EE2001
Circuit Analysis

Energy Storage Elements

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Energy Storage Elements

1 Capacitors

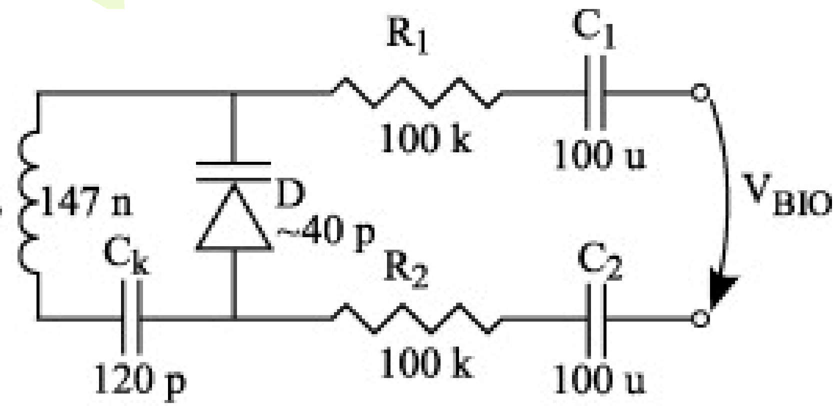
2 Series and Parallel Capacitors

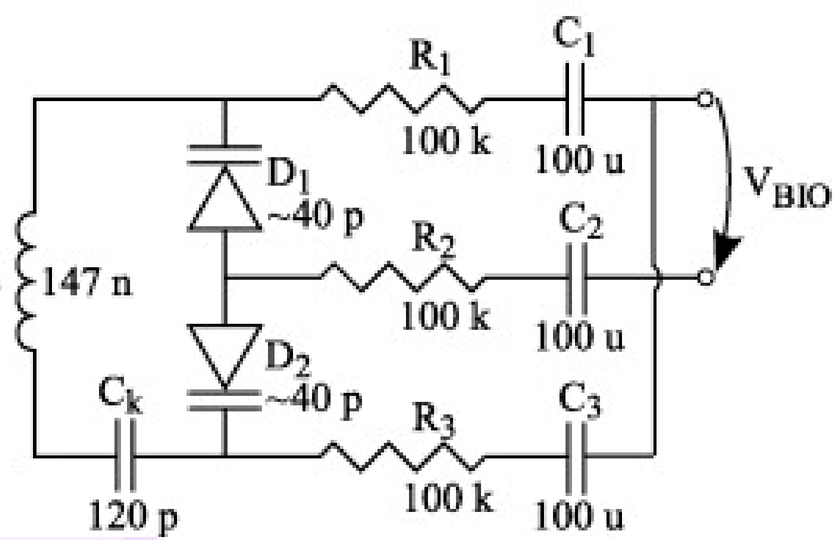
3 Inductors

4 Series and Parallel Inductors

5 Mutual Inductance

Example from Biomedical Engineering



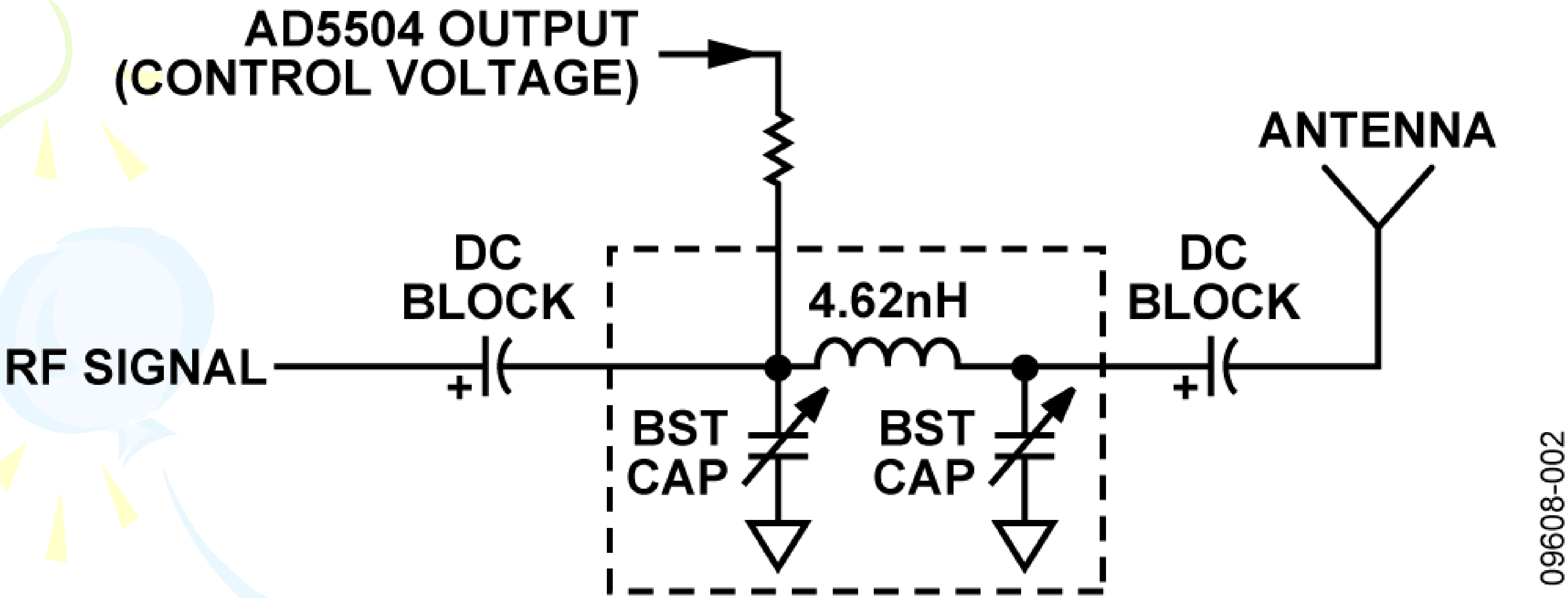


A measurement method to measure **biopotentials** with a passive LC resonator.

The sensor itself dissipates virtually no energy at all but of course, some energy is lost in form of resistive losses in the sensor wires and components. The sensor is inexpensive to manufacture and since it is a totally passive device, it is very suitable for implantable applications.

Totally passive wireless biopotential measurement sensor by utilizing inductively coupled resonance circuits, J. Riistama et al., Sensors and Actuators A: Physical, Feb 2010

Example from Communications Engineering

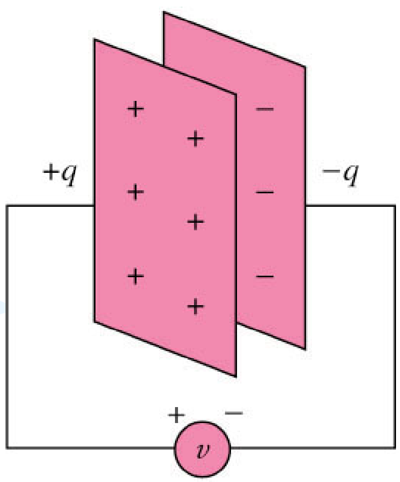


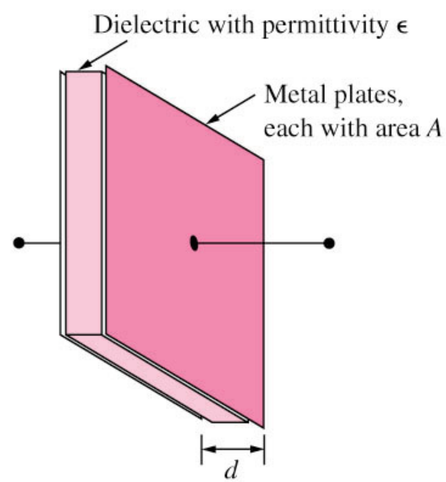
Tunable filter and antenna, for applications in communications, radar engineering, etc.

http://www.analog.com/en/circuits-from-the-lab/CN0193/vc.html

Capacitors

- A capacitor is a passive element designed to **store energy** in its **electric field.**
- It is a two terminal device that consists of two conducting plates separated by an insulator (or dielectric).

