Discrete Mathematics

MH1812

Topic 6.1 - Linear Recurrence Relations

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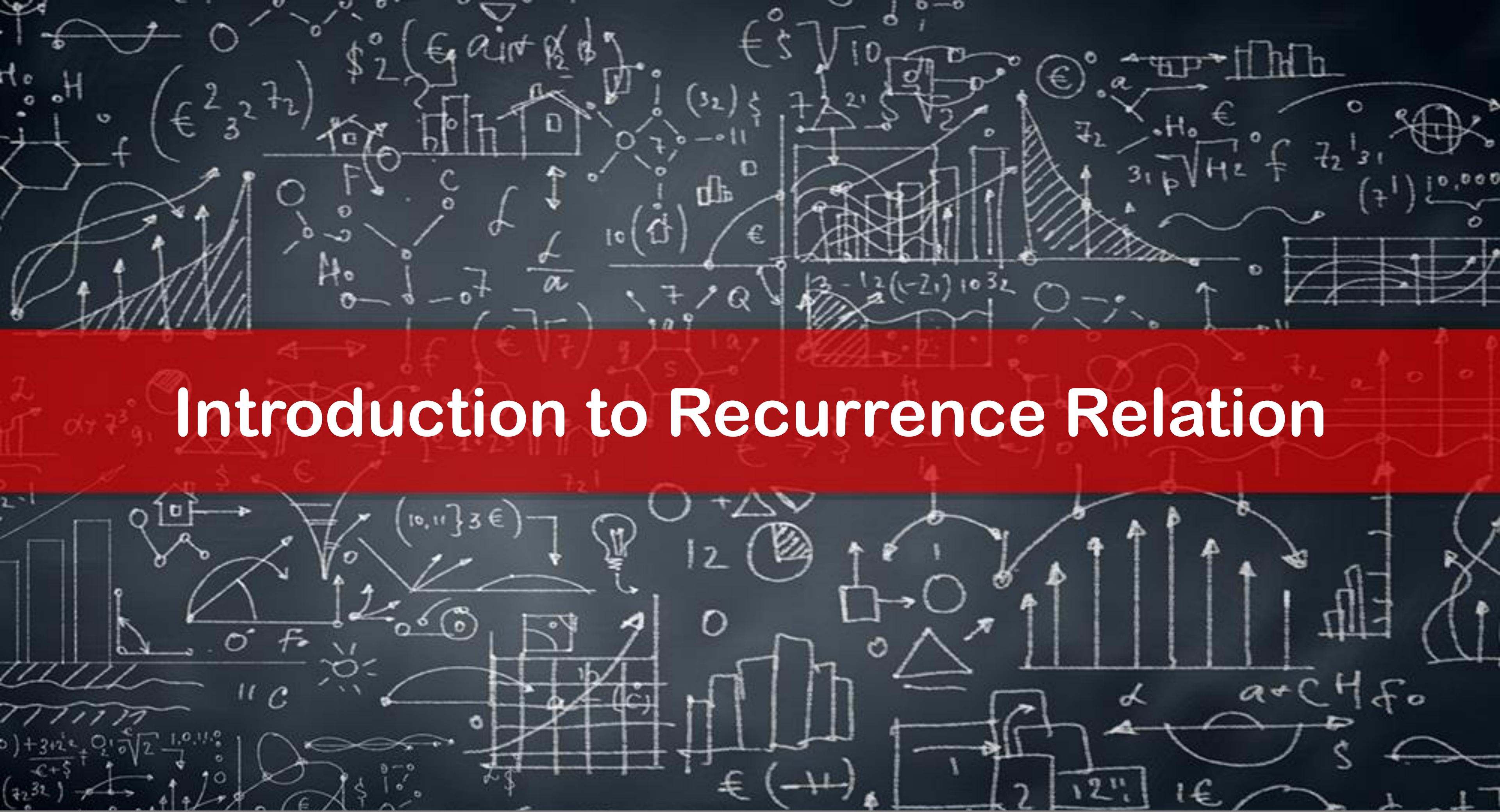
Topic Overview

What’s in store…

- **I**ntroduction to Recurrence Relation
- **B**acktracking
- **C**haracteristic Equation

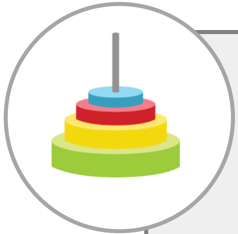
By the end of this lesson, you should be able to…

- Explain what is recurrence relation.
- Use the backtracking method to solve linear recurrences involving initial conditions.
- Use the characteristic equation to solve linear homogenous recurrence.



