Exercise 25. Decide whether the following argument is valid:

$(p \vee q) \to \neg r$;
$\neg r \to s$;
p;
$\therefore s$

Solution. We start by noticing that

p;
$\therefore (p \vee q)$

Then

$(p \vee q) \to \neg r$;
$p \vee q$;
$\therefore \neg r$

Finally

$\neg r \to s$;
$\neg r$;
$\therefore s$

and we conclude that the argument is valid.

We can come to the same conclusion using a truth table. Note that we care only about the critical rows, those for which the premises are true. Thus in the table below, we assume that p is always true.

s	q	r	$p \vee q$	$p \vee q \to \neg r$	$\neg r \to s$	
T	T	T	T	F		
T	T	F	T	T	T	critical
T	F	T	T	F		
T	F	F	T	T	T	critical
F	T	T	T	F		
F	T	F	T	T	F	
F	F	T	T	F		
F	F	F	T	T	F	

We see that there are only 2 critical rows, for which s is true, therefore the argument is valid.

Exercises for Chapter 3

Exercise 26. Consider the predicates $M(x, y) =$ "x has sent an email to y", and $T(x, y) =$ "x has called y". The predicate variables x, y take values in the domain $D = \{\text{students in the class}\}$. Express these statements using symbolic logic.

1. There are at least two students in the class such that one student has sent the other an email, and the second student has called the first student.

2. There are some students in the class who have emailed everyone.

Solution. 1. We need two predicate variables since at least 2 students are involved, say x and y. There are at least two students in the class becomes

$$\exists x \in D,\ \exists y \in D.$$

Then x sent an email to y, that is $M(x, y)$ and y has called x, that is $T(y, x)$, thus

$$M(x, y) \wedge T(y, x).$$

Furthermore, we need to take into account the fact that there are at least "two" students, so x and y have to be distinct! Thus the final answer is

$$\exists x \in D,\ \exists y \in D,\ ((x \neq y) \wedge M(x, y) \wedge T(y, x)).$$

2. There are students becomes

$$\exists x \in D,$$

then x has emailed everyone, that is

$$\exists x \in D, (\forall y \in D\ M(x, y)).$$

Note that the order of the quantifiers is important.

Exercise 27. Consider the predicate $C(x, y) =$ "x is enrolled in the class y", where x takes values in the domain $S = \{\text{students}\}$, and y takes values in the domain $D = \{\text{courses}\}$. Express each statement by an English sentence.

1. $\exists\ x \in S,\ C(x, \text{MH1812})$.

2. $\exists\ y \in D,\ C(\text{Carol}, y)$.

3. $\exists\, x \in S,\ (C(x, \text{MH1812}) \wedge C(x, \text{CZ2002}))$.

4. $\exists\, x \in S,\ \exists\, x' \in S,\ \forall y \in D,\ ((x \neq x') \wedge (C(x,y) \leftrightarrow C(x',y)))$.

Solution.

1. There exists a student such that this student is enrolled in the class MH1812, that is some student enrolled in the class MH1812.

2. There exists a course such that Carol is enrolled in this course, that is, Carol is enrolled in some course, or Carol is enrolled in at least one course.

3. There exists a student, such that this student is enrolled in MH1812 and in CZ2002, that is some student is enrolled in both MH1812 and CZ2002.

4. There exist two distinct students x and x', such that for all courses, x is enrolled in the course if and only if x' is enrolled in the course. In other words, there exist two students which are enrolled in exactly the same courses.

Exercise 28. Consider the predicate $P(x,y,z) =$ "$xyz = 1$", for $x, y, z \in \mathbb{R}$, $x, y, z > 0$. What are the truth values of these statements? Justify your answer.

1. $\forall\, x,\ \forall\, y,\ \forall\, z,\ P(x,y,z)$.

2. $\exists\, x,\ \exists\, y,\ \exists\, z,\ P(x,y,z)$.

