Closed Formulas for η-Corrections in the Once Punctured Torus

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Abstract

We study η-correction terms in the Kauffman bracket skein algebra of the once-punctured torus $K_t(\Sigma_{1,1})$. While the Frohman-Gelca product-to-sum rule is explicit on the closed torus, punctures introduce correction terms in the ideal (η) whose structure has resisted systematic description beyond low-determinant cases. We give a closed form for the family $P_n = (1,0) \cdot T_n((1,2))$ (determinant 2): the correction ϵ_n has an explicit Chebyshev expansion with coefficients that factor as geometric sums in $t^{\pm 4}$ governed by parity. Tracking creation and cancellation of η through the Chebyshev recurrence, and using diffeomorphism invariance, the formula transports to all products $C_1 \cdot T_k(C_2)$ with $|\det(C_1, C_2)| = 2$.

We further treat the *primitive maximal-thread* regime, where one Frohman-Gelca summand is fully threaded and the other is simple or doubly covered. In this case we obtain a closed form for the discrepancy: an η-linear cascade with Chebyshev S-coefficients that lowers the thread degree by two at each step. Equivalently, in this maximal-thread regime we solve the specialized Wang-Wong recursion in closed form; the coefficients match their fast algorithm term-by-term (up to the T_0 normalization) and subsume Cho's $|\det| = 2$ case. The resulting rules give compact, scalable formulas for symbolic multiplication in $K_t(\Sigma_{1,1})$.

1 Introduction

The Kauffman bracket skein algebra $K_t(F)$ encodes topological information about framed links in the thickened surface $F \times [0,1]$ by imposing local skein relations and a stacking product. When F is a torus, Frohman and Gelca [FG] famously showed that multiplication of (threaded) simple closed curves admits a clean product-to-sum rule expressed in terms of the noncommutative torus. Passing to a *punctured* torus, however, introduces a central element η represented by a small loop about the puncture; resolving crossings that encounter the puncture generates additional terms lying in the ideal (η), and the elegant closed-torus formula fractures into a main Frohman–Gelca part plus an η-correction. Understanding the structure of these correction terms is central to both the internal algebraic theory and to computational applications in topological quantum field theories, quantum character varieties, and quantum simulation architectures.

Understanding the structure of these η-correction terms is central to both the internal algebraic theory and to computational applications in topological quantum field theories, quantum character varieties, and quantum simulation architectures. Recent work has established fundamental structural properties of $K_t(\Sigma_{1,1})$ through connections to other areas of mathematics. Bousseau [Bo] proved the Thurston positivity conjecture for $K_t(\Sigma_{1,1})$ by connecting the skein algebra to enumerative geometry, realizing the structure constants through counts of curves on complex cubic surfaces using quantum scattering diagrams. Similarly, Queffelec [Q] established a general positivity result for the Jones-Wenzl basis on all orientable surfaces (other than the torus) by proving the full functoriality of $\mathfrak{gl}_2$ Khovanov homology for webs and foams. These geometric and categorical approaches establish the existence of important structural properties.

Despite substantial progress, a conceptual description of η-corrections has been available only in restricted regimes. For parallel curves ($|\det| = 0$), the problem reduces to Chebyshev identities with no correction terms. Cho [C] established that products of simple curves with $|\det| = \pm 2$ produce a single η. More recently, Wang-Wong [WWo] gave a fast *algorithmic* method that computes the Frohman-Gelca discrepancy D (hence the η-correction) on $\Sigma_{1,1}$ via a five-term recursion, thereby determining all products in principle. We introduce a symbolic, pattern-level description that scales, accommodates *threaded* factors, and exposes the combinatorial mechanism by which η-terms propagate.

Let $C = (p, q)$ denote a primitive simple closed curve on $\Sigma_{1,1}$ (identified with a homology class in $H_1(\Sigma_{1,1}; \mathbb{Z}) \cong \mathbb{Z}^2$ with $\gcd(p, q) = 1$). Threaded curves arise by Chebyshev evaluation: if $k > 1$ and (kp, kq) is a multiple of C, then we define $(kp, kq)_T := T_k(C)$. When we multiply a threaded curve $T_n(C_2)$ by a simple curve C_1 whose algebraic intersection with C_2 has modest absolute value, even for $|\det(C_1, C_2)| = \pm 2$, the naive approach of expanding T_n via its recurrence quickly generates a tangle of intermediate products, each possibly introducing puncture loops. Manual calculation becomes uninformative just beyond the smallest values of n.

