NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

Discrete Mathematics

MH1812

Topic 8.1 - Relations I

Dr. Guo Jian

Topic Overview

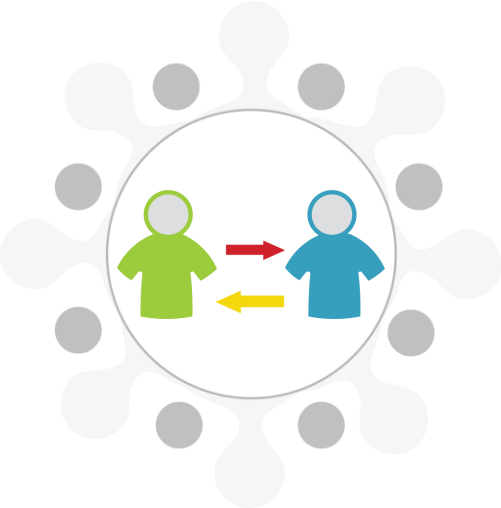
What's in store…

Binary Relations

Reflexivity

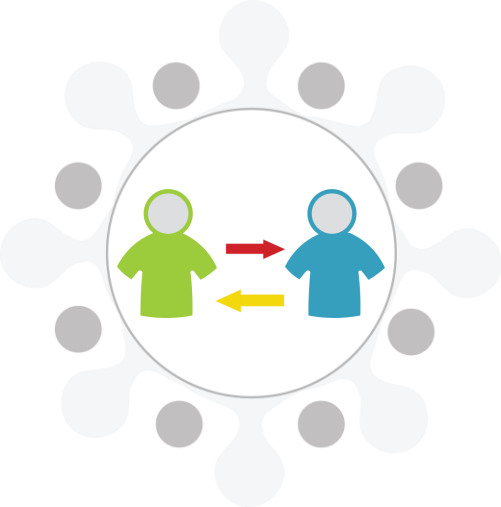
Symmetry

Transitivity



By the end of this lesson, you should be able to…

- Explain the different types of binary relations.
- Explain the concept of reflexivity.
- Explain the concept of symmetry.
- Explain the concept of transitivity.



Binary Relations

Binary Relations: Between Two Sets

Let A and B be sets. A **binary relation R** from A to B is a subset of $A \times B$. Given (x,y) in $A \times B$, **x is related to y by R** $(xRy) \leftrightarrow (x,y) \in R$.

Example

$A = \{1,2\}$, $B = \{1,2,3\}$, $(x,y) \in R \leftrightarrow (x - y)$ **is even**

$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

$(1,1) \in R$, $(1,3) \in R$, $(2,2) \in R$

$x > y$, x **owes** y, x **divides** y

Binary Relations: Between Two Sets (Graphically)

$A = \{1,2\}$, $B = \{1,2,3\}$, $(x,y) \in R \leftrightarrow (x - y)$ **is even**

$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

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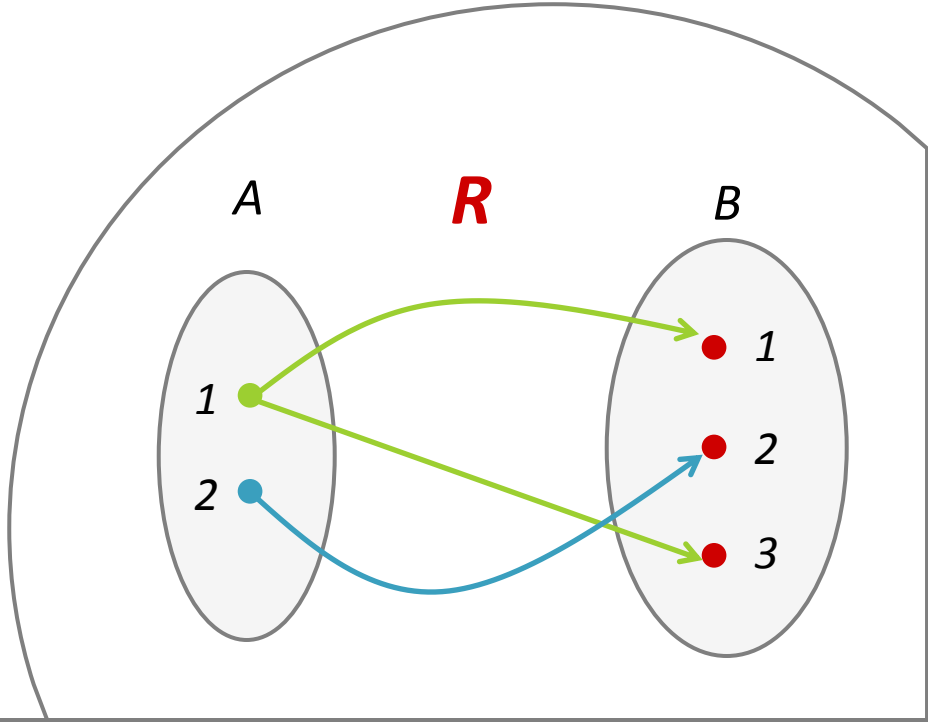
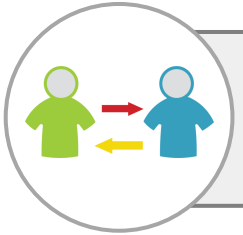


Binary Relations: Inverse of a Binary Relation



Let R be a relation from A to B. The **inverse relation** R^{-1} from B to A is defined as: $R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}$.

Binary Relations: Inverse of a Binary Relation (Example)

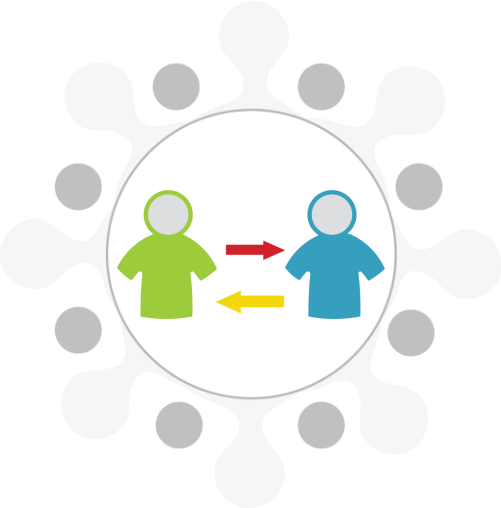
$A = \{2,3,4\}$, $B = \{2,6,8\}$, $(x, y) \in R \leftrightarrow x$ **divides** y

$A \times B = \{(2,2), (2,6), (2,8), (3,2), (3,6), (3,8), (4,2), (4,6), (4,8)\}$

$(2,2) \in R, (2,6) \in R, (2,8) \in R, (3,6) \in R, (4,8) \in R$

$(2,2) \in R^{-1}, (6,2) \in R^{-1}, (8,2) \in R^{-1}, (6,3) \in R^{-1}, (8,4) \in R^{-1}$

