NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2016-2017

EE2007 / IM2007 – ENGINEERING MATHEMATICS II

November / December 2016 | Time Allowed: 2½ hours

INSTRUCTIONS

1. This paper contains 4 questions and comprises 5 pages.
2. Answer ALL questions.
3. All questions carry equal marks.
4. This is a closed-book examination.
5. Unless specifically stated, all symbols have their usual meanings.
6. A list of useful formulae is given in the Appendix A on page 5.

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1. (a) Consider the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & a & 3 & c \\ 4 & 6 & b & 9 \\ 2 & 3 & 5 & 8 \end{array}\right].$$

Determine the conditions on a, b and c for

(i) unique solution,

(ii) many solutions, and

(iii) no solution.

(15 Marks)

(b) Let $\mathbf{C} = \begin{bmatrix} \mathbf{a}+3\mathbf{b} \\ 3\mathbf{b}+2\mathbf{a} \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}$, where $\mathbf{a}$ and $\mathbf{b}$ are 1 x n row vectors. If the determinant of $\mathbf{C}$ is 3, find the determinant of $\mathbf{D}$.

(5 Marks)

Note: Question No. 1 continues on page 2.

(c) Consider $\mathbf{A} = [\mathbf{u}+3\mathbf{v}+5\mathbf{w} \quad 3\mathbf{u}+\mathbf{w} \quad 6\mathbf{v}+9\mathbf{w}]$ and $\mathbf{B} = [\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}]$ where, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are n x 1 column vectors. If the determinant of $\mathbf{A}$ is 3, find the determinant of $\mathbf{B}$.

(5 Marks)

2. (a) Show that if $\mathbf{v}$ is an eigenvector of the matrix $\mathbf{A}$, $\mathbf{v}$ is also an eigenvector of $\mathbf{A}^n$. What is the corresponding eigenvalue of $\mathbf{A}^n$? Justify your answer.

(6 Marks)

(b) Consider the following coupled differential equation

$$\begin{bmatrix} \frac{d}{dt}y_1(t) \\ \frac{d}{dt}y_2(t) \end{bmatrix} = \mathbf{A}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix},$$

where

$$\mathbf{A} = \begin{bmatrix} -a & b \\ a & -b \end{bmatrix}, a \neq -b, \text{and } a, b \neq 0.$$

The initial conditions are $y_1(0) = C_1$ and $y_2(0) = C_2$.

(i) What are the eigenvalues and eigenvectors of the matrix A?

(10 Marks)

(ii) Hence, by diagonalising A, solve the coupled differential equations. Show clearly the key steps in arriving at the solution. Express your solution in terms of the eigenvectors of A.

(9 Marks)

3. (a) The complex hyperbolic cosine is defined by the formula

$$\cosh z = \frac{e^z + e^{-z}}{2},$$

where z is complex. Using the Cauchy-Riemann equations, comment on the differentiability and analyticity of $\cosh z$.

(7 Marks)

Note: Question No. 3 continues on page 3.

(b) Evaluate the line integral $\int_{i}^{2\pi+i} \cosh z \, dz$ along the path $C: y = \cos x$, where $z = x + iy$.

(8 Marks)

(c) Evaluate the following integral

$$\lim_{p\to\infty} \int_{-p}^{p} \frac{1}{\left(x^2 - 4x + 7\right)^2} dx .$$

(10 Marks)

4. (a) "The closed surface integral of the curl of a vector field is always equal to zero." Is this statement true or false? Justify your answer with proof. (You may use the formulas listed in Appendix A without proof.)

(6 Marks)

(b) Two particles, A and B, move along two separate paths, C_1 and C_2, respectively from (1, 1, 2) to (3, 3, 2) in a field

$$\mathbf{F}(x, y, z) = 6e^{2x} y \cos z \,\mathbf{i} + 3e^{2x} \cos z \,\mathbf{j} - 3e^{2x} y \sin z \,\mathbf{k} .$$

The two paths C_1 and C_2 are defined as follows:

$C_1: \ 2y = x^2 - 2x + 3, \quad z = 2$

$C_2: \ 2x = y^2 - 2y + 3, \quad z = 2$

Determine the particle for which less work is to be done by the force to move it from (1,1,2) to (3,3,2). Find the work done by the force to move this particle.

(9 Marks)

(c) The shelter of a MRT station has a semi-cylindrical surface shape S described by

$$y^2 + z^2 = 4, \quad z \ge 0$$
$$-10 \le x \le 10$$

Note: Question No. 4 continues on page 4.