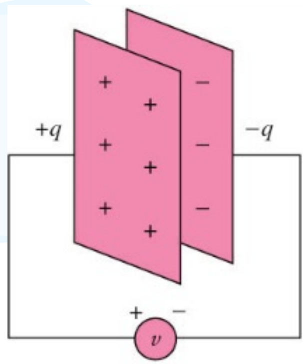




Alexander Charles K and Sadiku Matthew N O, ***Fundamentals of Electric Circuits,*** 4th Edition, McGraw-Hill, 2009. **CHAPTER 6**

Capacitors

- **Capacitance** *C* is the ratio of the charge *q* on one plate of a capacitor to the voltage difference *v* between the two plates, measured in farads (F).

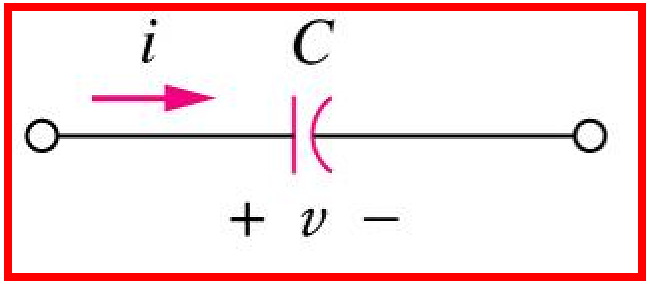


$$C = \frac{q}{v} \Longrightarrow q = C\,v \quad \text{or} \quad C = \frac{\varepsilon\,A}{d}$$

- where ε is the permittivity of the dielectric material between the plates, A is the surface area of each plate, d is the distance between the plates.
- Unit: F, pF (10^{-12}), nF (10^{-9}), and **μF (10^{-6})**

Capacitors

- If *i* is flowing into the +ve terminal of C
  - Charging => *i* is +ve
  - Discharging => *i* is –ve



- The current-voltage relationship of capacitor according to above convention is

$$\boxed{i = C\frac{dv}{dt}}$$

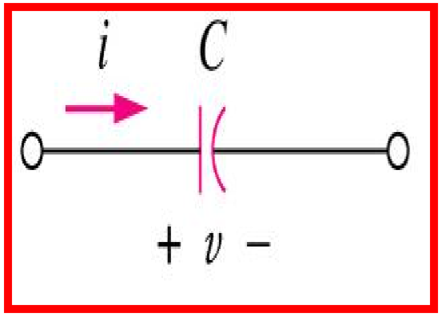
or

$$\boxed{v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)}$$

Example 1

The current through a 100-μF capacitor is

$i(t) = 50\sin(120\pi t)$ mA.



Calculate the voltage across it at t =1 ms and t = 5 ms. Take $v(0)$ =0.

$$v = \frac{1}{C}\int_0^t i\,dt = \frac{10^{-3}}{0.1\text{x}10^{-3}}\int_0^t 50\sin 120\pi t\,dt \text{ V}$$

$$= -\frac{500}{120\pi}\cos 120\pi t\Big|_0^t = \frac{50}{12\pi}(1-\cos 120\pi t)\text{V}$$

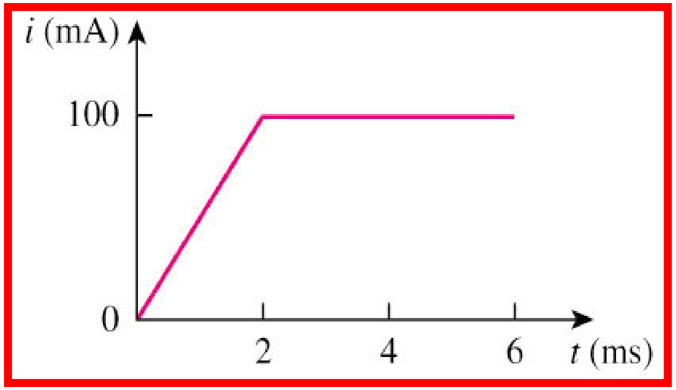
$$v(t = 1\text{ms}) = \frac{50}{12\pi}(1-\cos 0.12\pi) = \underline{\mathbf{93.14\ mV}}$$

$$v(t = 5\text{ms}) = \frac{50}{12\pi}(1-\cos 0.6\pi) = \underline{\mathbf{1.736\ V}}$$

Example 2

An initially uncharged 1-mF capacitor has the current shown below across it.

Calculate the voltage across it at *t* = 2 ms and *t* = 5 ms.



$$i(t) = \begin{cases} 50t, & 0 < t < 2 \\ 100, & 2 < t < 6 \end{cases} \qquad v = \frac{1}{C}\int i\,dt$$

For $0 < t < 2$, $\quad v = \frac{1}{C}\int_0^t 50t\,dt = 25t^2 \times 10^3$ V

At $t = 2$ms, $\quad v = \underline{\mathbf{100\ mV}}$

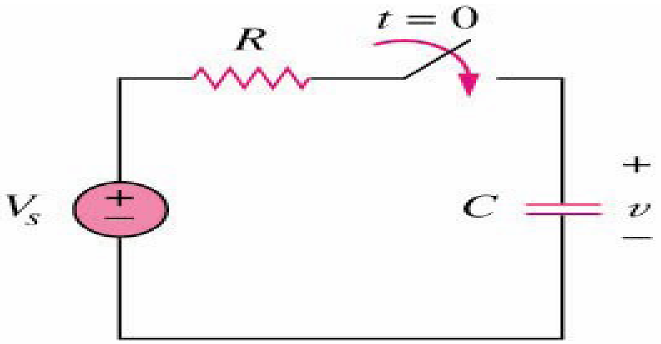
For $2 < t < 6$, $\quad v = \frac{1}{C}\int_2^t 0.1\,dt + v(2) \quad = (100t - 0.2 + 0.1) = (100t - 0.1)$V

At $t = 5$ms, $\quad v = (500 - 100)$mV

$= \underline{\mathbf{400\ mV}}$

Important notes on capacitor's behaviour:

•i = 0 when v is constant. Hence, a capacitor acts like an **open circuit** to a DC voltage.



DC steady-state operating condition

Suppose all the independent sources in a circuit are DC sources. For a sufficient long time, all currents and voltages in the circuit settle down to constant values. Then the circuit is said to be in a DC steady state.

- When there is a switching (or change in circuit) at time $t = 0$
  - Time instant just before the switching, $t = 0^-$
  - Time instant immediately after the switching, $t = 0^+$

$t = 0^-$: Open

$t = 0^+$: Closed

•When the voltage changes, the corresponding current is nonzero. The more rapidly *v* changes, the larger is *i*.

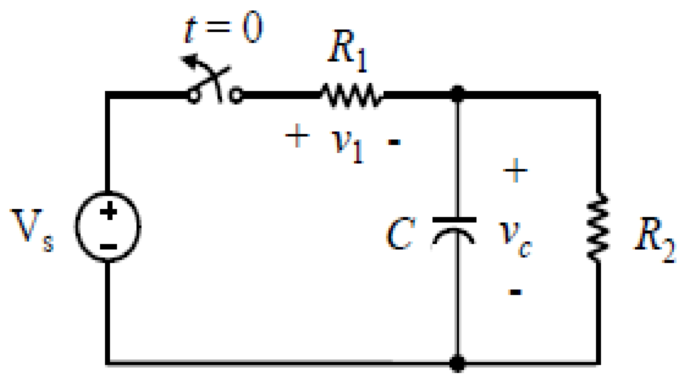
•An abrupt or instantaneous change in voltage requires an infinite current (meaning infinite power), which is physically impossible.

