EE2001 Circuit Theory

Introduction to the Laplace Transform

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- Reasons of Using Laplace Transform
- Definition of Laplace Transform
- Properties of Laplace Transform
- The Inverse Laplace Transform
- The Convolution Integral
- Application to Integro-differential Equation

Reasons of Using Laplace Transform (LT)

Main Reasons:

- Integro-differential time-domain complex analysis is avoided; freq-domain analysis involves simple calculations.
- LT can be applied to a wide variety of inputs.
- Initial charges (initial conditions) on capacitors and inductors can be easily incorporated.
- In one single operation, the LT enables us to obtain the total response of the circuit consisting of both natural and forced responses!

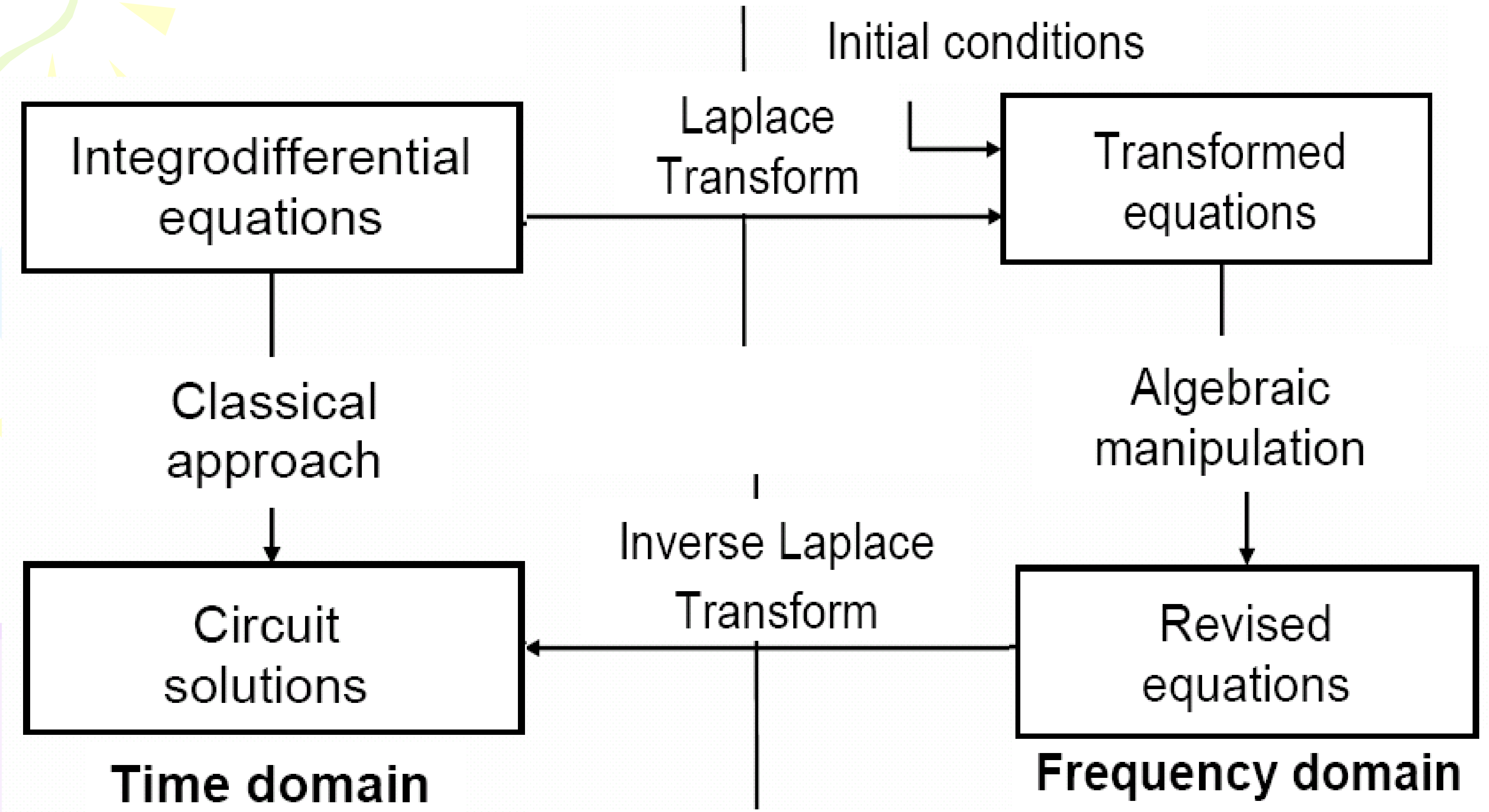
Main Idea of Using LT in Circuit Analysis:

- Transform the circuit from the time-domain to the frequency-domain.