It was suspected that some electromagnetic (EM) interference within this station was due to the curl of a field (curl **F**), intersecting the surface S of the MRT shelter with **F** defined as

$$\mathbf{F}(x, y, z) = ye^{z}\ \mathbf{i} + 2xe^{3z}\ \mathbf{j} + x^2y^3e^{2z}\ \mathbf{k}\,.$$

Evaluate

$$\iint_S curl\ \mathbf{F} \cdot d\mathbf{A}\,.$$

(Note: Surface S is the curve part of the semi-cylindrical shape, including the semi-circular planar ends but not including the rectangular planar base.)

(10 Marks)

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1/a)
$$\begin{bmatrix}1 & a & 3 & c\\ 4 & 6 & b & 9\\ 2 & 3 & 5 & 8\end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix}2 & 3 & 5 & 8\\ 4 & 6 & b & 9\\ 1 & a & 3 & c\end{bmatrix} \xrightarrow[R_3 \leftarrow 2R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix}2 & 3 & 5 & 8\\ 0 & 0 & b-10 & -7\\ 0 & 2a-3 & 1 & 2c-8\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix}2 & 3 & 5 & 8\\ 0 & 2a-3 & 1 & 2c-8\\ 0 & 0 & b-10 & -7\end{bmatrix}$$

i) unique solution $b-10 \neq 0$, $2a-3 \neq 0$ $c \in \mathbb{R}$

$\therefore b \neq 10$ & $a \neq \frac{3}{2}$ & $c \in \mathbb{R}$

ii) many solution $b-10 \neq 0$, $2a-3=0$

$b \neq 10$, $a = \frac{3}{2}$, $c \in \mathbb{R}$

iii) no solution $b-10=0$

$b=10$ & $a, c \in \mathbb{R}$

b) $$C = \begin{bmatrix}a+3b\\ 3b+2a\end{bmatrix} = \begin{bmatrix}1 & 3\\ 2 & 3\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}1 & 3\\ 2 & 3\end{bmatrix} D$$

$$\det(C) = \begin{vmatrix}1 & 3\\ 2 & 3\end{vmatrix} \det(D)$$

$$\det(D) = \frac{3}{-3} = -1$$

c) $$A^T = \begin{bmatrix}u+3v+5w\\ 3u+w\\ 6v+9w\end{bmatrix} = \begin{bmatrix}1 & 3 & 5\\ 3 & 0 & 1\\ 0 & 6 & 9\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix} = \begin{bmatrix}1 & 3 & 5\\ 3 & 0 & 1\\ 0 & 6 & 9\end{bmatrix} B^T$$

$$\det(A^T) = \det(A) = \begin{vmatrix}1 & 3 & 5\\ 3 & 0 & 1\\ 0 & 6 & 9\end{vmatrix} \det(B^T) = 3$$

$$\det(B) = \det(B^T) = \frac{3}{3} = 1$$

2/a) $A = PDP^{-1}$

$A^n = PD^nP^{-1}$

P is the eigenvectors for both A and A^n, thus A & A^n have same set of eigenvectors

D is eigenvalues for A, D^n is eigenvalues for A^n.
if x is eigenvalue for A, then eigenvalue of A^n is x^n.

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2/b) i) $\begin{bmatrix} -a & b \\ a & -b \end{bmatrix} = A \quad \det(\lambda I - A) = \begin{vmatrix} \lambda+a & -b \\ -a & \lambda+b \end{vmatrix}$

$= (\lambda+a)(\lambda+b) - (-a)(-b)$

$= \lambda^2 + \lambda(a+b) = 0$

$\lambda = 0, \quad \lambda = -(a+b)$

$\lambda = 0, \quad \lambda I - A = \begin{bmatrix} a & -b \\ -a & b \end{bmatrix} \quad ax_1 = bx_2$
$x_2 = t, \quad x_1 = \frac{b}{a} \quad \therefore [x] = t\begin{bmatrix} \frac{b}{a} \\ 1 \end{bmatrix}$

