity's component normal to the surface

C: Is related to the quantity's component tangential to the surface

D: Is not related in any way to the divergence of that vector quantity

Summary

- The divergence of a vector field is a measure of:
  - the net outward flux from an element of volume dV at a point in the field.

- In Cartesian coordinate system, the divergence of a vector field is:

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

- Divergence of a vector field may be zero, positive, or negative.

Summary

- By attaching a unit vector which is perpendicular to an element of area, we get the corresponding vector element of area; and

- The surface integral or flux of any vector field $\vec{D}$ through a surface S is:

$$\iint_S \vec{D} \cdot \vec{a}_n ds = \iint_S \vec{D} \cdot \vec{ds}$$

Divergence of a Vector Field: Definition

The **divergence** of a vector field is defined as the net outward flux of the vector $\vec{E}$ per unit volume as the volume shrinks to zero.

$$\nabla \cdot \vec{E} = \lim_{\Delta V \to 0} \frac{\oint\!\!\oint_S \vec{E} \cdot \vec{a}_n \, ds}{\Delta V}$$

Flux, and therefore divergence, is a scalar quantity

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EE3001 Engineering Electromagnetics

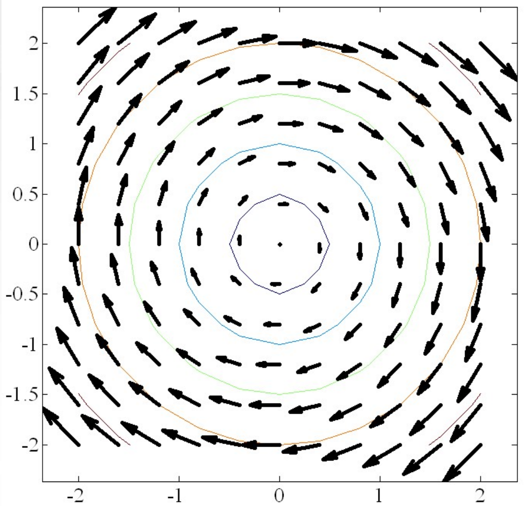
Session 6-3

Curl, Line Integral and Summary of ∇ Operations

Learning Objectives

- Explain the physical meaning of curl;
- State the formula for calculating curl in Cartesian coordinates;
- Apply the formula for calculating curl;
- Explain the concept of vector element of length;
- Explain the concept of line integral; and
- Summarise the del operations in Cartesian and cylindrical coordinate systems.

Curl



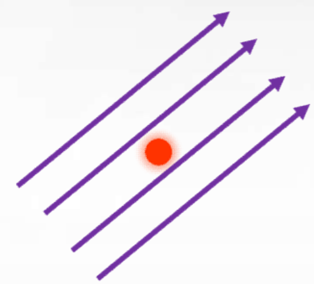
Consider a group of children sitting on a merry-go-round

What is the rate of rotation or ‘circulation’?

Physical Meaning of Curl Operation

The curl of a vector field at a point is a measure of the rotation or circulation of the field at that point.

Examples: Whirlpool; merry-go-round; etc.



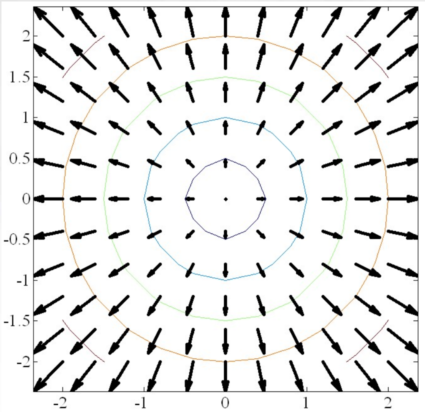
$$\nabla \times \vec{F} \neq 0 \qquad \qquad \nabla \times \vec{F} = 0$$

Curl Operation: Calculation

In Cartesian coordinates, we can use the following method for calculating the curl of a vector:

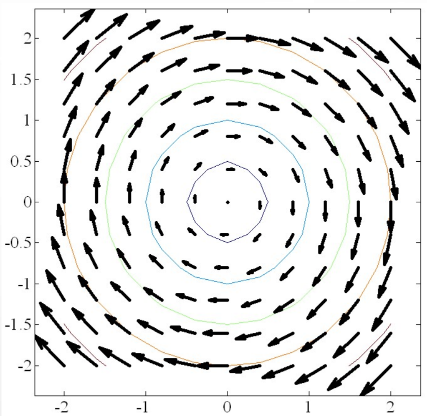
$$\nabla \times \vec{A} = \boxed{\begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}}$$

Curl Operation: Examples



- **Example 1:** Given $\vec{F} = x\vec{a}_x + y\vec{a}_y$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = 0$$



- **Example 2:** Given $\vec{F} = y\vec{a}_x - x\vec{a}_y$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\vec{a}_z$$

How do we get the formula for curl?

