NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

EE3001 Engineering Electromagnetics

Session 7-1

Electric Flux Density and Gauss's Law

Learning Objectives

- Define electric flux density;
- Explain relative permittivity;
- Obtain the relation between total electric flux and charge;
- Explain and state Gauss's law; and
- Explain and state differential form of Gauss's law.

Electric Flux Density

- Consider $\vec{E}$ for a point charge Q in free space:

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a}_r$$

- Value of $\vec{E}$ also depends upon the permittivity ε_0
- Let us define electric flux density $\vec{D}$ as

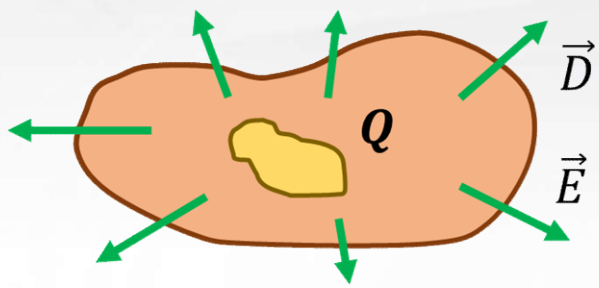
$$\boxed{\vec{D} = \varepsilon_0 \vec{E}}$$

- Then $\vec{D} = \frac{Q}{4\pi r^2}\vec{a}_r \quad \mathrm{C/m^2}$
- $\vec{D}$ is independent of permittivity. It is also called displacement density.

Nature of Electric Flux Density

- $\vec{D}$ begins on positive charge and terminates on negative charge and is in the same direction as that of $\vec{E}$

$$\boxed{\vec{D} = \varepsilon_0 \vec{E}}$$



- For a general medium:

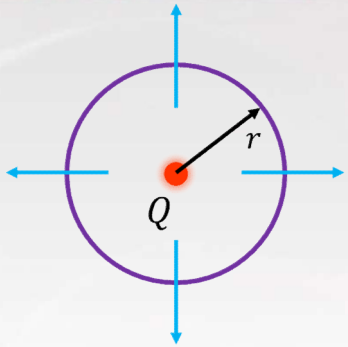
$$\boxed{\vec{D} = \varepsilon \vec{E} = \varepsilon_r \varepsilon_0 \vec{E}}$$

Relative Permittivity (ε_r)

- ε_r is the relative permittivity or dielectric constant; it is the ratio of the permittivity of the medium to that of free space.
- Insulators are called dielectric materials.
- The value of the dielectric constant depends upon the material's property and frequency, temperature etc.

Material	Dielectric constant at 3 GHz
Air	1.00059
Glass	2.5~6
Wood	1.82
Water	76.7~81

Relation Between Total Electric Flux and Charge



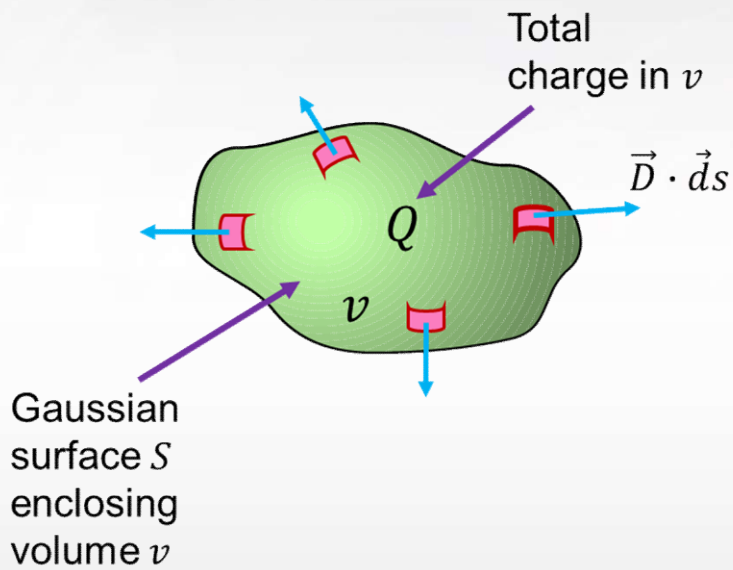
- Consider a point charge Q in free space located at the origin.
- The electric flux density at a point P at a distance r away from Q is $\vec{D} = \frac{Q}{4\pi r^2}\vec{a}_r$.
- The total electric flux Φ_e is the flux density D multiplied by the area of the spherical surface, $4\pi r^2$: $\Phi_e = Q$.
- The total electric flux flowing out of a spherical surface centered at the point charge is equal to the total charge inside the surface, Q.

Gauss's Law

- Gauss's Law: The total outward electric flux through any closed surface is equal to the total charge enclosed by the surface:

$$\oiint_S \vec{D} \cdot \vec{ds} = Q = \iiint_V \rho_v \, dv$$

One of the Maxwell's equations!



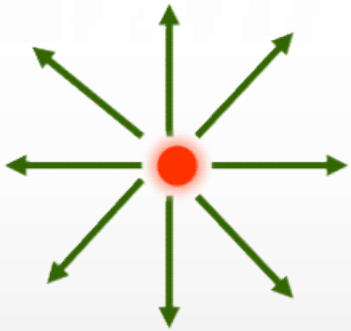
Differential Form of Gauss's Law

- From the integral form: $$\oiint_S \vec{D} \cdot \vec{ds} = Q = \iiint_V \rho_v \, dv$$

- Using the Divergence theorem, LHS equals: $$\oiint_S \vec{D} \cdot \vec{ds} = \iiint_V (\nabla \cdot \vec{D}) \, dv$$

which results in

$$\boxed{\nabla \cdot \vec{D} = \rho_v}$$



One of the Maxwell's equations!

Animation Display

- Hayt and Buck, Ch.3, Animation 1
- The total flux is plotted as a function of the number of unit positive charges in the sphere.

Quiz

Hayt and Buck, Ch 3, Q4:

The flux produced by a given set of fixed charges enclosed in a given closed region is :

A: Dependent on the surface shape of the region, but not the volume

B: Dependent on the total volume of the region, but not the surface shape

C: Dependent on the ratio of volume to surface area of the region

D: Not dependent on any of these as long as the charges are inside the region

Summary

- Electric flux density is defined as $\vec{D} = \varepsilon_0 \vec{E}$;
- The electric flux density for a general medium is defined as $\vec{D} = \varepsilon \vec{E} = \varepsilon_r \varepsilon_0 \vec{E}$; and
- Relative permittivity or dielectric constant is the ratio of the permittivity of the medium to that of free space.

Summary

- Gauss's Law states that the total outward electric flux through any closed surface is equal to the total charge enclosed by the surface. It is mathematically expressed as:

$$\oiint_S \vec{D} \cdot \vec{ds} = Q = \iiint_V \rho_v \, dv$$

- The differential form of Gauss's law is:

$$\nabla \cdot \vec{D} = \rho_v$$

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EE3001 Engineering Electromagnetics

Session 7-2

Introduction to Electric Potential

Learning Objectives

- Define scalar electric potential in terms of electrostatic field;
- Explain the physical meaning of electric potential; and
- Obtain and use the expression for electric potential due to a point charge.

Electric Potential

- An arbitrary electrostatic field can be written in the form:

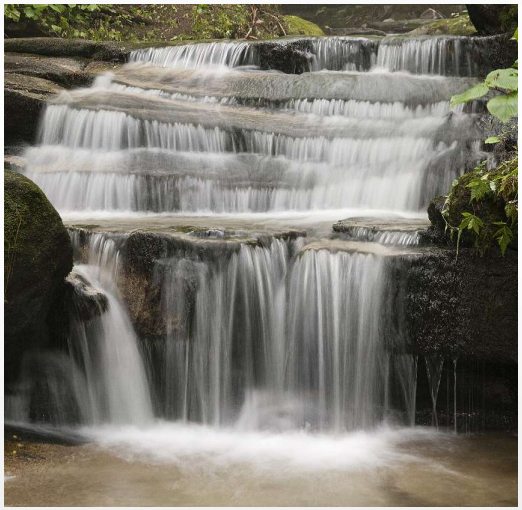
$$\boxed{\vec{E} = -\nabla V}$$

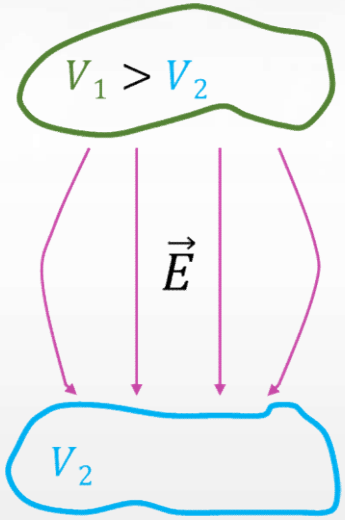
Here V, defined by $\vec{E} = -\nabla V$, is called scalar electric potential.

Why $\vec{E} = -\nabla V$?

- The negative sign means that the electric field is in the direction along which the electric potential is *decreasing most rapidly*.

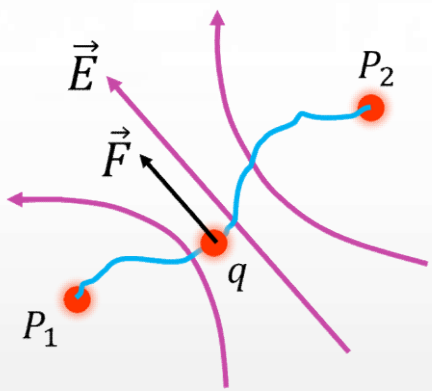
- This is similar to the situation that water flows from a higher place to a lower place.





Physical Meaning of Electric Potential

- Consider moving a point charge q from point P_1 to point P_2 in an electric field $\vec{E}$.
- An electric force $\vec{F} = q\vec{E}$ must be overcome to do this.
- It can be shown that (see Appendix) *work done per unit charge* to complete the movement of q from point P_1 to point P_2, is:

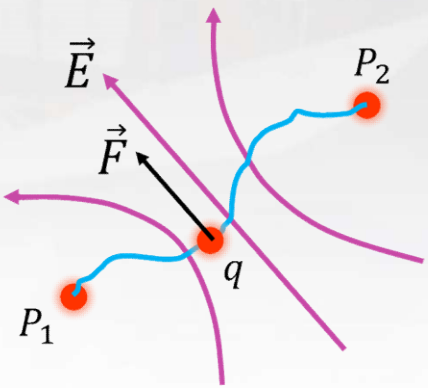


$$\frac{W}{q} = -\int_{P_1}^{P_2} \vec{E} \cdot \vec{dl} = V_2 - V_1$$

Appendix

Physical Meaning of Electric Potential

$$\frac{W}{q} = -\int_{P_1}^{P_2} \vec{E} \cdot \vec{d}l = V_2 - V_1$$



The electrostatic potential difference between P_2 and P_1 is the work done in moving a unit positive test charge from P_1 to P_2.

Quiz 1

Hayt and Buck, Ch 4, Q1:

An electron ($q_e = 1.602 \times 10^{-19}$ C) leaves the cathode of a cathode ray tube (CRT) and travels in a uniform electrostatic field towards the anode, which is at a potential $V_a = 500$ V with respect to the cathode. What is the work W done by the electrostatic field involved in moving the electron from the cathode to the anode?