$\lambda = -(a+b), \quad \lambda I - A = \begin{bmatrix} -b & -b \\ -a & -a \end{bmatrix} \quad x_1 = -x_2$
$x_2 = t, \quad x_1 = -t \quad \therefore [x] = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

eigenvalues $\lambda = 0$ & $\lambda = -(a+b)$
eigenvectors $\begin{bmatrix} \frac{b}{a} \\ 1 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

ii) $D = \begin{bmatrix} 0 & 0 \\ 0 & -(a+b) \end{bmatrix} \quad P = \begin{bmatrix} \frac{b}{a} & -1 \\ 1 & 1 \end{bmatrix} \quad P^{-1} = \frac{1}{\frac{b}{a}+1}\begin{bmatrix} 1 & 1 \\ -1 & \frac{b}{a} \end{bmatrix}$

$= \frac{a}{a+b}\begin{bmatrix} 1 & 1 \\ -1 & \frac{b}{a} \end{bmatrix}$

$[\dot{y}] = A[y]$
$= PDP^{-1}[y]$
$P^{-1}[\dot{y}] = DP^{-1}[y]$
$[\dot{W}] = D[W]$
$[W] = e^{Dt}[W(0)]$
$P^{-1}[y] = e^{Dt}P^{-1}[y(0)]$
$[y] = Pe^{DT}P^{-1}[y(0)]$

$= P\begin{bmatrix} e^{0t} & 0 \\ 0 & e^{-(a+b)t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & \frac{b}{a} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\frac{a}{a+b}$

$= \frac{a}{a+b}P\begin{bmatrix} 1 & 1 \\ -e^{-(a+b)t} & \frac{b}{a}e^{-(a+b)t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$= \frac{a}{a+b}\begin{bmatrix} \underset{\sim}{V_1} & \underset{\sim}{V_2} \end{bmatrix}\begin{bmatrix} c_1+c_2 \\ (-c_1+\frac{b}{a}c_2)e^{-(a+b)t} \end{bmatrix}$

$= \frac{a}{a+b}\left[(c_1+c_2)\underset{\sim}{V_1} + (-c_1+\frac{b}{a}c_2)e^{-(a+b)t}\underset{\sim}{V_2}\right]$

3/a) $\cosh z = \frac{e^{z}+e^{-z}}{2} = \frac{e^{x+iy}+e^{-x-iy}}{2}$

$= \cancel{\frac{1}{2}e^{x}(\cos y + i\sin y) + \frac{1}{2}e^{-x}(\cos y - i\sin y)}$

$= \frac{1}{2}e^{x}\cos y + \frac{1}{2}e^{-x}\cos y + i\frac{1}{2}e^{x}\sin y - i\frac{1}{2}e^{-x}\sin y$

$u = \frac{1}{2}e^{x}\cos y + \frac{1}{2}e^{-x}\cos y$ $\qquad$ $v = \frac{1}{2}e^{x}\sin y - \frac{1}{2}e^{-x}\sin y$

$u_x = \frac{1}{2}e^{x}\cos y - \frac{1}{2}e^{-x}\cos y$ $\qquad$ $v_x = \frac{1}{2}e^{x}\sin y + \frac{1}{2}e^{-x}\sin y$

$u_y = -\frac{1}{2}e^{x}\sin y - \frac{1}{2}e^{-x}\sin y$ $\qquad$ $v_y = \frac{1}{2}e^{x}\cos y - \frac{1}{2}e^{-x}\cos y$

$\therefore\ u_x = v_y,\quad v_x = -u_y$

It is differentiable everywhere

Thus it is fully analytic.

b) $z = x+iy = x + i\cos x$ (path: $y = \cos x$)

$dz = (1-i\sin x)\,dx$

z from i to $2\pi+i$, x from 0 to 2π.

$\int_{i}^{2\pi+i} \cosh z\, dz = \int_{0}^{2\pi} \frac{e^{x+i\cos x}+e^{-x-i\cos x}}{2}\cdot(1-i\sin x)\,dx$

$= \frac{1}{2}\int_{0}^{2\pi}(1-i\sin x)e^{x+i\cos x}dx + \frac{1}{2}\int_{0}^{2\pi}(1-i\sin x)e^{-x-i\cos x}dx$

$= \frac{1}{2}\left[e^{x+i\cos x}\right]_{0}^{2\pi} - \frac{1}{2}\left[e^{-x-i\cos x}\right]_{0}^{2\pi}$

$= \frac{1}{2}(e^{2\pi+i} - e^{0+i}) - \frac{1}{2}(e^{-2\pi-i} - e^{0-i})$

$= \frac{1}{2}\left(e^{2\pi+i} - e^{i} - e^{-2\pi-i} + e^{-i}\right)$

c). $\lim_{p\to\infty}\int_{-p}^{p}\frac{1}{(x^2-4x+7)^2}dx = \oint_{UHP}\frac{1}{(z^2-4z+7)^2}dz$