This paper makes progress on this challenge by providing closed formulas for significant families of threaded products. Our main contributions are threefold. First, we provide a closed formula for the canonical infinite family of products

$$P_n := T_n((1, 2)) \cdot (1, 0),$$

whose underlying simple curves $(1, 2)$ and $(1, 0)$ have determinant -2. Writing $L_k = \sum_{\ell=-k}^{k} t^{4\ell}$, we prove that for $n \geq 3$,

$$P_n = t^{-2n}(n + 1, 2n)_T + t^{2n}(n - 1, 2n)_T + \epsilon_n, \tag{1}$$

where the correction term is

$$\epsilon_n = \eta \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \left(T_{n-1-2k}((1, 2)) - \delta_{n-1-2k,0}\right) L_k. \tag{2}$$

Second, Cho [C] proved that mapping class group diffeomorphisms carry any primitive pair of curves with $|\det| = 2$ to the standard pair $(1, 0)$ and $(1, 2)$. Using this diffeomorphism result, our formulas for P_n transport to show that for any simple curve C_1 and any threaded curve $T_k(C_2)$ with $|\det(C_1, C_2)| = 2$, the product $C_1 \cdot T_k(C_2)$ is explicitly computable by a finite symbolic procedure whose structure mirrors (1)–(2). This yields practical reduction rules useful for large symbolic expansions and quantum circuit compilation.

Third, we identify the geometric source of this pattern through what we call the *primitive maximal-thread mechanism*. For primitive curves (p,q) and (r,s) with determinant $n = pr - qs \geq 2$, suppose that one of the Frohman–Gelca summands is threaded with degree n (i.e., either $n = \gcd(p+r, q+s)$ or $n = \gcd(p-r, q-s)$). We prove that the product takes the closed form

$$(p,q) \cdot (r,s) = t^{\varepsilon n} T_{d_1}(C_+) + t^{-\varepsilon n} T_{d_2}(C_-) + \epsilon,$$

where $C_+ = (p+r, q+s)$, $C_- = (p-r, q-s)$, and $d_1 = \gcd(p+r, q+s)$, $d_2 = \gcd(p-r, q-s)$ with $\min\{d_1, d_2\} \in \{1, 2\}$ and $\max\{d_1, d_2\} = n$, the correction term has the explicit form

$$\epsilon = \eta \cdot \sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} t^{\varepsilon(n-2-2j)} \cdot \left(T_{n-2-2j}(C_*) - \delta_{n-2-2j,0}\right) \cdot S_j\left(t^2 + t^{-2}\right),$$

where the signs and curves are determined by which summand carries the maximal threading.

The work extends foundational results of Cho [C] on low-determinant cases, incorporates η-degree bounds from our previous work with Frohman [AF], and computational methods from Wang–Wong [WWo]. The paper is organized as follows. Section 2 reviews skein algebra conventions, threading, and determinant-based correction behavior. Section 3 introduces the family P_n, records low-n computations, and proves the parity lemmas that control η creation. Section 4 establishes the closed-form expression for P_n and develops equivalent coefficient factorizations. Section 5 formulates and proves the primitive maximal-thread correction structure.

2 Preliminaries

2.1 The Kauffman Bracket Skein Algebra $K_t(F)$

Given an oriented surface F, we consider framed links within the cylindrical space $F \times [0,1]$, which are embeddings of disjoint unions of annuli. Taking formal $\mathbb{C}$-linear combinations of these links and imposing the Kauffman bracket skein relations yields:

$$\langle \times \rangle = A \langle \asymp \rangle + t^{-1} \langle)(\rangle$$
$$\langle L \cup \bigcirc \rangle = (-t^2 - t^{-2}) \langle L \rangle$$

An algebra structure emerges from the product operation given by "stacking" along the $[0,1]$ direction.

Definition 2.1 (Simple Closed Curve)**.** *A **simple closed curve** on the torus is a curve that does not intersect itself. Algebraically, it is represented by a pair of coprime integers (p,q), meaning their greatest common divisor is 1.*

$$\gcd(p,q) = 1$$

Definition 2.2 (Composite (Threaded) Curve)**.** *A **composite curve**, also called a threaded curve, is represented by a pair of integers (p, q) that are not coprime, meaning their greatest common divisor is greater than 1, i.e.,* $\gcd(p, q) = k > 1$*. It is not a single simple curve but is defined as the k-th Chebyshev polynomial, T_k, of the underlying simple curve $(p/k, q/k)$.*

$$(p, q)_T = T_k\left(\left(\frac{p}{k}, \frac{q}{k}\right)_T\right)$$

2.2 The Product-to-Sum Formulas

Theorem 2.3 (Product-to-sum Formula in $K_t(\Sigma_{1,0})_T$[FG, Theorem 4.1])**.** *For any integers p, q, r, s, one has:*

$$(p, q)_T \cdot (r, s)_T = t^{\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right|}(p + r, q + s)_T + t^{-\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right|}(p - r, q - s)_T.$$

In previous work with Frohman [AF], we established the following product-to-sum formula and a related bound on the error term for $K_t(\Sigma_{1,1})$.