- Obtain the solution of algebraic equations in the frequency-domain.

- Finally apply the inverse LT to the freq-domain results, in order to obtain the solution in the time-domain.

Solution Process of LP and Comparison with That of Classical Approach



Definition of Laplace Transform

Given a time-domain function *f(t)*, its Laplace transform is denoted as *F(s)* or *L[f(t)]* , and given by

$$\boxed{L[\,f(t)] = F(s) = \int_{0^-}^{\infty} f(t)\,e^{-st}\,dt}$$

where " s" is a complex variable given by

$$\boxed{s = \sigma + j\omega}$$

✓ "*s* " has dimensions of frequency and units of "***seconds inverse".***

✓Lower limit 0^{-} in LT equation is used as to indicate a time ***just before*** *t=0.* The idea is to capture singularity functions (which may be discontinuous) at t=0 (thus including initial conditions).

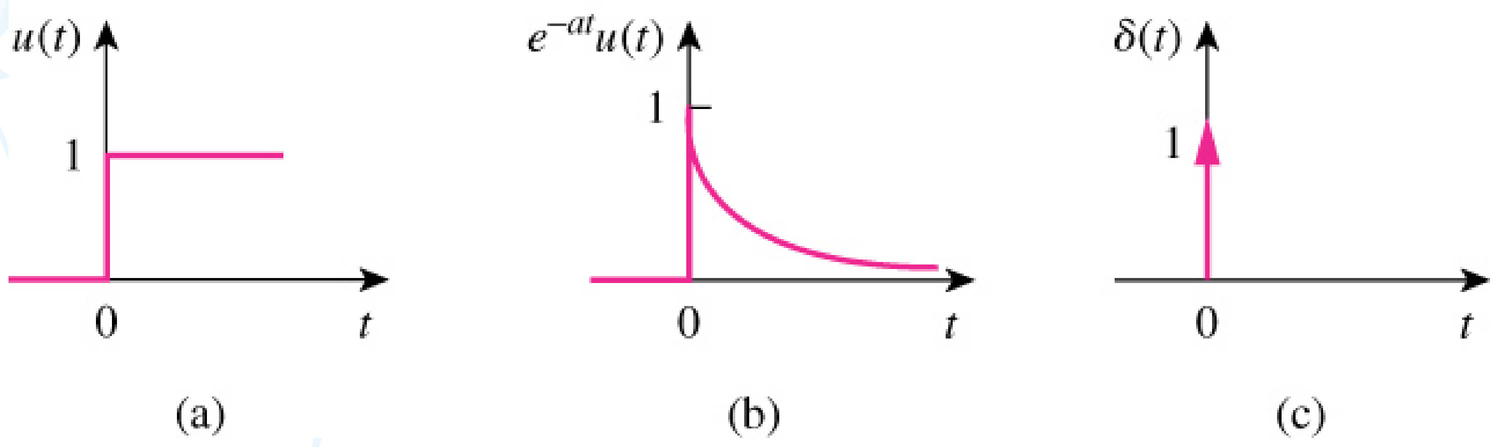
✓Since *f(t)* is ignored for *t<0*, the LT is also known as the unilateral *(1-sided) transform!*

✓Functions $f(t) \Leftrightarrow F(s)$ are called LT pair.

Definition of Laplace Transform

Example 1

Determine the Laplace transform of each of the following functions shown below:



Definition of Laplace Transform

Solution:

a) The Laplace Transform of unit step, *u(t)* is given by

$$L\left[u(t)\right]=\int_{0^-}^{\infty}1e^{-st}dt=-\frac{1}{s}e^{-st}\Big|_{0^-}^{\infty}=\frac{1}{s}$$

Definition of Laplace Transform

Solution:

b) The Laplace Transform of exponential function, $e^{-\alpha t}u(t), \alpha > 0$ is given by

$$L\left[e^{-\alpha t}u(t)\right] = \int_{0^-}^{\infty} e^{-\alpha t} e^{-st} dt = -\frac{1}{a+s} e^{-(a+s)t}\Big|_{0^-}^{\infty} = \frac{1}{s+\alpha}$$