$(y, x) \in R^{-1} \leftrightarrow y$ **is a multiple of** x

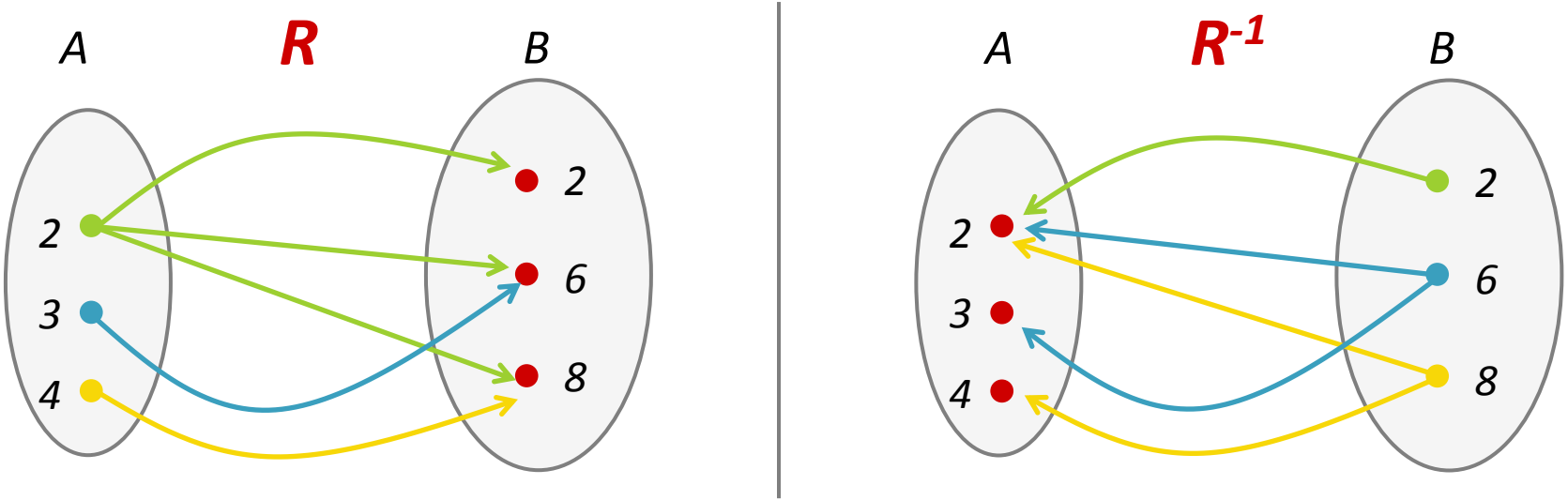


Binary Relations: Inverse of a Binary Relation (Graphically)

$A = \{2,3,4\}$, $B = \{2,6,8\}$, $(x, y) \in R \leftrightarrow x$ **divides** y

$(2,2) \in R, (2,6) \in R, (2,8) \in R, (3,6) \in R, (4,8) \in R$

$(2,2) \in R^{-1}, (6,2) \in R^{-1}, (8,2) \in R^{-1}, (6,3) \in R^{-1}, (8,4) \in R^{-1}$



Binary Relations: Matrix Representation

$A = (a_1, a_2, a_3),\ B = (b_1, b_2, b_3, b_4),$

$R = \{(a_1, b_2), (a_2, b_1), (a_3, b_1), (a_3, b_4)\}$

(i, j)th **entry is** T **if** $a_i R b_j$:

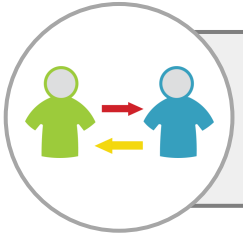
$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} F & T & F & F \\ T & F & F & F \\ T & F & F & T \end{bmatrix} \end{array}$$

Example

$A = \{2,3,4\},\ B = \{2,6,8\}, (x, y) \in R \leftrightarrow x$ **divides** y.

A\B	2	6	8
2	T	T	T
3	F	T	F
4	F	F	T

Binary Relations: Matrix Representation



R **relation** from A to B: $R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}$.

$A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3, b_4)$

$R = \{(a_1, b_2), (a_2, b_1), (a_3, b_1), (a_3, b_4)\}$

$R^{-1} = \{(b_2, a_1), (b_1, a_2), (b_1, a_3), (b_4, a_3)\}$

The matrix of R^{-1} is the transpose of the matrix of R.

$$a_i R b_j: \quad \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} F & T & F & F \\ T & F & F & F \\ T & F & F & T \end{bmatrix} \end{array}$$

$$b_i R^{-1} a_j: \quad \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} F & T & T \\ T & F & F \\ F & F & F \\ F & F & T \end{bmatrix} \end{array}$$

Binary Relations: Composition of Relations

Given R in $A \times B$, and S in $B \times C$, the **composition** of R and S is a relation on $A \times C$ defined by
$R \circ S = \{(a, c) \in A \times C \mid \exists\, b \in B, aRb \text{ and } bSc\}$.

Example

$A = \{a_1, a_2\}, B = \{b_1, b_2\}, C = \{c_1, c_2, c_3\}$

$R = \{(a_1, b_1), (a_1, b_2)\}$

$S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}$

What is $R \circ S$?

$R \circ S = \{(a_1, c_1), (a_1, c_3), (a_1, c_2)\}$

Binary Relations: Composition of Relations (Graphically)

$A = \{a_1, a_2\}, B = \{b_1, b_2\}, C = \{c_1, c_2, c_3\}$

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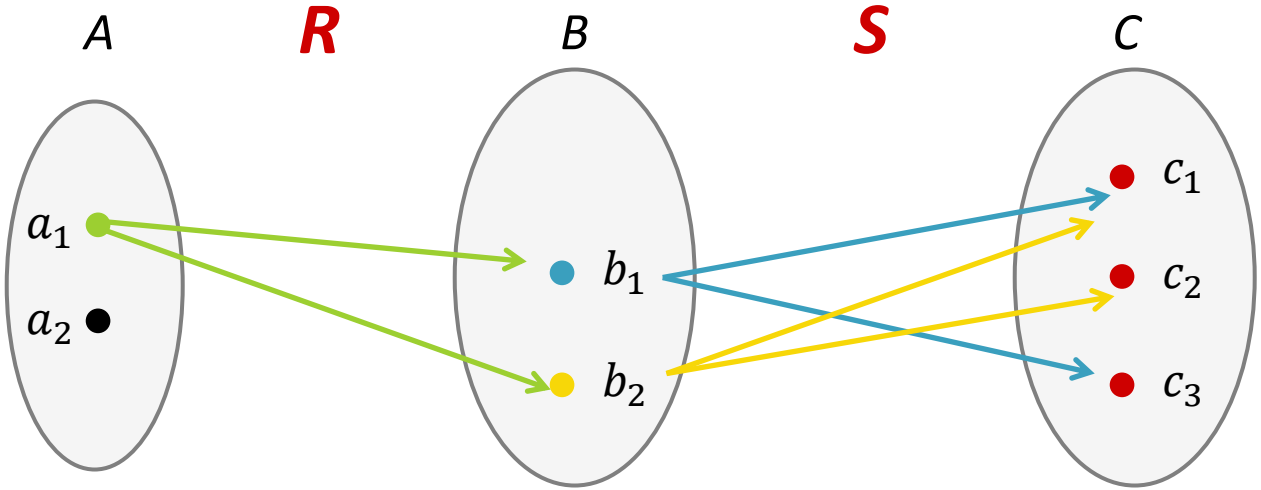

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$S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}$

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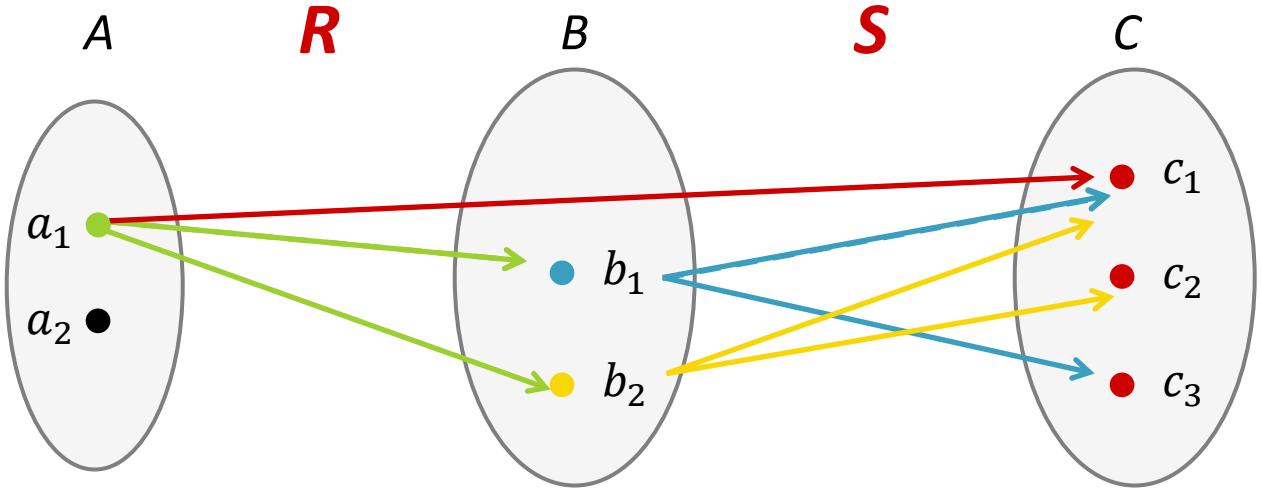