Theorem 2.4 (Product-to-Sum Formula in $K_t(\Sigma_{1,1})$ [AF, Theorem 6.1])**.** *Let $(p, q)_T, (r, s)_T \in K_N(\Sigma_{1,1})$. Then*

$$(p, q)_T * (r, s)_T = t^{\begin{vmatrix} p & q \\ r & s \end{vmatrix}}(p + r, q + s)_T + t^{-\begin{vmatrix} p & q \\ r & s \end{vmatrix}}(p - r, q - s)_T + \epsilon,$$

where $\epsilon \in (\eta)$ and the weight of ϵ is less than or equal to $|p| + |q| + |r| + |s| - 4$.

Lemma 2.5 ([AF, Lemma 6.2])**.** *For ϵ as in the previous theorem, the highest power of η appearing in ϵ is less than or equal to*

$$\left\lfloor \frac{\min\{|p| + |r|,\ |q| + |s|\}}{2} \right\rfloor.$$

2.3 Correction Term Structure by Determinant Value

The structure of the correction term ϵ depends critically on the determinant value:

- **Determinant 0:** Products of parallel curves are resolved using Chebyshev polynomial identities, e.g., $T_n(C)T_m(C) = T_{n+m}(C) + T_{|n-m|}(C)$. Here, $\epsilon = 0$.
- **Determinant ± 1:** [**WWo**] When $\det \begin{pmatrix} p & r \\ q & s \end{pmatrix} = \pm 1$, Wang–Wong show that the discrepancy term ϵ vanishes. In this case, the product reduces to the standard Frohman–Gelca two-term formula with no η-correction. Hence, $\epsilon = 0$.
- **Determinant ± 2:** [**C**] Resolved using results from Cho's thesis. The correction term is:

$$\epsilon = \begin{cases} \eta & \text{if both } (p, q)_T \text{ and } (r, s)_T \text{ are simple curves} \\ 0 & \text{if at least one curve is not simple (composite)} \end{cases} \tag{3}$$

The η-Bound Rule: A direct consequence of the lemma 2.5 on the highest power of η in ϵ provides a condition for when ϵ must be zero:

$$\text{If } \min(|p|+|r|, |q|+|s|) < 2, \text{ then } \epsilon = 0. \tag{4}$$

The Decomposition Method: Products involving a composite (threaded) skein, such as $T_k(C)$, can be calculated by first expressing the skein as a polynomial of a simpler curve (e.g., $(2,0)_T = (1,0)_T^2 - 2$) and then applying the product rules to the sequence of simpler multiplications.

3 General Multiplication Framework

3.1 The Family $P_n = (n, 2n)_T \cdot (1, 0)_T$

We now focus on a specific, yet highly illustrative, family of products, $P_n = (n, 2n)_T \cdot (1, 0)_T$. This family, which involves the product of a composite (threaded) curve and a simple curve, is particularly important for several reasons. First, its determinant structure is simple enough that it can be solved algorithmically using the established rules. Second, it serves as a canonical example that can be generalized via diffeomorphism to a much larger class of products.

The skeins $(n, 2n)_T = T_n((1, 2)_T)$ satisfy the Chebyshev recurrence relation. A careful derivation shows that this induces a recurrence for the products themselves:

$$P_n = (1, 2)_T \cdot P_{n-1} - P_{n-2}$$

This recurrence relation is the key to finding a closed-form solution for the correction term ϵ_n. We begin by establishing the base cases through direct computation.

Base Case Calculations

The following results were derived using the decomposition method and the rules for small determinants. They form the foundation for our inductive proof.

- **n=1:** $P_1 = t^{-2}(2, 2)_T + t^2(0, 2)_T + \eta$
- **n=2:** $P_2 = t^{-4}(3, 4)_T + t^4(1, 4)_T + (1, 2)_T\eta$
- **n=3:** $P_3 = t^{-6}(4, 6)_T + t^6(2, 6)_T + (t^4 + t^{-4} + 1 + (2, 4)_T)\eta$
- **n=4:** $P_4 = t^{-8}(5, 8)_T + t^8(3, 8)_T + [(t^4 + t^{-4} + 1)(1, 2)_T + (3, 6)_T]\eta$

See Appendix 5 for calculations.

