Part 14
Frenet Formulas, continued

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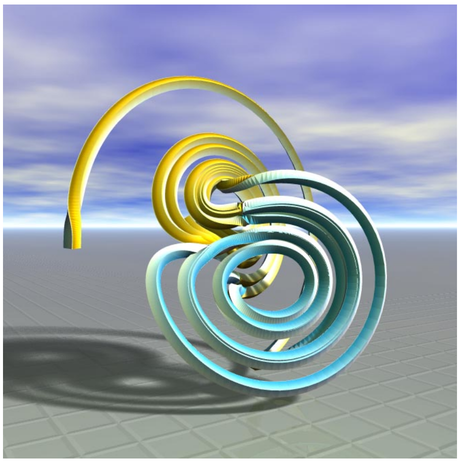
14.1

Overview

Contents



14.2

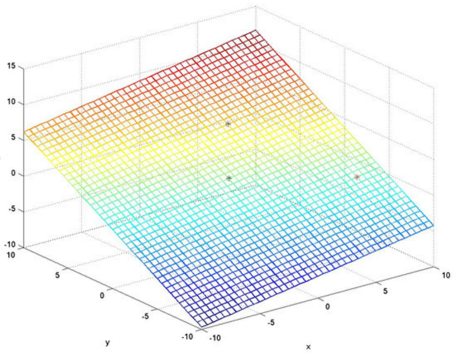
1 Frenet Approximation

Planes

We will think of a plane in $\mathbb{R}^3$ as the set of all points that are orthogonal to some tangent vector.

Definition
Let p, q be points in $\mathbb{R}^3$, with $q \neq \vec{0}$.
Then the *plane through p orthogonal to q* is the set of all points r in $\mathbb{R}^3$ such that

$$(r - p) \bullet q = 0.$$



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Taylor approximation of β

Taylor approximation of Euclidean coordinate functions
Given the unit-speed curve $\beta = (\beta_1, \beta_2, \beta_3)$,

$$\beta_i(s) \sim \beta_i(0) + \frac{d\beta_i}{ds}(0)s + \frac{d^2\beta_i}{ds^2}(0)\frac{s^2}{2!} + \frac{d^3\beta_i}{ds^3}(0)\frac{s^3}{3!}$$

are the first four terms of the Taylor series for β_i at the point $s = 0$ for $i = 1, 2, 3$.

The $\sim$ means that the difference between the left hand side and the right hand side converges to 0 as s tends to 0.

Taylor approximation of the curve
From the approximations of the Euclidean coordinate functions we get an approximation

$$\beta(s) = (\beta_1(s), \beta_2(s), \beta_3(s)) \sim \beta(0) + \beta'(0)s + \beta''(0)\frac{s^2}{2} + \beta'''(0)\frac{s^3}{6}.$$

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Applying the Frenet formulas to the approximation

Substituting derivatives
We defined $T(s) = \beta'(s)$, so

$$\beta'(0) = T(0) = T_0.$$

We also defined $N(s) = T'(s)/\kappa(s) = \beta''(s)/\kappa(s)$, so

$$\beta''(0) = \kappa(0)N(0) = \kappa_0 N_0.$$

To calculate $\beta'''(0)$ we can use the Leibniz rule:

$$\beta''' = \frac{d(\kappa N)}{ds} = \frac{d\kappa}{ds}N + \kappa N'.$$

From the Frenet formulas we have $N' = -\kappa T + \tau B$. Now

$$\beta'''(0) = -\kappa_0^2 T_0 + \frac{d\kappa}{ds}(0)N_0 + \kappa_0\tau_0 B_0,$$

where $\tau_0 = \tau(0)$ and $B_0 = B(0)$.

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The Frenet approximation

We have calculated:

$$\beta(s) \sim \beta(0) + \beta'(0)s + \beta''(0)\frac{s^2}{2} + \beta'''(0)\frac{s^3}{6},$$

$$\beta'(0) = T_0, \;\; \beta''(0) = \kappa_0 N_0, \;\; \beta'''(0) = \kappa_0^2 T_0 + \frac{d\kappa}{ds}(0)N_0 + \kappa_0\tau_0 B_0.$$

Approximating $\beta(s)$ as a linear combination of T_0, N_0, B_0 :

$$\beta(s) \sim \beta(0) + (s + \kappa_0^2\frac{s^3}{6})T_0 + (\kappa_0\frac{s^2}{2} + \frac{d\kappa}{ds}(0)\frac{s^3}{6})N_0 + \kappa_0\tau_0\frac{s^3}{6}B_0.$$

Definition

$$\hat{\beta}(s) = \beta(0) + sT_0 + \kappa_0\frac{s^2}{2}N_0 + \kappa_0\tau_0\frac{s^3}{6}B_0$$

is called *Frenet approximation* of β near $s = 0$.