•Thus, abrupt or instantaneous changes in voltages across capacitor are not possible, even though the current may be discontinuous.

- At $t = 0$, time instant of switching (or change in circuit)
  - As voltage across capacitor cannot change abruptly, $v_C(0^-) = v_C(0) = v_C(0^+)$

Example 3

For the circuit shown, compute the voltage across the resistor R_2 at time $t = 0^+$.
Given that at $t = 0^-$, $V_s = 6$ V, $v_c(0^-) = 4$ V and $v_1(0^-) = 2$ V.



– At $t = 0^+$, no voltage source since switch is open.

– No current through R_1 and thus $v_1(0^+) = 0$.

– Due to continuity, $v_c(0^+) = v_c(0^-) = 4$ V

– Voltage across R_2 at $t = 0^+$ is equal to 4V.

Power and Energy

The power in a capacitor is

$$p = vi \quad = Cv\frac{dv}{dt}$$

The energy in a capacitor at time *t* is

$$w_C(t) = \int_{-\infty}^{t} p\,dt = \int_{-\infty}^{t} vi\,dt = \int_{-\infty}^{t} C\frac{dv}{dt}v\,dt = C\int_{-\infty}^{t} v\,dv$$

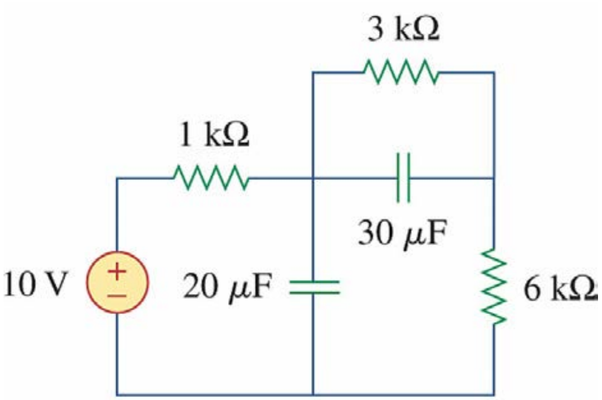
$$\therefore \quad w_C(t) = \frac{1}{2}Cv^2(t) - \frac{1}{2}Cv^2(-\infty)$$

$$= \frac{1}{2}Cv^2(t), \qquad \text{as } \mathrm{v}(-\infty) = 0$$

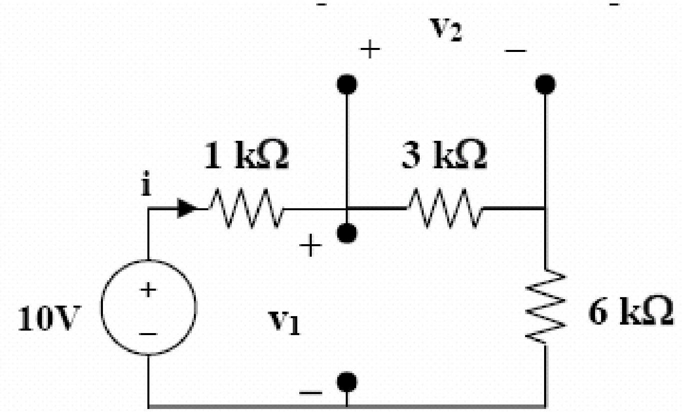
Example 4:
For a capacitor of 1F, with 10 V across it, the energy stored is

$$w_C = \frac{1}{2}Cv^2 = \frac{1}{2}\times 1\times 10^2 = 50\mathrm{J}$$

Example 5:
Obtain the energy stored in each capacitor in the figure shown under dc conditions.



Under dc conditions, the capacitors act like open-circuits as shown below:



$$i = \frac{10}{1+3+6} = 1\text{mA}$$

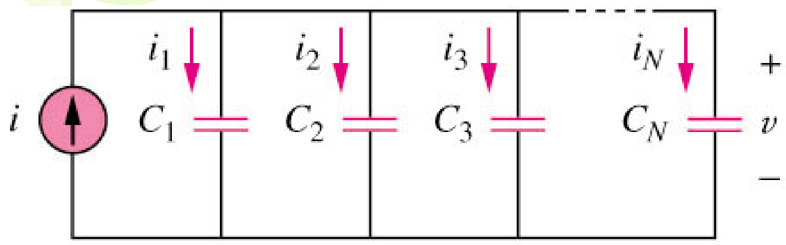
$$v_1 = (3k + 6k)i = 9V$$

$$v_2 = (3k)i = 3V$$

$$w_1 = \frac{1}{2}C_1 v_1^2 = \frac{1}{2}\left(20x10^{-6}\right)(9)^2 = \underline{\mathbf{810\ \mu J}}$$

$$w_2 = \frac{1}{2}C_2 v_2^2 = \frac{1}{2}\left(30x10^{-6}\right)(3)^2 = \underline{\mathbf{135\ \mu J}}$$

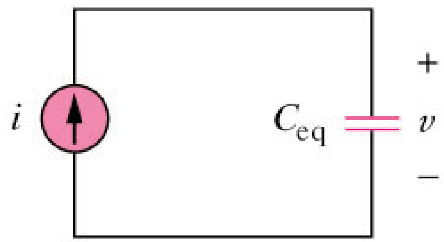
Series and Parallel Capacitors



(a)

$$i = i_1 + i_2 + \ldots + i_N$$

$$= C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + \ldots + C_N \frac{dv}{dt}$$



(b)

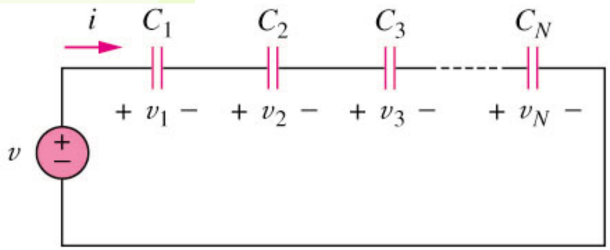
$$i = (C_1 + C_2 + \ldots + C_N)\frac{dv}{dt}$$

$$= C_{eq} \frac{dv}{dt}$$

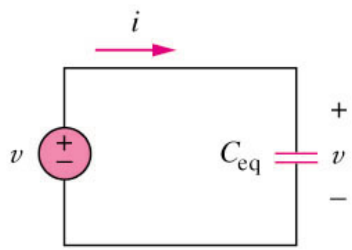
- The equivalent capacitance of *N* **parallel-connected** capacitors is the sum of the individual capacitances.

$$C_{eq} = C_1 + C_2 + \ldots + C_N$$

Series and Parallel Capacitors



(a)



(b)

$$
\begin{aligned}
v &= v_1 + v_2 + \ldots + v_N \\
&= \frac{1}{C_1}\int_{t_0}^{t} i\,dt + v_1(t_0) + \frac{1}{C_2}\int_{t_0}^{t} i\,dt + v_2(t_0) \\
&\quad + \ldots + \frac{1}{C_N}\int_{t_0}^{t} i\,dt + v_N(t_0)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{C_{eq}}\int_{t_0}^{t} i\,dt + v_{eq}(t_0) &= \left(\frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}\right)\int_{t_0}^{t} i\,dt \\
&\quad + v_1(t_0) + v_2(t_0) + \ldots + v_N(t_0)
\end{aligned}
$$

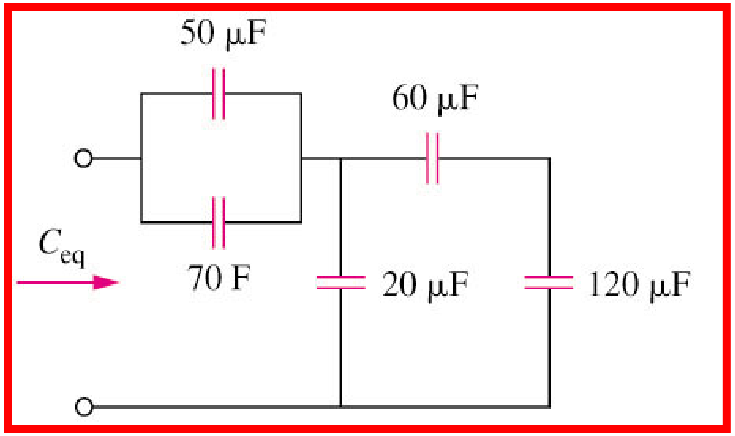
- The equivalent capacitance of *N* **series-connected** capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}$$