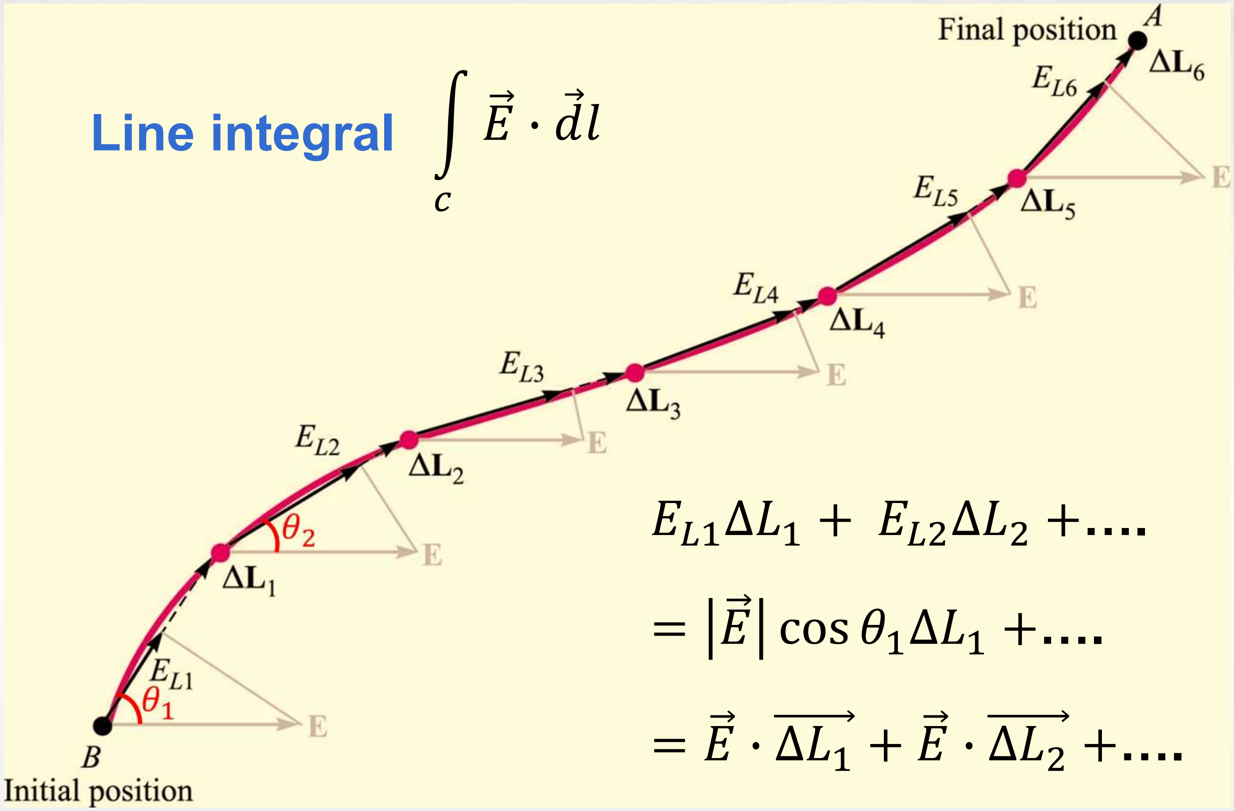
$$\nabla \times \vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Through the concept of circulation!

Circulation is related to line integral.