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The Frenet approximation at point s_0

We found

$$\hat{\beta}(s) = \beta(0) + sT_0 + \kappa_0 \frac{s^2}{2} N_0 + \kappa_0 \tau_0 \frac{s^3}{6} B_0$$

as approximation of β near $s = 0$.

As approximation at a point s_0 we have in general

$$\hat{\beta}_{s_0}(s) \sim \beta(s_0) + (s - s_0)T_{s_0} + \kappa_{s_0} \frac{(s-s_0)^2}{2} N_{s_0} + \kappa_{s_0} \tau_{s_0} \frac{(s-s_0)^3}{6} B_{s_0}$$

as the Frenet approximation of β near $s = s_0$. 14.7

Tangent line and osculating plane

The Frenet approximation

$$\hat{\beta}(s) = \beta(0) + sT_0 + \kappa_0 \frac{s^2}{2} N_0 + \kappa_0 \tau_0 \frac{s^3}{6} B_0$$

Definition
The curve defined by $s \mapsto \beta(0) + sT_0$ is the *tangent line* to β at $s = 0$.
It is the best linear approximation to β near $\beta(0)$.

Definition
The curve defined by $s \mapsto \beta(0) + sT_0 + \kappa_0 \frac{s^2}{2} N_0$ is a parabola which is the best quadratic approximation to β at $s = 0$.
This parabola lies in the plane through $\beta(0)$ orthogonal to B_0.
This plane is called the *osculating plane* of β at $s = 0$.

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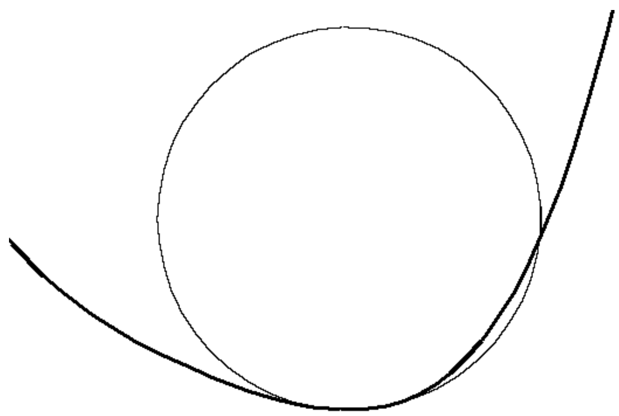
Osculate



The curves meet in a point where they have a common tangent. 14.9

2 Plane Curves

Plane curves

Definition
A *plane curve* in $\mathbb{R}^3$ is a curve in $\mathbb{R}^3$ such that all its points lie in the same plane in $\mathbb{R}^3$.

If the torsion τ of β at $s = 0$ is zero, $\tau(0) = 0$, then the Frenet approximation becomes

$$\hat{\beta}(s) = \beta(0) + sT_0 + \kappa_0 \frac{s^2}{2} N_0,$$

so $\hat{\beta}$ is a plane curve in the osculating plane of β at $s = 0$.

We will prove that if $\tau(s) = 0$ holds for every s, then β itself is a plane curve.

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Torsion-free curves

Corollary 3.5
Let β be a unit-speed curve in $\mathbb{R}^3$ with positive curvature $\kappa > 0$.
Then β is a plane curve if and only if $\tau = 0$.

Proof of the "only if" part of Corollary 3.5
Assume that β is a plane curve.
So there are p, q in $\mathbb{R}^3$ such that $(\beta(s) - p) \bullet q = 0$ for all s.
Differentiating $(\beta(s) - p) \bullet q = 0$ gives

$$\beta' \bullet q + (\beta - p) \bullet 0 = \beta' \bullet q = 0 \text{ and } \beta'' \bullet q = 0.$$

So also $T \bullet q = 0$ and $N \bullet q = 0$.
By orthonormal expansion, $q = (q \bullet B)B$.
From $||B|| = 1$ follows $B = q/||q||$ or $B = -q/||q||$.
Since B is constant, we get $B' = 0$.
The definition of τ is given by $B' = -\tau N$.
It follows from $N \neq 0$ that $\tau = 0$. □