A: $W = 5$ kJ

B: $W = 5$ J

C: $W = 8 \times 10^{-17}$ J

D: $W = 8 \times 10^{-19}$ J

The electrostatic potential difference between two points P_1 and P_2 is the work done in moving a unit positive test charge from P_1 to P_2.

Quiz 2

Hayt and Buck, Ch 4, Q2:

In Problem CH4,Q1, what is the electric field strength $E = |E|$ if the distance between the cathode and anode is 10 cm?

$$\boxed{V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \vec{E} \cdot \vec{dl}}$$

A: $\boldsymbol{E} = 5\ \mathrm{V/m}$

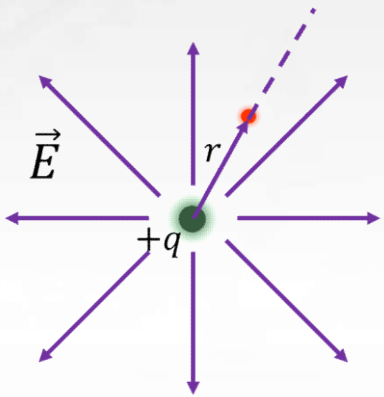
C: $\boldsymbol{E} = 500\ \mathrm{V/m}$

B: $\boldsymbol{E} = 50\ \mathrm{V/m}$

D: $\boldsymbol{E} = 5\ \mathrm{kV/m}$

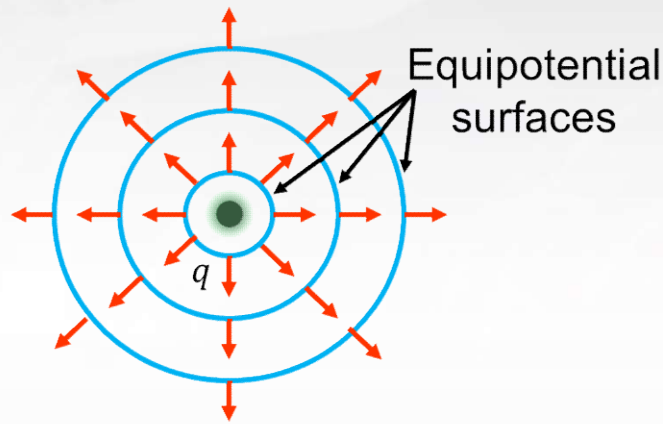
Electric Potential Due to a Point Charge

For a single point charge q located at the origin of the coordinate system, we have (since $V_\infty = 0$):



Note:

$$V \alpha \frac{1}{r}$$



$$\boxed{V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \vec{E} \cdot \vec{d}l}$$

$$V = V_r - V_\infty = -\int_{\infty}^{r} \left(\vec{a}_r \frac{q}{4\pi\varepsilon_0 r^2}\right) \cdot (\vec{a}_r dr) = \frac{q}{4\pi\varepsilon_0 r}\bigg|_{\infty}^{r} = \frac{q}{4\pi\varepsilon_0 r}$$

Quiz 3

Hayt and Buck, Ch 4, Q3:

The electrostatic potential due to a point charge Q_1 at a distance $R_1 = 1\ \mathrm{cm}$ away from it is $V_1 = 1\ \mathrm{V}$.
What is the potential V_2 for a charge $Q_2 = 4Q_1$ at a distance $R_2 = 2\ \mathrm{cm}$ from it?

$$V = \frac{q}{4\pi\varepsilon_0 r}$$

A: $\boldsymbol{V_2} = 0.5\ \mathrm{V}$

C: $\boldsymbol{V_2} = 4\ \mathrm{V}$

B: $\boldsymbol{V_2} = 1\ \mathrm{V}$

D: $\boldsymbol{V_2} = 2\ \mathrm{V}$

Summary

- The scalar electric potential is related to electrostatic field as $\vec{E} = -\nabla V$.

- The physical meaning of electric potential is that the electrostatic potential difference between P_2 and P_1 is the work done in moving a unit positive test charge from P_1 to P_2. It is defined as:

$$\frac{W}{q} = -\int_{P_1}^{P_2} \vec{E} \cdot \vec{d}l = V_2 - V_1$$

- The electric potential due to a point charge is:

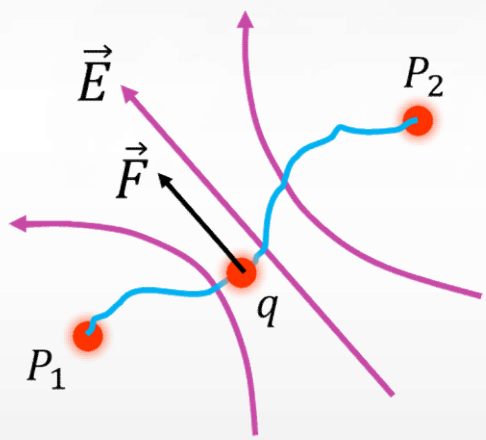
$$V = \frac{q}{4\pi\varepsilon_0 r}$$

Physical Meaning of Electric Potential

- Consider moving a point charge q from point P_1 to point P_2 in an electric field $\vec{E}$.
- An electric force $\vec{F} = q\vec{E}$ must be overcome to do this.
- Work done to complete the movement of q from point P_1 to point P_2, is:

$$W = -\int_{P_1}^{P_2} \vec{F} \cdot \vec{dl} = -q \int_{P_1}^{P_2} \vec{E} \cdot \vec{dl}$$

$$\frac{W}{q} = -\int_{P_1}^{P_2} \vec{E} \cdot \vec{dl}$$



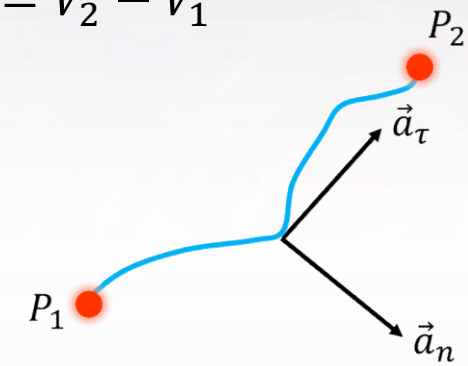
Next

Physical Meaning of Electric Potential (cont.)

- Since $\vec{E} = -\nabla V$, $\vec{dl} = \vec{a}_\tau dl$, we obtain

$$\frac{W}{q} = \int_{P_1}^{P_2} (\nabla V) \cdot \vec{a}_\tau dl = \int_{P_1}^{P_2} \frac{\partial V}{\partial l} dl = \int_{P_1}^{P_2} dV = V_2 - V_1$$

The electrostatic potential difference between two points P_1 and P_2 is the work done in moving a unit positive test charge from P_1 to P_2.



$$\nabla V = \frac{\partial V}{\partial x}\vec{a}_x + \frac{\partial V}{\partial y}\vec{a}_y + \frac{\partial V}{\partial z}\vec{a}_z = \frac{\partial V}{\partial l}\vec{a}_\tau + \frac{\partial V}{\partial n}\vec{a}_n + \frac{\partial V}{\partial z}\vec{a}_z$$

Next

Physical Meaning of Electric Potential (cont.)

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- An **equipotential surface** is defined as a surface on which $V(x, y, z) = Const$.

- From the properties of ∇V, and using $\vec{E} = -\nabla V$, the electric field is perpendicular to the surface $V(x, y, z) = Const$.

- $\vec{E} = -\nabla V$ produces ambiguous solutions for V within an arbitrary constant. This means that if $V_1(x, y, z)$ is a solution, $V_2(x, y, z) = V_1(x, y, z) + C$ is also a solution (since $\nabla C = 0$). For convenience, the reference point is usually taken to be at infinity.

- This is to say that the potential at infinity is usually taken to be zero, $V_\infty = 0$.

Back

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EE3001 Engineering Electromagnetics

Session 7-3

Nature of Electric Field and Electric Potential due to Charge Distribution

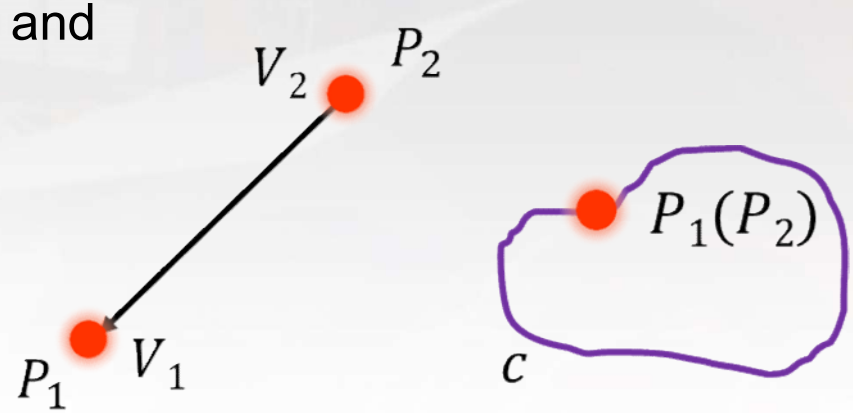
Learning Objectives

- Explain the nature of electrostatic field;
- Obtain the electric potential due to line, surface and volume charge distributions; and
- Apply the source and field point concept to determine electric potential due to a given line charge distribution.

Nature of Electric Field

The potential difference between two points P_1 and P_2 can be calculated as

$$\boxed{V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \vec{E} \cdot \vec{dl} = \int_{P_2}^{P_1} \vec{E} \cdot \vec{dl}}$$



- The above integral is independent of the path taken.
- If $P_1 = P_2$, then $V_{21} = V_2 - V_1 = 0$.
- For any closed loop C, we have

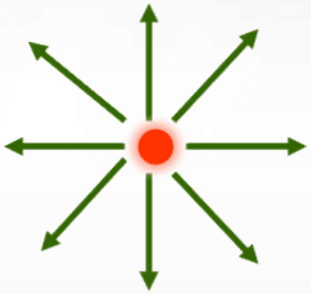
$$\oint_C \vec{E} \cdot \vec{dl} = 0$$

Nature of Electric Field

$$\oint_C \vec{E} \cdot \vec{dl} = 0 \quad \textit{But} \quad \oint_C \vec{E} \cdot \vec{dl} = \iint_S \nabla \times \vec{E} \cdot \vec{ds}$$

- Hence for an arbitrary electro*static* field, we have

$$\boxed{\nabla \times \vec{E} = 0}$$



- Static electric field (i.e., when there is no time variation) is irrotational!

Appendix

Relation with Kirchhoff's Voltage Law

- Kirchhoff's Voltage Law (KVL): Sum of the potential differences around any closed circuit is zero.
- $\oint_C \vec{E} \cdot d\vec{l} = 0$ is just a more general form of KVL!