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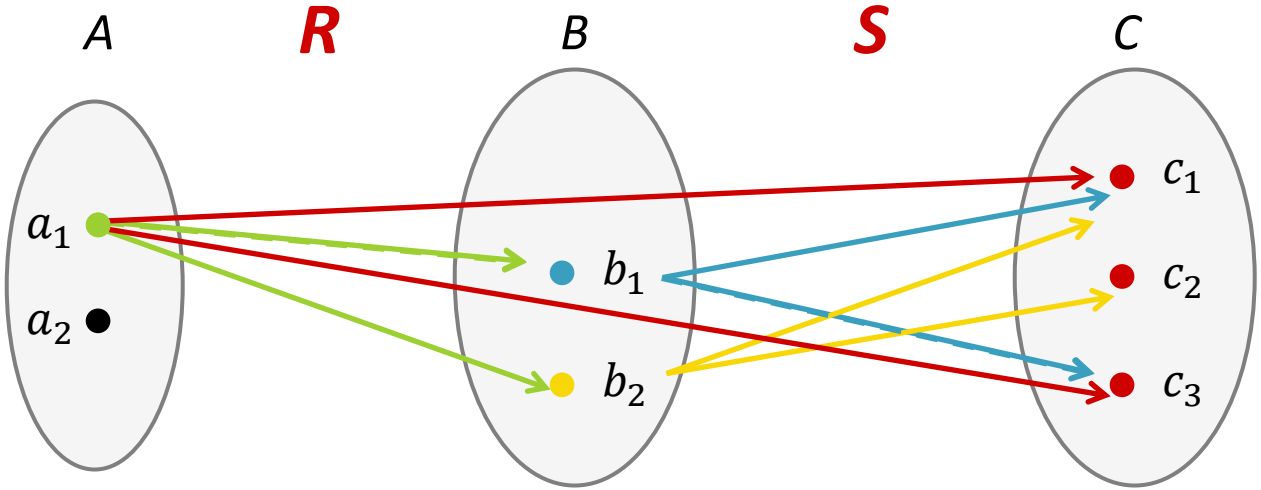

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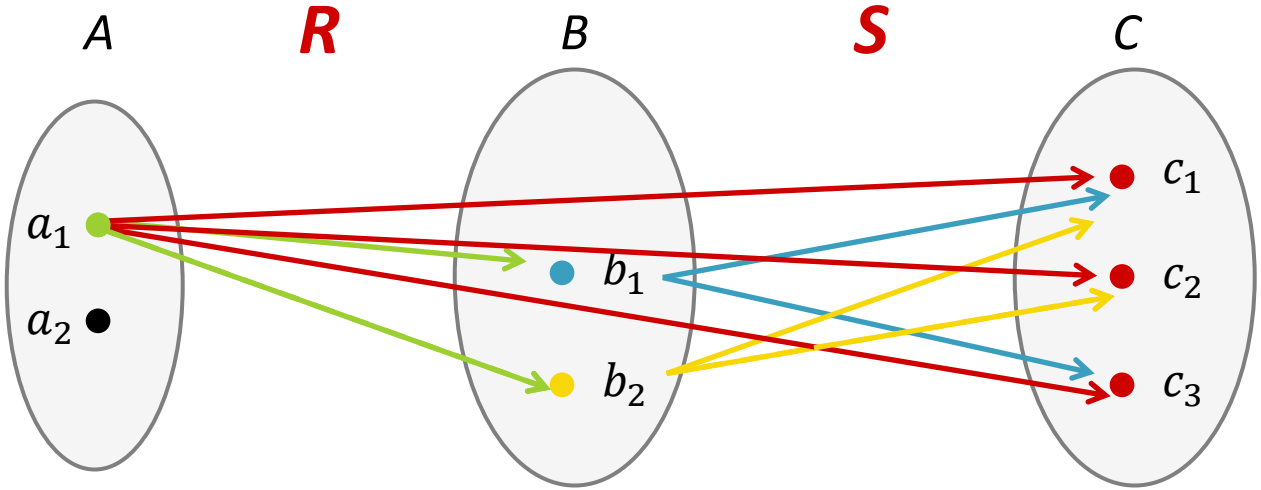


Reflexivity

Reflexivity: Definition

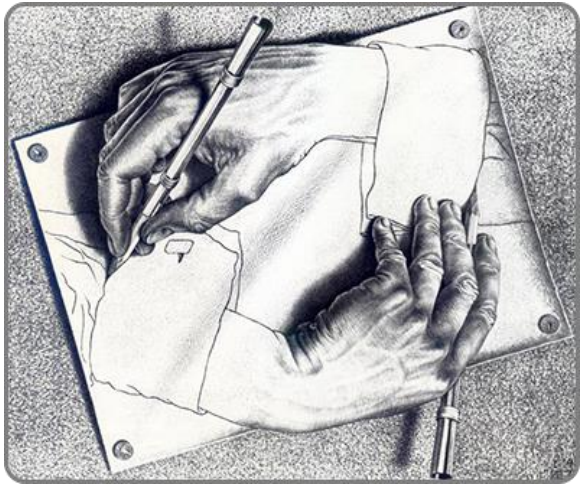
A relation R on a set A is **reflexive** if every element of A is related to itself: $\forall x \in A, xRx$.

Example

$A = \mathbb{Z}$, $xRy \leftrightarrow x = y$: reflexive

$A = \mathbb{Z}$, $xRy \leftrightarrow x > y$: not reflexive

What is the reflexivity on the matrix representing R?



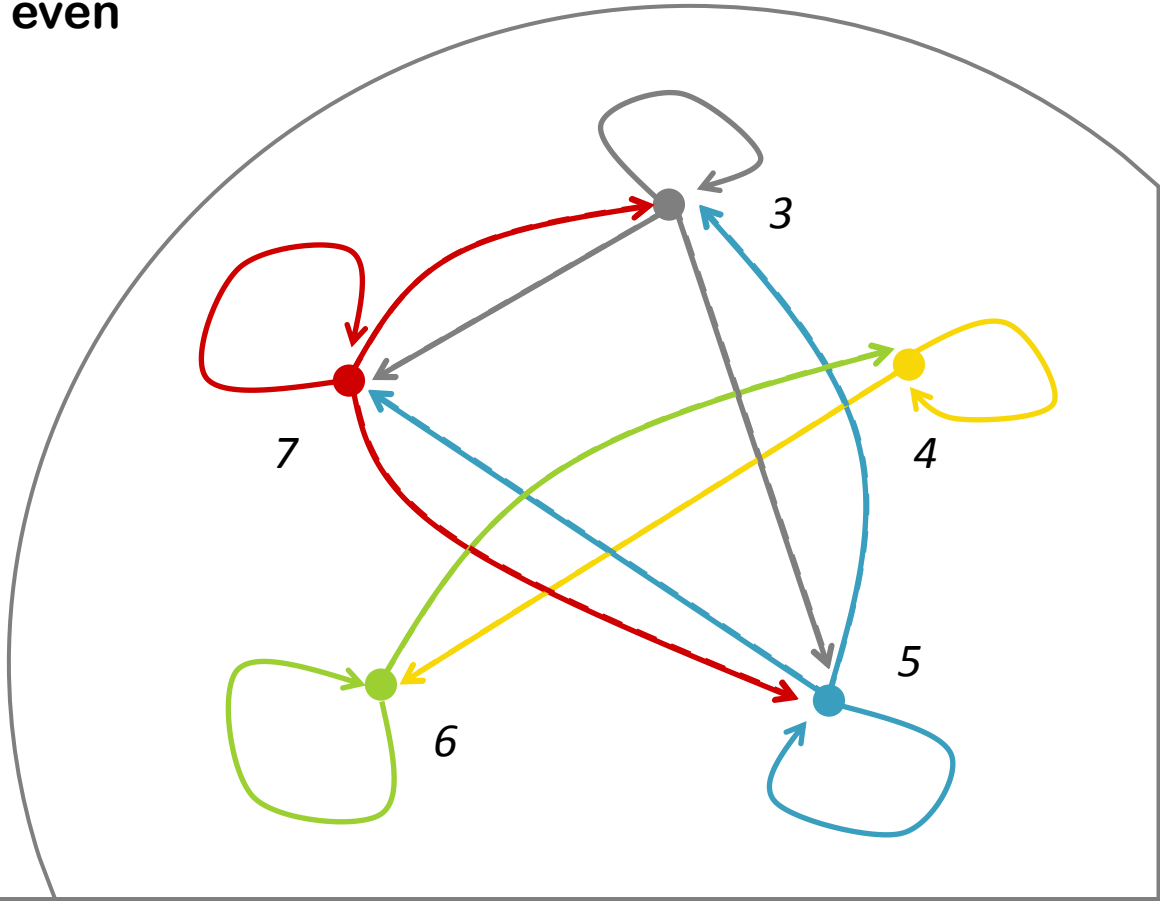
Drawing Hands (M.C. Escher)