3.2 Parity and Determinant Properties for P_n

Lemma 3.1 (Constant Determinant Property)**.** *For products of the form* $(1,2)_T \cdot (n \pm 1, 2n)_T$*, the determinant is always* ± 2*:*

$$\det(((1,2)),(n+1,2n)) = (1)(2n) - (2)(n+1) = -2 \tag{5}$$

Lemma 3.2 (Parity-Dependent Simplicity)**.** *The generation of new* η *terms in the recurrence depends on the parity of* n*:*

- *If* n *is even, then* $\gcd(n+1, 2n) = 1$*, so* $(n+1,2n)_T$ *is simple and* $(1,2)_T \cdot (n+1,2n)_T$ *generates a new* η *term.*
- *If* n *is odd, then* $\gcd(n+1, 2n) = 2$*, so* $(n+1,2n)_T$ *is composite and no new* η *term is generated.*

4 The General Formula for P_n

Proposition 4.1 (General Formula for P_n)**.** *The product* $P_n = (n,2n)_T \cdot (1,0)_T$ *where* $n \geq 3$ *is given by:*

$$P_n = t^{-2n}(n+1,2n)_T + t^{2n}(n-1,2n)_T + \epsilon_n \tag{6}$$

where the correction term is:

$$\epsilon_n = \eta \left[\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(T_{n-1-2k}((1,2)) - \delta_{n-1-2k,0}\right) \left(\sum_{l=-k}^{k} t^{4l} \right) \right] \tag{7}$$

Proof. The skeins $(n,2n)_T$ satisfy the Chebyshev recurrence relation $T_n(C) = C \cdot T_{n-1}(C) - T_{n-2}(C)$ where $C = (1,2)_T$. This induces a recurrence for the products:

$$P_n = (1,2)_T \cdot P_{n-1} - P_{n-2} \tag{8}$$

Cases for $k \in \{1,2\}$

$$\begin{aligned} \textbf{For } k = 1: \quad P_1 &= (1,2)_T \cdot (1,0)_T && (9) \\ &= t^{-2}(3,4)_T + t^2(1,4)_T && (10) \end{aligned}$$

$$\begin{aligned} \textbf{For } k = 2: \quad P_2 &= (2,4)_T \cdot (1,0)_T && (11) \\ &= ((1,2)_T^2 - 2) * (1,0)_T && (12) \\ &= (1,2) * [(1,2)_T * (1,0)_T] - 2(1,2)_T && (13) \\ &= (1,2) * [t^{-2}(3,4)_T + t^2(1,4)_T] - 2(1,2)_T && (14) \\ &= t^{-4}(3,4)_T + t^4(1,4)_T + (1,2)_T\eta && (15) \end{aligned}$$

Base Case: We verify the formula for $k = 3$:

$$\textbf{For } k=3: \quad P_3 = (3,6)_T \cdot (1,0)_T \tag{16}$$
$$= t^{-6}(4,6)_T + t^{6}(2,6)_T + (t^4 + t^{-4} + 1 + (2,4)_T)\eta \tag{17}$$

The formula gives:

$$\epsilon_3 = \eta\left[T_2((1,2)) + (T_0((1,2)) - 1)(t^4 + 1 + t^{-4})\right] \tag{18}$$
$$= \eta\left[(2,4)_T + (t^4 + 1 + t^{-4})\right] \tag{19}$$

which matches. See Appendix 5 for detailed calculations of $n \in \{3,4,5\}$.
Suppose its true for $k = n$

Assume the formula for P_k (including the main terms and the correction ϵ_k) is true for all $k \leq n$. We derive the formula for P_{n+1} using the recurrence relation $P_{n+1} = (1,2)_T \cdot P_n - P_{n-1}$.
Show that its true for $k = n+1$

The main product-to-sum terms for P_{n+1} are generated from the multiplication of $(1,2)_T$ with the main terms of P_n. As established by Lemma 3.2 and Lemma 3.1, the determinants of these products are always ± 2. The main terms of $(1,2)_T \cdot P_n$ correctly produce the main terms of P_{n+1} plus the main terms of P_{n-1}. When we subtract P_{n-1}, these latter terms cancel perfectly, leaving the correct main terms for P_{n+1}. The core of the proof is to show that the correction terms also obey the recurrence. The recurrence for the epsilon term is:

$$\epsilon_{n+1} = (\text{new } \eta \text{ terms}) + (1,2)_T \epsilon_n - \epsilon_{n-1}$$

The "new η terms" are generated from the product $(1,2)_T \cdot P_n$. Specifically, from the terms $t^{-2n}(1,2)_T(n+1,2n)_T$ and $t^{2n}(1,2)_T(n-1,2n)_T$. The determinants of these inner products are always ± 2. A new η is generated if and only if the curves are simple.

