

# Stochastic variational inference and VAEs

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# Outline

- Probabilistic PCA
- Variational auto-encoder
- REINFORCE and Reparameterization trick
- Stochastic computation graphs

# Continuous latent variables

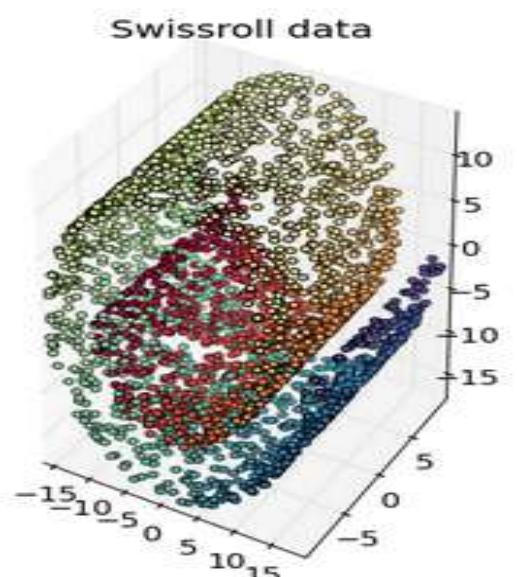
- Continuous variables can be regarded as a mixture of a continuum of distributions

$$p(x_i|\theta) = \int p(x_i, z_i|\theta) dz_i = \int p(x_i|z_i, \theta)p(z_i|\theta) dz_i$$

- E-step can be done in closed form only in case of **conjugate distributions**, otherwise the true posterior is intractable

$$q(z_i) = p(z_i|x_i, \theta) = \frac{p(x_i|z_i, \theta)p(z_i|\theta)}{\int p(x_i|z_i, \theta)p(z_i|\theta) dz_i}$$

- Typically continuous latent variables are used for dimension reduction also known as **representation learning**



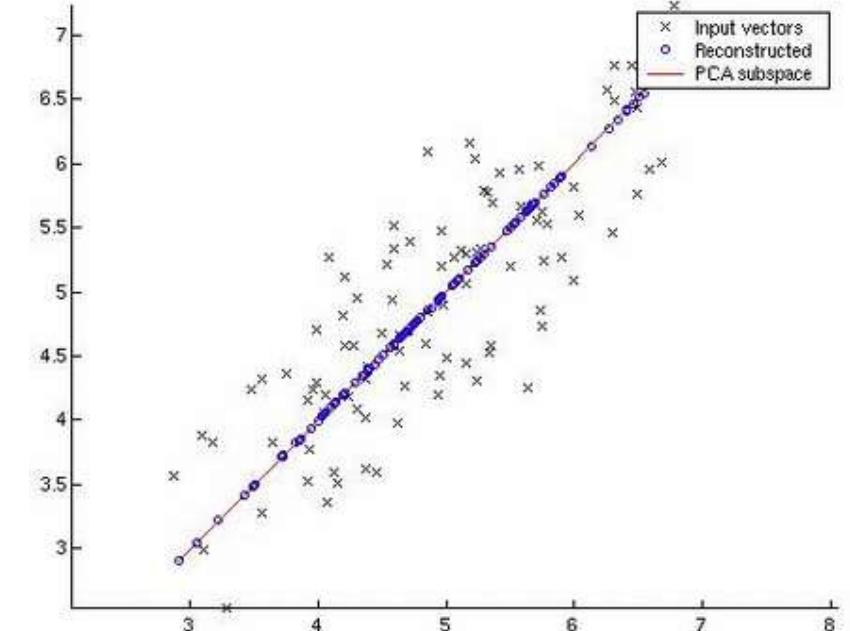
# Example: PCA model

- Consider  $x \in \mathbb{R}^D$ ,  $z \in \mathbb{R}^d$ , such that  $D \gg d$
- Joint distribution

$$p(X, Z|\theta) = \prod_{i=1}^n p(x_i|z_i, \theta)p(z_i|\theta) = \prod_{i=1}^n \mathcal{N}(x_i|Vz_i + \mu, \sigma^2 I)\mathcal{N}(z_i|0, I)$$

- $\theta$  consists of  $D \times d$  matrix  $V$ ,  $D$ -dimensional vector  $\mu$  and scalar  $\sigma$
- Can use EM-algorithm to find  $\arg \max_{\theta} p(X_{tr}|\theta)$

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# Advantages of EM PCA

In PCA the explicit equation for  $\theta$  can be obtained analytically. Then why use EM?..

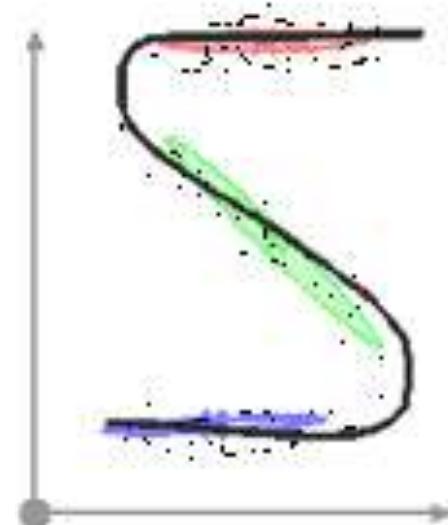
- EM updates have complexity  $O(nDd)$  instead of  $O(nD^2)$  in analytic solution
- Can process missing parts in  $X$  and present parts in  $Z$
- Can determine  $d$  if  $p(\theta)$  is established
- Can be extended to more general models such as mixture of PCA

# Mixture of PCA

- Two types of latent variables: discrete  $t \in \{1, \dots, K\}$  and continuous  $z \in \mathbb{R}^d$
- Joint distribution

$$p(X, Z, T | \theta) = \prod_{i=1}^n p(x_i | t_i, z_i, \theta) p(z_i | \theta) p(t_i | \theta) = \prod_{i=1}^n \mathcal{N}(x_i | V_{t_i} z_i + \mu_{t_i}, \sigma_{t_i}^2 I) \mathcal{N}(z_i | 0, I) \pi_{t_i}$$

- $\theta$  consists of matrices  $\{V_k\}$ , vectors  $\mu_k$ , scalars  $\{\sigma_k\}$ , and vector of probabilities  $\pi$  such that  $p(t_i = k) = \pi_k$
- Can be used for non-linear dimension reduction



# EM-algorithm for mixture of PCA

- Two types of latent variables: discrete  $t \in \{1, \dots, K\}$  and continuous  $z \in \mathbb{R}^d$
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- E-step: Estimate

$$q(Z, T) = \prod_{i=1}^n q(z_i, t_i) = \prod_{i=1}^n p(z_i, t_i | x_i, \theta)$$

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- M-step: Optimize

$$\mathbb{E}_{Z, T} \log p(X, Z, T | \theta) = \sum_{i=1}^n \mathbb{E}_{z_i, t_i} (\log p(x_i | t_i, z_i, \theta) + \log p(z_i | \theta) + \log p(t_i | \theta)) \rightarrow \max_{\theta}$$

# Non-linear model

- PCA constructs only linear subspaces while data often lives in non-linear manifolds
- When there is sufficient training data one may try more complicated probabilistic models for dimension reduction
- We may establish either simple prior on latent representations or simple generator

$$p(x, z|\theta) = \boxed{p(x|z, \theta)} \boxed{p(z|\theta)}$$

Generator      Prior on latents

- Why not to use deep neural networks that are ideal for working with large datasets?

