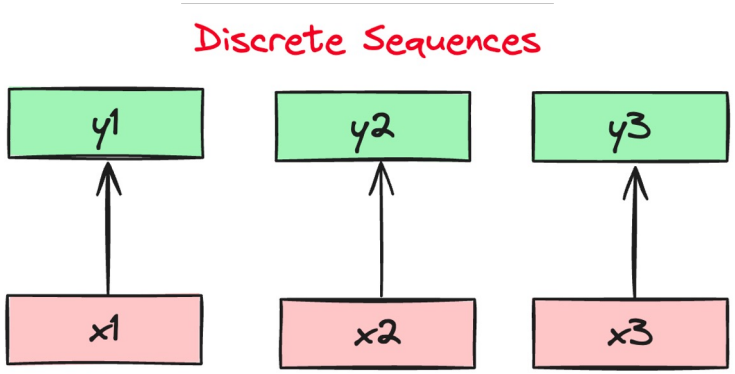
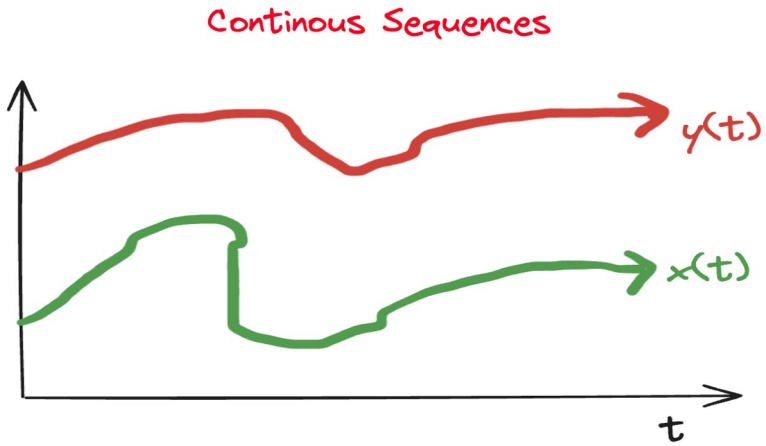
State Space Models 101

Shubham Gupta
in/shubhamgupta2208 shubhamg.in

Sequence Modelling: Recap

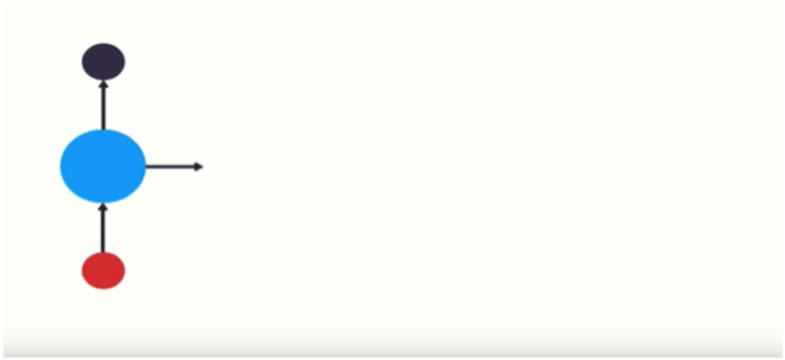
Goal: Map an input sequence to an output sequence





Sequence Modelling: Models

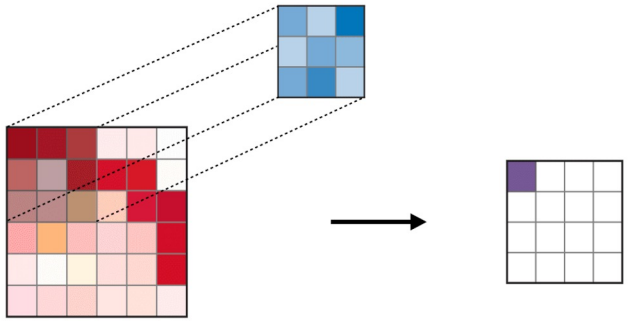
RNN



Source: Medium

Training: O(N) 👎 ‖
Inference: O(N) 😊
Performance: ☹️

CNN

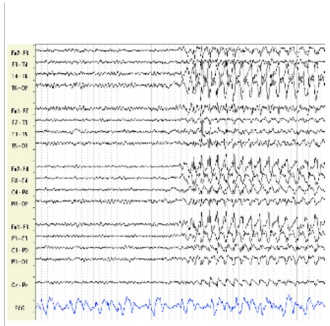


Source: Medium

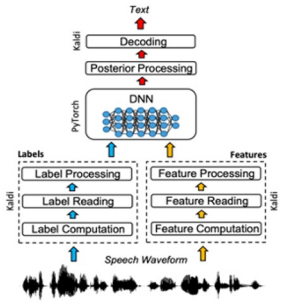
Training: O(N) 😊 ‖
Inference: O(N) 😊
Performance: ☹️

Sequence Modelling: Transformers

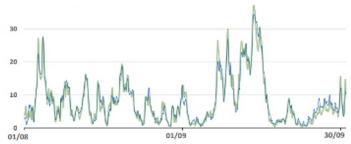
- SoTA on sequence modelling tasks
- Struggle to scale over long sequences
- Why do we care?
  - Enable new capabilities
  - Model other sequential data



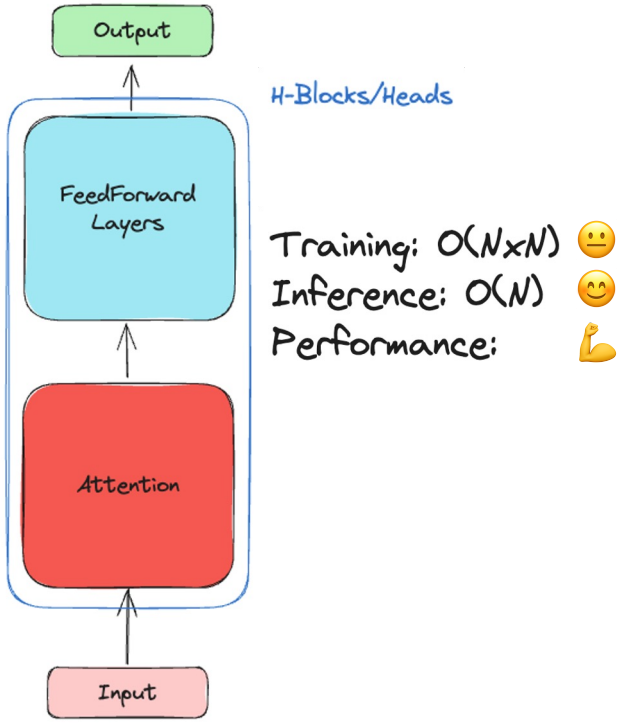
EEG/ECG



Audio



Energy Forecasting



Long Range Arena Benchmark

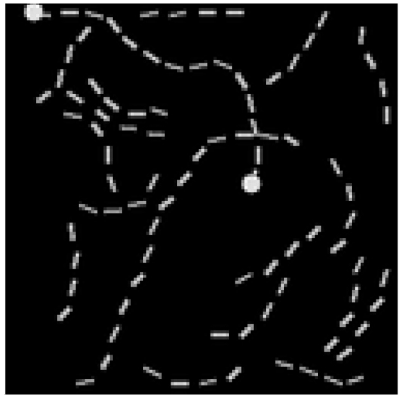
- LRA [Yi Tay et al., 2020]
- Measure long-context model quality
- Multiple input modalities
- Total tasks: 6
- Input sequence length: 1k-16k
- Transformer Performance:
  - Avg. Accuracy: **52%** 👎
  - Unable to solve Path-X

ListOps

[MAX 4 3 [MIN 2 3] 1 0 [MEDIAN 1 5 8 9, 2]]

Output: 5

Path-Finding



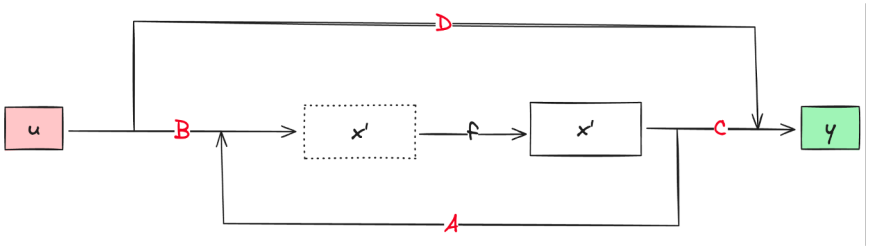
Ideal Model

Training: O(N) 😊 ||
Inference: O(N) 😊
Performance: 💪

Obtain best of all models for long-context?

State Space Models

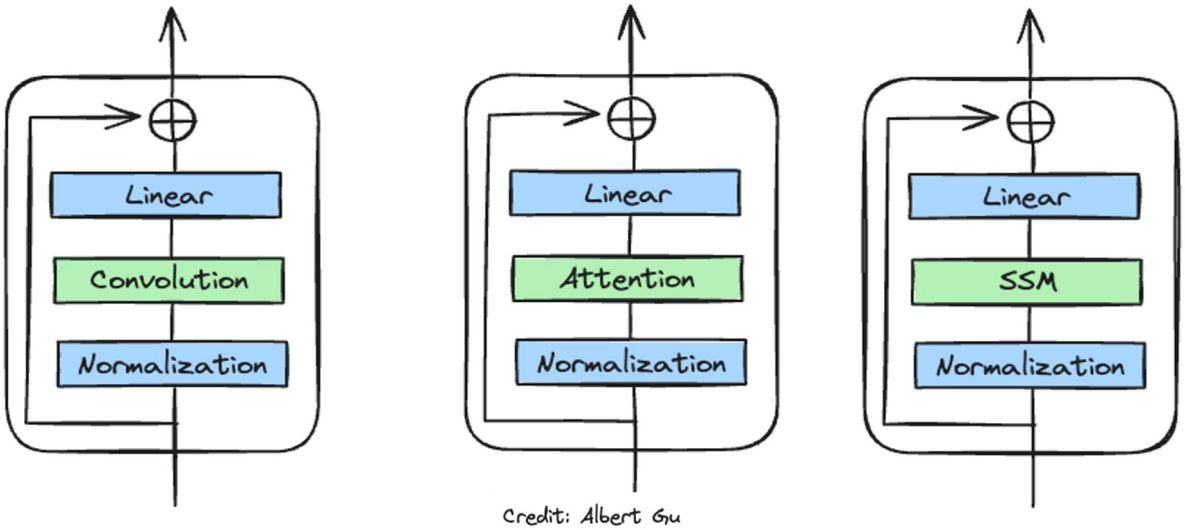
- Kalman, 1960
- Used in control theory/signal processing
- Modelled as *continuous-timed* differential equation
- 1-layer, Linear Model
- Time - Invariant