Retrieved Mar 10, 2018 from https://www.wikiart.org/en/m-c-escher/drawing-hands

Reflexivity: Graphically

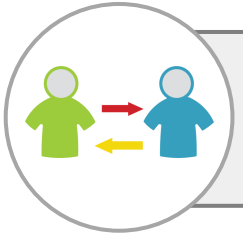
$A = \{3,4,5,6,7\}$, $xRy \leftrightarrow (x - y)$ **is even**

R **reflexive**

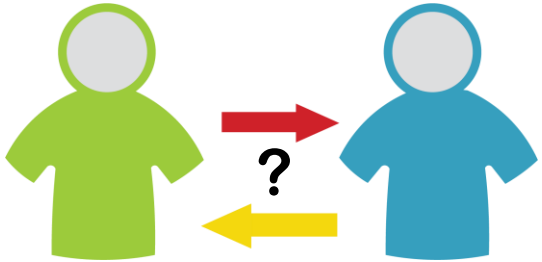


Symmetry

Symmetry: Definition

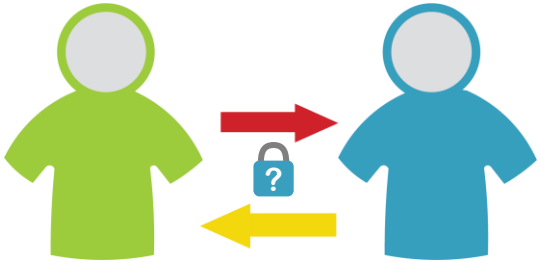


A relation R on a set A is symmetric if $(x,y) \in R$ implies $(y,x) \in R$: $\forall x \in A \;\; \forall y \in A,\; xRy \rightarrow yRx$.



Not Symmetric Relationship

E.g., $A = \mathbb{Z}$, $xRy \leftrightarrow x > y$: not symmetric



Symmetric Relationship

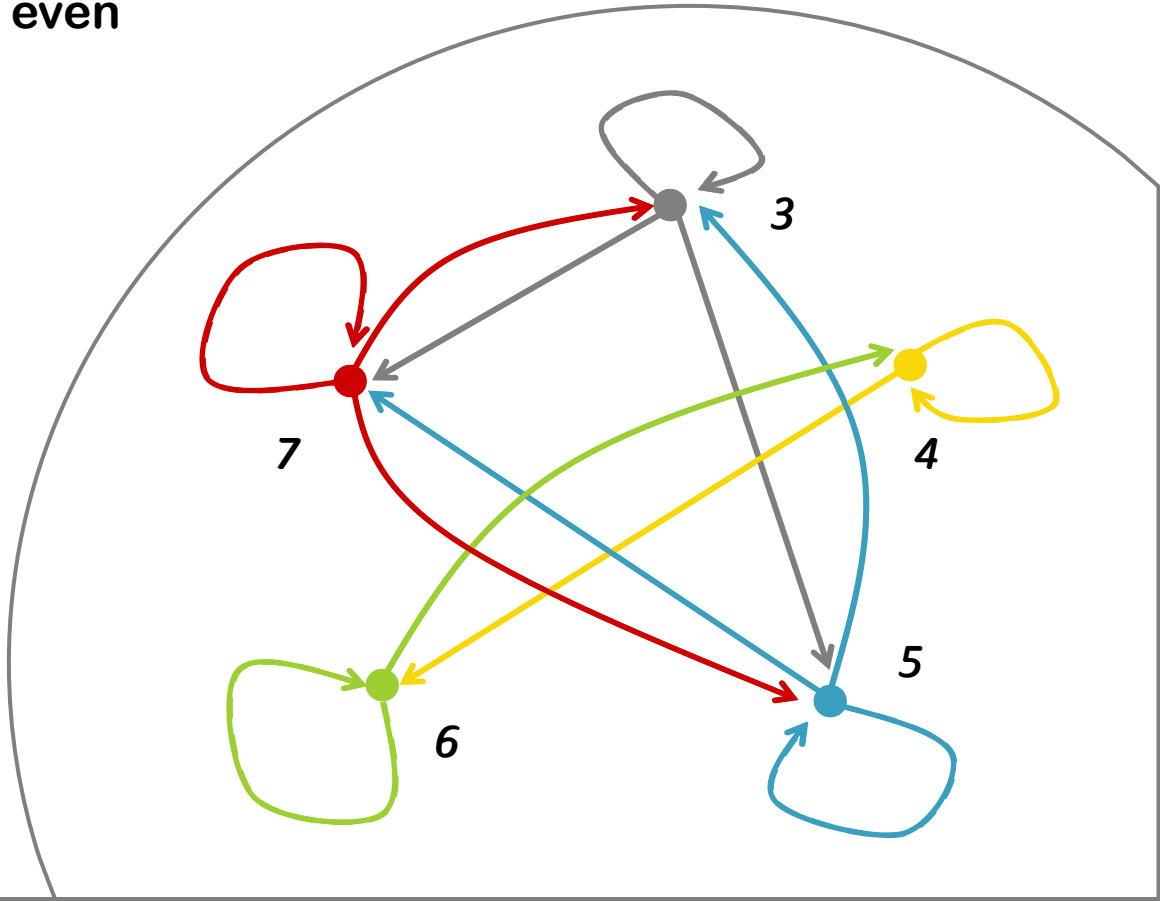
E.g., $A = \mathbb{Z}$, $xRy \leftrightarrow x = y$: symmetric

Symmetry: Graphically

$A = \{3,4,5,6,7\}$, $xRy \leftrightarrow (x - y)$ **is even**

R **reflexive**

R **symmetric**



Transitivity

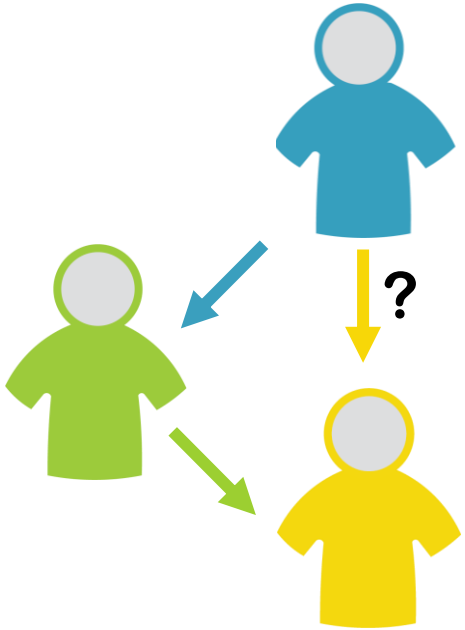
Transitivity: Definition

A relation R on a set A is **transitive** if $(x,y) \in R$ and $(y,z) \in R$ implies $(x,z) \in R$: $\forall x\ \forall y\ \forall z\ xRy \wedge yRz \rightarrow xRz$.