(One needs to calculate line integral to calculate potential and to apply Ampere's Law)

Line Integral



Line integral $\int_c \vec{E} \cdot \vec{d}l$

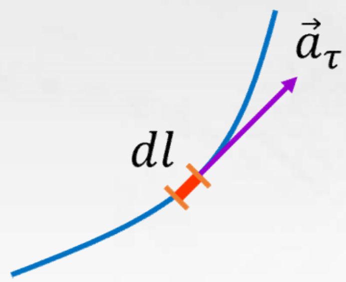
$$E_{L1}\Delta L_1 + E_{L2}\Delta L_2 + \ldots.$$

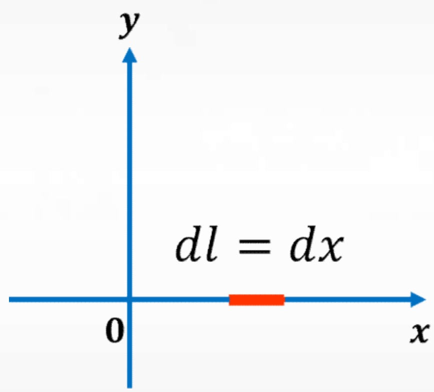
$$= |\vec{E}| \cos \theta_1 \Delta L_1 + \ldots.$$

$$= \vec{E} \cdot \overrightarrow{\Delta L_1} + \vec{E} \cdot \overrightarrow{\Delta L_2} + \ldots.$$

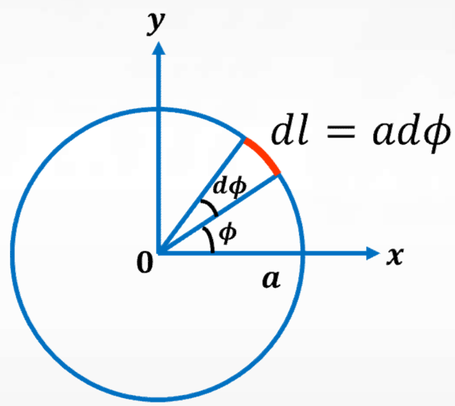
Vector $\vec{dl}$ and Scalar dl

$$\vec{dl} = \vec{a}_\tau dl$$





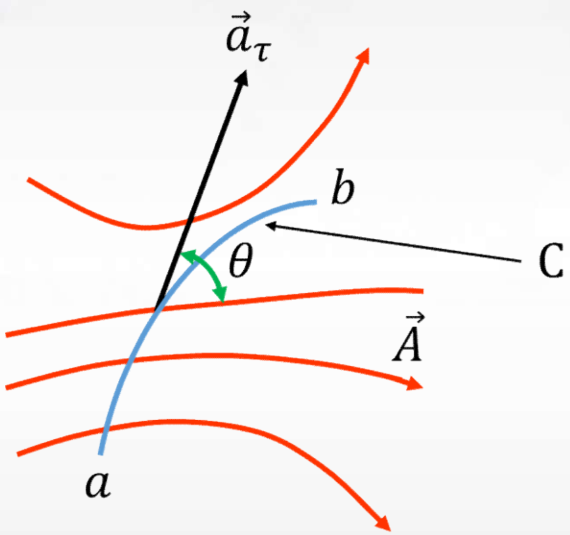
$$\vec{dl} = \vec{a}_x dx$$



$$\vec{dl} = \vec{a}_\phi ad\phi$$

Line Integral

The **line integral** $\int_c \vec{A} \cdot \vec{d}l$ is the integral of the tangential component of $\vec{A}$ along curve C.



$$\int_c \vec{A} \cdot \vec{d}l = \int_a^b |\vec{A}| \cos\theta dl$$

where $\vec{a}_\tau$ is the unit vector tangential to the contour at the differential element dl

Line Integral

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If curve C is closed, the line integral becomes a closed contour integral

$$\oint_c \vec{A} \cdot \vec{d}l$$

Property:

$$\int_a^b \vec{A} \cdot \vec{d}l = -\int_b^a \vec{A} \cdot \vec{d}l$$

Summary of the Three ∇ operations

- Gradient $\nabla V(x,y,z) = \vec{a}_x \frac{\partial V}{\partial x} + \vec{a}_y \frac{\partial V}{\partial y} + \vec{a}_z \frac{\partial V}{\partial z}$

- Divergence $\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

- Curl

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla = \vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z}$$

$$\vec{A} = \vec{a}_x A_x + \vec{a}_y A_y + \vec{a}_z A_z$$

Summary of the Three ∇ operations

Note: In Cartesian coordinates, the above three operations can be easily carried out by considering the ∇ operator as a vector:

$$\nabla = \vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z}$$

∇ operations in Cylindrical Coordinates

- ❑ Gradient $\nabla V(r,\phi,z) = \vec{a}_r \frac{\partial V}{\partial r} + \vec{a}_\phi \frac{1}{r}\frac{\partial V}{\partial \phi} + \vec{a}_z \frac{\partial V}{\partial z}$

