EE3001 Engineering Electromagnetics

Session 13-1

Displacement Current

Learning Objectives

- ❑ State complete Ampere's Law; and
- ❑ Define and explain displacement current.

NASA Juno to Jupiter (July 2016)

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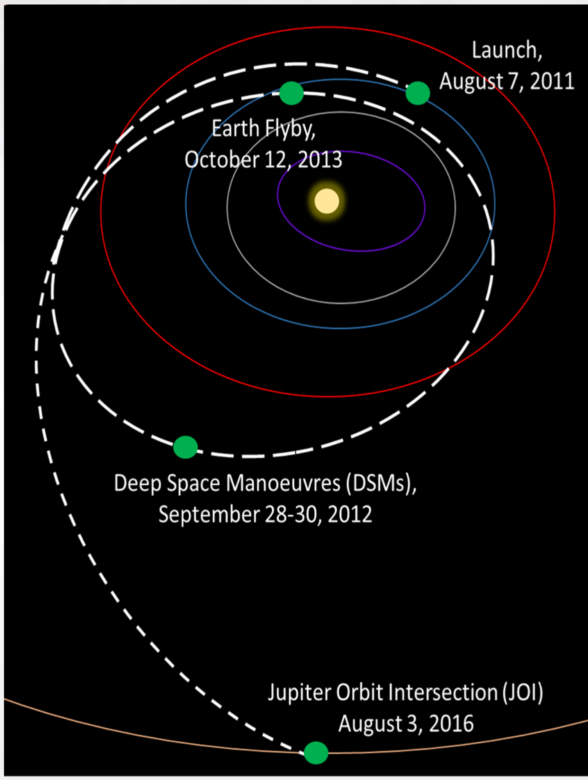
Juno to Jupiter



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Summary of Equations Introduced So Far

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{(Faraday's law)}$$

$$\nabla \times \vec{H} = \vec{J} \quad \text{(Ampere's law)}$$

$$\nabla \cdot \vec{D} = \rho \quad \text{(Gauss' law)}$$

$$\nabla \cdot \vec{B} = 0 \quad \text{(No isolated magnetic charge)}$$

- The first equation is Faraday's law; it relates the induced electric field to a **time-varying** magnetic field.
- The other three equations arise from **static** electric field and **static** magnetic field.
- $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$ Principle of conservation of charge

Is Ampere's Law Complete?

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- From the Ampere's law for static magnetic field

$$\nabla \times \vec{H} = \vec{J}$$

we get $$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} = 0$$

- From the principle of conservation of charge, we have

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

- The above form of Ampere's law for **static** magnetic field and the principle of conservation of charge **contradict** each other!

- Ampere's law for static magnetic field should be ***modified*** when the charge density is varying with time.

Complete Ampere's Law

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- James Clerk Maxwell modified (completed) the Ampere's law for static case by adding a new term to resolve the contradiction.
- The complete Ampere's law has the form

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

One of the Maxwell's equations

Complete Ampere's Law (continued)

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

- Let us perform the divergence operation on the modified Ampere's law:

$$\begin{aligned}
\nabla \cdot (\nabla \times \vec{H}) &= \nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \\
&= \nabla \cdot \vec{J} + \frac{\partial (\nabla \cdot \vec{D})}{\partial t} \\
&= \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0
\end{aligned}$$

which is now in agreement with the principle of conservation of charge.

Displacement Current Density

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

- The new term is known as the **displacement current density**

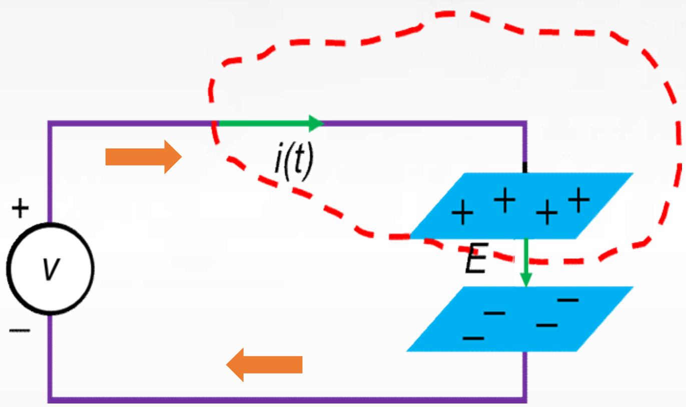
$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$$

- The most brilliant contribution of Maxwell was to **imagine** that the displacement current must also occur in vacuum, i.e., even **in the complete absence of material media**.

- Based on this concept, Maxwell ***predicted*** propagating waves in free space and ***calculated*** the speed of light.

Displacement Current

- The experimental proof[1] of the presence of displacement current **and waves** was provided by Hertz in 1888 (**23 years after** Maxwell formulated the complete electromagnetic theory).



During charging (and discharging), displacement current exists in the region between the parallel plates!

http://www.juliantrubin.com/bigten/hertzexperiment.html

Quiz

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Hayt and Buck, Ch 10, Q4:

The magnetic field intensity of a time-varying field is given by $\vec{H} = \pi\left(-z\hat{a}_y + y\hat{a}_z\right)\cos(2\pi 10^9 t)\ \mathrm{A/m}$. What is the electric flux density vector $\vec{D}(x, y, z, t)$? Assume $\vec{J} = 0$.

A: $\vec{D} = 10^{-9}\sin(2\pi 10^9 t)\hat{a}_x\ \mathrm{C/m^2}$

B: $\vec{D} = -2\pi\sin(2\pi 10^9 t)\hat{a}_x\ \mathrm{C/m^2}$

C: $\vec{D} = 2\pi\cos(2\pi 10^9 t)\hat{a}_z\ \mathrm{C/m^2}$

D: $\vec{D} = -10^{-9}\cos(2\pi 10^9 t)\hat{a}_y\ \mathrm{C/m^2}$

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

$$\nabla \times \vec{H} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

Summary

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- Ampere's Law for static magnetic field should be modified when there is time variation.
- The complete Ampere's Law is:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

- The additional term in the Ampere's Law is called displacement current density. Displacement current can also flow in vacuum.

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Session 13-2

Maxwell's Equations

Learning Objectives

- List Maxwell's equations and identify the independent ones;
- Explain coupling of electric and magnetic fields in time-varying case; and
- Explain applications of Maxwell's equations in electrical and electronic engineering.

Maxwell's Equations

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<u>Differential Form</u>	<u>Integral Form</u>	
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_C \vec{E} \cdot \vec{dl} = -\frac{d}{dt} \iint_S \vec{B} \cdot \vec{ds}$	(1)
$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\oint_C \vec{H} \cdot \vec{dl} = \iint_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{ds}$	(2)
$\nabla \cdot \vec{D} = \rho_v$	$\oiint_S \vec{D} \cdot \vec{ds} = \iiint_V \rho_v dv$	(3)
$\nabla \cdot \vec{B} = 0$	$\oiint_S \vec{B} \cdot \vec{ds} = 0$	(4)

- ❑ Why are these called Maxwell's Equations?
- ❑ So many equations ☹

Integral VS. Differential Form

- Using Stokes's theorem or Divergence theorem, one can derive the integral form of Maxwell's equations from their differential form and vice versa.