Definition of Laplace Transform

Solution:

c) The Laplace Transform of impulse function, $\delta(t)$ is given by

$$L\left[\delta(t)\right] = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = e^{-0^-} = 1$$

Properties of Laplace Transform

Linearity:

If $F_1(s)$ and $F_2(s)$ are, respectively, the Laplace Transforms of $f_1(t)$ and $f_2(t)$

$$\boxed{L\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1 F_1(s) + a_2 F_2(s)}$$

Example:

$$L\left[\cos(\omega t)u(t)\right] = L\left[\frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)u(t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s - j\omega} + \frac{1}{s + j\omega}\right] = \frac{s}{s^2 + \omega^2}$$

Example:

$$L\left[\sin(\omega t)u(t)\right] = L\left[\frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)u(t)\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right]$$

$$= \frac{\omega}{s^2 + \omega^2}$$

Properties of Laplace Transform

Scaling:

If *F (s)* is the Laplace Transforms of *f (t)*, then

$$\boxed{L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)}$$

Example:

$$L[\sin(2\omega t)u(t)] = \frac{1}{2} \frac{\omega}{\left(\frac{s}{2}\right)^2 + \omega^2}$$

$$= \frac{2\omega}{s^2 + 4\omega^2}$$

Properties of Laplace Transform

Time Shift:

If *F (s)* is the Laplace Transforms of *f (t)*, then

$$\boxed{L\left[f(t-a)u(t-a)\right]=e^{-as}F(s)}$$

Example:

$$L\left[\cos(\omega(t-a))u(t-a)\right]=e^{-as}\frac{s}{s^2+\omega^2}$$

Properties of Laplace Transform

Frequency Shift:

If $F(s)$ is the Laplace Transforms of $f(t)$, then

$$\boxed{L\left[e^{-at} f(t)u(t)\right] = F(s+a)}$$

Example:

$$\boxed{L\left[e^{-at}\cos(\omega t)u(t)\right] = \frac{s+a}{(s+a)^2+\omega^2}}$$

Properties of Laplace Transform

Time Differentiation:

If *F (s)* is the Laplace Transforms of *f (t)*, then the Laplace Transform of its derivative is

$$\boxed{L\left[\frac{df}{dt}u(t)\right] = sF(s) - f(0^-)}$$

Example:

$$L\left[\cos(\omega t)u(t)\right] = L\left[\frac{1}{\omega}\frac{d}{dt}\sin(\omega t)u(t)\right]$$

$$= \frac{1}{\omega}[s\frac{\omega}{s^2+\omega^2} - \sin(0)]$$

$$= \frac{s}{s^2+\omega^2}$$

Properties of Laplace Transform

Time Integration:

If *F (s)* is the Laplace Transforms of *f (t)*, then the Laplace Transform of its integral is

$$\boxed{L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)}$$

Example:

$$L[t] = L\left[\int_0^t u(t)dt\right]$$

$$= \frac{1}{s}\frac{1}{s} = \frac{1}{s^2}$$

In General

$$\boxed{L\left[t^n\right] = \frac{n!}{s^{n+1}}}$$

Properties of Laplace Transform

Frequency Differentiation:

If *F(s)* is the Laplace Transforms of *f (t)*, then the derivative with respect to s, is

$$\boxed{L[tf(t)] = -\frac{dF(s)}{ds}}$$

Example:

$$L\left[te^{-at}u(t)\right] = -\frac{d}{ds}\frac{1}{(s+a)}$$

$$= \frac{1}{(s+a)^2}$$

Properties of Laplace Transform

Initial and Final Values:

The initial-value and final-value properties allow us to find the initial value *f(0)* and *f(∞)* of *f(t)* directly from its Laplace transform *F(s)*.

Initial-value theorem
$$\boxed{f(0) = \lim_{s\to\infty} sF(s)}$$

Final-value Theorem:
$$\boxed{f(\infty) = \lim_{s\to 0} sF(s)}$$

If *f(t)* has a finite limit as *t* tends to ∞, then

Example: Note that for $f(t) = e^{-at}\cos\omega t u(t)$

$$L\left[e^{-at}\cos(\omega t)u(t)\right] = \frac{s+a}{(s+a)^2+\omega^2}$$

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} s\frac{s+a}{(s+a)^2+\omega^2} = 0, \quad if \quad a>0$$