Equivalent initial voltage $\quad v_{eq}(t_0) = v_1(t_0) + v_2(t_0) + \ldots + v_N(t_0)$

Example 6

Find the equivalent capacitance seen at the terminals of the circuit in the circuit shown :



Combining 60 and 120μF in series = $\dfrac{60 \times 120}{180} = 40\mu F$

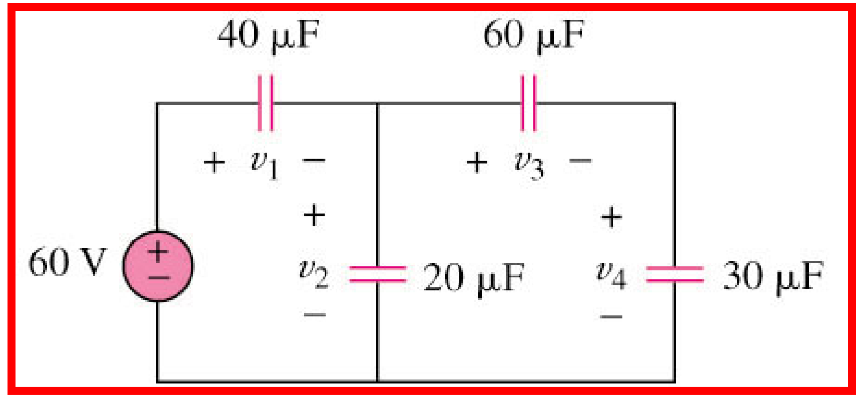
40μF in parallel with 20μF = 40 + 20 = 60μF

50μF in parallel with 70μF = 50 + 70 = 120μF

60μF in series with 120μF = $\dfrac{60 \times 120}{180} =$ **<u>40μF</u>**

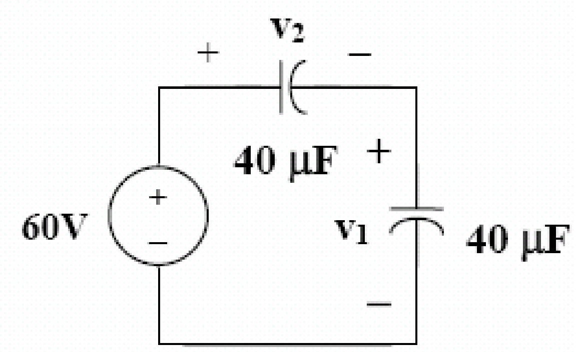
Example 7

Find the voltage across each of the capacitors in the circuit shown



$$60\mu F \text{ in series with } 30\mu F = \frac{60x30}{90} = 20\mu F$$

$$20\mu F \text{ in parallel with } 20\mu F = 40\mu F$$



$$\text{From the Figure, } v_1 = v_2 = \frac{60}{2} = 30V$$

Note that

$$q_3 = C_3 v_3 = q_4 = C_4 v_4 \Rightarrow \frac{v_3}{v_4} = \frac{C_4}{C_3} = \frac{1}{2}$$

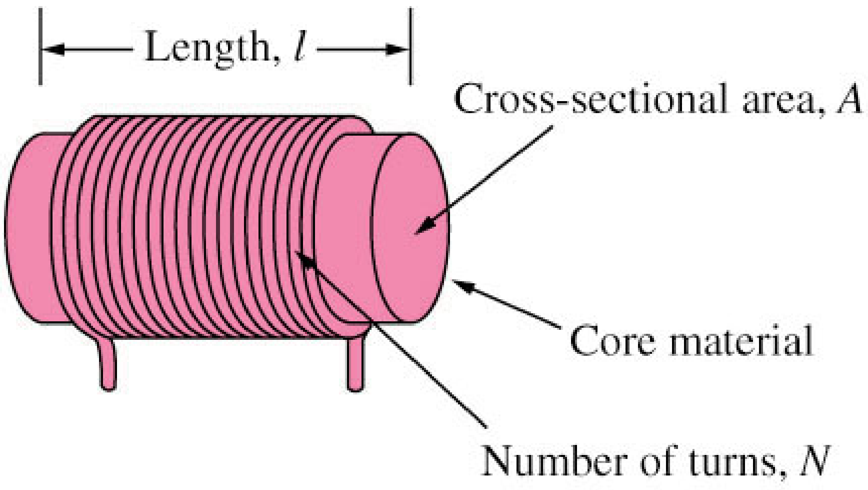
Then

$$v_3 + v_4 = 3v_3 = 30 \Rightarrow v_3 = 10 \Rightarrow v_4 = 20$$

Inductors

- An inductor is a two terminal device designed to store energy in its magnetic field.

- An inductor consists of a coil of conducting wire.



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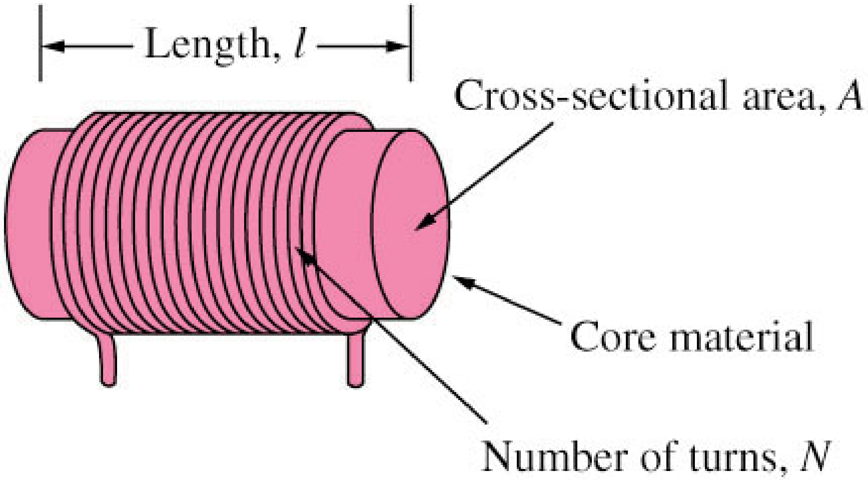
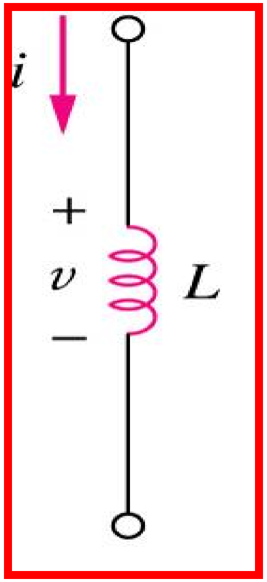


$$L = \frac{N^2 \mu A}{l}$$

Inductors

- The current-voltage relationship of an inductor:

$$v = L\frac{d\,i}{d\,t} \qquad \text{or} \qquad i = \frac{1}{L}\int_{t_0}^{t} v(t)\,dt + i(t_0)$$



where $i(t_0)$ is the current buildup from $-\infty$ to t_0 and L is the inductance

- The unit of inductors is Henry (H), mH (10^{-3}) and μH (10^{-6}).