# Variational auto-encoder

- Deep generalization of probabilistic PCA

$$p(X, Z|\theta) = \prod_{i=1}^n p(x_i|z_i, \theta)p(z_i) = \prod_{i=1}^n \left( \prod_{j=1}^D \mathcal{N}(x_{ij}|\mu_j(z_i), \sigma_j^2(z_i)) \right) \mathcal{N}(z_i|0, I)$$

- Generator returns factorized gaussian whose mean and variance are non-linear functions of latent variable modelled by deep neural network parameterized by  $\theta$
- Simple prior over latent representations and complicated generator
- Allows to convert complicated large-dimensional distributions into simple low-dimensional ones

# Intractability problem

- Since the latent variables  $Z$  are unknown we cannot train the model directly
- If we could integrate out  $Z$  this would be possible but we cannot

$$\int p(X, Z|\theta) dZ = \prod_{i=1}^n \boxed{\int p(x_i|z_i, \theta)p(z_i) dz_i}$$

INTRACTABLE

- What about using EM-algorithm?..

# EM for VAE

- At E-step we need to compute a distribution over latent variables given data

$$q(Z) = \prod_{i=1}^n q(z_i) = \prod_{i=1}^n p(z_i|x_i, \theta) = \prod_{i=1}^n \frac{p(x_i|z_i, \theta)p(z_i)}{\int p(x_i|z_i, \theta)p(z_i)dz_i}$$

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- Denominator is still intractable
- But now we may use ideology of variational Bayes and convert inference problem to optimization!

INTRACTABLE

# Variational inference

- Instead of direct inferring of  $p(z_i|x_i, \theta)$  let us define flexible variational approximation

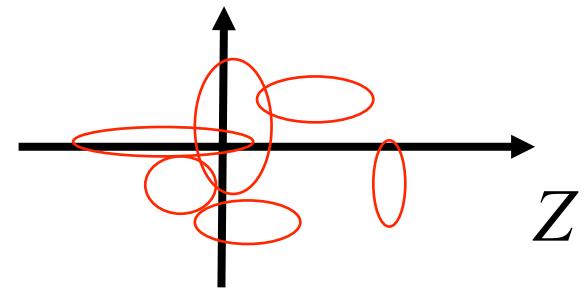
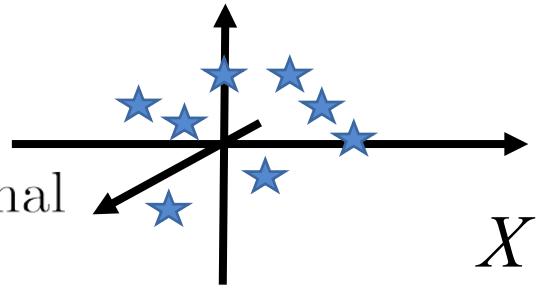
$$q(z_i|x_i, \phi) \approx p(z_i|x_i, \theta)$$

- The variational approximation has a form of factorized gaussian

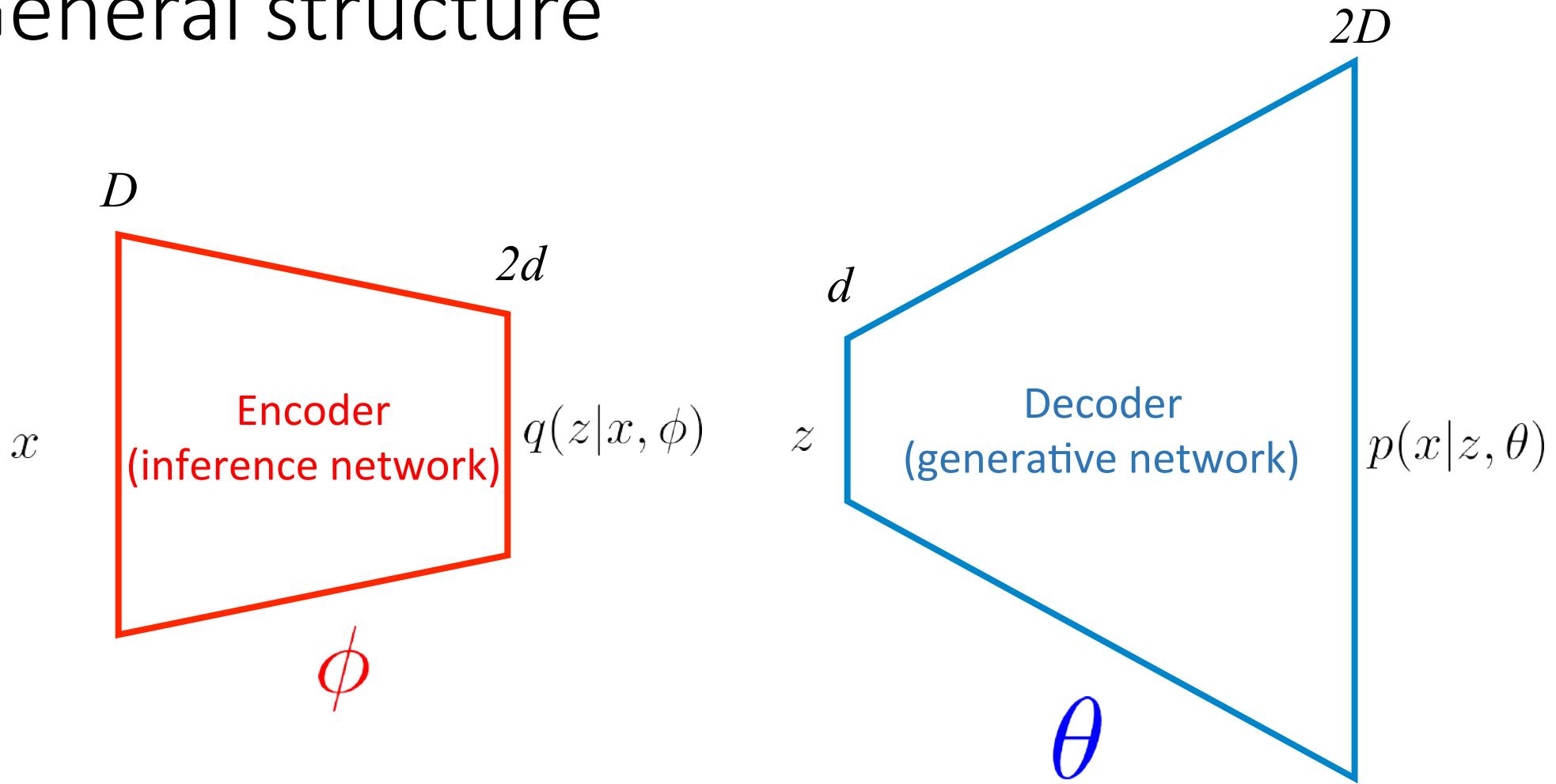
$$q(z_i|x_i, \phi) = \prod_{j=1}^d \mathcal{N}(z_{ij}|\mu_j(x_i), \sigma_j^2(x_i)),$$

whose the mean and variance are given by **another neural network** parameterized by weights  $\phi$

- This additional neural network ensures tractability of the distribution while being very flexible