Example

$A = \mathbb{Z}$, $xRy \leftrightarrow x = y$: transitive

$A = \mathbb{Z}$, $xRy \leftrightarrow x > y$: transitive



Transitivity: Graphically

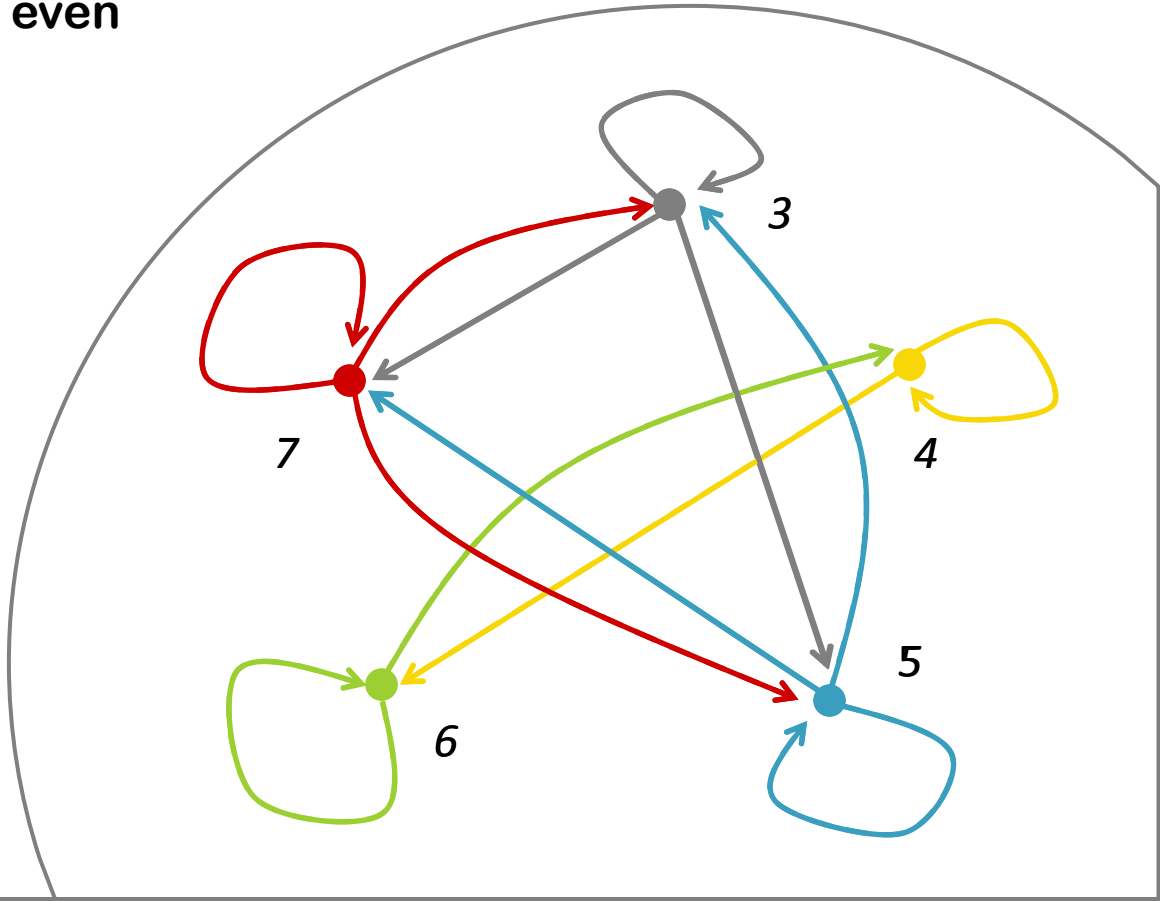
$A = \{3,4,5,6,7\}$, $xRy \leftrightarrow (x - y)$ **is even**

$[3] = \{3,5,7\}$, $[4] = \{4,6\}$

R **reflexive**

R **symmetric**

R **transitive**



Topic Summary

Let's recap…

- Binary relations:
  - Inverse and composition
  - Graphical representation
- Properties:
  - Reflexivity
  - Symmetry
  - Transitivity

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Topic 8.2 - Relations II

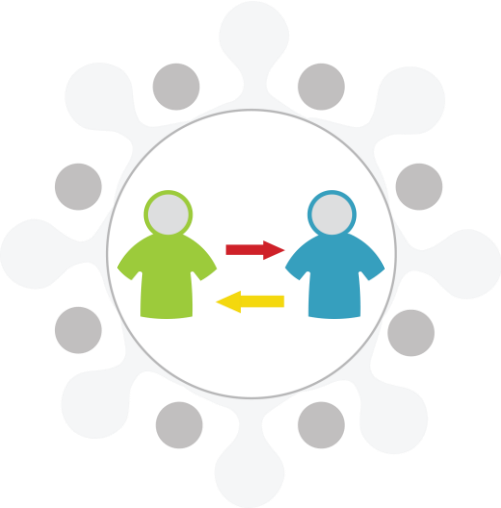
Dr. Guo Jian

Topic Overview

What's in store…

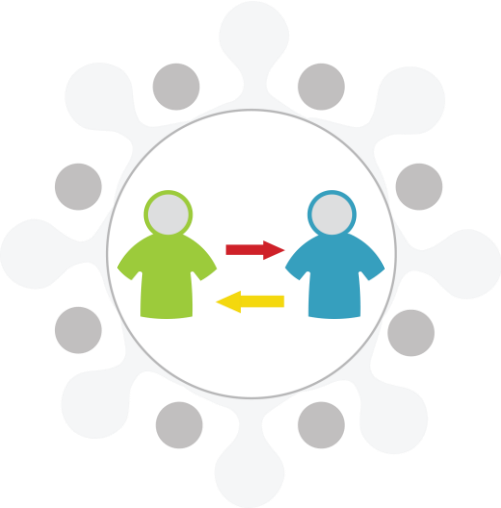
E quivalence Relations

A ntisymmetry



By the end of this lesson, you should be able to…

- Explain the conditions for an equivalence relation.
- Explain the concept of antisymmetry.



Equivalence Relations

Equivalence Relations: Definition

A relation R on a set A is an equivalence relation if:

1. R **is reflexive:** $\forall x \in A, xRx$
2. R **is symmetric:** $\forall x\ \forall y\ xRy \rightarrow yRx$
3. R **is transitive:** $\forall x\ \forall y\ \forall z\ xRy \wedge yRz \rightarrow xRz$

Equivalence class of a in A: $[a] = \{x \in A \mid aRx\}$ **for** R **an equivalence relation.**

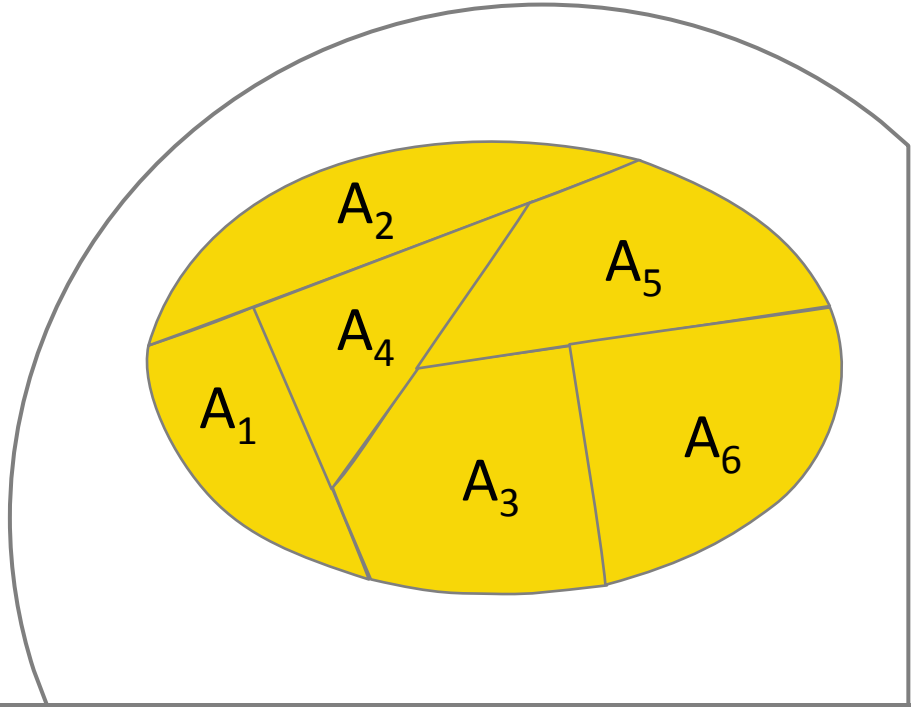
Equivalence Relations: Equivalence Classes

Partition of a set A:

$A_i \cap A_j = \varphi$ **whenever** $i \neq j$

$A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 = A$

Equivalence classes of A form a partition of A.