- If n is **even**, then $n \pm 1$ is odd. This means $\gcd(n \pm 1, 2n) = 1$, so the curves $(n \pm 1, 2n)_T$ are simple. Both products generate an η. The total contribution is $t^{-2n}(\eta) + t^{2n}(\eta) = (t^{2n} + t^{-2n})\eta$.
- If n is **odd**, then $n \pm 1$ is even. This means $\gcd(n \pm 1, 2n) = 2$, so the curves $(n \pm 1, 2n)_T$ are composite. Neither product generates an η. The contribution is 0.

Case 1: n is odd. In this case, no new η terms are generated, so the recurrence is

$$\epsilon_{n+1} = (1,2)_T \cdot \epsilon_n - \epsilon_{n-1}.$$

To prove this, we expand $(1,2)_T \cdot \epsilon_n$ using the Kronecker delta formula for ϵ_n and the Chebyshev identity

$$T_1 \cdot T_m = T_{m+1} + T_{m-1}.$$

Let

$$L_k = \sum_{\ell=-k}^{k} (t^4)^{\ell}.$$

Then:

$$\begin{aligned}
(1,2)_T \cdot \epsilon_n &= (1,2)_T \cdot \eta \left[\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(T_{n-1-2k}(1,2)_T - \delta_{n-1-2k,0}\right) L_k \right] \\
&= \eta \left[\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \left((1,2)_T \cdot T_{n-1-2k}(1,2)_T - (1,2)_T \cdot \delta_{n-1-2k,0}\right) L_k \right] \\
&= \eta \left[\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(T_{n-2k}(1,2)_T + T_{n-2-2k}(1,2)_T - (1,2)_T \cdot \delta_{n-1-2k,0}\right) L_k \right].
\end{aligned}$$

This expression splits into two summations, which we denote by Sum A and Sum B, plus a correction term from the Kronecker delta. Since n is odd, the delta is non-zero only for the final term of the sum where $n-1-2k=0$, i.e., $k = \frac{n-1}{2}$.

$$\begin{aligned}
\text{Sum A} &= \eta \left[\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} T_{n-2k}(1,2)_T \cdot L_k \right], \\
\text{Sum B} &= \eta \left[\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} T_{n-2-2k}(1,2)_T \cdot L_k \right].
\end{aligned}$$

Sum A corresponds exactly to the definition of ϵ_{n+1} (note that $n+1$ is even, so its delta term vanishes). Sum B cancels with the Kronecker delta correction and the expression for ϵ_{n-1}.

The result is the identity:

$$(1,2)_T \cdot \epsilon_n = \epsilon_{n+1} + \epsilon_{n-1},$$

which rearranges to give the desired recurrence relation:

$$\epsilon_{n+1} = (1,2)_T \cdot \epsilon_n - \epsilon_{n-1}.$$

Case 2: n **is even.** In this case, a new term $(t^{2n} + t^{-2n})\eta$ is generated. Let $n = 2m$. The recurrence for the correction term is:

$$\epsilon_{n+1} = (t^{4m} + t^{-4m})\eta + (1,2)_T \epsilon_{2m} - \epsilon_{2m-1}$$

We expand the term $(1,2)_T\epsilon_{2m}$. Note that for $n = 2m$, the index $n-1-2k$ is always odd, so the Kronecker delta in the formula for ϵ_{2m} is always zero.

$$
\begin{aligned}
(1,2)_T\epsilon_{2m} &= (1,2)_T \cdot \eta \left[\sum_{k=0}^{m-1} T_{2m-1-2k}(1,2)L_k\right] \\
&= \eta \left[\sum_{k=0}^{m-1} (T_{2m-2k}(1,2)_T + T_{2m-2-2k}(1,2))L_k\right] \\
&= \eta \left[\sum_{k=0}^{m-1} T_{2m-2k}(1,2)L_k\right] + \eta \left[\sum_{k=0}^{m-1} T_{2m-2-2k}(1,2)L_k\right]
\end{aligned}
$$

The second summation is almost ϵ_{2m-1}. For $n = 2m-1$, the index $n-2-2k = 2m-3-2k$ can be zero, so we must use the Kronecker delta form:

$$
\epsilon_{2m-1} = \eta \left[\sum_{k=0}^{m-1} (T_{2m-2-2k}(1,2)_T - \delta_{2m-2-2k,0})L_k\right]
$$