Useful LT pairs

	$f(t)$	$F(s)$
Unit impulse	$\delta(t)$	1
Unit step	$u(t)$	$\frac{1}{s}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Unit ramp	t	$\frac{1}{s^2}$
Polynomial	t^n	$\frac{n!}{s^{n+1}}$
Multiplicative }	te^{-at}	$\frac{1}{(s+a)^2}$
Polynomial }	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
Sine } Zero phase	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
Cosine } Zero phase	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
Sine } Phase θ	$\sin(\omega t+\theta)$	$\frac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
Cosine } Phase θ	$\cos(\omega t+\theta)$	$\frac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
Damped sine	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
Damped cosine	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
	* Defined for $t \geq 0$, $f(t) = 0$ for $t < 0$.	

The Inverse Laplace Transform

Example

Find the inverse Laplace transform of

$$F(s) = \frac{3}{s} - \frac{5}{s+1} + \frac{6}{s^2+4}$$

Solution:

$$f(t) = L^{-1}\left(\frac{3}{s}\right) - L^{-1}\left(\frac{5}{s+1}\right) + L^{-1}\left(\frac{6}{s^2+4}\right)$$

$$= [3 - 5e^{-t} + 3\sin(2t)]u(t)$$

The Inverse Laplace Transform

Suppose F(s) has the general form of

$$F(s) = \frac{\textit{numerator polynomial}}{\textit{denominator polynomial}} = \frac{N(s)}{D(s)}$$

Then finding the inverse Laplace transform of *F(s)* involves two steps:

1. Decompose F(s) into simple terms using partial fraction expansion.
2. Find the inverse of each term by matching entries in Laplace Transform Table.

If $F(s)=\frac{N(s)}{D(s)}$, then solutions of $N(s)=0$ are called zeros of $F(s)$, and solutions of $D(s)=0$ are called poles of $F(s)$

- **Order of Numerator ≥ Order of Denominator**

Use long division to obtain

$$F(s)=\frac{N(s)}{D(s)}=Q(s)+\frac{R(s)}{D(s)}$$

where the degree of $R(s)$ *is less than the degree of* $D(s)$

- **Order of Numerator < Order of Denominator**

Case 1: Simple Real Poles

$$F(s) = \frac{N(s)}{(s+p_1)(s+p_2)\dots(s+p_N)}$$

If $p_i, i = 1, 2, \dots n,$ are all different real numbers, then $-p_i, i = 1, 2, \dots n,$ are simple real poles of $F(s)$

$$\Rightarrow F(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \dots + \frac{k_n}{s+p_n}$$

where

$$k_i = (s+p_i)F(s)\big|_{s=-p_i}$$

$$\Rightarrow f(t) = (k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \dots + k_n e^{-p_n t})u(t)$$

Example

$$F(s)=\frac{96(s+5)(s+12)}{s(s+8)(s+6)}$$

Find inverse LT of *F(s)*, i.e. find *f(t)*.

$$F(s)=\frac{k_1}{s}+\frac{k_2}{s+8}+\frac{k_3}{s+6}$$

$$k_1=sF(s)\Big|_{s=0}=s\left.\frac{96(s+5)(s+12)}{s(s+8)(s+6)}\right|_{s=0}$$

$$=\frac{96\times(0+5)\times(0+12)}{(0+8)\times(0+6)}=120$$

$$k_2 = (s+8)F(s)\Big|_{s=-8} = (s+8)\frac{96(s+5)(s+12)}{s(s+8)(s+6)}\Bigg|_{s=-8}$$

$$= \frac{96\times(-8+5)\times(-8+12)}{-8\times(-8+6)} = -72$$

$$k_3 = (s+6)F(s)\Big|_{s=-6} = (s+6)\frac{96(s+5)(s+12)}{s(s+8)(s+6)}\Bigg|_{s=-6}$$

$$= \frac{96\times(-6+5)\times(-6+12)}{-6\times(-6+8)} = 48$$

$$\Rightarrow F(s) = \frac{120}{s} - \frac{72}{s+8} + \frac{48}{s+6}$$

$$\therefore\ f(t) = (120 - 72e^{-8t} + 48e^{-6t})u(t)$$

Case 2: Repeated Real Poles

If *F(s)* contains *n* repeated poles at *s=-p*, then in its partial fraction expansion, there will be the following *n* terms:

$$F(s) = \ldots + \frac{k_n}{(s+p)^n} + \frac{k_{n-1}}{(s+p)^{n-1}} \ldots + \frac{k_1}{s+p} + \ldots$$

where

$$k_i = \frac{1}{(n-i)!} \frac{d^{(n-i)}}{ds^{(n-i)}} \left[(s+p)^n F(s) \right] \Big|_{s=-p}$$