$$x'(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx'(t) + Du(t)$$

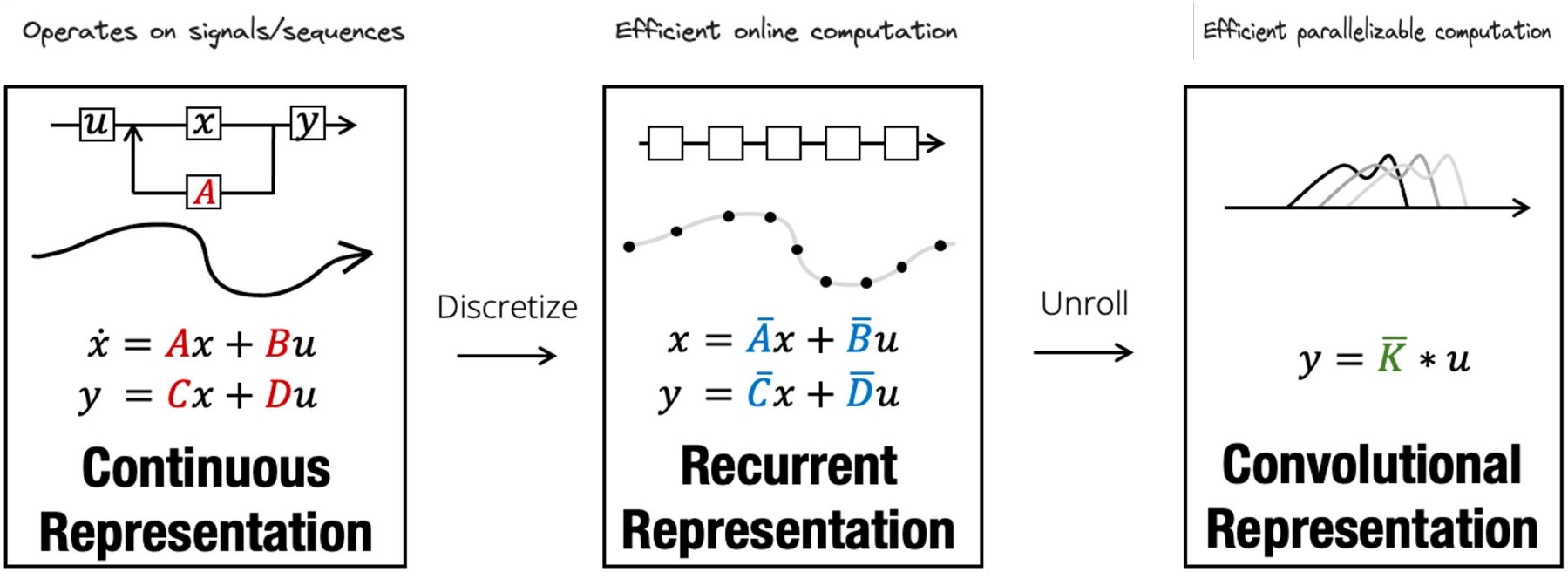
Deep SSMs

- Deep, non-linear model
- Deterministic transformation



Credit: Albert Gu

SSM Properties



Source: Albert Gu

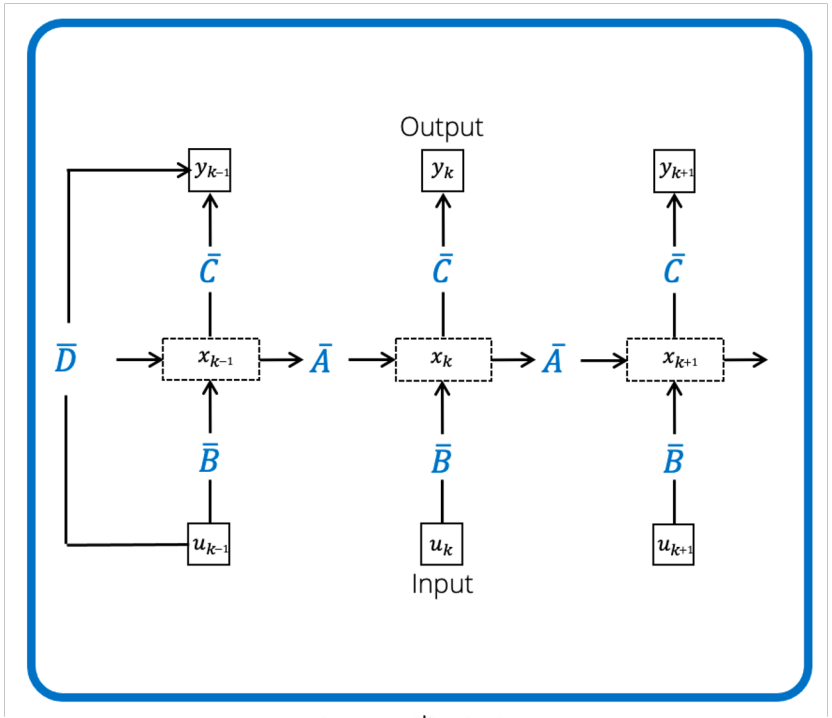
SSM: Recurrent

$$x'(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx'(t) + Du(t)$$

Discretize: $\bar{A} = I + \Delta A$

Recurrent "hidden state" $x_k = \bar{A}x_{k-1} + \bar{B}u_k$

Out Projection $y_k = \bar{C}x_k + \bar{D}u_k$



Source: Albert Gu

SSM: Convolutional

$$x_k = \bar{A}x_{k-1} + \bar{B}u_k \qquad y_k = \bar{C}x_k + \bar{D}u_k$$

Expand the terms

$$x_0 = \bar{B}u_0 \quad x_1 = \bar{A}\bar{B}u_0 + \bar{B}u_1 \quad x_2 = \bar{A}^2\bar{B}u_0 + \bar{A}\bar{B}u_1 + \bar{B}u_2$$

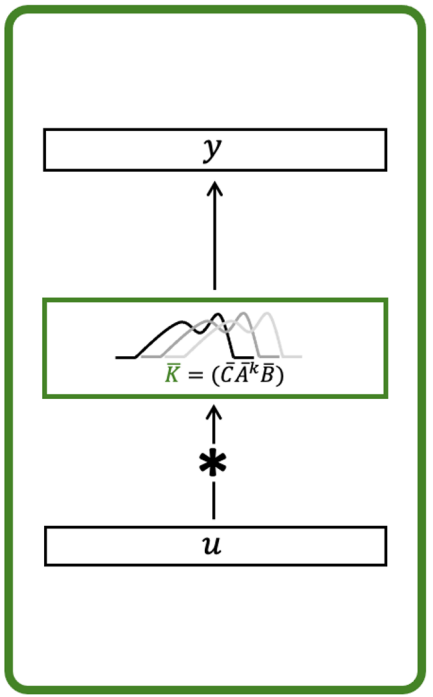
Output will be a linear projection of state

$$y_0 = \bar{C}\bar{B}u_0 + \bar{D}u_0$$
$$y_1 = \bar{C}\bar{A}\bar{B}u_0 + \bar{C}\bar{B}u_1 + \bar{D}u_1$$
$$y_2 = \bar{C}\bar{A}^2\bar{B}u_0 + \bar{C}\bar{A}\bar{B}u_1 + \bar{C}\bar{B}u_2 + \bar{D}u_2$$

No non-linearity => Simple computation

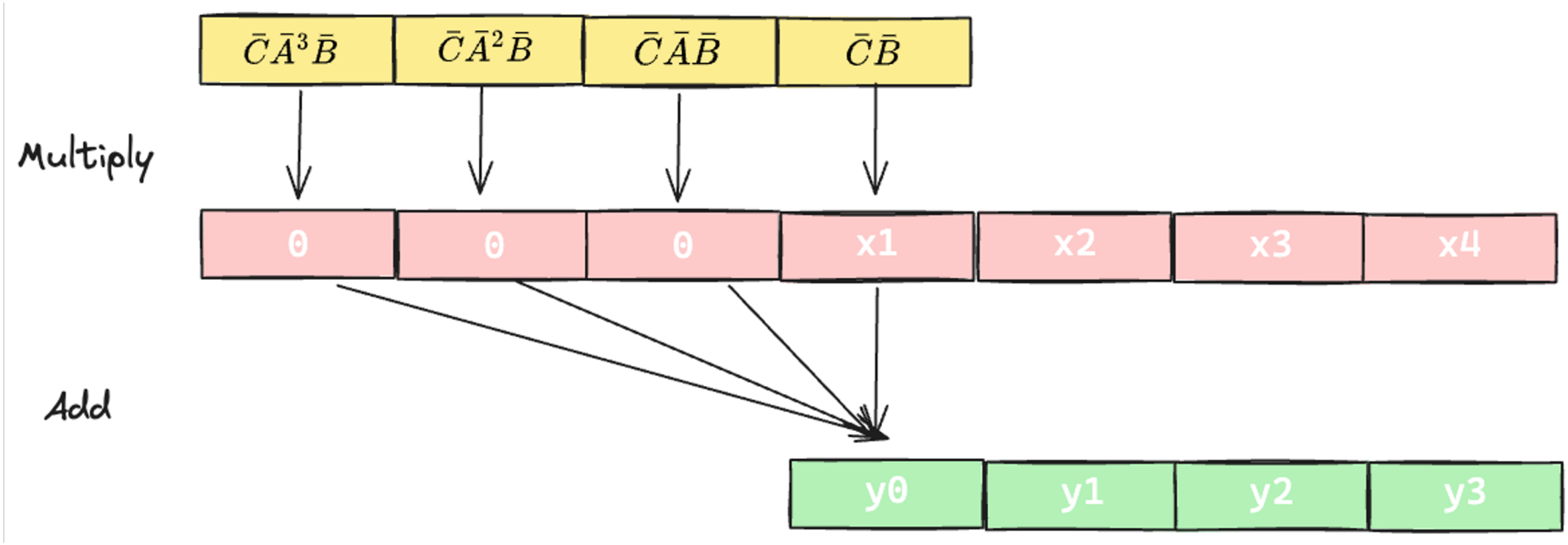
Extracting common coefficients, we get the SSM Kernel