3. $\forall\, x,\ \forall\, y,\ \exists\, z,\ P(x,y,z)$.

4. $\exists\, x,\ \forall\, y,\ \forall\, z,\ P(x,y,z)$.

Solution.

1. $\forall\, x,\ \forall\, y,\ \forall\, z,\ P(x,y,z)$ is false: take $x = 1$ and $y = 1$, then whenever $z \neq 1$, $xyz = z \neq 1$.

2. $\exists\, x,\ \exists\, y,\ \exists\, z,\ P(x,y,z)$ is true: take $x = y = z = 1$.

3. $\forall\, x,\ \forall\, y,\ \exists\, z,\ P(x,y,z)$ is true: choose any x and any y, then there exists a z, namely $z = \frac{1}{xy}$ such that $xyz = 1$.

4. $\exists\, x,\ \forall\, y,\ \forall\, z,\ P(x,y,z)$ is false: one cannot find a single x such that $xyz = 1$ no matter what are y and z. This is because once yz are chosen, then x is completely determined, so x changes whenever yz does.

Exercise 29. Consider the domains $X = \{2, 3\}$ and $Y = \{2, 4, 6\}$, and the predicate $P(x, y) =$ "x divides y". What are the truth values of these statements:

1. $\exists x \in X,\ \forall y \in Y,\ P(x, y)$.
2. $\neg(\exists x \in X,\ \exists y \in Y,\ P(x, y))$.

Solution.

1. This is true, there exists an $x \in X$, namely $x = 2$, such that this x divides y no matter which y you pick in Y, that is $x = 2$ divides 2,4 and 6.
2. This is false. One way to look at it is to say that since there exists x in X, say $x = 2$, for which there exists a y in Y, say $y = 4$ for which x divides y, then what is inside the parenthesis is true, therefore its negation is false. Another way is to write

$$\forall x \in X,\ \forall y \in Y, \neg P(x, y).$$

This is also false.

Exercise 30.

1. Express

$$\neg(\forall\ x,\ \forall\ y,\ P(x, y))$$

in terms of existential quantification.

2. Express

$$\neg(\exists\ x,\ \exists\ y,\ P(x, y))$$

in terms of universal quantification.

Solution.

1. We see that $\neg(\forall\, x,\ \forall\, y,\ P(x, y))$ is a negation of two universal quantifications. Denote $Q(x) =$ "$\forall\ y,\ P(x, y)$", then $\neg(\forall x, Q(x))$ is $(\exists x, \neg Q(x))$, thus

$$\neg(\forall\ x,\ \forall\ y,\ P(x, y)) \equiv \exists x, \neg(\forall\ y,\ P(x, y))$$

and now we iterate the same rule on the next negation, to get

$$\neg(\forall\ x,\ \forall\ y,\ P(x, y)) \equiv \exists x,\ \exists y,\ \neg P(x, y).$$

2. We repeat the same procedure with the negation of two existential quantifications, by setting this time $Q(x) =$ "$\exists\ y,\ P(x,y)$":

$$\begin{aligned}\neg(\exists\ x,\ \exists\ y,\ P(x,y)) &\equiv \neg(\exists x Q(x))\\ &\equiv \forall x \neg Q(x)\\ &\equiv \forall x \neg(\exists\ y,\ P(x,y))\\ &\equiv \forall x \forall y \neg P(x,y).\end{aligned}$$

Exercise 31. Consider the predicate $C(x,y) =$ "x is enrolled in the class y", where x takes values in the domain $S = \{\text{students}\}$, and y takes values in the domain $C = \{\text{courses}\}$. Form the negation of these statements:

1. $\exists\ x,\ (C(x, \text{MH1812}) \wedge C(x, \text{CZ2002}))$.
2. $\exists\ x\ \exists\ y,\ \forall z,\ ((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z)))$.

