MBAS 821: Topic 1

Time Value of Money

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Money now vs Money in the future

CANADA

Peterborough man opts for $7M instead of $1,000 a day for life in lottery win

By Greg Davis · Global News
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John Wedlock of Peterborough won the $1,000 a day for life grand prize in the June 28th Daily Grand draw. **OLG**

https://globalnews.ca/news/4314259/peterborough-man-7m-lottery-win/

One-Period Future Value

If you were to invest $10,000 at 5% interest for one year, how much would you have at the end of the year?

One-Period Future Value

If you were to invest \$10,000 at 5% interest rate for one year, your investment would grow to **\$10,500**.

\$10,000 is the principal repayment

\$500 would be interest: \$10,000 $\times$ 0.05

\$10,500 is the total due. It can be calculated as:

$$\$10,500 = \$10,000 \times (1 + 0.05)$$

The total amount due at the end of the investment is called the **Future Value (FV)**. What is the formula for **FV**?

$$FV = PV(1 + r)$$

One-Period Present Value

If you were promised \$10,000, due in one year, when interest rate is 5%. How much would your investment be worth today?

One-Period Present Value

If you were promised \$10,000, *due in one year*, when interest rate is 5%, your investment would worth \$9,523.81 in *today's dollars*.

$$\frac{\$10,000}{1+0.05} = \$9,523.81$$

The amount that a borrower would need to set aside today to be able to meet the promised payment of \$10,000 in one year is called the **Present Value (PV)** of \$10,000. What is the formula for **PV**?

$$PV = \frac{FV}{1+r}$$

Net Present Value (NPV)

- The **Net Present Value (NPV)** of an investment is the present value of the expected (future) cash flows, less the cost of the investment.

$$NPV = -\text{investment} + \frac{CF}{1+r} > 0$$

- Q: Suppose an investment that promises to pay $10,000 in one year is offered for sale for $9,500. Your interest rate is 5%. Should you invest?

Net Present Value (NPV)

Q: Suppose an investment that promises to pay \$10,000 in one year is offered for sale for \$9,500. Your interest rate is 5%. Should you invest?

$$NPV = -\$9,500 + \frac{\$10,000}{1 + 0.05} = \$23.81 > 0$$

A: Yes!

Note: "Your interest rate" is your **required rate of return**, it is possible that different individuals/entities will have **different** required rate of return.

Multiperiod Future Value

You have \$100 to invest in a bank account that pays an interest rate $r = 6\%$ per year, compounding annually. What is the:

- Future value of investment in year 1?
- Future value of investment in year 2?

Multiperiod Future Value

You have \$100 to invest in a bank account that pays an interest rate $r = 6\%$ per year, compounding annually.

- Future value of investment in year 1:

$$FV_1 = \$100 \times (1 + 0.06) = \$106$$

- Future value of investment in year 2:

$$FV_2 = FV_1 \times (1 + 0.06) = \$106 \times 1.06 = \$112.36$$

Generalized formula for future value after t-periods:

$$FV_t = PV \times (1 + r)^t$$

$FV_t =$ Future Value at time t
$PV =$ Present value at time 0
$r =$ Interest rate per compounding period
$t =$ Number of compounding periods

Multiperiod Present Value

You would like to save to purchase a Ferrari 5 years from now, which costs $400,000. With interest rate of 8% per year, compounding annually, how much do you need to save today?

Multiperiod Present Value

$$PV = \frac{FV_t}{(1+r)^t}$$

$PV =$ Present Value at time 0
$FV_t =$ Future Value at time t
$r =$ Interest rate per period
$t =$ Number of compounding periods

$$PV = \frac{\$400,000}{(1+0.08)^5} = \$272,233$$

Discount Factor (DF)

Discount Factor (DF) is the Present Value of a future $1.

Therefore,

$$DF = \frac{1}{(1+r)^t}$$

and

$$PV = FV_t \times DF$$

Exercise: Solve for number of periods

If we deposit $5,000 today in an account paying 10% interest rate, compounding annually. How long does it take to double our money?

