Tridendriform and dendriform Zeta Values from Schroeder trees

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Abstract

To build new generalisations of Multiple Zeta Values, we define new spaces of formal series and formal integrals. We show that they are tridendriform and dendriform algebras. This allows us to reinterpret the fact that Multiple Zeta Values are algebra morphisms for shuffles of words in terms of finer tridendriform and dendriform structures. Applying universal properties of Schroeder trees we obtain generalisations of Multiple Zeta Values that are algebra morphisms for associative products. Hence we find new properties of Arborified Zeta Values and state how this enables the computation of some Shintani Zeta Values.

Keywords. Shuffle products, multizeta function, mathematical physics, trees, Schroeder trees, dendriform, tridendriform.

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Introduction

Notations

For any set Ω , we will denote:

- for any set X, we denote by $\mathbb{K}X$ the vector space whose basis is the set X.
- Ω^* the set of finite words on the alphabet Ω ;
- \mathcal{W}_{Ω} the unique \mathbb{Q} -vector space whose basis is given by elements of Ω^{\star} ;
- We write $\emptyset \in \mathcal{W}_{\Omega}$ the empty word;
- For $\Omega = \mathbb{N}^*$, $\mathcal{W}^{conv}_{\mathbb{N}^*}$ is the sub-vector space of $\mathcal{W}_{\mathbb{N}^*}$ spanned by the empty word and words that do not start by 1;
- For $\Omega = \{x, y\}$, $\mathcal{W}^{\text{conv}}_{\{x, y\}}$ is the sub-vector space of $\mathcal{W}_{\{x, y\}}$ spanned by the empty word and words that start with x and end with y;
- for any $n \in \mathbb{N}^*$, we denote $\mathcal{W}_{\Omega,n}$ the sub-vector space generated by all the words of length exactly n;

Multiple Zeta Values and their generalisations

The Multiple Zeta Values¹ where introduced by Leonard Euler in 1775 [14] and have been rediscovered many times since then. The modern interest of this topic was sparked by the work of Ecalle[13] and the systematic study of Multiple Zeta Values started with the works of Hoffman [19] and Zagier [35]. There are many open conjectures and known results regarding the general theory of Multiple Zeta Values and it is not the purpose of this introduction to present them all. We refer the interested readers to, for example, [10] for a presentation of this exciting field of research.

Let us instead start by giving the relevant definitions of Multiple Zeta Values:

Definition (Multiple Zeta Value function). We define the following map:

We then have another representation of the numbers in the image of ζ , given by iterated integrals.

Definition (Integral representations). We introduce the following map:

$$\zeta_{\text{int}} : \begin{cases}
\mathcal{W}_{\{x,y\}}^{\text{conv}} & \longrightarrow \mathbb{R}, \\
\omega_{1} \dots \omega_{n} & \longmapsto \int_{0 < u_{n} < \dots < u_{1} < 1} \prod_{i=1}^{n} g_{\omega_{i}}(u_{i}) \, du_{i} \\
\emptyset & \longmapsto 1.
\end{cases} (2)$$

where:

$$g_x: \left\{ \begin{array}{ccc}]0,\,1[& \to & \mathbb{R}, \\ & t & \mapsto & \frac{1}{t}, \end{array} \right. \qquad g_y: \left\{ \begin{array}{ccc}]0,\,1[& \to & \mathbb{R}, \\ & t & \mapsto & \frac{1}{1-t}. \end{array} \right.$$

A result usually attributed to Kontsevitch but published first in [35]² is that these iterated series and integrals are different ways of writing the same real numbers. These two representations are linked by the following map:

Definition. The binarization map is the unique linear map defined by:

$$\mathfrak{s}: \left\{ \begin{array}{ccc} \mathcal{W}_{\mathbb{N}^*} & \to & \mathcal{W}_{\{x,y\}}, \\ n_1 \dots n_k & \mapsto & x^{n_1-1}yx^{n_2-1}y \dots x^{n_k-1}y. \end{array} \right.$$

It restricts to a bijection between $\mathcal{W}^{\text{conv}}_{\mathbb{N}^*}$ and $\mathcal{W}^{\text{conv}}_{\{x,y\}}$. This map is a bridge between the two representations of Multiple Zeta Values:

Proposition (Kontsevitch's relation [35]). We have $\zeta = \zeta_{\text{int}} \circ \mathfrak{s}|_{\mathcal{W}_{\text{nut}}^{\text{conv}}}$.

 $^{^{1}}$ also called polyzeta or other names by various authors, but we use here what seems to be now the most widespread designation.

²although, as pointed out in [27], a remark of [13] might refer to this fact.

The spaces $W_{\{x,y\}}$ and $W_{\mathbb{N}^*}$ have the structure of commutative algebras³ respectively with the *shuffle* and *quasi-shuffle* as products respectively denoted \coprod and \coprod ⁴. These products will play important roles in this paper and are defined in details in definitions 1.8 and 1.7 below. Then, a crucial but well-known result is

Theorem. The maps ζ and ζ_{int} are respectively algebra morphisms for the quasi-shuffle and the shuffle product: for any two elements u and v of $\mathcal{W}_{\mathbb{N}^*}^{conv}$ and any two elements u' and v' of $\mathcal{W}_{\{x,y\}}^{conv}$, we have

$$\zeta(u \boxtimes v) = \zeta(u) \cdot \zeta(v), \qquad \zeta_{\text{int}}(u' \sqcup v') = \zeta_{\text{int}}(u') \cdot \zeta_{\text{int}}(v'). \tag{3}$$

The quasi-shuffle version seems to be due to Hoffmann [20] but some properties of MZVs appeared earlier in [12, 13]. See for example [33] for a more detailed presentation of these results.

Since, Multiple Zeta Values have been generalised in many directions. Again, it is not the purpose of this introduction to be a review, so let us just mention the generalisations that will play a role in this paper. Relevant definitions will be introduced within the main text of this paper.

- Since words can be seen as rooted trees without branching, a natural generalisation of Multiple Zeta Values is to write them as maps with (convergent) rooted trees and forests as domain. This was introduced in [27] and independently in [34]. A systematic study of these arborified zeta values (see definition 4.6) was started in [8] and further developed in [9]. The iterated series version of these arborified zeta values is given below in definition 4.6, while the iterated integrals version in written in definition 7.1.
- Another generalisation of Multiple Zeta Values that will play a role here are the *Shintani Zeta Values* (definition 8.1). The (single valued) Shintani zeta were introduced in [31] and their multiple counterpart in [3, 1]. These were further studied in [28, 26].
- Notice that some results of [5] are close to ours (for the dendriform part), since it studies the Zinbiel structure behind Multiple Zeta Values. The paper [5] belongs to the theory of *motivic zeta values*, which is thus related to our results. For this approach to Multiple Zeta Value see for example [17].

Scope and main results

The first goal of this paper is to use the freeness property of a (tri)dendriform structures of rooted trees to build a new generalisation of Multiple Zeta Values to rooted trees. Dendriform algebras were introduced by Loday [21, 22] and their free objects were described in [23]. Tridendriform algebras were introduced in [25] and their free objects were built in [36, 4]. Dendriform and tridendriform algebras have been since then quite an active field of research and applied to many areas of mathematics and even physics. We refer the reader to the introduction of [36] for an overview of these applications.

For this, we introduce spaces of formal series in definition 2.1 and of formal integrals in definition 5.1. We then show that some subspaces of formal series and formal integrals admit respectively a natural tridendriform algebra structure and dendriform algebra structure. Applying the universal properties for these spaces of formal series and formal integrals we obtain

³actually, of Hopf algebras

⁴provided Ω as a commutative semigroup structure, both algebra structures over W_{Ω} given by \square or $\overline{\square}$ are isomorphic [20]

generalisations of Multiple Zeta Values to two distinct spaces of Schroeder trees in definitions 3.6 (tridendriform zeta values) and 6.5 (dendriform zeta values). The tridendriform version (definition 3.6) is a generalisation of equation (1) while the dendriform version (definition 6.5) is a generalisation of equation (2). Thus definitions 3.6 and 6.5 completely fulfill our first objective.

Our second objective is to study the properties of the aforementioned tridendriform and dendriform zeta values, both algebraic and number-theoretic. Regarding the algebraic part, from their definitions we directly have that the dendriform and tridendriform zeta values are algebra morphisms for the quasi-shuffle and shuffle of Schroeder trees respectively. These products are associative and not commutative generalisations of the quasi-shuffle and shuffle products of words of equation (3).

For the number-theoretic aspects of this question, we show that tridendriform and dendriform zeta values are closely related to arborified zeta values, both in the iterated series version (corollary 4.12) and in the iterated integral one (corollary 7.6). This in turn implies that dendriform and tridendriform zeta values can be written as linear combinations of Multiple Zeta Values with rational coefficients, in a completely explicit way. Thus our second goal is also reached.

The third and final goal of this paper is to apply the unravelled tridendriform and dendriform structures to obtain new results regarding Multiple Zeta Values and some of their generalisations. First, we show that the map $\zeta: \mathcal{W}^{\text{conv}}_{\mathbb{N}^*} \longrightarrow \mathbb{R}$ and $\zeta_{\text{int}}: \mathcal{W}^{\text{conv}}_{\{x,y\}} \longrightarrow \mathbb{R}$ of equations (1) and (2) can be decomposed as

$$\zeta = \text{ev} \circ \zeta_{\text{FS}}, \qquad \zeta_{\text{int}} = \text{ev} \circ \zeta_{\text{FI}}.$$

In these decompositions, ev are two evaluation maps from spaces of formal series and integrals to real numbers. However, strikingly, $\zeta_{\rm FS}$ and $\zeta_{\rm FI}$ are morphisms of tridendriform and dendriform algebras respectively. In other words, evaluating the Multiple Zeta Values makes us lose some of their algebraic properties.

We also show that arborified zeta values in their iterated series (resp. integral) form are an algebra morphism for a quasi-shuffle (resp. shuffle) product of decorated Schroeder trees. These results are stated in corollaries 4.13 and 7.7. Notice that these products are associative and not commutative. As such, they are more natural generalisations to (Schroeder) trees of the quasi-shuffle and shuffle products of words than the ones previously built in [8, 9].

Our last applications is for Shintani Zeta Values. We show in theorem 8.4 that dendriform zeta values can be written as iterated series. This series are Shintani zetas, so the previous results implies that a large family of Shintani Zeta Values can be written as Multiple Zeta Values with rational coefficients.

Content

The paper is composed of four parts:

- 1. The first one composed of section 1 presents the objects underlining most of this paper: tridendriform and dendriform algebras. They are introduced in definitions 1.1 and 1.2. We then describe the classical dendriform algebra spanned by words written over a set, and the tridendriform algebra words written over \mathbb{N}^* . These structures serve as examples but are also important in the reminder of this work.
- 2. The second part deals with tridendriform structures and generalisations of Multiple Zeta Values as iterated series (equation (1)). It goes from section 2 where we introduce *formal series* (definition 2.1) to section 4 establishing a bridge with usual Multiple Zeta Values (corollary 4.12).

In section 2, some of the standard properties of usual series are shown to hold in this framework in proposition 2.2. We then introduce in definition 2.3 a subspace of formal series together with an evaluation map (definition 2.4) which sends some convergent (in some sense) formal series to real numbers. This subspace of formal series is then shown to have the structure of a tridendriform algebra (proposition 2.6). We then introduce a formal series version of Multiple Zeta Values (definition 2.9) which, when composed with the evaluation map gives back the usual Multiple Zeta Values of equation (1). We further show in proposition 2.10 that this formal series version of Multiple Zeta Values forms a morphism of tridendriform algebras.

We start section 3 by recalling some definitions, namely of Schroeder trees (definition 3.1) and their tridendriform structures (definition 3.2 and theorem 3.3). Their universal property in the category of tridendriform algebras is recalled in theorem 3.5. Applying this universal property to the tridendriform structure of formal series, we obtain a map dubbed tridendriform zeta values in definition 3.6, which is by construction a morphism of tridendriform algebras, and in particular an algebra morphism for the quasi-shuffle of Schroeder trees (equation (17)).

Section 4 is the last of the aforementioned second part. It starts with a definition of Schroeder trees decorated on their vertices (definition 3.1) and of a map ι relating these decorated Schroeder trees to the ones previously used (definition 4.2). Thanks to this map, this space of decorated Schroeder trees inherits a tridendriform algebra structure from the one of usual Schroeder trees (proposition 4.5). We then recall the definition of arborified zeta values as iterated series (definition 4.6), of the flattening map (definition 4.7) and state in theorem 4.8 some known properties of arborified zeta values. We show in theorem 4.10 that these various spaces and maps nicely fit together. Specialising this theorem to convergent Schroeder trees (definition 4.11) and applying the evaluation map mentioned above we obtain that tridendriform zeta values are related to arborified zeta values and to the usual Multiple Zeta Values (corollary 4.12). This implies a new algebraic property for arborified zeta values (corollary 4.13).

3. The third part follows roughly the same structure than the first one, but uses dendriform structures rather than tridendriform ones. It starts in section 5 by the introduction of formal integrals (definition 5.1) and ends with the connection with Multiple Zeta Values in section 7 (corollary 7.6).

Section 5 introduces formal integrals and shows some usual properties of integrals are still true for their formal version. We then look at a particular subspace of formal integrals, namely formal Chen integrals (definition 5.3). It allows us to find back the usual integral representations of Multiple Zeta Values via a formal version of Multiple Zeta Values (definition 5.9). Then, the evaluation map (definition 5.4) sends convergent words to real numbers. The subspace of Chen integrals together with the evaluation map gives back the usual integral Multiple Zeta Values of equation (2). Moreover, we show that the space of Chen's formal integrals has a dendriform structure (proposition 5.6) and the formal version of Multiple Zeta Values is a morphism of dendriform algebras (proposition 5.10).

Section 6 states how the free dendriform algebra is built over decorated binary trees (definitions 6.1) and its universal property (theorem 6.4). We get a *dendriform zeta values* applying the universal property of the free dendriform algebra. In particular, we obtain a dendriform morphism for binary trees decorated by $\{x,y\}$ (equation (20)).

In section 7, the last of this third part, we recall the definition of integral Arborified Zeta Values (definition 7.1) as iterated integrals, the flattening map (definition 7.2) and state

in theorem 7.3 some known results about integral Arborified Zeta Values. We show in theorem 7.5 that all those maps behave well with each other. Specialising this theorem to convergent binary trees (first point of definition 7.1) and applying the evaluation map of formal integrals we get that dendriform zeta values are related to usual Multiple Zeta Values (corollary 7.6).

4. The last part is composed of section 8 dedicated to the study of Shintani Zeta Values (definition 8.1). We obtain that dendriform zeta values are Shintani Zeta Values (theorem 8.4).

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1 Dendriform and Tridendriform algebras

1.1 Description of algebraic structures

Let us start by defining the central structures of this paper, namely tridendriform and dendriform algebras. These definitions are taken from [2, 25, 16, 30].

Definition 1.1. Let A be an vector space endowed with three bilinear operations \prec , \cdot , \succ . We say that (A, \succ, \prec, \cdot) is a *tridendriform algebra* if for all $(x, y, z) \in A^3$:

$$(x \prec y) \prec z = x \prec (y * z),\tag{4}$$

$$(x \succ y) \prec z = x \succ (y \prec z), \tag{5}$$

$$(x*y) \succ z = x \succ (y \succ z), \tag{6}$$

$$(x \succ y) \cdot z = x \succ (y \cdot z),\tag{7}$$

$$(x \prec y) \cdot z = x \cdot (y \succ z), \tag{8}$$

$$(x \cdot y) \prec z = x \cdot (y \prec z), \tag{9}$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \tag{10}$$

where for all $x, y \in A$, we set $x * y := x \prec y + x \cdot y + x \succ y$. We respectively call \prec, \succ, \cdot the *left*, right and middle products.

