Computer Vision, Homework 4
Leon Lixing Yu

1 Parametrizing Curves

1.1 Find $\alpha(s)$

$$\alpha(s) = (\frac{1}{k}cos(s), \frac{1}{k}sin(s))$$

$$s \in [-\pi, \pi]$$

$$\alpha'(s) = (-\frac{1}{k}sin(s), \frac{1}{k}cos(s))$$

$$\|\alpha'(s)\| = \sqrt{sin^2(s) + cos^2(s)} = \frac{1}{k}$$

1.2 Find Frenet approximation to the circle

$$\alpha''(s) = (-\frac{1}{k}cos(s), -\frac{1}{k}sin(s))$$

$$\alpha(s_0 + s) = \alpha(s_0) + s(\alpha'(s_0)) + \frac{s^2}{2}\alpha''(s_0)$$

From here we can sub in α, α', and α''.

1.3 Plot magnitude error

Before plotting the magnitude, I wants to plot the scatter graph to get a better scene of what's going on, see figure below.

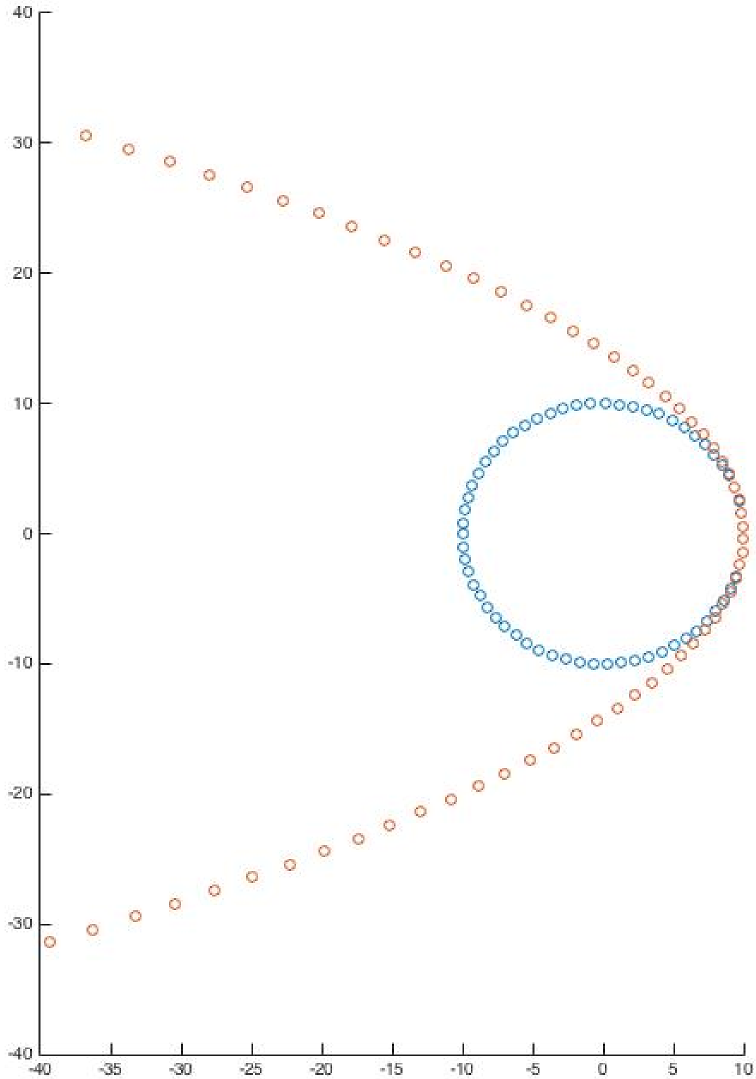

Figure 1: The blue circle is our scatter of $\alpha(s)$, and the red parabola is our scatter of frenet approximation. $s \in [-\pi, \pi]$ In the picture shown, the parabola lies in the plane through $\alpha(s_0)$ orthogonal to $\alpha(s_0)$