$$\bar{K} = (\bar{C}\bar{A}^i\bar{B})_{i \in L}$$

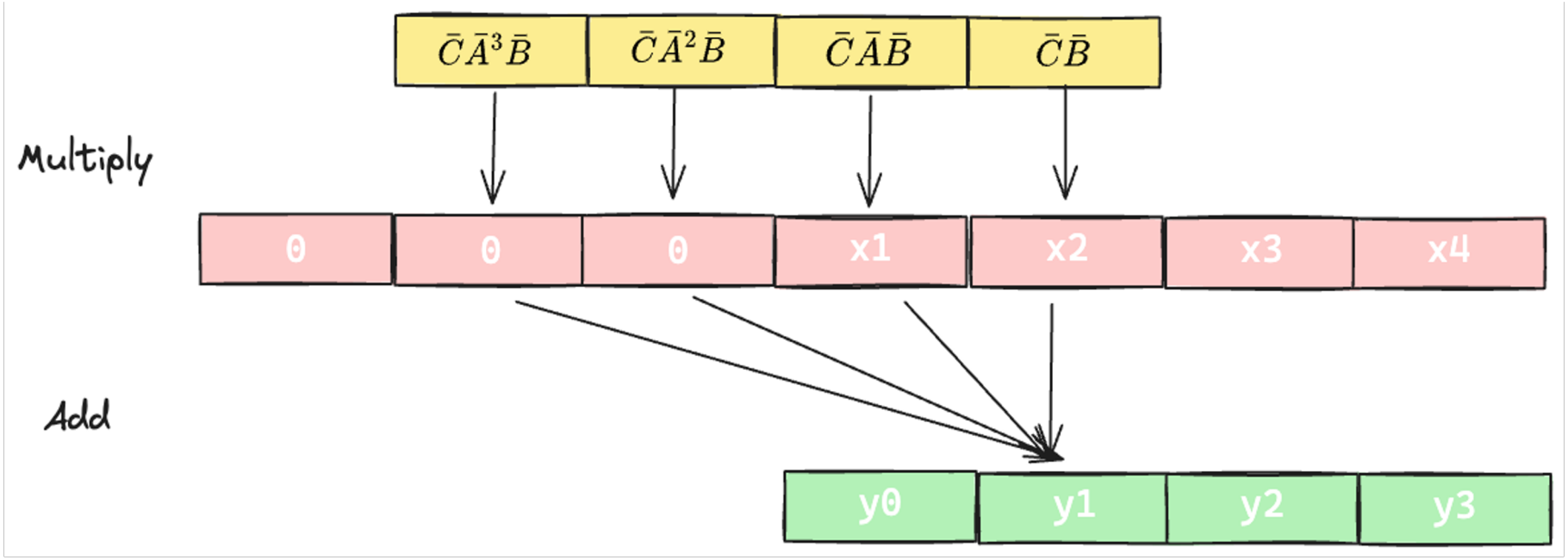


Source: Albert Gu

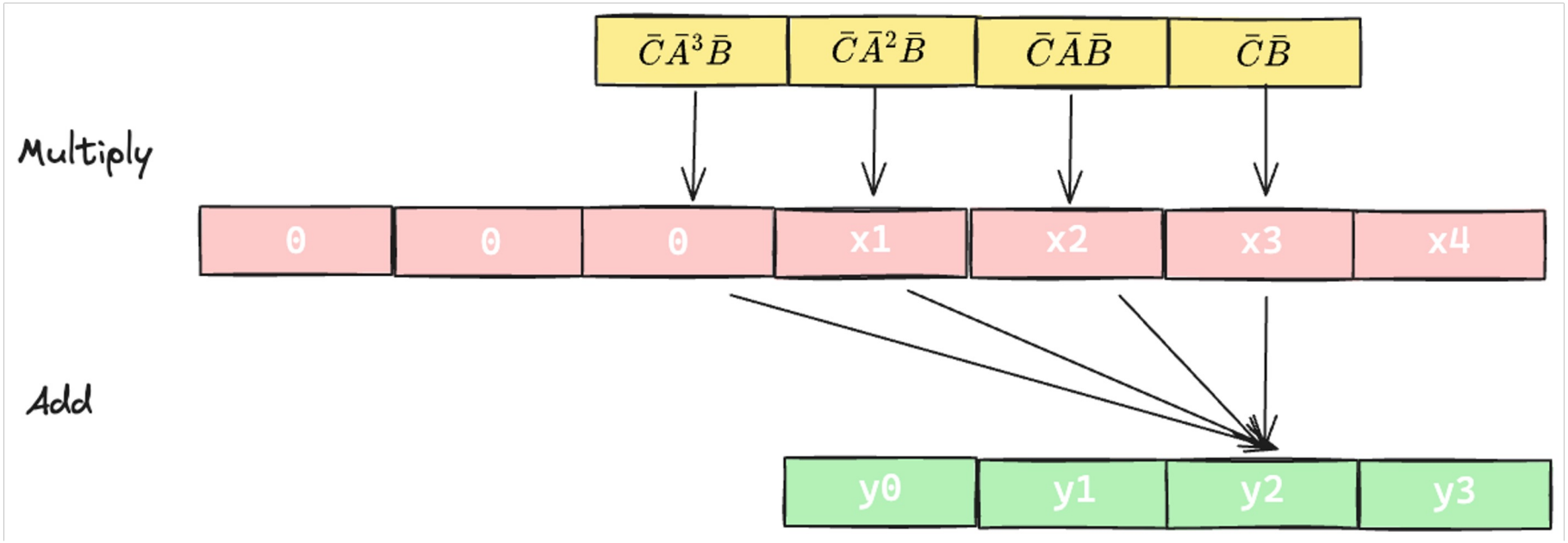
SSM: Convolutional Kernel



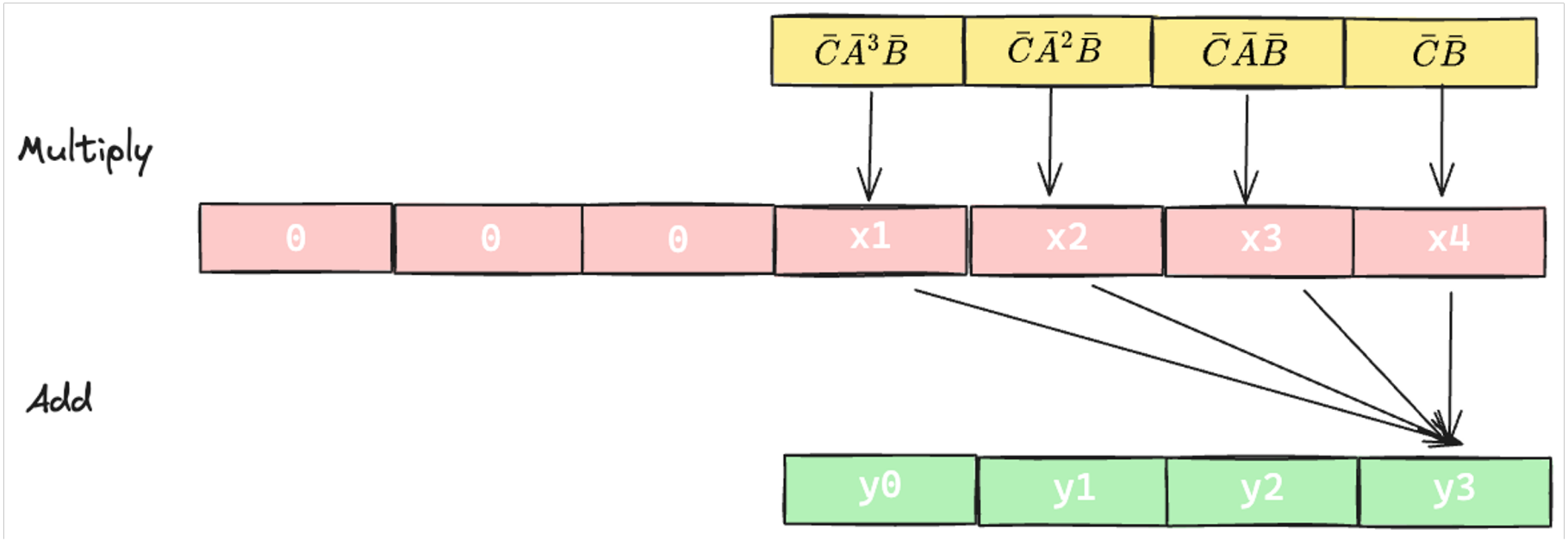
SSM: Convolutional Kernel



SSM: Convolutional Kernel



SSM: Convolutional Kernel



Deep SSM: Challenges

- **Modelling Challenge**
  - SSMs inherit problems of CNN,RNN on LRA
  - Random init A
    - **60%** acc sequential MNIST 😕
- **Computation Challenge**
  - SSM has nice properties if $\bar{A}$ and $\bar{K}$ are known
  - Computing them is **hard**!
- **Computing the Kernel**
  - A power 🔼-> vanishing gradient?
  - A power 🔼-> $O(D^2N)$ computation
    - Ideal -> $O(N)$

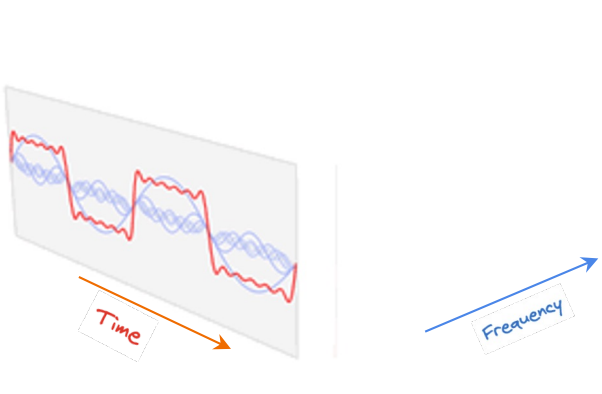
$$x_k = \bar{A}x_{k-1} + \bar{B}u_k$$

$$y_k = \bar{C}x_k + \bar{D}u_k$$

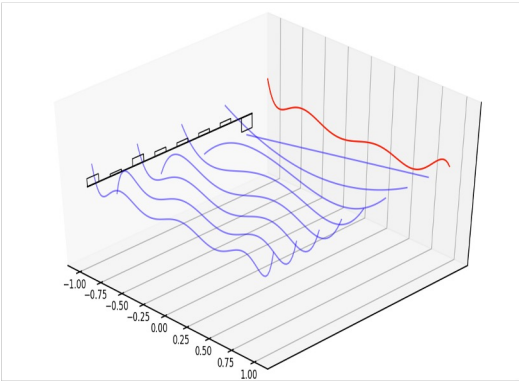
$$\bar{K} = (\bar{C}\bar{A}^i\bar{B})_{i\in L}$$

S4: Structured SSM

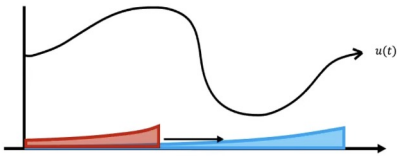
SSM + HiPPO + Structured Matrices



Fourier Transform



Legendre Polynomial Coefficients



$$x'(t) = \boldsymbol{A}x(t) + \boldsymbol{B}u(t)$$

$$\boldsymbol{A}_{nk} = \begin{cases} (2n+1)^{1/2}(2k+1)^{1/2} & \text{if } n > k \\ n+1 & \text{if } n = k \\ 0 & \text{if } n < k \end{cases}$$

HiPPO Matrix

S4: LRA Results

Model	ListOps	Text	Retrieval	Image	Pathfinder	Path-X	Avg
Transformer	36.37	64.27	57.46	42.44	71.40	✗	53.66
Reformer	37.27	56.10	53.40	38.07	68.50	✗	50.56
BigBird	36.05	64.02	59.29	40.83	74.87	✗	54.17
Linear Trans.	16.13	65.90	53.09	42.34	75.30	✗	50.46
Performer	18.01	65.40	53.82	42.77	77.05	✗	51.18
FNet	35.33	65.11	59.61	38.67	77.80	✗	54.42
Nyströmformer	37.15	65.52	79.56	41.58	70.94	✗	57.46
Luna-256	37.25	64.57	79.29	47.38	77.72	✗	59.37
S4	**59.60**	**86.82**	**90.90**	**88.65**	**94.20**	**96.35**	**86.09**

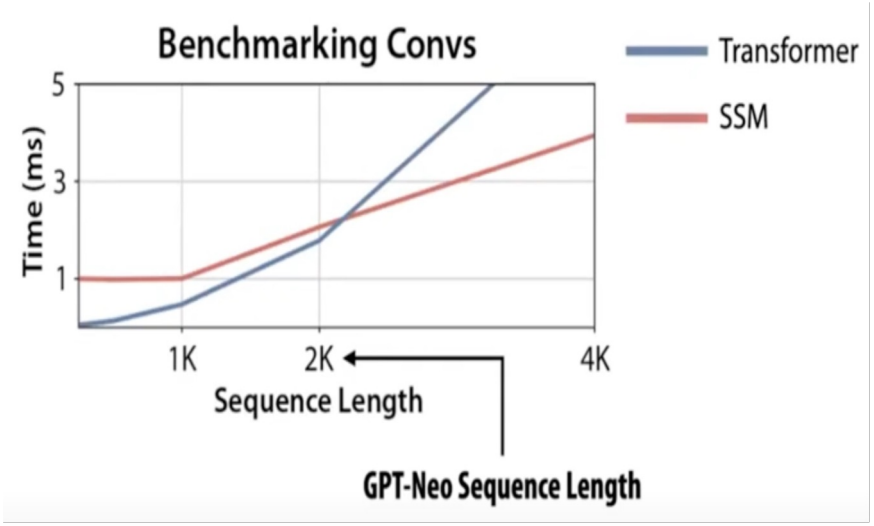
S4: LRA Results

S4: Problems

Language Modelling

Model	PPL(OWT)
Transformer	**20.6**
S4D	24.9
GSS	24.0

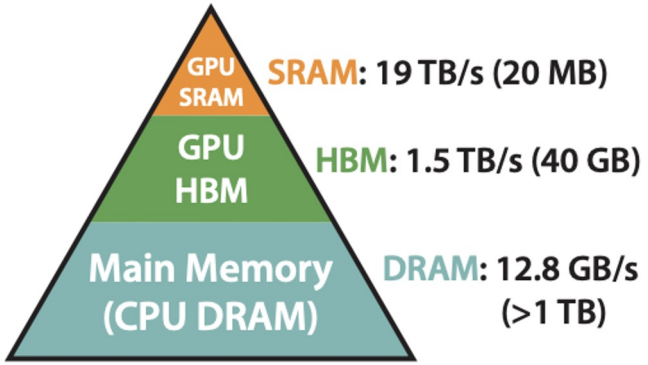
Short Sequence Efficiency



Source: Albert Gu

Background: GPU Memory

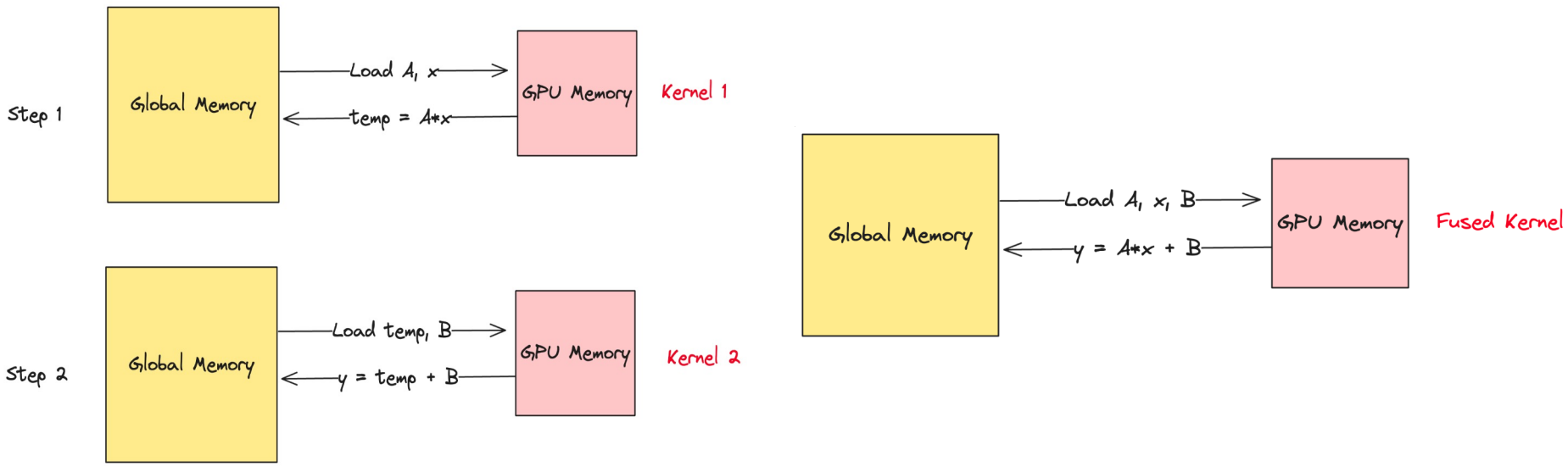
- Components:
  - DRAM
  - HBM
  - SRAM
- Problems:
  - Size
  - Speed
- Optimizations:
  - Kernel Fusion
  - Recomputation
  - Parallelism



