NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2020-2021

MH1812 - DISCRETE MATHEMATICS

December 2020 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the end of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT** an **OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. **(20 marks)**

For a finite set A of real numbers, we define $\Pi(A)$ to be the product of all elements in A. For example, $\Pi(\{-2, 3, \pi, 5\}) = (-2) \cdot 3 \cdot \pi \cdot 5 = -30\pi$. Additionally we define $\Pi(\emptyset) = 1$.

(a) Define $\Sigma = \{\Pi(A) \bmod 12 \mid A \subseteq \{1, \ldots, 10\}\}$. Determine whether Σ is closed under "addition modulo 12". Justify your answer. **(5 marks)**

(b) Find the number of subsets $A \subseteq \{1, \ldots, 100\}$ such that $\Pi(A)$ is <u>not</u> divisible by 5. Justify your answer. **(5 marks)**

(c) Find the number of subsets $A \subseteq \{1, \ldots, 100\}$ such that $\Pi(A)$ is <u>not</u> divisible by 8. Justify your answer. **(10 marks)**

[Solution:]

(a) We claim that the answer is yes. It is clear that Σ contains $1, \ldots, 10$ (from the corresponding singleton A). Since $\Pi(\{3, 4\}) = 12$ and $\Pi(\{5, 7\}) = 35$, we also have $0 \in \Sigma$ and $11 \in \Sigma$. Therefore $\Sigma = \{0, 1, \ldots, 11\}$ and is closed under "addition modulo 12".

(b) The sets A are exactly those which do not contain any integer multiple of 5. There are 80 integers in $\{1, \ldots, 100\}$ which are not integer multiples of 5. Therefore, the answer is 2^{80}.

(c) Let $S = \{2, 6, 10, 12 \ldots, 98, 100\}$ (integer multiples of 2 but not of 8) and $T = \{2, 6, 10, \ldots, 98\}$ (integer multiples of 2 but not of 4). We have $|S| = 38$ and $|T| = 25$.

The desired sets A are exactly those:

(i) A contains only odd numbers (could be none) — there are 2^{50} such sets;

(ii) A contains odd numbers (could be none) plus exactly one element in S — there are $2^{50} \cdot 38$ such sets;

(iii) A contains odd numbers (could be none) plus exactly two elements in T — there are $2^{50} \cdot \binom{25}{2}$ such sets.

Therefore, the answer is

$$2^{50}\left(1 + 38 + \binom{25}{2}\right) = 2^{50} \cdot 339.$$

QUESTION 2. **(20 marks)**

On a set $S = \{a, b, c, d, e\}$ we define a relation $R = \{(a, a), (a, b), (b, c), (d, e)\}$.

(a) What is the transitive closure of R ? **(6 marks)**
[Solution:] $R^t = \{(a, a), (a, b), (b, c), (a, c), (d, e)\}$.

(b) What is the smallest equivalence relation containing R ? **(7 marks)**
[Solution:] $\{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c), (d, d), (d, e), (e, d), (e, e)\}$

(c) What is the smallest partial order containing R ? **(7 marks)**
[Solution:] $R^t = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c), (d, d), (d, e), (e, e)\}$.

QUESTION 3. **(10 marks)**

Show that

$$\frac{n}{2} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n - 1} < n$$

for all integers $n \geq 2$.

[Solution:] It is easy to prove the result via mathematical induction, where the inductive step relies on the following inequality

$$\frac{1}{2} < \frac{1}{2^n} + \frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1} - 1} < 1,$$

which is easy to verify by summing up

$$\frac{1}{2^{n+1}} < \frac{1}{2^n + i} \leq \frac{1}{2^n}, \quad i = 0, \ldots, 2^n - 1.$$

Note that the right equality is attained only when $i = 0$ and thus we shall have strict inequality in the sum.

[Grading:] In general, each side of the inequality is worth 5 marks. For an induction proof, the base case is worth 2 marks (each side of the inequality is worth 1 mark) and the inductive step 8 marks (each side of the inequality is worth 4 marks).

QUESTION 4. (30 marks)

(a) How many surjective functions are there from set A to B, where $|A| = 5$ and $|B| = 3$? Justify your answer. **(10 marks)**

[Solution:] We can group the 5 elements into 3 non-empty sub-groups, and then map them to the 3 elements in B. Ordered by sizes of the sub-groups, we can have 2 different cases:

(i) $3 + 1 + 1$: there are $\binom{5}{3}$ ways to select the sub-group of size 3, then the rest 2 elements each forms one sub-group.