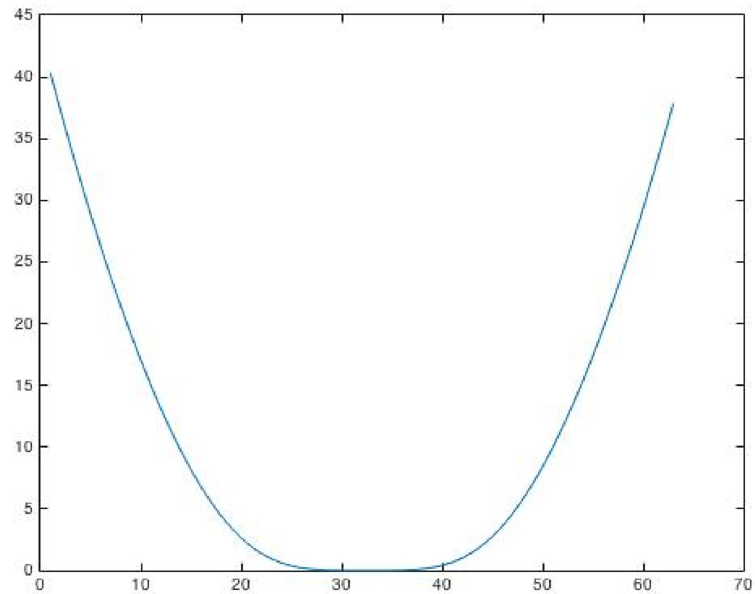

Figure 2: The x-axis is the number of data points, and y-axis is the magnitude of error, we see that the peak error is around 40 when curvature is 0.1, meaning radius is 10

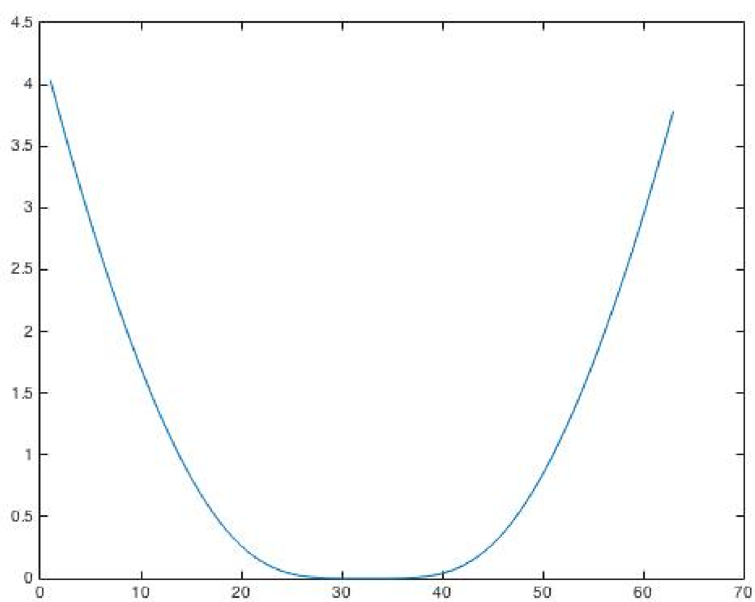

Figure 3: The x-axis is the number of data points, and y-axis is the magnitude of error, we see that the peak error is around 4 when curvature is 1, meaning radius is 1