A dendriform algebra can be seen as a tridendriform algebra with a vanishing middle product.

Definition 1.2. We say that a tridendriform algebra (A, \prec, \succ, \cdot) is a *dendriform algebra* if $\cdot = 0$. Only three relations remains:

$$(x \prec y) \prec z = x \prec (y \star z), \tag{11}$$

$$(x \succ y) \prec z = x \succ (y \prec z), \tag{12}$$

$$(x \star y) \succ z = x \succ (y \succ z), \tag{13}$$

where $\star := \prec + \succ$.

The following simple proposition indicates that (tri)dendriform structures are endowed with a classical associative algebra structure.

Proposition 1.3. Let (A, \succ, \prec, \cdot) (resp (A, \succ, \prec)) be a tridendriform algebra (resp. a dendriform algebra). Then the product * on A defined by $x * y := x \prec y + x \cdot y + x \succ y$ (resp. $x \star y := x \prec y + x \succ y$) is associative.

Proof. One just needs to write the associator for * (or *) and develop it with the definition using \prec , \succ and \cdot .

The definition of tridendriform algebra does not allow the existence of an unit. Hence we need to add it by hand:

Definition 1.4. Let (A, \succ, \prec, \cdot) be a tridendriform algebra. We expand the definitions of the products over the space $(\mathbb{K} \otimes A) \oplus (A \otimes \mathbb{K}) \oplus (A \otimes A)$ putting :

$$\forall a \in A, \quad 1 \prec a \coloneqq 0 = a \succ 1, \quad a \prec 1 \coloneqq a = 1 \succ a \quad \text{and} \quad a \cdot 1 \coloneqq 0 = 1 \cdot a. \quad (14)$$

We put $\overline{A} := A \oplus \mathbb{K} \cdot 1$. We call $(\overline{A}, \prec, \cdot, \succ, 1)$ the augmented tridendriform algebra.

The notion of tridendriform (and dendriform putting $\cdot = 0$) comes with its notion of morphisms and subalgebras:

Definition 1.5. A tridendriform morphism of augmented tridendriform algebra between $(\overline{A}, \prec, \succ, \cdot)$ and $(\overline{B}, \prec, \succ, \cdot)$ is a linear map $f: A \to B$ such that for any $x, y \in A$:

$$f(x \prec y) = f(x) \prec f(y), \qquad f(x \succ y) = f(x) \succ f(y), \qquad f(x \cdot y) = f(x) \cdot f(y),$$

and f(1) = 1.

Definition 1.6 (tridendriform subalgebra). Let (A, \prec, \succ, \cdot) be an augmented tridendriform algebra. Let $B \subseteq A$, we say it is a tridendriform subalgebra if it is a subalgebra for (A, *, 1) and it is closed for the operations \prec, \succ and \cdot .

Remark 1.1. For sake of simplicity, when we talk about tridendriform or dendriform algebra later, it will mean it is an augmented tridendriform or dendriform algebra.

1.2 Dendriform and tridendriform structures for words

We recall classical examples of (tri)dendriform structures on words. These examples are crucial in the theory of Multiple Zeta Values (MZVs), see for example [33, 20].

Definition 1.7 (shuffle product for words). Let Ω be a set. Let $u = u_1 \dots u_n$ and $v = v_1 \dots v_k$ be two words in Ω^* . We define inductively over n + k the shuffle product by $u \sqcup \emptyset = \emptyset \sqcup u = u$ for all u in \mathcal{W}_{Ω} and:

$$u \sqcup v \coloneqq u_1(u_2 \ldots u_n \sqcup v) + v_1(u \sqcup v_2 \ldots v_k).$$

We extend by linearity this product to a linear map $\sqcup : \mathcal{W}_{\Omega}^{\otimes 2} \to \mathcal{W}_{\Omega}$. We can split this shuffle product into two smaller products \succ and \prec defined inductively by:

$$u \prec v = u_1(u_2 \dots u_n \coprod v), \qquad \qquad u \succ v = v_1(u \coprod v_2 \dots v_k).$$

Example 1.1. Let xy be an element of $\mathcal{W}_{\{x,y\}}$, then the shuffle product with itself is:

$$xy \coprod xy = 4 \ xxyy + 2 \ xyxy.$$

Definition 1.8 (quasi-shuffle product for words). Let $u = u_1 \dots u_n$ and $v = v_1 \dots v_k$ be two words in $\mathcal{W}_{\mathbb{N}^*}$. We define inductively over k and l the quasi-shuffle product by $u \, \Box \!\!\Box \!\!\Box = u$ for all u in $\mathcal{W}_{\mathbb{N}^*}$ and:

$$u \, \overline{\coprod} \, v = u_1(u_2 \dots u_n \, \overline{\coprod} \, v) + v_1(u \, \overline{\coprod} \, v_2 \dots v_k) + (u_1 + v_1)(u_2 \dots u_n \, \overline{\coprod} \, v_2 \dots v_k).$$

We can split this shuffle product into three smaller products \prec , \succ and \cdot defined inductively by:

$$u \prec v = u_1(u_2 \dots u_n \, \overline{\coprod} \, v), \quad u \succ v = v_1(u \, \overline{\coprod} \, v_2 \dots v_k), \quad u \cdot v = (u_1 + v_1)(u_2 \dots u_n \, \overline{\coprod} \, v_2 \dots v_k).$$

Remark 1.2. Note that with these definitions neither $\emptyset \prec v$, $u \succ \emptyset$ nor $\emptyset \cdot v$ and $u \cdot \emptyset$ are defined, for \succ and \prec the left and right parts of the shuffle or quasi-shuffle product. For more details see [4].

Example 1.2 (for the quasi-shuffle product). Consider **1 2** and **3 2** two elements of $W_{\mathbb{N}^*}$, the quasi-shuffle product of those two objects is:

Remark 1.3. The construction of tridendriform algebras is here described on the monoid $(\mathbb{N}^*, +)$. One could do this for any monoid (M, \cdot) where the letters of our words are elements of M. We will not need this level of generality here and omit it for the sake of simplicity.

Shuffle and quasi-shuffle, with the decompositions presented above, are classical examples of dendriform and tridendriform algebras [11].

Proposition 1.9. Let Ω be a set. Then $(W_{\Omega}, \prec, \succ)$ (resp. $(W_{\mathbb{N}^*}, \prec, \cdot, \succ, \emptyset)$) is a dendriform algebra (resp. an augmented tridendriform algebra).

We know from [4] that the product of any tridendriform algebra is described by the following specific permutations that we now introduce.

Definition 1.10. • (Quasi-shuffles/sticky shuffles) Let $k, l \in \mathbb{N} \setminus \{0\}$. A (k, l)-quasi-shuffle is a surjective map $\sigma : [\![1, k + l]\!] \rightarrow [\![1, n]\!]$ such that:

$$\sigma(1) < \cdots < \sigma(k)$$
 and $\sigma(k+1) < \cdots < \sigma(k+l)$.

We write Qsh(k, l) the set of (k, l)-quasi-shuffle.

• (Shuffle) Let $k, l \in \mathbb{N} \setminus \{0\}$. A (k, l)-shuffle is an element from S_{k+l} such that:

$$\sigma(1) < \cdots < \sigma(k)$$
 and $\sigma(k+1) < \cdots < \sigma(k+l)$.

We write Sh(k, l) the set of (k, l)-shuffle.

Thank to these objects, we can describe the two shuffle products in terms or shuffles/quasi-shuffles (see [32] for the shuffle product and [18] for the quasi-shuffle):

Lemma 1.11. Let Ω be a set. Consider $w = w_1 \dots w_k$ and $w' = w_{k+1} \dots w_{k+l}$ two elements either of Ω^* or $W_{\mathbb{N}^*}$. The shuffle product of two elements w and w' of Ω^* can be described as:

$$w \sqcup w' = \sum_{\sigma \in Sh(k,l)} w_{\sigma^{-1}(\{1\})} \dots w_{\sigma^{-1}(\{\max(\sigma)\})}.$$

The quasi-shuffle product of two elements w and w' of $W_{\mathbb{N}^*}$ can be described as:

$$w \boxtimes w' = \sum_{\sigma \in Q\operatorname{sh}(k,l)} w_{\sigma^{-1}(\{1\})} \dots w_{\sigma^{-1}(\{\max(\sigma)\})},$$

where $w_{\{i,j\}} := w_i + w_j$

In particular, the pieces of each product corresponds to a condition over the first letter of the words appearing in the shuffle:

Lemma 1.12. Let $(k,l) \in \mathbb{N}^2$, $w = w_1 \dots w_k$ and $w' = w_{k+1} \dots w_{k+l}$. A non-inductive formula for the products of definition 1.8 is given below:

$$w \prec w' \coloneqq \sum_{\substack{\sigma \in \mathrm{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1\} }} w_{\sigma^{-1}(\{1\})} \dots w_{\sigma^{-1}(\{\max(\sigma)\})},$$

$$w \succ w' \coloneqq \sum_{\substack{\sigma \in \mathrm{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{k+1\} \}}} w_{\sigma^{-1}(\{1\})} \dots w_{\sigma^{-1}(\{\max(\sigma)\})},$$

$$w \cdot w' \coloneqq \sum_{\substack{\sigma \in \mathrm{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\} }} w_{\sigma^{-1}(\{1\})} \dots w_{\sigma^{-1}(\{\max(\sigma)\})}.$$

In the rest of the paper, we will often use the gradations of words by their lengths that we now recall.

Definition 1.13. Let Ω be a set, and $w = w_1 \dots w_k \in \mathcal{W}_{\Omega}$ with $k \geq 1$. We set l(w) := k the length of w. We also set $l(\emptyset) = 0$. We write $\mathcal{W}_{\Omega,k}$ the vector space generated by words of length k written in the alphabet Ω .

In our applications, we will focus on the cases $\Omega = \{x, y\}$ and $\mathcal{W}_{\mathbb{N}^*}$. Let us introduce some objects that are relevant in those situations.

Definition 1.14. We define $\mathcal{W}_{\mathbb{N}^*}^{\text{conv}}$ the sub-vector space of $\mathcal{W}_{\mathbb{N}^*}$ generated by the empty word and words whose first letters are an element of \mathbb{N}^* different from 1.

Moreover, we introduce $\mathcal{W}_{\{x,y\}}^{\text{conv}}$ the sub-vector space of $\mathcal{W}_{\{x,y\}}$ such that its basis is given by the empty word and by elements that can be written as $x\mathcal{W}_{\{x,y\}}y$.

Remark 1.4. The structure $(W_{\mathbb{N}^*}^{\text{conv}}, \prec, \succ, \cdot)$ is a tridendriform subalgebra of $W_{\mathbb{N}^*}$. Moreover, $(W_{\{x,y\}}^{\text{conv}}, \prec, \succ)$ is a dendriform subalgebra of $W_{\{x,y\}}$.

2 The tridendriform formal series algebra

2.1 Formal series

In this subsection, the tool to introduce a tridendriform structure is formal series. Our construction will mainly involve the monoid of positive integers $(\mathbb{N}^*, +)$. But tridendriform structures can be usually defined over \mathcal{W}_{Ω} for any monoid (Ω, \cdot) .

Notation 2.1. We denote $(\mathbb{N}^* \to \mathbb{R})^{\mathbb{N}^*}$ the set of maps from \mathbb{N}^* to the set of functions from \mathbb{N}^* to \mathbb{R} . It as a natural structure of vector space over \mathbb{R} .

Definition 2.1 (Formal series over \mathbb{R}). Consider the commutative monoid structure $(\mathbb{N}^*, +)$. Let $w \in \mathcal{W}_{\mathbb{N}^*}$ and a map $F : \mathbb{N}^* \to (\mathbb{N}^* \to \mathbb{R})$. We denote for any $n \in \mathbb{N}^*$, $f_n := F(n)$. Then for $w = \omega_1 \dots \omega_k \in \mathcal{W}_{\mathbb{N}^*}$ let us set $F_w : (\mathbb{N} \setminus \{0\})^k \to \mathbb{R}$ the function defined by:

$$F_w(n_1,\ldots,n_k) \coloneqq \prod_{i=1}^k f_{w_i}(n_i).$$

We call a *formal series* a pair $(A \otimes w, F)$ of the set:

$$\left\langle \mathbb{R}\mathcal{P}(\mathbb{N}^{\ell(w)}) \middle/ J_{\ell(w)} \otimes w \middle| w \in \mathcal{W}_{\mathbb{N}^*} \right\rangle \times (\mathbb{N}^* \to \mathbb{R})^{\mathbb{N}^*},$$
 (15)

where $\mathcal{P}(X)$ is the power set of X and for all $n \in \mathbb{N}$, J_n is the ideal of the algebra $(\mathbb{K}\mathcal{P}(\mathbb{N}^n), +, \cup, 0, \emptyset)$ generated by:

$${A \cup B - A - B + A \cap B \mid A, B \in \mathcal{P}(\mathbb{N}^n)}.$$

A formal series $(A \otimes w, F)$ with $w = w_1 \dots w_k$ will be written:

$$\sum_{A} \left(\prod_{i=1}^{k} f_{w_i} \right).$$

The omission of the summation variable is here to distinguish formal series from usual ones. We denote by $FS_{\mathbb{N}}$ the set of formal series over \mathbb{N} .

Remark 2.1. The notation of formal series is made such that a specification of F gives a concrete series to compute. Note that we are not evaluating it, we only consider it as a formal object.

Notation 2.2. For sake of simplicity, a formal series will be denoted $A \otimes w \times F$ instead of $(A \otimes w, F)$.

The aim of this construction is to consider a formal object previous to any evaluation of Multiple Zeta Value.

Example 2.1. In this section, we will mainly apply this construction to $F: k \to (n \to n^{-k})$. Hence, a formal version of the following Multiple Zeta Values

$$\sum_{1 \le n_2 < n_1} \frac{1}{n_1^2} \frac{1}{n_2}$$

is given by:

$$\{1 \le n_2 < n_1\} \otimes \mathbf{2} \ \mathbf{1} \times F.$$

This is why we use the series notation.

Our formal series have some of the usual properties of series.

Proposition 2.2. Let $d \in \mathbb{N}^*$ and $F \in (\mathbb{N}^* \to \mathbb{R})^{\mathbb{N}^*}$. The following properties of series hold for our formal series:

• let $A, B \in \mathcal{P}(\mathbb{N}^d)$ and $w = \omega_1 \dots \omega_d \in \mathcal{W}_{\mathbb{N}^*}$:

$$\sum_{A \cup B} \left(\prod_{i=1}^d f_{\omega_i} \right) = \sum_{A} \left(\prod_{i=1}^d f_{\omega_i} \right) + \sum_{B} \left(\prod_{i=1}^d f_{\omega_i} \right) - \sum_{A \cap B} \left(\prod_{i=1}^d f_{\omega_i} \right).$$

• let $w_1 = \omega_1^{(1)} \dots \omega_1^{(d)}$, $w_2 = \omega_2^{(1)} \dots \omega_2^{(d)} \in \mathcal{W}_{\mathbb{N}^*}$ and $A \in \mathcal{P}(\mathbb{N}^d)$, then:

$$\sum_{A} \left(\prod_{i=1}^d f_{\omega_1^{(i)}} + \prod_{j=1}^n f_{\omega_2^{(j)}} \right) = \sum_{A} \left(\prod_{i=1}^d f_{\omega_1^{(i)}} \right) + \sum_{A} \left(\prod_{j=1}^d f_{\omega_2^{(j)}} \right).$$

Proof. Let $d \in \mathbb{N}^*$, $w = \omega_1 \dots \omega_d \in \mathcal{W}_{\mathbb{N}^*}$ and $A, B \in \mathcal{P}(\mathbb{N}^d)$, hence by definition 2.1:

$$\sum_{A \cup B} \left(\prod_{i=1}^{d} f_{\omega_{i}} \right) = (A \cup B) \otimes w \times F$$

$$= A \otimes w \times F + B \otimes w \otimes F - A \cap B \otimes w \times F$$

$$= \sum_{A} \left(\prod_{i=1}^{d} f_{\omega_{i}} \right) + \sum_{B} \left(\prod_{i=1}^{d} f_{\omega_{i}} \right) - \sum_{A \cap B} \left(\prod_{i=1}^{d} f_{\omega_{i}} \right).$$

Finally, let us consider two words $w_1 = \omega_1^{(1)} \dots \omega_1^{(d)}, \ w_2 = \omega_2^{(1)} \dots \omega_2^{(d)}$ and $A \in \mathcal{P}(\mathbb{N}^d)$. Hence, by definition 2.1:

$$\sum_{A} \left(\prod_{i=1}^{d} f_{\omega_{1}^{(i)}} + \prod_{j=1}^{n} f_{\omega_{2}^{(j)}} \right) = A \otimes (w_{1} + w_{2}) \times F$$

$$= A \otimes w_{1} \times F + A \otimes w_{2} \times F$$

$$= \sum_{A} \left(\prod_{i=1}^{d} f_{\omega_{1}^{(i)}} \right) + \sum_{A} \left(\prod_{i=1}^{d} f_{\omega_{2}^{(j)}} \right).$$

So, it fulfils those usual properties of the series.