# General structure



# ELBO for VAE

- By performing variational approximation we minimize the corresponding KL-divergence

$$q(Z|X, \phi) = \arg \min_{\phi} KL(q(Z|X, \phi) || p(Z|X, \theta))$$

- This is equivalent to optimizing ELBO

$$\mathcal{L}(\phi, \theta) = \int q(Z|X, \phi) \log \frac{p(X|Z, \theta)p(Z)}{q(Z|X, \phi)} dZ \rightarrow \max_{\phi, \theta}$$

- Block-coordinate optimization w.r.t. encoder parameters  $\phi$  corresponds to E-step
- Block-coordinate optimization w.r.t. decoder parameters  $\theta$  corresponds to M-step

# Stochastic optimization

$$\mathcal{L}(\phi, \theta) = \int q(Z|X, \phi) \log \frac{p(X|Z, \theta)p(Z)}{q(Z|X, \phi)} dZ$$

- Problem 1. The training data is assumed to be large  $\Rightarrow$  iterations might be expensive
- Problem 2. The integral in ELBO is still intractable
- Solution: Compute stochastic gradients by using **mini-batching** and **Monte-Carlo** estimation

# Optimization w.r.t. $\theta$

$$\mathcal{L}(\phi, \theta) = \int q(Z|X, \phi) \log \frac{p(X|Z, \theta)p(Z)}{q(Z|X, \phi)} dZ$$

- Stochastic gradients w.r.t.  $\theta$  can be obtained quite straightforwardly
- Mini-batching:

$$\begin{aligned}\nabla_{\theta} \mathcal{L}(\phi, \theta) &= \nabla_{\theta} \sum_{i=1}^n \int q(z_i|x_i, \phi) \log \frac{p(x_i|z_i, \theta)p(z_i)}{q(z_i|x_i, \phi)} dz_i = \\ &\sum_{i=1}^n \int q(z_i|x_i, \phi) \nabla_{\theta} \log p(x_i|z_i, \theta) dz_i \approx n \int q(z_i|x_i, \phi) \nabla_{\theta} \log p(x_i|z_i, \theta) dz_i, \quad i \sim \mathcal{U}\{1, \dots, n\}\end{aligned}$$

# Optimization w.r.t. $\theta$

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$$\sum_{i=1}^n \int q(z_i|x_i, \phi) \nabla_{\theta} \log p(x_i|z_i, \theta) dz_i \approx n \int q(z_i|x_i, \phi) \nabla_{\theta} \log p(x_i|z_i, \theta) dz_i, \quad i \sim \mathcal{U}\{1, \dots, n\}$$

- Monte-Carlo estimation

$$n \int q(z_i|x_i, \phi) \nabla_{\theta} \log p(x_i|z_i, \theta) dz_i \approx n \nabla_{\theta} \log p(x_i|z_i^*, \theta), \quad z_i^* \sim q(z_i|x_i, \phi)$$

5 min break

# Optimization w.r.t. $\phi$

- Stochastic gradient w.r.t.  $\theta$

$$\text{stoch. grad}_\theta \mathcal{L}(\phi, \theta) = n \nabla_\theta \log p(x_i | z_i^*, \theta), \quad i \sim \mathcal{U}\{1, \dots, n\}, \quad z_i^* \sim q(z_i | x_i, \phi)$$

- For differentiation w.r.t.  $\phi$  the situation is different
- The density function itself depends on the parameters w.r.t. which we want to differentiate the expression
- Can no longer move gradient inside integral

$$\nabla_x \int p(y|x) h(x, y) dy \neq \int p(y|x) \nabla_x h(x, y) dy$$

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$$\nabla_x \int p(y|x) h(x, y) dy \neq \int p(y|x) \nabla_x h(x, y) dy$$

- Question: Why can't we simply do

$$\nabla_x \int p(y|x) h(x, y) dy = \int \nabla_x (p(y|x) h(x, y)) dy?$$

# Log-derivative trick

Consider differentiation of  $\mathbb{E}_{y|x} h(x, y)$  in details

$$\begin{aligned}\frac{\partial}{\partial x} \int p(y|x)h(x, y)dy &= \int \frac{\partial}{\partial x} (p(y|x)h(x, y)) dy = \\ \int \left( h(x, y) \frac{\partial}{\partial x} p(y|x) + p(y|x) \frac{\partial}{\partial x} h(x, y) \right) dy &= \int p(y|x) \frac{\partial}{\partial x} h(x, y) dy + \int h(x, y) \frac{\partial}{\partial x} p(y|x) dy\end{aligned}$$

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The first term is ok since it can be replaced with a Monte Carlo estimate of expectation. To deal with second term we need to use **log-derivative trick**

$$\frac{\partial}{\partial x} p(y|x) = p(y|x) \frac{\partial}{\partial x} \log p(y|x)$$

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Appying the trick yields to

$$\frac{\partial}{\partial x} \int p(y|x)h(x, y)dy = \int p(y|x) \frac{\partial}{\partial x} h(x, y) dy + \int p(y|x)h(x, y) \frac{\partial}{\partial x} \log p(y|x) dy$$

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Appying the trick yields to **Expectations can be estimated via Monte Carlo**

$$\frac{\partial}{\partial x} \int p(y|x)h(x, y)dy = \boxed{\int p(y|x) \frac{\partial}{\partial x} h(x, y) dy} + \boxed{\int p(y|x)h(x, y) \frac{\partial}{\partial x} \log p(y|x) dy}$$

# Log-derivative trick for ELBO

$$\frac{\partial}{\partial x} \int p(y|x)h(x,y)dy \approx \frac{\partial}{\partial x} h(x,y_0) + h(x,y_0)\frac{\partial}{\partial x} \log p(y_0|x)$$

- Now remember our ELBO

$$\begin{aligned}\mathcal{L}(\phi, \theta) &= \int q(Z|X, \phi) \log \frac{p(X|Z, \theta)p(Z)}{q(Z|X, \phi)} dZ = \\ &\quad \int q(Z|X, \phi) \log p(X|Z, \theta) dZ - KL(q(Z|X, \phi)||p(Z))\end{aligned}$$

- Note that second term is simply a sum of KL-divergences between factorized gaussians  $q(z_i|x_i, \phi)$  and  $p(z_i)$  that can be computed and differentiated in closed form
- Consider the first term and apply mini-batching and log-derivative trick

$$\frac{\partial}{\partial \phi} \int q(Z|X, \phi) \log p(X|Z, \theta) dZ \approx n \log p(x_i|z_i^*, \theta) \frac{\partial}{\partial \phi} \log q(z_i^*|x_i, \phi),$$

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**ACHTUNG!**

$$\frac{\partial}{\partial \phi} \int q(Z|X, \phi) \log p(X|Z, \theta) dZ \approx \boxed{n \log p(x_i|z_i^*, \theta)} \frac{\partial}{\partial \phi} \log q(z_i^*|x_i, \phi),$$

# Score function

Consider so-called **score function**

$$\frac{\partial \log q(z_i|x_i, \phi)}{\partial \phi}$$

Let's prove that it has zero mean

$$\begin{aligned} \int q(z_i|x_i, \phi) \frac{\partial \log q(z_i|x_i, \phi)}{\partial \phi} dz_i &= \int q(z_i|x_i, \phi) \frac{1}{q(z_i|x_i, \phi)} \frac{\partial q(z_i|x_i, \phi)}{\partial \phi} dz_i = \\ &\quad \frac{\partial}{\partial \phi} \int q(z_i|x_i, \phi) dz_i = \frac{\partial}{\partial \phi} 1 = 0 \end{aligned}$$