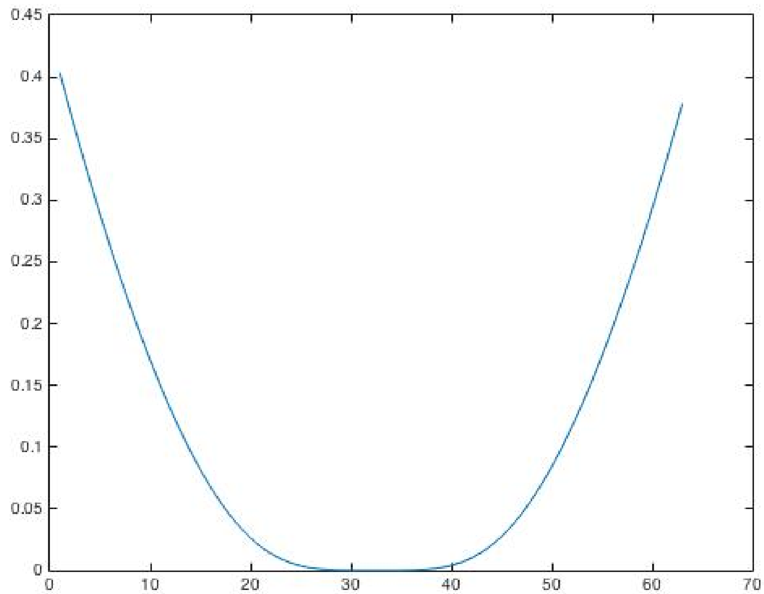

Figure 4: The x-axis is the number of data points, and y-axis is the magnitude of error, we see that the peak error is around 0.4 when curvature is 10, meaning radius is 0.1

The results show that the error is inverse-proportional to the curvature. However, I think the magnitude does not change the fact that the frenet approximation is the best quadratic approximation to α at s_0.

2 Geometric Insights into Shape from Texture

2.1 calculate F_{*p}

We have:

$$F_{*p}(\boldsymbol{v}) = \frac{d}{du}(F(\alpha(u))), u = 0$$

Given:

$$F(\boldsymbol{p}) = r(\boldsymbol{p})\boldsymbol{p}$$

$$\Rightarrow \frac{d}{du}(F(\alpha(u))) = \frac{d}{du}(r(\alpha(u))\alpha(u))$$

$$= r'((\alpha(u))\alpha(u) + r(\alpha(u))\alpha'(u)$$

We also know that:

$$\alpha(u) = \boldsymbol{p}, \alpha'(u) = \boldsymbol{v}, u = 0$$

Therefore, we have:

$$\Rightarrow F_{*p}(\boldsymbol{v}) = \nabla r \, \boldsymbol{p} + r \, \boldsymbol{v}$$

2.2 Prove argmax

If $\boldsymbol{v} = \nabla r / \|\nabla r\|$, we know that $\boldsymbol{v}$ is tangential to r, so the angle between r and $\boldsymbol{v}$ is 0. Plug it back to the equation above:

$$\|F_{*p}(\boldsymbol{v})\| = \|\nabla r\ \boldsymbol{p} + r(\nabla r/\|\nabla r\|)cos(0)\|$$

This form will clearly reach optima as $cos(0)$ equal to 1.

If v is tangential to r, we can just move v from viewsphere at p to our surface s.

2.3 Show $F_*(t)$ is orthogonal to $F_*(b)$

Given that $b = p \times t$, we know for sure that b is orthogonal to t.

We have representation of $F_*(\boldsymbol{t})$:

$$F_*(\boldsymbol{t}) = \nabla r \boldsymbol{p} + r\ \boldsymbol{t}$$

Since F_* is a tangent map from viewsphere to the surface, and $F_*(\boldsymbol{t})$signifies the direction it is moving, when $t = \nabla\ r/\|\nabla\ r\|$, F_* is moving tangentially to the surface.
Similarly, $F_*(b)$ is moving in the direction of b. Since b is orthogonal to t, we can say that $F_*(t)$ is orthogonal to $F_*(b)$.

2.4 Show N = T x B

Given $\boldsymbol{T}$, $\boldsymbol{B}$, and $\boldsymbol{t}$ as described,

$$N = \boldsymbol{T} \times \boldsymbol{B}$$

We know that:

$$F_*(t) = \nabla r\ \boldsymbol{p} + r\ \boldsymbol{t}$$

$$F_*(b) = \nabla r\ \boldsymbol{p} + r\ \boldsymbol{b}$$

We also know that:

$$\boldsymbol{b} = \boldsymbol{p} \times \boldsymbol{t} = \|\boldsymbol{p}\|\ \|\boldsymbol{t}\|\ sin(\pi/2) = \|\boldsymbol{p}\|\ \|\boldsymbol{t}\|$$