Electric Potential Due to a Charge Distribution

The electric potential due to a continuous distribution of charge can be calculated as follows:

- **Line charge distribution**

$$V = \frac{1}{4\pi\varepsilon_0}\int_c \frac{\rho_l}{R}\,dl$$

- **Surface charge distribution**

$$V = \frac{1}{4\pi\varepsilon_0}\iint_S \frac{\rho_s}{R}\,ds$$

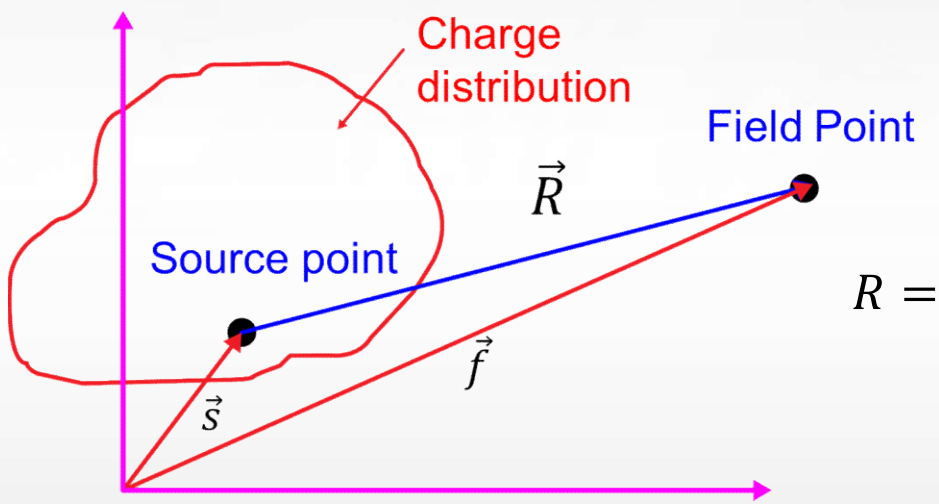
- **Volume charge distribution**

$$V = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho_v}{R}\,dv$$

Source Point and Field Point

- **Source point** ($\vec{S}$) is the point where the small element of electric charge (source) is located.

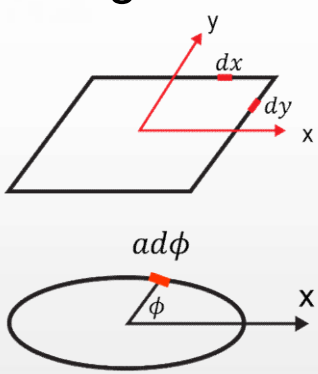
- **Field point** ($\vec{f}$) is the point where the field is to be determined .



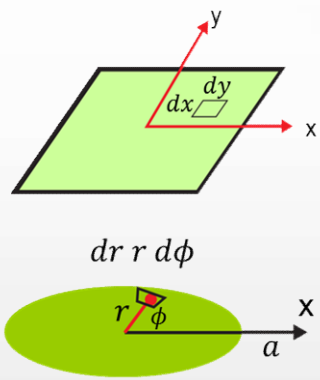
$$R = |\vec{f} - \vec{s}|$$

How to Choose the Source Point

- **Source point** is the point where the differential element of charge (source) is located.
- **Source point** should always be on or within the charge distribution.
- **Source element** must include coordinates that are varying so that it can represent any point on the entire charge distribution.



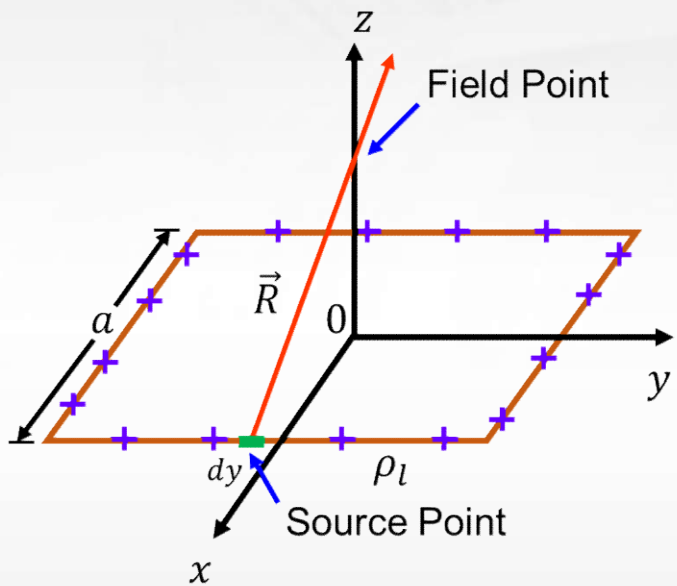
Charged loops



Charged plates

Example: A Charged Square Loop

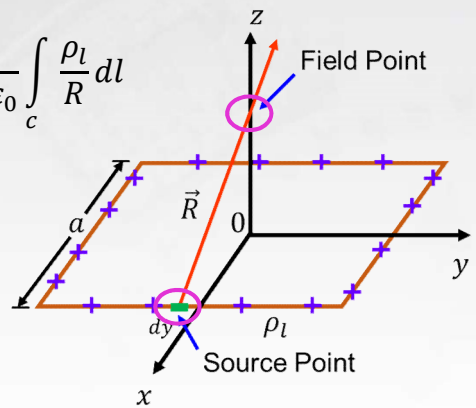
A square loop of side length a, positioned in the $x - y$ plane and centered at the origin, is uniformly charged with a line charge density ρ_l.



Determine the electric potential V at a point $(0,0,z)$.

Solution

$$V = \frac{1}{4\pi\varepsilon_0}\int_c \frac{\rho_l}{R}\,dl$$



Choosing line section at $x = \frac{a}{2}$

Field point: $(0,0,z)$

Source point: $(\frac{a}{2}, y, 0)$

Source element: $dl = dy$

Then $R = \sqrt{\left(\frac{a}{2}\right)^2 + y^2 + z^2}$

The electric potential at $(0,0,z)$ due to this line section is

$$V_1 = \frac{1}{4\pi\varepsilon_0}\int_{one\ side}\frac{\rho_l}{R}\,dl = \frac{\rho_l}{4\pi\varepsilon_0}\int_{-\frac{a}{2}}^{\frac{a}{2}}\frac{dy}{\sqrt{y^2 + \left(\frac{a}{2}\right)^2 + z^2}}$$

$$\int \frac{dx}{\sqrt{x^2+b}} = \ln(x + \sqrt{x^2+b})$$

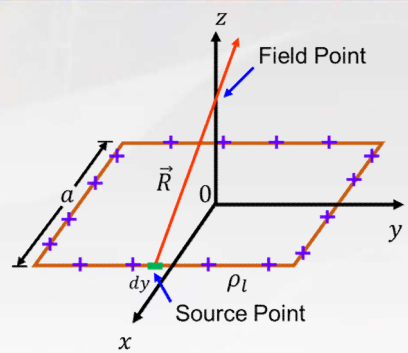
$$= \frac{\rho_l}{4\pi\varepsilon_0}\ln\left(y + \sqrt{y^2 + \left(\frac{a}{2}\right)^2 + z^2}\right)\Bigg|_{-\frac{a}{2}}^{\frac{a}{2}}$$

$$= \frac{\rho_l}{4\pi\varepsilon_0}\left[\ln\left(\frac{a}{2} + \sqrt{\frac{a^2}{2} + z^2}\right) - \ln\left(-\frac{a}{2} + \sqrt{\frac{a^2}{2} + z^2}\right)\right]\text{V}$$

Solution

For the entire loop,

$$V = \frac{\rho_l}{\pi\varepsilon_0}\left[\ln\left(\frac{a}{2}+\sqrt{\frac{a^2}{2}+z^2}\right)-\ln\left(-\frac{a}{2}+\sqrt{\frac{a^2}{2}+z^2}\right)\right]V$$



The electric field at $(0,0,z)$ is then

$$\vec{E} = -\nabla V \rightarrow -\vec{a}_z\frac{\partial V}{\partial z} = -\vec{a}_z\frac{\rho_l}{\pi\varepsilon_0}\frac{d}{dz}\left[\ln\left(\frac{a}{2}+\sqrt{\frac{a^2}{2}+z^2}\right)-\ln\left(-\frac{a}{2}+\sqrt{\frac{a^2}{2}+z^2}\right)\right]$$

$$= -\vec{a}_z\frac{\rho_l}{\pi\varepsilon_0}\frac{\frac{1}{2}(2z)}{\sqrt{\frac{a^2}{2}+z^2}}\left[\frac{1}{\frac{a}{2}+\sqrt{\frac{a^2}{2}+z^2}}-\frac{1}{-\frac{a}{2}+\sqrt{\frac{a^2}{2}+z^2}}\right]$$

$$= -\vec{a}_z\frac{\rho_l}{\pi\varepsilon_0}\frac{z}{\sqrt{\frac{a^2}{2}+z^2}}\frac{-a}{\frac{a^2}{4}+z^2} = \frac{\rho_l}{\pi\varepsilon_0}\frac{z}{\sqrt{\frac{a^2}{2}+z^2}}\frac{a}{\frac{a^2}{4}+z^2}\vec{a}_z \quad \text{V/m}$$

Summary

- For any closed loop the line integral of E equals to 0 and the electrostatic field is irrotational; and

- The calculation of electric potential is easier than that for electric field intensity since potential is a scalar quantity. If required, one can calculate the electric field as the negative gradient of the potential function.

Summary

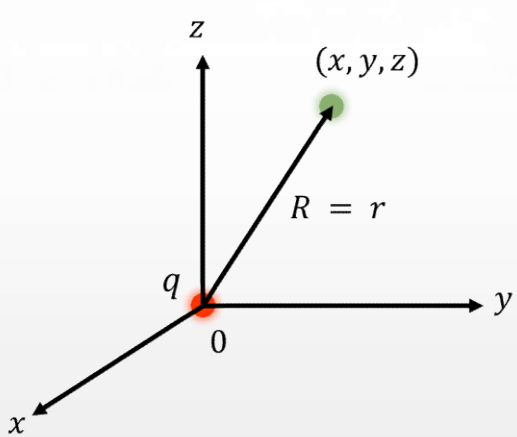
- The electric potential due to a continuous distribution of charge can be calculated as follows:
  - Line charge distribution $$V = \frac{1}{4\pi\varepsilon_0}\int_c \frac{\rho_l}{R}\,dl$$
  - Surface charge distribution $$V = \frac{1}{4\pi\varepsilon_0}\iint_S \frac{\rho_s}{R}\,ds$$
  - Volume charge distribution $$V = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho_v}{R}\,dv$$

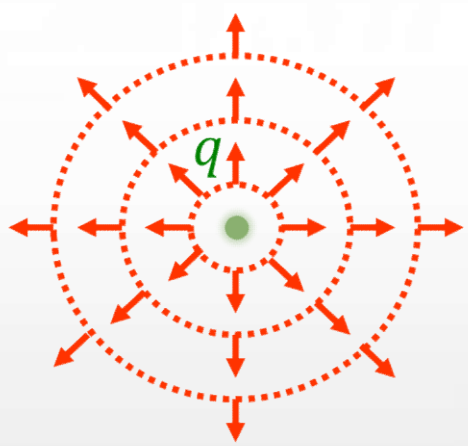
Appendix: Nature of Electric Field

Electric Field Due to a Point Charge

- The electric field due to a point charge q located at the origin is

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 R^3}\vec{R} = \frac{q}{4\pi\varepsilon_0}\frac{x\vec{a}_x + y\vec{a}_y + z\vec{a}_z}{(x^2+y^2+z^2)^{3/2}}$$





Next

Appendix: Nature of Electric Field

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 R^3}\vec{R} = \frac{q}{4\pi\varepsilon_0}\frac{x\vec{a}_x + y\vec{a}_y + z\vec{a}_z}{(x^2+y^2+z^2)^{3/2}}$$

- The curl of this electric field is: $\nabla \times \vec{E} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$

$$= \frac{q}{4\pi\varepsilon_0}\begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{(x^2+y^2+z^2)^{3/2}} & \frac{y}{(x^2+y^2+z^2)^{3/2}} & \frac{z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix}$$

$$= 0$$

Next

Appendix: Nature of Electric Field

Thus

$$\nabla \times \vec{E} = 0$$

Static electric field due to a point charge (i.e., when there is no time variation) is **irrotational**!