Power and energy

– The power in an inductor is

$$p = vi = Li\frac{di}{dt}$$

– The energy stored in an inductor is

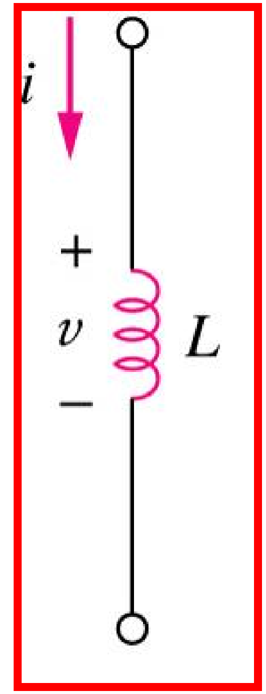
$$w_L(t) = \int_{-\infty}^{t} p\, dt = \int_{-\infty}^{t} vi\, dt = \int_{-\infty}^{t} L\frac{di}{dt}i\, dt = L\int_{-\infty}^{t} i\, di$$

$$\therefore \quad w_L(t) = \frac{1}{2}Li^2(t) \text{ J}, \quad \text{as } i(-\infty) = 0$$

Example 8

The terminal voltage of a 2-H inductor is

v = 10(1-t) V



Find the current flowing through it at t = 4 s and the energy stored in it within **0 < t < 4 s**. Assume *i(0)* = 2 A.

$$i = \frac{1}{L}\int_{t_0}^{t} v(t)dt + i(t_0) = \frac{1}{2}\int_{0}^{t} 10(1-t)dt + 2 = 5\left(t - \frac{t^2}{2}\right) + 2$$

At t = 4, $\quad i = 5(4 - 8) + 2 = \underline{\mathbf{-18\ A}}$

$$p = vi = 10(1 - t)\left[5t - \frac{5}{2}t^2 + 2\right] = 20 + 30t - 75t^2 + 25t^3$$

$$w = \int_0^4 p\,dt = \left[20t + 15t^2 - 25t^3 + 25t^4/4\right]\Big|_0^4 = 80 + 15 \text{ x } 16 - 1600 + 1600$$

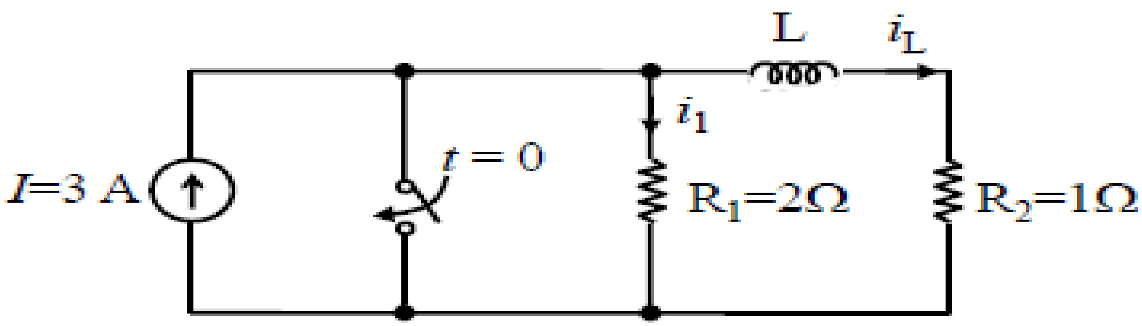
$$= \underline{\mathbf{320\ J}}$$

- If i is constant, $\dfrac{di}{dt} = 0 \quad \Rightarrow \quad v = 0$

- Important notes on inductor's behaviour:
  - An inductor acts as a **short circuit** to DC. The more rapidly the current changes, the greater is the voltage that appears across inductor's terminals.
  - Abrupt changes in the current require infinite voltages to appear across the terminals of the inductor, which means infinite power, which is a physical impossibility.
  - As a result, instantaneous changes in a current through an inductor are not possible. **Current** through an inductor is **continuous** even though the voltage may be discontinuous.
  - As current through inductor cannot change abruptly, $i_L(0^-) = i_L(0) = i_L(0^+)$

Example 9

Determine $i_1(0^+)$ and $i_L(0^+)$



– The switch is closed at $t = 0$

$I = 3A;\ i_L(0^-) = 2\ A \rightarrow i_1(0^-) = 1\ A$

– At $t = 0^+$, the current source is shorted

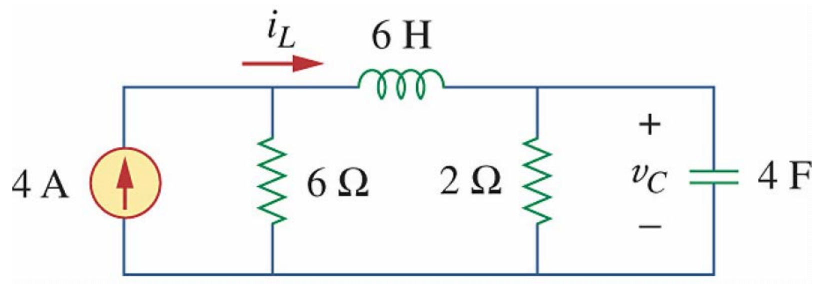
$i_1(0^+) = 0\ A$

$i_L(0^+) = i_L(0^-) = 2\ A$

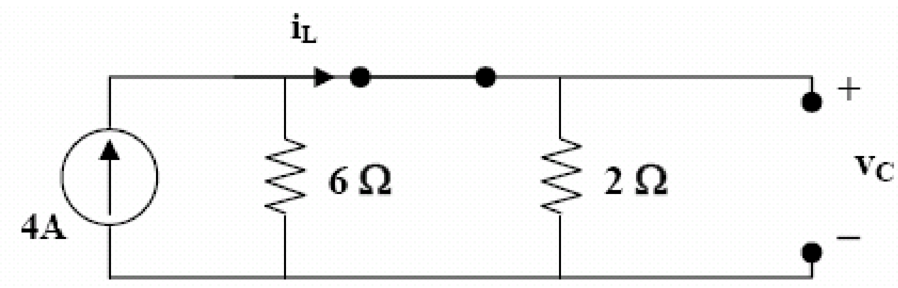
Exercise: determine the current flowing through the switch at $t = 0^+$

Example 10

Determine v_c, i_L, and the energy stored in the capacitor and inductor in the circuit of circuit shown under dc conditions.



Under dc conditions, the circuit is equivalent to



$$i_L = \frac{6}{6+2} \times 4 = 3A$$

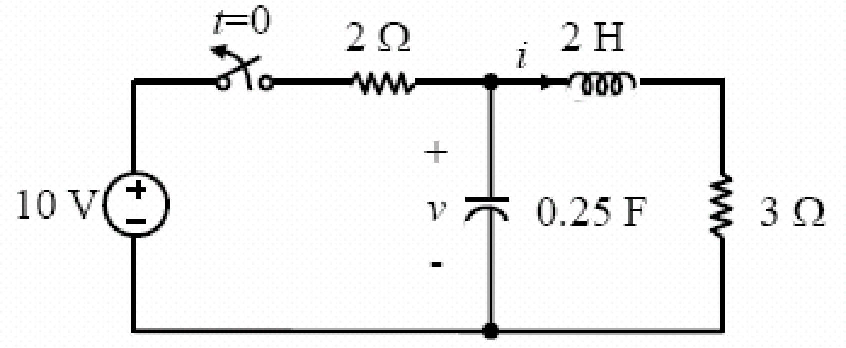
$$v_C = 2i_C = \underline{\mathbf{6\ V}}$$

$$w_C = \frac{1}{2}Cv_C^2 = \frac{1}{2}(4)(6)^2 = \underline{\mathbf{72\ J}}$$

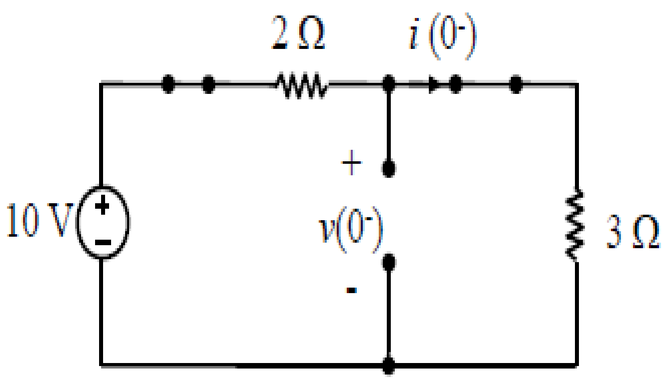
$$w_L = \frac{1}{2}Li_L^2 = \frac{1}{2}(6)(3)^2 = \underline{\mathbf{27\ J}}$$

Example 11

Suppose the switch in the RLC circuit has been closed for long time before it is open at *t= 0. Find the voltage v and current i at t= 0+.*



At $t = 0^-$, capacitor acts as open circuit and inductor acts as short circuit.