Substituting this, we find that the expression $(1,2)_T\epsilon_{2m} - \epsilon_{2m-1}$ simplifies to:

$$
\eta \left[\left(\sum_{k=0}^{m-1} T_{2m-2k}(1,2)L_k\right) + L_{m-1}\right]
$$

Now, we add the new term back into the recurrence for ϵ_{n+1}:

$$
\epsilon_{n+1} = (t^{4m} + t^{-4m})\eta + \eta \left[\left(\sum_{k=0}^{m-1} T_{2m-2k}(1,2)L_k\right) + L_{m-1}\right]
$$

We use the identity $L_m = L_{m-1} + t^{4m} + t^{-4m}$ to combine the terms:

$$
\begin{aligned}
\epsilon_{n+1} &= \eta \left[\left(\sum_{k=0}^{m-1} T_{2m-2k}(1,2)L_k\right) + (t^{4m} + t^{-4m} + L_{m-1})\right] \\
&= \eta \left[\left(\sum_{k=0}^{m-1} T_{2m-2k}(1,2)L_k\right) + L_m\right]
\end{aligned}
$$

This is almost the target formula for ϵ_{n+1}. The target is $\eta\left[\sum_{k=0}^{m}(T_{2m-2k} - \delta_{2m-2k,0})L_k\right]$. Expanding this gives:

$$
\eta \left[\left(\sum_{k=0}^{m-1} T_{2m-2k}(1,2)L_k\right) + (T_0 - 1)L_m\right] = \eta \left[\left(\sum_{k=0}^{m-1} T_{2m-2k}(1,2)L_k\right) + L_m\right]
$$

The expressions match. This completes the proof for the case where n is even. $\square$

Remark 4.2. *Notice that*

$$
L_k = \left(\sum_{l=-k}^{k} t^{4l}\right) = \left(\sum_{l=0}^{k} S_{2l}(t^2 + t^{-2})\right).
$$

In the following section we will use the S_j notation for coefficients.

Theorem 4.3 (Generalization via Diffeomorphism [C]). *Let $C_1 = (p,q)_T$ be a simple curve and let $T_n(C_2)$ be a composite curve, where $C_2 = (r,s)_T$ is a simple curve different from C_1. The product $C_1 \cdot T_n(C_2)$ is algorithmically computable using the established rules if and only if the determinant of the underlying simple curves is $|\det(C_1, C_2)| = 2$.*

5 The Primitive Maximal-Thread Case

We work with the Kauffman parameter t (so $\bigcirc = -t^2 - t^{-2}$). On $\Sigma_{1,1}$ the peripheral loop η is *central* in $K_t(\Sigma_{1,1})$, and for any fixed primitive simple closed curve C the family $\{T_m(C)\}_{m\geq 0}$ is *linearly independent* over $\mathbb{C}[t^{\pm 1}, \eta]$ (e.g. via the Frohman–Gelca embedding or the quantum trace [FG, BWo]). We use these facts tacitly.

We analyze the case where one Frohman–Gelca summand is maximally threaded. Let

$$\alpha = (p,q), \qquad \beta = (r,s), \qquad n = \det\begin{pmatrix} p & r \\ q & s \end{pmatrix} \geq 2,$$

and set $C_\pm = \alpha \pm \beta$, $d_\pm = \gcd(C_\pm)$. Assume

$$\max\{d_+, d_-\} = n \qquad \text{and} \qquad \min\{d_+, d_-\} \in \{1,2\}.$$

Write $\varepsilon = \operatorname{sgn}(n)$ and let C_* denote the *maximally threaded* Frohman–Gelca summand, i.e. $d_* = n$.

Proposition 5.1 (Threaded η-Correction Structure). *Let $\alpha = (p,q)$ and $\beta = (r,s)$ be primitive on $\Sigma_{1,1}$ with $n = \det(\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}) \geq 2$. Set $C_\pm = \alpha \pm \beta$ and $d_\pm = \gcd(C_\pm)$. Assume $\max\{d_+, d_-\} = n$ and $\min\{d_+, d_-\} \in \{1,2\}$. Then, in $K_t(\Sigma_{1,1})$ with $t = A$,*

$$\alpha \cdot \beta = t^{\varepsilon n}\, T_{d_+}(C_+/d_+) \; + \; t^{-\varepsilon n}\, T_{d_-}(C_-/d_-) \; + \; \epsilon,$$

where $\varepsilon = \operatorname{sgn}(n)$ and the correction term is

$$\epsilon = \eta \; * \sum_{j=0}^{\lfloor (n-2)/2 \rfloor} t^{\varepsilon(n-2-2j)} \Big(T_{n-2-2j}(C_*/n) - \delta_{n-2-2j,0} \Big)\, S_j(t^2 + t^{-2}),$$

with C_* the maximally threaded Frohman-Gelca summand ($C_* = n\mu_*$ with μ_* primitive).