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NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

EE3001 Engineering Electromagnetics

Session 8-1

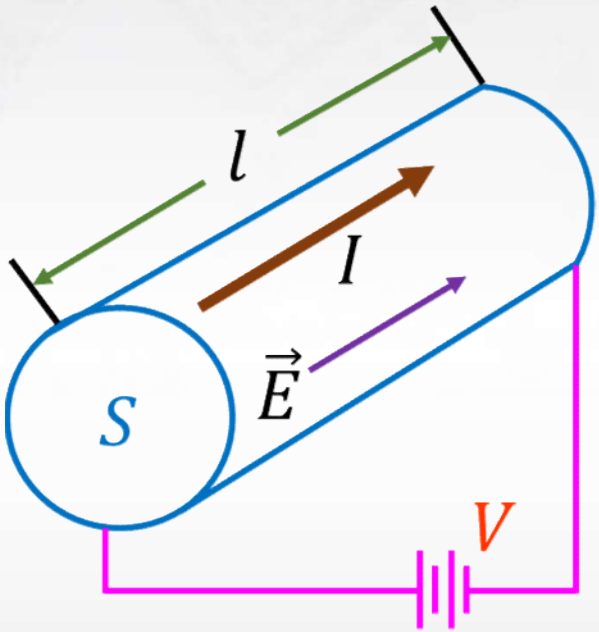
Review of Ohm's Law and Introduction to Poisson's and Laplace's Equations

Learning Objectives

- State the differential form of Ohm's law;
- State Poisson's equation;
- Explain the Laplacian operator; and
- State Laplace's equation.

Ohm's Law: A Revisit

- Consider a cylindrical conductor of length l and cross-section S.



$V = IR$ (Ohm's Law)

$$V = \left| \int \vec{E} \cdot \vec{d}l \right| = El$$

$I = JS$

J is the current density

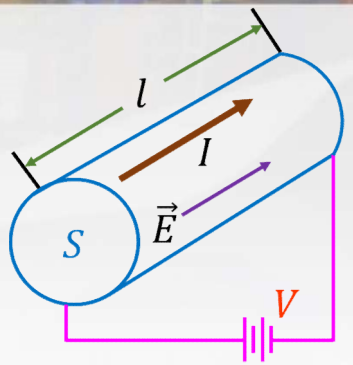
Differential Form of Ohm's Law

- Using the formula for calculating resistance,

$$R = \rho \frac{l}{S} = \frac{l}{\sigma S}$$

$\boldsymbol{\rho}$ is the resistivity

$\boldsymbol{\sigma}$ is the conductivity



we obtain

$$\mathrm{V} = El = IR = JSR = JS\frac{l}{\sigma S} = J\frac{l}{\sigma}$$

which results in $J = \sigma E$

or $\boxed{\vec{J} = \sigma \vec{E}}$

This is the differential form of Ohm's law.

Quiz 1

Hayt and Buck, Ch 5, Q5:

The resistance of a $1609\ \mathrm{m}$ (1 mile) length of copper wire with diameter 0.0254 inches and conductivity of $5.8 \times 10^7\ \mathrm{S/m}$ is

A: $21.21\ \Omega$

B: $10.60\ \Omega$

C: $42.43\ \Omega$

D: $84.86\ \Omega$

$$R = \rho \frac{l}{S} = \frac{l}{\sigma S}$$

$$S = \pi \frac{d^2}{4}$$

Poisson's Equation

- From the two equations for static electric field,

$$\nabla \times \vec{E} = 0$$

$$\nabla \cdot \vec{D} = \rho_v$$

and using

$$\vec{D} = \varepsilon \vec{E} = -\varepsilon \nabla V$$

we obtain:

$$\nabla \cdot \vec{D} = \nabla \cdot \left(\varepsilon \vec{E}\right) = -\varepsilon \nabla \cdot (\nabla V) = -\varepsilon \nabla^2 V = \rho_v$$

or

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}}$$

Poisson's equation

Laplacian Operator ∇^2

- Definition: The Laplacian operator $\nabla^2 = \nabla \cdot \nabla$
- The Laplacian operator ∇^2 has different expansions in different systems of coordinates.
  - In Cartesian coordinates,

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$$

  - Poisson's equation in Cartesian coordinates is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}$$

Laplace's Equation

- A special case of Poisson's equation, with $\rho_v = 0$

$$\boxed{\nabla^2 V = 0}$$

Laplace's equation

Quiz 2

Hayt and Buck, Ch 7, Q1:

Poisson's and Laplace's equations are different in terms of

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

$$\nabla^2 V = 0$$

A: Definition of potential

B: Presence of non-zero charge

C: Boundary conditions on potential

D: No difference

Quiz 3

Hayt and Buck, Ch 7, Q2:

For arbitrary constants A, B, and C, if the function V satisfies $\frac{\partial^2 V}{\partial x^2} = 0$, then V must be of the form:

A: $Ax + B$

B: Ax^3

C: $Ax^2 + Bx + C$

D: None of the above

Summary

- Differential form of Ohm's law is stated as $J = \sigma E$ or $\vec{J} = \sigma \vec{E}$;
- Poisson's equation is stated as $\nabla^2 V = -\frac{\rho_v}{\varepsilon}$;
- Laplacian operator in Cartesian coordinates is expressed as:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \ ;$$

- Laplace's equation is stated as $\nabla^2 V = 0$.

NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

EE3001 Engineering Electromagnetics

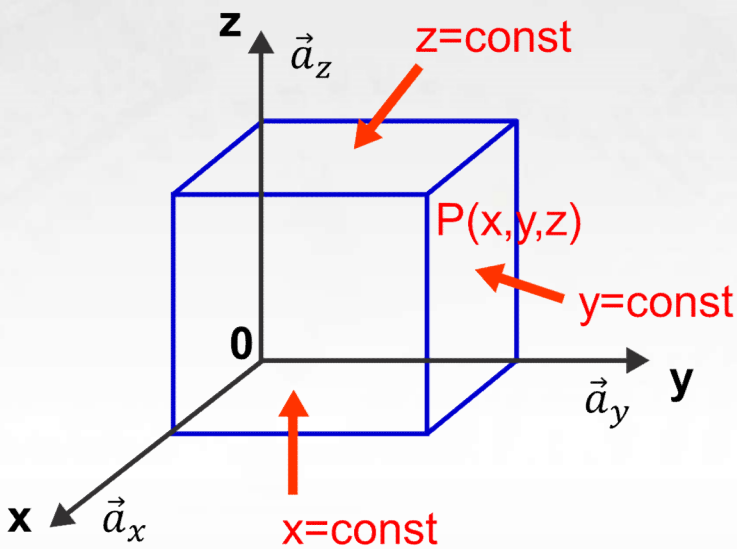
Session 8-2

Cylindrical Coordinates

Learning Objectives

- Explain the coordinates and the unit vectors in the Cylindrical coordinate system; and
- Obtain relations between the coordinates and the unit vectors in the Cartesian coordinates and the Cylindrical coordinates.

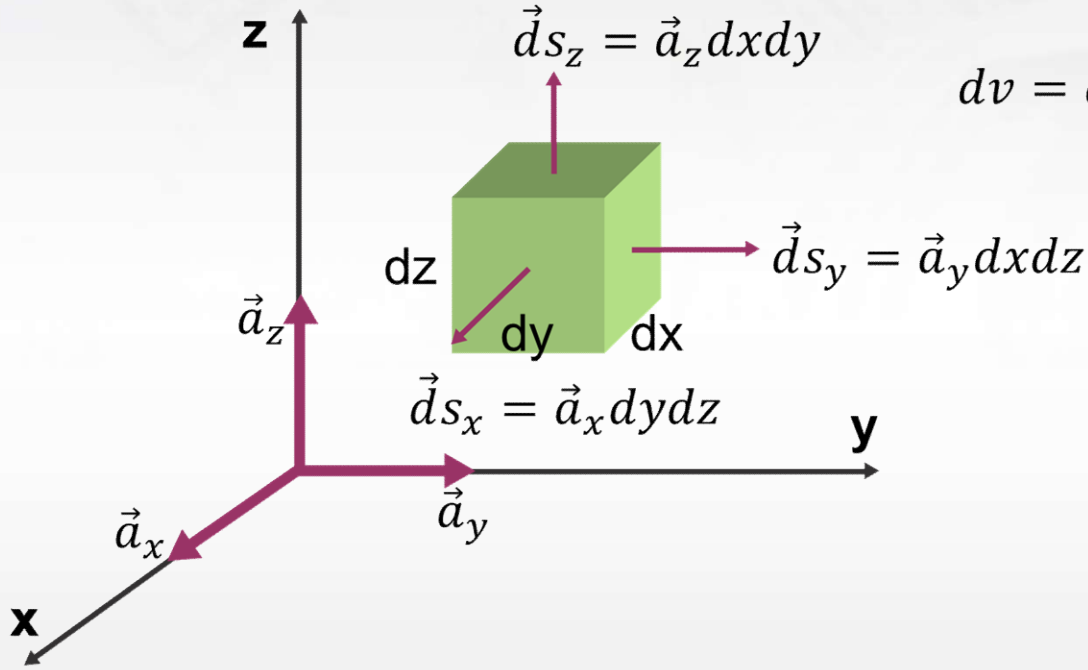
Cartesian Coordinate System



- Three coordinates (x, y, z)
- Three unit vectors $\vec{a}_x$ $\vec{a}_y$ $\vec{a}_z$
- Same as $\hat{i}$ $\hat{j}$ $\hat{k}$

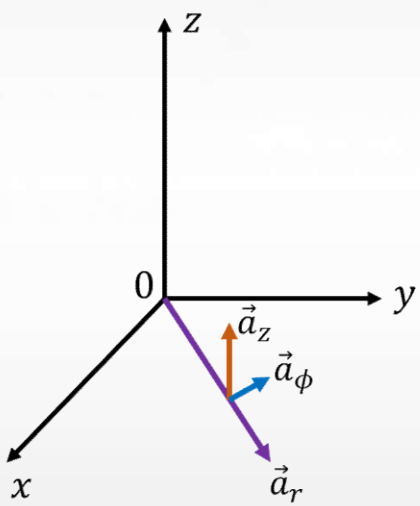
Cartesian Coordinates: Differential Elements