- ❑ Divergence $\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$

- ❑ Curl

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \vec{a}_r & r\vec{a}_\phi & \vec{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

Summary

- The physical meaning of curl operation:
  - $\nabla \times \vec{F} \neq 0$: There is circulation or rotation.
  - $\nabla \times \vec{F} = 0$: There is no rotation.
- To calculate curl, we use:

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Summary

- The formula for line integral is:

$$\int_{c} \vec{A} \cdot \vec{d}l = \int_{a}^{b} |\vec{A}| \cos\theta dl$$

- The formula for line integral of a closed contour is:

$$\oint_{c} \vec{A} \cdot \vec{d}l$$

Summary

- The property for line integral is:

$$\int_a^b \vec{A} \cdot \vec{d}l = -\int_b^a \vec{A} \cdot \vec{d}l$$

- The important point to note when calculating gradient, divergence and curl is that the formulas are different in different coordinate systems.

EE3001 Engineering Electromagnetics

Session 6-4

Stokes' Theorem, Divergence Theorem and Null Identities

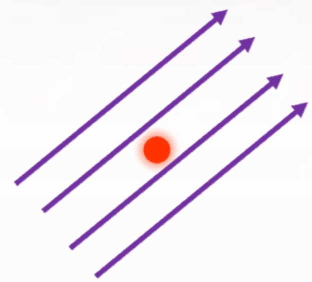
Learning Objectives

- Explain the difference between solenoidal and irrotational fields;
- Explain Stoke's and Divergence theorems; and
- Explain null identities.

Solenoidal and Irrotational Fields

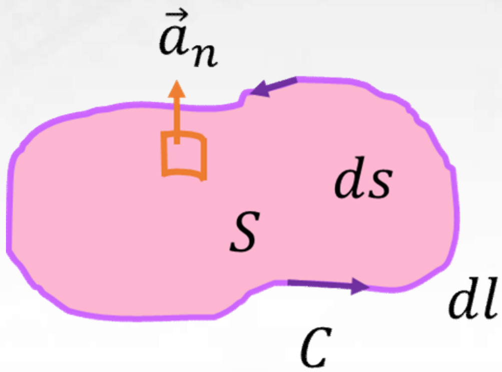
- **Solenoidal Field:** $\nabla \cdot \vec{A} = 0$
  For a solenoidal field, the divergence is zero and there is no source associated with its flux lines.
  Therefore, its flux lines form continuous loops.

- **Irrotational Field:** $\nabla \times \overrightarrow{\mathrm{A}} = 0$
  The curl of a vector describes the rotational property of the vector field.
  $\nabla \times \overrightarrow{\mathrm{A}} = 0$ means that there is no rotation associated with the vector field $\vec{A}$.



Stokes' Theorem

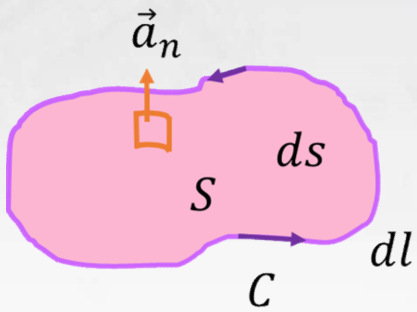
The theorem works with reference to a **closed contour** C that bounds a **surface** S.



$$\boxed{\iint_S (\nabla \times \vec{A}) \cdot \vec{a}_n ds = \oint_C \vec{A} \cdot \vec{a}_\tau dl}$$

*The theorem converts a **two-dimensional surface integral** of the curl of a vector into a **one-dimensional closed contour line integral** of the vector and vice versa.*

Stokes' Theorem



$$\boxed{\iint_S (\nabla \times \vec{A}) \cdot \vec{a}_n ds = \oint_C \vec{A} \cdot \vec{a}_\tau dl}$$

$$\vec{ds} = \vec{a}_n ds$$

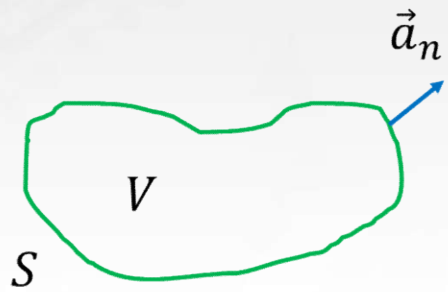
The net flux of $\nabla \times \vec{A}$ through any open surface S.	=	The line integral of $\vec{A}$ along the closed contour C that bounds S.

