Dependent Random Weighting

Katherine Goode and Shan Yu

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Introduction

We were interested in learning about resampling methods for irregularly spaced time series data. This led us to read the paper

> *"The Dependent Random Weighting" (2015) by Srijan Sengupta, Xiaofeng Shao, and Yingchuan Wang.*

The paper:

- Introduces a method that assigns random weights to the irregular time series data
- Weights are created using a dependence structure that mimics that of the observed data

Irregular Time Series Data

Irregular time series data can occur in two ways.

1. **Missing Values**: Time series occurs at equally space intervals but not all data points are observed

2. **Unequal Intervals**: Times when the data are observed are generated from a 1-D point process

Dependent Random Weighting

Assign a random weight to each observation

- A stationary time series $\{X_t\}_{t\in\mathbb{Z}}$. And the parameter of interest is $\theta = T(F)$, where T is a given function and F is the marginal distribution of $\{X_t\}$.
- The estimator of θ is $\widehat{\theta_n} = T(F_n)$, where F_n is the empirical distribution function based on observations $\{X_{t_j}\}_{j=1}^n$ and t_j are the time points at which the data are observed.
- The random weighted empirical distribution F_n^* is defined as

$$F_n^*(x) = \sum_{i=1}^{n} w(t_i) I(X_{t_i} \leq x),$$

where $\{w(t_i)\}_{i=1}^n$ are the random weights.

Dependent Random Weighting

- The bth random weighted empirical distribution is

$$F^*_{n,b}(x) = \sum_{i=1}^{n} w_b(t_i) I(X_{t_i} \leq x)$$

  where $\{w_b(t_i)\}_{i=1}^n$ are the bth realization from $w(t)$.
- The bootstrap sample is

$$\widehat{\theta}^*_{n,b,DRW} = T(F^*_{n,b}).$$

- We get $\{\widehat{\theta}^*_{n,b,DRW}\}_{b=1}^B$.
- **Example:** If we are interest in the marginal expectation of X_t, then we have

$$\bar{X}^*_{n,b,DRW} = \sum_{j=1}^{n} w_b(t_j) X_{t_j}.$$

Generating Weights

- Assume the random weights $\{w(t_i)\}_{i=1}^n$ take the form

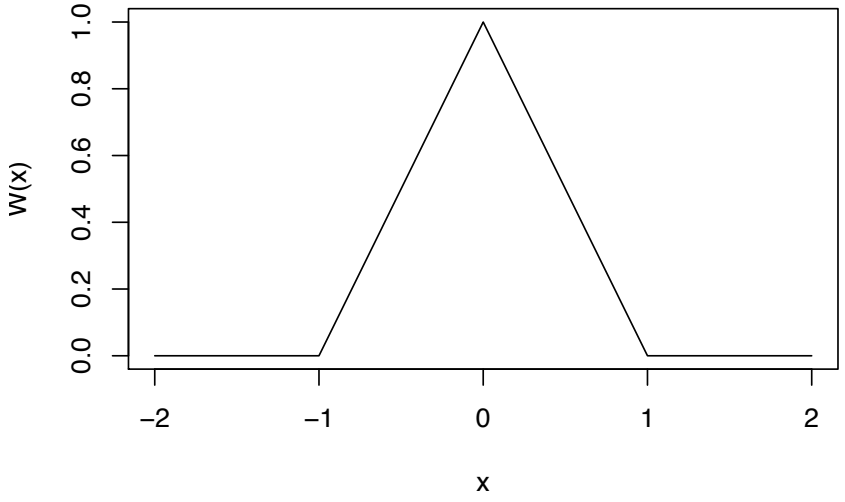
$$w(t_i) = \frac{Z(t_i)}{\sum_{i=1}^n Z(t_i)}.$$

- $Z(t_i)$ are a realization from a **non-negative** and $l-$**dependent** process $Z(t),\ t \in R$.

- l plays a similar role as the block size in the moving block bootstrap.

Generating Weights

Example: $Z(t_i) = (Y(t_i) + c)^2$, where $\{Y(t_i)\}_{i=1}^n \sim N(0, \Sigma)$. Σ is a $n \times n$ matrix with $\Sigma(i,j) = W\left(\frac{t_i - t_j}{l}\right)$, where $W(\cdot)$ is a symmetric kernel function.