2.2 The tridendriform algebra of formal series

2.2.1 Formal series for Multiple Zeta Values

In this section, we focus on the map:

$$F: \left\{ \begin{array}{cc} \mathbb{N}^* & \longrightarrow (\mathbb{N}^* \to \mathbb{R}), \\ k & \longmapsto (n \mapsto n^{-k}). \end{array} \right.$$
 (16)

We will omit the dependency on F in this section to lighten the notation. Note that some of the results are still true with other appropriate F.

Definition 2.3. We define $\mathcal{S}_{\mathbb{N}^*}$ the subspace of $FS(\mathbb{N}^*)$ generated by:

$$\bigcup_{r \ge 0} \bigcup_{\omega_1 \dots \omega_r \in \mathcal{W}_{\mathbb{N}^*}} \left\{ \sum_{1 \le n_r < \dots < n_1} \left(\prod_{i=1}^r f_{\omega_i} \right) \right\}.$$

Since F is fixed $\mathcal{S}_{\mathbb{N}^*}$ inherits the vector space structure from the left part of equation (15). Hence, $\mathcal{S}_{\mathbb{N}^*}$ is the set of formal series needed to build Multiple Zeta Values in their series version (equation (1)).

It is well-known, from standard analysis tools, that a formal series $A \otimes \omega \times F \in \mathcal{S}_{\mathbb{N}^*}$ is actually equivalent to a convergent series provided the first letter of ω is not one. Then the following map will encode all the analysis job to get any real value.

Definition 2.4 (Evaluation map). A formal series $A \otimes \omega \times F \in \mathcal{S}_{\mathbb{N}^*}$ is said to be *convergent* if $\omega = \emptyset$ or if its first letter is not one. We write $\mathcal{S}_{\mathbb{N}^*}^{\text{conv}}$ these formal series. Then we define the *evaluation map*, denoted ev, as:

$$\operatorname{ev}: \left\{ \begin{array}{ccc} \mathcal{S}^{\operatorname{conv}}_{\mathbb{N}^*} & \to & \mathbb{R}, \\ A \otimes \omega_1 \dots \omega_k \times F & \mapsto & \sum_{(n_1, \dots, n_k) \in A} \left(\prod_{i=1}^k f_{\omega_i}(n_i) \right), \end{array} \right.$$

where F is the function of equation (16).

Remark 2.2. Notice that ev could a priori be defined for other domains than the ones of definition 2.3. This extended map is a linear map for the first variable and is homogenous on the second variable. For instance, take for example $A = \{(2,2)\}, w = \mathbf{2} \ \mathbf{2}$ with $F(k) = (n \mapsto n^{-k})$. Then $\operatorname{ev}(A \otimes w \times F) = \frac{1}{2^2} \cdot \frac{1}{2^2} = \frac{1}{16}$. But it is not linear for the second argument, for instance let $\lambda \in \mathbb{R}, \lambda F(k) = (n \mapsto \lambda n^{-k})$, thus

$$\operatorname{ev}(A \otimes w \times \lambda F) = \lambda \frac{1}{2^2} \cdot \lambda \frac{1}{2^2} = \frac{1}{16} \lambda^2.$$

More precisely, for $A \otimes w$ fixed it satisfies for any $\lambda \in \mathbb{R}$:

$$\operatorname{ev}(A \otimes w \times \lambda F) = \lambda^{|w|} \operatorname{ev}(A \otimes w \times F).$$

Example 2.2. More formally, to evaluate such a formal integral, one needs to get rid of the indeterminate forms. For instance, for F such that for any $k \in \mathbb{N}^*$, $f_k : n \mapsto n^{-k}$, we have:

$$\frac{\pi^2}{6} = \operatorname{ev}\left(\sum_{n\geq 1} \frac{1}{n^2}\right) = \operatorname{ev}(\{n\geq 1\} \otimes \mathbf{2} \times F)$$

$$= \operatorname{ev}\left(\sum_{n\geq 1} \frac{1-n}{n^2} + \sum_{n\geq 1} \frac{1}{n}\right) = \operatorname{ev}\left(\sum_{n\geq 1} \frac{1}{n^2} - \sum_{n\geq 1} \frac{1}{n} + \sum_{n\geq 1} \frac{1}{n}\right)$$

$$= \operatorname{ev}(\{n\geq 1\} \otimes \mathbf{2} \times F - \{n\geq 1\} \otimes \mathbf{1} \times F + \{n\geq 1\} \otimes \mathbf{1} \times F).$$

Note that as $\sum_{n\geq 1}\frac{1-n}{n^2}$ and $\sum_{n\geq 1}\frac{1}{n}$ are not convergent formal series one can not apply the linearity property of ev. Hence, it prevents us from writing $\infty-\infty$.

2.2.2 The tridendriform structure

Using an analogous of Chen's iterated integrals lemma [6] but for sums, we define the unique product $\overline{\coprod}$ over $\mathcal{S}_{\mathbb{N}^*}$ such that ev is a morphism for the multiplicative structure of \mathbb{R} :

Definition 2.5. We define three new linear operators \prec , \succ and \cdot defined for any $\alpha, \beta \in \mathcal{S}_{\mathbb{N}}$ such that:

$$\alpha = \sum_{1 \le n_k < \dots < n_1} \prod_{i=1}^k f_{\omega_i}, \qquad \beta = \sum_{1 \le n_{k+l} < \dots < n_{k+1}} \prod_{i=k+1}^{k+l} f_{\omega_i},$$

by:

$$\begin{split} \alpha \prec \beta &\coloneqq \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1\}}} \{1 \leq n_{\max(\sigma)} < \dots < n_1\} \otimes \omega_{\sigma^{-1}(\{1\})} \dots \omega_{\sigma^{-1}(\{\max(\sigma)\})} \times F, \\ &= \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1\}}} \sum_{1 \leq n_{\max(\sigma)} < \dots < n_1} \prod_{i=1}^{\max(\sigma)} f_{\omega_{\sigma^{-1}(\{i\})}}, \\ \alpha \succ \beta &\coloneqq \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{k+1\}}} \{1 \leq n_{\max(\sigma)} < \dots < n_1\} \otimes \omega_{\sigma^{-1}(\{1\})} \dots \omega_{\sigma^{-1}(\{\max(\sigma)\})} \times F, \\ &= \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{k+1\}}} \sum_{1 \leq n_{\max(\sigma)} < \dots < n_1} \prod_{i=1}^{\max(\sigma)} f_{\omega_{\sigma^{-1}(\{i\})}}, \\ \alpha \cdot \beta &\coloneqq \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \{1 \leq n_{\max(\sigma)} < \dots < n_1\} \otimes \omega_{\sigma^{-1}(\{1\})} \dots \omega_{\sigma^{-1}(\{\max(\sigma)\})} \times F, \\ &= \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \sum_{1 \leq n_{\max(\sigma)} < \dots < n_1} \prod_{i=1}^{\max(\sigma)} f_{\omega_{\sigma^{-1}(\{i\})}}, \\ &= \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \sum_{1 \leq n_{\max(\sigma)} < \dots < n_1} \prod_{i=1}^{\max(\sigma)} f_{\omega_{\sigma^{-1}(\{i\})}}, \\ &= \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \sum_{1 \leq n_{\max(\sigma)} < \dots < n_1} \prod_{i=1}^{\max(\sigma)} f_{\omega_{\sigma^{-1}(\{i\})}}, \\ &= \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \sum_{1 \leq n_{\max(\sigma)} < \dots < n_1} \prod_{i=1}^{\max(\sigma)} f_{\omega_{\sigma^{-1}(\{i\})}}, \\ &= \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \sum_{1 \leq n_{\max(\sigma)} < \dots < n_1} \prod_{i=1}^{\max(\sigma)} f_{\omega_{\sigma^{-1}(\{i\})}}, \\ &= \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \sum_{1 \leq n_{\max(\sigma)} < \dots < n_1} \prod_{i=1}^{\max(\sigma)} f_{\omega_{\sigma^{-1}(\{i\})}}, \\ &= \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \sum_{1 \leq n_{\max(\sigma)} < \dots < n_1} \prod_{i=1}^{\max(\sigma)} f_{\omega_{\sigma^{-1}(\{i\})}}, \\ &= \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \sum_{1 \leq n_{\max(\sigma)} < \dots < n_1} \prod_{i=1}^{\max(\sigma)} f_{\omega_{\sigma^{-1}(\{i\})}}, \\ &= \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \sum_{1 \leq n_{\max(\sigma)} < \dots < n_1} \prod_{i=1}^{\max(\sigma)} f_{\omega_{\sigma^{-1}(\{i\})}}, \\ &= \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \sum_{1 \leq n_{\max(\sigma)} < \dots < n_1} \prod_{i=1}^{\max(\sigma)} f_{\omega_{\sigma^{-1}(\{i\})}}, \\ &= \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \sum_{1 \leq n_{\max(\sigma)} < \dots < n_1} \prod_{i=1}^{\max(\sigma)} f_{\omega_{\sigma^{-1}(\{i\})}}, \\ &= \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \sum_{1 \leq n_{\max(\sigma)} < \dots < n_1} \prod_{i=1}^{\max(\sigma)} f_{\omega_{\sigma^{-1}(\{1\})}}, \\ &= \sum_{\substack{\sigma \in \operatorname{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \sum_{1 \leq n_{\max(\sigma$$

where for any $(i,j) \in [\![1,k]\!] \times [\![k+1,k+l]\!], \omega_{\{i,j\}} = \omega_i + \omega_j$. Moreover, we denote:

$$\begin{split} \alpha \, \overline{\coprod} \, \beta &\coloneqq \alpha \prec \beta + \alpha \succ \beta + \alpha \cdot \beta \\ &= \sum_{\sigma \in \operatorname{Qsh}(k,l)} \sum_{1 \leq n_{\max(\sigma)} < \dots \leq n_1} \prod_{i=1}^{\max(\sigma)} f_{\omega_{\sigma^{-1}(\{i\})}}. \end{split}$$

Proposition 2.6. The structure $(S_{\mathbb{N}^*}, \prec, \succ, \cdot)$ is a tridendriform algebra.

Proof. Let α, β and γ be three elements of $S_{\mathbb{N}^*}$:

$$\alpha = \sum_{1 \leq n_k < \dots < n_1} \prod_{i=1}^k f_{\omega_i}, \quad \beta = \sum_{1 \leq n_{k+l} < \dots < n_{k+1}} \prod_{i=k+1}^{k+l} f_{\omega_i}, \quad \gamma = \sum_{1 \leq n_{k+l+s} < \dots < n_{k+l+1}} \prod_{i=k+1}^{k+l} f_{\omega_i}.$$

We need to show those seven relations:

$$(\alpha \prec \beta) \prec \gamma = \alpha \prec (\beta \overline{\sqcup} \gamma),$$

$$(\alpha \succ \beta) \prec \gamma = \alpha \succ (\beta \prec \gamma),$$

$$(\alpha \overline{\sqcup} \beta) \succ \gamma = \alpha \succ (\beta \succ \gamma),$$

$$(\alpha \succ \beta) \cdot \gamma = \alpha \succ (\beta \cdot \gamma),$$

$$(\alpha \prec \beta) \cdot \gamma = \alpha \cdot (\beta \succ \gamma),$$

$$(\alpha \prec \beta) \prec \gamma = \alpha \cdot (\beta \prec \gamma),$$

$$(\alpha \cdot \beta) \prec \gamma = \alpha \cdot (\beta \prec \gamma).$$

Note that applying a quasi-shuffle to those quantities is just equivalent to quasi-shuffle the words $\omega_1 \dots \omega_k$, $\omega_{k+1} \dots \omega_{k+l}$ and $\omega_{k+l+1} \dots \omega_{k+l+s}$. Actually, considering the map F of equation (16) there is the following bijection:

$$\Phi_F: \left\{ \begin{array}{ccc} \mathcal{W}_{\mathbb{N}^*} & \to & \mathcal{S}_{\mathbb{N}^*}, \\ \omega_1 \dots \omega_k & \mapsto & \{1 \leq n_k < \dots < n_1\} \otimes \omega_1 \dots \omega_k \times F \end{array} \right.$$

extended by linearity. This map is clearly a morphism for the products \prec , \succ and \cdot in words and the same products in $\mathcal{S}_{\mathbb{N}}$. Thus the relations hold thanks to this bijection.

Proposition 2.7. Then, $(S_{\mathbb{N}^*}, \prec, \succ, \cdot)$ is generated as a tridendriform algebra by the following set:

 $\left\{ \sum_{1 \le n} f_t \right\}_{t \in \mathbb{N}}.$

Proof. This is a consequence of the fact that the set of words is generated by its letters as a tridendriform algebra and the existence of the isomorphism of tridendriform algebras Φ_F build above.

Finally, the following well-known result states that the evaluation map of definition 2.4 respects the associative algebra structure:

Proposition 2.8. The space $\mathcal{S}_{\mathbb{N}^*}^{\mathrm{conv}}$ is a tridendriform subalgebra of $\mathcal{S}_{\mathbb{N}^*}$, and $\mathrm{ev}:\mathcal{S}_{\mathbb{N}^*}^{\mathrm{conv}}\longrightarrow\mathbb{R}$ is an algebra morphism for the quasi-shuffle product $\overline{\sqcup}$.

Proof. If w and w' are two words that are either empty or do not start with 1, then clearly so are $w \succ w', w \cdot w'$ and $w \prec w'$. Thus $\mathcal{S}_{\mathbb{N}^*}^{\text{conv}}$ is a tridendriform subalgebra. The fact that ev is an algebra morphism for the quasi-shuffle product is a well-known fact of the theory of Multiple Zeta Values that can be proved using standard analytical techniques. It is essentially a reformulation of the first equality in equation (3).

2.3 Tridendriform structure and Multiple Zeta Values

In this subsection, F will still be the map of equation (16).

Definition 2.9 (Multiple Zeta Value series representation). We define the linear map $\zeta_{FS}: \mathcal{W}_{\mathbb{N}^*}^{conv} \to \mathcal{S}_{\mathbb{N}^*}$ defined for any word $w = \omega_1 \dots \omega_k \in \mathcal{W}_{\mathbb{N}^*}$ by:

$$\zeta_{\text{FS}}(w) = \sum_{0 \le n_k \le \dots \le n_1} \left(\prod_{i=1}^k f_{\omega_i} \right) = A \otimes w \times F$$

with $\zeta_{FS}(\emptyset) = \emptyset \otimes \emptyset \times F$ and with A given by the domain in the definition of $\mathcal{S}_{\mathbb{N}^*}$ (definition 2.3). Hence, the classical Multiple Zeta Value of definition 1 satisfies:

$$\zeta(w) = \text{ev} \circ \zeta_{\text{FS}}.$$

Our motivation for introducing formal series as an intermediate step for building Multiple Zeta Values comes from the following simple yet important proposition.