Equivalence Relations: Integers mod n

$$a \equiv b \pmod{n} \leftrightarrow a = qn + b$$

$\equiv \pmod{n}$ **is an equivalence relation:**

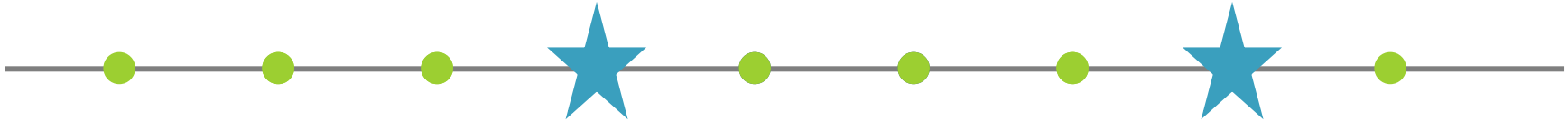
1. $\equiv \pmod{n}$ **is reflexive:** $\forall x \in A, x \equiv x \bmod n$
2. $\equiv \pmod{n}$ **is symmetric:** $\forall x\, \forall y\, x \equiv y \pmod{n} \rightarrow y \equiv x \pmod{n}$
3. $\equiv \pmod{n}$ **is transitive:** $\forall x\, \forall y\, \forall z\, x \equiv y \pmod{n} \wedge y \equiv z \pmod{n} \rightarrow x \equiv z \pmod{n}$

Equivalence Relations: Integers mod n

Equivalence class of $[0] = \{0, n, 2n, 3n, \ldots, -n, -2n, -3n \ldots\}$

Equivalence class of $[1] = \{1, n + 1, 2n + 1, 3n + 1, \ldots, -n + 1, -2n + 1 \ldots\}$

Example: Integers mod 4



Integers mod n **can be represented as elements between** 0 **and** $n - 1$:
$\{0, 1, 2, \ldots, n - 1\}$

Antisymmetry

Antisymmetry: Definition

A relation R on a set A is antisymmetric if $(x,y) \in R$ and $(y,x) \in R$ implies $x = y$: $\forall x\, \forall y\; xRy \wedge yRx \rightarrow x = y$.

Example

$A = \mathbb{Z}$, $xRy \leftrightarrow x = y$: **antisymmetric**

$A = \mathbb{Z}$, $xRy \leftrightarrow x \geq y$: **antisymmetric**

$BRC \leftrightarrow B \subseteq C$: **antisymmetric**

Antisymmetry: Graphically

$A = \{3,4,5,6,7\}$, $xRy \leftrightarrow (x - y)$ **is even**

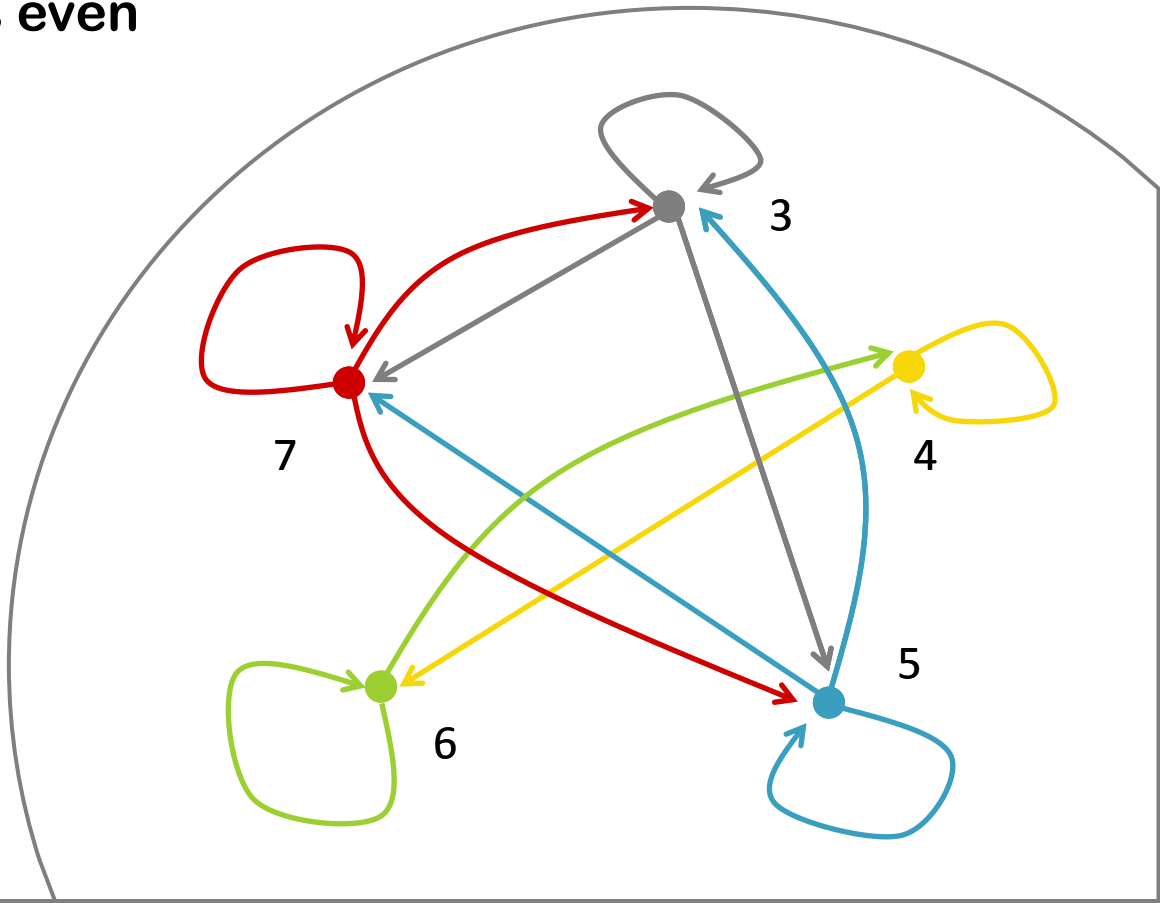
$[3] = \{3,5,7\}$, $[4] = \{4,6\}$

R **reflexive**

R **symmetric**

R **transitive**

R **is not antisymmetric**



Topic Summary

Let's recap…

- Equivalence relations: equivalence class
- Partial order: antisymmetry

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Topic 8.3 - Relations III

Dr. Guo Jian

Topic Overview

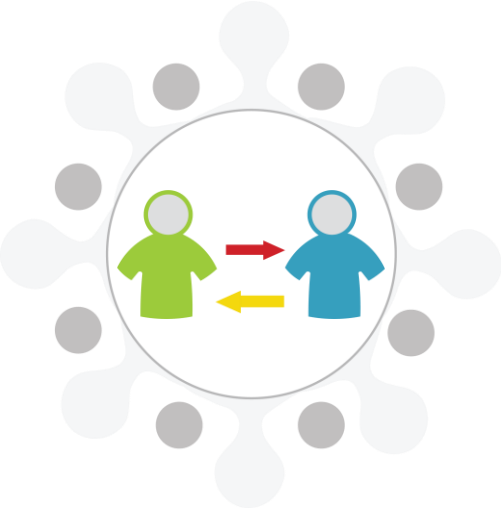
What's in store…

Partial Order

Transitive Closure

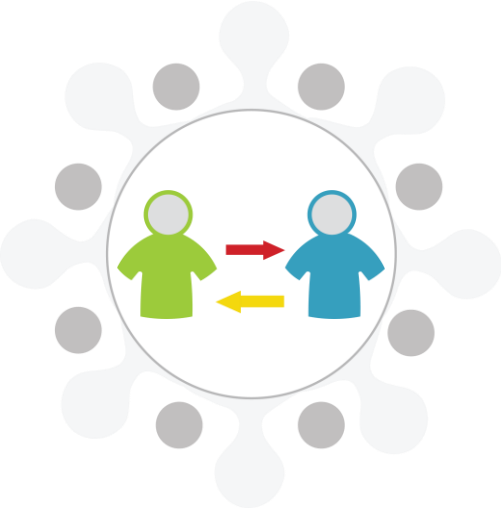
Beyond Binary Relations

Operations on Relations



By the end of this lesson, you should be able to…

- Explain the concept of partial order.
- Explain the three properties of transitive closure.
- Explain the concept of non-binary relations.
- Explain the different operations on relations.