(ii) $2+2+1$: there are $\binom{5}{1}$ ways to select the sub-group of size 1, then 3 ways to split the rest 4 into $2 + 2$.

Overall: we have $(\binom{5}{3} + \binom{5}{1} \cdot 3) \cdot 3! = 150$

(b) How many surjective functions are there from set $A = \{1, 2, \ldots, m\}$ to $B = \{1, 2, \ldots, n\}$ with positive integers $m \geq n$, such that $f(1) \leq f(2) \leq \cdots \leq f(m)$? Justify your answer. **(10 marks)**

[Solution:] We split the m numbers in sequence into n non-empty blocks, by inserting $n - 1$ separators into $m - 1$ possible positions (positions in between numbers), then map these n blocks into the n numbers in the co-domain in sequence, hence there are $\binom{m-1}{n-1}$ ways.

(c) For an injective function $f : D \to R$, prove or disprove $f(A \cap B) = f(A) \cap f(B)$, where $A, B \subseteq D$ and $f(X)$ is defined as $f(X) = \{f(x) \mid x \in X\}$ for any $X \subseteq D$. **(10 marks)**

[Solution:] The conclusion is true.
First of all, it is easy to see LHS is a subset of RHS, since for each $x \in A \cap B$, $x \in A$ hence $f(x) \in f(A)$, similarly $f(x) \in f(B)$, so $f(x) \in f(A) \cap f(B)$.
Then we are to prove RHS is also a subset of LHS, and we prove by contradiction, i.e., assume there exists some y such that $y \in f(A) \cap f(B)$ but $y \notin f(A \cap B)$. Since $y \in f(A) \cap f(B)$, y has at least one pre-image from A and B, and denote them as $x_1 \in A$ and $x_2 \in B$. Then we have $x_1 \neq x_2$, otherwise $x_1 = x_2 \in A \cap B$, then $y = f(x_1) = f(x_2) \in f(A \cap B)$ contradicting with "$y \notin f(A \cap B)$". $x_1 \neq x_2$ means y has at least 2 different pre-images, contradicting with the definition of "injective function".

QUESTION 5. (20 marks)

A quinary string is a string whose characters are 0, 1, 2, 3 or 4. It is clear that there are 5^n quinary strings of length n for integers $n \geq 1$.

For each integer $n \geq 1$, let a_n be the number of quinary strings of length n that do not contain adjacent 2s. Find an explicit formula for a_n.

[Solution:] Depending on whether the last digit is 2, we see that such a string of length n consists of: (1) such a string of length $n-1$ followed by 0, 1, 3 or 4; (2) such a string of length $n-2$ followed by 02, 12, 32 or 42. This leads to

$$a_n = 4a_{n-1} + 4a_{n-2}.$$

The characteristic equation is

$$x^2 - 4x - 4 = 0,$$

which has two roots $x = 2(1 \pm \sqrt{2})$. Hence a_n can be written as

$$a_n = 2^n(c_1(1+\sqrt{2})^n + c_2(1-\sqrt{2})^n).$$

for some constants c_1 and c_2 to be determined.

The initial values are $a_1 = 5$ and $a_2 = 24$, from which we can formally assign $a_0 = 1$. It follows that

$$\begin{aligned} 1 = a_0 &= c_1 + c_2, \\ 5 = a_1 &= 2(c_1(1+\sqrt{2}) + c_2(1-\sqrt{2})) \\ &= 2(c_1 + c_2) + 2\sqrt{2}(c_1 - c_2). \end{aligned}$$

Plugging in the first equation into the second, we obtain that $c_1 - c_2 = 3\sqrt{2}/4$. It is then easy to solve for c_1 and c_2,

$$c_1 = \frac{4+3\sqrt{2}}{8}, c_2 = \frac{4-3\sqrt{2}}{8}.$$

Therefore we conclude that

$$a_n = 2^{n-3}((4-3\sqrt{2})(1-\sqrt{2})^n + (4+3\sqrt{2})(1+\sqrt{2})^n).$$

[Grading:] The recurrent relation $a_n = 4a_{n-1} + 4a_{n-2}$ is worth 8 points; the characteristic equation 2 points, two roots 2 points; the general form of a_n is worth 2 points; the linear system (after plugging in the initial values) is worth 2 points, its solution 2 points; the final answer is worth 2 points.

END OF PAPER