Proposition 2.10. The map $\zeta_{FS}: \mathcal{W}_{\mathbb{N}^*} \to \mathcal{S}_{\mathbb{N}}$ is a tridendriform morphism.

Proof. Using the fact that $\prec = \succ^{\text{op}}$, it is enough to prove that ζ_{FS} is a morphism for \prec and \cdot . Let $v, u \in (\mathbb{N} \setminus \{0\})^*$ such that $v = \omega_1 \dots \omega_k$ and $u = \omega_{k+1} \dots \omega_{k+l}$. Hence:

$$\zeta_{\text{FS}}(v \prec u) = \sum_{\substack{\sigma \in \text{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1\}}} \zeta_{\text{FS}}\left(\omega_{\sigma^{-1}(\{1\})} \dots \omega_{\sigma^{-1}(\{\max(\sigma))\}}\right)$$

$$= \sum_{\substack{\sigma \in \text{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1\}}} \sum_{\substack{1 \le n_{\max(\sigma)} < \dots < n_1 \ i = 1}} \prod_{i=1}^k f_{\omega_{\sigma^{-1}(\{i\})}}$$

$$= \left(\sum_{1 \le n_k < \dots < n_1} \prod_{i=1}^k f_{\omega_i}\right) \prec \left(\sum_{1 \le n_{k+1} < \dots < n_{k+l}} \prod_{j=k+1}^{k+l} f_{\omega_j}\right)$$

$$= \zeta_{\text{FS}}(v) \prec \zeta_{\text{FS}}(u).$$

The third equality comes from the definition of the product \prec in $\mathcal{S}_{\mathbb{N}^*}$.

We need to check that $\zeta_{\rm FS}$ is also a morphism for \cdot . With the same notations, we compute:

$$\begin{split} \zeta_{\mathrm{FS}}\left(v \cdot u\right) &= \sum_{\substack{\sigma \in \mathrm{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \zeta_{\mathrm{FI}}\left(\omega_{\sigma^{-1}(\{1\})} \dots \omega_{\sigma^{-1}(\{\max(\sigma)\})}\right) \\ &= \sum_{\substack{\sigma \in \mathrm{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \sum_{\substack{1 \le n_{\max(\sigma)} < \dots < n_1 \\ 1 \le n_{j} = 1}} \prod_{i=1}^{\max(\sigma)} f_{\omega_{\sigma^{-1}(\{i\})}} \\ &= \left(\sum_{\substack{1 \le n_k < \dots < n_1 \\ 1 \le n_k < \dots < n_1}} \prod_{i=1}^{k} f_{\omega_i}\right) \cdot \left(\sum_{\substack{1 \le n_{k+l} < \dots < n_{k+1} \\ 1 \le k+1}} \prod_{j=k+1}^{k+l} f_{\omega_j}\right) \\ &= \zeta_{\mathrm{FS}}(v) \cdot \zeta_{\mathrm{FS}}(u). \end{split}$$

This achieves the proof.

This proposition indicates that the well-known property that Multiple Zeta Values form an algebra morphism for the quasi-shuffle product comes from a deeper tridendriform structure. This structure is lost when one uses the evaluation map to obtain real numbers since $\mathbb R$ has no tridendriform structure.

In the following, using the universal properties of the free tridendriform algebra over Schroeder trees and binary trees, we extend the definition of the formal Multiple Zeta Values we have just introduced.

3 Multiple Zeta Values and Tridendriform structures

3.1 Schroeder trees

Let us quickly introduce the combinatorial view of the free tridendriform algebra. Its definition over one generator can be found in the first author's article [4]. In order to describe the free tridendriform algebra with several generators, we introduce Schroeder trees.

Definition 3.1. Recall that a *planar rooted tree* is a planar connected directed acyclic graph such that there is at most one path between any two vertices, and such that there is a unique minimal element for the relation on its set of vertices

$$v \leq v' : \iff$$
 there is path from v to v' .

For any tree t, a *leaf* of t is a vertex that is maximal for the poset relation \leq . We say one of its vertex is *internal* if it is not a leaf.

So for any tree, the minimal element is called the *root* of the tree and its edges are oriented from the root to the leave.

A *Schroeder* tree is a planar rooted tree such that any of its internal vertices has at least two children. We write T the set of Schroeder trees.

Let $t \in T$ a Schroeder tree. Let $\nu(t)$ be the set of internal vertices of t. An angle of v is a pair (e, e') of consecutive (for the reading order from left to right) edges starting from v. For any $v \in \nu(t)$, we define $\operatorname{Ang}(v)$ the set of angles of v in t.

We will denote
$$\operatorname{Ang}(t) := \bigcup_{v \in \nu(t)} \operatorname{Ang}(v)$$
.

Let t be a Schroeder tree and Ω a set. An angle decoration map for t is a map $D: \operatorname{Ang}(t) \to \Omega$. A Ω -decorated Schroeder tree is a pair (t, D) with t a Schroeder tree and D an angle decoration map for t. We write $T(\Omega)$ the set of Ω -decorated Schroeder trees.

Remark 3.1. In general, we omit to write down the decoration map of a decorated Schroeder tree, and instead write the decorations on the angles. Alternatively, if $|\operatorname{Ang}(v)| = n$, we can write v as being decorated by the word $\omega_1 \cdots \omega_n$ instead of ω_i being the *i*th decoration of the *i*th angle of v (read from left to right).

We give below some examples of decorated and undecorated Schroeder trees and refer the reader to [36] for a detailed presentation.

Example 3.1. Here are some examples of such trees where T_n is the set of Schroeder trees with exactly n+1 leaves.

$$T_0 = \{|\}, \qquad T_1 = \{\}, \qquad T_2 = \{\}, \qquad T_3 = \{\}.$$

Setting $T_n(\Omega)$ the set of Ω -decorated Schroeder trees with exactly n+1 leaves we have

$$T_0(\Omega) = \{|\}\,, \quad T_1(\Omega) = \left\{ \begin{array}{c} \swarrow \\ \\ \end{array} \middle| \omega \in \Omega \right\}, \quad T_2(\Omega) = \left\{ \begin{array}{c} \alpha \\ \\ \end{array} \middle| \begin{array}{c} \alpha \\ \\ \\ \\ \end{array} \middle| \begin{array}{c} \alpha \\ \\ \\ \end{array} \middle| \begin{array}{c} \alpha \\ \\ \\ \end{array} \middle| \begin{array}{c} \alpha \\ \\ \\ \\ \\ \end{array} \middle| \begin{array}{c} \alpha \\ \\ \\ \\ \\ \end{array} \middle| \begin{array}{c} \alpha \\ \\ \\ \\ \\ \end{array} \middle| \begin{array}{c} \alpha \\ \\ \\ \\ \\ \end{array} \middle| \begin{array}{c} \alpha \\ \\ \\ \\ \\ \end{array} \middle| \begin{array}{c$$

3.2 Free tridendriform algebras

The free tridendriform algebra over \mathbb{N}^* has for underlined vector space the space $T(\mathbb{N}^*)$.

Let t, s be two trees different from |. We see t as a right comb and s as a left comb. In other words, we put:



where for all $i \in [1, k+l]$, F_i is a forest with decorated angles and the w_i is the word of integer labels decorating the angles between the elements of the forest F_i . Here, k represents the number of nodes on the rightmost branch of t and t is the number of nodes on the leftmost branch of t.

Definition 3.2. Let t and s be two decorated trees whose respective right comb and left comb representations are given as above. Put k (respectively l) the number of nodes on the rightmost branch (respectively the leftmost) of t (respectively of s). Let $\sigma \in \text{Qsh}(k, l)$ whose image is [1, n]. We define $\sigma(t, s)$ the tree obtained the following way:

1. let us start from the tree:



- 2. For $i \in [1, k]$, we graft F_i as the *left* son of the node $\sigma(i)$;
- 3. For $i \in [k+1, k+l]$, we graft F_i as a right son of the node $\sigma(i)$;
- 4. For the decorations, we proceed the following way:
 - all the forests F_i keep their original decorations;
 - Let $m \in [1, n]$. The node m of the ladder is decorated by w_i if i is the unique preimage of m by σ . Otherwise, there exists a unique $(i, j) \in [1, k] \times [k + 1, k + l]$ such that $m = \sigma(i) = \sigma(j)$, then we decorate the node m of the ladder with $w_i w_j$.

Example 3.2. Consider $\sigma=(1,3,2,3)$ a (2,2)-quasi-shuffle. Take $t=F_1$ v_1 and $s=v_4$ v_2 and $s=v_4$ v_3 v_4 v_4 v_4 v_5 v_6 v_8 v_8 v_8 v_9 v_9

$$\sigma(t,s) = F_{1} \underbrace{\begin{array}{c} F_{2} \\ w_{2}w_{4} \\ F_{3} \\ w_{1} \end{array}}_{w_{1}} F_{4}$$

An explicit, non-inductive description of the tridendriform structure of Schroeder trees was given in [4].

Theorem 3.3 ([4]). Let t, s be two trees different from | as described above. Then for any set Ω , $T(\Omega)$ has the structure of a tridendriform algebra given by:

$$t*s = \sum_{\sigma \in \mathrm{Qsh}(k,l)} \sigma(t,s).$$

Moreover:

$$t \prec s = \sum_{\substack{\sigma \in \text{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1\}}} \sigma(t,s), \quad t \cdot s = \sum_{\substack{\sigma \in \text{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1,k+1\}}} \sigma(t,s), \quad t \succ s = \sum_{\substack{\sigma \in \text{Qsh}(k,l) \\ \sigma^{-1}(\{1\}) = \{k+1\}}} \sigma(t,s).$$

The three products \prec , \succ and \cdot also admit an inductive description that we now present.

Notation 3.1. Let $k \in \mathbb{N}$, $k \geq 2$ and t_1, \ldots, t_k a family of k trees. We denote by $t_1 \vee_{\omega_1} \cdots \vee_{\omega_{k-1}} t_k$ the tree obtained grafting in this order t_1, \ldots, t_k to a common root decorated by $\omega_1 \ldots \omega_{k-1}$.

Example 3.3. With this notation, all trees can be represented only with Y-shaped trees and the tree |:

$$\bigvee_{1}^{1}\bigvee_{2}|=\bigvee_{2}^{1}.$$

Lemma 3.4. Let t and s be two Schroeder trees different from | with:

$$t = t^{(1)} \vee_{\omega_1} \dots \vee_{\omega_{k-1}} t^{(k)}, \qquad s = s^{(1)} \vee_{\mu_1} \dots \vee_{\mu_{l-1}} s^{(l)}.$$

Then:

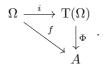
$$t \prec s = t^{(1)} \vee_{\omega_1} \dots \vee_{\omega_{k-1}} \left(t^{(k)} * s \right), t \succ s = \left(t * s^{(1)} \right) \vee_{\mu_1} s^{(2)} \vee_{\mu_2} \dots \vee_{\mu_{l-1}} s^{(l)},$$
$$t \cdot s = t^{(1)} \vee_{\omega_1} \dots \vee_{\omega_{k-1}} \left(t^{(k)} * s^{(1)} \right) \vee_{\mu_1} s^{(2)} \vee_{\mu_2} \dots \vee_{\mu_{l-1}} s^{(l)}.$$

This lemma allows us to do proofs by induction over the number of leaves of a tree.

3.3 Tridendriform zeta values

The tridendriform algebra of theorem 3.3 is the free tridendriform algebra over Ω .

Theorem 3.5 ([36]). Let Ω be a set. $(T(\Omega), \prec, \succ, .)$ is the free tridendriform algebra over Ω : for any tridendriform algebra A and any map $f: \Omega \longrightarrow A$, there exists a unique morphism of tridendriform algebras $\Phi: T(\Omega) \longrightarrow A$ such that the diagram below commutes, with $i: \Omega \longrightarrow T(\Omega)$ the canonical embedding defined by $i(n) := \bigcap_{i=1}^{n} A_i$



Notation 3.2. We will write Tridend(Ω) the free tridendriform algebra over Ω described above.

Applying the freeness property of Tridend(\mathbb{N}^*) to the tridendriform algebra $\mathcal{S}_{\mathbb{N}^*}$ (definition 2.3), we obtain tridendriform zeta values.

Definition 3.6. Let $\sum: \mathbb{N}^* \longrightarrow \mathcal{S}_{\mathbb{N}^*}$ be the map defined by $\sum(n) \coloneqq \sum_{k \geq 1} f_n$, where $f_n(m) \coloneqq m^{-n}$ as in definition 2.3. Then the $tridendriform\ zeta\ values\ map$ is the unique morphism of tridendriform algebras $\zeta^{\mathrm{Tri}}: \mathrm{Tridend}(\mathbb{N}^*) \longrightarrow \mathcal{S}_{\mathbb{N}^*}$ whose existence and unicity is given by the universal property of $\mathrm{Tridend}(\mathbb{N}^*)$ stated in theorem 3.3.

Note that the map that we built is by definition a morphism of tridendriform algebras. Therefore, it is also a morphism for the quasi-shuffle product * of Schroeder trees which is explicitly described in theorem 3.3:

$$\forall (t, s) \in \text{Tridend}(\mathbb{N}^*), \ \zeta^{\text{Tri}}(t * s) = \zeta^{\text{Tri}}(t) * \zeta^{\text{Tri}}(s). \tag{17}$$

Composing a suitably restricted ζ^{Tri} with the evaluation map of definition 2.4 gives thanks to proposition 2.8 a map with values in \mathbb{R} which is an algebra morphism. The aforementioned restriction needs to be taken to avoid divergent series, and this will be expanded upon later on. In other words, we have build a new generalisation of Multiple Zeta Values which is an algebra morphism for a generalisation of the quasi-shuffle product on words.

In the rest of this section, we will aim at relating ζ^{Tri} to other known objects, and in particular Arborified Zeta Values and Multiple Zeta Values. This will also solve the issue of convergence of Tridendriform Zeta Values, which we have left pending so far.

4 From tridendriform to Zeta Values

4.1 From the free algebra to usual trees

As we have seen the free vector space of Schroeder trees with angles decorated by non-negative integers has a tridendriform structure explained above making it the *free tridendriform algebra* over \mathbb{N}^* denoted Tridend(\mathbb{N}^*). Our aim is to recover classical trees for the Multiple Zeta Values theory.

Now, we introduce the main objects that we will care about:

Definition 4.1. We define $\text{Tree}(\mathbb{N}^*)$ the vector space generated by Schroeder trees whose internal vertices are decorated by an element of \mathbb{N}^* greater than the number of angles of this vertex.

Hence, one can see an element of Tree(\mathbb{N}^*) as the data (t,d) of a Schroeder tree t and a decoration map $d: \nu(t) \to \mathbb{N}^*$.

Example 4.1. For instance:
$$\sqrt{1}$$
, $\sqrt{3}$, $\sqrt{1}$, $\sqrt{2}$ and $\sqrt{2}$ are elements of Tree(\mathbb{N}^*) whereas $\sqrt{1}$ is not.

Definition 4.2. We define the map ι : Tridend(\mathbb{N}^*) \to Tree(\mathbb{N}^*) keeping the same tree structure but getting from the decoration of any angles a decoration of internal vertices by:

$$\forall v \in \nu(t), d(v) = \sum_{a \in \text{Ang}(v)} D(a).$$

In other words, $\iota(t, D) = (t, d)$.

Example 4.2. For instance:

$$\iota\left(\begin{array}{c}2\\1\end{array}\right)=\begin{array}{c}3\end{array}$$

Consider t and s be two elements of Tree(\mathbb{N}^*):

where for all $i \in [1, k+l]$, F_i is a decorated forest of elements of $\text{Tree}(\mathbb{N}^*)$ and the d_i 's are vertex decorations in \mathbb{N}^* . Then, k represents the number of nodes on the rightmost branch of t and l is the number of nodes on the leftmost branch of s.