Partial Order

Partial Order: Definition

R is **a partial order** on A if R is reflexive, antisymmetric and transitive.

Example

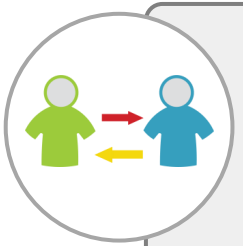
$A = \mathbb{Z}$, $xRy \leftrightarrow x \leq y$

Notion of partial order is useful for scheduling problems across possibly different domains.

Transitive Closure

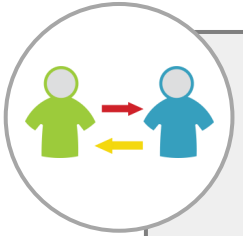
Transitive Closure: What is Closure?

Let A be a set and R a binary relation on A.



The **closure of a relation** $R \subseteq A \times A$ with respect to a property P (P being reflexive, symmetric, or transitive) is the relation obtained by adding the minimum number of ordered pairs to R to obtain property P.

Transitive Closure: Definition



Let A be a set and R a binary relation on A. The **transitive closure** of R is the binary relation R^t on A that satisfies the following three properties:

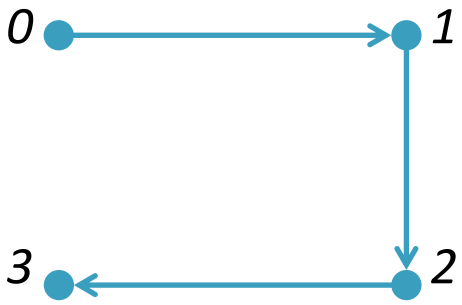
1. R^t is transitive
2. $R \subseteq R^t$
3. If S is any other transitive relation that contains R then $R^t \subseteq S$

Transitive Closure: Example

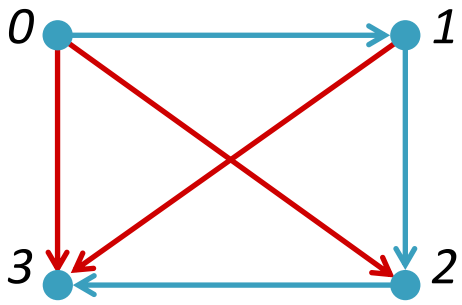
Let $A = \{0,1,2,3\}$

Consider a relation $R = \{(0,1),(1,2),(2,3)\}$ **on** A

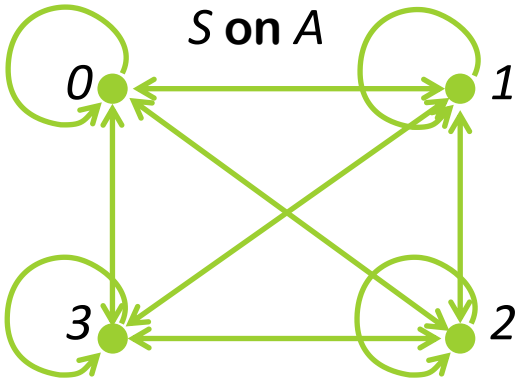
R **on** A



R^t **on** A



S **on** A



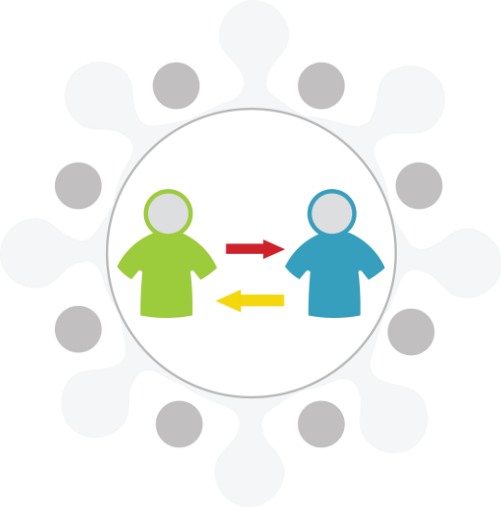
S **is transitive and** $R \subseteq S$

Thus $R^t \subseteq S$

$$R^t = \{(0,1),(1,2),(2,3),(0,2),(0,3),(1,3)\}$$

Transitive Closure: Construction

- Let A be a set and R a binary relation on A.
- Start with R, and do the following: $\forall x, y, z \in A$, if $(xRy \wedge yRz \wedge x\not{R}z)$ then add (x,z).
- Repeat until the obtained relation is transitive (will stop if $|A|$ is finite).
- The ordering in which the edges are added does not matter.

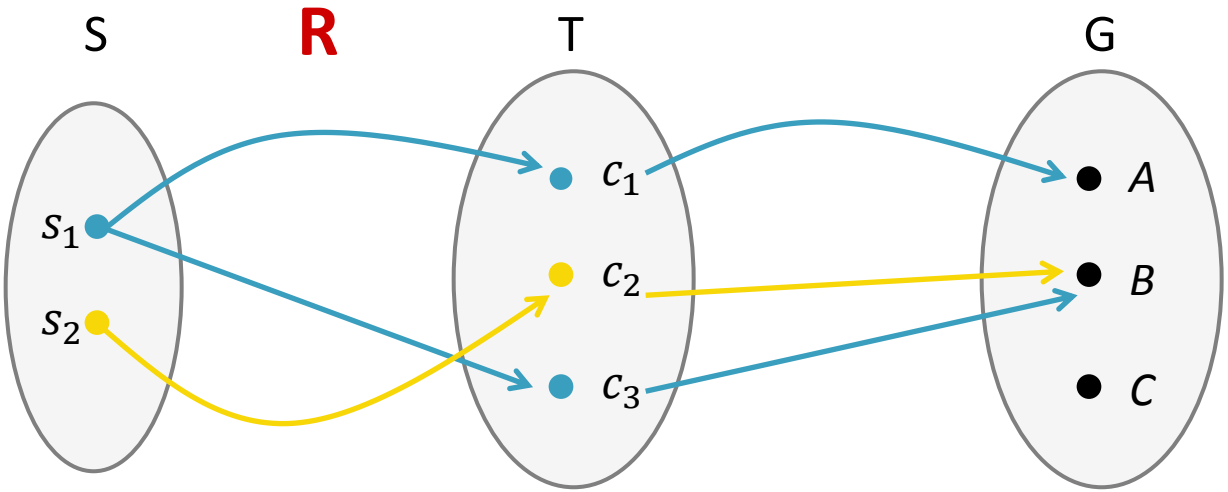


Beyond Binary Relations

Beyond Binary Relations: Non-binary Relations (Example)

$S = \{s_1, s_2\}$ **students,** $T = \{c_1, c_2, c_3\}$ **courses**

$G = \{A, B, C\}$ **grades,** (s_1, c_1, A), (s_1, c_3, B), (s_2, c_2, B)



Beyond Binary Relations: *n-ary* Relations

Let $A_1, \ldots, A_n$ be sets. A *n-ary* relation R is a subset of $A_1 \times \cdots \times A_n$. $a_1, \ldots, a_n$ are related if $(a_1, \ldots, a_n) \in R$.

Example

$S = \{s_1, s_2\}$ **students,** $T = \{c_1, c_2, c_3\}$ **courses**

$G = \{A, B, C\}$ **grades,** (s_1, c_1, A), (s_1, c_3, B), (s_2, c_2, B)

Operations of Relations: Complement of a Relation

Let $R \subseteq A_1 \times \cdots \times A_n$ be a relation.
$\bar{R} = (A_1 \times \cdots \times A_n - R)$ **is the relational complement of R,**
i.e., $(a_1, a_2, a_3, \ldots, a_n) \in \bar{R} \Leftrightarrow (a_1, a_2, a_3, \ldots, a_n) \notin R$.

Example

$A = \{1,2\}$, $B = \{3,5\}$ **and** $R = \{(1,3), (2,5)\}$

Then $\bar{R} = A \times B - R = \{(1,5), (2,3)\}$

Operations on Relations

Operations of Relations: Union of Relations

Let $R, S \subseteq A_1 \times \cdots \times A_n$ be two relations. $R \cup S$ is the relation such that $(a_1, a_2, a_3, \ldots, a_n) \in R \cup S \Leftrightarrow (a_1, a_2, a_3, \ldots, a_n) \in R \vee (a_1, a_2, a_3, \ldots, a_n) \in S$.