Formula Setup

Recall the Present Value formula:

$$PV = \frac{FV_t}{(1+r)^t}$$

We can rearrange the formula to solve for t:

$$\begin{aligned} (1+r)^t &= \frac{FV_t}{PV} \\ \Rightarrow \quad t\log(1+r) &= \log(\frac{FV_t}{PV}) \\ \Rightarrow \quad t &= \frac{\log(\frac{FV_t}{PV})}{\log(1+r)} \end{aligned}$$

Answer

Using the derived formula for t, we can get:

$$\begin{aligned} t &= \frac{\log(\$10,000/\$5,000)}{\log(1+0.1)} \\ &= \log 2/\log 1.1 \\ &= 7.27 \text{ years} \end{aligned}$$

Fun fact:
Compare the answer to the well known *Rule-of-72*, which states that:

$$\text{Years to double} \approx \frac{72}{\text{Interest Rate}}$$

Real Life Example: Solve for Interest Rate

In 1626, Dutch Settlers who were representatives of the West India Company, which was publicly traded at the time, purchased the Island Manhattes from the Indians for a value of 60 guilders (equivalent to $24 at the time).

According to Bloomberg, commercial and residential properties in Manhattan for 2018 was $483.6 billion.

What was the rate of return on this purchase?

Was it a good deal?

Formula Setup

Recall the Present Value formula:

$$PV = \frac{FV_t}{(1+r)^t}$$

We can rearrange the formula to solve for r:

$$\begin{aligned} (1+r)^t &= \frac{FV_t}{PV} \\ \Rightarrow \quad 1+r &= \left(\frac{FV_t}{PV}\right)^{\frac{1}{t}} \\ \Rightarrow \quad r &= \left(\frac{FV_t}{PV}\right)^{\frac{1}{t}} - 1 \end{aligned}$$

Answer

Using the derived formula for t, we can get:

$$
\begin{aligned}
r &= \left(\frac{FV_t}{PV}\right)^{\frac{1}{t}} - 1 \\
&= \left(\frac{\$483,600,000,000}{\$24}\right)^{\frac{1}{2018-1626}} - 1 \\
&= 6.24\%
\end{aligned}
$$

So... Was it worth?

Things we did not account for:

- Cost of developing the Manhattan area.
- Income/Benefits from the Manhattan area.
- Many other stuff...

APR vs EAR

Uptil now we have been assuming that interests are compounded annually. But what if interests are compounded more (or less) frequently?

- **Annual Percentage Rate (APR)** is the stated or quoted annual interest rate without consideration of compounding.
  - e.g. An APR of 12% compounding quarterly means that it will pay 3% per quarter and each compounding period is also 3 months.
  - $1 investment in a 12% APR with quarterly compounding would be worth $(1 + 0.03)^4 = 1.1255$ next year.
- **Effective Annual Rate (EAR)** is the interest rate that is annualized using compound interest.
  - e.g. In the example above, the EAR is 12.55%.

APR vs EAR (conversion)

We can convert between APR and EAR using the following formula:

$$
\begin{aligned}
& (1+EAR)^t = (1+\frac{r}{m})^{mt} \\
\Rightarrow \quad & (1+EAR) = (1+\frac{r}{m})^{m}
\end{aligned}
$$

Where:

- r = Nominal interest rate per year (APR)
- m = Number of compounding periods per year
  - m = 2 if semi-annual, m = 4 if quarterly, m = 12 if monthly, etc.

APR vs EAR Exercise

If you invest $500 for 5 years at 12% APR compounded semi-annually, what will be the future value of your investment be?

- What is the effective annual rate of this investment?

APR vs EAR Answer

$$
\begin{aligned}
FV_5 &= PV \times (1 + \frac{r}{m})^{mt} \\
&= 500 \times (1 + \frac{0.12}{2})^{2 \times 5} \\
&= 500 \times 1.06^{10} \\
&= 895.42
\end{aligned}
$$

The investment will be worth \$895.42 after 5 years.

$$
\begin{aligned}
EAR &= (1 + \frac{r}{m})^{m} - 1 \\
&= 1.06^{2} - 1 \\
&= 12.36\%
\end{aligned}
$$

The effective annual rate is 12.36%.