Moreover, quasi-shuffles act on the set of pairs of elements of $\text{Tree}(\mathbb{N}^*)$ by:

Definition 4.3. Let $t, s \in \text{Tree}(\mathbb{N}^*)$ and $\sigma \in \text{Qsh}(k, l)$ where k (respectively l) is the number of forests in the right (respectively left) comb representation of t (respectively of s). Put $n = \max(\sigma)$. We define $\sigma(t, s)$ the element of $\text{Tree}(\mathbb{N}^*)$ defined by:

- 1. the underlying tree is the same built in definition 3.2;
- 2. the decoration function is unchanged for any forests expect for the vertices v_i for $i \in [1, n]$ of the ladder used to build the new tree in definition 3.2:

$$d(v_i) \coloneqq \begin{cases} d_t(v_j) & \text{if } \sigma^{-1}(\{i\}) = \{j\} \text{ and } v_j \in \nu(t), \\ d_s(v_j) & \text{if } \sigma^{-1}(\{i\}) = \{j\} \text{ and } v_j \in \nu(s), \\ d_t(v_{j_1}) + d_s(v_{j_2}) & \text{if } \sigma^{-1}(\{i\}) = \{j_1, j_2\} \text{ and } v_{j_1} \in \nu(t), v_{j_2} \in \nu(s), \end{cases}$$

where d_t and d_s are respectively the decorations maps of t and s.

Example 4.3. Let $\sigma = (1, 3, 2, 3)$ be (2, 2)-quasi-shuffle.

Take
$$t=F_1$$
 A_2 and $s=A_4$ A_3 A_4 A_4 A_5 A_4 A_5 A_6 two elements of Tree(\mathbb{N}^*). Then:

$$\sigma(t,s) = \begin{cases} F_2 & F_4 \\ f_2 + d_4 & F_3 \\ F_1 & d_3 \\ d_1 & \end{cases}.$$

With this construction we get:

Lemma 4.4. Let t and s having the same combs representations as equation (18). The map ι is surjective and for any $\sigma \in \operatorname{Qsh}(k,l)$ and $t,s \in \operatorname{Tridend}(\mathbb{N}^*)$:

$$\iota(\sigma(t,s)) = \sigma(\iota(t),\iota(s)).$$

Proof. Let $t \in \text{Tree}(\mathbb{N}^*)$. By definition of $\text{Tree}(\mathbb{N}^*)$ the decoration on any $v \in \nu(t)$ is greater than its number of angles. We define t' an element of $\text{Tridend}(\mathbb{N}^*)$ with the same tree structure such that for any $v \in \nu(t)$ the angles of v are decorated from left to right by a partition of the decoration of v of length the number of its angles. Hence, $\iota(t') = t$ so ι is surjective.

By construction of the action, ι sends the concatenation of the decorations of angles in $\operatorname{Tridend}(\mathbb{N}^*)$ to the sum of decorations in $\operatorname{Tree}(\mathbb{N}^*)$. Thus, it fulfils the equation required to achieve the proof of the lemma.

As a consequence from this lemma, we deduce that $\text{Tree}(\mathbb{N}^*)$ has a tridendriform structure inherited from the one of $\text{Tridend}(\mathbb{N}^*)$.

Proposition 4.5. For any trees $(t,s) \in \text{Tree}(\mathbb{N}^*)^2$, let us define the products

$$t \prec s \coloneqq \sum_{\substack{\sigma \in \mathrm{Qsh}(k,l), \\ \sigma^{-1}(\{1\}) = \{1\}}} \sigma(t,s), \quad t \succ s \coloneqq \sum_{\substack{\sigma \in \mathrm{Qsh}(k,l), \\ \sigma^{-1}(\{k+1\}) = \{k+1\}}} \sigma(t,s), \quad t \cdot s \coloneqq \sum_{\substack{\sigma \in \mathrm{Qsh}(k,l), \\ \sigma^{-1}(\{1,k+1\}) = \{1,k+1\}}} \sigma(t,s).$$

Then $(\operatorname{Tree}(\mathbb{N}^*), \prec, \succ, .)$ is a tridendriform algebra and ι is a morphism of tridendriform algebras. We will denote the sum of these three products by *.

Proof. Let us first show that ι is an algebra morphism for each of the products \prec , \succ and \cdot . For any trees $(t,s) \in \text{Tridend}(\mathbb{N}^*)^2$ we have

$$\iota(s \prec t) = \iota \left(\sum_{\substack{\sigma \in \text{Qsh}(k,l), \\ \sigma^{-1}(\{1\})}} \sigma(t,s) \right)$$
by definition of \prec

$$= \sum_{\substack{\sigma \in \text{Qsh}(k,l), \\ \sigma^{-1}(\{1\}) = \{1\}}} \iota(\sigma(t,s))$$
by linearity of ι

$$= \sum_{\substack{\sigma \in \text{Qsh}(k,l), \\ \sigma^{-1}(\{1\})}} \sigma(\iota(t),\iota(s))$$
by lemma 4.4
$$= \iota(t) \prec \iota(s)$$

by definition of \prec in Tree(N*). We show similarly that $\iota(s \succ t) = \iota(t) \succ \iota(s)$ and $\iota(s \cdot t) = \iota(t) \cdot \iota(s)$. We can now show that the seven equalities of definition 1.1 hold for the products of the proposition. Take $(r,s,t) \in \text{Tree}(\mathbb{N}^*)^3$, let us show that $(r \prec s) \prec t = r \prec (s*t)$. From the surjectivity of ι , there exist $(r',s',t') \in \text{Tridend}(\mathbb{N}^*)^3$ such that $\iota(r') = r$, $\iota(s') = s$, and $\iota(t') = t$. Then

$$(r \prec s) \prec t = (\iota(r') \prec \iota(s')) \prec \iota(t')$$

= $\iota((r' \prec s') \prec t')$ by the previous point
= $\iota(r' \prec (s' * t'))$ since Tridend(N*) is tridendriform
= $r \prec (s * t)$

again since ι is a morphism for the \prec and \cdot products. The other six equalities are shown in the same way.

Then, by lemma 4.4, ι is a morphism of tridendriform algebras.

Let us make some computations in the tridendriform structure of $Tree(\mathbb{N}^*)$:

Example 4.4. Working with low decorations we find

For the other products we have

$$\frac{1}{2}$$
 \times $\frac{1}{2}$ \times

Remark 4.1. By the freeness property of Tridend(\mathbb{N}^*), it shows that ι is the unique tridendriform morphism such that the diagram 1 commutes with i_1 and i_2 are the natural injections. Here, i_2 is the map i of theorem 3.5.

Notice that the properties of $\operatorname{Tree}(\mathbb{N}^*)$ can be generalised to $\operatorname{Tree}(\Omega)$ for any set Ω with a commutative monoid structure. We will not need this level of generality in this paper and we omit these statements, whose proofs can be copied mutatis mutandis for those wrote above.

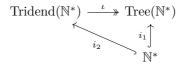


Figure 1: The ι map

4.2 Arborified Zeta Values

In the literature [27, 34], we can find a definition of an arborification of the Multiple Zeta Value. We introduce a similar concept adapted to the trees we are considering:

Definition 4.6 (Arborification of the integral Multiple Zeta Values). We define the following linear map $\zeta^T : \text{Tree}(\mathbb{N}^*)^{\text{conv}} \to \mathbb{R}$ such that for any tree t of $\text{Tree}(\mathbb{N}^*)^{\text{conv}}$, we define:

$$\zeta^{T}(t) = \sum_{\mathbf{k} \in D_{t}} \prod_{v \in \nu(t)} \frac{1}{k_{v}^{n_{v}}},$$

where D_t is the set defined by:

$$\left\{ (k_v)_{v \in \nu(t)} \in \mathbb{N}^{|\nu(t)|} \mid \forall (v, w) \in \nu(t)^2, k_v < k_w \text{ if } w \le v \right\},\,$$

where for any $(u, v) \in \nu(t)^2$, $u \leq v$ reading the tree as a Hasse diagram where the root is the minimum of the poset. Here, $\text{Tree}(\mathbb{N}^*)^{\text{conv}}$ is the subset of $\text{Tree}(\mathbb{N}^*)$ where the series is convergent, and will be specified below.

Notice that in [27, 34, 8], these Arborified Zeta Values are defined on non-planar, not necessarily Schroeder, forests. The requirement that internal vertices have at least two direct descendants does not restraint our results since one can simply add spurious leaves related to internal vertices without changing the values of the associated arborified zeta value. Working with planar trees does not change the series, nor their properties as it can be readily checked.

Example 4.5. For instance:

$$\zeta^T \left(2 \sqrt{2 \atop 2} \right) = \sum_{\substack{0 < k_2 < k_1 \\ 0 < k_3 < k_1}} \frac{1}{k_1^2} \frac{1}{k_2^2} \frac{1}{k_3}.$$

Notation 4.1. Let $k \in \mathbb{N}^*$ and t_1, \ldots, t_k be k trees of $\text{Tree}(\mathbb{N}^*)$. Let $n \in \mathbb{N}^*$. We denote the tree with root decorated by n which has t_1 to t_k for sons from right to left with:

$$B_n^+(t_1,...,t_k).$$

Moreover, there is also the known map:

Definition 4.7. We define the map flat : Tree(\mathbb{N}^*) $\to \mathcal{W}_{\mathbb{N}^*}$ by flat(|) = \emptyset and for any $B_n^+(t_1, \ldots, t_k) = t$ with $n \in \mathbb{N}^*$ by:

$$flat(t) = n \cdot (flat(t_1) \, \overline{\coprod} \dots \, \overline{\coprod} \, flat(t_k)).$$

Then we have the following properties of ζ^T .

Theorem 4.8 ([8]). A tree $t \in \text{Tree}(\mathbb{N}^*)$ lies in $\text{Tree}(\mathbb{N}^*)^{\text{conv}}$ if, and only if t = | or its root is not decorated by 1.

Then $\operatorname{flat}(\operatorname{Tree}(\mathbb{N}^*)^{\operatorname{conv}}) = \mathcal{W}^{\operatorname{conv}}_{\mathbb{N}^*}$ and ζ^T factorises through the flattening: $\zeta^T = \zeta \circ \operatorname{flat}$.

Finally, the map flat have an important property for our context that we will later use.

Lemma 4.9. The map flat: $\text{Tree}(\mathbb{N}^*) \to \mathcal{W}_{\mathbb{N}^*}$ is a morphism of tridendriform algebras.

Proof. We prove that flat is a morphism of tridendriform algebras with an induction over v the sum of the numbers of internal vertices of the trees we are multiplying.

Initialisation: v = 2, we have for any $n, m \in \mathbb{N}^*$:

$$\operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = n \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = n \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = n \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \end{array}\right) = m \operatorname{flat}\left(\begin{array}{c} \\ \\ \end{array}\right) = m \operatorname{fl$$

Heredity: suppose that there exists v such that for all pair of trees $(t_1, t_2), |\nu(t_1)| + |\nu(t_2)| \le v$, flat behaves like a tridendriform morphism. Consider t, s two elements of $\operatorname{Tree}(\mathbb{N}^*)$ with $|\nu(t)| + |\nu(s)| = v + 1$. Let us show that $\operatorname{flat}(t \ltimes s) = \operatorname{flat}(t) \ltimes \operatorname{flat}(s)$ for $\kappa \in \{\prec, \succ, \cdot\}$. If $|\nu(t)| = 0$ or $|\nu(s)| = 0$, the result trivially holds. Otherwise, we can write

$$t = B_n^+(t_1, \dots, t_k), \quad s = B_m^+(s_1, \dots, s_l).$$

Then, using the inductive descriptions of the tridendriform products from lemma 3.4 (which hold for the tridendriform products of $\text{Tree}(\mathbb{N}^*)$ by lemma 4.4) we have:

$$\operatorname{flat}(t \prec s) = \operatorname{flat}\left(\mathrm{B}_{n}^{+}\left(t_{1}, \ldots, \left(t_{k} * s\right)\right)\right)$$

$$= n\left(\operatorname{flat}\left(t_{1}\right) \, \overline{\sqcup} \cdots \, \overline{\sqcup} \, \operatorname{flat}\left(t_{k} * s\right)\right) \qquad \text{by definition of flat}$$

$$= n\left(\operatorname{flat}\left(t_{1}\right) \, \overline{\sqcup} \cdots \, \overline{\sqcup} \, \operatorname{flat}\left(t_{k}\right) \, \overline{\sqcup} \, \operatorname{flat}\left(s\right)\right)$$

from the associativity of $\overline{\coprod}$ and the induction hypothesis. Hence:

$$\operatorname{flat}(t \prec s) = [n \left(\operatorname{flat}(t_1) \, \overline{\coprod} \cdots \, \overline{\coprod} \, \operatorname{flat}(t_k) \right)] \prec \operatorname{flat}(s)$$

= $\operatorname{flat}(t) \prec \operatorname{flat}(s)$.

Moreover, using the same ideas:

$$\operatorname{flat}(t \succ s) = m \left(\operatorname{flat}(t * s_1) \, \overline{\coprod} \cdots \, \overline{\coprod} \, \operatorname{flat}(s_l) \right)$$
$$= \operatorname{flat}(s) \succ \left[m \left(\operatorname{flat}(s_1) \, \overline{\coprod} \cdots \, \overline{\coprod} \, \operatorname{flat}(s_l) \right) \right]$$
$$= \operatorname{flat}(s) \succ \operatorname{flat}(t) .$$

Finally:

$$\operatorname{flat}(t \cdot s) = \operatorname{flat}\left(\operatorname{B}_{n+m}^{+}(t_{1}, \dots, t_{k} * s_{1}, s_{2}, \dots, s_{l})\right)$$

$$= (n+m)\left(\operatorname{flat}\left(t_{1}\right) \, \overline{\sqcup} \cdots \, \overline{\sqcup} \, \operatorname{flat}\left(t_{k} * s_{1}\right) \, \overline{\sqcup} \cdots \, \overline{\sqcup} \, \operatorname{flat}\left(s_{l}\right)\right)$$

$$= (n+m)\left(\operatorname{flat}\left(t_{1}\right) \, \overline{\sqcup} \cdots \, \overline{\sqcup} \, \operatorname{flat}\left(t_{k}\right) \, \overline{\sqcup} \, \operatorname{flat}\left(s_{1}\right) \, \overline{\sqcup} \cdots \, \overline{\sqcup} \, \operatorname{flat}\left(s_{l}\right)\right)$$

$$= \left[n\left(\operatorname{flat}\left(t_{1}\right) \, \overline{\sqcup} \cdots \, \overline{\sqcup} \, \operatorname{flat}\left(t_{k}\right)\right)\right] \cdot \left[m \, \operatorname{flat}\left(s_{1}\right) \, \overline{\sqcup} \cdots \, \overline{\sqcup} \, \operatorname{flat}\left(s_{l}\right)\right]$$

$$= \operatorname{flat}(t) \cdot \operatorname{flat}(s).$$

Hence, by the induction principle, it shows that flat is a tridendriform morphism between $\text{Tree}(\mathbb{N}^*)$ and $\mathcal{W}_{\mathbb{N}^*}$.