Therefore, we have:

$$F_*(t) \times F_*(b) = (\nabla r\ \boldsymbol{p} + r\ \boldsymbol{t}) \times (\nabla r\ \boldsymbol{p} + r\|\boldsymbol{p}\|\ \|\boldsymbol{t}\|)$$

Since $\boldsymbol{p}$ and $\boldsymbol{t}$ are unite vectors, their norm is 1.

$$F_*(t) \times F_*(b) = (\nabla r\ \boldsymbol{p} + r\ \boldsymbol{t}) \times (\nabla r\ \boldsymbol{p} + r)$$

$$= \|\nabla r\ \boldsymbol{p} + r\ \boldsymbol{t}\|\ \|\nabla r \boldsymbol{p} + r\|$$

$$= \|\nabla r\|\ \boldsymbol{t} + r\boldsymbol{p}$$

Since t and p are unit vectors, the norm of them are going to be 1, hence:

$$\|F_*(t) \times F_*(b)\| = \sqrt{\|\nabla\ r\|^2 + r^2}$$

$$\Rightarrow \boldsymbol{N} = \frac{\|\nabla r\|\ \boldsymbol{t} + r\boldsymbol{p}}{\sqrt{\|\nabla\ r\|^2 + r^2}}$$

2.5 examine the angle σ

Given:

$$\boldsymbol{N} = \frac{\|\nabla r\| \ \boldsymbol{t} + r\boldsymbol{p}}{\sqrt{\|\nabla \ r\|^2 + r^2}}$$

Since $\boldsymbol{N}$ and $\boldsymbol{p}$ are unit vectors, norm of them are 1. Therefore, we have:

$$sin\sigma = \|\boldsymbol{N} \times \boldsymbol{p}\|$$

$$= \frac{(\|\nabla r\| \ \boldsymbol{t} + r\boldsymbol{p}) \times \boldsymbol{p}}{\sqrt{\|\nabla \ r\|^2 + r^2}}$$

We know that $\boldsymbol{t} \times \boldsymbol{p} = 1$ and $\boldsymbol{p} \times \boldsymbol{p} = 0$, so this yields to:

$$\Rightarrow sin\sigma = \frac{(\|\nabla r\|}{\sqrt{\|\nabla \ r\|^2 + r^2}}$$

We know that dot product of two orthogonal vectors is 0, and dot product of two identical vector is the 2 norm.

$$cos\sigma = \boldsymbol{N} \cdot \boldsymbol{p} = \frac{(\|\nabla r\| \ \boldsymbol{t} + r\boldsymbol{p}) \cdot \boldsymbol{p}}{\sqrt{\|\nabla \ r\|^2 + r^2}}$$

$$\Rightarrow cos\sigma = \frac{r}{\sqrt{\|\nabla \ r\|^2 + r^2}}$$

With sinσ and cosσ solved, we can easily get tan σ:

$$tan\sigma = \frac{sin\sigma}{cos\sigma}$$

$$\Rightarrow tan\sigma = \frac{\|\nabla r\|}{r}$$

2.6 Discuss slant and tilt

It is much easier to show the distortion between the viewshere and our surface via slant and tilt. The angle, σ acts as a map to convert p and t to N and T. It is computationally much easier than using $\mathbb{R}^3$ coordinates. The ellipse in Figure 1 can be represented as a collection of N and T that is coverted via σ.

2.7 F_*

$$F_* = \nabla r \ \boldsymbol{p} + r \ \boldsymbol{t}$$

$$= \nabla r \ (\boldsymbol{t} \times \boldsymbol{b}) + r \ \boldsymbol{t}$$

$$N = (\boldsymbol{b} \times \boldsymbol{t})cos\sigma - \boldsymbol{t}sin\sigma$$

$$T = (\boldsymbol{b} \times \boldsymbol{t})sin\sigma + \boldsymbol{t}cos\sigma$$

Sorry, I don't quite understand the question, I understand the panelty.

$$m = cos\sigma/r$$

$$M = 1/r$$

2.8 t, b, m, and M

The larger M and m is, the more distortion we will obtain from viewsphere.