$= \oint_{UHP}\frac{1}{[z-(2+\sqrt{3}i)]^2[z-(2-\sqrt{3}i)]^2}dz$

$= \frac{2\pi i}{(2-1)!}\frac{d}{dz}\left[z-(2-\sqrt{3}i)\right]^{-2}\Big|_{z=2+\sqrt{3}i}$

$= 2\pi i\,(-2)\cdot\left[z-(2-\sqrt{3}i)\right]^{-3}\Big|_{z=2+\sqrt{3}i}$

$= -4\pi i\,(2+\sqrt{3}i-2+\sqrt{3}i)^{-3}$

$= -4\pi i\,(2\sqrt{3}i)^{-3}$

$= \frac{\sqrt{3}}{18}\pi$

4(a) $\oint\!\!\oint_S \nabla \times \vec{F} \cdot d\vec{A} = 0$?

$= \iiint_T \nabla \cdot (\nabla \times \vec{F})\, dV$

$$\nabla \cdot (\nabla \times \vec{F}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}F_z - \frac{\partial}{\partial z}F_x\right) + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)$$

$$= \frac{\partial}{\partial x}\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial x}\frac{\partial}{\partial z}F_y - \frac{\partial}{\partial x}\frac{\partial}{\partial y}F_z + \frac{\partial}{\partial y}\frac{\partial}{\partial z}F_x + \frac{\partial}{\partial x}\frac{\partial}{\partial z}F_y - \frac{\partial}{\partial y}\frac{\partial}{\partial z}F_x$$

$= 0$

$\therefore \oint\!\!\oint \nabla \times \vec{F} \cdot d\vec{A} = \iiint_T 0\, dV$

$= 0$

$\therefore$ the statement is true.

b) $$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6e^{2x}y\cos z & 3e^{2x}\cos z & -3e^{2x}y\sin z \end{vmatrix}$$

$$= \begin{bmatrix} -3e^{2x}\sin z + 3e^{2x}\sin z \\ -6e^{2x}y\sin z + 6e^{2x}y\sin z \\ 6e^{2x}\cos z - 6e^{2x}\cos z \end{bmatrix} = 0$$

$\therefore \vec{F}$ is conservative. Work done along two paths are the same.

$\frac{\partial}{\partial x}V = 6e^{2x}y\cos z \qquad V = 3e^{2x}y\cos z + f(y,z)$

$\frac{\partial}{\partial y}V = 3e^{2x}\cos z \qquad V = 3e^{2x}y\cos z + f(x,z) \qquad (\nabla V = \vec{F})$

$\frac{\partial}{\partial z}V = -3e^{2x}y\sin z \qquad V = 3e^{2x}y\cos z + f(x,y) \quad \therefore V = 3e^{2x}y\cos z + C$

wok done $= V(3,3,2) - V(1,1,2)$

$= 3e^6 \times 3\cos 2 - 3e^2 \times 1\cos 2$

$= 9e^6\cos 2 - 3e^2\cos 2$

c) $\iint_S \nabla \times \vec{F} \cdot d\vec{A} = \oint_C \vec{F} \cdot d\underline{r} = \iint_{S_1} \nabla \times \vec{F} \cdot d\vec{A}$

S: sheltor of the train

C: path along base (closed)

S_1: base of the train enclosed by the path C.

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{z} & 2xe^{3z} & x^2y^3e^{2z} \end{vmatrix} = \begin{bmatrix} 3x^2y^2e^{2z} - 6xe^{3z} \\ ye^{z} - 2xy^3e^{2z} \\ 2e^{3z} - e^{z} \end{bmatrix}$$

$$\cancel{d\vec{A} = \hat{n}\,dx\,dy = \hat{k}\,dx\,dy}$$

$$\nabla \times \vec{F} \cdot d\vec{A} = (2e^{3z} - e^{z})\,dx\,dy$$

$$\therefore \iint_{S_1} \nabla \times \vec{F} \cdot d\vec{A} = \int_{-2}^{2} \int_{-10}^{10} 2e^{3z} - e^{z}\,dx\,dy \Big|_{z=0}$$

$$= (2-1) \times 4 \times 20$$

$$= 80$$

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