Then the value

$$\frac{\partial \log q(z_i^*|x_i, \phi)}{\partial \phi},$$

where  $z_i^* \sim q(z_i|x_i, \phi)$  oscillates around zero

# REINFORCE

Has large absolute value

$$\frac{\partial}{\partial \phi} \int q(Z|X, \phi) \log p(X|Z, \theta) dZ \approx n \log p(x_i|z_i^*, \theta) \frac{\partial}{\partial \phi} \log q(z_i^*|x_i, \phi),$$

- Term  $n \log p(z_i^*|x_i, \theta)$  can be arbitrary large negative that leads to very unstable stochastic gradients
- A partial solution is to use **baselines**
- Consider a function  $b(z_i, \phi)$  such that

$$\int q(z_i|\phi) b(z_i, \phi) dz_i = 0$$

- A good example is

$$b(z_i, \phi) = B(\phi) \frac{\partial}{\partial \phi} \log q(z_i|\phi)$$

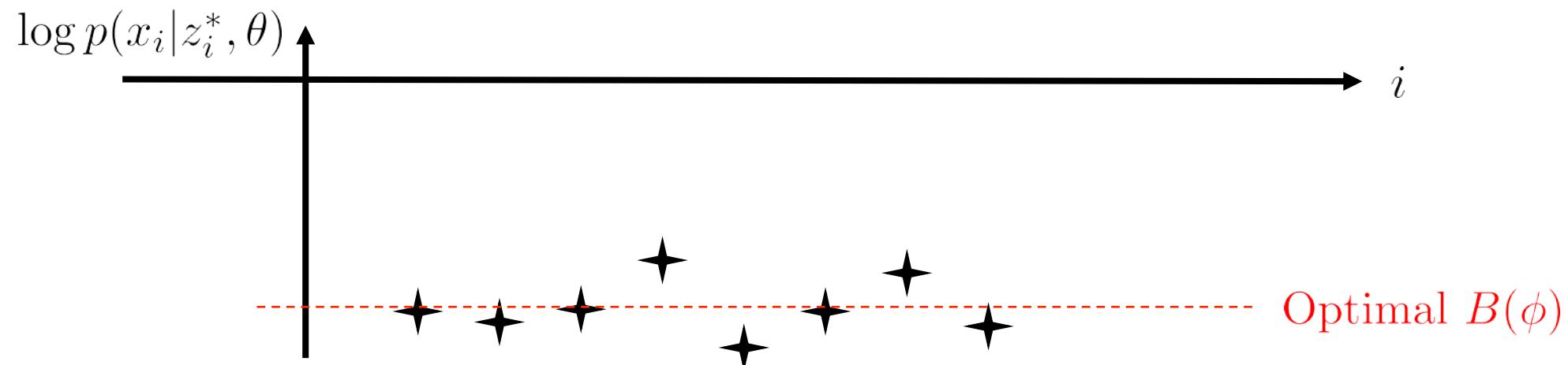
the so-called **score function**

# REINFORCE

Then

$$\begin{aligned} \frac{\partial}{\partial \phi} \int q(z_i|x_i, \phi) \log p(x_i|z_i, \theta) dz_i &= \frac{\partial}{\partial \phi} \int q(z_i|x_i, \phi) (\log p(x_i|z_i, \theta) - b(z_i, \phi)) dz_i \approx \\ &(\log p(x_i|z_i^*, \theta) - B(\phi)) \frac{\partial}{\partial \phi} \log q(z_i^*|x_i, \phi), \quad z_i^* \sim q(z_i|x_i, \phi) \end{aligned}$$

- If baseline  $B(\phi)$  is close to  $\mathbb{E}_i \mathbb{E}_{z_i|x_i, \phi} \log p(x_i|z_i, \theta)$  then it may significantly reduce the variance of stochastic gradient



# Stochastic gradients

Function	Stochastic gradient
$f(x) = \sum_{i=1}^N f_i(x)$	$N \nabla f_i(x), \quad i \sim \mathcal{U}\{1, \dots, N\}$
$f(x) = \mathbb{E}_y h(x, y) = \int p(y) h(x, y) dy$	$\frac{\partial}{\partial x} h(x, y_0), \quad y_0 \sim p(y)$
$f(x) = \sum_{i=1}^N \mathbb{E}_y h(x_i, y) = \sum_{i=1}^N \int p(y) h_i(x, y) dy$	$N \frac{\partial}{\partial x} h_i(x, y_0), \quad y_0 \sim p(y)$
$f(x) = \mathbb{E}_{y x} h(x, y) = \int p(y x) h(x, y) dy$	$\frac{\partial}{\partial x} h(x, y_0) + h(x, y_0) \frac{\partial}{\partial x} \log p(y_0 x)$

Last example has extremely large variance!

Variance reduction is needed

# Reparameterization trick

- Consider differentiation of complex expectation

$$\frac{\partial}{\partial x} \int p(y|x)h(x,y)dy$$

- Express  $y$  as a deterministic function  $g(\cdot)$  of random  $\epsilon$  and  $x$  and perform change-of-variables rule

$$\int p(y|x)h(x,y)dy = \int r(\epsilon)h(x,g(\epsilon,x))d\epsilon$$

- Then stochastic differentiation is simply

$$\frac{\partial}{\partial x} \int p(y|x)h(x,y)dy = \frac{\partial}{\partial x} \int r(\epsilon)h(x,g(\epsilon,x))d\epsilon \approx$$

$$\frac{d}{dx} h(x, g(x, \hat{\epsilon})) = \frac{\partial}{\partial x} h(x, g(x, \hat{\epsilon})) + \frac{\partial}{\partial g} h(x, g(x, \hat{\epsilon})) \frac{\partial}{\partial x} g(x, \hat{\epsilon})$$

where  $\hat{\epsilon} \sim r(\epsilon)$

# Examples of reparameterization

$p(x y)$	$r(\epsilon)$	$g(\epsilon, y)$
$\mathcal{N}(x \mu, \sigma^2)$	$\mathcal{N}(\epsilon 0, 1)$	$x = \sigma\epsilon + \mu$
$\mathcal{G}(x 1, \beta)$	$\mathcal{G}(\epsilon 1, 1)$	$x = \beta\epsilon$
$\mathcal{E}(x \lambda)$	$\mathcal{U}(\epsilon 0, 1)$	$x = -\frac{\log \epsilon}{\lambda}$
$\mathcal{N}(x \mu, \Sigma)$	$\mathcal{N}(\epsilon 0, I)$	$x = A\epsilon + \mu$ , where $AA^T = \Sigma$

- Not all continuous distributions can be effectively reparameterized
- Discrete distributions **cannot** be reparameterized

# Reparameterization trick for ELBO

- Return to the first term in our ELBO

$$\int q(Z|X, \phi) \log p(X|Z, \theta) dZ$$

- To get stochastic gradient w.r.t.  $\phi$  first apply mini-batching

$$\frac{\partial}{\partial \phi} \int q(Z|X, \phi) \log p(X|Z, \theta) dZ \approx n \frac{\partial}{\partial \phi} \int q(z_i|x_i, \phi) \log p(x_i|z_i, \theta),$$