Quiz

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Hayt and Buck, Ch 10, Q5:

Is it possible to derive the Maxwell's divergence equation $\nabla \cdot \vec{B} = 0$ from any other Maxwell's equation? If yes, which one?

A: No.

B: Yes: $\nabla \times \vec{E} = -\partial \vec{B} / \partial t$

C: Yes: $\nabla \times \vec{H} = \partial \vec{D} / \partial t + \vec{J}$

D: Yes: $\nabla \cdot \vec{D} = \rho_v$

Maxwell's Equations: Curl Equations

- Applying the divergence operation to Faraday's law, $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$, we obtain $\nabla \cdot \left(\nabla \times \vec{E}\right) = -\frac{\partial(\nabla \cdot \vec{B})}{\partial t} = 0$

  which leads to $\nabla \cdot \vec{B} = \text{Constant} = 0$

  because we can assume that the field components were zero at the beginning of the universe $(t = 0)$.
  This is also consistent with Gauss' Law for magnetic flux.

- Only the two curl equations in Maxwell's equations are independent. The two divergence equations can be derived from the curl equations.

Coupling of E and H Field: Static Case

- If the electromagnetic fields are static ($\frac{\partial}{\partial t} = 0$), for static electric field one gets

$$\nabla \times \vec{E} = 0 \qquad \nabla \cdot \vec{D} = \rho_v$$

and for static magnetic fields one gets

$$\nabla \times \vec{H} = \vec{J} \qquad \nabla \cdot \vec{B} = 0$$

E and H fields are decoupled!

Coupling of E and H Field: Time-varying Case

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Stationary Charges

Charge moving at a constant velocity

Direct current

Static electric field

$$\nabla \times \vec{E} = 0$$
$$\nabla \cdot \vec{D} = \rho$$

Charge or current varying with time

Static magnetic field

$$\nabla \times \vec{H} = \vec{J}$$
$$\nabla \cdot \vec{B} = 0$$

E and H fields are coupled!

Time-varying electromagnetic fields

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$
$$\nabla \cdot \vec{D} = \rho \qquad \nabla \cdot \vec{B} = 0$$

Interaction Between E and H

(1) Faraday's law $\oint_C \vec{E} \cdot \vec{dl} = -\frac{d}{dt} \iint_S \vec{B} \cdot \vec{ds}$ describes

the creation of an electric field by a time-varying magnetic field.

(2) Ampere's law $\oint_C \vec{H} \cdot \vec{dl} = \iint_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{ds}$ describes

the creation of a magnetic field by an electric current and/or a time-varying electric field.

Key Personalities in EM Theory

James Clerk Maxwell (1831-1879)
- Formulated the electromagnetic theory
- Predicted the existence of EM waves

Heinrich Hertz (1847-1896)
- Verified Maxwell's equations
- Transmitted and received radio waves in lab

Guglielmo Marconi (1874-1937)
- Received radio waves transmitted across Atlantic Ocean
- Opened the era of wireless communications

Life of James Clerk Maxwell

- http://presentur.ntu.edu.sg/aculearn-idm/opr/clientredirection.asp?sessionid=acl04nov15%5F05%5F21&author=taskforce&cat=acustudio

- Lecturer: James Rautio
- Title: Life of James Clerk Maxwell
- Date of Creation: 07-November-2005

Applications of Maxwell's Equations

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- Machines and power systems: Transformers, DC & AC motors, generators, power transmission lines, wireless power transfer
- Communications: Antennas, wave propagation, telecommunication, microwave circuits, radar, remote sensing, GPS
- Electronic circuits: Circuit theory, semiconductor devices, high speed circuits
- Computers: Computer networks, internet, internet of things
- Photonics: Optoelectronic devices (e.g., lasers), optical fibres and waveguides
- Biomedical Engineering: Imaging, detection and treatment of tumors

Summary

- The Maxwell's equations and identified the independent ones in these equations;
- Electric and magnetic fields get coupled in time-varying cases; this can lead to electromagnetic waves; and
- There are numerous applications of Maxwell's equations in electrical and electronic engineering.

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Session 14-1

Phasor Notation

Learning Objectives

- Develop phasor notation for sinusoidally time varying signals; and
- Convert sinusoidally time varying signals into phasor form and vice versa.

Sinusoidal Time Variation

- In the general case, the electric and magnetic fields ($\vec{E}$, $\vec{D}$, $\vec{H}$ and $\vec{B}$) and their sources, the charge density ρ and the current density $\vec{j}$, are an arbitrary function of time variable t.

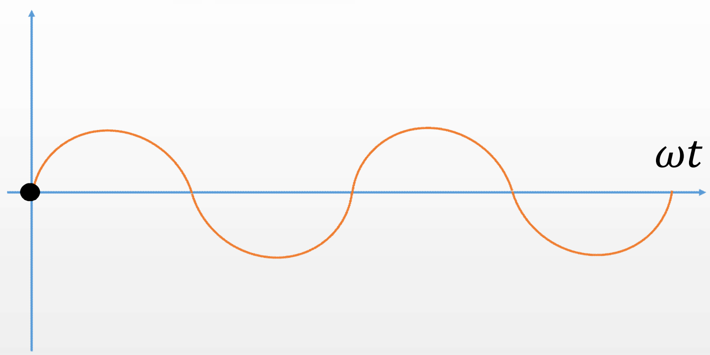
- However, numerous practical applications (broadcast radio and TV, radar, microwave and optical applications) operate in a narrow band of frequencies.

- In such a case, the behaviour of all the field components is very similar to that of a single-frequency sinusoid.

Sinusoidal Time Variation (continued)

When the fields vary sinusoidally with respect to time (i.e., with a single frequency), it is convenient to use phasor notation.

Time-varying quantity	Phasor Form
$\tilde{\vec{A}}(x,y,z,t) = \vec{A}(x,y,z)\cos(\omega t + \phi)$	$\vec{A}(x,y,z)e^{j\phi}$



Phasor Notation

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Considering the time variation to be $\cos \omega t$, a vector phasor is defined as follows:

$$\boxed{\tilde{\vec{A}}(x,y,z,t) = \vec{A}(x,y,z)\cos(\omega t + \phi) = \mathrm{Re}[\vec{A}(x,y,z)e^{j\phi}e^{j\omega t}]}$$

where $\vec{A}(x,y,z)e^{j\phi}$ is a vector phasor that contains information on direction, magnitude and phase.

The equation above provides a relationship between a vector's phasor form and its time-varying form.

Phasor Notation: Example 1

Write the phasor expression for $\tilde{\vec{A}}(t) = -\vec{A}_0 \cos(\omega t - 30°)$.

