Deep|Bayes 2019

Theoretical Assignment

Shubham Gupta

shubhamgblr@gmail.com

Q1

The random variable ξ has Poisson distribution with the parameter λ. If ξ = k we perform k Bernoulli trials with the probability of success p. Let us define the random variable η as the number of successful outcomes of Bernoulli trials. Prove that η has Poisson distribution with the parameter pλ.

Answer

We know,

$$\xi = Poisson(\lambda)$$

A poisson distribution for k events in an interval is given by:

$$P(\text{k events in an interval}) = \frac{e^{-\lambda}\lambda^k}{k!}$$

For a bernoulli trail, the conditional probability distribution of obtaining x successes in k trails and success probability p is given by:

$$P(\eta = k | \xi = x) = (\genfrac{}{}{0pt}{}{k}{x}) p^x (1-p)^{k-x}$$

We need to find the marginal probability of $P(\eta = x)$. We know the formula for marginal probability is given by:

$$P(\eta = x) = \sum_{k \geq x} P(\eta = x | \xi = k) P(\xi = k)$$

Substituting the values, we get:

$$P(\eta = x) = \sum_{k \geq x} \binom{k}{x} p^x (1-p)^{k-x} \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \sum_{k \geq x} \frac{k!}{x!(k-x)!} \binom{k}{x} p^x (1-p)^{k-x} \frac{e^{-\lambda}\lambda^k}{k!}$$

Removing the constants out and cancelling the common factor $k!$, we get:

$$P(\eta = x) = \frac{p^x e^{-\lambda}}{x!} \sum_{k \geq x} \frac{(1-p)^{k-x}\lambda^k}{(k-x)!}$$

Let us substitute $z = k - x$. We get

$$P(\eta = x) = \frac{p^x e^{-\lambda}}{x!} \sum_{z \geq 0} \frac{(1-p)^z \lambda^{z+x}}{z!}$$

$$= \frac{(p\lambda)^x e^{-\lambda}}{x!} \sum_{z \geq 0} \frac{((1-p)\lambda)^z}{z!}$$

$$= \frac{(p\lambda)^x e^{-\lambda}}{x!} e^{\lambda(1-p)}$$

$$= \frac{(p\lambda)^x e^{-p\lambda}}{x!}$$

$$= Poisson(p\lambda)$$

Hence, η has a Poisson distribution with parameter $p\lambda$.

Q2

A strict reviewer needs t1 minutes to check assigned application to DeepBayes summer school, where t1 has normal distribution with parameters μ1 = 30, σ1 = 10. While a kind reviewer needs t2 minutes to check an application, where t2 has normal distribution with parameters μ2 = 20, σ2 = 5. For each application the reviewer is randomly selected with 0.5 probability. Given that the time of review t = 10, calculate the conditional probability that the application was checked by a kind reviewer.

Answer

We are given the following:

$$t1 = N(30, 10)$$

$$t2 = N(20, 5)$$

$$P(t1) = P(t2) = 0.5$$

Using Bayes theorm, we get:

$$P(r = kind|t = 10) = \frac{P(t = 10|r = kind)P(r = kind)}{P(t = 10)}$$

We can subsitute the denominator with the actual value i.e

$$P(r = kind|t = 10) = \frac{P(t = 10|r = kind)P(r = kind)}{P(r = kind|t = 10)P(r = kind) + P(r = strict|t = 10)P(r = strict)}$$

Now, for normal distributions, we know the probability mass function at any given point x is given by:

$$P(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Substituting the above formula, we get:

$$P(r = kind|t = 10) = \frac{\frac{1}{\sqrt{2\pi\sigma_{kind}^2}} e^{-\frac{\left(t-\mu_{kind}\right)^2}{2\sigma_{kind}^2}} * 0.5}{0.5 * (\frac{1}{\sqrt{2\pi\sigma_{kind}^2}} e^{-\frac{\left(t-\mu_{kind}\right)^2}{2\sigma_{kind}^2}} + \frac{1}{\sqrt{2\pi\sigma_{strict}^2}} e^{-\frac{\left(t-\mu_{strict}\right)^2}{2\sigma_{strict}^2}})}$$

Now, $e^{-\frac{\left(t-\mu_{kind}\right)^2}{2\sigma_{kind}^2}}$ and $e^{-\frac{\left(t-\mu_{strict}\right)^2}{2\sigma_{strict}^2}}$ both evaluate to 2 when we substitute the values. Furthermore, we can cancel our the common values of 0.5 and $\frac{1}{\sqrt{2\pi}}$. Therefore, we get:

$$= \frac{\frac{1}{5}}{\frac{1}{5} + \frac{1}{10}}$$

$$= \frac{2}{3}$$

Hence, the probability of the reviewer being kind at $t = 10$ is $\frac{2}{3}$.