	A100 40GB PCIe	A100 80GB PCIe	A100 40GB SXM	A100 80GB SXM
GPU Memory	40GB HBM2	80GB HBM2e	40GB HBM2	80GB HBM2e
GPU Memory Bandwidth	1,555GB/s	1,935GB/s	1,555GB/s	2,039GB/s

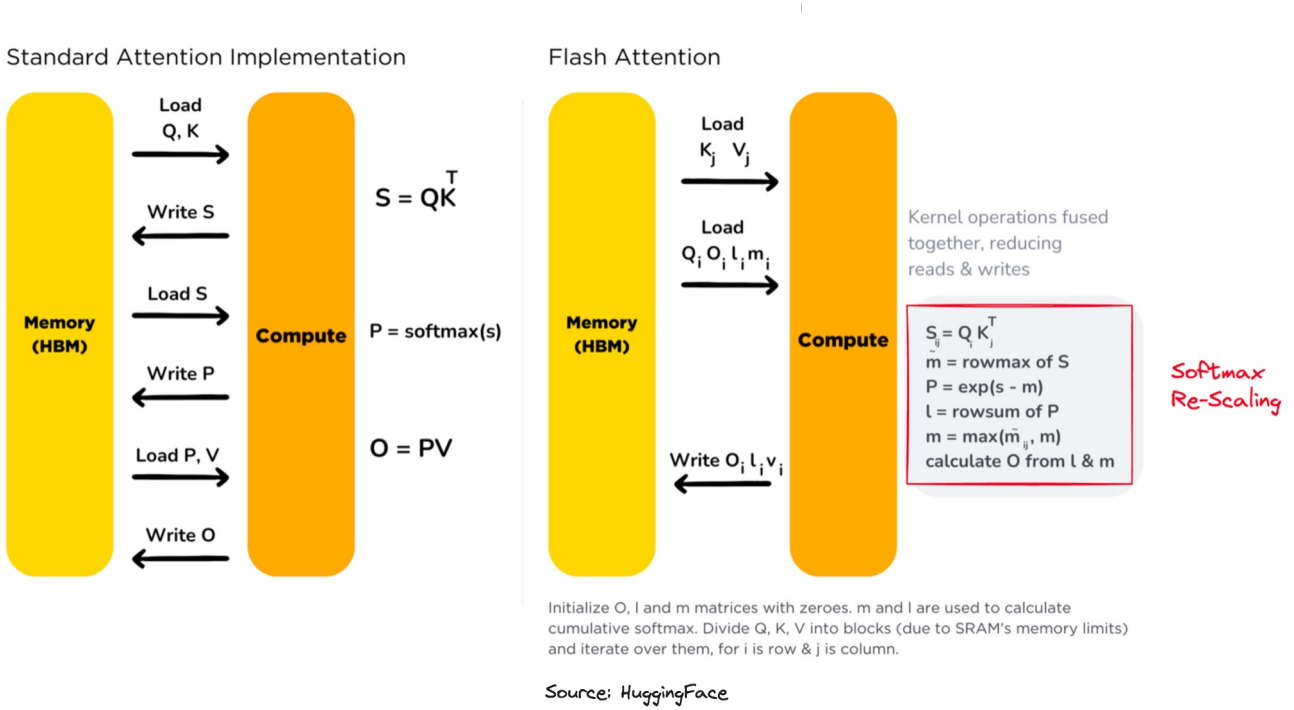
GPU: Kernel Fusion

$$y = Ax + B$$



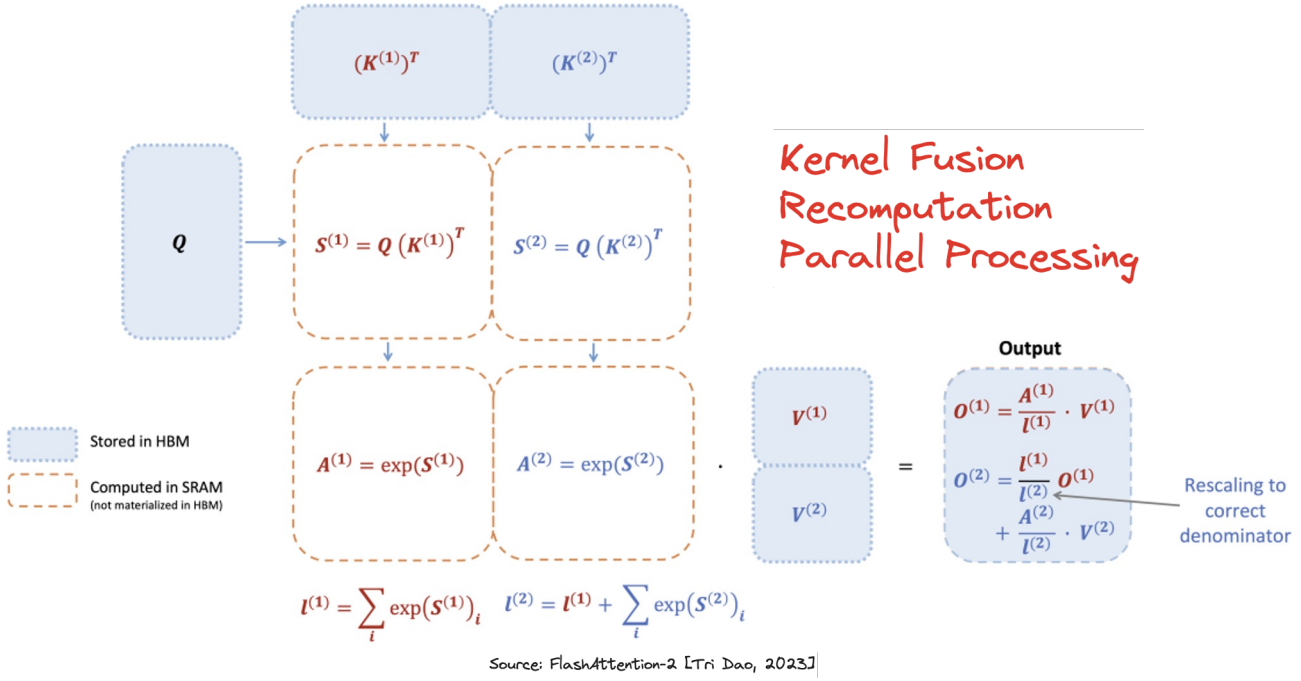
Flash Attention: Kernel Fusion

Attention = softmax(Q * K^T) * V



Source: HuggingFace

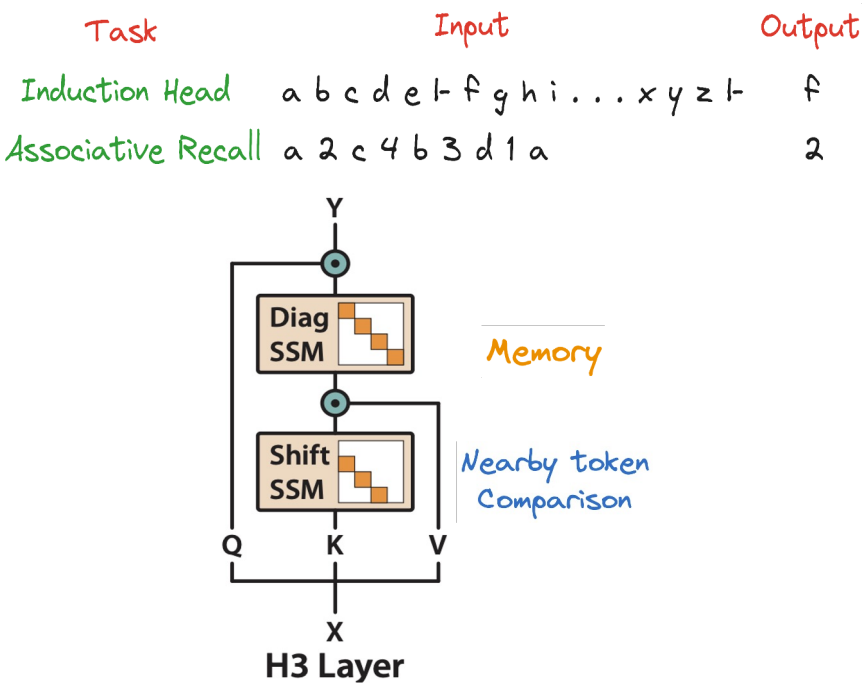
Flash Attention: Implementation



Source: FlashAttention-2 [Tri Dao, 2023]

Hungry-Hungry HiPPOs (H3)

- Improve SSM on synthetic tasks
- Combine two SSMs in a single network
- Replaces all Transformer layers, except two
- Trained using FlashConv + Fusion



H3: Results

Language Modelling

Model	PPL(OWT)
Transformer	**20.6**
S4D	24.9
GSS	24.0
H3	**21.0**
H3 (with two attention blocks)	**19.6**

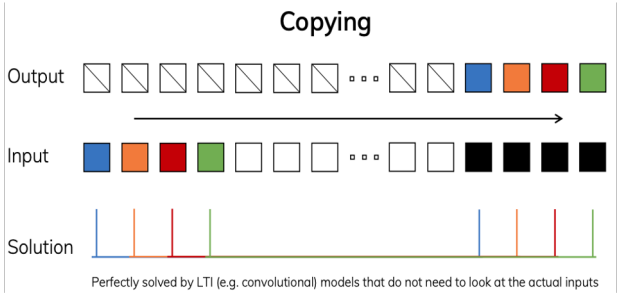
Source: HazyResearch

H3: Scaling Results

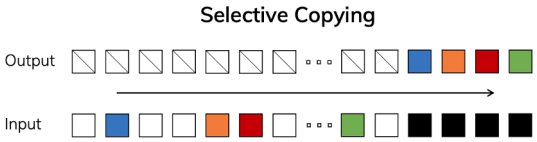
Model	Pile	OpenWebText	WikiText103
GPT-2 small (125M)	19.0*	22.6	29.9
GPT-Neo-125M	9.4	22.6	26.3
Hybrid H3-125M	**8.8**	**20.9**	**23.7**
GPT-2 medium (355M)	13.9*	17.0	21.8
Hybrid H3-355M	**7.1**	**15.9**	**16.9**
GPT-2 XL (1.5B)	12.4*	12.9	17.0
GPT-Neo-1.3B	6.2	13.1	13.3
Hybrid H3-1.3B	**6.0**	**12.4**	**12.5**
GPT-Neo-2.7B	5.7	11.7	11.5
Hybrid H3-2.7B	**5.4**	**11.0**	**10.6**

Model Perplexity [Dan Fu et al., 2022]

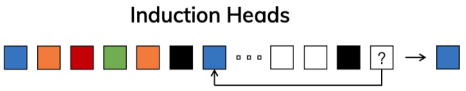
S4: Additional Problems



Task: Rewrite input one token at a time, but time-shifted



Task: Given a comment, rewrite it by filtering certain words.