Continuous Compounding

The general formula for the future value of an investment compounded continuously over t years can be written as:

$$FV_t = PV \times e^{rt}$$

Where:

- PV is the present value of the cash flow at time 0.
- r is the APR.
- t is the number of years over which the cash is invested.
- e is Euler's number ($e \approx 2.718$). e^x is the exponential function and also a key on your calculator.

Note: For those who are interested in the mathematical derivation of the formula, here is a youtube video for that. (Not required for the course)

Short Recap

We learned to:

- Compare cash flows that occur at different points in time.
- Determine economically equivalent future values from values that occur in previous periods through compounding.
- Determine economically equivalent present values from cash flows that occur in the future through discounting.

Back to the lottery

CANADA

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John Wedlock of Peterborough won the $1,000 a day for life grand prize in the June 28th Daily Grand draw. **OLG**

Which one would you pick?

Perpetuities and Annuities

Perpetuity

- A constant stream of cash flows that continues forever.

Growing Perpetuity

- A stream of cash flows that grows at a constant rate and continues forever.

Annuity

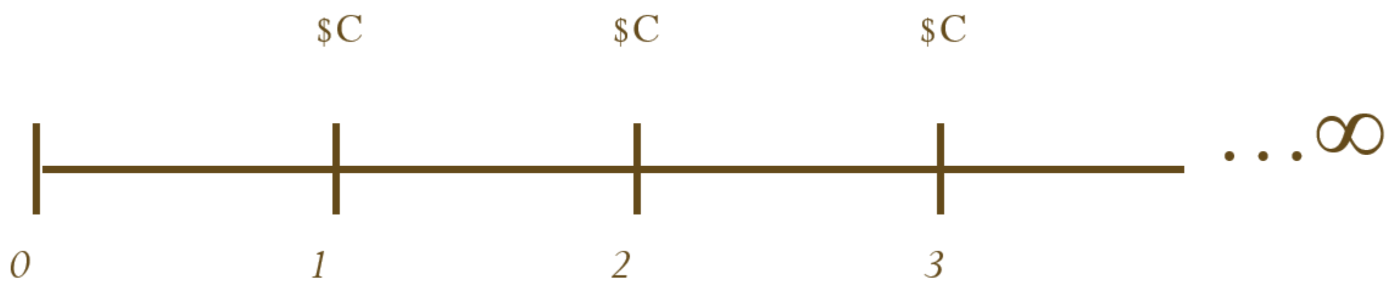
- A stream of constant cash flows that lasts for a fixed number of periods.

Growing Annuity

- A stream of cash flows that grows at a constant rate for a fixed number of periods.

Perpetuities

- A constant stream of cash flows that continues forever.

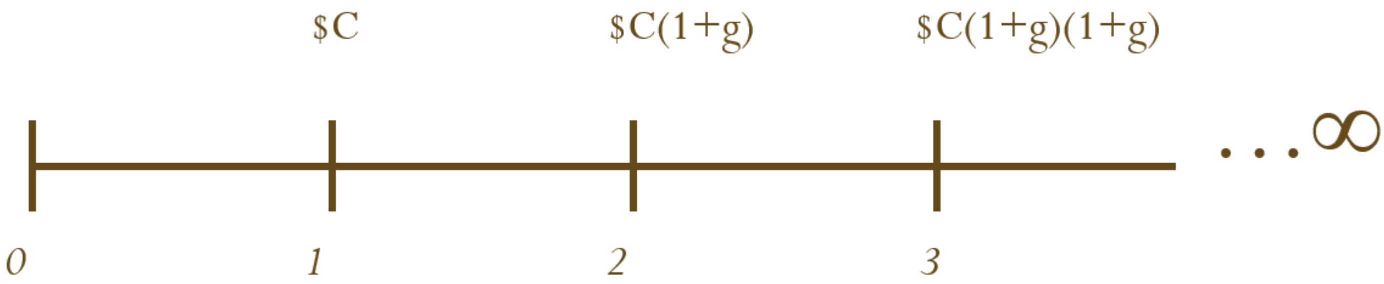


$$PV_0 = \frac{C}{r}$$

Note: We are calculating the present value at *year 0*, but the first cash flow comes in *year 1*.