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- Now perform reparameterization trick  $z_i = g(\epsilon, x_i, \phi)$

$$n \frac{\partial}{\partial \phi} \int q(z_i|x_i, \phi) \log p(x_i|z_i, \theta) = n \frac{\partial}{\partial \phi} \int r(\epsilon) \log p(x_i|g(\epsilon, x_i, \phi)z_i, \theta) d\epsilon \approx$$
$$n \frac{\partial}{\partial \phi} \log p(x_i|g(\hat{\epsilon}, x_i, \phi)z_i, \theta), \quad \hat{\epsilon} \sim r(\epsilon)$$

# VAE: final algorithm

- Input: Training data  $X$ , dimension of latent space  $d$
- Pick random  $i \sim \mathcal{U}\{1, \dots, n\}$  and compute stochastic gradients of ELBO w.r.t.  $\theta$  and  $\phi$ 
  - Differentiate w.r.t.  $\theta$

$$\text{stoch.grad}_\theta \mathcal{L}(\phi, \theta) = n \frac{\partial}{\partial \theta} \log p(x_i | z_i^*, \theta),$$

where  $z_i^* \sim q(z_i | x_i, \phi)$

- Differentiate w.r.t.  $\phi$

$$\text{stoch.grad}_\phi \mathcal{L}(\phi, \theta) = n \frac{\partial}{\partial \phi} \log p(x_i | g(\hat{\epsilon}, x_i, \phi), \theta) - \frac{\partial}{\partial \phi} KL(q(z_i | x_i, \phi) || p(z_i)),$$

where  $\hat{\epsilon} \sim r(\epsilon)$

- Update  $\theta$  and  $\phi$  according to selected stochastic optimization method

# VAE vs. GAN

Pros:

- No mode-collapsing
- More robust training
- May restore latent representation of real object
- Better interpolation between instances

Cons:

- GANs generate more realistic objects
- The divergence measure is trainable in GANs



Both models seem to be particular cases of a more general (hopefully Bayesian) model.

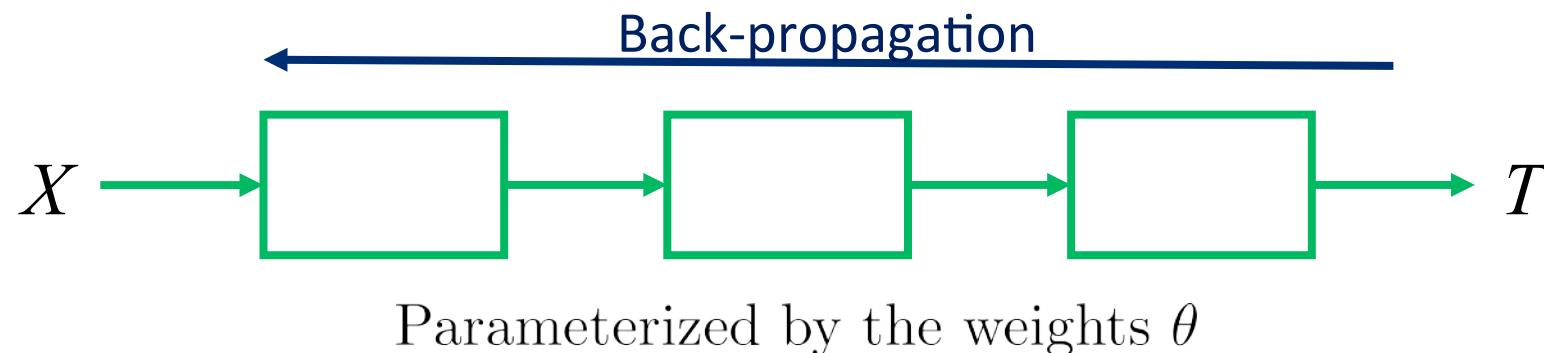
Work in progress...

# Automatic differentiation

- When given a **training sample**  $(X_{tr}, T_{tr})$  one may try to minimize the error on training sample
- For example in case of **probabilistic model**  $p(T|X, \theta)$  we compute **maximum likelihood** estimate

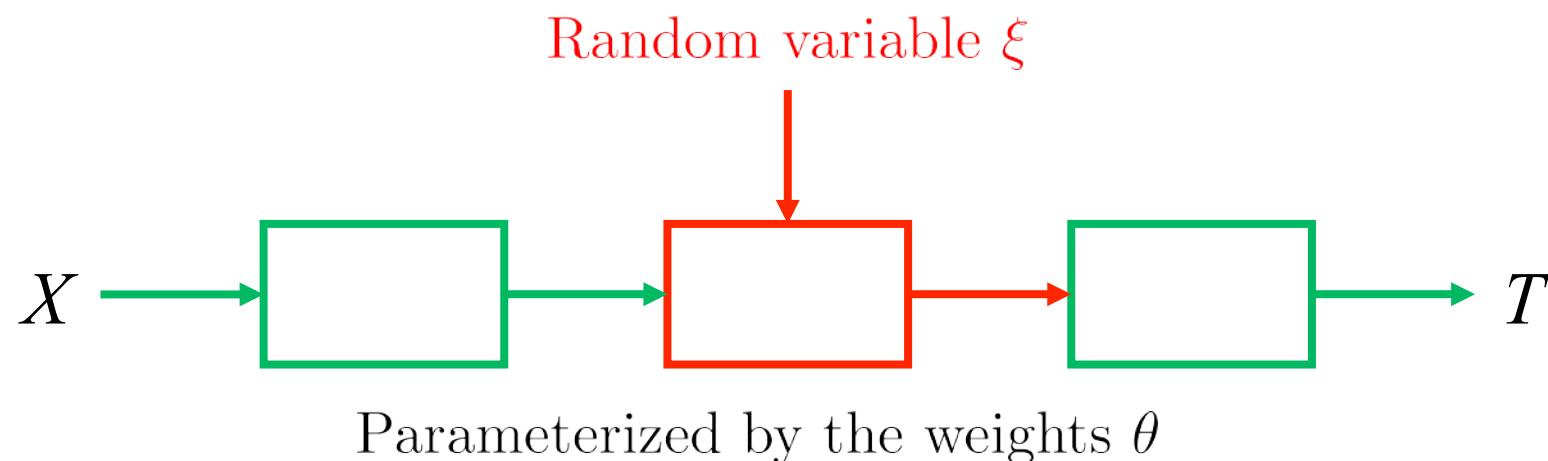
$$\theta_{ML} = \arg \max_{\theta} p(T_{tr}|X_{tr}, \theta)$$

- Automatic differentiation allows us to propagate gradients and compute all derivatives w.r.t.  $\theta$  efficiently via chain rule