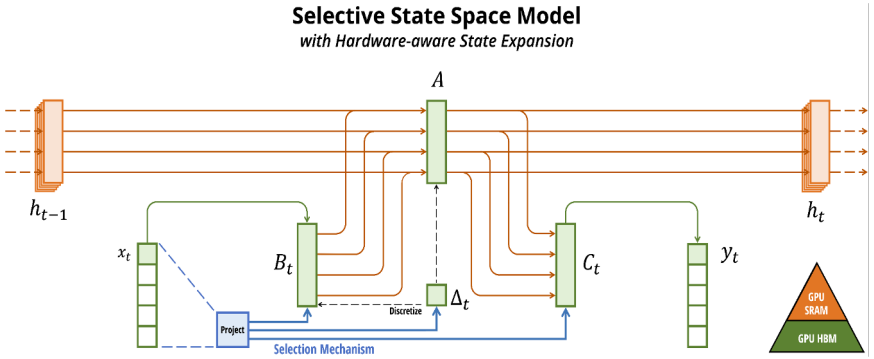


Task: Solve new objective given "few-shot" examples

$\bar{A}$, $\bar{B}$ and $\bar{C}$ are independent of input

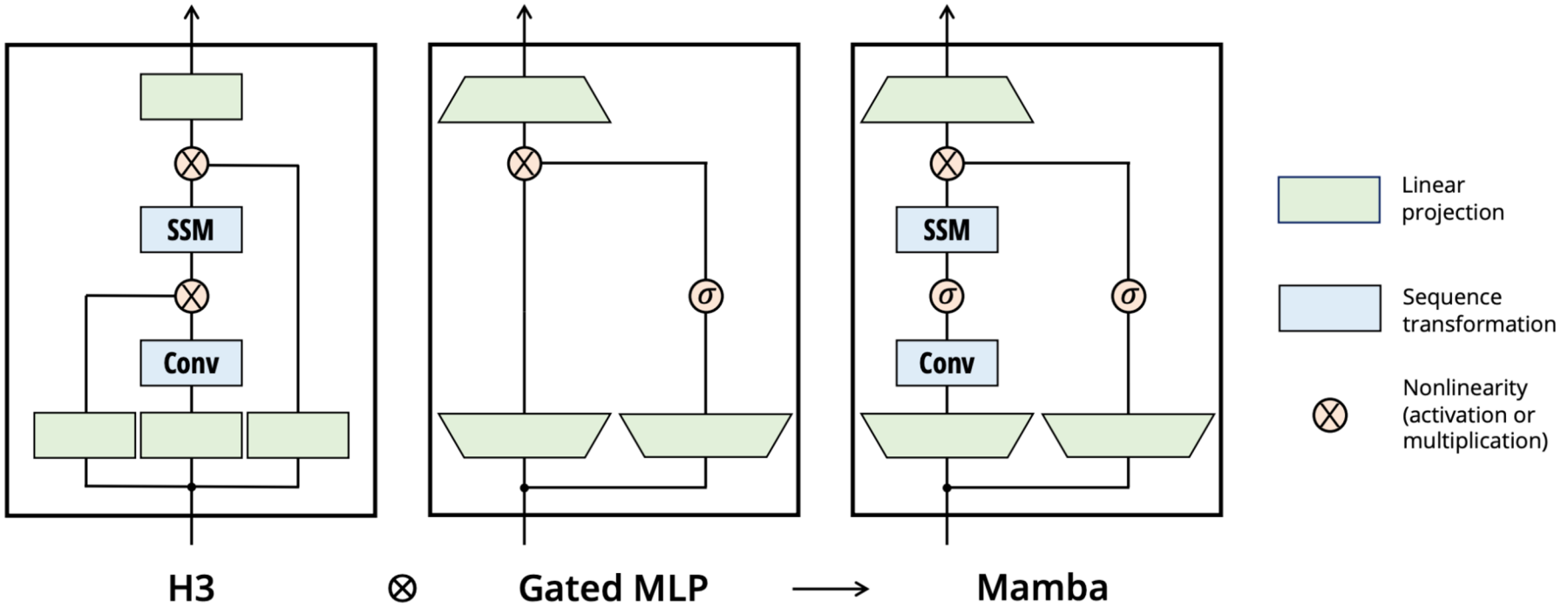
Mamba(S6): Selective SSM

- SSSSSS...🐍
- Improves SSM performance on copying tasks
- Handles input data that is varying in time
- Only supports recurrent form



SSM(H3) + Selection Mechanism + Parallel Scan

Mamba: Architecture



Source: Mamba[Albert Gu, Tri Dao 2023]

Mamba: Implementation

Algorithm 1 SSM (S4)

Input: $x : (\mathtt{B}, \mathtt{L}, \mathtt{D})$
Output: $y : (\mathtt{B}, \mathtt{L}, \mathtt{D})$
1: $\boldsymbol{A} : (\mathtt{D}, \mathtt{N}) \leftarrow$ Parameter
▷ Represents structured $N \times N$ matrix
2: $\boldsymbol{B} : (\mathtt{D}, \mathtt{N}) \leftarrow$ Parameter
3: $\boldsymbol{C} : (\mathtt{D}, \mathtt{N}) \leftarrow$ Parameter
4: $\Delta : (\mathtt{D}) \leftarrow \tau_\Delta(\text{Parameter})$
5: $\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}} : (\mathtt{D}, \mathtt{N}) \leftarrow \text{discretize}(\Delta, \boldsymbol{A}, \boldsymbol{B})$
6: $y \leftarrow \text{SSM}(\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}, \boldsymbol{C})(x)$
▷ Time-invariant: recurrence or convolution
7: **return** y

Algorithm 2 SSM + Selection (S6)

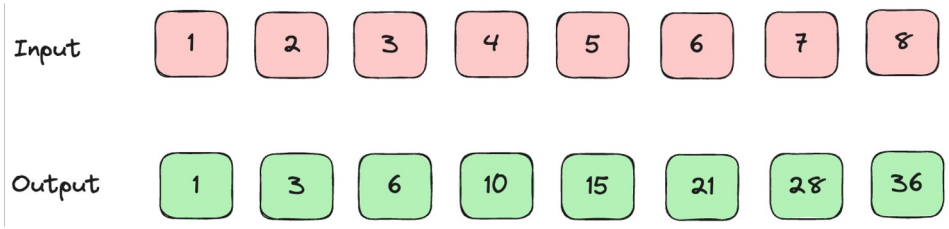
Input: $x : (\mathtt{B}, \mathtt{L}, \mathtt{D})$
Output: $y : (\mathtt{B}, \mathtt{L}, \mathtt{D})$
1: $\boldsymbol{A} : (\mathtt{D}, \mathtt{N}) \leftarrow$ Parameter
▷ Represents structured $N \times N$ matrix
2: $\boldsymbol{B} : (\mathtt{B}, \mathtt{L}, \mathtt{N}) \leftarrow s_B(x)$
3: $\boldsymbol{C} : (\mathtt{B}, \mathtt{L}, \mathtt{N}) \leftarrow s_C(x)$
4: $\Delta : (\mathtt{B}, \mathtt{L}, \mathtt{D}) \leftarrow \tau_\Delta(\text{Parameter} + s_\Delta(x))$
5: $\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}} : (\mathtt{B}, \mathtt{L}, \mathtt{D}, \mathtt{N}) \leftarrow \text{discretize}(\Delta, \boldsymbol{A}, \boldsymbol{B})$
6: $y \leftarrow \text{SSM}(\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}, \boldsymbol{C})(x)$
▷ Time-varying: recurrence (*scan*) only
7: **return** y

Linear layers

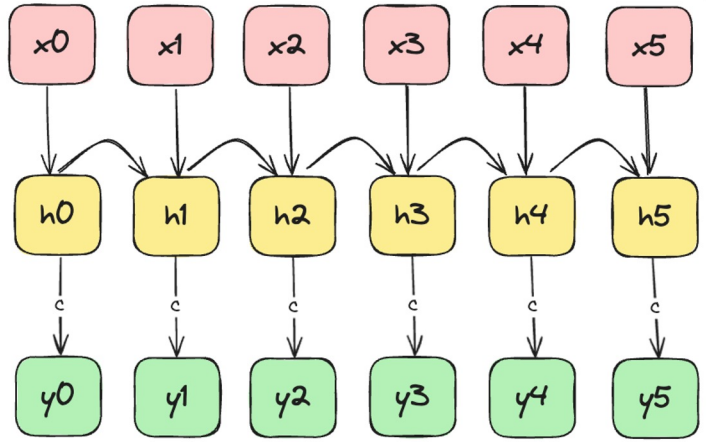
No parallel training 👎

Source: Mamba[Albert Gu, Tri Dao 2023]