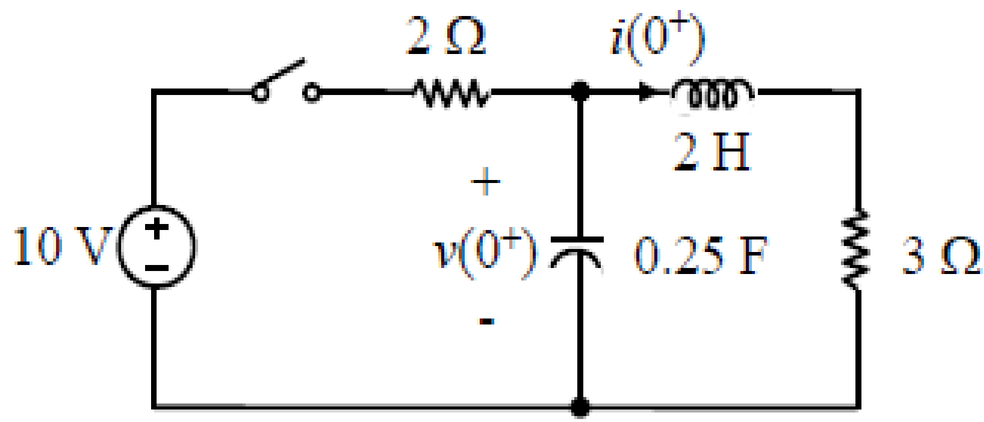


– $i(0^-) = 10/5 = 2$ A; $v(0^-) = 3\times2 = 6$ V

The switch is opened at $t = 0$. Immediately after switching at $t = 0^+$,

– $i(0^+) = i(0^-) = 2$ A

– $v(0^+) = v(0^-) = 6$ V



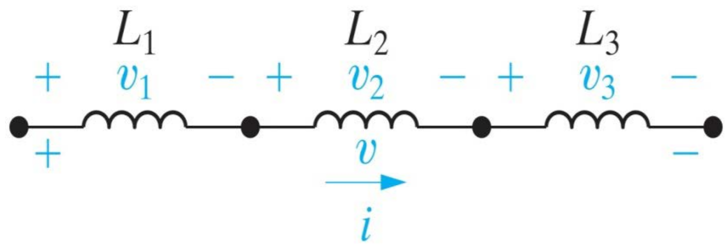
Series-parallel connections of inductance

Inductors in series share the same current,

$$v_1 = L_1 \frac{di}{dt} \quad ; \quad v_2 = L_2 \frac{di}{dt} \quad ; \quad v_3 = L_3 \frac{di}{dt}$$

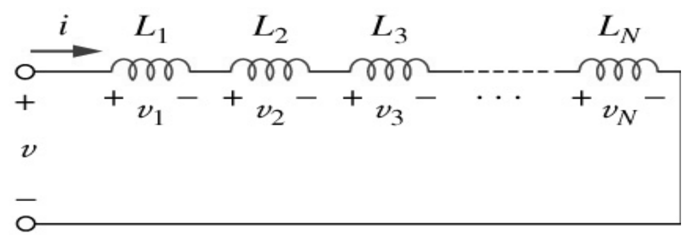
$$v = v_1 + v_2 + v_3 = \left(L_1 + L_2 + L_3\right)\frac{di}{dt}$$

$$L_{eq} = L_1 + L_2 + L_3$$

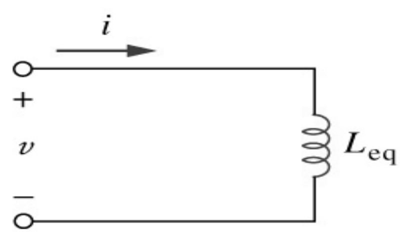


For *n* inductors in series,

$$\boxed{L_{eq} = L_1 + L_2 + \ldots + L_N}$$

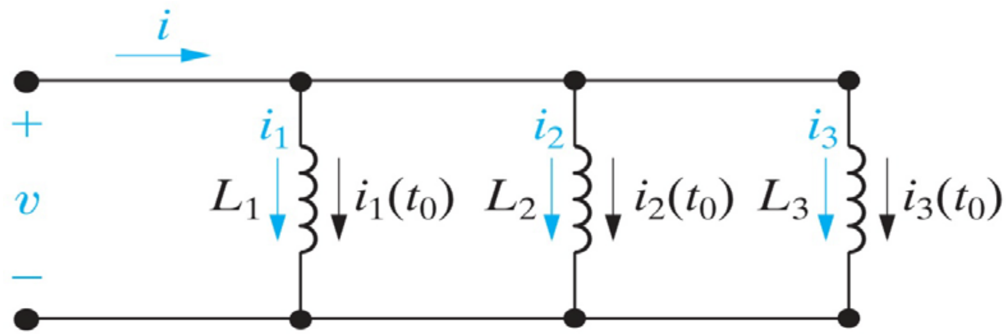


(a)



(b)

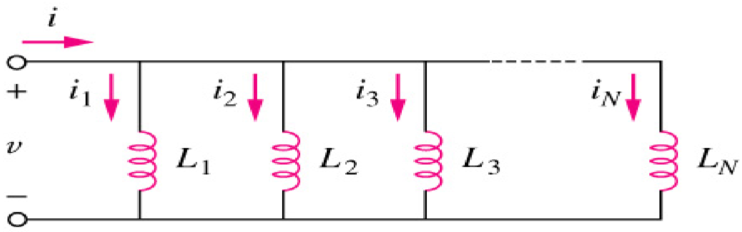
- Inductors in parallel have the same voltage,



$$i_1 = \frac{1}{L_1}\int_{t_0}^{t} v\, d\tau + i_1(t_0) \quad ; \quad i_2 = \frac{1}{L_2}\int_{t_0}^{t} v\, d\tau + i_2(t_0) \quad ; \quad i_3 = \frac{1}{L_3}\int_{t_0}^{t} v\, d\tau + i_3(t_0)$$

$$i = i_1 + i_2 + i_3 = \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)\int_{t_0}^{t} v\, d\tau + i_1(t_0) + i_2(t_0) + i_3(t_0),$$

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$



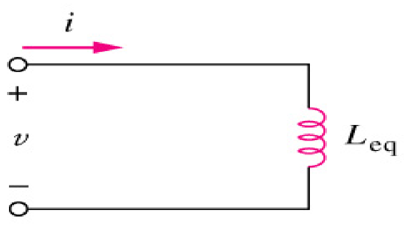
(a)