$l = 2$

$$\begin{pmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{pmatrix}$$

$l = 3$

$$\begin{pmatrix} 1 & 2/3 & 1/3 & 0 \\ 2/3 & 1 & 2/3 & 1/3 \\ 1/3 & 2/3 & 1 & 2/3 \\ 0 & 1/3 & 2/3 & 1 \end{pmatrix}$$

Equally Spaced Time Series

- **Assumption 1.** $\{Z(t)_j\}_{j=1}^n$ are independent of data and a realization of a stationary process with $cov(Z(t_j), Z(t_j')) = a\{(t_j - t_j')/l\}$, where a is continuous, symmetric and has compact support on $[-1, 1]$.
- **Assumption 2.** There exists a $\delta \geq 2$ such that $\sum_{j=1}^{\infty} \alpha(j)^{\delta/(2+\delta)} < \infty$ and $E\|X_1\|^{2+\delta} < \infty$.
- **Assumption 3.** $\sum_{t_1,t_2,t_3=-\infty}^{\infty} |cum(X_0, X_{t_1}, X_{t_2}, X_{t_3})| \leq \infty$.

Theorem 1. Assume that the function H is differentiable in a neighborhood of μ, under the assumptions in paper, then

$$\sup_{x\in\mathbb{R}} |P[\sqrt{n}\{H(\bar{X}_n) - H(\mu)\} \leq x] - P^*[\sqrt{n}\{H(\bar{X}_{n,DRW}) - H(\bar{X}_n)\}S_Z \leq x]| \to o_p(1),$$

where $S_Z = E(Z(1))/\sqrt{var(Z(1))}$

Irregularly Spaced Time Series

Based on a stochastic sampling design, $t_j = \lambda_n v_j$, where v_j takes values in a Borel subset of $(-1/2, 1/2]$. Let $k = \lim_{n\to\infty} n/\lambda_n$.

Theorem 2. Under the assumptions in the paper, we have

1. if $k \in (0, \infty)$, then

$$\sup_{x\in\mathbb{R}} |P[\sqrt{n}\{\bar{X}_n - \mu\} \le x] - P^*[\sqrt{n}\{\bar{X}_{n,DRW} - \bar{X}_n\}S_Z \le x]| \to o_p(1).$$

2. if $k = \infty$, then

$$\sup_{x\in\mathbb{R}} |P[\sqrt{\lambda_n}\{\bar{X}_n - \mu\} \le x] - P^*[\sqrt{\lambda_n}\{\bar{X}_{n,DRW} - \bar{X}_n\}S_Z \le x]| \to o_p(1).$$

Our Simulations: Overview

We wanted to apply and compare DRW to methods learned in STAT 651. We decided to compare the following situations.

- **Methods**: DRW versus MBB
- **Data**: MA versus AR time series
- **Estimators**: mean versus median
- **Bandwidth**: block size versus l-dependence

Note on irregular data type:

- Paper used unequal time intervals (type 2)
- We used equal time intervals with missing values (type 1)

Our Simulations: The Procedure

We used the following procedure for our simulations.

1. **Generate irregular time series of size $n = 400$.**

   (i) Simulate Y_t for $t = 1, ..., n$ from
   - an MA(2) process with $\mu = 0$, $\theta_1 = -1$, and $\theta_2 = 0.7$ or
   - an AR(2) process with $\mu = 0$, $\phi_1 = -0.1$, or $\phi_2 = 0.6$.

   (ii) Assign a weight ω_t to Y_t where

$$\omega_t = \sin\left(\frac{\pi \cdot t}{n}\right).$$

   (iii) Generate $Z_t \sim \textit{binomial}(\omega_t)$ for $t = 1, ..., n$.

   (iv) Let

$$X_t = \begin{cases} Y_t & \text{if } Z_t = 1 \\ \text{missing} & \text{if } Z_t = 0 \end{cases}$$

   for $t = 1, ..., n$.

   (v) Re-index the non-missing X_t as X_i for i from 1 to n_j and use as the observed sample.

Our Simulations: The Procedure

2. **Let $\ell = 1$, and apply the resampling method to $K = 1000$ samples.**
   - MBB: Draw block bootstrap samples from $X_1, ..., X_{n_j}$ with blocks of size $b = \ell$. (ignores missing values)
   - DRW: Randomly assign weights to $X_1, ..., X_{n_j}$ using the method from the paper assuming m-dependence with $m = \ell$.
3. **Compute the mean and median from the K samples.**
4. **Use the distributions of means and medians to compute evaluative measures.**
   - Determine if the 95% confidence interval contains the true value. (True process medians were approximated using 100,000 Monte Carlo simulations.)
   - Compute the standard deviation of the distribution. (Denote this as $\sigma_{n_j}^{(j)}/\sqrt{n_j}$.)