Since $\tilde{\vec{A}}(t) = \mathrm{Re}(\vec{A}e^{j\omega t})$

$$
\begin{aligned}
&= -\vec{A}_0 \cos(\omega t - 30°) \\
&= \vec{A}_0 \cos(\omega t - 30° + 180°) \\
&= \vec{A}_0 \cos(\omega t + 150°) \\
&= \vec{A}_0 \mathrm{Re}[e^{j(\omega t + 150°)}] \\
&= \mathrm{Re}[\vec{A}_0 e^{j150°} e^{j\omega t}]
\end{aligned}
$$

The corresponding phasor is

$$\vec{A} = \vec{A}_0 e^{j150°}$$

Phasor Notation: Example 2

Write the time-varying expression for the phasor

$$\vec{E} = |E_0| e^{j\phi} e^{-j\beta z} \vec{a}_E$$

Using the conversion formula, we obtain:

$$\tilde{\vec{E}}(t) = \mathrm{Re}(\vec{E} e^{j\omega}\ )$$

$$= \mathrm{Re}\left(|E_0| e^{j\phi} e^{-j\beta z} \vec{a}_E e^{j\omega}\ \right) = \mathrm{Re}[|E_0| e^{j(\omega t - \beta z\quad)}] \vec{a}_E$$

$$= |E_0| \cos(\omega t - \beta z + \phi)\, \vec{a}_E$$

Conversion between Phasor and Time-Varying Forms

- Conversion from phasor $\vec{A}$ to time-varying form $\tilde{\vec{A}}(t)$

$$\tilde{\vec{A}}(t) = \mathrm{Re}[\vec{A}e^{j\omega}\ ]$$

- Conversion from time-varying form $\tilde{\vec{A}}(t)$ to phasor $\vec{A}$
  - First, put the real-time form into the following form:

$$\tilde{\vec{A}} = \vec{A}_0 \cos(\omega t + \phi)$$

  - Second, directly write down the phasor form:

$$\vec{A} = \vec{A}_0 e^{j\phi}$$

Phasor Notation: Time Derivative

The time derivative of a phasor vector field is

$$\frac{\partial}{\partial t}\tilde{\vec{A}}(x,y,z,t)=\frac{\partial}{\partial t}\mathrm{Re}\left[\vec{A}(x,y,z)e^{j\omega t}\right]=\mathrm{Re}\left[j\omega\vec{A}(x,y,z)e^{j\omega t}\right]$$

therefore,

$$\frac{\partial\tilde{\vec{A}}(x,y,z,t)}{\partial t} ==> j\omega\vec{A}(x,y,z)$$

Time-varying form $\Longleftrightarrow$ Phasor form

Maxwell's Equations in the Frequency Domain (Phasor Form)

Maxwell's equations for time-harmonic fields (sinusoidal time variation) have the form:

Phasor Form

$$\nabla \times \tilde{\vec{E}} = -\frac{\partial \tilde{\vec{B}}}{\partial t}$$

$$\boxed{\nabla \times \vec{E} = -j\omega \vec{B}}$$

$$\nabla \times \tilde{\vec{H}} = \vec{J} + \frac{\partial \tilde{\vec{D}}}{\partial t}$$

$$\boxed{\nabla \times \vec{H} = \vec{J} + j\omega \vec{D}}$$

$$\nabla \cdot \tilde{\vec{D}} = \rho_v$$

$$\boxed{\nabla \cdot \vec{D} = \rho_v}$$

$$\nabla \cdot \tilde{\vec{B}} = 0$$

$$\boxed{\nabla \cdot \vec{B} = 0}$$

Advantages of Using Phasor Notation

- Using the phasor notation, one can 'hide' the time dependence; this simplifies the notation.

- Partial derivatives with respect to time can be readily converted to algebraic operations; this simplifies the analysis in phasor notation.

- Phasor notation directly gives the frequency response (steady-state response) of a system or component, which is desirable for many engineering applications.

Summary

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- A sinusoidally time-varying function can be expressed in the phasor form which is more compact;
- A sinusoidal function can be converted from phasor form to time-varying form and vice versa; and
- Time derivative in a time-varying function simplifies to multiplication by a factor of $j\omega$ in the phasor notation.

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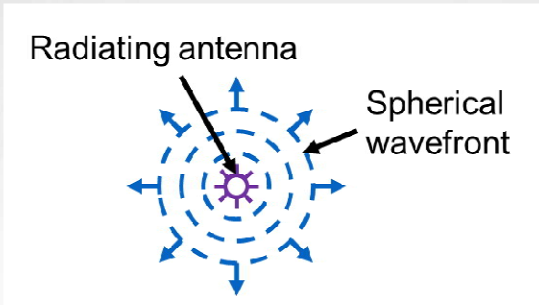
Session 14-2

Introduction to Plane Waves

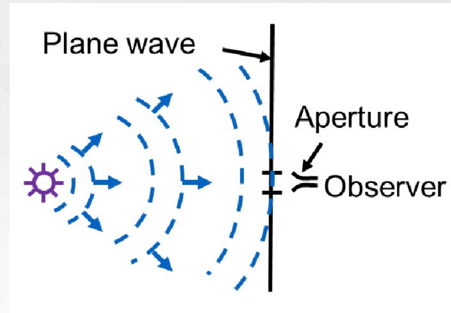
Learning Objectives

- ❑ Define plane waves and uniform plane waves;
- ❑ State the practical and theoretical importance of plane waves; and
- ❑ Represent a wave mathematically.

Practical Importance of Plane Waves



a) Spherical wave

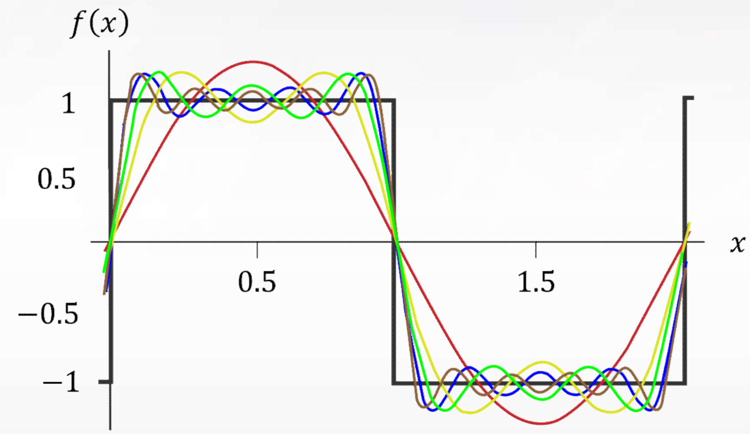


b) Plane-wave approximation

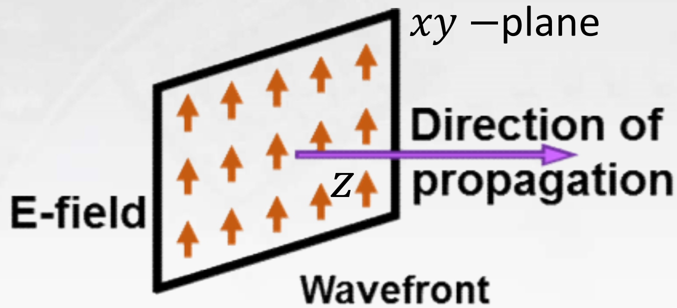
- Waves radiated by an EM source, such as a light bulb or a transmitting antenna, have a spherical wavefront.
- When the field point is far enough from the source, a small portion of the large spherical wavefront can be represented by a plane wavefront.