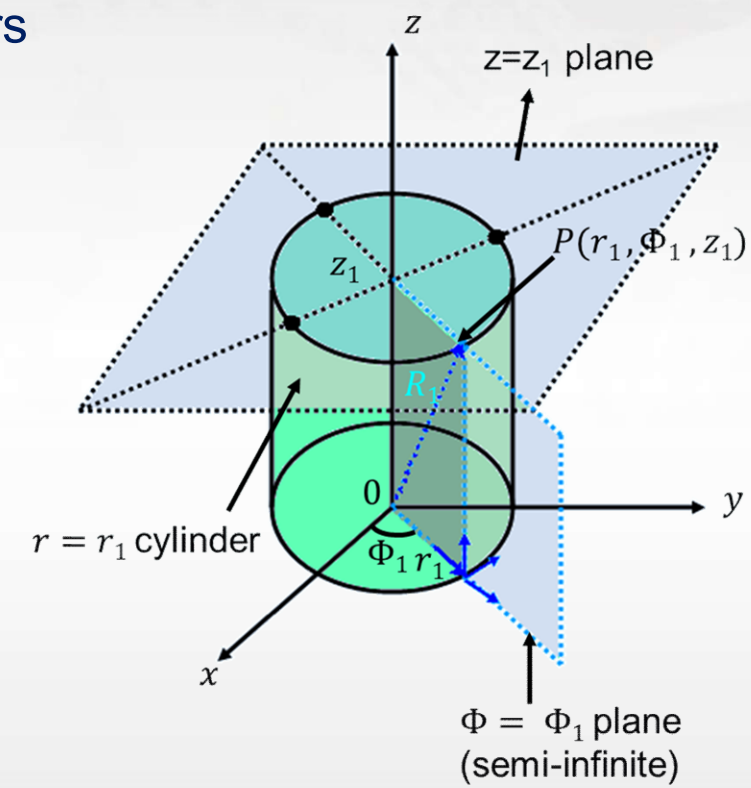
- Differential elements of length : dx, dy, dz



Cylindrical Coordinate System

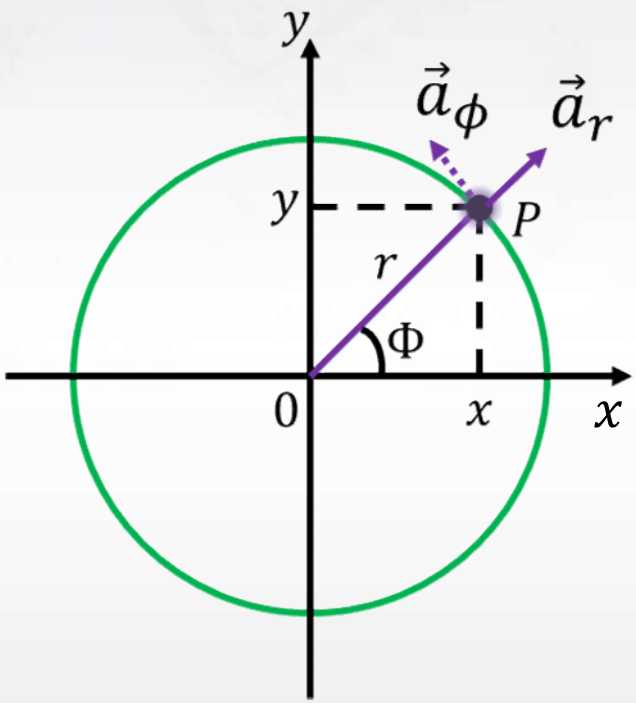
- Three coordinates (r, ϕ, z) or (ρ, ϕ, z)
- Three unit vectors





Relations Between Cartesian and Cylindrical Coordinates

$P(x, y)$ or $P(r, \phi)$



$$r = \sqrt{x^2 + y^2} \qquad \tan\phi = \frac{y}{x}$$

$$x = r\cos\phi \qquad y = r\sin\phi$$

$$\vec{a}_r = \cos\phi\,\vec{a}_x + \sin\phi\,\vec{a}_y$$

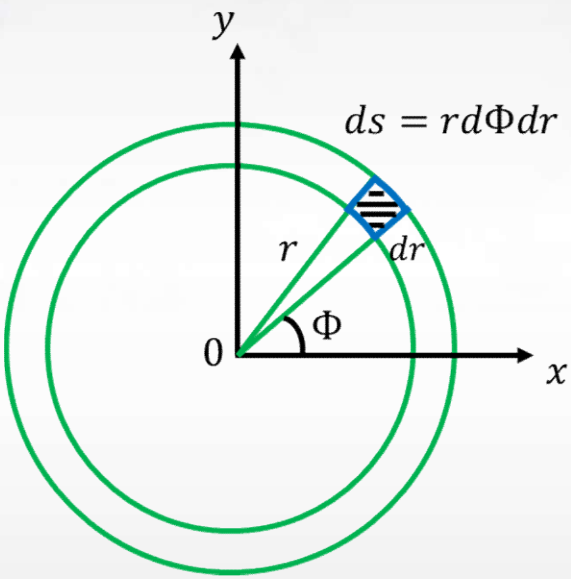
$$\vec{a}_\phi = -\sin\phi\,\vec{a}_x + \cos\phi\,\vec{a}_y$$

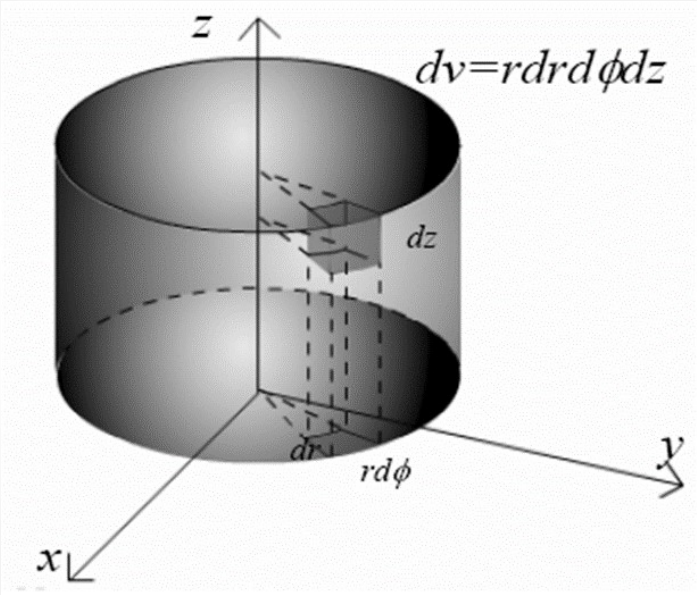
Appendix

Cylindrical Coordinates: Differential Elements

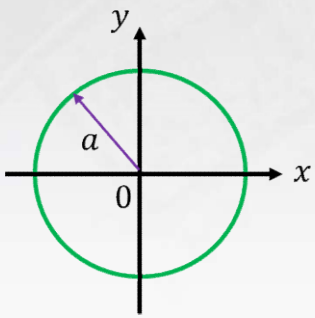
- Three coordinates (r, ϕ, z)

  Differential elements of length: dr, $rd\phi$, dz

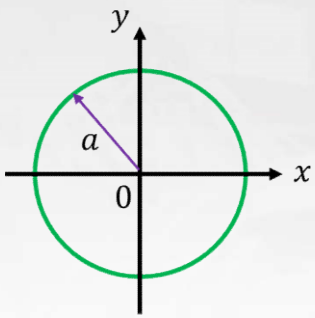




Importance of Suitable Coordinate System: Area of Circle



$$x^2 + y^2 = a^2$$



Cartesian system

$$area = \iint dxdy$$

$$= \int_{-a}^{+a} dx \int_{-y}^{+y} dy \qquad y = \pm\sqrt{a^2 - x^2}$$

$$= \int_{-a}^{+a} dx \times 2y = 2\int_{-a}^{+a} \sqrt{a^2 - x^2}dx$$

Using the substitution

$x = asin\theta; \qquad dx = a\cos\theta d\theta$

$area = \pi a^2$

Cylindrical system

$$area = \iint rd\phi dr$$

$$= \int_0^{2\pi} d\phi \int_0^a rdr$$

$$= 2\pi \times \frac{a^2}{2} = \pi a^2$$

Much simpler!

Summary

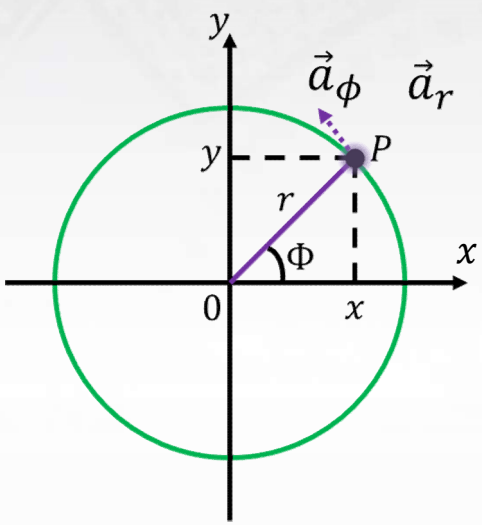
- The coordinates in the cylindrical coordinate system are r, ϕ,z and the unit vectors are $\vec{a}_r$ $\vec{a}_\varphi$ $\vec{a}_z$.

- The coordinates and the unit vectors in the Cartesian coordinates and the Cylindrical coordinates are related in a simple manner.

- A problem that has circular symmetry is easier to solve in the cylindrical coordinate system.

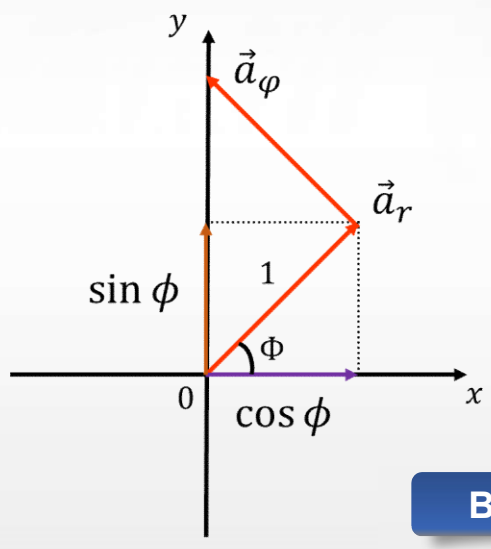
Relations Between Cartesian and Cylindrical Unit Vectors

$P(x, y)$ or $P(r, \phi)$



$$r = \sqrt{x^2 + y^2} \qquad \tan\phi = \frac{y}{x}$$

$$x = r\cos\phi \qquad y = r\sin\phi$$



$$\vec{a}_r = \cos\phi\,\vec{a}_x + \sin\phi\,\vec{a}_y$$

$$\vec{a}_\phi = -\sin\phi\,\vec{a}_x + \cos\phi\,\vec{a}_y$$

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NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

EE3001 Engineering Electromagnetics

Session 8-3

Applications of Magnetic Field

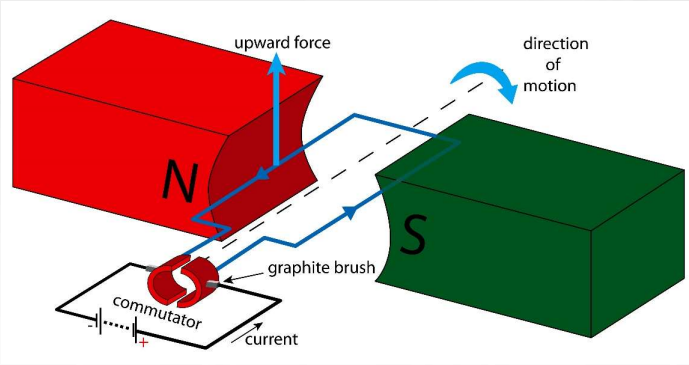
Learning Objectives

- Explain the basic principle of operation of some example applications of magnetic field. These examples are electric motors, multimeters and speakers;

- Explain the principle of operation of a magnetically levitated and propulsed train, as well as its advantages and disadvantages.