Example

$A = \{1,2,3\}$, $B = \{1,2,3,4\}$ **and** $R = \{(1,1), (2,2), (3,3)\}$

$S = \{(1,1), (1,2), (1,3), (1,4)\}$

Then $R \cup S = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$

Operations of Relations: Intersection of Relations

Let $R, S \subseteq A_1 \times \cdots \times A_n$ be two relations. $R \cap S$ is the relation such that $(a_1, a_2, a_3, \ldots, a_n) \in R \cap S \Leftrightarrow (a_1, a_2, a_3, \ldots, a_n) \in R \wedge (a_1, a_2, a_3, \ldots, a_n) \in S$.

Example

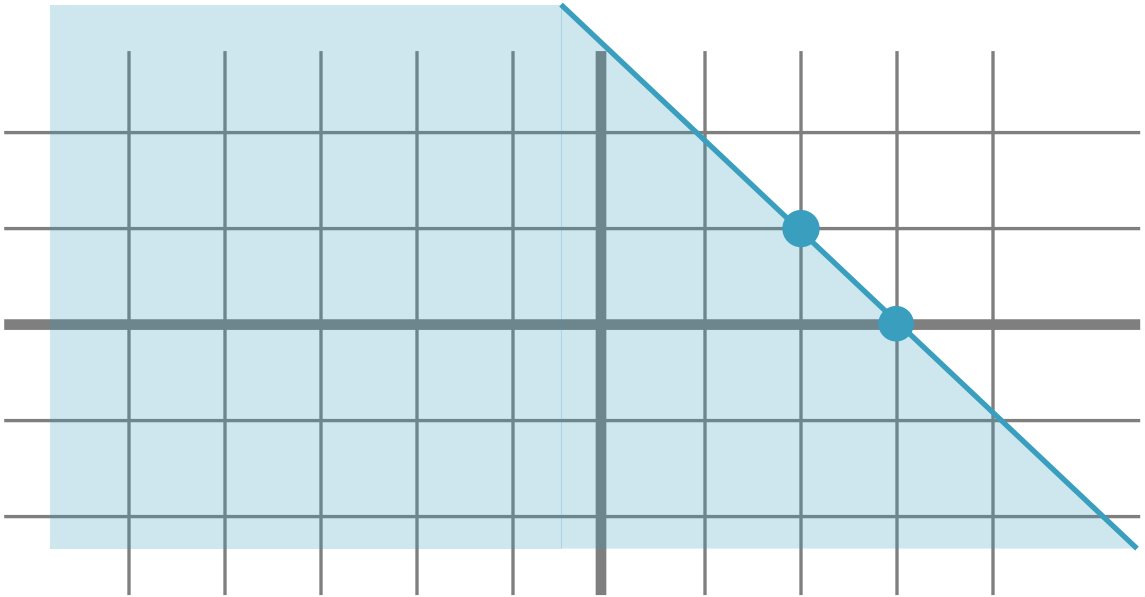
$A = \{1,2,3\}$, $B = \{1,2,3,4\}$ **and** $R = \{(1,1), (2,2), (3,3)\}$

$S = \{(1,1), (1,2), (1,3), (1,4)\}$

Then $R \cap S = \{(1,1)\}$

Operations of Relations: Example

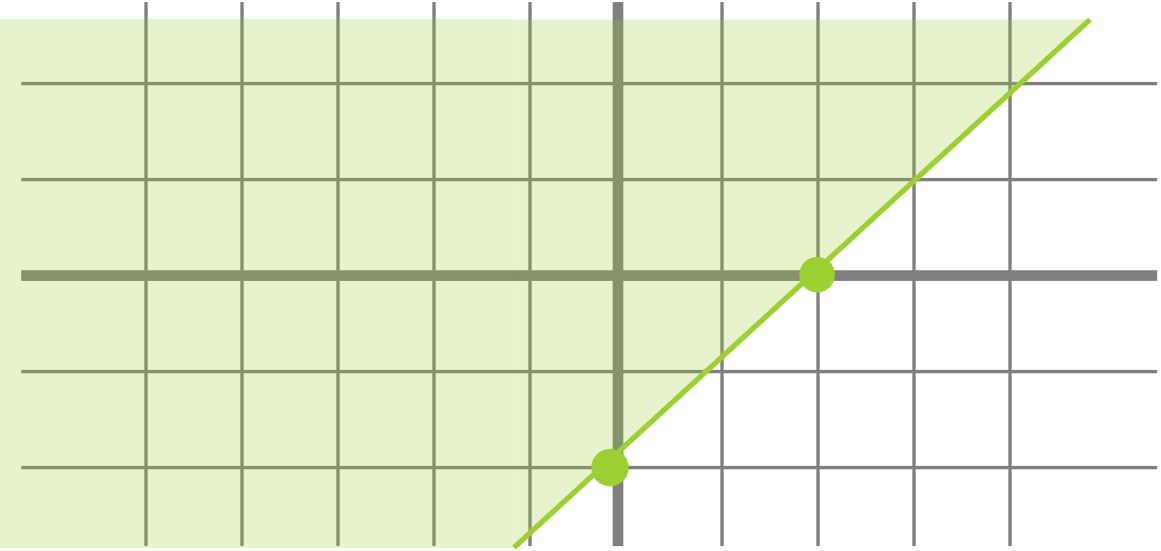
$T = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid x + y \leq 3 \}$



Operations of Relations: Example

$T = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid x + y \leq 3 \}$

$S = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid x - y \leq 2 \}$

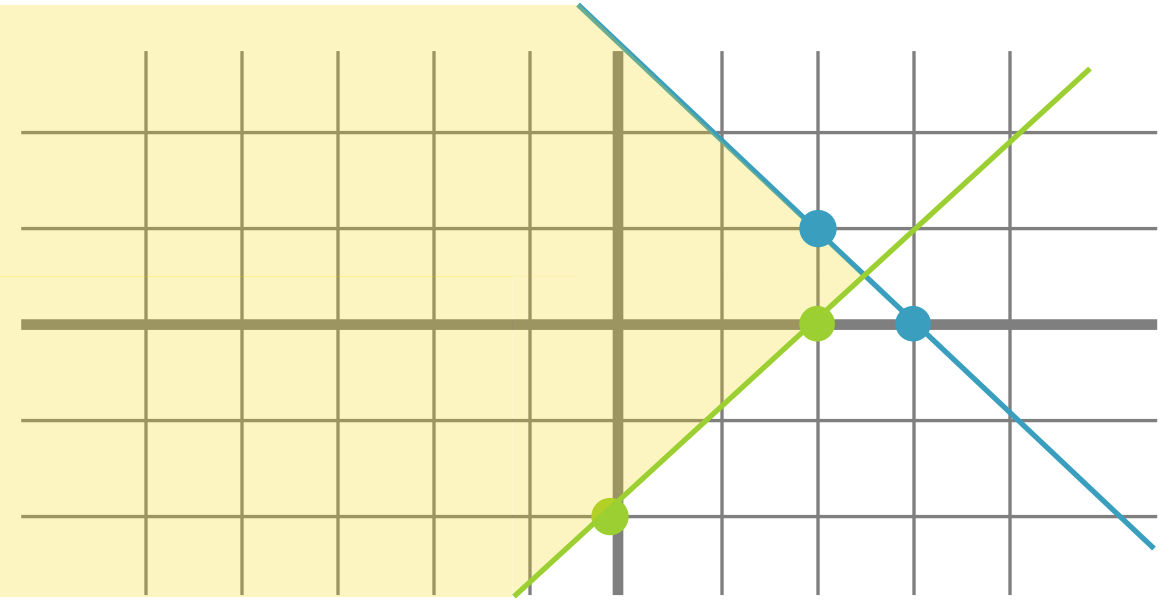


Operations of Relations: Example

$T = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid x + y \leq 3 \}$

$S = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid x - y \leq 2 \}$

$T \cap S = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid (x + y \leq 3) \wedge (x - y \leq 2) \}$



Topic Summary

Let's recap…

- Partial Order
- Transitive Closure
- Beyond binary relations
- Operations on relations
  - Complement
  - Union
  - Intersection