Theoretical Importance of Plane Waves

- ❑ Plane wave solution to the Maxwell's equations is the simplest one.
- ❑ One can appreciate the solution process by seeking plane wave solution to the Maxwell's equations.
- ❑ Plane waves play a role similar to that of sinusoids.
- ❑ Non-planar waves (cylindrical waves, spherical waves, guided waves) can be considered a superposition of plane waves.



Definition of a Uniform Plane Wave



Note: Wavefront is defined as a virtual surface on which the phase is constant. Wavefront is perpendicular to the direction of propagation.

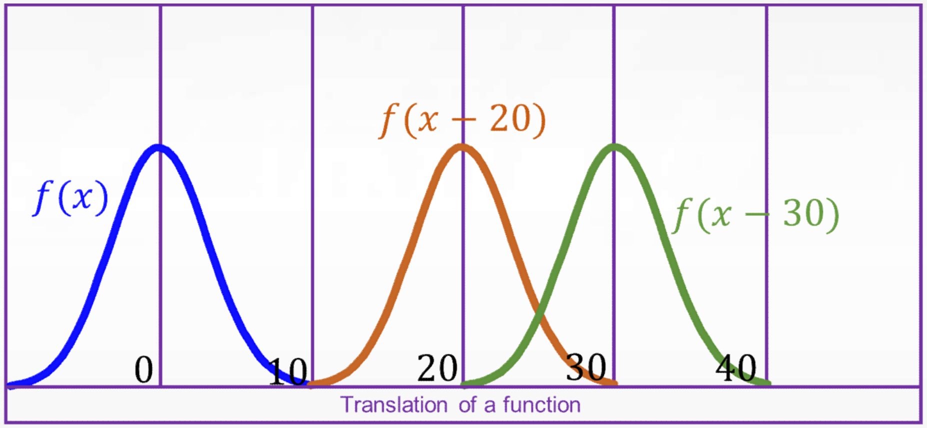
A uniform plane wave is a particular solution to Maxwell's equations with the electric field assuming:

- ❑ the same direction
- ❑ the same magnitude
- ❑ the same phase

in infinite planes perpendicular to the direction of propagation.

Translation of a Function

- A function of a single, real variable is translated to the right on the abscissa if we subtract a positive number from its argument. The amount of translation is equal to this subtracted number:

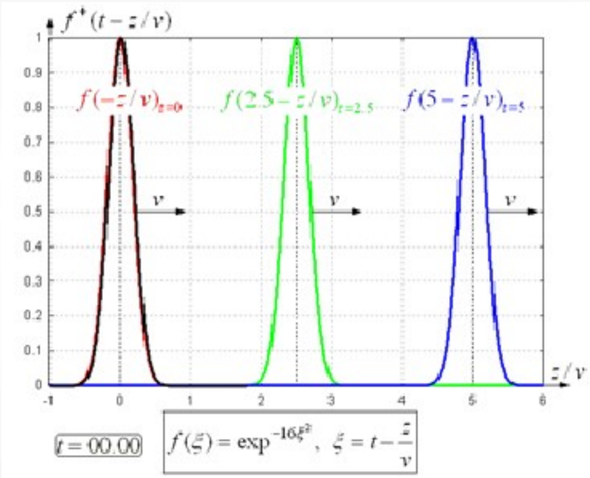


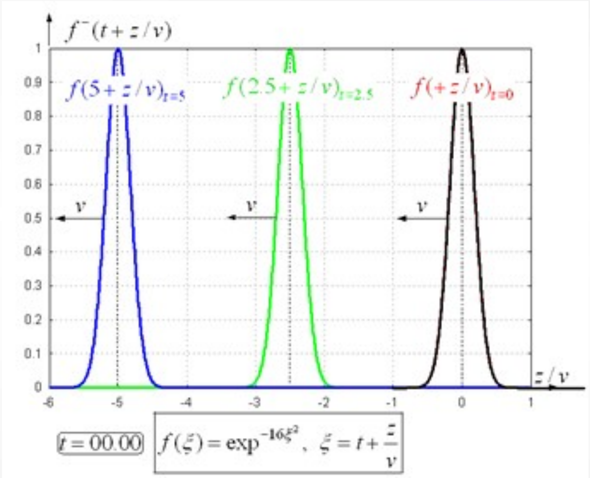
- If a positive number is added to the argument, the function would translate to the left.

Hayt and Buck, Ch. 12, Animation 1

One-dimensional (1-D) Waves

- Observe the 1-D Gaussian-pulse as an incident (left plot) and reflected (right plot) waves.





Representation of a Wave

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- A wave can be represented by a function of two variables:
  - one of these variables is a spatial coordinate $(x, y, \text{or } z)$; and
  - the second variable is time (t).
- Examples: $f(t - z/v)$ or $f(t + z/v)$

  $\cos(\omega t - kz)$ or $\cos(\omega t + kz)$

Summary

- A uniform plane wave has a plane wavefront and the fields have the same direction and magnitude over the wavefront;
- Far away from its source the wavefront of a wave can be approximated as a plane wavefront;
- Any type of electromagnetic wave can be expressed as a superposition of plane waves; and
- A wave can be represented by a function of two variables:
  - one of these variables is a spatial coordinate $(x, y, \text{or } z)$; and
  - the second variable is time (t).

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Session 14-3

Wave Equation

Learning Objectives

- State the steps for derivation of Wave Equation;
- Obtain plane wave solution from wave equation; and
- Obtain expressions for wave number, phase velocity, and wavelength for plane waves.

Wave Equation

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Maxwell's Equations in a Lossless Medium

- For a lossless medium, one has
  - Permittivity ε
  - Permeability μ
  - Conductivity $\sigma = 0$

- Maxwell's equations in a source-free ($\rho = 0$ and $\vec{J} = 0$) lossless medium.

Wave Equation

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Maxwell's Equations in a Lossless Medium

- Maxwell's equations in a source-free ($\rho = 0$ and $\vec{J} = 0$) lossless medium:

$$\nabla \times \tilde{\vec{E}} = -\frac{\partial \tilde{\vec{B}}}{\partial t} \rightarrow \boxed{\nabla \times \tilde{\vec{E}} = -\mu \frac{\partial \tilde{\vec{H}}}{\partial t}}$$

$$\nabla \times \tilde{\vec{H}} = \vec{J} + \frac{\partial \tilde{\vec{D}}}{\partial t} \rightarrow \boxed{\nabla \times \tilde{\vec{H}} = \varepsilon \frac{\partial \tilde{\vec{E}}}{\partial t}}$$

$$\boxed{\nabla \cdot \tilde{\vec{E}} = 0} \leftarrow \nabla \cdot \tilde{\vec{D}} = \rho$$

$$\boxed{\nabla \cdot \tilde{\vec{H}} = 0} \leftarrow \nabla \cdot \tilde{\vec{B}} = 0$$

Wace Equation (contd.)

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- Using these equations (see Appendix), we arrive at

$$\boxed{\nabla^2 \tilde{E} - \mu\varepsilon \frac{\partial^2 \tilde{E}}{\partial t^2} = 0}$$

Wave Equation

or

$$\nabla^2 \tilde{E} - \frac{1}{u_p^2}\frac{\partial^2 \tilde{E}}{\partial t^2} = 0$$

- $u_p = \frac{1}{\sqrt{\mu\varepsilon}}$ is the velocity of an electromagnetic wave traveling in the lossless medium.