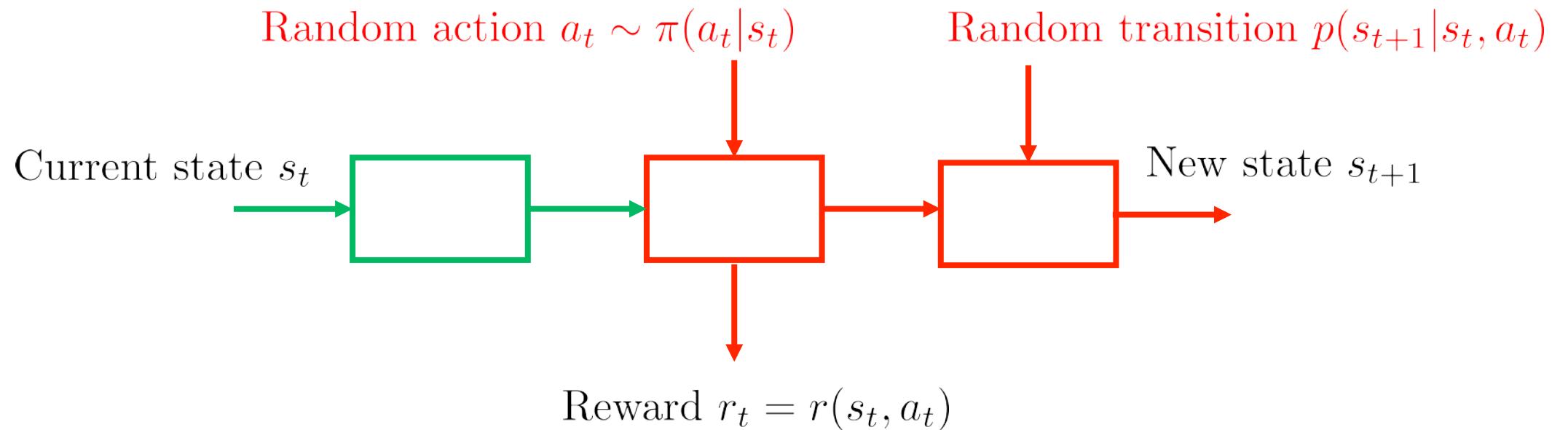
# Stochastic computational graphs

- In many popular models we insert random elements in our computational graph
- The ML model becomes more complex but obtains many useful properties
- Switching to stochastic computational graphs is current trend in modern ML



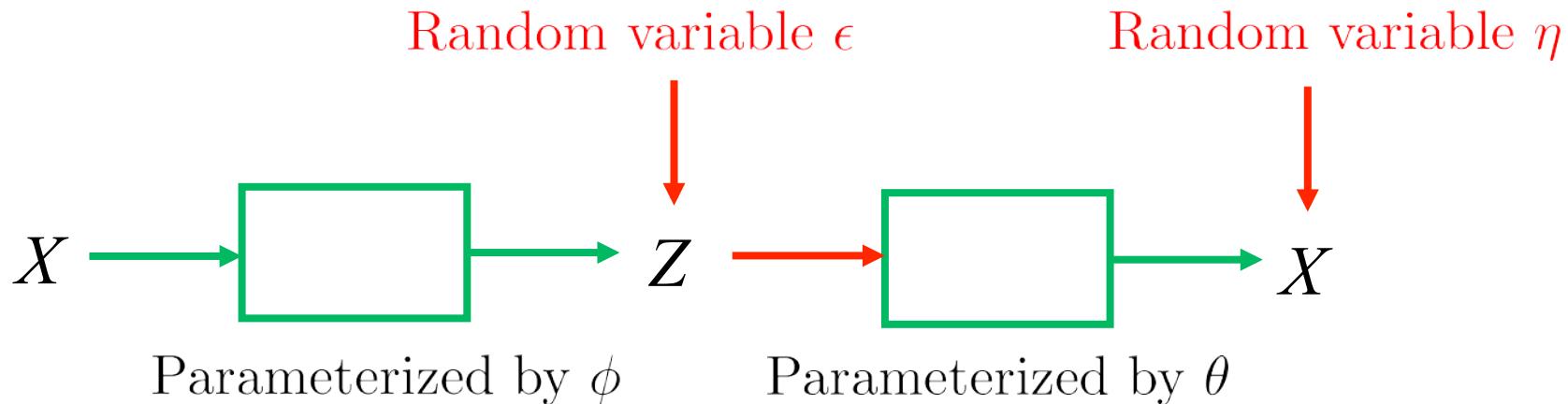
# Reinforcement learning

- States  $s_t$  are defined as Markov decision process:  $s_t \sim p(s_t|s_{t-1}, a_{t-1})$
- Actions are defined by current policy:  $a_t \sim \pi(a_t|s_t)$
- Rewards are defined as a function of state and action:  $r_t = r(s_t, a_t)$



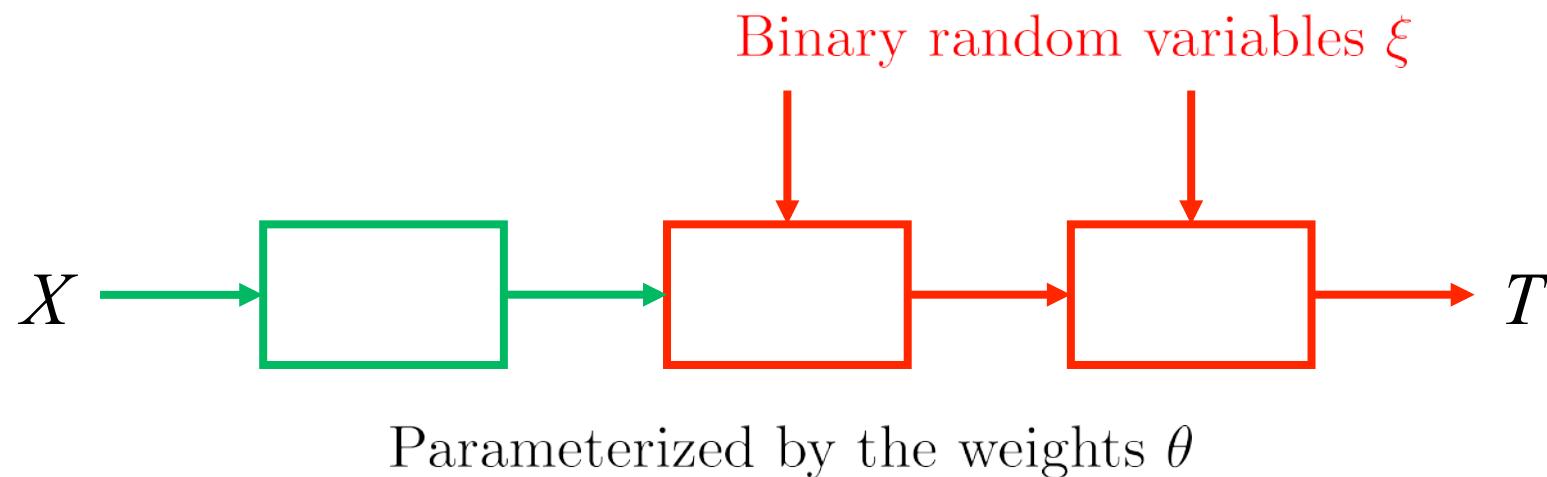
# Variational auto-encoders

- New way for dimension reduction (learning representations)
- High-dimensional data  $X$  are generated from low-dimensional latent representations  $Z$  via deep neural network (decoder) that models  $p(X|Z, \theta)$
- Inference is made by another deep network (encoder) that returns a distribution  $q(Z|X, \phi)$  in the space of latent variables rather than a point estimate