Applications of Magnetic Field

- Electric motor



- Multimeter

- Speaker

How a Speaker Works

- http://www.google.com.sg/url?sa=t&rct=j&q=how%20speaker%20works%20animation&source=web&cd=2&cad=rja&ved=0CDQQtwIwAQ&url=http%3A%2F%2Fwww.youtube.com%2Fwatch%3Fv%3DLKuHuyaRiHg&ei=puT0UauHCMmOrQfIEw&usg=AFQjCNFLfFSAVXrKnZe1wI37CHy7gbosSg
  (no sound, 1:36 minutes, Uploaded by OcSTeam)

- http://www.google.com.sg/url?sa=t&rct=j&q=how%20speaker%20works%20animation&source=web&cd=1&cad=rja&ved=0CC4QtwIwAA&url=http%3A%2F%2Fwww.youtube.com%2Fwatch%3Fv%3DmFiZZyV-YCo&ei=puT0UauHCMmOrQfIEw&usg=AFQjCNEmir0T3hHX1GqQocnBsskZbt8RUA
  (3:04 minutes, Uploaded by lunchboxinthahouse)

Magnetic Levitation[1] (MagLev) - 1

- Levitated and propulsed by interaction between electromagnets in the train and current in the guidance rails.
- No moving parts
- Air cushion
- High cost
- More energy-efficient?

Transrapid: Shanghai Maglev Train
Top speed: 501 km/h
Typical speed: 431 km/h
In operation since 2004

1. http://magnetbahnforum.de/index.php?en_aboutimb

Magnetic Levitation (MagLev) - 2

- http://www.youtube.com/watch?gl=SG&hl=en-GB&v=GHtAwQXVsuk (2:37 minutes)
- Uploaded by WOWLab

Summary

- Some example applications of magnetic field and understood their basic principle of operation. These examples are electric motors, multimeters and speakers.

- The principle of operation of a magnetically levitated and propulsed train, as well as its advantages and disadvantages.

NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

EE3001 Engineering Electromagnetics

Session 9-1

Current and Principle of Conservation of Charge

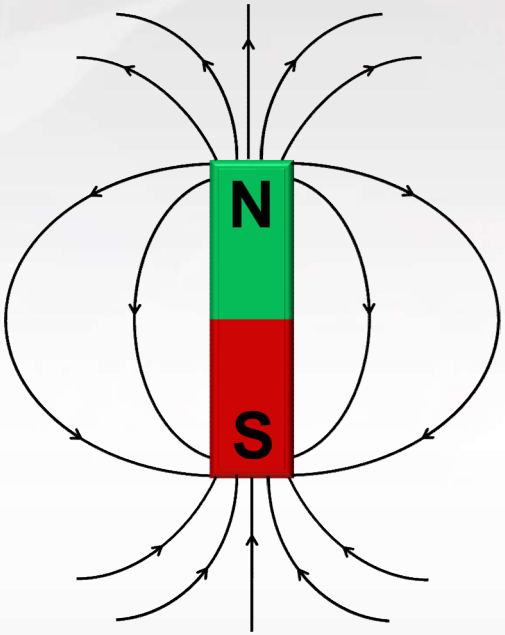
Learning Objectives

- Define current and calculate line current and volume current; and
- State and explain the principle of conservation of charge.

Sources of Static (Steady) Magnetic Fields

- **Permanent magnet:** The Earth itself acts as a huge magnet.

- **Direct Current:** Movement of free charge constitutes electric current; direct current produces a steady (or, static) magnetic field.



Note: This lecture and the next one will be devoted to the study of static magnetic fields arising from direct current distributions.

Current Produces Magnetic Field: Oersted's Experiment

- Oersted[1] (1777-1851) discovered that electric current produces magnetic field (action at a distance!)
- http://www.youtube.com/watch?v=p_bU2CInQDE (2:11)
- Uploaded by TutorVista

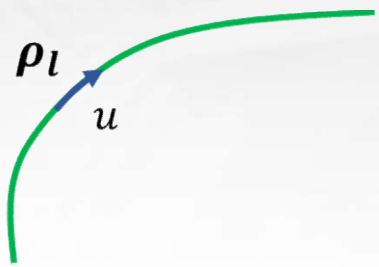
1. http://en.wikipedia.org/wiki/Hans_Christian_%C3%98rsted

Electric Current

Current is a flow of electric charge; it is measured by the rate (u) at which charge (ρ_l) passes any specified surface area.

- **Line current**

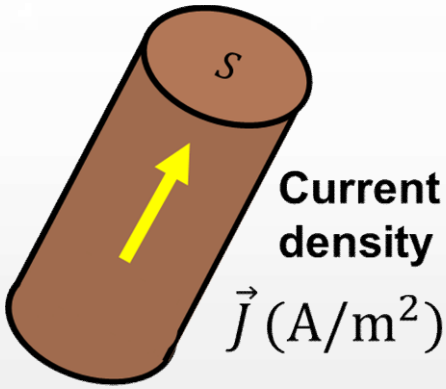
$$I = u\,\rho_l \text{ (A)}$$



- **Volume current**

The total current flowing through the cross-section of the cylinder is:

$$\boxed{I_{total} = \iint_S \vec{J} \cdot \vec{d}s}$$

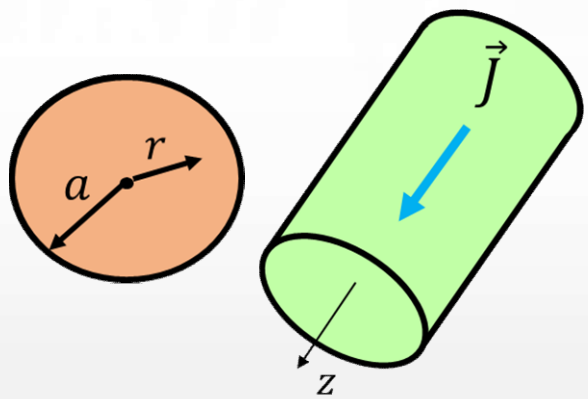


Example

An infinitely long conducting cylinder of radius a, oriented with its axis along the z-direction, carries a current density:

$$\vec{J} = \vec{a}_z \frac{3}{\pi a^2}\left(1 - \frac{r}{a}\right) \text{ A/m}^2$$

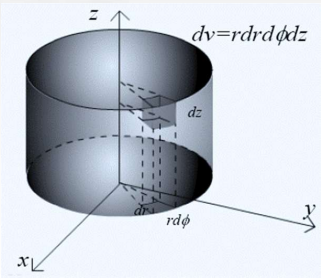
Determine the total current I flowing through the entire cross-section of the conducting cylinder.



Solution

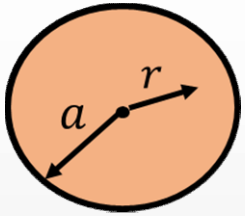
The total current passing through the entire cross-section of the cylinder is the integral of the current density over the cross-section.

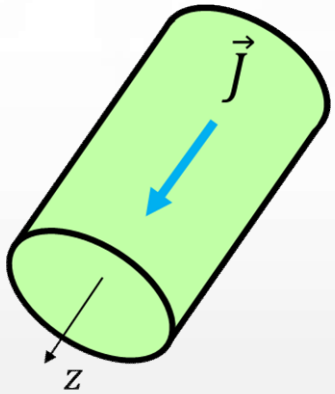
$$I = \iint_S \vec{J} \cdot \vec{ds}$$



$$= \int_0^{2\pi} \int_0^a \frac{3}{\pi a^2}\left(1 - \frac{r}{a}\right) \vec{a}_z \cdot \vec{a}_z r \, dr d\phi$$

$$I = \frac{3}{\pi a^2} 2\pi \int_0^a \left(r - \frac{r^2}{a}\right) dr$$



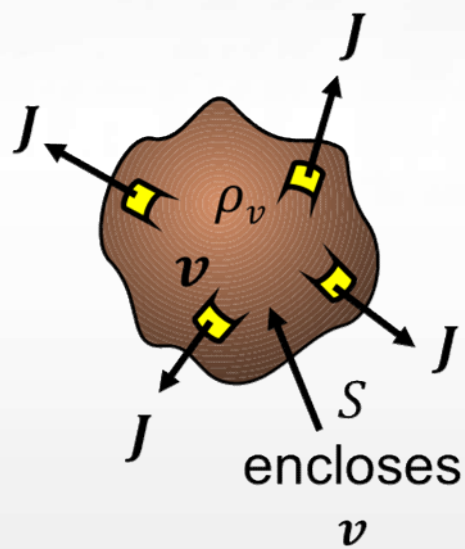


$$I = \frac{3}{\pi a^2} 2\pi \left(\frac{a^2}{2} - \frac{a^3}{3a}\right) = 1 \text{ A}$$

Principle of Conservation of Charge

Electric charge can neither be created nor destroyed.

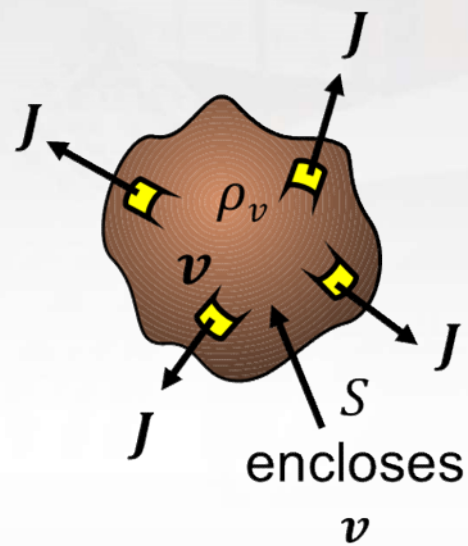
For a volume V enclosed by a surface S, the rate of decrease of electric charge within the volume V is equal to the current flowing out of the closed surface S.