Introduction to Recurrence Relation

Introduction to Recurrence Relation: Definition

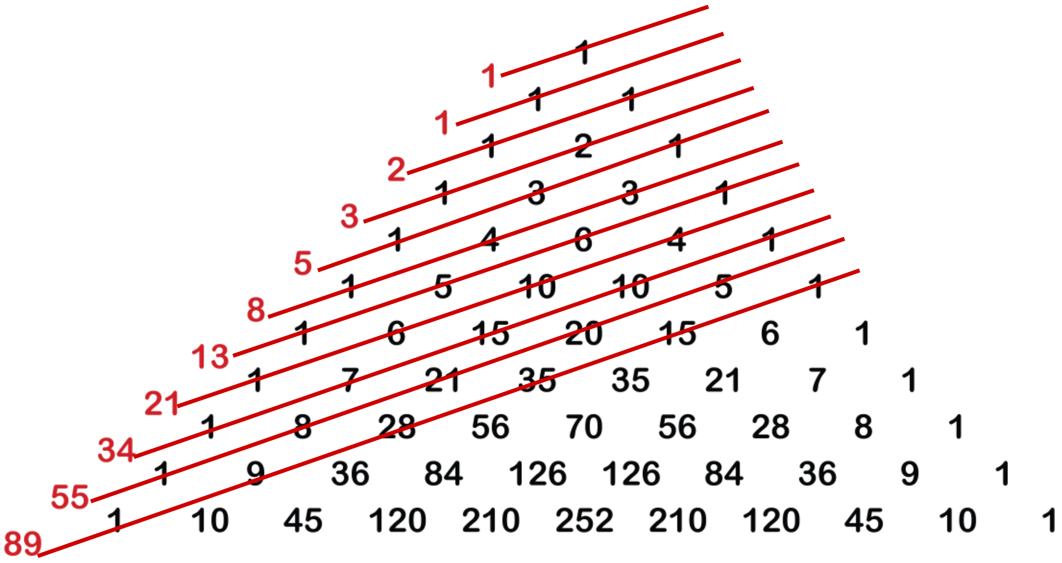


A **recurrence relation** is an equation that **recursively defines a sequence**, i.e., each term of the sequence is defined as a function of the preceding terms.

A recursive formula must be accompanied by **initial conditions** (information about the beginning of the sequence).

Introduction to Recurrence Relation: Fibonacci Sequence

$$f_n = f_{n-1} + f_{n-2} \text{ with } f_0 = 0, f_1 = 1$$



Leonardo Pisano Bigolio
(c. 1170 - c. 1250)

Leonardo Pisano Bigolio under WikiCommons (PD-OLD)

Backtracking

Backtracking: Solving Recurrence Relation

Backtracking is a technique for finding explicit formula for recurrence relation.

Example

$a_n = a_{n-1} + 3$ and $a_1 = 2$

$$
\begin{aligned}
a_n = a_{n-1} + 3 &= (a_{n-2} + 3) + 3 = a_{n-2} + 2*3 \\
&= (a_{n-3} + 3) + 2*3 = a_{n-3} + 3*3 \\
&= (a_{n-4} + 3) + 3*3 = a_{n-4} + 4*3 \\
&\ldots \\
&= a_1 + (n - 1)*3
\end{aligned}
$$

$$a_n = 2 + (n - 1)*3$$

Backtracking: Example

Solve $a_n = 2a_{n-1} - a_{n-2}$ **with initial conditions** $a_1 = 3$, $a_0 = 0$

Characteristic Equation

Characteristic Equation: Homogenous Relation of Degree *d*

A **linear homogeneous relation** of degree d is of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_d a_{n-d}$$

E.g.,:

- The Fibonacci sequence
- The relation: $a_n = 2a_{n-1}$ (degree 1)
- But **not** the relation: $a_n = 2a_{n-1} + 1$

The **characteristic equation** of the above relation is:

$$x^d = c_1 x^{d-1} + c_2 x^{d-2} + \ldots + c_d$$

Characteristic Equation: Theorem

If the characteristic equation $x^2 - c_1x - c_2 = 0$ (of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$) has:

- two distinct roots s_1, s_2, then the explicit formula for the sequence a_n is

$$u*s_1^n + v*s_2^n$$

- a single root s, then the explicit formula for a_n is

$$u*s^n + v*n*s^n$$

where u and v are determined by initial conditions.

Characteristic Equation: Example

Solve $a_n = 2a_{n-1} - a_{n-2}$ **with initial conditions** $a_1 = 3,\ a_0 = 0$

Characteristic Equation: Example

Determine the number of bit strings (i.e., comprising *0*s and *1*s) of length n that contains no adjacent *0*s.

- C_n = the number of such bit strings
- A binary string with no adjacent *0*s is constructed by:
  - Adding "*1*" to any string w of length $n - 1$ satisfying the condition, or
  - Adding "*10*" to any string v of length $n - 2$ satisfying the condition
- Thus $C_n = C_{n-1} + C_{n-2}$ where $C_1 = 2$ $(0,1)$, $C_2 = 3$ $(01, 10, 11)$

Characteristic Equation: Example

Now solve $C_n = C_{n-1} + C_{n-2}$ where $C_1 = 2$, $C_2 = 3$

- **Characteristic equation:** $x^2 - x - 1 = 0$
- **Its roots are:**

$$(1 + \sqrt{5})/2$$

$$(1 - \sqrt{5})/2$$

- **Thus**

$$C_n = u * \left(\frac{1+\sqrt{5}}{2}\right)^n + v * \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Recall roots of quadratic equation

$$a * x^2 + b * x + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Characteristic Equation: Example

$$C_n = \left(\frac{\sqrt{5}+3}{2\sqrt{5}}\right) * \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{\sqrt{5}-3}{2\sqrt{5}}\right) * \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Initial conditions give us:

$$C_1 = u * \left(\frac{1+\sqrt{5}}{2}\right) + v * \left(\frac{1-\sqrt{5}}{2}\right) = 2$$

i.e., $\dfrac{u+v}{2} + \dfrac{(u-v)\sqrt{5}}{2} = 2$

$$C_2 = u * \left(\frac{1+\sqrt{5}}{2}\right)^2 + v * \left(\frac{1-\sqrt{5}}{2}\right)^2 = 3$$

i.e., $\dfrac{3(u+v)}{2} + \dfrac{(u-v)\sqrt{5}}{2} = 3$

Solving this we get:

$$u = \frac{\sqrt{5}+3}{2\sqrt{5}}$$

$$v = \frac{\sqrt{5}-3}{2\sqrt{5}}$$

Topic Summary

Let’s recap…

- Definition of linear recurrence
- Backtracking
- Characteristic equation