4.3 Relating the Zetas

Let us recall the following maps, which were respectively introduced in theorem 3.5 and definition 3.6:

$$i: \left\{ \begin{array}{ccc} \mathbb{N}^* & \to & \operatorname{Tridend}(\mathbb{N}^*), \\ n & \mapsto & \stackrel{n}{\longrightarrow}, \end{array} \right. \qquad \sum: \left\{ \begin{array}{ccc} \mathbb{N}^* & \to & \mathcal{S}_{\mathbb{N}^*}, \\ n & \mapsto & \sum_{1 \le k} f_n. \end{array} \right.$$

Let us also denote by j the natural inclusion from \mathbb{N}^* into $\mathcal{W}_{\mathbb{N}^*}$. Then we recall the maps of tridendriform algebras: $\zeta^{\operatorname{Tri}}$: Tridend(\mathbb{N}^*) $\longrightarrow \mathcal{S}_{\mathbb{N}^*}$ and Ψ : Tridend(\mathbb{N}^*) $\longrightarrow \mathcal{W}_{\mathbb{N}^*}$ whose existence and unicity is given by the universal property of Tridend(\mathbb{N}^*) (theorem 3.5) as we can see in diagrams 2a and 2b. These tridendriform maps and algebras are related.

$$\mathbb{N}^* \xrightarrow{i} \operatorname{Tridend}(\mathbb{N}^*) \qquad \mathbb{N}^* \xrightarrow{i} \operatorname{Tridend}(\mathbb{N}^*) \\ \downarrow^{j} \downarrow^{\exists ! \, \Psi} \\ \mathcal{W}_{\mathbb{N}^*} \qquad \mathcal{S}_{\mathbb{N}^*}$$

(a) First freeness property of $Tridend(\mathbb{N}^*)$ (b) Second freeness property of $Tridend(\mathbb{N}^*)$

Figure 2: Freeness diagrams

Theorem 4.10. The following diagram commutes:

$$\operatorname{Tree}(\mathbb{N}^*) \xrightarrow{\iota} \operatorname{Tridend}(\mathbb{N}^*) \xrightarrow{\zeta^{\operatorname{Tri}}} \mathcal{S}_{\mathbb{N}^*} . \tag{19}$$

$$\downarrow^{\Psi} \qquad \qquad \downarrow^{\zeta_{\operatorname{FS}}}$$

Proof. We know that ι and flat are tridendriform algebras morphisms (proposition 4.5 and lemma 4.9 respectively). Thus flat $\circ\iota$: Tridend(\mathbb{N}^*) $\longrightarrow \mathcal{W}_{\mathbb{N}^*}$ is a morphism of tridendriform algebras. But from the definition of maps i and j in the diagram 2a, replacing Ψ by flat $\circ\iota$ also commutes. Thus, by the unicity of Ψ in this diagram, we have $\Psi = \text{flat } \circ\iota$.

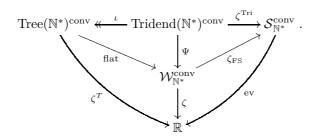
The same argument also holds for the commutation of the second triangle since $\zeta_{FS}: \mathcal{W}_{\mathbb{N}^*} \longrightarrow \mathcal{S}_{\mathbb{N}^*}$ is a morphism of tridendriform algebras (proposition 2.10) as well as $\Psi: \operatorname{Tridend}(\mathbb{N}^*) \longrightarrow \mathcal{W}_{\mathbb{N}^*}$ by definition. Furthermore, replacing $\zeta^{\operatorname{Tri}}$ by $\Psi \circ \zeta_{FS}$ in diagram 2b also commutes. So $\zeta^{\operatorname{Tri}} = \zeta_{FS} \circ \Psi$ by unicity of $\zeta^{\operatorname{Tri}}$.

Let us now define convergent elements of Tridend(\mathbb{N}^*).

Definition 4.11. Let $t \in \text{Tridend}(\mathbb{N}^*)$ be a decorated Schroeder tree. Then t is convergent if t = | or the root of t does not have a unique angle decorated by 1. Let $\text{Tridend}(\mathbb{N}^*)^{\text{conv}}$ be the subvector space of $\text{Tridend}(\mathbb{N}^*)$ generated by convergent trees.

Then specializing the diagram of theorem 4.10 to convergent trees we obtain

Corollary 4.12. The following diagram commutes:



Proof. From the definitions of the maps and the spaces in this diagram, it is clear that the maps send convergent spaces to convergent spaces. The two upper triangles are a special case of theorem 4.10. The left lower triangle is theorem 4.8. The right lower triangle is a direct consequence of the definition of ζ_{FS} (definition 2.9).

In particular, we obtain that tridendriform zeta values are Arborified Zeta Values, and also linear combinations (with integer coefficients) of Multiple Zeta Values, given by

$$\forall t \in \text{Tridend}(\mathbb{N}^*)^{\text{conv}}, \text{ ev } \circ \zeta^{\text{Tri}}(t) = \zeta^T(\iota(t)) = \zeta \circ \text{flat } \circ \iota(t).$$

Finally, notice that this construction implies a new property for Arborified Zeta Values.

Corollary 4.13. The map $\zeta^T : \operatorname{Tree}(\mathbb{N}^*)^{\operatorname{conv}} \to \mathbb{R}$ is an algebra morphism for the quasi-shuffle product of $\operatorname{Tree}(\mathbb{N}^*)$: for any couple of trees $(t_1, t_2) \in \left(\operatorname{Tree}(\mathbb{N}^*)^{\operatorname{conv}}\right)^2$ we have

$$\zeta^{T}(t_1 * t_2) = \zeta^{T}(t_1)\zeta^{T}(t_2).$$

Proof. It is clear from the definition that $\text{Tree}(\mathbb{N}^*)^{\text{conv}}$ is a tridendriform subalgebra of $\text{Tree}(\mathbb{N}^*)$. Then for any couple of trees $(t_1, t_2) \in \left(\text{Tree}(\mathbb{N}^*)^{\text{conv}}\right)^2$ we have

$$\zeta^{T}(t_{1} * t_{2}) = \zeta \circ \operatorname{flat}(t_{1} * t_{2}) \qquad \text{by corollary 4.12}$$

$$= \zeta \circ (\operatorname{flat}(t_{1}) \, \overline{\coprod} \, \operatorname{flat}(t_{2})) \qquad \text{by lemma 4.9}$$

$$= \zeta \circ \operatorname{flat}(t_{1})\zeta \circ \operatorname{flat}(t_{2}) \qquad \text{since } \zeta \text{ is an algebra morphism for } \overline{\coprod}$$

$$= \zeta^{T}(t_{1})\zeta^{T}(t_{2}) \qquad \text{by corollary 4.12.}$$

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Hence, we have proved the theorem.

Let us write down an explicit example of new relations amongst arborified zeta values using the computation of example 4.4

Example 4.6. We have

$$\zeta^{T} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \zeta^{T} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$= \zeta^{T} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \zeta^{T} \begin{pmatrix}$$

This relation can be checked using corollary 4.12. It allows us to reformulate this equation as

$$\begin{split} \left(2\zeta(2,1,1) + \zeta(2,2)\right)\zeta(2) &= \zeta(2,1,2,1) + 2\zeta(2,1,1,2) + \zeta(2,1,3) + \zeta(2,2,2) \\ &+ 2\zeta(2,2,1,1) + \zeta(2,1,2,1) + \zeta(2,3,1) + \zeta(2,2,2) \\ &+ \zeta(2,3,1) + \zeta(2,1,3) + \zeta(2,4) \\ &+ 2\zeta(2,2,1,1) + \zeta(2,2,2) + 2\zeta(4,1,1) + \zeta(4,2) \end{split}$$

which can be checked using the first equality in equation (3).

We have solved the issue we sat ourselves upon and related our tridendriform zeta values to the usual Multiple Zeta Values and their arborified counterparts. As a consequence of the structures underlining our approach, we found an associative product for Schroeder trees such that the arborified Multiple Zeta Value is a morphism for this product. In the next section, we are doing the same construction with the integral representation of Multiple Zeta Values using a dendriform structure. It corresponds to a tridendriform structure where the · product is equal to 0. The proofs are similar and easier in this part. So, we will give less details in order to save space.

5 Dendriform algebra of formal integrals

We first built a tridendriform algebra to recover the arborified version of Multiple Zeta Values. We now perform a quite similar construction to section 2 with formal integrals. One can see the previous work as a specification of this one modifying the Lebesgue measure with another one with Radon-Nikodym decomposition $\nu + \mu$ where $\mu \neq 0$ and μ and λ are singular. As a consequence, the "diagonal" terms do not vanish any more.

5.1 Formal integrals

Notation 5.1. Let $n \in \mathbb{N}^*$. We denote by λ the Lebesgue measure over \mathbb{R}^n . We denote by $\mathcal{B}(\mathbb{R}^n)$ the borelian set of \mathbb{R}^n for λ . We also denote by $\mathcal{N}(\mathbb{R}^n)$ the subset of $\mathcal{B}(\mathbb{R}^n)$ containing all negligible sets for the Lebesgue measure of \mathbb{R}^n . Let $n \in \mathbb{N}^*$ and set $B_{int}(\mathbb{R}^n) := \mathcal{B}(\mathbb{R}^n)$. Hence, $B_{int}(\mathbb{R}^n)$ is our notation for the set of borelian set up to sets of measure 0.

Remark 5.1. One can take any measure to define formal integrals. For the sake of simplicity we take the Lebesgue measure since it is largely enough for our purposes.

Definition 5.1. For any $n \in \mathbb{N}^*$, we consider the formal vector space $\mathbb{K} B_{\text{int}}(\mathbb{R}^n)$ whose basis is $B_{\text{int}}(\mathbb{R}^n)$ and we extend linearly the definition of the union \cup . We define I the ideal of the algebra $(\mathbb{K} B_{\text{int}}(\mathbb{R}^n), +, \cup)$ generated by the set of elements:

$${A \cup B - A - B + A \cap B \mid A, B \in B_{int}(\mathbb{R}^n)}.$$

We finally set

$$\mathfrak{B}(\mathbb{R}^n) = \mathbb{K} \operatorname{B}_{\operatorname{int}}(\mathbb{R}^n) / I$$
.

Let Ω be a set, $w \in \Omega^*$ and a map $G \in (\mathbb{R} \to \mathbb{R})^{\Omega}$. We denote for any $\omega \in \Omega$, $g_{\omega} := G(\omega)$. Set $G_w : \mathbb{R}^k \to \mathbb{R}$ with $w = w_1 \dots w_k$ the function defined by:

$$G_w(x_1,\ldots,x_k) \coloneqq \prod_{i=1}^k g_{w_i}(x_i).$$

We call a formal integral an element $(B \otimes w, G)$ of

$$\langle \mathfrak{B}(\mathbb{R}^n) \otimes w \,|\, w \in \mathcal{W}_{\Omega}, n = \ell(w) \rangle \times (\mathbb{R} \to \mathbb{R})^{\Omega}.$$
 (20)

A formal integral $(B \otimes w, G)$ with $w = w_1 \cdots w_n$ will be written

$$\int_A \left(\prod_{i=1}^k g_{w_i}\right) dt_1 \dots dt_k.$$

The omission of the variable in which the function is taken will distinguish formal integrals from usual integrals. We will denote FI_{Ω} the set of definite formal integrals over Ω .

Notation 5.2. Following notation 2.2, we will denote a formal integral by $B \otimes w \times G$.

Remark 5.2. The notation of formal integrals is made such that a specification of F gives a concrete integral to compute.

As said in the previous section, our aim is to consider formal objects before any evaluation of the integral.

Example 5.1. In this section, we will mainly apply this construction with $\Omega = \{x, y\}$ and G is the map:

$$G: \begin{cases} x \mapsto \left(g_x : \left\{\begin{array}{ccc}]0, 1[& \to & \mathbb{R}, \\ & t & \mapsto & \frac{1}{t}, \end{array}\right), \\ y \mapsto \left(g_y : \left\{\begin{array}{ccc}]0, 1[& \to & \mathbb{R}, \\ & t & \mapsto & \frac{1}{1-t}. \end{array}\right) \end{cases}$$

Hence a formal version of the following Multiple Zeta Value

$$\int_{\substack{0 < t_2 < t_1 < 1, \\ 0 < t_3 < t_1 < 1}} \frac{\mathrm{d}t_1}{t_1} \frac{\mathrm{d}t_2}{1 - t_2} \frac{\mathrm{d}t_3}{1 - t_3}$$

is given by:

$$\{0 < t_2 < t_2 < 1 \text{ or } 0 < t_3 < t_1 < 1\} \otimes xyy \times G.$$

Remark 5.3. Note that in the previous example omitted the limit step. One should get the formal version for any 0 < a < b < 1 then taking the limit $a \to 0$ and $b \to 1$. For sake of simplicity, we will always omit this step later on.

The previous construction enables the following manipulations of formal integrals, which mimic the properties of usual integrals.

Proposition 5.2. Let $n \in \mathbb{N}^*$. Let Ω be a set, $A, B \in \mathfrak{B}(\mathbb{R}^n)$, $w \in \mathcal{W}_{\Omega}$ with $\ell(w) = n$, and $G \in (\mathbb{R} \to \mathbb{R})^{\Omega}$.

• If $(A \cup B) \otimes w \times G$ is a formal integral over Ω , then so are $A \otimes w \times G$ and $B \otimes w \times G$.

Furthermore we have

$$\int_{A \cup B} \left(\prod_{i=1}^n g_{\omega_i} \right) \prod_{i=1}^n \mathrm{dt}_i = \int_A \left(\prod_{i=1}^n g_{\omega_i} \right) \prod_{i=1}^n \mathrm{dt}_i + \int_B \left(\prod_{i=1}^n g_{\omega_i} \right) \prod_{i=1}^n \mathrm{dt}_i - \int_{A \cap B} \left(\prod_{i=1}^n g_{\omega_i} \right) \prod_{i=1}^n \mathrm{dt}_i.$$

• Let $w_1 = \omega_1^{(1)} \dots \omega_1^{(k)}, w_2 = \omega_2^{(1)} \dots \omega_2^{(k)}$ and $A \in \mathfrak{B}(\mathbb{R}^k)$. Then:

$$\int_{A} \left(\prod_{i=1}^{k} g_{\omega_{1}^{(i)}} + \prod_{j=1}^{k} g_{\omega_{2}^{(j)}} \right) \prod_{i=1}^{k} dt_{i} = \int_{A} \left(\prod_{i=1}^{k} g_{\omega_{1}^{(i)}} \right) \prod_{i=1}^{k} dt_{i} + \int_{A} \left(\prod_{j=1}^{n} g_{\omega_{2}^{(j)}} \right) \prod_{j=1}^{k} dt_{j}.$$

Proof. A similar proof strategy to proposition 2.2 applies here.

5.2 The dendriform algebra of formal integrals

In this subsection, we will mainly focus on the case $\Omega = \{x, y\}$ and:

$$G: \begin{cases} x \mapsto \left(g_x : \left\{\begin{array}{ccc}]0, 1[& \to & \mathbb{R}, \\ & t & \mapsto & \frac{1}{t}, \end{array}\right), \\ y \mapsto \left(g_y : \left\{\begin{array}{ccc}]0, 1[& \to & \mathbb{R}, \\ & t & \mapsto & \frac{1}{1-t}. \end{array}\right) \end{cases}$$
 (21)

In order to lighten the notations, we will omit the dependencies on G in the remaining of this section. Note that some results are still true other appropriate G and Ω .

Definition 5.3. Define the space of formal Chen integrals over Ω as the subspace of formal integrals with G fixed of \mathcal{A}_{Ω} generated by

$$\bigcup_{r \ge 0} \bigcup_{w \in \mathcal{W}_{\Omega, r}} \left\{ \int_{0 < t_1 < \dots < t_r < 1} \left(\prod_{i=1}^r g_{\omega_i} \right) \prod_{i=1}^r \mathrm{d}t_i \right\}$$

where we set $w = \omega_1 \dots \omega_r$ as usual. In fact, \mathcal{A}_{Ω} inherits the vector space structure of left space in equation (20). Hence, \mathcal{A}_{Ω} is the set of formal integrals needed to build Multiple Zeta Values in their integral version (equation (2)).

It is well known, from standard analysis tools, that a formal integral $A \otimes w \times G \in \mathcal{A}_{\Omega}$ is equivalent to a convergent integral provided that the first letter of the word is an x and the last one is a y. Then, the following map will encode all the analysis job to get any real value.