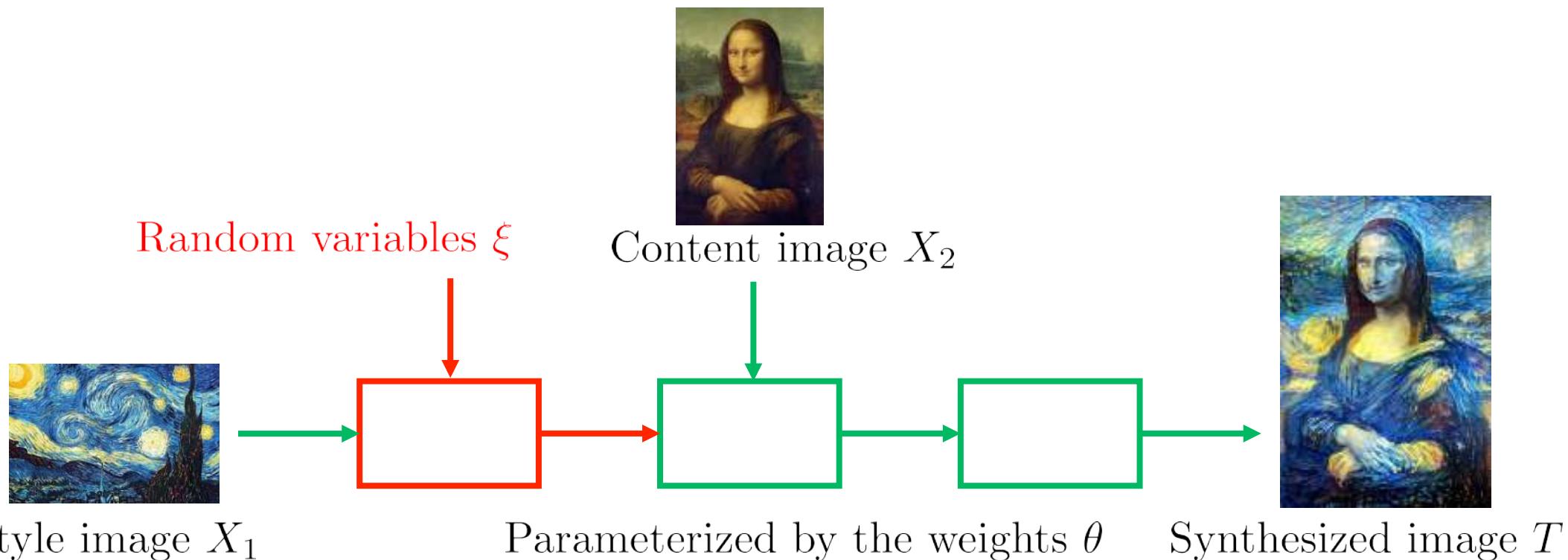
# Dropout

- Dropout is an heuristic regularization mechanism for deep neural networks
- Some neurons are randomly nullified during training
- It corresponds to training an ensemble of networks



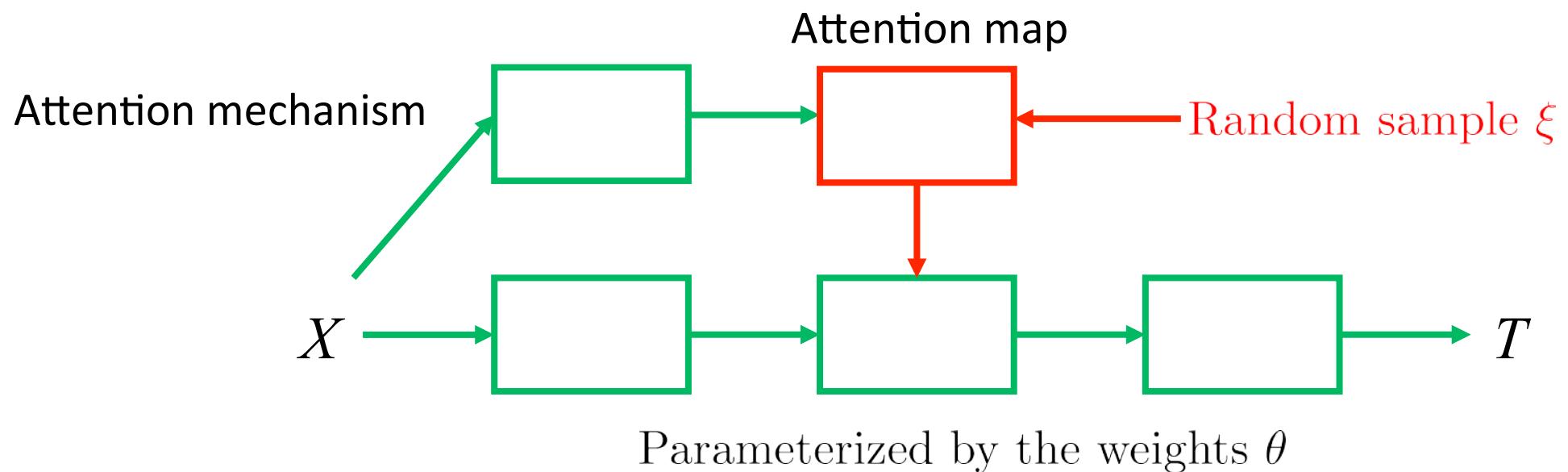
# Artistic style

- Artistic style allows to transfer style from one image and apply it to another image
- The content is kept fixed



# Attention mechanism

- New framework for detecting what computations should be made for a particular data
- Attention mechanism is modeled by an additional neural network that returns a distribution on the space of computations
- Surprisingly trainable scheme

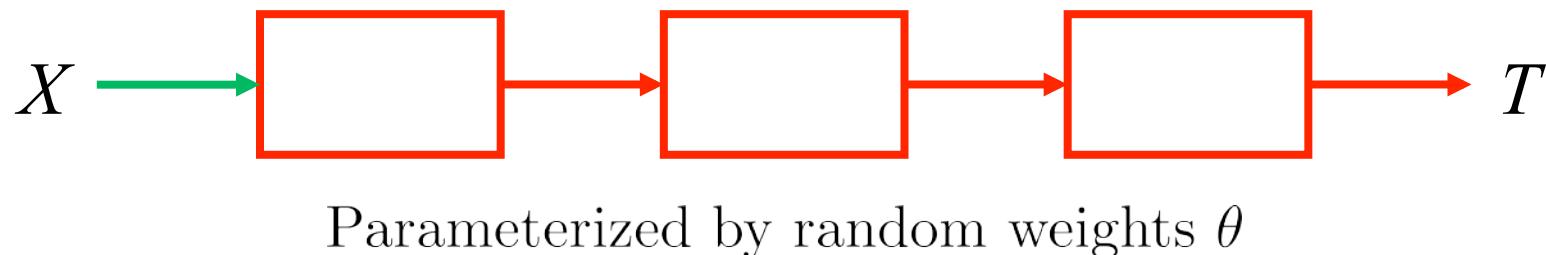


# Bayesian regularization

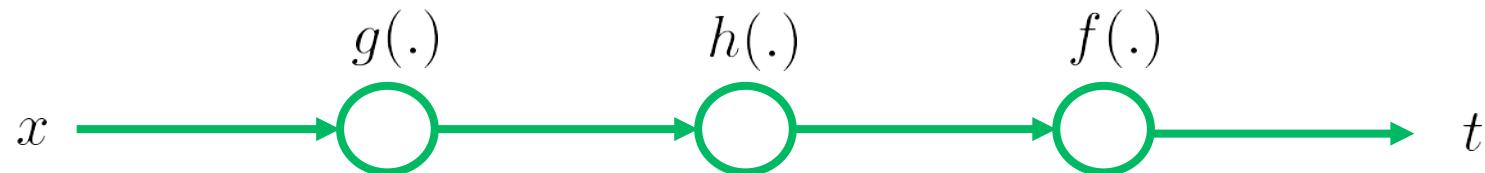
- In Bayesian framework the weights  $\theta$  are treated as random variables that have a distribution

$$p(T, \theta | X) = p(T | X, \theta)p(\theta)$$

- Prior  $p(\theta)$  allows to **regularize** the model thus preventing overfitting on training set
- We need to take into account the randomness of  $\theta$  during training