Mamba: Scan Operation



Current val = Sum of previous val + input

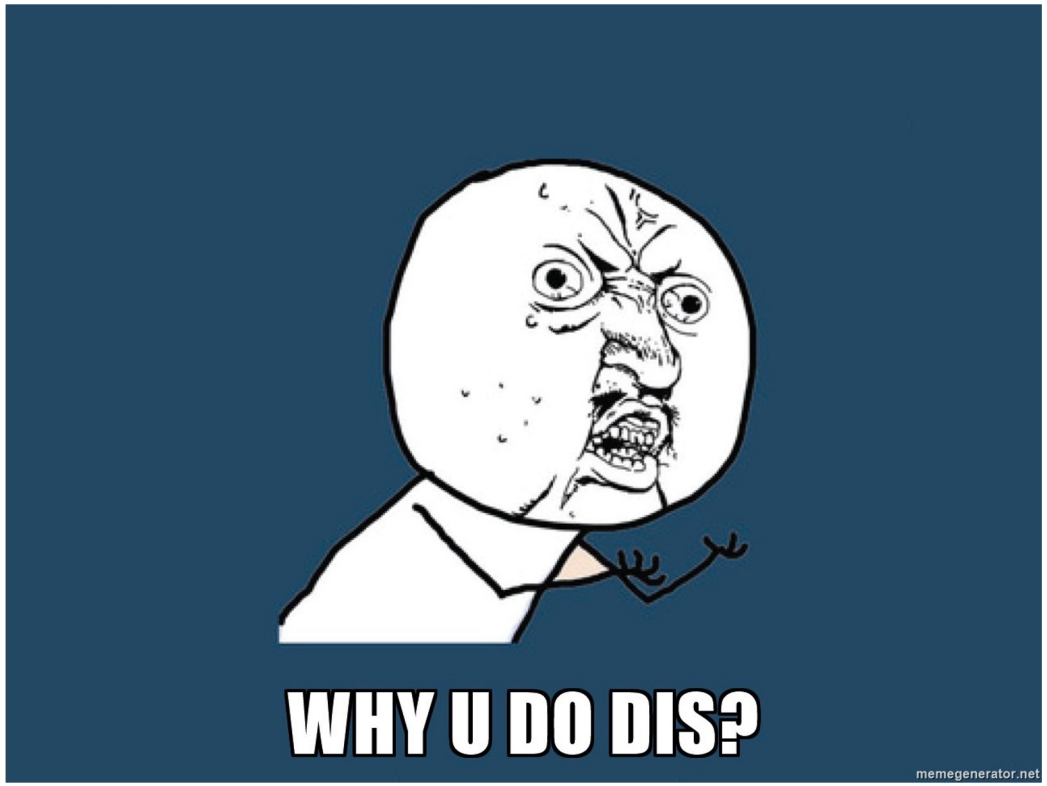


$$x_t = \bar{A}x_{t-1} + \bar{B}u_t$$

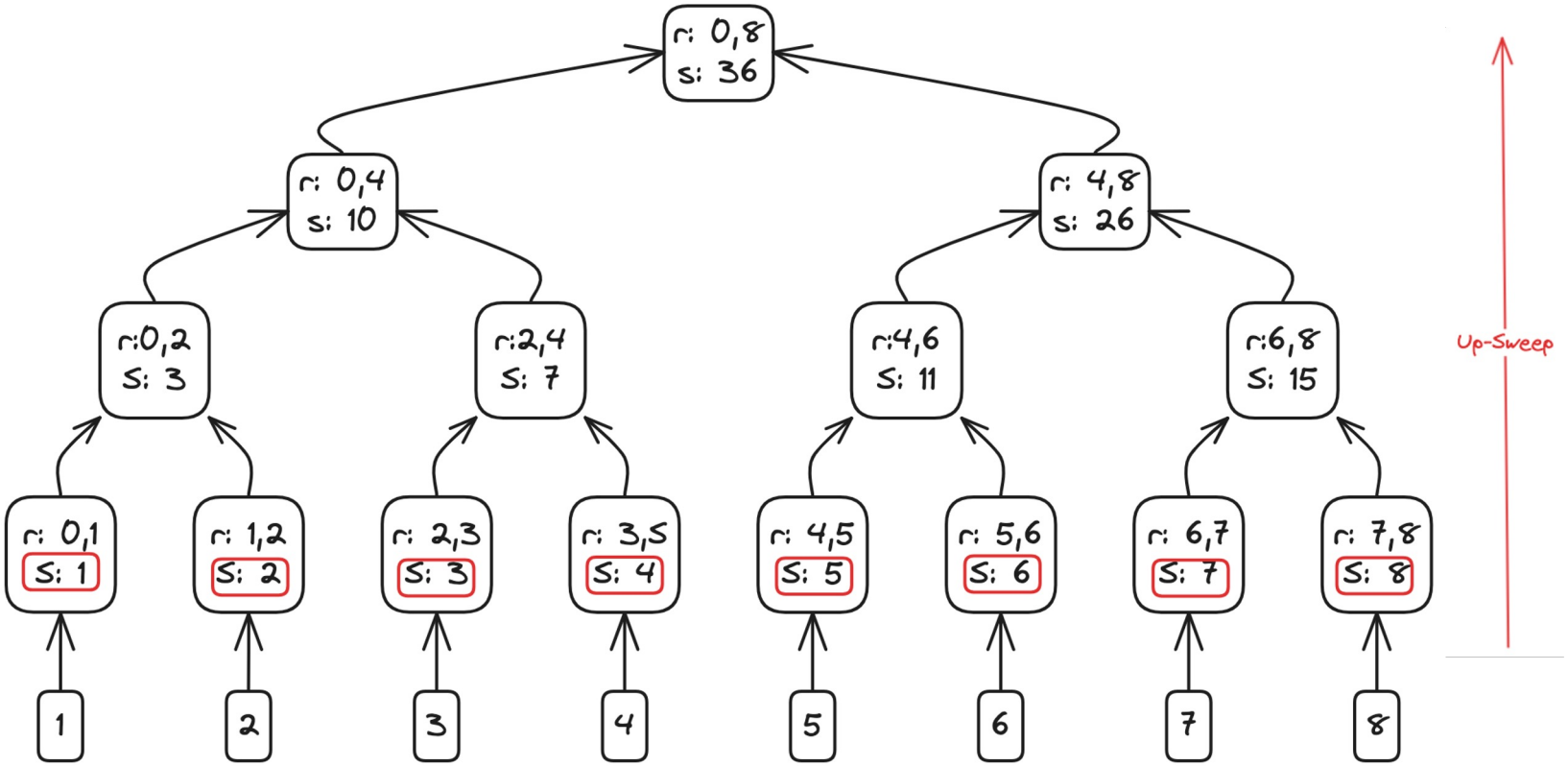
$$y_t = Cx_t$$

Current State: Sum of previous state + input

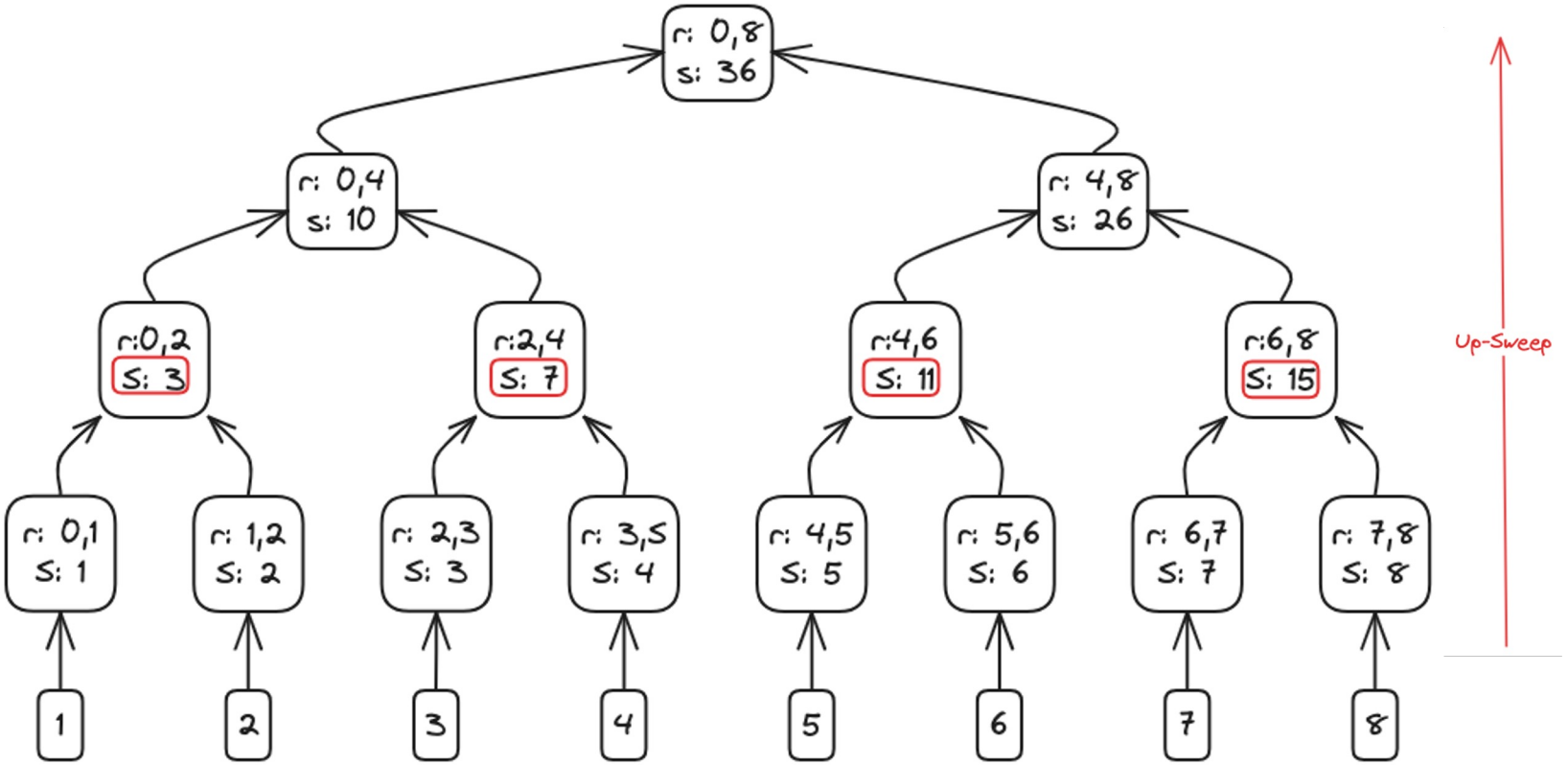
Parallelize a recurrence relation?