Definition 5.4 (Evaluation map). A formal integral $A \otimes w \times G$ is said to be convergent if $w = \emptyset$ or if w begins with a x and ends with a y (see definition 1.14). We will denote the space of convergent formal integrals by $\operatorname{FI}_{\Omega}^{\operatorname{conv}}$. We define the evaluation map, denoted ev, as follows:

$$\operatorname{ev}_{\Omega} : \left\{ \begin{array}{ccc} \operatorname{FI}_{\Omega}^{\operatorname{conv}} & \to & \mathbb{R}, \\ \int_{A} \left(\prod_{i=1}^{k} f_{\omega_{i}} \right) \prod_{i=1}^{k} \operatorname{dt}_{i} & \mapsto & \int_{A} \left(\prod_{i=1}^{k} f_{\omega_{i}}(t_{i}) \right) \prod_{i=1}^{k} \operatorname{dt}_{i}. \end{array} \right.$$

Remark 5.4. This evaluation map satisfies similar properties to remark 2.2.

Using Chen's lemma for iterated integrals [7], we introduce the unique shuffle product \sqcup on formal integrals such that the evaluation map is a morphism for the multiplicative structure of \mathbb{R} .

Definition 5.5. We introduce two new operators \prec and \succ defined for any $\alpha, \beta \in \mathcal{A}_{\Omega}$ written as follows:

$$\alpha = \int_{0 < t_1 < \dots < t_k < 1} \prod_{i=1}^k f_{\omega_i} dt_i, \qquad \beta = \int_{0 < t_{k+1} < \dots < t_{k+l} < 1} \prod_{i=k+1}^{k+l} f_{\omega_i} dt_i,$$

by:

$$\alpha \prec \beta := \sum_{\substack{\sigma \in Sh(k,l) \\ \sigma^{-1}(\{1\}) = \{1\}}} \int_{0 < t_1 < \dots < t_{k+l} < 1} \prod_{i=1}^{k+l} f_{\omega_{\sigma^{-1}(i)}} dt_i,$$

$$\alpha \succ \beta := \sum_{\substack{\sigma \in Sh(k,l) \\ \sigma^{-1}(\{1\}) = \{k+1\}}} \int_{0 < t_1 < \dots < t_{k+l} < 1} \prod_{i=1}^{k+l} f_{\omega_{\sigma^{-1}(i)}} dt_i.$$

Moreover, we have $\alpha \sqcup \beta := \alpha \prec \beta + \alpha \succ \beta$.

Then, we have analogous results to the tridendriform part.

Proposition 5.6. With this structure $(A_{\Omega}, \prec, \succ)$ is a dendriform algebra.

Proof. A similar proof strategy to proposition 2.6 proves the proposition.

Proposition 5.7. Let Ω be a set and $G: \Omega \to \mathbb{R}^{\mathbb{R}}$. Then, $(\mathcal{A}_{\Omega}, \prec, \succ)$ is generated as a dendriform algebra by the following set:

$$\left\{ \int_0^1 g_\omega \, \mathrm{dt} \right\}_{\omega \in \Omega}.$$

Proof. The proof is similar to proposition 2.7.

Finally, we have the dendriform version of proposition 2.8, which is proved in exactly the same fashion:

Proposition 5.8. $\operatorname{FI}_{\Omega}^{\operatorname{conv}}$ is a dendriform subalgebra of \mathcal{A}_{Ω} and ev is an algebra morphism for the product \sqcup .

Definition 5.9 (Multiple Zeta Value integral representation). We define the following linear map $\zeta_{\text{FI}}: \mathcal{W}^{\text{conv}}_{\{x,y\}} \to \mathcal{A}_{\{x,y\}}$ for any word $w = \omega_1 \dots \omega_k \in \mathcal{W}_{\{x,y\}}$ by:

$$\zeta_{\mathrm{FI}}(w) = \int_{0 < t_1 < \dots < t_k < 1} \left(\prod_{i=1}^k g_{\omega_i} \right) \prod_{i=1}^k \mathrm{d}t_i$$

and $\zeta_{\text{FI}}(\emptyset) := \emptyset \otimes \emptyset \times G$. An element in the image of ζ_{FI} is called a formal integral Multiple Zeta Value.

Hence, the classical integral Multiple Zeta Value of proposition 2 satisfies:

$$\zeta_{\rm int} = {\rm ev} \circ \zeta_{\rm FI}$$
.

As a first advantage of the formal integral version of Multiple Zeta Values, we have that ζ_{FI} is a morphism of dendriform algebra. This is the dendriform counterpart to proposition 2.10.

Proposition 5.10. The map $\zeta_{\text{FI}}: \mathcal{W}_{\{x,y\}} \to \mathcal{A}_{\{x,y\}}$ is a morphism of dendriform algebras.

Proof. Using the fact that $\prec = \succ^{\text{op}}$, it is enough to prove that ζ_{FI} is a morphism for \prec . Let v, u be two elements of $\mathcal{W}_{\{x,y\}}$ such that $v = \omega_1 \dots \omega_k$ and $u = \omega_{k+1} \dots \omega_{k+l}$. Then:

$$\zeta_{\mathrm{FI}}(v \prec u) = \sum_{\substack{\sigma \in \mathrm{Sh}(k,l) \\ \sigma^{-1}(1)=1}} \zeta_{\mathrm{FI}} \left(\omega_{\sigma^{-1}(1)} \dots \omega_{\sigma^{-1}(k+l)} \right)$$

$$= \sum_{\substack{\sigma \in \mathrm{Sh}(k,l) \\ \sigma^{-1}(1)=1}} \int_{0 < t_1 < \dots < t_{k+l} < 1} \prod_{i=1}^{k+l} f_{\omega_{\sigma^{-1}(i)}} \, \mathrm{d}t_i$$

$$= \left(\int_{0 < t_1 < \dots < t_k < 1} \prod_{i=1}^{k} f_i \, \mathrm{d}t_i \right) \prec \left(\int_{0 < t_{k+1} < \dots < t_{k+l} < 1} \prod_{j=k+1}^{k+l} f_j \, \mathrm{d}t_j \right)$$

$$= \zeta_{\mathrm{FI}}(v) \prec \zeta_{\mathrm{FI}}(u).$$

This concludes the proof.

This proposition indicates that the property of integral representation of Multiple Zeta Values being a morphism for the shuffle product of $W_{\{x,y\}}$ comes from a deeper dendriform structure. Studying the link between the evaluation map and the dendriform structure is beyond the scope of this article since we are not aiming at deriving new relations among Multiple Zeta Values and is therefore left for future research.

6 Multiple Zeta Values and the dendriform structure

The construction in this section is similar to the one in the tridendriform case section 3. In this particular case, the construction is simpler than in the previous one. We introduce the combinatorics of the free dendriform algebra.

Definition 6.1 (binary tree). A binary tree is a Schroeder tree where all of its internal vertices has exactly two children. We will denote the set of binary trees union | by \mathcal{BT} . Let us denote by $\mathcal{BT}(\{x,y\})$ the set of binary trees whose angles are decorated by an element of $\{x,y\}$.

Remark 6.1. Given a binary tree, a labelling of its angles is equivalent to do a labelling of its internal vertices.

Example 6.1. As before, we write \mathcal{BT}_n (resp. $\mathcal{BT}_n(\{x,y\})$) the set of binary trees with n+1 leaves (resp. decorated by $\{x,y\}$). Here are some examples of binary trees:

$$\mathcal{BT}_0 = \{|\}, \quad \mathcal{BT}_1 = \{\}, \quad \text{and } \mathcal{BT}_2 = \{\}, \}$$

Moreover, we list here all the trees of $\mathcal{BT}(\{x,y\})$ with at most 3 leaves:

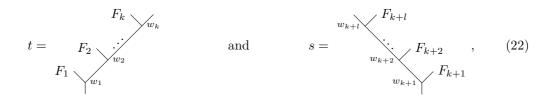
$$\mathcal{BT}_0(\{x,y\}) = \{|\}, \quad \mathcal{BT}_1(\{x,y\}) = \left\{ \begin{array}{c} x \\ \\ \end{array}, \begin{array}{c} y \\ \\ \end{array} \right\}, \text{ and}$$

$$\mathcal{BT}_2(\{x,y\}) = \left\{ \begin{array}{c} x \\ \\ \end{array}, \begin{array}{c} y \\ \\ \end{array}, \begin{array}{c} y \\ \\ \end{array}, \begin{array}{c} y \\ \\ \end{array}, \begin{array}{c} x \\ \\ \end{array}, \begin{array}{c} y \\ \\ \end{array}, \begin{array}{c} x \\ \\ \end{array}, \begin{array}{c} y \\ \\ \end{array}, \begin{array}{c} x \\ \\ \end{array}, \begin{array}{c} x$$

We recall the notations for Schroeder trees that are still valid for binary trees.

Notation 6.1. Let t be a tree in $Dend(\Omega)$. We denote by $\nu(t)$ the set of internal vertices of t. We also define the map $d: \nu(t) \to \Omega$ such that v is decorated by d_v .

The free dendriform algebra over $\{x,y\}$ has for underlying vector $\mathbb{K}\mathcal{BT}(\{x,y\})$. The products are defined via an action of the shuffles over the combs representation of these trees, see [4] in the case where $\cdot = 0$. The combinatorial description is a simpler version of the one in the free tridendriform algebra involving only *shuffles* introduced in definition 1.10. Let t, s be two elements of $\mathcal{BT}(\{x,y\})$ |. We see t as a right comb and s as a left comb. In other words, we put:



where for all $i \in [1, k+l]$, F_i is a binary tree and w_i is the label of the interval vertex on which F_i is grafted. Restricting ourselves to shuffles in definition 3.2, we have an action of shuffles on any pair (t, s). As a consequence, we also have the analogous of theorem 3.3 in the dendriform case:

Theorem 6.2 ([4]). Let $(t,s) \in \mathcal{BT}(\{x,y\})^2$ different from | as in equation (22). Then for any set, $\mathcal{BT}(\{x,y\})$ has the structure of a dendriform algebra given by:

$$t\star s = \sum_{\sigma\in\operatorname{Sh}(k,l)} \sigma(t,s).$$

Moreover:

$$t \prec s = \sum_{\substack{\sigma \in \mathrm{Sh}(k,l) \\ \sigma^{-1}(\{1\}) = \{1\}}} \sigma(t,s), \qquad t \succ s = \sum_{\substack{\sigma \in \mathrm{Sh}(k,l) \\ \sigma^{-1}(\{1\}) = \{k+1\}}} \sigma(t,s).$$

Moreover, it also fulfils a similar lemma to lemma 3.4:

Lemma 6.3. Let t and s be two trees of $\mathcal{BT}(\{x,y\})$ different from | with:

$$t = t^{(1)} \vee_{\omega_1} \dots \vee_{\omega_{k-1}} t^{(k)}, \qquad s = s^{(1)} \vee_{\mu_1} \dots \vee_{\mu_{l-1}} s^{(l)}.$$

Then:

$$t \prec s = t^{(1)} \vee_{\omega_1} \cdots \vee_{\omega_{k-1}} \left(t^{(k)} \star s \right) \text{ and } t \succ s = \left(t \star s^{(1)} \right) \vee_{\mu_1} s^{(2)} \vee_{\mu_2} \cdots \vee_{\mu_{l-1}} s^{(l)}.$$

The freeness property of $\mathrm{Dend}(\{x,y\})$, similar to the one of $\mathrm{Tree}(\Omega)$ stated in theorem 3.5, is given below.

Theorem 6.4 ([29, 24]). The structure (Dend($\{x,y\}$), \prec , \succ) is the free dendriform algebra over $\{x,y\}$: for any dendriform algebra A and any map $L:\{x,y\} \longrightarrow A$, there exists a unique morphism of dendriform algebras $\Phi: \mathrm{Dend}(\{x,y\}) \longrightarrow A$ such that the diagram below commutes, with $i:\{x,y\} \longrightarrow \mathrm{Dend}(\{x,y\})$ the canonical embedding defined by i(z):=

$$\{x,y\} \xrightarrow{i} \text{Dend}(\{x,y\})$$

$$\downarrow L \qquad \downarrow_{\Phi} \qquad .$$

Applying the freeness property of $\mathrm{Dend}(\{x,y\})$ to the dendriform algebra $\mathcal{A}_{\{x,y\}}$, we obtain dendriform zeta values:

Definition 6.5. Let $\mathfrak{G}: \{x,y\} \to \mathcal{A}_{\{x,y\}}$ be the map defined by $z \mapsto \int_0^1 g_z \, dt$ where g_z are chosen in equation (21). Then the *dendriform zeta values* is the unique morphism of dendriform algebras $\zeta^{\mathrm{Dend}}: \mathrm{Dend}(\{x,y\}) \to \mathcal{A}_{\{x,y\}}$ whose existence and unicity is given by the universal property of $\mathrm{Dend}(\{x,y\})$ stated in theorem 6.4 for $L=\mathfrak{G}$.

Let us recall that $\mathcal{A}_{\{x,y\}}$ is a space of *formal* integrals, as the lack of the integration parameter in $\int_0^1 g_z dt$ indicates. Thus, the reader should not be concerned with the fact that the integrals $\int_0^1 g_z(t) dt$ do not converge.

Now, by definition, ζ^{Dend} is a morphism of dendriform algebras, therefore it is also a morphism for the shuffle product \star in $\mathcal{BT}(\{x,y\})$ which is explicitly described in theorem 6.2:

$$\forall (t, s) \in \mathcal{BT}(\{x, y\}), \zeta^{\mathrm{Dend}}(t \star s) = \zeta^{\mathrm{Dend}}(t) \cdot \zeta^{\mathrm{Dend}}(s).$$

Hence, we have build (once again, after using the evaluation map of definition 5.4) a new generalisation of integral Multiple Zeta Values which is a morphism for a generalisation of the shuffle products of words. In the rest of the section, we will aim at relating ζ^{Dend} to other known objects, and in particular integral Arborified Zeta Values and integral Multiple Zeta Values. This will also solve the issue of convergence of dendriform zeta values that we have omitted so far.

7 From dendriform to usual Multiple Zeta Values

Unlike in section 4, we are directly working with the adequate objects.

7.1 Arborified integrals zeta values

In the literature [27, 34], we can a find a definition of an Arborified Zeta Value for the integral representation of Multiple Zeta Value. We introduce a similar concept adapted to the trees we are considering:

Definition 7.1 (Arborification of the Multiple Zeta Values). Let $Dend(\{x,y\})^{conv}$ be the subspace of $Dend(\{x,y\})$ generated by the tree t=| and trees whose roots are decorated by x and whose leaves are decorated by y.

We introduce the linear map $\zeta_{\text{int}}^T: \text{Dend}(\{x,y\})^{\text{conv}} \to \mathbb{R}$ such that for any tree t of $\text{Dend}(\{x,y\})^{\text{conv}}$, it is defined by:

$$\zeta_{\text{int}}^T(t) = \int_{u \in \Delta_t} \prod_{v \in \nu(t)} g_{d_v}(t_v) \, dt_v,$$

where $\Delta_t \subseteq [0, 1]^{|\nu(t)|}$ stands for:

$$\{(t_{v_1}, \dots, t_{v_{|\nu(t)|}}) \mid \forall (i,j) \in [1, |\nu(t)|]^2, t_{v_i} > t_{v_j} \iff v_i \le v_j \},$$

where for any $(u, v) \in \nu(t)^2$, $u \leq v$ reading the tree as a Hasse diagram where the root is the minimum of the poset.