# Gradient propagation in CG



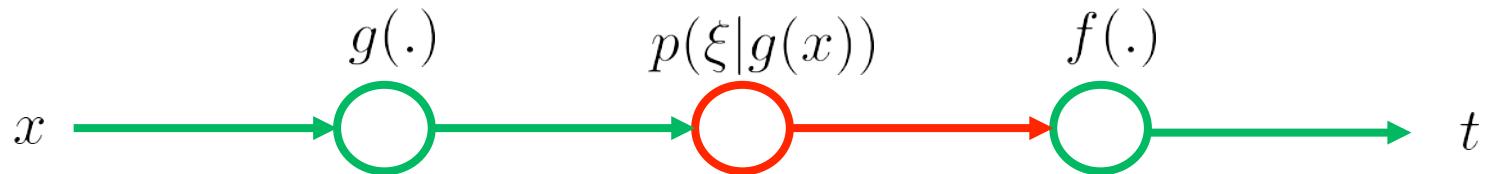
- Any computational graph defines **superposition** of functions to be computed
- Remember how we propagate the gradients in computational graphs  $t = f(h(g(x)))$

- Using chain rule we have

$$\frac{\partial t}{\partial x} = \frac{\partial f(h)}{\partial h} \frac{\partial h(g)}{\partial g} \frac{\partial g(x)}{\partial x}$$

- Also known as **back-propagation**

# Gradient propagation in Stochastic CG

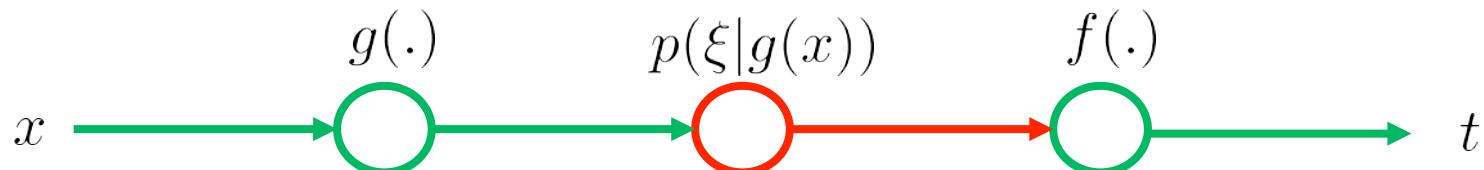


- Now the situation has changed

$$t = \mathbb{E}f(\xi) = \int p(\xi|g(x))f(\xi)d\xi$$

- Using either log-derivative or reparameterization tricks we are still able to propagate the (stochastic) gradients

# Gradient propagation in Stochastic CG



- Now the situation has changed

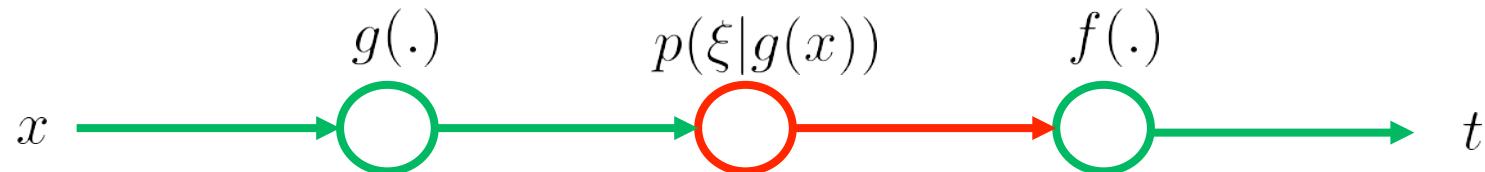
$$t = \mathbb{E}f(\xi) = \int p(\xi|g(x))f(\xi)d\xi$$

- Log-derivative trick**

$$\begin{aligned}\frac{\partial t}{\partial x} &= \frac{\partial}{\partial x} \int p(\xi|g(x))f(\xi)d\xi = \int \frac{\partial p(\xi|g(x))}{\partial x} f(\xi)d\xi = \\ &\int p(\xi|g(x)) \frac{\partial \log p(\xi|g(x))}{\partial x} f(\xi)d\xi \approx \frac{\partial \log p(\hat{\xi}|g(x))}{\partial x} f(\hat{\xi}) = \frac{\partial \log p(\hat{\xi}|g(x))}{\partial g} \frac{\partial g(x)}{\partial x} f(\hat{\xi}),\end{aligned}$$

where  $\hat{\xi} \sim p(\xi|g(x))$

# Gradient propagation in Stochastic CG



- Now the situation has changed

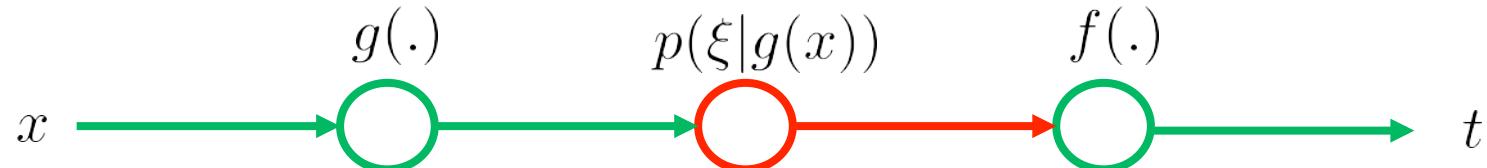
$$t = \mathbb{E}f(\xi) = \int p(\xi|g(x))f(\xi)d\xi$$

- Reparameterization trick.** Let  $\xi = h(\epsilon, g(x))$ . Then

$$\begin{aligned}\frac{\partial t}{\partial x} &= \frac{\partial}{\partial x} \int p(\xi|g(x))f(\xi)d\xi = \frac{\partial}{\partial x} \int p(\epsilon)f(h(\epsilon, g(x)))d\epsilon = \\ &\quad \int p(\epsilon) \frac{\partial f(h(\epsilon, g(x)))}{\partial x} d\epsilon \approx \frac{\partial f(h(\hat{\epsilon}, g(x)))}{\partial x} = \frac{\partial f(h)}{\partial h} \frac{\partial h(g)}{\partial g} \frac{\partial g(x)}{\partial x},\end{aligned}$$

where  $\hat{\epsilon} \sim p(\epsilon)$

# Gradient propagation in Stochastic CG



- Now the situation has changed

$$t = \mathbb{E}f(\xi) = \int p(\xi|g(x))f(\xi)d\xi$$

- Reparameterization trick.** Let  $\xi = h(\epsilon, g(x))$ . Then

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- Note that unlike previous slide here we use the information about gradient of  $f(\cdot)$

# Conclusion

- Variational Bayes is highly scalable inference technique
- Deep neural networks may be used for approximating intractable distributions via stochastic variational inference
- Stochastic computational graphs allow much more flexibility when modeling dependencies within the data
- May transform complex high-dimensional distributions into simple lower-dimensional ones and vice versa
- Most promising mean for generative models
- May process and generate “raw data”(texts, images, videos, graphs, etc.)
- **OPEN PROBLEM:** How to deal with discrete latent random variables?