Growing Perpetuities

- A stream of cash flows that grows at a constant rate g forever.



$$PV_0 = \frac{C}{r-g}$$

Perpetuity Exercise

The expected dividend next year is $1.30 and the same amount of dividend is expected to be paid every year in the future. The discount rate is 10%.

- What is the present value of this dividend stream?
- If the dividends are expected to grow at 5% every year forever (instead of being the same every year), what is the present value of the dividend stream?

Perpetuity Exercise Answers

$1.30 every year:

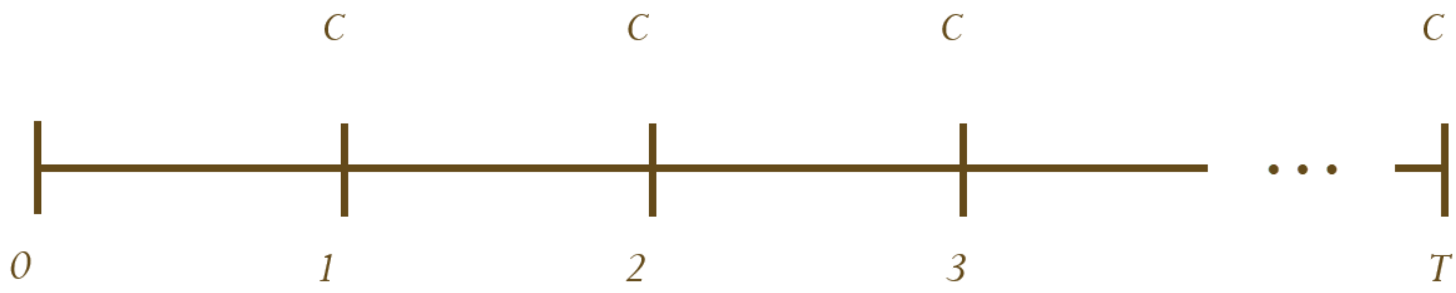
$$PV = \frac{C}{r} = \frac{1.30}{0.1} = \$13$$

$1.30 at year 1 and grows at 5% each year forever:

$$PV = \frac{C}{r-g} = \frac{1.30}{0.1-0.05} = \$26$$

Annuity

A constant stream of cash flows with a fixed maturity t:



$$PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots + \frac{C}{(1+r)^t}$$

The formula for the present value of an annuity is:

$$PV = \frac{C}{r}\left[1 - \frac{1}{(1+r)^t}\right]$$

Growing Annuity

A growing annuity is a stream of cash flows growing at rate g with a fixed maturity t.

$$PV = \frac{C}{1+r} + \frac{C \times (1+g)}{(1+r)^2} + \ldots + \frac{C \times (1+g)^{t-1}}{(1+r)^t}$$

The formula for the present value of a growing annuity is:

$$PV = \frac{C}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^t\right]$$

Annuity Exercise

A retirement plan will payout $20,000 at the end of every year for the next 40 years. The discount rate is 10%.

- What is the present value of the retirement plan?
- If the retirement plan payout increases by 3% every year, starting at $20,000 in the first year, what is the present value of the retirement plan?

Annuity Exercise Answers

PV of annuity retirement plan:

$$
\begin{aligned}
PV &= \frac{C}{r}\left[1-\frac{1}{(1+r)^t}\right] \\
&= \frac{20,000}{0.1}\left[1-\frac{1}{(1+0.1)^{40}}\right] \\
&= \$195,581
\end{aligned}
$$

PV of growing annuity retirement plan:

$$
\begin{aligned}
PV &= \frac{C}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^t\right] \\
&= \frac{20,000}{0.1-0.03}\left[1-\left(\frac{1+0.03}{1+0.1}\right)^{40}\right] \\
&= \$265,122
\end{aligned}
$$

Tips for Solving Time Value of Money Problems

- Draw a timeline that precisely identifies when cash flows occur.
- Find an anchor - either the present value or future value of the cash flow stream.
- Make sure that your discount rate matches the compounding and payment frequencies.
  - Do not confuse APR with EAR
  - Maintain consistency between the interest rate and the number of periods.
- Be careful: The perpetuity and annuity formulas assume that the first payment occurs at time $t = 1$.