Then corresponding these *n* terms, *f(t)* contains

$$f(t) = \left(\ldots + k_n \frac{t^{n-1}}{(n-1)!} e^{-pt} + k_{n-1} \frac{t^{n-2}}{(n-2)!} e^{-pt} + \ldots + k_1 e^{-pt} + \ldots \right) u(t)$$

Example

$$F(s) = \frac{100(s+25)}{s(s+5)^3}$$

$$= \frac{k}{s} + \frac{k_3}{(s+5)^3} + \frac{k_2}{(s+5)^2} + \frac{k_1}{s+5}$$

$$k = sF(s)\Big|_{s=0} = s\frac{100(s+25)}{s(s+5)^3}\Bigg|_{s=0} = 20$$

$$k_3 = [(s+5)^3 F(s)]\Big|_{s=-5}$$

$$=(s+5)^3 \frac{100(s+25)}{s(s+5)^3}\Bigg|_{s=-5} = \frac{100(-5+25)}{-5} = -400$$

$$k_2 = \frac{d}{ds}[(s+5)^3 F(s)]\Big|_{s=-5} = \frac{d}{ds}\frac{100(s+25)}{s}\Big|_{s=-5}$$

$$= \frac{d}{ds}[100 + 100\frac{25}{s}]_{s=-5} = -\frac{2500}{s^2}\Big|_{s=-5} = -100$$

$$k_1 = \frac{1}{2!}\frac{d^{(2)}}{ds^{(2)}}[(s+5)^3 F(s)]\Big|_{s=-5} = \frac{1}{2}\frac{d}{ds}[-\frac{2500}{s^2}]\Big|_{s=-5} = \frac{2500}{s^3}\Big|_{s=-5} = -20$$

$$\therefore F(s) = \frac{20}{s} - \frac{400}{(s+5)^3} - \frac{100}{(s+5)^2} - \frac{20}{s+5}$$

$$\Rightarrow f(t) = (20 - 400\frac{t^2}{2!}e^{-5t} - 100\frac{t}{1!}e^{-5t} - 20e^{-5t})u(t)$$

$$= (20 - 200t^2 e^{-5t} - 100te^{-5t} - 20e^{-5t})u(t)$$

Case 3: Distinct Complex Poles

Suppose *F(s)* contains *one* pair of simple complex conjugate poles at $s=-\alpha \pm j\beta$. Then its denominator contains a quadratic factor $(s+\alpha)^2+\beta^2$.

$$F(s)=\frac{A_1 s+A_2}{(s+\alpha)^2+\beta^2}+\ldots$$

$$=\frac{A_1(s+\alpha)}{(s+\alpha)^2+\beta^2}+\frac{A_2-A_1\alpha}{(s+\alpha)^2+\beta^2}$$

Then corresponding these *two* terms, *f(t)* contains

$$f(t)=[A_1 e^{-\alpha t}\cos\beta t+B_1 e^{-\alpha t}\sin\beta t]u(t)+\ldots$$

where

$$B_1=\frac{A_2-A_1\alpha}{\beta}$$

Example

$$F(s)=\frac{20}{(s+3)(s^2+8s+25)}$$

Note that F(s) has a pair of complex poles at *-4±j3*, *i.e* *α=4* *and* *β=3*

$$F(s)=\frac{k_1}{s+3}+\frac{A_1 s+A_2}{(s+4)^2+3^2}$$

$$k_1=(s+3)F(s)\Big|_{s=-3}$$

$$=(s+3)\frac{20}{(s+3)(s^2+8s+25)}\Bigg|_{s=-3}=2$$

To get A_1 and A_2, take two specific values of s, say, 0 and 1, respectively:

When s=0, we have

$$\frac{20}{75} = \frac{2}{3} + \frac{A_2}{25} \quad \Rightarrow 20 = 50 + 3A_2 \quad \Rightarrow A_2 = -10$$