- An identical equation can be derived for the magnetic field :

$$\boxed{\nabla^2 \tilde{H} - \frac{1}{u_p^2}\frac{\partial^2 \tilde{H}}{\partial t^2} = 0}$$

Wave Equation

Appendix

Wave Equation in the Frequency Domain

For time-harmonic fields, we have

$$\boxed{\nabla^2 \tilde{\vec{E}} - \mu\varepsilon \frac{\partial^2 \tilde{\vec{E}}}{\partial t^2} = 0} \qquad \frac{\partial \tilde{\vec{E}}}{\partial t} => j\omega\vec{E} \qquad \frac{\partial^2 \tilde{\vec{E}}}{\partial t^2} => -\omega^2\vec{E}$$

and $$\nabla^2 \vec{E} + \frac{\omega^2}{u_p^2}\vec{E} = 0 \qquad u_p = \frac{1}{\sqrt{\mu\varepsilon}}$$

or $$\nabla^2 \vec{E} + k^2\vec{E} = 0$$ **Helmholtz Equation**

where $\boxed{k = \frac{\omega}{u_p} = \omega\sqrt{\mu\varepsilon}}$ is the **wave number** for the lossless medium (also called **phase constant**).

Simplifying Assumptions to Solve Helmholtz Eq.

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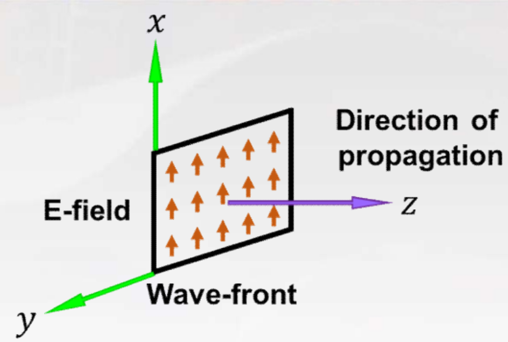
$$\nabla^2\vec{E} + k^2\vec{E} = 0 \qquad \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0$$



- See Appendix for expansion of Helmholtz equation
- Simplifying assumptions:
  - Uniform plane wave propagating along the z −axis
  - Electric field $\vec{E}$ has only x-directed component

$$\frac{\partial}{\partial x} = 0 \qquad \frac{\partial}{\partial y} = 0 \qquad \frac{\partial}{\partial z} \neq 0$$

$$\vec{E} = \vec{a}_x E_x \qquad E_y = E_z = 0$$

Appendix

Simplified Helmholtz's Equation and its Solution

- The Helmholtz's equation $\nabla^2\vec{E} + k^2\vec{E} = 0$ simplifies to

$$\frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0$$

- This is a homogeneous second-order differential equation and has a solution of the form

$$E_x(z) = E_0^+ e^{-jk} \ + E_0^- e^{+jkz}$$

E_0^+ and E_0^- are arbitrary constants, $k = \omega\sqrt{\mu\varepsilon}$ and it is called the wave number or phase constant.

Simplified Helmholtz's Equation and its Solution

$$E_x(z) = E_0^+ e^{-jkz} + E_0^- e^{+jkz}$$

Note:

The first term $(E_0^+ e^{-jkz})$ can be interpreted as *a wave traveling in the $+z$ direction*

The second term $(E_0^- e^{+jk}\ )$ can be interpreted as *a wave traveling in the $-z$ direction*.

Plane Wave Propagation

- Let us consider the first term

$$\vec{E}(z) = \vec{a}_x E_x^+(z) = \vec{a}_x E_0^+ e^{-jkz}$$

$$= \vec{a}_x |E_0^+| e^{j\phi} e^{-jkz} \qquad \textit{(phasor form)}$$

- The time-varying expression for $\vec{E}(z)$ is

$$\tilde{\vec{E}}(z,t) = \vec{a}_x \mathrm{Re}\left[E_x^+(\mathrm{z}) e^{j\omega t}\right]$$

$$= \vec{a}_x |E_0^+| \cos(\omega t - kz + \phi) \qquad (14.1)$$

Plane Wave Propagation

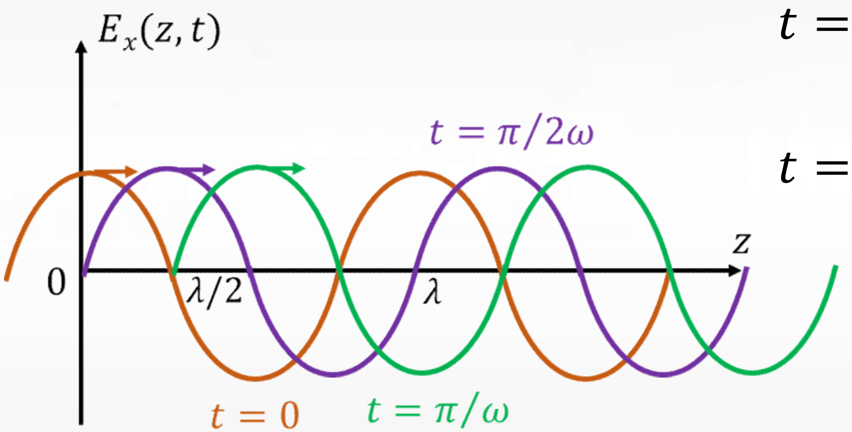
$$\tilde{\vec{E}}(z,t) = \vec{a}_x |E_0^+| \cos(\omega t - kz + \phi)$$

Let $\phi = 0$ $\quad \cos(\omega t - kz)$

$t = 0 \quad \cos\{-kz\}$

$t = \dfrac{\pi}{2\omega} \quad \cos\{\dfrac{\pi}{2} - kz\}$

$t = \dfrac{\pi}{\omega} \quad \cos\{\pi - kz\}$



Phase Velocity

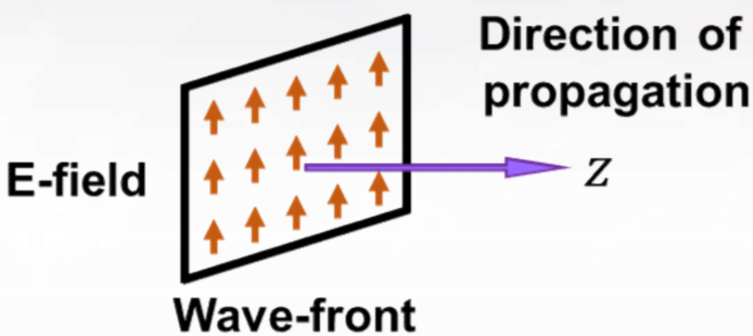
- From (14.1), we see that the phase of the wave propagating in the $+z$ direction is $\omega t - kz + \phi$.