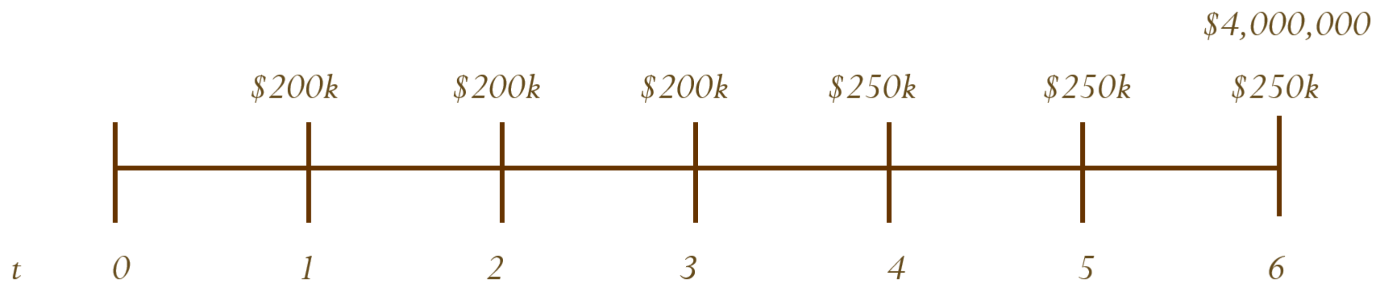
Example: More Complicated Cash Flows

You are planning to buy a property. You think that you can sell it in six years for $4,000,000. You also expect to earn rent of $200,000 per year for the first 3 years (starting at the end of this year), and $250,000 for the following 3 years. The interest rate is 8%. How much would you be willing to pay for it today?

Hint: you can value any complicated stream of cash flows by splitting that stream into separate cash flows, annuities, and/or perpetuities.

Answer

First, draw a timeline for the cash flows:



The brute force way (Sum up the PV of each cash flow):

$$PV = \frac{200,000}{(1+0.08)} + \frac{200,000}{(1+0.08)^2} + \frac{200,000}{(1+0.08)^3} + \frac{250,000}{(1+0.08)^4} + \frac{250,000}{(1+0.08)^5} + \frac{4,250,000}{(1+0.08)^6}$$

Answer (cont'd)

Or, we can break down the cash flows into:

1. An annuity from year 1 to 3 for $200,000 each year.
2. An annuity from year 4 to 6 for $250,000 each year.
3. A cash flow of $4,000,000 in year 6.

$$PV = \frac{200,000}{0.08}\left[1 - \frac{1}{(1+0.08)^3}\right] + \frac{250,000}{0.08}\left[1 - \frac{1}{(1+0.08)^3}\right]\left[\frac{1}{(1+0.08)^3}\right] + \frac{4,000,000}{(1+0.08)^6}$$

Both methods should yield the same result:

$$PV = \$3,547,544$$

In Summary

We now know how to:

- Compare cash flows that occur at different points in time.
- Determine economically equivalent future values from values that occur in previous periods through compounding.
- Determine economically equivalent present values from cash flows that occur in the future through discounting.
- Find present value of perpetuities and growing perpetuities.
- Find present value and future values of annuities and growing annuities.

Formula Sheet

$$PV = \frac{FV_t}{(1+r)^t}$$

$$FV_t = PV(1+r)^t$$

$$PV(\text{Perpetuity}) = \frac{C}{r}$$

$$PV(\text{Annuity}) = \frac{C}{r}\left[1 - \frac{1}{(1+r)^t}\right]$$

$$PV(\text{Growing Perpetuity}) = \frac{C}{r-g}$$

$$PV(\text{Growing Annuity}) = \frac{C}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^t\right]$$

$$1 + EAR = (1 + \frac{APR}{m})^m$$

$$FV_t(\text{Continuous Compounding}) = e^{rt}$$