Solution. 1. We have

$$\begin{aligned}&\neg(\exists\ x,\ (C(x, \text{MH1812}) \wedge C(x, \text{CZ2002})))\\ \equiv\ &\forall x \neg(C(x, \text{MH1812}) \wedge C(x, \text{CZ2002}))\\ \equiv\ &\forall x \neg C(x, \text{MH1812}) \vee \neg C(x, \text{CZ2002})\end{aligned}$$

where the first equivalence is the negation of quantification, and the second equivalence De Morgan's law.

2. We have

$$\begin{aligned}&\neg(\exists\ x\ \exists\ y,\ \forall z,\ ((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z))))\\ \equiv\ &\forall x \neg(\exists\ y,\ \forall z,\ ((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z))))\\ \equiv\ &\forall x \forall y \neg(\forall z,\ ((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z))))\\ \equiv\ &\forall x \forall y \exists z \neg((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z)))\\ \equiv\ &\forall x \forall y \exists z \neg(x \neq y) \vee \neg(C(x,z) \leftrightarrow C(y,z)))\end{aligned}$$

using three times the negation of quantification, and lastly the Morgan's law. Next $\neg(x \neq y) = (x = y)$ and using that

$$C(x,z) \leftrightarrow C(y,z) \equiv (C(x,z) \to C(y,z)) \wedge (C(y,z) \to C(x,z))$$

we get

$$\neg(C(x,z) \leftrightarrow C(y,z)) \equiv \neg(C(x,z) \to C(y,z)) \vee \neg(C(y,z) \to C(x,z))$$

so that, using the Conversion theorem to get $\neg(\neg C(x,z) \vee C(y,z))$ and $\neg(\neg C(y,z) \vee C(x,z))$

$$\begin{aligned} & \neg(\exists\ x\ \exists\ y,\ \forall z,\ ((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z)))) \\ \equiv\ & \forall x \forall y \exists z((x = y) \vee [(C(x,z) \wedge \neg C(y,z)) \vee (C(y,z) \wedge \neg C(x,z))]). \end{aligned}$$

The last term can be further modified using distributivity:

$$\begin{aligned} & (C(x,z) \wedge \neg C(y,z)) \vee (C(y,z) \wedge \neg C(x,z)) \\ \equiv\ & [(C(x,z) \wedge \neg C(y,z)) \vee C(y,z)] \wedge [(C(x,z) \wedge \neg C(y,z)) \vee \neg C(x,z)] \\ \equiv\ & (C(x,z) \vee C(y,z)) \wedge (\neg C(x,z) \vee \neg C(y,z)) \end{aligned}$$

to finally get

$$\begin{aligned} & \neg(\exists\ x\ \exists\ y,\ \forall z,\ ((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z)))) \\ \equiv\ & \forall x \forall y \exists z((x = y) \vee [(C(x,z) \vee C(y,z)) \wedge (\neg C(x,z) \vee \neg C(y,z))]). \end{aligned}$$

When many steps are involved, it is often a good idea to check the sanity of the answer. If we look at $\neg(C(x,z) \leftrightarrow C(y,z))$, it is false exactly when $C(x,z)$ and $C(y,z)$ are taking the same truth value (either both true or both false). Now we look at $(C(x,z) \vee C(y,z)) \wedge (\neg C(x,z) \vee \neg C(y,z))$: when $C(x,z)$ and $C(y,z)$ are taking the same value, we get false, and true otherwise. This makes sense!

Exercise 32. Show that $\forall x \in D,\ P(x) \rightarrow Q(x)$ is equivalent to its contrapositive.

