NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2019-2020

MH1812 - DISCRETE MATHEMATICS

December, 2019 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **SIX (6)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the end of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This **IS NOT** an **OPEN BOOK** exam.

5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. **(10 marks)**

Decide whether the following argument is valid:

$$\begin{aligned}&T \to (E \vee M);\\&S \to \neg E;\\&T \wedge S;\\&\therefore M.\end{aligned}$$

Solution:This is a valid argument. Since $T \wedge S$ is true, it follows both T and S are true.
Since S and $S \to \neg E$ are true, it follows $\neg E$ is true. Hence E is false.
Since T and $T \to (E \vee M)$ are true, it follows $(E \vee M)$ true.
But E is false and $(E \vee M)$ is true, M must be true. Therefore the conclusion is true.
Hence the argument is valid.

Alternatively, one can use the truth table and look at the critical rows. $\square$

QUESTION 2.

(a) Find the solution of the recurrence relation $a_n = 2a_{n-1} + 1$ with $a_1 = 1$.
(10 marks)

Solution:

$$
\begin{aligned}
a_n &= 2a_{n-1}+1 \\
&= 2(2a_{n-2}+1)+1 \\
&= 2^2a_{n-2}+2+1 \\
&= 2^2(2a_{n-3}+1)+2+1 \\
&= 2^3a_{n-3}+2^2+2+1 \\
&\vdots \\
&= 2^ia_{n-i}+2^{i-1}+2^{i-2}+\cdots+2+1 \\
&\vdots \\
&= 2^{n-1}a_1+2^{n-2}+\cdots+2+1 \\
&= 2^{n-1}+2^{n-2}+\cdots+2+1 \\
&= \frac{2^n-1}{2-1} \\
&= 2^n-1.
\end{aligned}
$$

□

(b) For all $n \geq 1$, prove the following by mathematical induction:

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n} = 2-\frac{n+2}{2^n}.$$

(10 marks)

Solution: Let $P(n): \frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n} = 2-\frac{n+2}{2^n}$.
For $n=1$. $LHS = RHS = 1/2$.
Assume $P(k)$ is true, that is, $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{k}{2^k} = 2-\frac{k+2}{2^k}$. We prove $P(k+1)$ is also true.

$$
\begin{aligned}
LHS &= \frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{k}{2^k}+\frac{k+1}{2^{k+1}} \\
&= 2-\frac{k+2}{2^k}+\frac{k+1}{2^{k+1}} \\
&= 2-\left(\frac{2(k+2)-(k+1)}{2^{k+1}}\right) \\
&= 2-\frac{k+3}{2^{k+1}} \\
&= 2-\frac{(k+1)+2}{2^{k+1}} \\
&= RHS.
\end{aligned}
$$

This proves that $P(k+1)$ is also true. □

QUESTION 3.
In how many ways can a committee of 5 be formed from a group of 11 people consisting of 4 teachers and 7 students if

(a) there is no restriction in the selection? **(5 marks)**

Solution:If no restriction, the number of ways is $\binom{11}{5} = 11!/(5!6!) = 462$. $\square$

(b) the committee must include exactly 2 teachers? **(5 marks)**

Solution:We first select 2 teachers from 4 and then $(5-2)$ students from 7. The number of ways is

$$\binom{4}{2}\binom{7}{3} = 6 \times 35 = 210.$$

$\square$

(c) the committee must include at least 3 teachers? **(5 marks)**

Solution:There are two cases: either 3 teachers or 4 teachers are in the comittee. In the former case, the number of ways is

$$\binom{4}{3}\binom{7}{2} = 4 \times 21 = 84,$$

while in the latter, the number of ways is

$$\binom{4}{4}\binom{7}{1} = 7.$$

Thus, the total number of ways is $84 + 7 = 91$. $\square$

(d) a particular teacher and a particular student cannot be both in the committee? **(5 marks)**

Solution:Let T be the particular teacher and S the particular student. We first find the number of ways to form a committee of 5 which includes both T and S. Such a committee of 5 can be formed by taking $\{T, S\}$ and a subset of 3 from the remaining 9 people. Thus, the number of ways to from a committee of 5 including T and S is $\binom{9}{3} = 84$. Hence the number of ways to form a committee of 5 which does not include both T and S is

$$\binom{11}{5} - \binom{9}{3} = 462 - 84 = 378.$$

$\square$

QUESTION 4. **(12 marks)**

Prove for three sets A, B, and C, if $A \cap C = B \cap C$ and $A \cup C = B \cup C$, then $A = B$.

Solution:

$$\begin{aligned}
A &= A \cap (A \cup C) && \text{(absorption law)}\\
&= A \cap (B \cup C) && \text{(since } A \cup C = B \cup C\text{)}\\
&= (A \cap B) \cup (A \cap C) && \text{(distributive law)}\\
&= (B \cap A) \cup (B \cap C) && \text{(associative law, } A \cap C = B \cap C\text{)}\\
&= B \cap (A \cup C) && \text{(distributive law)}\\
&= B \cap (B \cup C) && \text{(since } A \cup C = B \cup C\text{)}\\
&= B && \text{(absorption law)}
\end{aligned}$$

□

QUESTION 5. **(16 marks)**

The relation R is defined on the set of integers $\mathbb{Z}$ as follows. For all $x, y \in \mathbb{Z}$,

$$x \; R \; y \iff 3 \mid (x^2 - y^2).$$

Determine if R is an equivalence relation, and if so, show the equivalence classes.

Solution:

- Reflexive: $\forall x \in \mathbb{Z}$, $3 \mid x^2 - x^2 = 0$, hence $(x, x) \in R$
- Symmetric: $\forall x, y \in \mathbb{Z}$, if $(x, y) \in R$, then $3 \mid x^2 - y^2$, this implies $x^2 - y^2 = 3k$ for some $k \in \mathbb{Z}$, and $y^2 - x^2 = -3k$ multiple of 3, i.e., $3 \mid y^2 - x^2 \Longrightarrow (y, x) \in R$.
- Transitive: $\forall x, y, z \in \mathbb{Z}$, if $(x, y), (y, z) \in R$, then $3 \mid x^2 - y^2$ and $3 \mid y^2 - z^2$, this implies $x^2 - y^2 = 3k_1$ and $y^2 - z^2 = 3k_2$ for some $k_1, k_2 \in \mathbb{Z}$, hence $x^2 - z^2 = (x^2 - y^2) + (y^2 - z^2) = 3k_1 + 3k_2 = 3(k_1 + k_2)$ multiple of 3, i.e., $3 \mid x^2 - z^2 \Longrightarrow (x, z) \in R$.

Hence R is an equivalence relation, and the equivalence classes are:
$[0] = \{3k \mid k \in \mathbb{Z}\}$, and $[1] = \{3k + 1, 3k + 2 \mid k \in \mathbb{Z}\}$ □

QUESTION 6.
Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, and $f : A \to B$.

(a) How many such functions f are there ? **(4 marks)**

(b) How many such onto functions f are there ? **(6 marks)**

(c) How many such one-to-one functions f are there ? **(4 marks)**

Solution:

(a) $3^4 = 81$.

(b) $\binom{4}{2} \cdot 3! = 36$.

(c) 0.

$\square$

QUESTION 7. **(8 marks)**
In Sam's messy dresser drawer, there is a jumble of 6 red socks, 7 blue socks, 9 green socks, and 5 yellow socks. If Sam grabs a handful of socks without looking at what he's taking, what is the minimum number of socks Sam has to grab in order to guarantee that he has at least 4 socks of the same color?

Solution: $3 \cdot 4 + 1 = 13$. $\square$

END OF PAPER