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Proof of the "if" part of Corollary 3.5
Assume that β is a curve with $\tau = 0$. So $B' = 0$, and B is parallel, that is, a constant $B = (b_1, b_2, b_3)$.
We want to show that β lies in the plane through $\beta(0)$ orthogonal to B.
So consider the function $f : \mathbb{R} \to \mathbb{R}, s \mapsto (\beta(s) - \beta(0)) \bullet B$.
We immediately see that $f(0) = 0$.
Since B is constant,

$$\frac{df}{ds} = \beta' \bullet B = T \bullet B = 0.$$

Hence f is a constant, which implies $f = 0$.
The equation

$$(\beta(s) - \beta(0)) \bullet B = 0$$

precisely means that β lies in the plane through $\beta(0)$ orthogonal to B. □

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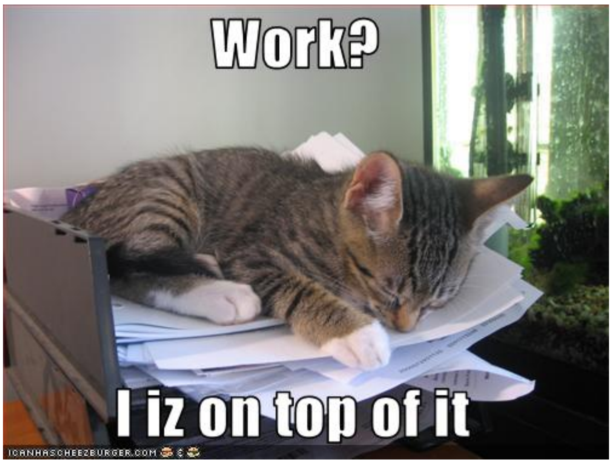

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Torsion-free curves with constant curvature

Lemma 3.6
If β is a unit-speed curve with constant curvature $\kappa > 0$ and torsion zero, $\tau = 0$, then β moves on a circle of radius $1/\kappa$.

Proof of Lemma 3.6
The assumption $\tau = 0$ implies that β is a plane curve, by Corollary 3.5.
Define a new curve $\gamma = \beta + N/\kappa$.
Differentiate using that κ is constant, and using $N' = -\kappa T$:

$$\gamma' = \beta' + N'/\kappa = T + (-\kappa T)/\kappa = T - T = 0.$$

Which implies that γ is constant.
Assume $\gamma(s) = c$ for all s. Then $\beta(s) + N(s)/\kappa = c$ for all s.
The distance from c to $\beta(s)$ is equal to

$$d(c, \beta(s)) = ||c - \beta(s)|| = ||N(s)/\kappa|| = 1/\kappa,$$

so it is constant $1/\kappa$. □

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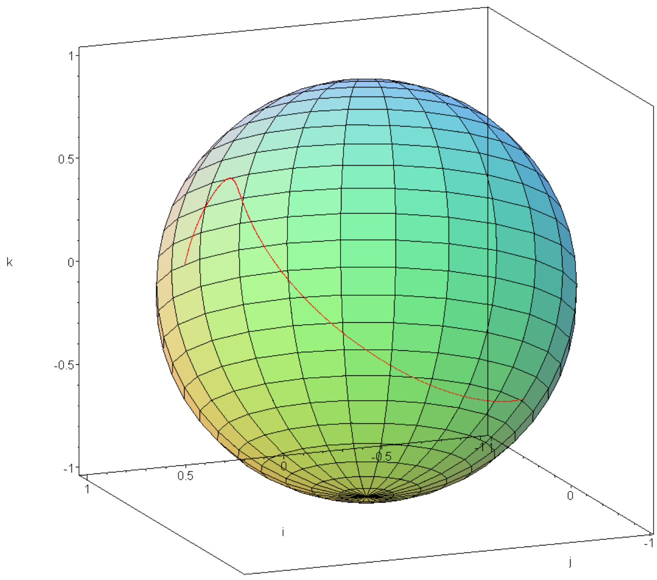
Spherical curves



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3 Conclusion

The End

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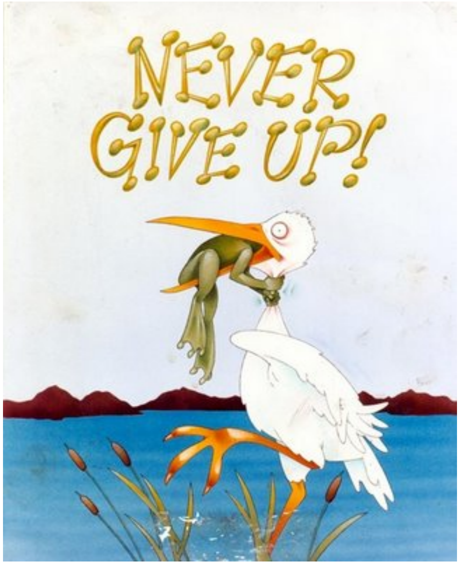

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Next time:

Solving equation by

$$\frac{1}{n}\sin x = ?$$

$$\frac{1}{\cancel{n}}\mathrm{si}\cancel{n}\,x =$$

$$six = 6$$

Friday: *Midterm test*

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