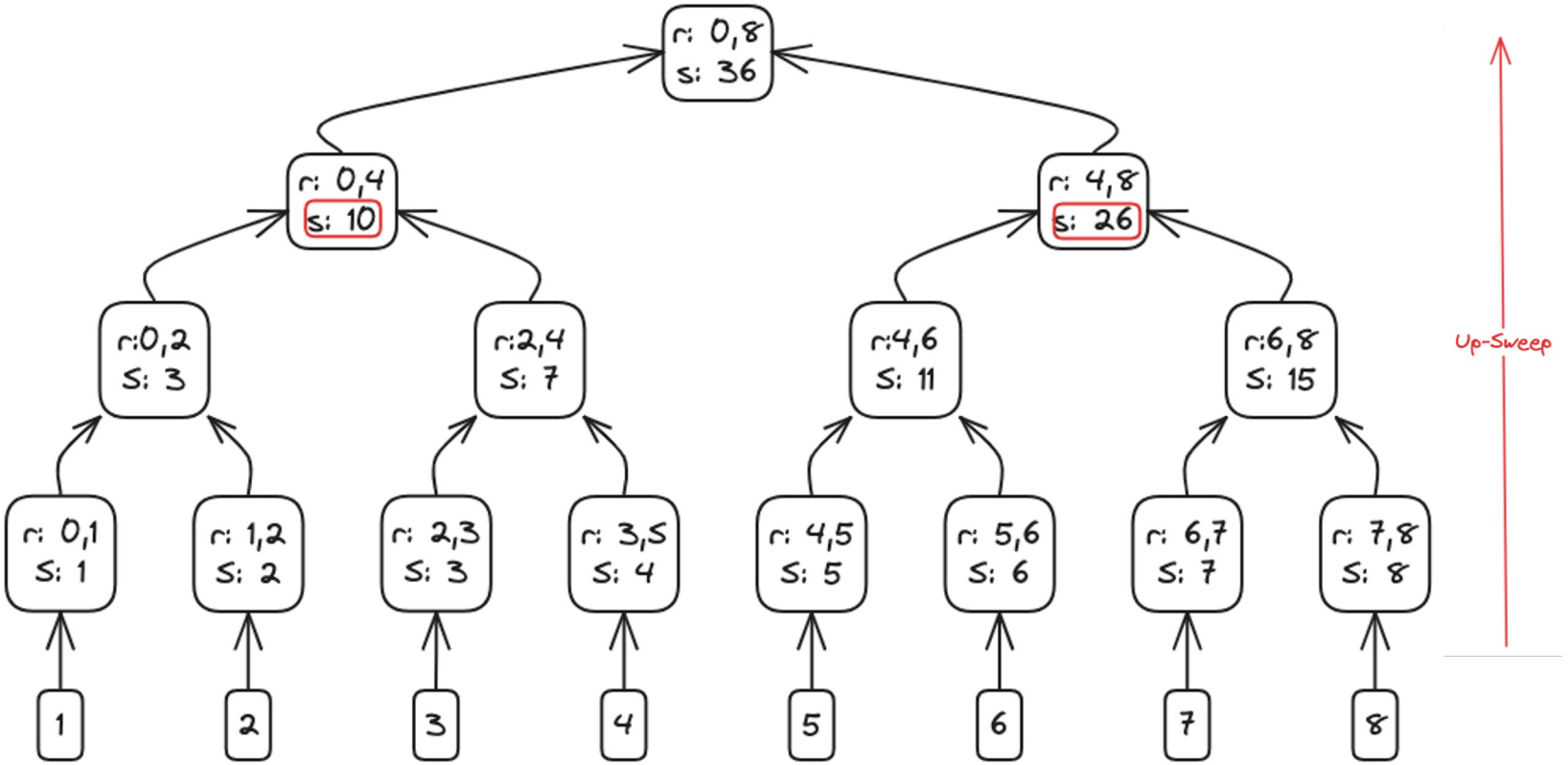
Parallel Scan: Up-Sweep



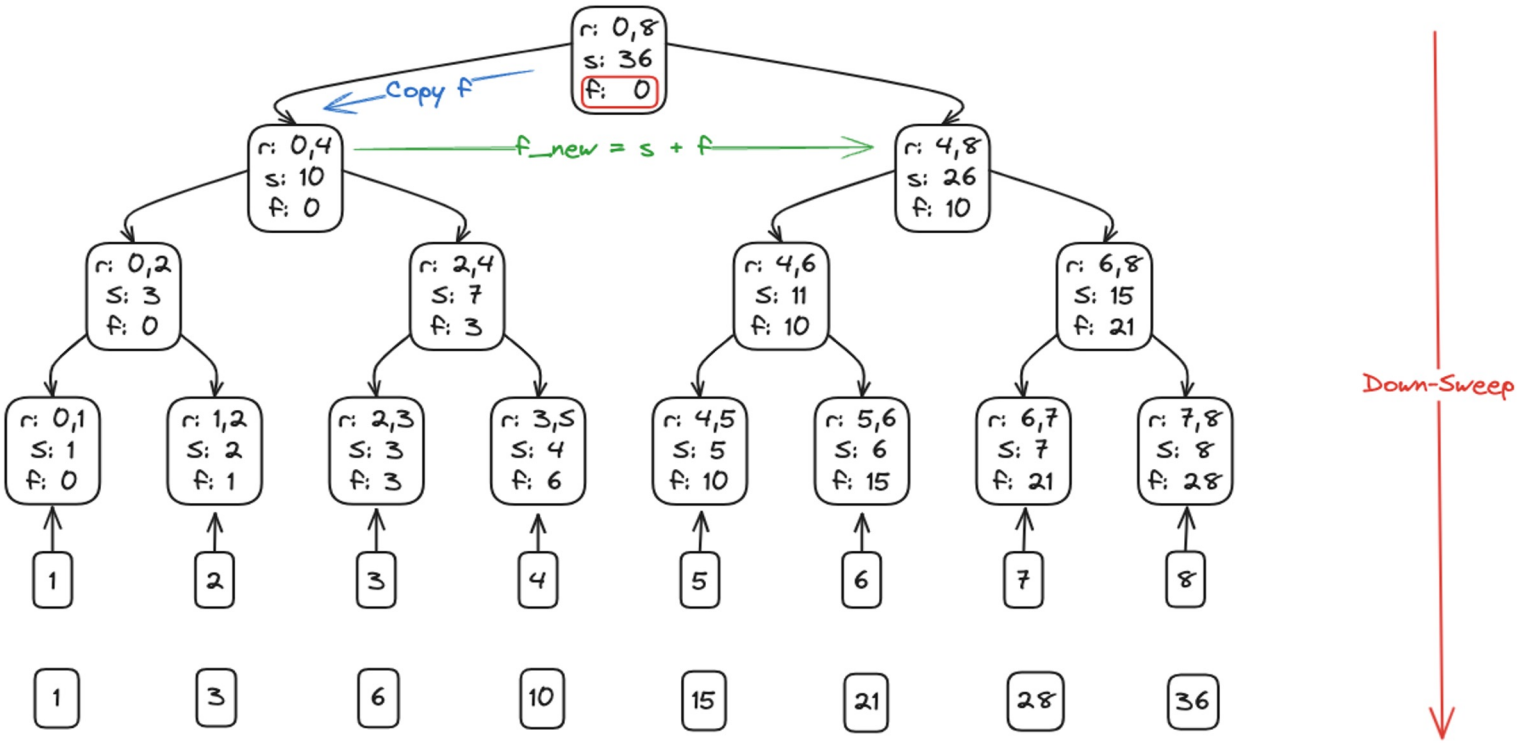
Parallel Scan: Up-Sweep



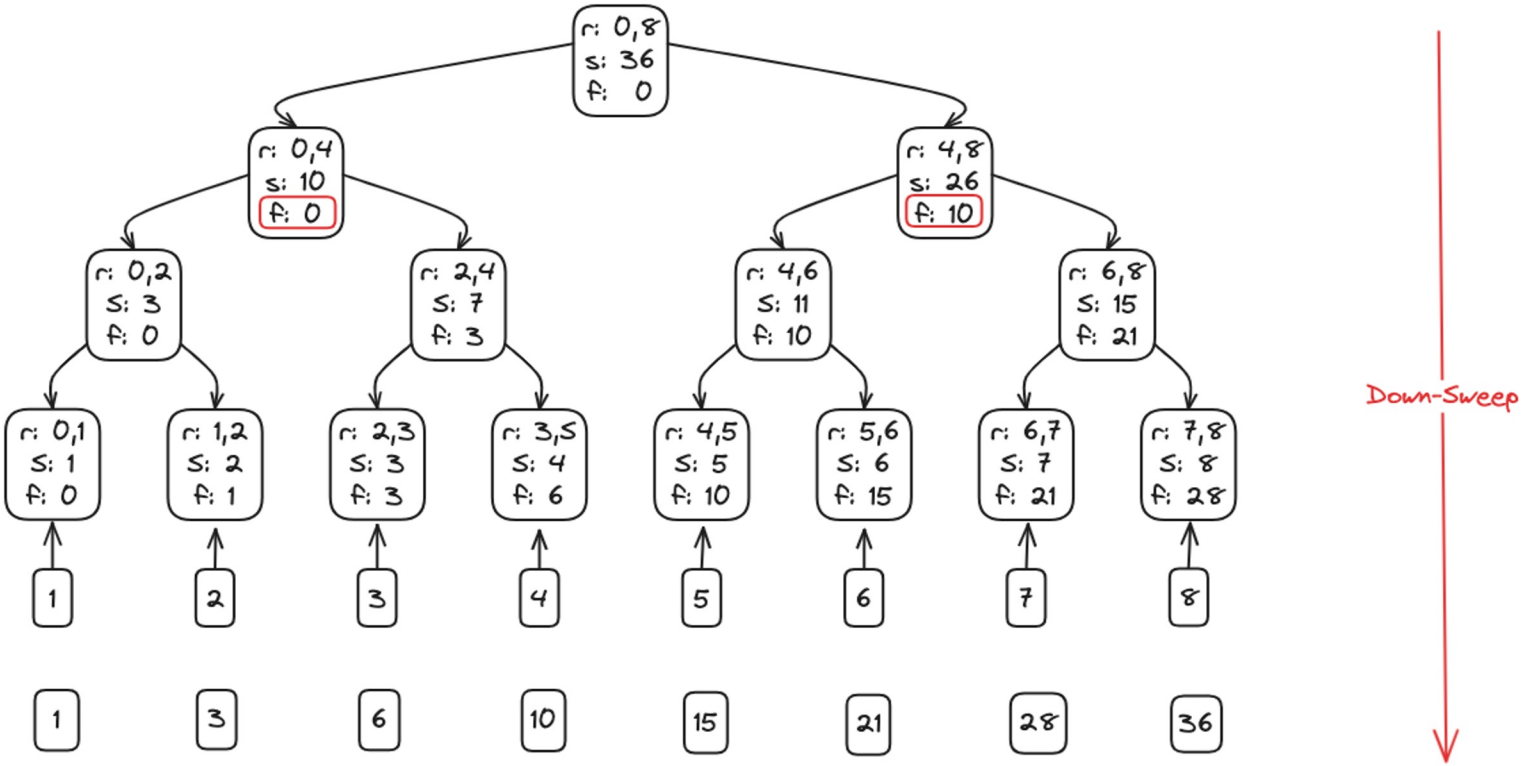
Parallel Scan: Up-Sweep



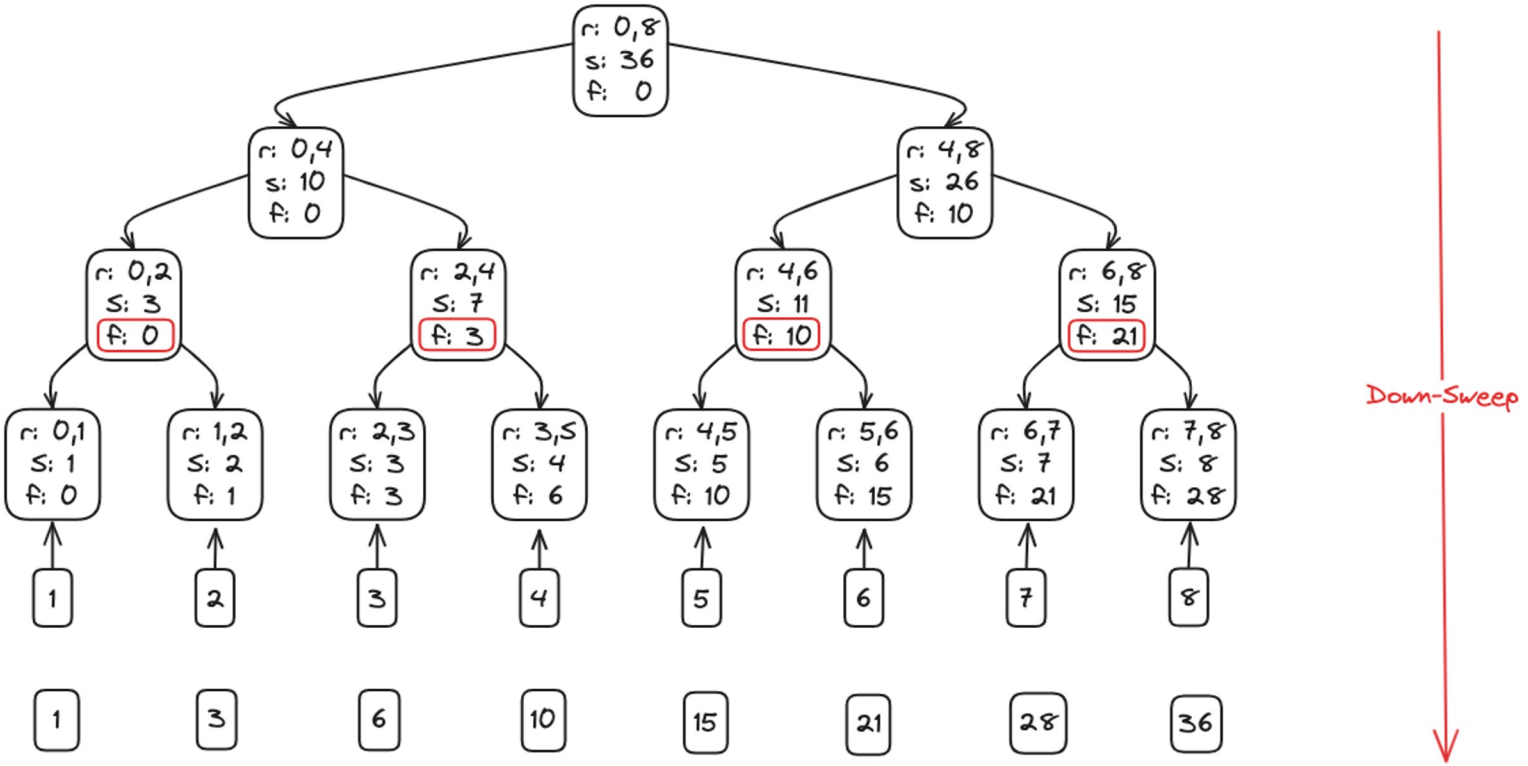
Parallel Scan: Down-Sweep



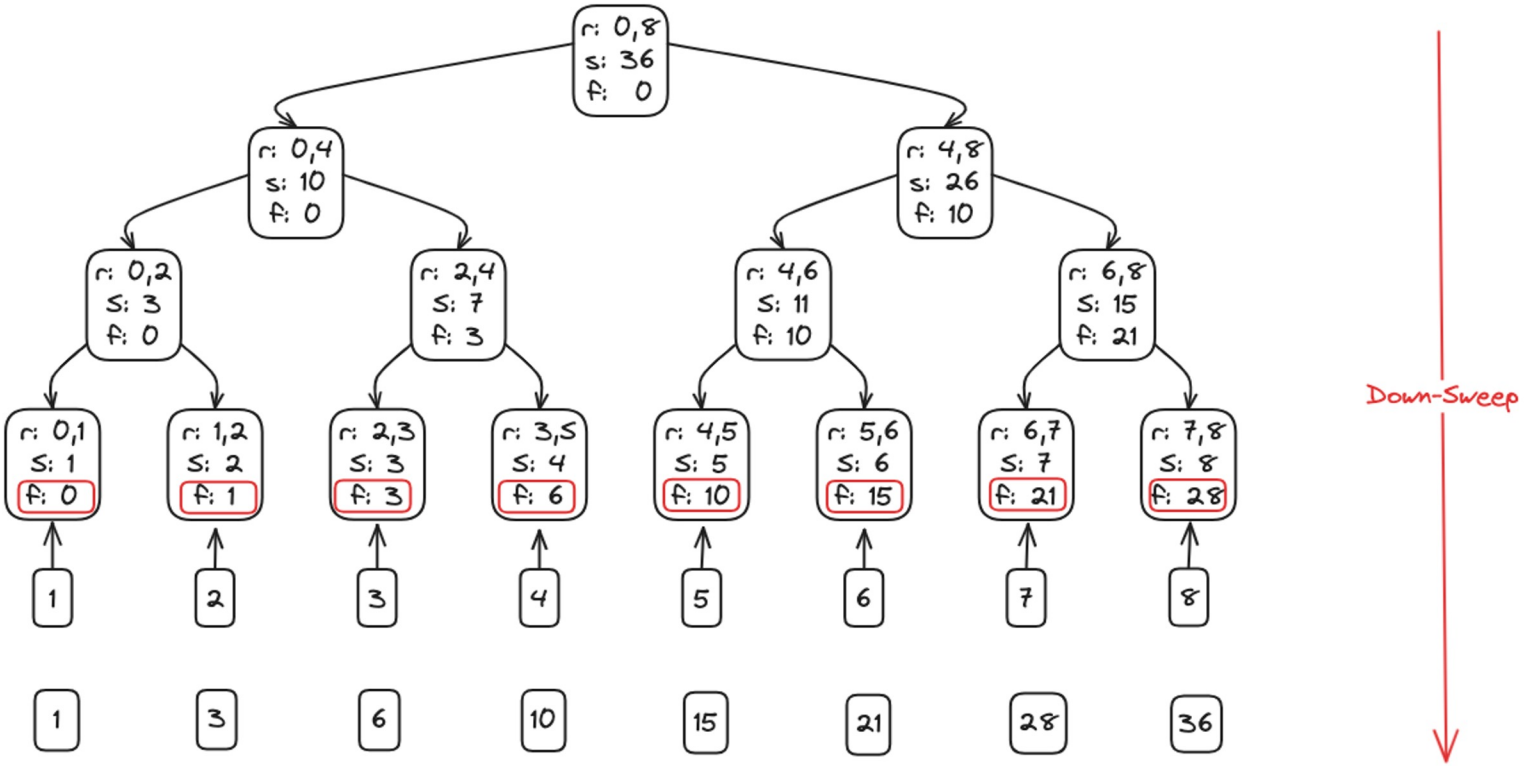
Parallel Scan: Down-Sweep



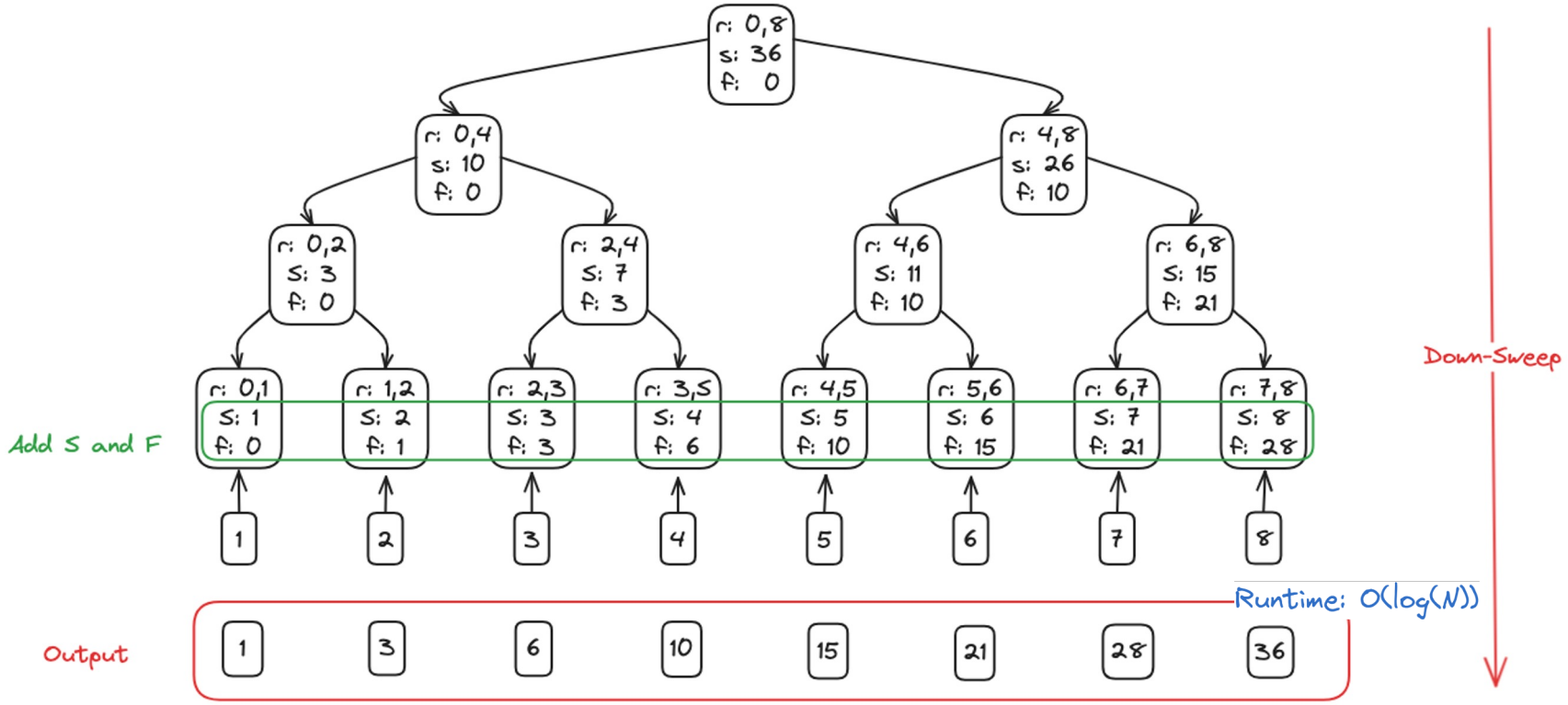
Parallel Scan: Down-Sweep



Parallel Scan: Down-Sweep



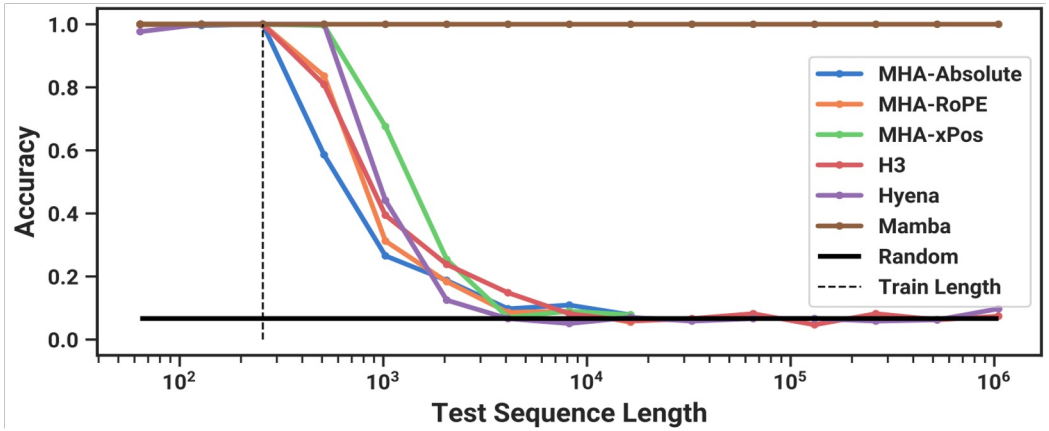
Parallel Scan: Final Result



Mamba: Synthetic Tasks Results

Model	Arch.	Layer	Acc.
S4	No gate	S4	18.3
-	No gate	S6	**97.0**
H3	H3	S4	57.0
Hyena	H3	Hyena	30.1
-	H3	S6	**99.7**
-	Mamba	S4	56.4
-	Mamba	Hyena	28.4
Mamba	Mamba	S6	**99.8**

Selective Copying



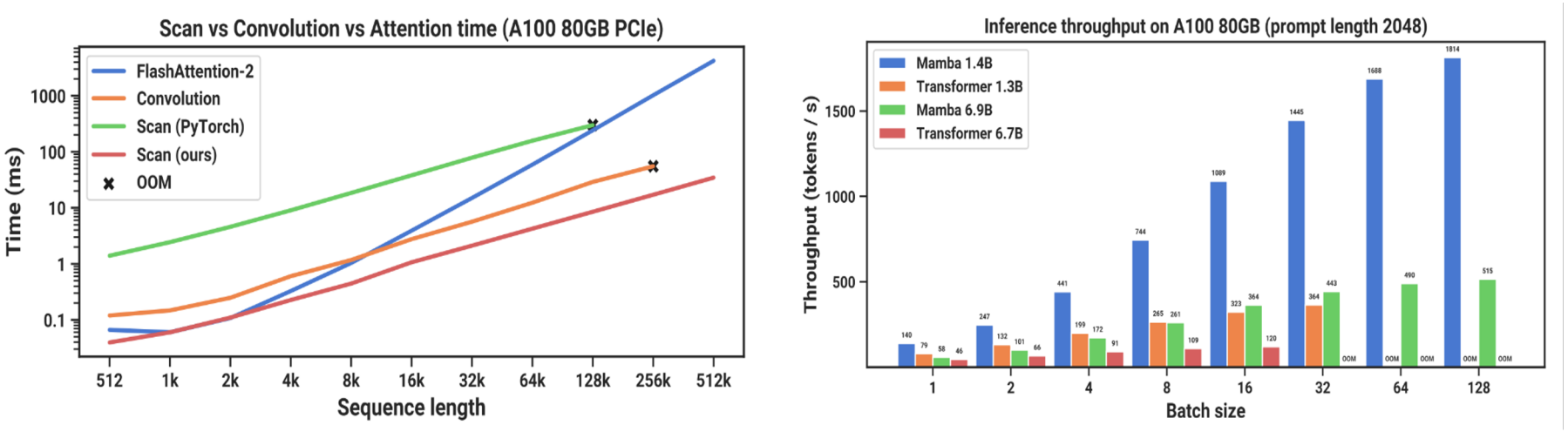
Induction Heads

Mamba: LM Results

Model	Token.	Pile ppl ↓	LAMBADA ppl ↓	LAMBADA acc ↑	HellaSwag acc ↑	PIQA acc ↑	Arc-E acc ↑	Arc-C acc ↑	WinoGrande acc ↑	Average acc ↑
GPT-Neo 1.3B	GPT2	—	7.50	57.2	48.9	71.1	56.2	25.9	54.9	52.4
Hybrid H3-1.3B	GPT2	—	11.25	49.6	52.6	71.3	59.2	28.1	56.9	53.0
OPT-1.3B	OPT	—	6.64	58.0	53.7	72.4	56.7	29.6	59.5	55.0
Pythia-1.4B	NeoX	7.51	6.08	61.7	52.1	71.0	60.5	28.5	57.2	55.2
RWKV-1.5B	NeoX	7.70	7.04	56.4	52.5	72.4	60.5	29.4	54.6	54.3