For *n* inductors in parallel,

$$\boxed{\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N}}$$

- Equivalent initial current,

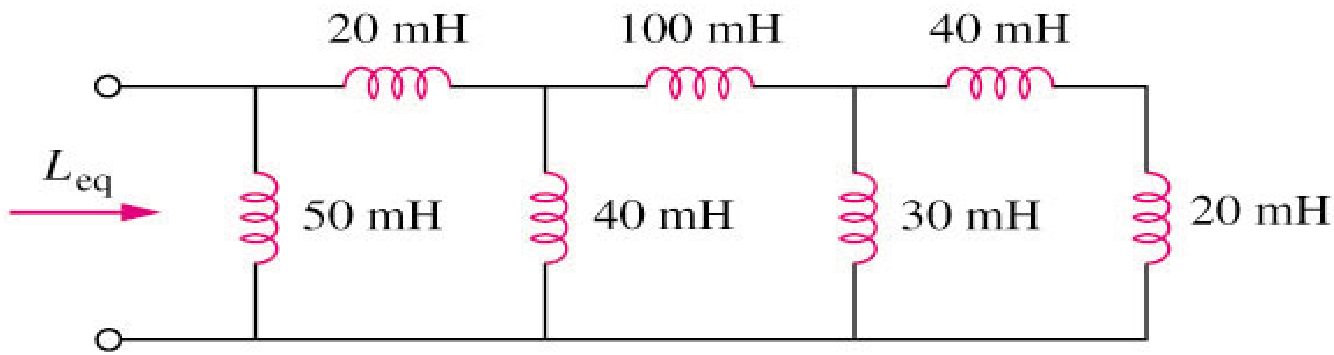
$$i(t_0) = i_1(t_0) + i_2(t_0) + i_3(t_0) + \cdots + i_n(t_0)$$



(b)

Example 12

Calculate the equivalent inductance for the inductive ladder network in the circuit shown below:



40mH in series with 20mH = 40 + 20 = 60mH

60mH in parallel with 30mH = 30 x 60/(90) = 20mH

20mH in series with 100mH = 120mH

120mH in parallel with 40mH = 40 x 120/(160) = 30mH

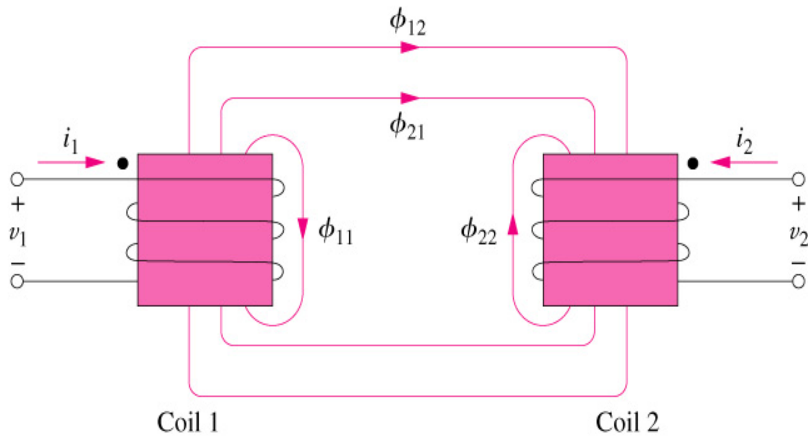
30mH in series with 20mH = 50mH

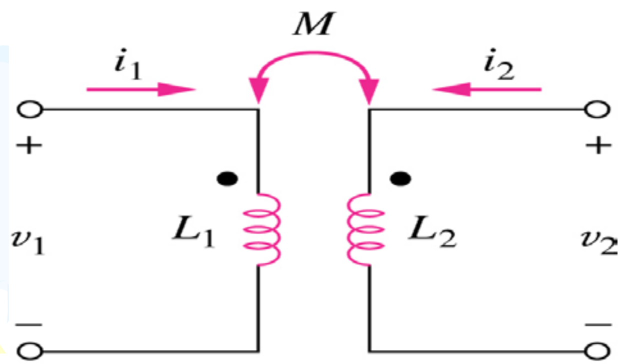
50mH in parallel with 50mH = 25mH

L_{eq} = **25mH**

Mutual Inductance

When two coils are linked by magnetic field, the voltage induced across one coil is related to the time-varying current in the other coil due to mutual inductance



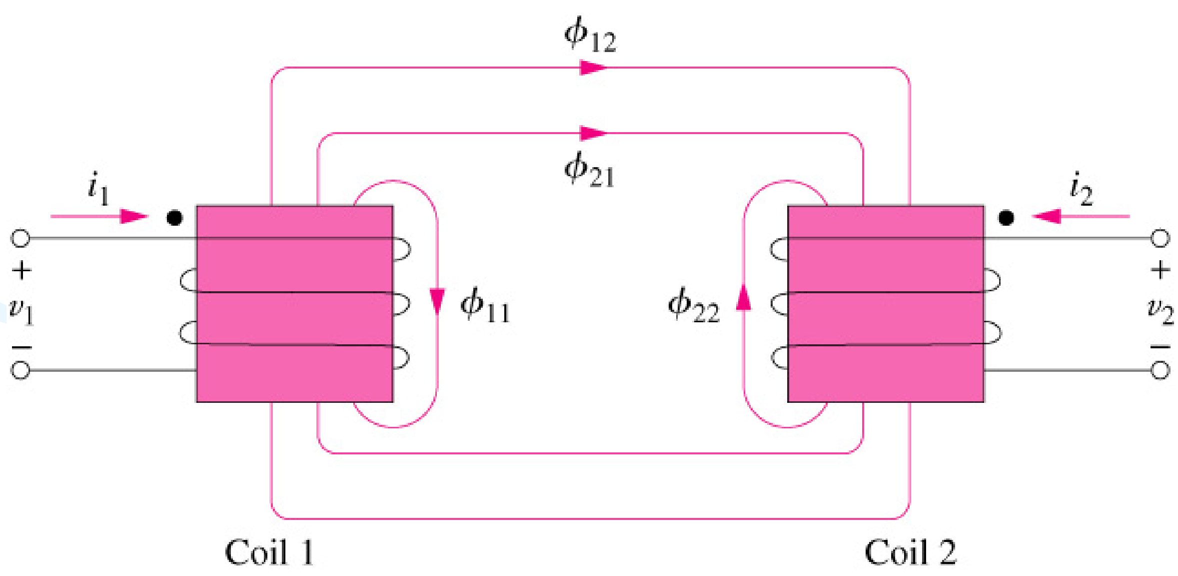


– M is the mutual inductance between the two circuits

– L_1 and L_2 are self-inductances of circuits 1 and 2 respectively

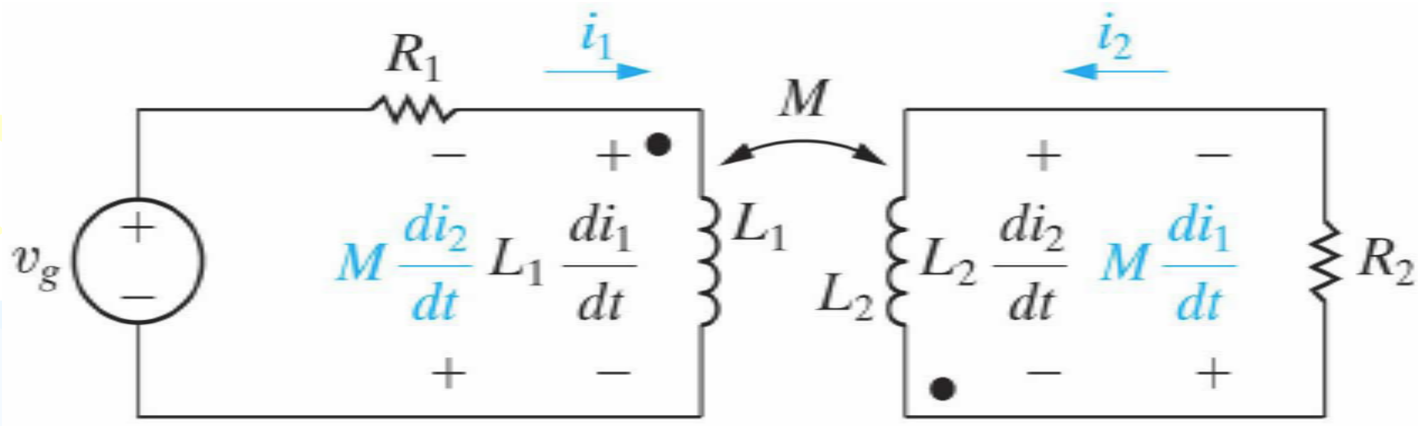
- The total induced voltage v_1 across circuit 1 comprises of two parts:
  – Self induced voltage $v_{11} = L_1 \dfrac{di_1}{dt}$ and mutually induced voltage $v_{12} = M \dfrac{di_2}{dt}$
- Similarly, induced voltage v_2 across circuit 2 also comprises of two parts:
  – Self induced voltage $v_{22} = L_2 \dfrac{di_2}{dt}$ and mutually induced voltage $v_{21} = M \dfrac{di_1}{dt}$

- How to determine the polarity of the induced voltages?
  - Depends on way the coils are wound and reference direction of the coil currents
  - Keep track of polarities by a method known as **dot convention**, where a dot is placed on one terminal of each winding
- If a current enters the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is positive at the dotted terminal of the second coil.