Solution. For every instantiation of x, $(\forall x \in D,\ P(x) \rightarrow Q(x))$ is a proposition, thus we can use the conversion theorem:

$$\begin{aligned} & (\forall x \in D,\ P(x) \rightarrow Q(x)) \\ \equiv\ & (\forall x \in D, \neg P(x) \vee Q(x)) \\ \equiv\ & (\forall x \in D, Q(x) \vee \neg P(x)) \\ \equiv\ & (\forall x \in D, \neg\neg Q(x) \vee \neg P(x)) \\ \equiv\ & (\forall x \in D, \neg Q(x) \rightarrow \neg P(x)). \end{aligned}$$

Exercise 33. Show that

$$\neg(\forall\ x, P(x) \rightarrow Q(x)) \equiv \exists x,\ P(x) \wedge \neg Q(x).$$

Solution.

$$
\begin{aligned}
& \neg(\forall\ x, P(x) \rightarrow Q(x)) \\
\equiv\ & \exists x,\ \neg(P(x) \rightarrow Q(x)) \\
\equiv\ & \exists x,\ \neg(\neg P(x) \vee Q(x)) \\
\equiv\ & \exists x,\ (P(x) \wedge \neg Q(x))
\end{aligned}
$$

where the first equivalence is the negation of quantifications, the second equivalence is the conversion theorem, and the third one is De Morgan's law.

Exercise 34. Let y, z be positive integers. What is the truth value of "$\exists y,\ \exists z,\ (y = 2z \wedge (y \text{ is prime}))$".

Solution. The truth value is true, take $y = 2$ and $z = 1$.

Exercise 35. Consider the domains $X = \{2, 4, 6\}$ and $Y = \{2, 3\}$, and the predicate $P(x, y)$ = "x is a multiple of y". What are the truth values of these statements:

1. $\forall x \in X,\ \exists y \in Y,\ P(x, y)$.
2. $\neg(\forall x \in X,\ \forall y \in Y,\ P(x, y))$.

Solution. a) The first one is true. We check all values in X. For $x = 2$, there exists $y = 2$ such that $x = 2$ is a multiple of $y = 2$. For $x = 4$, there exists $y = 2$ such that $x = 4$ is a multiple of $y = 2$. For $x = 6$, there exists $y = 2$ such that $x = 6$ is a multiple of 2.

b) $\neg(\forall x \in X,\ \forall y \in Y,\ P(x, y))$ can be rewritten as

$$\exists x \in X,\ \exists y \in Y, \neg P(x, y).$$

So it is true. There exists an x, take $x = 2$, and there exists a y, take $y = 3$, such that $x = 2$ is not a multiple of $y = 3$.

Exercise 36. Write in symbolic logic "Every SCE student studies discrete mathematics. Jackson is an SCE student. Therefore Jackson studies discrete mathematics".

Solution. Consider the domain $D = \{$ SCE students $\}$. Set $P(x)$="x studies discrete mathematics. Then every SCE student studies discrete mathematics becomes

$$\forall x \in D,\ P(x).$$

Now Jackson is a SCE student means Jackson belongs to D. This gives

$$\forall x \in D,\ P(x); \text{Jackson} \in D; \therefore P(\text{Jackson}).$$

Exercise 37. Here is an optional exercise about universal generalization. Consider the following two premises: (1) for any number x, if $x > 1$ then $x - 1 > 0$, (2) every number in D is greater than 1. Show that therefore, for every number x in D, $x - 1 > 0$.

Solution. Set $P(x)$ = "$x > 1$" and $Q(x)$ = "$x - 1 > 0$". Let us formalize what we want to prove:

$$[\forall x\ (P(x) \to Q(x)) \wedge \forall x \in D\ P(x)] \to \forall x \in D,\ Q(x).$$

1. $\forall x\ (P(x) \to Q(x))$, by hypothesis

2. $\forall x \in D\ P(x)$, also by hypothesis

3. $P(y) \to Q(y)$, by universal instantiation on the first hypothesis

4. $P(y)$, by universal instantiation on D in the second hypothesis

5. $Q(y)$, using modus ponens

6. $\forall x \in D,\ Q(x)$, using universal generalization.

Exercises for Chapter 4

Exercise 38. Let q be a positive real number. Prove or disprove the following statement: if q is irrational, then $\sqrt{q}$ is irrational.