Example 7.1. For instance:

$$\zeta_{\text{int}}^T \left(\bigvee_{\mathbf{y}}^{\mathbf{y}} \bigvee_{\mathbf{y}}^{\mathbf{y}} \right) = \int_{\substack{0 < t_2 < t_1 < 1, \\ 0 < t_3 < t_1 < 1}} \frac{\mathrm{d}t_1}{t_1} \frac{\mathrm{d}t_2}{1 - t_2} \frac{\mathrm{d}t_3}{1 - t_3}.$$

Remark 7.1. As in the tridendriform case, we are now working with Schroeder trees instead of non-planar ones and this has no impact on the arborified zeta values that we are studying. A new difference is that these trees have to be binary (in [27, 34, 8] they are not), and their branching vertices do not have to be decorated by y, as in [8]. However, one can easily check that the proofs of [8] are still valid in this case: the planar binary Schroeder forests still form an $\{x, y\}$ -operated algebra. All the results follow from this simple fact.

We adapt the definition of the flattening to our binary trees using notation 4.1:

Definition 7.2. We define the map flat : Dend($\{x,y\}$) $\to \mathcal{W}_{\{x,y\}}$ by flat(|) = \emptyset and for any $B_z^+(t_1,t_2)=t$ with $z\in\{x,y\}$ by:

$$flat(t) = z \cdot (flat(t_1) \coprod flat(t_2)).$$

Then, we have the following properties of ζ_{int}^T

Theorem 7.3 ([8]). We have flat(Dend($\{x,y\}$)^{conv}) = $\mathcal{W}^{\text{conv}}_{\{x,y\}}$ and ζ^T_{int} factorises through the flattening: $\zeta^T_{\text{int}} = \zeta_{\text{int}} \circ \text{flat}$.

Finally, we have an important property for our use:

Lemma 7.4. The map flat : Dend($\{x,y\}$)^{conv} $\to \mathcal{W}^{\text{conv}}_{\{x,y\}}$ is a morphism of dendriform algebras. Proof. The proof is similar to the one of lemma 4.9.

7.2 Relating the integral zetas

We recall the following three maps:

$$i: \left\{ \begin{array}{ccc} \{x,y\} & \to & \mathrm{Dend}(\{x,y\}), \\ x & \mapsto & & \\ y & \mapsto & & \\ \end{array} \right., \qquad L: \left\{ \begin{array}{ccc} \{x,y\} & \to & \mathcal{A}_{\{x,y\}}, \\ x & \mapsto & \int_{0 < t < 1} g_x \, \mathrm{dt}, \\ y & \mapsto & \int_{0 < t < 1} g_y \, \mathrm{dt}, \end{array} \right.$$

and j is the natural inclusion from $\{x,y\}$ into $\mathcal{W}_{\{x,y\}}$. Then, let us recall (resp. define) the map of dendriform algebras ζ^{Dend} : Dend($\{x,y\}$) $\to \mathcal{A}_{\{x,y\}}$ (resp. $\Psi : \text{Dend}(\{x,y\}) \to \mathcal{W}_{\{x,y\}}$) whose existence and unicity is given by the universal property of theorem 6.4 as we can see in diagrams 3a (resp. 2b).

(a) First freeness property
$$\{x,y\} \xrightarrow{i} \mathrm{Dend}(\{x,y\}) \qquad \qquad \{x,y\} \xrightarrow{i} \mathrm{Dend}(\{x,y\}) \\ \downarrow^{j} \downarrow^{\exists ! \, \Psi} \qquad \qquad \downarrow^{L} \downarrow^{\exists ! \, \zeta^{\mathrm{Dend}}} \\ \mathcal{W}_{\{x,y\}} \qquad \qquad \mathcal{A}_{\{x,y\}}$$

Figure 3: Diagrams for dendriform structures

Proof. We have shown in proposition 5.10 that $\zeta_{\rm FI}$ is a dendriform morphism. Hence, $\zeta_{\rm FI} \circ$ flat and $\zeta^{\rm Dend}$ are both dendriform morphisms. Moreover, $\zeta_{\rm FI} \circ$ flat $\circ i = L$. So, we deduce by the unicity of $\zeta^{\rm Dend}$ of the universal property of theorem 5.10 that $\zeta_{\rm FI} \circ$ flat $= \zeta^{\rm Dend}$.

Example 7.2. For instance, considering the usual family of functions, we have:

$$\zeta^{\text{Dend}} \left(\begin{array}{c} \mathbf{y} \\ \mathbf{y} \\ \end{array} \right) = \int_{\substack{0 < t_2 < t_1 < 1, \\ 0 < t_3 < t_1 < 1}} g_x g_y g_y \, \mathrm{dt}_1 \, \mathrm{dt}_2 \, \mathrm{dt}_3 \,.$$

Then, evaluating this quantity gives the Arborified Zeta Values of example 7.1.

Then, specializing the diagram of theorem 7.5 to convergent trees we obtain

Corollary 7.6. The following diagram commutes:

$$\begin{array}{ccc}
\operatorname{Dend}(\{x,y\})^{\operatorname{conv}} & \xrightarrow{\zeta^{\operatorname{Dend}}} \mathcal{A}_{\{x,y\}} \\
\downarrow^{\operatorname{flat}} & \downarrow^{\operatorname{ev}} \\
\mathcal{W}^{\operatorname{conv}}_{\{x,y\}} & \xrightarrow{\zeta_{\operatorname{int}}} & \mathbb{R}
\end{array}$$

In particular, we obtain that dendriform zeta values are integral Arborified Zeta Values, and also linear combinations (with integer coefficients) of Multiple Zeta Values, given by

$$\forall t \in \text{Dend}(\{x, y\})^{\text{conv}}, \text{ev} \circ \zeta^{\text{Dend}}(t) = \zeta_{\text{int}}^T(t).$$

Finally, notice that this construction implies a new property for integral Arborified Zeta Values

Corollary 7.7. The map $\zeta_{\text{int}}^T : \text{Dend}(\{x,y\})^{\text{conv}} \to \mathbb{R}$ is a morphism of dendriform algebras. In particular, for any couple of tree $(t_1,t_2) \in (\text{Dend}(\{x,y\})^{\text{conv}})^2$:

$$\zeta_{\text{int}}^T(t_1 \star t_2) = \zeta_{\text{int}}^T(t_1) \cdot \zeta_{\text{int}}^T(t_2).$$

Proof. It is clear from the definition that $Dend(\{x,y\})^{conv}$ is a dendriform subalgebra of $Dend(\{x,y\})$. Then for any couple $(t_1,t_2) \in (Dend(\{x,y\})^{conv})^2$ we have:

$$\zeta_{\text{int}}^{T}(t_{1} \star t_{2}) = \zeta_{\text{int}} \circ \text{flat}(t_{1} \star t_{2}) \qquad \text{by corollary 7.6}$$

$$= \zeta_{\text{int}} \circ (\text{flat}(t_{1}) \sqcup \text{flat}(t_{2})) \qquad \text{by lemma 7.4}$$

$$= \zeta_{\text{int}} \circ \text{flat}(t_{1})\zeta_{\text{int}} \circ \text{flat}(t_{2}) \qquad \text{since } \zeta_{\text{int}} \text{ is an algebra morphism for } \sqcup$$

$$= \zeta_{\text{int}}^{T}(t_{1})\zeta_{\text{int}}^{T}(t_{2}) \qquad \text{by corollary 7.6.}$$

Hence the corollary is proven.

So, we have solved the issue of finding an associative product generalizing the shuffle of words that is still compatible for integral Arborified Zeta Values. Then, we sum up our results in diagram 4 given in the conclusion.

8 Shintani Zeta Values

Let us start by introducing (mutlivariate) Shintani Zeta Values. The following definition is borrowed from [28].

Definition 8.1. • Let $\Sigma_{n\times r}(\mathbb{R}_{\geq 0})$ be the set of $n\times r$ matrices with real non negatives entries, and with, at least, one non zero argument in each row and each column.

• Given a matrix $A = \{a_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq r} \in \Sigma_{n \times r}(\mathbb{R}_{\geq 0})$ and a word $\omega \in \mathbb{N}^*$, we define the Shintani zeta function associated to A as

$$\zeta_A(\omega) = \sum_{m_1 \ge 1} \cdots \sum_{m_r \ge 1} (a_{11}m_1 + \cdots + a_{1r}m_r)^{-\omega_1} \cdots (a_{n1}m_1 + \cdots + a_{nr}m_r)^{-\omega_r}.$$
 (23)

The sum (23) is absolutely convergent if $Re(\omega_i) > r$ for every $1 \le i \le n$. A better convergence result was obtained in [26] but is rather technical and we will omit here.

Given a tree $t \in \text{Dend}(\{x,y\})^{\text{conv}}$ we will associate its arborified zeta value $\zeta_{\text{int}}^T(t)$ (definition 7.1) to a Shintani Zeta Value $\zeta_{A_t}(\omega_t)$. For that, we need the following definitions, which are from [9].

Definition 8.2. Given $t \in \text{Dend}(\{x, y\})$ we define

$$\nu_{\nu}(t) = \{ v \in \nu(t) \mid d(v) = y \}; \quad \nu_{\nu}(t) = \{ v \in \nu(t) \mid d(v) = x \}.$$

A bifurcated vertex is a vertex of t that has more two descendants that are internal vertices. Let

$$B(t) := \{ v \in \nu(t) \mid v \in \nu_v(t) \text{ or } v \in \nu_x(t) \text{ and } v \text{ is a bifurcated vertex} \}.$$

Let (t, d_t) be binary tree decorated by $\{x, y\}$ and $v \in \nu(t)$. A segment $s_v = (v_1, \dots, v_n = v)$ is a non-empty path in t such that $v_n = v \in B(t)$, $v_i \notin B(t)$, for all $i \in \{1, \dots, n-1\}$ and $a(v_1) \in B(t)$, where $a(v_1)$ is the direct ancestor of v_1 . We call $|s_v| = n$ the length of s_v . We say that $s_v \leq s_{v'}$ if $v \leq v'$ in t.

We set
$$S(t) = \{s_v \mid v \in B(t)\}.$$

In words, a segment s_v of t is a path that ends in an element of B(t) and contains every node decorated by x that are not bifurcated between v and it's first ancestor that is also in B(t).

We will associate the tree $t \in \text{Dend}(\{x,y\})^{\text{conv}}$ to a pair (A_t, ω_t) with $A_t \in \Sigma_{|B(t)| \times |\nu_y(t)|}(\mathbb{R}_{\geq 0})$ and $\omega_t \in \mathcal{W}_{\mathbb{N}^*, |B(t)|}$.

Definition 8.3. Let $t \in \text{Dend}(\{x,y\})^{\text{conv}}$ be a tree different from |. First enumerate the elements of B(t) from 1 to |B(t)| and the element of $|\nu_y(t)|$ from 1 to $|\nu_y(t)|$. Then the matrix $A_t := (a_{ij})_{(i,j) \in \{1,...,|B(t)|\} \times \{1,...,|\nu_y(t)|\}}$ is defined by

$$a_{ij} := \begin{cases} 1 \text{ if the vertex in B}(t) \text{ numbered by } i \text{ is below the vertex in } \nu_y(t) \text{ numbered by } j, \\ 0 \text{ otherwise,} \end{cases}$$

for each $(i, j) \in \{1, \dots, |B(t)|\} \times \{1, \dots |\nu_{\nu}(t)|\}.$

For the word ω_t , for any $i \in \{1, \dots, |B(t)|\}$, set the *i*-th letter of ω_t to be $|s_v|$, where $|s_v|$ is the is the length of s_v .

Example 8.1. Consider the following tree

$$t = \underbrace{y}_{x} \underbrace{y}_{y}^{y}, \tag{24}$$

from the construction above we have

$$A_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \omega_t = 111211.$$

Remark 8.1. Permuting the rows of the matrix does not change the value of the Shintani Zeta Value.

The main result of this section is the existence of a series representation of $\operatorname{ev} \circ \zeta^{\operatorname{Dend}}(t)$. We will omit the proof since it is an adaptation of [9, Theorem 1.19]. Indeed, the only difference is that bifurcated vertices can now be decorated by x. It is easy to see that this does not change the proof. It does change the matrix A_t we built above, hence we state this result with Shintani Zeta Values rather than with conical zeta values.

Theorem 8.4. For any tree $t \in \text{Dend}(\{x,y\})^{\text{conv}}$ we have

$$\operatorname{ev} \circ \zeta^{\operatorname{Dend}}(t) = \zeta_{\operatorname{int}}^{T}(t) = \sum_{\substack{n_v \ge 1 \\ v \in \nu_y(t)}} \prod_{\substack{v \in B(t) \\ v' \succeq v}} \left(\sum_{\substack{v' \in \nu_y(t) \\ v' \succeq v}} n_{v'} \right)^{-|s_v|} = \zeta_{A_t}(\omega_t). \tag{25}$$

where the matrix A_t and the word ω_t are constructed in definition 8.3.

Example 8.2. From the tree in equation (24) we have

$$\zeta(t) = \sum_{m_1 \ge 1} \sum_{m_2 \ge 1} \sum_{m_3 \ge 1} \sum_{m_4 \ge 1} \sum_{m_5 \ge 1} \frac{1}{m_1 m_2 (m_1 + m_2) m_3^2 (m_3 + m_4) (m_1 + m_2 + m_3 + m_4 + m_5)}.$$

Remark 8.2. We found in this section a class of Shintani Zeta Values that are convergent but do not follow the sufficient convergent condition of [28]. Furthermore, this result (together with theorem 7.3 also implies that a family of Shintani Zeta Values can be written as linear combinations of Multiple Zeta Values with rational coefficients.

Conclusion

The diagram 4 below summarizes all the known results (with the \mathfrak{s} map is the Kontsevitch map detailed in the introduction) and the ones from our construction in only one picture.

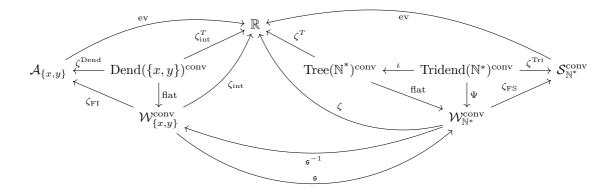


Figure 4: Picture of the general situation

With our construction, we have:

- introduced two intermediate objects, namely formal series and formal integrals, in order to factorize ζ and $\zeta_{\rm int}$ with respectively a tridendriform morphism and a dendriform morphism;
- built two new generalisations of ζ_{int} and ζ using the universal properties of Schroeder trees in the categories of dendriform and tridendriform algebras and related these generalisations to usual Multiple Zeta Values and arborified zeta values ζ_{int}^T and ζ^T ;
- built two associative products \star for binary trees and * for Schroeder trees (in a non-inductive way) such that ζ_{int}^T and ζ^T are respectively morphisms using universal properties of the free dendriform and free tridendriform algebras;
- shown that any iterated integral ev $\circ \zeta^{\mathrm{Dend}}(t)$ can be written as a multiple series which is a Shintani's Multiple Zeta Values.

From this work some new perspectives arise, which could be tackled in future projects:

• Could we find an analogous of the Kontsevitch's map $\mathfrak s$ for trees such that the diagram is still commutative ?

At first it may be not so simple. For instance,

$$\zeta_{\text{int}}^T \left(\bigvee_{y=1}^y \bigvee_{m>1}^y \right) = \sum_{\substack{n\geq 1\\m>1}} \frac{1}{n \cdot m \cdot (n+m)} = 2\zeta(2\ 1),$$

but one cannot find a *single* tree $t \in \text{Tree}(\mathbb{N}^*)$ giving rise to the above sum as the tree should have three vertices v_1, v_2 and v_3 satisfying $v_1 < v_3$ and $v_2 < v_3$. So if it exists, it should be $\wedge \bullet$ but this obviously not a tree.

The recent work [15] proposes a new generalisation of Kontsevitch's map that has some interesting properties. Seeing how this map fits into the picture displayed in figure 4 could solve this problem.

- We noticed with this construction that we are able to use universal properties of free dendriform and tridendriform algebras. Could we do this game for other type of algebras?
- Both free dendriform and free tridendriform algebras have a coalgebra structure. Is it possible to interpret these coalgebra structures for Multiple Zeta Values?

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