Example 13:

Write the mesh equations of the circuit below



Mesh-current equations for both coils are:

$$-v_g + i_1R_1 + v_1 = 0$$

$$-v_g + i_1R_1 + v_{11} - v_{12} = 0$$

$$-v_g + i_1R_1 + L_1\frac{di_1}{dt} - M\frac{di_2}{dt} = 0$$

$$i_2R_2 + v_2 = 0$$

$$i_2R_2 + v_{22} - v_{21} = 0$$

$$i_2R_2 + L_2\frac{di_2}{dt} - M\frac{di_1}{dt} = 0$$

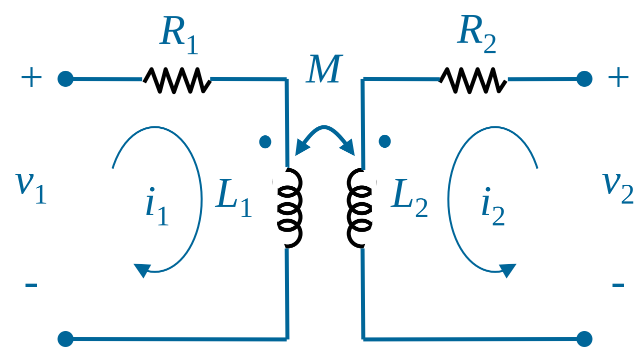
Example 14:

T-equivalent circuit for magnetically coupled coils

For coil 1,

$$v_1 = i_1 R_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

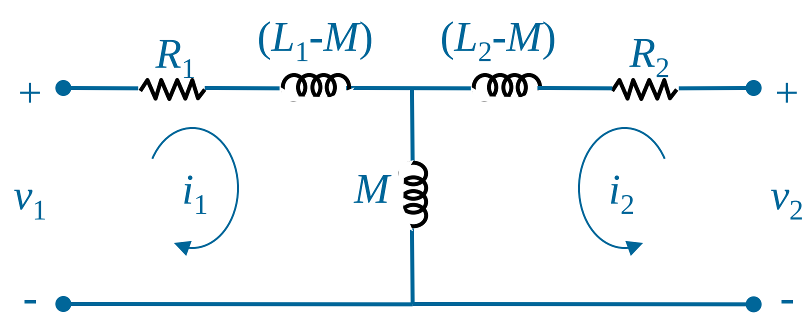
$$v_1 = i_1 R_1 + (L_1 - M)\frac{di_1}{dt} + M \frac{d}{dt}(i_1 + i_2)$$



For coil 2,

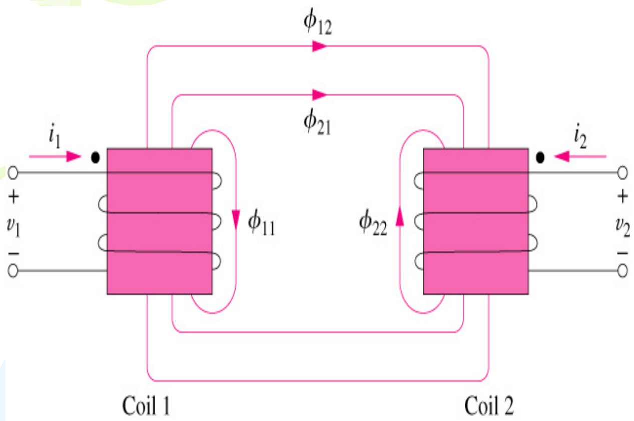
$$v_2 = i_2 R_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

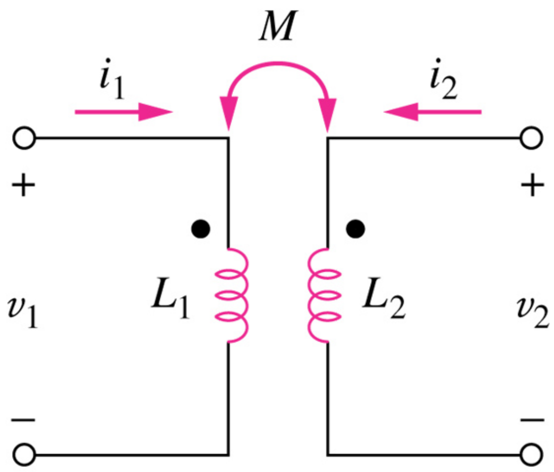
$$v_2 = i_2 R_2 + (L_2 - M)\frac{di_2}{dt} + M \frac{d}{dt}(i_1 + i_2)$$



The circuit is equivalent to that on the right

Coupling Coefficient





$$k = \frac{M}{\sqrt{L_1 L_2}}$$

$$M = k\sqrt{L_1 L_2}$$

- The coupling coefficient, *k*, is a measure of the magnetic coupling between two coils;
- *0≤k≤1*.

Example 15:

k = 0 when two coils have no common flux, i.e. $\phi_{12} = \phi_{21} = 0$

k = 1 when flux that links coil 1 also links coil 2, i.e. $\phi_{11} = \phi_{22} = 0$

Summary:

- Current, voltage and energy relationships for L and C

Circuit Element	Voltage	Current	Energy
Capacitor	$v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$	$i = C\frac{dv}{dt}$	$w_C(t) = \frac{1}{2}Cv^2(t)$
Inductors	$v = L\frac{di}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} v(t)\,dt + i(t_0)$	$w_L(t) = \frac{1}{2}Li^2(t)$

- DC steady-state operating condition

- If all the independent sources in a circuit are DC sources, such as batteries or current sources, for a sufficient long time, as all currents and voltages settle down to constant values, the circuit is said to be in a DC steady state.

  - At DC steady state:
    - Capacitors act like Open Circuits, $i_c = 0$
    - Inductors act like Short Circuits, $v_L = 0$

- Note, when a switch is opened or closed at $t = 0$,

  - $v_C(0^-) = v_C(0) = v_C(0^+)$

  - $i_L(0^-) = i_L(0) = i_L(0^+)$

Series and Parallel Capacitors

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}$$

$$C_{eq} = C_1 + C_2 + \ldots + C_N$$

Series and Parallel Inductors

$$L_{eq} = L_1 + L_2 + \ldots + L_N$$

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N}$$

Mutual Inductance

dot convention
T-equivalent circuit for magnetically coupled coils
coupling coefficient *k*

Exercises (Problems in Chapter 6 of the textbook)

6.4 Ans: 1.75 – 0.75 cos 4t V

6.9 Ans: 13.624 V, 70.66 W

6.16 Ans: $20\ \mu\text{F}$

6.46 Ans: 0V, 4A, 0J, 4J

6.56 Ans: $\frac{5}{8}\text{L}$

~~6.60, Ans: -30e^{−2t} V, (3e^{−2t} + 1) A~~, $-15e^{-2t}$ V, $(1.5e^{-2t} + 0.5)$ A

6.64 Ans: 120V, 0V

Also try problems 6.39 to 6.41 in Reference 1.