Our Simulations: The Procedure

5. **Repeat steps 1 to 4 for $M = 500$ times.**

6. **Compute final evaluative measures.**

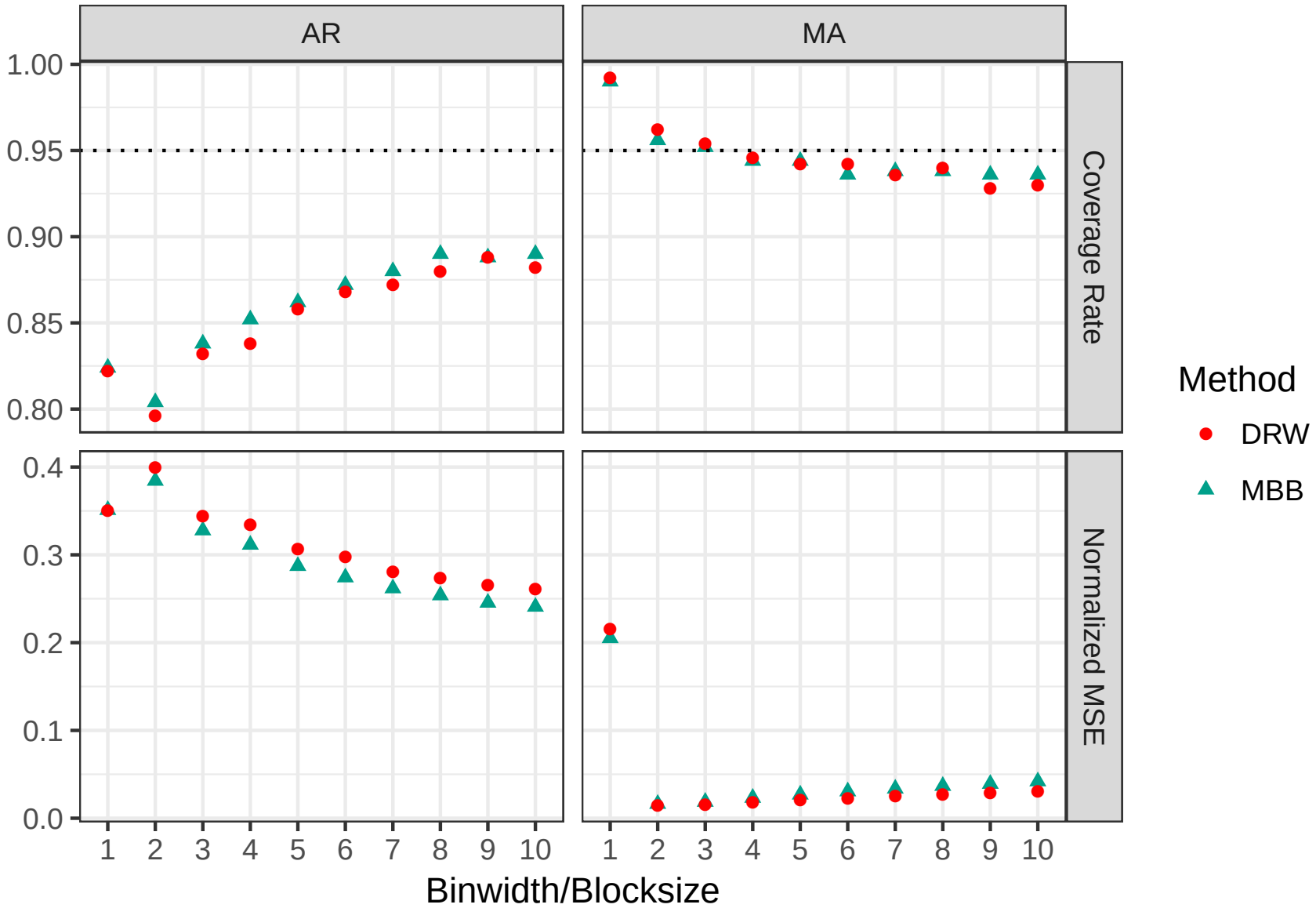
   - Coverage rate for both the mean and median
   - Normalized MSE:

$$\frac{1}{M}\sum_{j=1}^{M}\left(\frac{n_j\sigma_{n_j}^{(j)}}{n\sigma_n}-1\right)^2$$