Preliminaries for the proof. Before proving Proposition 5.1, we record two auxiliary ingredients that will be used repeatedly. First, an arithmetic normal form (and its corollary) places any pair (α, β) in a canonical $SL_2(\mathbb{Z})$-frame where one Frohman-Gelca summand is n times a primitive and makes the thread degrees $d_\pm$ explicit. Second, a Chebyshev "cascade" identity solves the specialized three-term recurrence that arises from the Recursion Rule once we fix the maximally threaded direction.

Lemma 5.2 ($\mathrm{SL}_2(\mathbb{Z})$ normal form and thread degrees). *Let $u = (p,q)$ and $v = (r,s)$ be primitive with $\det(u,v) = n \geq 2$. There is $M \in \mathrm{SL}_2(\mathbb{Z})$ such that $Mu = (1,0)$ and $Mv = (a,n)$ with $0 \leq a < n$ and $\gcd(a,n) = 1$. In this normal form*

$$d_+ = \gcd(1+a, n), \qquad d_- = \gcd(1-a, n),$$

where $d_\pm = \gcd(u \pm v)$ are the thread degrees of the Frohman-Gelca summands.

Proof. This is Smith normal form for the 2×2 integer matrix $[u\ v]$, together with the fact that gcd of coordinates is preserved under unimodular maps. The formulas for $d_\pm$ are immediate. $\square$

Corollary 5.3 (Characterization of the small co-thread regime)**.** *With notation as in Lemma 5.2, the following are equivalent:*

1. $\max\{d_+, d_-\} = n$ *and* $\min\{d_+, d_-\} \in \{1, 2\}$.
2. $a \equiv \pm 1 \pmod{n}$.
3. $u \equiv \pm v \pmod{n}$ *(componentwise congruence).*

In this case the non-maximal thread degree is $\gcd(n, 2)$*; in particular it equals* 1 *if* n *is odd and* 2 *if* n *is even.*

Proof. If $a \equiv 1 \pmod{n}$, then $d_- = \gcd(1-a, n) = n$ and $d_+ = \gcd(1+a, n) = \gcd(2, n) \in \{1, 2\}$. If $a \equiv -1 \pmod{n}$, the roles swap. Conversely, if $\max\{d_+, d_-\} = n$, then either $1 + a \equiv 0$ or $1 - a \equiv 0 \pmod{n}$, i.e. $a \equiv \pm 1 \pmod{n}$. For (ii)⇒(iii), if $Mu = (1, 0)$ and $Mv = (a, n)$, then $(1, 0) \equiv \pm(a, n) \pmod{n}$ is equivalent to $u \equiv \pm v \pmod{n}$ by unimodularity of M. The reverse implication is identical. $\square$

Remark 5.4 (Quick test and parametrization)**.** *Practically: given primitive* u, v *with* $n = \det(u, v) \geq 2$*, compute* $u \pm v$ *modulo* n*. If one of* $u \pm v$ *vanishes mod* n*, we are in the regime of Proposition 5.1 with the other thread degree* $= \gcd(n, 2)$*. Equivalently, every such pair is* $SL_2(\mathbb{Z})$*-equivalent to* $\big((1, 0), (\pm 1, n)\big)$*; a convenient parametrization is* $v = \pm u + nw$*, where* w *is any primitive vector with* $\det(u, w) = 1$.

Lemma 5.5 (Chebyshev cascade identity)**.** *Let* $\mu = (0, 1)$ *and put* $\varepsilon = \operatorname{sgn}(n)$*,* $x := t^2 + t^{-2}$*. Define*

$$G_n \;:=\; \sum_{j=0}^{\lfloor (n-2)/2 \rfloor} t^{\varepsilon(n-2-2j)} \Big(T_{n-2-2j}(\mu) - \delta_{n-2-2j,0} \Big)\, S_j(x).$$

Then

$$\mu_T \cdot G_n \;=\; t^{\varepsilon}\, G_{n-1} \;+\; t^{-\varepsilon}\, G_{n+1} \;-\; t^{-\varepsilon n}\,. \tag{20}$$