***Note**: The relative directions of $\vec{a}_\tau$ and $\vec{a}_n$ (normal direction) follow the right-hand rule.*

Appendix

Divergence (Gauss's) Theorem

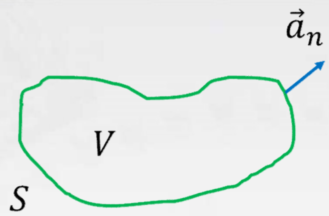
The theorem works with reference to a **volume V** enclosed by the **closed surface S**.



$$\boxed{\iiint_V (\nabla \cdot \vec{A})\,dv = \oiint_S \vec{A} \cdot \vec{a}_n\,ds}$$

The theorem converts a **three-dimensional volume integral** of the divergence of a vector to a **two-dimensional closed surface integral** of the vector and vice versa.

Divergence (Gauss's) Theorem



$$\iiint_V (\nabla \cdot \vec{A})\, dv = \oint\!\!\oint_s \vec{A} \cdot \vec{a}_n\, ds$$

The integral of the divergence of a vector $\vec{A}$ throughout the volume V enclosed by the surface S. = **The total outward flux of the vector field $\vec{A}$ (i.e., the integral of the normal component) over the *closed* surface S.**

****Note*: The theorem **converts** a three-dimensional volume integral of the divergence of a vector to a two-dimensional closed surface integral of the vector and vice versa.*

Quiz

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Hayt and Buck, Ch 3, Q2:

The Divergence theorem:

A: Relates a line integral to a surface integral

B: Holds for specific vector fields only

C: Works only for open surfaces

D: Relates a surface integral to a volume integral

Two Null Identities

- For any scalar function V, we have

$$\boxed{\nabla \times (\nabla V) = 0}$$

- For any vector function $\vec{A}$, we have

$$\boxed{\nabla \cdot \left(\nabla \times \vec{A}\right) = 0}$$

Note: *The proof of these two null identities is left as a tutorial question.*

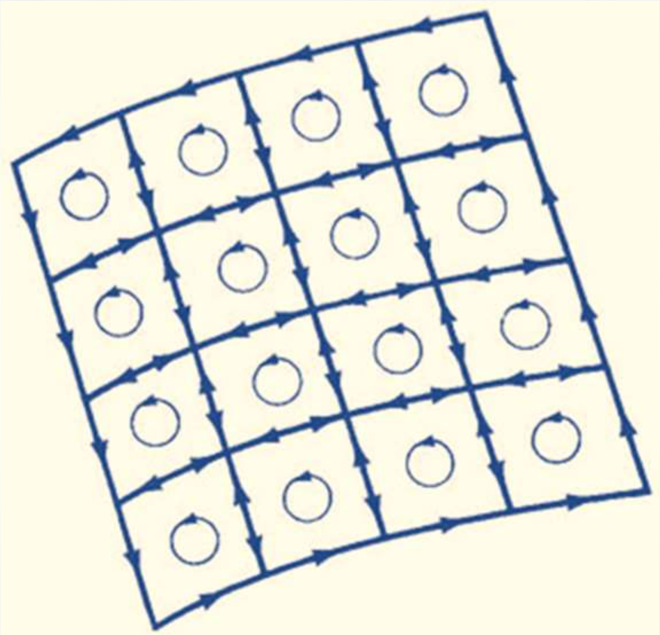
Summary

- When divergence equals to zero, it represents a solenoidal field. When curl equals to zero, it represents an irrotational field.
- Stokes' theorem helps to convert a two-dimensional surface integral into a one-dimensional line integral and vice versa.
- Divergence theorem helps to relate a three-dimensional volume integral to a two-dimensional surface integral and vice versa.
- Null identity has result as zero and it always holds true, no matter what is the function.

Simple Explanation for Stokes' Theorem

The line integrals cancel over every interior path

$$\iint_S (\nabla \times \vec{A}) \cdot \vec{a}_n ds = \oint_C \vec{A} \cdot \vec{a}_\tau dl$$



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