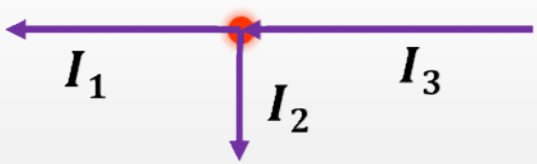
Principle of Conservation of Charge (continued)

$$\oiint_S \vec{J} \cdot \vec{ds} = -\frac{\partial}{\partial t} \iiint_V \rho_v dv$$

$$\boxed{\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}}$$



$$I_1 + I_2 - I_3 = 0$$



Note: Kirchhoff's current law is just a consequence of conservation of charge.

Summary

- Current is a flow of electric charge. Line current is calculated as $I = u\,\rho_l$. Volume current is calculated as $I_{total} = \iint_S \vec{J} \cdot \vec{ds}$;

- The principle of conservation of charge states that for a volume V enclosed by a surface S, the rate of decrease of electric charge within the volume V is equal to the current flowing out of the closed surface S. It is defined as:

$$\oiint_S \vec{J} \cdot \vec{ds} = -\frac{\partial}{\partial t} \iiint_V \rho_v dv \quad \text{or} \quad \nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}$$

NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

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Session 9-2

Biot-Savart Law

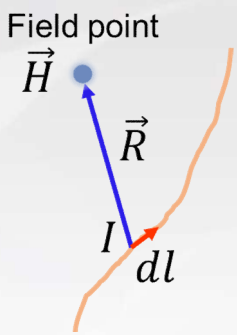
Learning Objectives

- State and apply Biot-Savart law.

Biot-Savart Law

- Oersted[1] (1777-1851) discovered that electric current produces magnetic field (action at a distance!)

- Biot and Savart, French physicists, found in 1820 that the magnetic field intensity $\vec{H}$ produced by a line current is



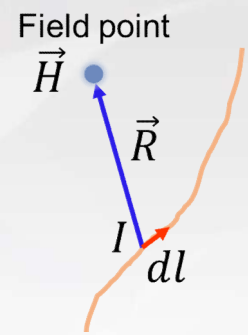
$$\boxed{\vec{H} = \frac{1}{4\pi}\int_C \frac{I\vec{dl} \times \vec{a}_R}{R^2} = \frac{1}{4\pi}\int_C \frac{I\vec{dl} \times \vec{R}}{R^3}}$$

where $\vec{R} = \vec{f} - \vec{s}$ $\quad$ $\mathrm{R} = |\vec{f} - \vec{s}|$.

1. http://en.wikipedia.org/wiki/Hans_Christian_%C3%98rsted

Biot-Savart Law

$$\vec{H} = \frac{1}{4\pi}\int_C \frac{I\vec{dl} \times \vec{a}_R}{R^2} = \frac{1}{4\pi}\int_C \frac{I\vec{dl} \times \vec{R}}{R^3}$$



- The above formula is known as the **Biot-Savart law**.
- Units of the magnetic field intensity are ***Amperes/metre*** $(\mathrm{A/m})$.

Quiz

Hayt and Buck, Ch 8, Q1:

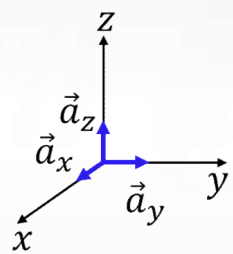
Determine the magnetic field $(\Delta H(P))$ due to a very short current element at the origin of the coordinate system. The current element is oriented along the positive x-axis, its current is $I = 1000\text{A}$, and its length is $\Delta L = 10^{-3}\text{m}$. The observation point P has coordinates $(0,2,0)\text{m}$.

A: $\Delta\vec{H} = 1/(8\pi)\,\hat{a}_x\ \text{A/m}$

B: $\Delta\vec{H} = 1000\hat{a}_y\ \text{A/m}$

C: $\Delta\vec{H} = 1/(16\pi)\,\hat{a}_z\ \text{A/m}$

D: $\Delta\vec{H} = 0.25\hat{a}_y\ \text{A/m}$



$$\vec{I}\Delta L = 1\vec{a}_x$$

$$\vec{R} = 2\vec{a}_y$$

$$\boxed{\vec{H} = \frac{1}{4\pi}\int_C \frac{I\vec{dl} \times \vec{a}_R}{R^2} = \frac{1}{4\pi}\int_C \frac{I\vec{dl} \times \vec{R}}{R^3}}$$

Summary

- Biot-Savart law states that the magnetic field intensity $\vec{H}$ produced by a line current is

$$\vec{H} = \frac{1}{4\pi}\int_C \frac{I\vec{d}l \times \vec{a}_R}{R^2} = \frac{1}{4\pi}\int_C \frac{I\vec{d}l \times \vec{R}}{R^3}$$

where $\vec{R} = \vec{f} - \vec{s}$ $\quad \mathrm{R} = \left|\vec{f} - \vec{s}\right|$.

NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

EE3001 Engineering Electromagnetics

Session 9-3

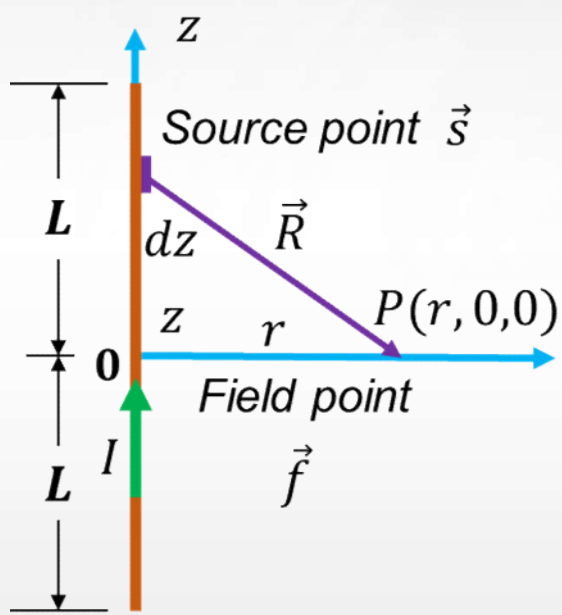
Magnetic Field Due to Line Currents Using Biot-Savart Law

Learning Objectives

- Apply Biot-Savart law to obtain magnetic field due to:
- A straight line current; and
- A circular loop current.

Magnetic Field Due to a Straight Line Current

Example: A direct current / flows in a straight thin wire of length $2L$. The wire is oriented along the z-axis and is symmetrically located about the origin. Find the magnetic field intensity $\vec{H}$ at a distance r from the wire in the xy-plane.



Magnetic Field Due to a Straight Line Current (Solution)

Solution:

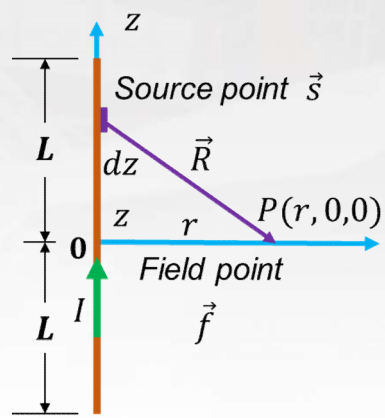
Field point: $\vec{f} = r\vec{a}_r$

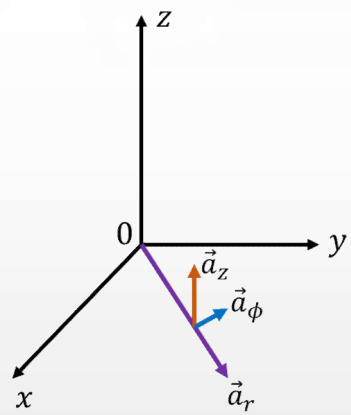
Source point: $\vec{s} = z\vec{a}_z$

Source element: $\vec{dl} = dz\vec{a}_z$

We obtain: $\vec{R} = \vec{f} - \vec{s} = \vec{a}_r r - \vec{a}_z z$

$$\vec{dl} \times \vec{R} = \vec{a}_z dz \times (\vec{a}_r r - \vec{a}_z z) = \vec{a}_\phi r dz$$



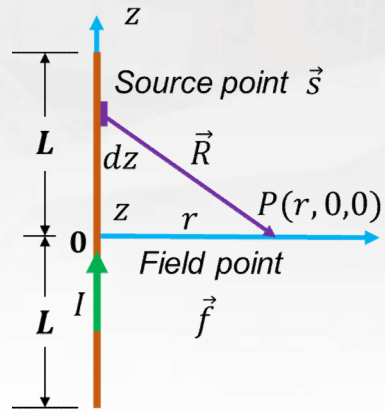


Magnetic Field Due to a Straight Line Current (Solution)

$$\vec{H} = \frac{1}{4\pi}\int_C \frac{I\vec{dl} \times \vec{R}}{R^3}$$

$$\vec{R} = \vec{f} - \vec{s} = \vec{a}_r r - \vec{a}_z z$$

$$R = \sqrt{r^2 + z^2}$$



Using Biot-Savart law, we obtain:

$$\vec{H} = \vec{a}_\phi \frac{I}{4\pi}\int_{-L}^{+L} \frac{rdz}{(z^2 + r^2)^{3/2}} = \vec{a}_\varphi \frac{IL}{2\pi r\sqrt{L^2 + r^2}} \quad \text{A/m}$$

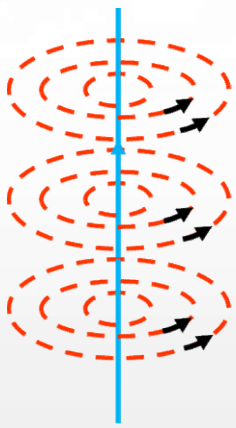
Here we use the integral formula:

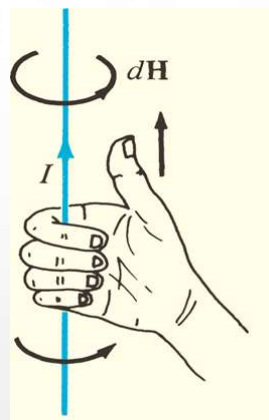
$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 + x^2}}$$

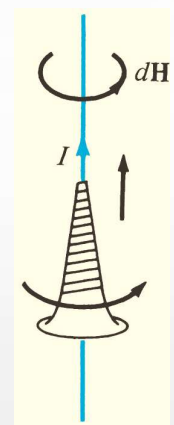
Right-Hand Rule

- Lines of magnetic field due to a line current encircle the line current.
- The direction of the magnetic field and the direction of current flow are related by the right-hand rule.

$$\vec{H} = \vec{a}_{\varphi} \frac{IL}{2\pi r\sqrt{L^2 + r^2}}$$







IEEE Logo

The right-hand rule can be seen in the **IEEE** (Institute of Electrical and Electronic Engineers) logo.