When s=1, we have

$$\frac{20}{4 \times 34} = \frac{2}{4} + \frac{A_1 + (-10)}{34} \quad \Rightarrow 4A_1 = 20 - 2 \times 34 + 4 \times 10, \;\; A_1 = -2$$

$$F(s) = \frac{2}{s+3} - \frac{2s+10}{(s+4)^2 + 3^2} = \frac{2}{s+3} - \frac{2(s+4)}{(s+4)^2 + 3^2} - \frac{2}{3}\frac{3}{(s+4)^2 + 3^2}$$

$$\Rightarrow f(t) = [2e^{-3t} - 2e^{-4t}\cos 3t - \frac{2}{3}e^{-4t}\sin 3t]u(t)$$

Application to Integro-differential Equation

- The Laplace transform is useful in solving linear integro-differential equations.
- Each term in the integro-differential equation is transformed into s-domain.
- Initial conditions are automatically taken into account.
- The resulting algebraic equation in the s-domain can then be solved easily.
- The solution is then converted back to time domain.

Application to Integro-differential Equation

Example:

Use the Laplace transform to solve the differential equation

$$\frac{d^2v(t)}{dt^2}+6\frac{dv(t)}{dt}+8v(t)=2u(t)$$

Given: *v(0)* = 1; *v'(0)* = -2

Solution:

Taking the Laplace transform of each term in the given differential equation and obtain

$$\left[s^2V(s)-sv(0)-v'(0)\right]+6\left[sV(s)-v(0)\right]+8V(s)=\frac{2}{s}$$

Solution:

Taking the Laplace transform of each term in the given differential equation and obtain

$$\left[s^2V(s) - sv(0) - v'(0)\right] + 6\left[sV(s) - v(0)\right] + 8V(s) = \frac{2}{s}$$

Substituting $v(0) = 1;\ \ v'(0) = \text{-}2$, we have

$$\left[s^2V(s) - s + 2\right] + 6\left[sV(s) - 1\right] + 8V(s) = \frac{2}{s}$$

$$(s^2 + 6s + 8)V(s) = s + 4 + \frac{2}{s} = \frac{s^2 + 4s + 2}{s}$$

$$\Rightarrow V(s) = \frac{s^2 + 4s + 2}{s(s^2 + 6s + 8)} = \frac{s^2 + 4s + 2}{s(s+2)(s+4)} = \frac{\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s+2} + \frac{\frac{1}{4}}{s+4}$$

By the inverse Laplace Transform,

$$v(t) = \frac{1}{4}(1 + 2e^{-2t} + e^{-4t})u(t)$$

Summary:

$$\boxed{L[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt}$$

Properties

summarized in Table 15.1 of the textbook

Laplace Transform Pairs

summarized in Table 15.2 of the textbook

Inverse Laplace transform

Using partial fraction and Laplace transform pairs

- Case 1: Simple Real Poles
- Case 2: Repeated Real Poles
- Case 3: Distinct Complex Poles

Application to Integro-differential Equation

Exercises (Problems in Chapter 15 of the textbook)

15.7 ans: $(a)\ \frac{2}{s^2}+\frac{4}{s},(b)\frac{4}{s}+\frac{3}{s+2},(c)\frac{8s+18}{s^2+9},(d)\frac{s+2}{s^2+4s-12}$

15.9 ans: **a)** $\frac{\mathbf{e^{-2s}}}{\mathbf{s^2}}-\frac{\mathbf{2e^{-2s}}}{\mathbf{s^2}}$ **b)** $\frac{2e^{-s}}{e^4(s+4)}$ **c)** $\frac{2.702s}{s^2+4}+\frac{8.415}{s^2+4}$ **d)** $\frac{6}{s}e^{-2s}-\frac{6}{s}e^{-4s}$

15.30 ans: $f_1(t)=\underline{\left[\frac{3}{5}+\frac{27}{5}e^{-t}\cos 2t+\frac{7}{10}e^{-t}\sin 2t\right]u(t)}$ $f_2(t)=\underline{\left[\frac{7}{9}e^{-t}+\frac{2}{3}te^{-t}+\frac{2}{9}e^{-4t}\right]u(t)}$

$f_3(t) = (2e^{-t} - 2e^{-t}\cos(2t) - 2e^{-t}\sin(2t))u(t).$

15.51 ans: $v(t)=\underline{(3e^{-t}+4e^{-2t}-5e^{-3t})u(t)}$

15.55 ans: $\left(\frac{1}{40}+\frac{1}{20}e^{-2t}-\frac{3}{104}e^{-4t}-\frac{3}{65}e^{-t}\cos(2t)-\frac{2}{65}e^{-t}\sin(2t)\right)u(t)$