Mamba-1.4B	NeoX	**6.80**	**5.04**	**64.9**	**59.1**	**74.2**	**65.5**	**32.8**	**61.5**	**59.7**
GPT-Neo 2.7B	GPT2	—	5.63	62.2	55.8	72.1	61.1	30.2	57.6	56.5
Hybrid H3-2.7B	GPT2	—	7.92	55.7	59.7	73.3	65.6	32.3	61.4	58.0
OPT-2.7B	OPT	—	5.12	63.6	60.6	74.8	60.8	31.3	61.0	58.7
Pythia-2.8B	NeoX	6.73	5.04	64.7	59.3	74.0	64.1	32.9	59.7	59.1
RWKV-3B	NeoX	7.00	5.24	63.9	59.6	73.7	67.8	33.1	59.6	59.6
Mamba-2.8B	NeoX	**6.22**	**4.23**	**69.2**	**66.1**	**75.2**	**69.7**	**36.3**	**63.5**	**63.3**

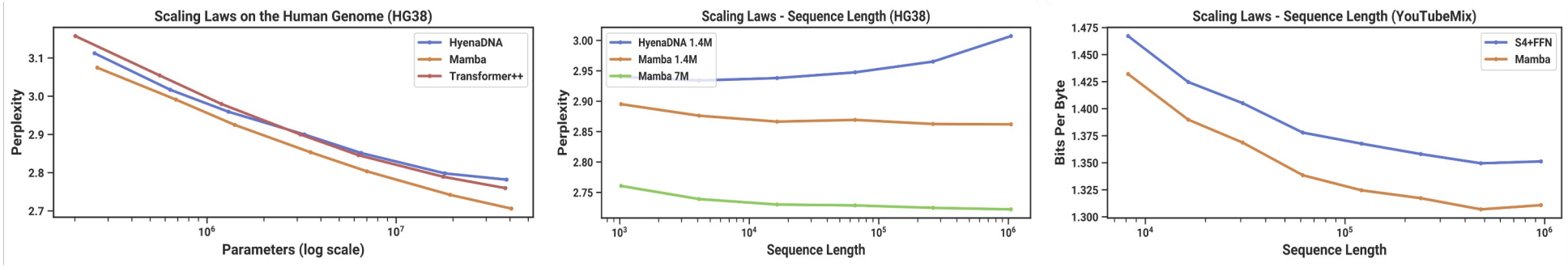
Zero-shot Eval on popular benchmarks

Mamba: Scaling Results



Source: Mamba[Albert Gu, Tri Dao 2023]

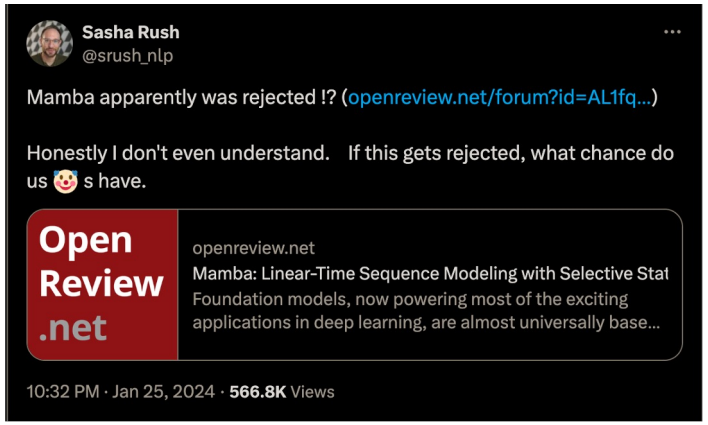
Mamba: Other Modalities



Zero-shot Eval on popular benchmarks

Mamba: Derivatives

- MambaByte [J Wang et al., 2024]
- Vision Mamba [Zhu, L, et al. , 2024]
- Mamba MoE [Pióro, Maciej, 2024]
- U-Mamba [Ma, Jun et al., 2024]
- Mamba-Morph [Guo, Tao et al., 2024]
- Swin-UMamba [Liu, J, et al., 2024]
- Graph Mamba [Behrouz et al., 2023]



Mamba rejected from ICLR 2024 🙄

Conclusion

- SSMs are a promising alternative to attention.
- Inference according to input was realized by Mamba.
- Hardware-Aware algorithms are the key to scale.
- Promising other research:
  - RWKV [Peng, Bo et al., 2023]
  - Gemini 1.5 [Gemini Team, 2024]

Thank you!