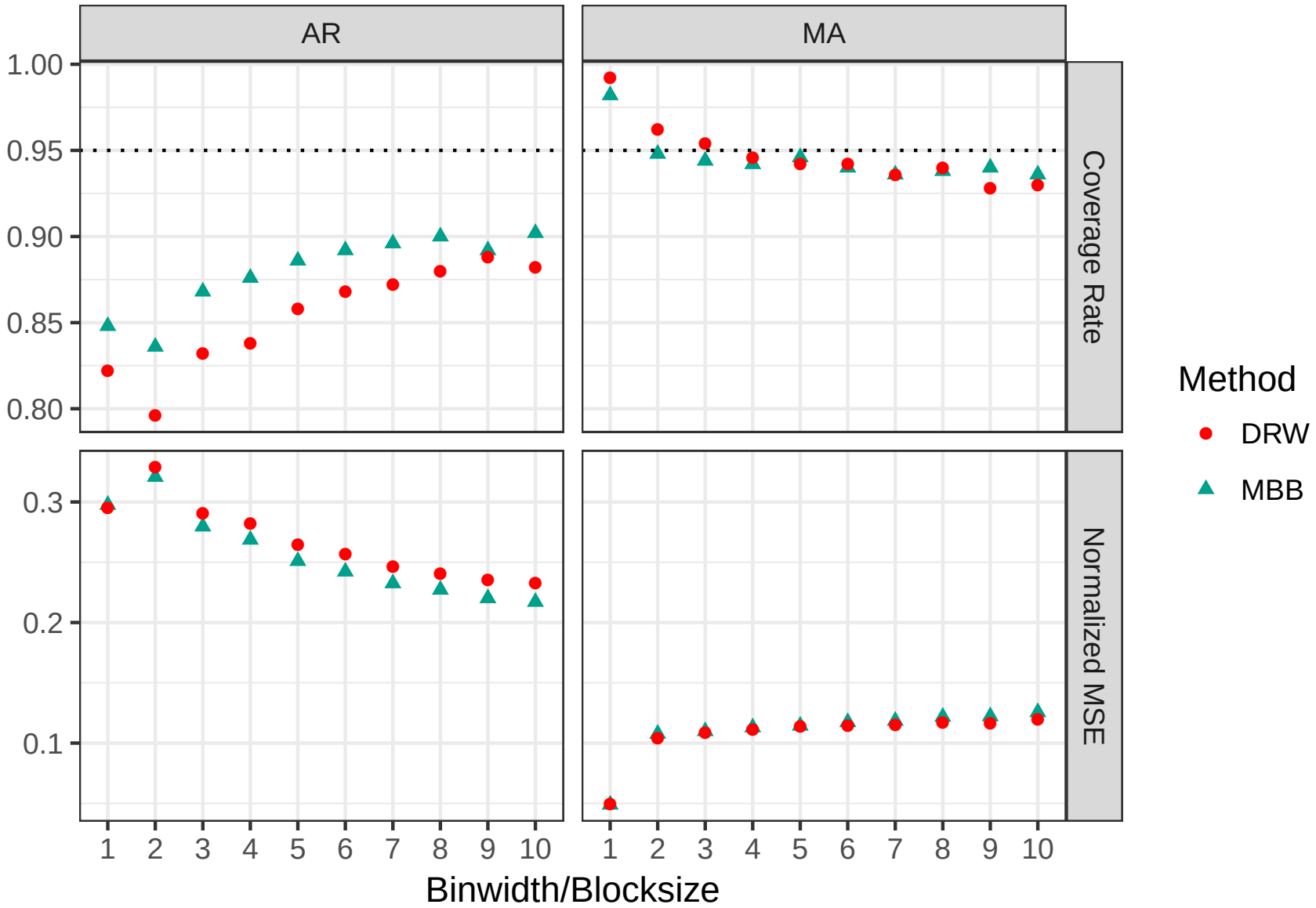
   where $\sigma_n = \sqrt{n}Var(\hat{\theta}_n)$ with $\hat{\theta}_n$ denoting the estimator of interest, was approximated using 100,000 Monte Carlo simulations for both the mean and median

7. **Repeat steps 1 to 6 for $\ell = 2, ..., 10$.**

Our Simulations: Results for Means



Our Simulations: Results for Medians



Our Simulations: Results for Computing Time

We wanted to compare computing times since the paper mentioned that DRW should be easier to implement.

- MBB simulations run on a personal computer (1 core)
- DRW simulations run on the ISU Condo Cluster (10 cores)

We found that the process took much longer for the DRW than the MBB even when run on a more powerful computer.

	MBB (personal computer)	DRW (ISU Condo)
AR	0.42	6.63
MA	0.41	6.48

Table 1: Computing times (in hours) for full simulation process within a category

Conclusions

Our simulations provided us with the following information.

- DRW is a new way to conduct bootstrap
- DRW results were usually similar or worse than MBB results
- DRW took more time than MBB

It would be interesting to run more simulations to consider:

- Would results change if different parameters were used to simulate AR and MA processes?
- How would different amounts or locations of missingness affect the results?
- How much would different sample sizes affect the results?