- The wavefront is the plane that satisfies

$$\omega t - kz + \phi = Const.$$

$$\omega dt - kdz = 0$$

which yields $$\boxed{u_p = \frac{dz}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}}$$



- $u_p = \frac{1}{\sqrt{\mu\varepsilon}}$ is the velocity of propagation of an equiphase front, also termed as the **phase velocity**.

Animation Display

- Hayt and Buck, Ch. 12, Animation 4 (access in NTU Learn, self and peer assessment)

- Sinusoidal Traveling Plane Wave.

Wavelength

- Wavelength is defined as the distance over which the phase changes by $2\pi^c$.
- The relation between k and the wavelength λ is then

$$k\lambda = 2\pi$$

$$k = \frac{2\pi}{\lambda} \qquad \boxed{\lambda = \frac{2\pi}{k}}$$

Summary

- The steps for derivation of Wave Equation.
- How to obtain plane wave solution from wave equation.
- Wave number, or phase constant, is given as $k = \omega\sqrt{\mu\varepsilon}\ \ \mathrm{rad/s}$.
- Phase velocity is given as $u_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}\ \ \mathrm{m/s}$.
- Wavelength is given as $\lambda = \frac{2\pi}{k}\ \ \mathrm{m}$.

Derivation of Wave Equation

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- From Faraday's Law, applying curl operation

$$\nabla \times \nabla \times \tilde{E} = \nabla \times \left(-\mu \frac{\partial \tilde{H}}{\partial t}\right) = -\mu \frac{\partial}{\partial t}(\nabla \times \tilde{H}) = -\mu \frac{\partial}{\partial t}\left(\varepsilon \frac{\partial \tilde{E}}{\partial t}\right) = -\mu\varepsilon \frac{\partial^2 \tilde{E}}{\partial t^2}$$

- Using the vector identity $\nabla \times \nabla \times \tilde{E} = \nabla(\nabla \cdot \tilde{E}) - \nabla^2 \tilde{E}$ and $\nabla \cdot \tilde{E} = 0$, we get

$$\nabla \times \nabla \times \tilde{E} = -\nabla^2 \tilde{E}$$

- Hence

$$\boxed{\nabla^2 \tilde{E} - \mu\varepsilon \frac{\partial^2 \tilde{E}}{\partial t^2} = 0}$$

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Solution of Helmholtz Equation

- For a uniform plane wave propagating along the positive z −direction, we can solve the wave equation to determine its electric field components

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0 \quad \Rightarrow \quad E_x(z) = E_{x0} e^{-jk}$$

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} + k^2 E_y = 0 \quad \Rightarrow \quad E_y(z) = E_{y0} e^{-jkz}$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} + k^2 E_z = 0 \quad \Rightarrow \quad E_z(z) = E_{z0} e^{-jkz}$$?

Next

Solution of Helmholtz Equation

- For a plane EM wave propagating in the $+z$ direction and with $\vec{E} = \vec{a}_x E_x$,

$$\frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0 \quad \Rightarrow \quad E_x(z) = E_{x0} e^{-jkz}$$

$$\frac{\partial^2 E_y}{\partial z^2} + k^2 E_y = 0 \quad \Rightarrow \quad E_y(z) = E_{y0} e^{-jk}$$

$$\frac{\partial^2 E_z}{\partial z^2} + k^2 E_z = 0 \quad \Rightarrow \quad E_z(z) = E_{z0} e^{-jkz}$$

$$\Downarrow$$

$$0$$

Next

Solution of Helmholtz Equation

- For a plane wave traveling in the z –direction,

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} + k^2 E_z = 0 \Longrightarrow E_z(z) = E_{z0} e^{-jkz}$$

$$\vec{E} = \vec{a}_z E_{z0} e^{-jk}$$

$$\vec{H} = \frac{\nabla \times \vec{E}}{-j\omega\mu} = \frac{1}{-j\omega\mu} \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_{z0} e^{-jkz} \end{vmatrix}$$

$$\vec{E} = \frac{1}{j\omega\varepsilon} \nabla \times \vec{H} = 0 \Longrightarrow E_{z0} = 0$$

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Session 15-1

Examples of Plane Waves (Lossless Media)

Learning Objectives

- Apply the expressions for wave number, phase velocity and wavelength to solve problems related to plane waves propagating in lossless media.

Quiz 1

Hayt and Buck, Ch 12, Q2:

The electric field of a plane wave propagating in a non-magnetic medium is given by $\tilde{\vec{E}} = 3\sin(2\pi 10^7 t - 0.4\pi x)\hat{a}_y$ V/m.
Determine the wavelength making use of the phase constant (wave number).

A: $\lambda = 0.5\ \mathrm{m}$

B: $\lambda = 30\ \mathrm{m}$

C: $\lambda = 1\ \mathrm{m}$

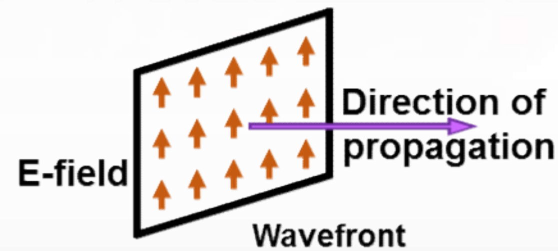
D: $\lambda = 5\ \mathrm{m}$

$$\tilde{\vec{E}}^{+}(z,t) = \vec{a}_x |E_0^+| \cos(\omega t - kz + \varphi)$$

$$\tilde{\vec{E}}^{+}(x,t) = \vec{a}_y |E_0^+| \cos(\omega t - kx + \varphi)$$

$$k = 0.4\pi$$

$$\lambda = \frac{2\pi}{k}$$



Plane Wave Propagation: Example

A uniform plane wave of frequency $1.8\ \mathrm{GHz}$ is traveling in free space. Determine the wave number k, phase velocity u_p, and the wavelength λ.

Plane Wave Propagation: Example Solution

Since the plane wave is propagating in free space, we know that

$$\varepsilon = \varepsilon_0 = \frac{1}{36\pi\times10^9}\ \mathrm{F/m} \text{ and } \mu = \mu_0 = 4\pi\times10^{-7}\ \mathrm{H/m}$$

The angular frequency is

$$\omega = 2\pi f = 2\pi\times1.8\times10^9 = 3.6\pi\times10^9\ \mathrm{rad/s}$$

The wave number k is

$$k = \omega\sqrt{\varepsilon_0\mu_0} = \frac{\omega}{c} = \frac{3.6\pi\times10^9}{3\times10^8} = 12\pi\ \mathrm{rad/m}$$

$$\sqrt{\varepsilon_0\mu_0} = \frac{1}{c}$$

Plane Wave Propagation: Example Solution

The phase velocity u_p is

$$\frac{1}{\sqrt{\mu\varepsilon}} = u_p = \frac{\omega}{k} = \frac{2\pi \times 1.8 \times 10^9}{12\pi} = 3 \times 10^8 \text{ m/s}$$

which is the speed of light in vacuum.

The wavelength λ is $\lambda = \frac{2\pi}{k} = \frac{2\pi}{12\pi} = \frac{1}{6}$ m

Question

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When can we apply the condition $\lambda = c/f$?