https://www.ieee.org/about/toolkit/masterbrand/index.html

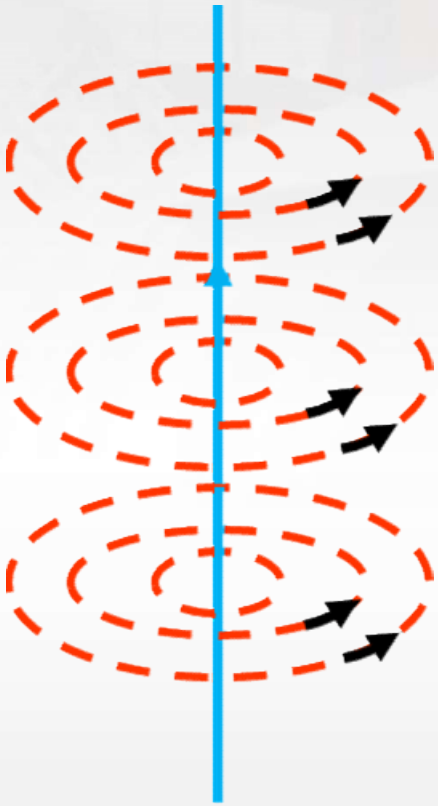
Magnetic Field Due to Infinitely Long Line Current

When L is finite:

$$\vec{H} = \vec{a}_\phi \frac{IL}{2\pi r\sqrt{L^2 + r^2}}$$

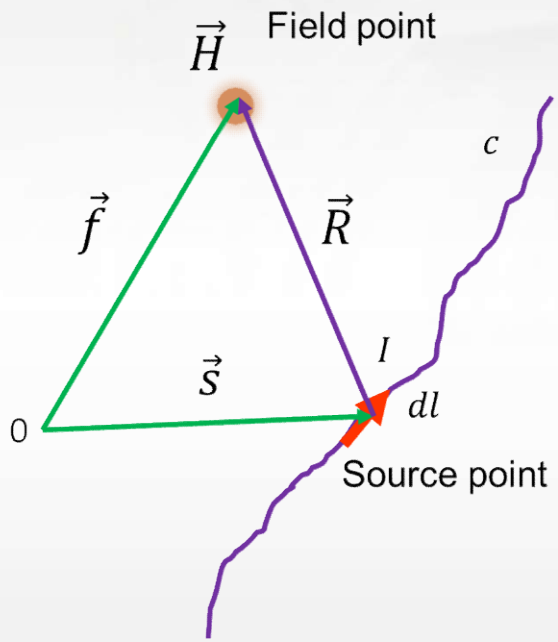
When L is infinite, or $L \gg r$:

$$\vec{H} = \vec{a}_\phi \frac{I}{2\pi r}$$



Question

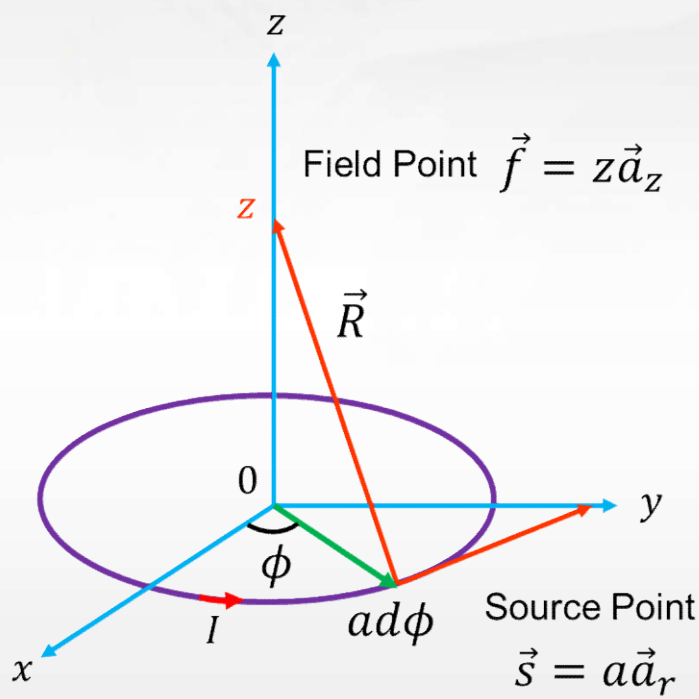
What is the line integral of magnetic field over a closed path around the line current?



Magnetic Field Due to a Circular Loop Current

Example: A circular loop of radius a, positioned in the $x - y$ plane and centered at the origin, is carrying a direct current I.

Determine the magnetic field intensity at $(0,0,z)$.

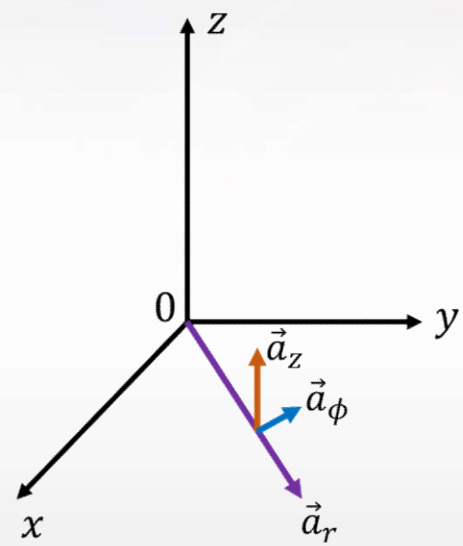


Magnetic Field Due to a Circular Loop Current (Solution)

Solution: For the problem considered, we have

$$\vec{R} = z\vec{a}_z - a\vec{a}_r \qquad R = \sqrt{a^2 + z^2} \qquad \vec{d}l = ad\phi\vec{a}_\phi \qquad \vec{H} = \frac{1}{4\pi}\int_C \frac{I\vec{d}l \times \vec{R}}{R^3}$$

$$\vec{H} = \frac{I}{4\pi}\int_0^{2\pi} \frac{ad\phi\vec{a}_\phi \times (z\vec{a}_z - a\vec{a}_r)}{(a^2 + z^2)^{3/2}} = \frac{I}{4\pi}\int_0^{2\pi} \frac{ad\phi(z\vec{a}_r + a\vec{a}_z)}{(a^2 + z^2)^{3/2}}$$



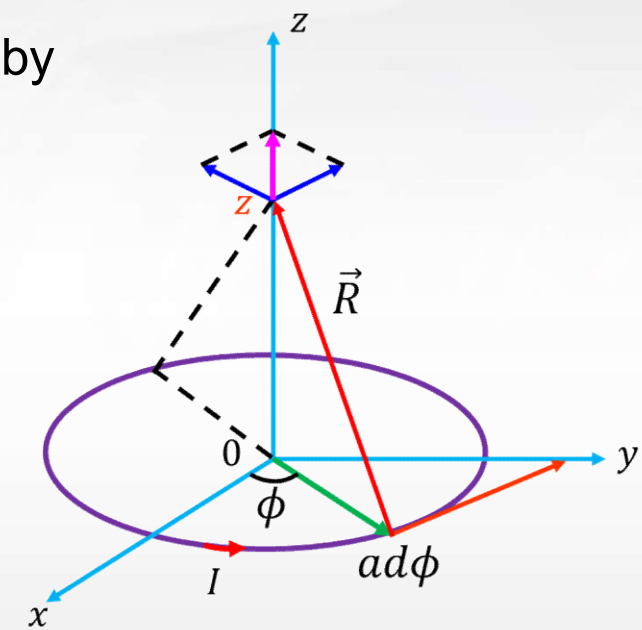
Magnetic Field Due to a Circular Loop Current (Solution)

Due to symmetry of the current distribution, the $\vec{a}_r$ component in the expression vanishes.

This can be easily seen from the diagram by considering two symmetrical differential current elements.

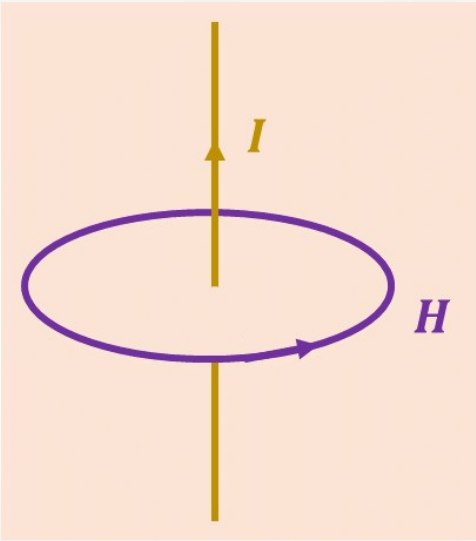
$$\vec{H} = \frac{I}{4\pi}\int_0^{2\pi} \frac{a^2 d\phi \vec{a}_z}{(a^2+z^2)^{3/2}}$$

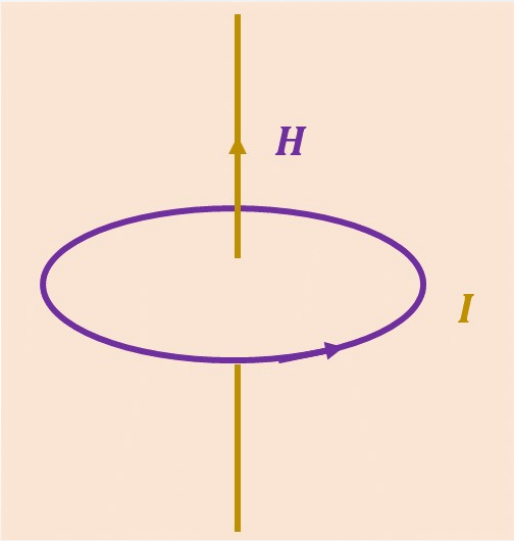
$$= \vec{a}_z \frac{a^2 I}{2(a^2+z^2)^{3/2}} \quad \text{A/m}$$



Right-Hand Rule

The direction of current flow and the direction of the corresponding magnetic field follow the right-hand rule:





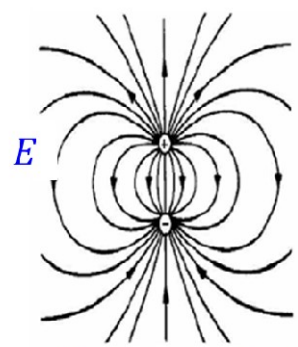
Illustrative Display

- Hayt and Buck, Ch. 8, Illustration 1 (access in NTU Learn, self and peer assessments)

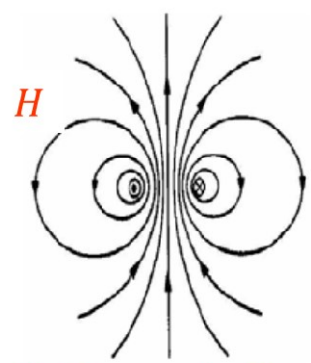
- Magnetic field lines for a number of configurations.

Comparison of Magnetic Field and Electric Field Lines

- The electric field/ flux lines for an electric dipole are shown in diagram (a) (dipole: a pair of equal and opposite charges).
- The magnetic field/ flux lines for a small circular loop current (magnetic dipole) are shown in diagram (b).
- Compare the magnetic field lines with electric field lines.



a) Electric dipole



b) Magnetic dipole

Summary

- Apply Biot-Savart law and obtain the magnetic field due to a straight line current:

$$\vec{H} = \vec{a}_{\varphi} \frac{IL}{2\pi r\sqrt{L^2+r^2}}$$

and obtain the magnetic field due to circular loop current:

$$\vec{H} = \vec{a}_{z} \frac{a^2 I}{2(a^2+z^2)^{3/2}} \ .$$