- From the discussion above, we know

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega/u_p} = \frac{2\pi u_p}{2\pi f} = \frac{u_p}{f}$$

- It is seen that we can apply $\boldsymbol{\lambda} = \boldsymbol{c}/\boldsymbol{f}$ only when $\boldsymbol{u_p} = \boldsymbol{c}$ which holds only for unbounded free space or air.

Quiz 2

Hayt and Buck, Ch 12, Q3:

The electric field of a plane wave propagating in a non-magnetic medium is given by $\tilde{E} = 3\sin(2\pi 10^7 t - 0.4\pi x)\hat{a}_y$ V/m.
Determine the relative permittivity of the medium.

A: $\varepsilon_r = 36$

B: $\varepsilon_r = 1$

C: $\varepsilon_r = 6$

D: $\varepsilon_r = 10$

$$k = \omega\sqrt{\mu\varepsilon}$$

Summary

- Wave number, or phase constant, is given as $k = \omega\sqrt{\mu\varepsilon}$ rad/s.
- Phase velocity is given as $u_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$ m/s.
- Wavelength is given as $\lambda = \frac{2\pi}{k}$ m.

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Session 15-2

Relation Between $\vec{E}$ and $\vec{H}$

Learning Objectives

- Explain the relationship between $\vec{E}$, $\vec{H}$, and direction of propagation;
- Define intrinsic impedance; and
- Write $\vec{E}$ and $\vec{H}$ fields in phasor and time-varying forms.

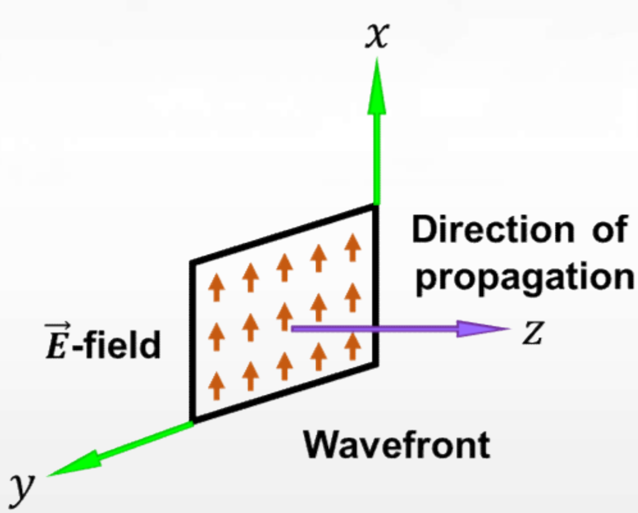
Plane Wave Solution for $\overrightarrow{H}$

In the previous lecture, we obtained

$\vec{E} = \vec{a}_x E_0^+ e^{-jk}$

$\tilde{\vec{E}}(z,t) = \vec{a}_x |E_0^+| \cos(\omega t - kz + \phi)$

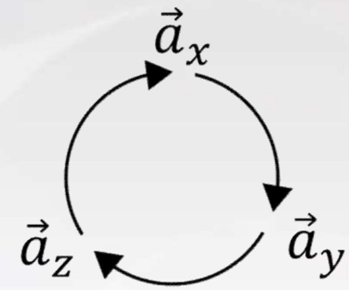
for a uniform plane wave traveling along the $+z$ direction and having E_x only.



What about $\vec{H}$?

Relation Between $\vec{E}$ and $\vec{H}$

From the first Maxwell's equation (Faraday's law)



$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

and $\vec{E} = \vec{a}_x E_0^+ e^{-jk}$, we obtain (see Appendix)

$$\vec{H} = \frac{k}{\omega\mu}\vec{a}_z \times \vec{E}$$

Question: What is the direction of $\vec{H}$ in this case?

Appendix

Intrinsic (or Wave) Impedance

Therefore, the magnetic field of a plane wave traveling in a lossless medium along the positive z direction is

$$\vec{H} = \frac{k}{\omega\mu}\vec{a}_z \times \vec{E} \qquad \boxed{\vec{H} = \vec{a}_z \times \frac{\vec{E}}{\eta}} \qquad \eta[eta]$$

where $\eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}}$ is the **intrinsic (wave) impedance** (!) of the medium.

The value of a medium's intrinsic impedance is dependent upon its material properties (*permittivity* and *permeability* for a lossless medium).

Intrinsic Impedance: Examples

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- Intrinsic impedance $$\eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}}\ \Omega$$

- Example 1: Free-space or air $$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi\ = 377\ \Omega$$

- Example 2: $\varepsilon_r = 9 \qquad \mu_r = 2.25$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{\mu_r}{\varepsilon_r}} = 120\pi\sqrt{\frac{2.25}{9}} = 60\pi\ \Omega$$

Note: η is a real number for lossless media

$\vec{E}$ and $\vec{H}$ Relation in Time-varying Form

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- Phasor Form

$$\vec{E} = E_0^+ e^{-jkz}\vec{a}_x = |E_0^+|e^{j\phi}e^{-jkz}\vec{a}_x$$

$$\vec{H} = \vec{a}_z \times \frac{\vec{E}}{\eta}$$

- Time-Varying Form

$$\tilde{\vec{E}}(z,t) = \vec{a}_x|E_0^+|\cos(\omega t - kz + \phi)$$

$$\tilde{\vec{H}}(z,t) = \vec{a}_z \times \frac{\tilde{\vec{E}}(z,t)}{\eta}$$

Note: Since η is a real number for a lossless medium, the relation $\vec{H} = \vec{a}_k \times \frac{\vec{E}}{\eta}$ can be used for both the phasor form and the time-varying form.

Phasor and Time-varying Forms for $\vec{E}$ and $\vec{H}$

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❑ Phasor Form

$$\vec{E}^{+}(z) = \vec{a}_x |E_0^+| e^{j\phi} e^{-jkz}$$

$$\vec{H}^{+}(z) = \vec{a}_y |H_0^+| e^{j\phi} e^{-jkz}$$

- Complex numbers
- Time dependence is hidden

❑ Time-Varying (Real-Time) Form

$$\tilde{\vec{E}}^{+}(z,t) = \vec{a}_x |E_0^+| \cos(\omega t - kz + \varphi)$$

$$\tilde{\vec{H}}^{+}(z,t) = \vec{a}_y |H_0^+| \cos(\omega t - kz + \varphi)$$

- Real numbers
- Time dependence is explicit

Propagation of Uniform Plane EM Waves

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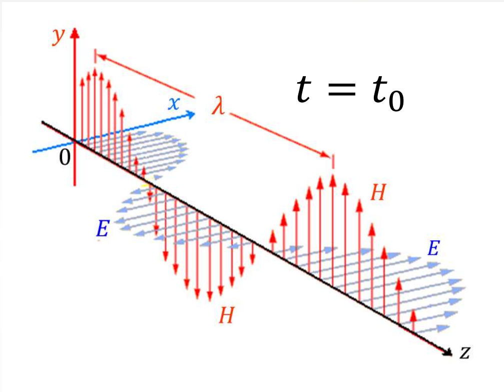
$$\vec{\tilde{E}}^{+}(z,t) = \vec{a}_x |E_0^+| \cos(\omega t - kz + \varphi)$$

$$\vec{\tilde{H}}^{+}(z,t) = \vec{a}_y |H_0^+| \cos(\omega t - kz + \varphi)$$



Notes:

- ❑ $\vec{E}$, $\vec{H}$ and the direction of propagation are mutually perpendicular in a plane wave (**TEM wave**)
- ❑ $\vec{a}_E \times \vec{a}_H = \vec{a}_k =$ direction of propagation

Summary

- The relationship between $\vec{E}$, $\vec{H}$, and direction of propagation: $\vec{a}_E \times \vec{a}_H = \vec{a}_k$;

- Intrinsic impedance for lossless media is $\eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}}$;

- Phasor form of $\vec{E}$ and $\vec{H}$:
  - $\vec{E}^+(z) = \vec{a}_x |E_0^+| e^{j\phi} e^{-jk}$
  - $\vec{H}^+(z) = \vec{a}_y |H_0^+| e^{j\phi} e^{-jkz}$

- Time-varying form of $\vec{E}$ and $\vec{H}$:
  - $\tilde{\vec{E}}^+(z,t) = \vec{a}_x |E_0^+| \cos(\omega t - kz + \varphi)$
  - $\tilde{\vec{H}}^+(z,t) = \vec{a}_y |H_0^+| \cos(\omega t - kz + \varphi)$

Relation Between $\vec{E}$ and $\vec{H}$

From the first Maxwell's equation (Faraday's law)

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

and

$$\vec{E} = E_0^+ e^{-jkz}\vec{a}_x$$

we obtain

Next

$\vec{E}$ and $\vec{H}$ Relation (cont'd)

$$\nabla \times \vec{E} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_0^+ e^{-jk} & 0 & 0 \end{vmatrix} = \vec{a}_y \frac{\partial}{\partial z}\left(E_0^+ e^{-jkz}\right)$$

$$= \vec{a}_y(-jk)E_0^+ e^{-jkz} = -j\omega\mu\vec{H}$$

$$\vec{H} = \frac{-jk\vec{a}_y E_0^+ e^{-jkz}}{-j\omega\mu} = \frac{k}{\omega\mu}\vec{a}_y\ E_0^+ e^{-jkz}$$

$$= \frac{k}{\omega\mu}\vec{a}_z \times \vec{a}_x E_0^+ e^{-jkz} = \frac{k}{\omega\mu}\vec{a}_z \times \vec{E}$$

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Session 15-3

Examples of Plane Waves (Lossless Media) - 2

Learning Objectives

- Use the relationship between $\vec{E}$ and $\vec{H}$ fields to solve examples on plane waves travelling in lossless media.

Quiz

Hayt and Buck, Ch 12, Q4:

The electric field of a plane wave propagating in a non-magnetic medium with $\varepsilon_r = 36$ is given by $\tilde{\vec{E}} = 3\sin(2\pi 10^7 t - 0.4\pi x)\hat{a}_y$ V/m. Determine the magnitude of the magnetic field vector $\tilde{\vec{H}}_m$.

A: $H_m = 0.008\,\mathrm{A/m}$

B: $H_m = 0.024\,\mathrm{A/m}$

C: $H_m = 0.016\,\mathrm{A/m}$

D: $H_m = 0.048\,\mathrm{A/m}$

$$\tilde{\vec{E}}^+(z,t) = \vec{a}_x|E_0^+|\cos(\omega t - kz + \varphi)$$

$$\tilde{\vec{E}}^+(x,t) = \vec{a}_y|E_0^+|\cos(\omega t - kx + \varphi)$$

$$\tilde{\vec{H}} = \vec{a}_k \times \frac{\tilde{\vec{E}}}{\eta}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Plane Wave in a Lossless Medium: Example

A uniform plane wave with $\vec{E} = \vec{a}_x E_x$ travels in a lossless medium $(\boldsymbol{\varepsilon_r = 4, \mu_r = 1, \sigma = 0})$ in the $+z$ direction. Assume E_x is sinusoidally varying with time, with a frequency of $100\ \mathrm{MHz}$, and has a maximum value of $+10^{-4}\ \mathrm{V/m}$ at $t = 0$ and $z = \frac{1}{8}\ \mathrm{m}$. Determine $\tilde{\vec{E}}(z,t)$ and $\tilde{\vec{H}}(z,t)$.

The electric field has the general form:

$$\tilde{\vec{E}}\ (z,t) = \vec{a}_x |E_0^+| \cos(\omega t - kz + \phi)$$

Plane Wave in a Lossless Medium: Solution

$$\omega = 2\pi f = 2\pi \times 10^8 \text{ rad/s}$$

$$k = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r} = \frac{\omega}{c}\sqrt{\mu_r\varepsilon_r} = \frac{2\pi \times 10^8}{3 \times 10^8}\sqrt{4} = \frac{4\pi}{3} \text{ rad/m}$$

The electric field has the general form

$$\tilde{\vec{E}}(z,t) = \vec{a}_x|E_0^+|\cos(\omega t - kz + \phi)$$

$$\tilde{\vec{E}}(z,t) = \vec{a}_x 10^{-4}\cos\left(2\pi \times 10^8 t - \frac{4\pi}{3}z + \phi\right) \quad \text{V/m}$$

where ϕ is an unknown to be determined.

Plane Wave in a Lossless Medium: Solution

$$\tilde{\vec{E}}(z,t) = \vec{a}_x 10^{-4} \cos\left(2\pi \times 10^8 t - \frac{4\pi}{3} z + \phi\right)$$

Since $E(z,t)$ attains its maximum value at $t = 0$ and $z = \frac{1}{8}$ m , we get

$$-\frac{4\pi}{3} z\Big|_{z=1/8} + \phi = 0 \qquad\qquad \phi = \frac{4\pi}{3} \times \frac{1}{8} = \frac{\pi}{6} \text{ rad}$$

Thus

$$\tilde{\vec{E}}(z,t) = \vec{a}_x 10^{-4} \cos\left(2\pi \times 10^8 t - \frac{4\pi}{3} z + \frac{\pi}{6}\right) \text{ V/m}$$

The intrinsic impedance is

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_r}} = \eta_0 \sqrt{\frac{1}{4}} = 60\pi\ \Omega$$

Plane Wave in a Lossless Medium: Solution

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Since $\vec{E}$ is along the x −direction and the wave is travelling in the z-direction, then $\vec{H}$ must be along the y −direction.

Thus, the magnetic field is

$$\tilde{\vec{H}}(z,t) = \vec{a}_y |H_0^+| \cos(\omega t - kz + \phi)$$

$$\tilde{\vec{H}}(z,t) = \vec{a}_y \frac{10^{-4}}{60} \cos\left(2\pi \times 10^8 t - \frac{4\pi}{3} z + \frac{\pi}{6}\right) \text{ A/m}$$

$$\tilde{\vec{H}}(z,t) = \vec{a}_y 5.3 \times 10^{-7} \cos\left(2\pi \times 10^8 t - \frac{4\pi}{3} z + \frac{\pi}{6}\right) \text{ A/m}$$

Summary

- Use the relationship between $\vec{E}$ and $\vec{H}$ fields to solve examples on plane waves travelling in lossless media.