Proof. Write $k = n - 2 - 2j$ (so $k \geq 0$ and $k \equiv n \bmod 2$). Using

$$\mu_T T_k(\mu) = T_{k+1}(\mu) + T_{k-1}(\mu) \quad (k \geq 1), \qquad S_{j+1}(x) = x S_j(x) - S_{j-1}(x), \quad S_0 = 1,\ S_1 = x,$$

we have, termwise,

$$\mu_T \big(t^{\varepsilon k}\, [T_k(\mu) - \delta_{k,0}]\, S_j(x) \big) = t^{\varepsilon k} \big(T_{k+1} + T_{k-1} - \delta_{k,0} \big) S_j(x).$$

Reindex the j-sum: the T_{k+1} part is the jth summand of $t^{\varepsilon} G_{n-1}$; the T_{k-1} part is the $(j+1)$st summand of $t^{-\varepsilon} G_{n+1}$. The only residue is when $k = 0$, where $T_0(\mu) = 2$ and we have subtracted 1; this yields the unit $1 = T_0'$ and produces the final $-t^{-\varepsilon n}$ in (20). Summing over j gives the claim. $\square$

Proof of Proposition 5.1. We use the Frohman-Gelca splitting in the Chebyshev-T basis:

$$(\alpha)_T (\beta)_T = t^{|n|} (C_+)_T + t^{-|n|} (C_-)_T \;+\; \eta\, D(\alpha, \beta), \tag{21}$$

where D is the discrepancy (see [WWo, Def. 2.5/Eq. (1)]).

Normalization. By Corollary 5.3, one of $C_\pm$ equals n times a primitive direction. Replacing β by $-\beta$ if needed, assume $C_* = C_+ = n\mu_*$. By the mapping-class action (e.g. [WWo, Lem. 3.2]), there exists $\phi \in SL_2(\mathbb{Z})$ with $\phi_*(\mu_*) = (0,1) =: \mu$; the action preserves (21) and the discrepancy. Hence it suffices to prove the claim for $(\tilde{\alpha}, \tilde{\beta}) := \phi_*(\alpha, \beta)$ with $\tilde{C}_+ = n\mu$ and $d_- := \gcd(\tilde{C}_-) \in \{1, 2\}$, then undo ϕ_*.

Specialized recurrence. Let $F_n := D(\tilde{\alpha}, \tilde{\beta})$ and $B := D(\mu; \tilde{C}_-)$. Apply the Recursion Rule ([WWo, Thm. 3.1]) with $u = \mu$. Since $D(\mu; \tilde{C}_+) = 0$ for det $= 0$ ([WWo, Lem. 2.6]), we obtain

$$\mu_T \cdot F_n \;=\; t^{\varepsilon} F_{n-1} \;+\; t^{-\varepsilon} F_{n+1} \;-\; t^{-\varepsilon n}\, B. \tag{22}$$

Inhomogeneous term. Because $d_- \in \{1, 2\}$, the small closed forms ([WWo, Prop. 4.1-4.3]) give $B = T'_0 = 1$ in the T'-normalization. With our $T_0 = 2$, we will compensate by the Kronecker subtraction in G_n below.

Identification of the discrepancy. Define G_n as in Lemma 5.5. By that lemma, G_n satisfies (22) with $B = 1$. For $|n| \le 1$ the discrepancy vanishes ([WWo, Lem. 2.6-2.8]); for $n = 2$ our subtraction at T_0 makes $G_2 = 0$. Thus F_n and G_n agree on bases, and by the algorithm's uniqueness/termination ([WWo, Prop. 5.1, Rem. 5.2, Prop. 5.4]) we conclude $F_n \equiv G_n$.

Insert $D = G_n$ into (21) for $(\tilde{\alpha}, \tilde{\beta})$ to obtain

$$(\tilde{\alpha})_T (\tilde{\beta})_T = t^{\varepsilon n}\, T_n(\mu) \;+\; t^{-\varepsilon n}\, T_{d_-}(\tilde{C}_-/d_-) \;+\; \eta\, G_n.$$

Undoing ϕ_* yields the displayed formula with $C_*/n = \mu_*$, i.e. with
$\epsilon = \eta G_n$ expressed in the direction of the maximally threaded summand. □

Remark 5.6 (Coefficient dictionary)**.** *For $x = t^2 + t^{-2}$,*

$$S_j(x) = \frac{t^{2(j+1)} - t^{-2(j+1)}}{t^2 - t^{-2}},$$

hence each summand of $\epsilon(\alpha, \beta)$ carries coefficient $t^{\varepsilon(n-2-2j)} S_j(x)$, matching the rational coefficients that appear in the fast algorithm simplifications in [WWo, § 4].

See Section C for full computations and the comparison with the Wang-Wong algorithm.

A Detailed Calculations

Calculation for n=3

The product is $P_3 = (3,6)_T \cdot (1,0)_T$. We use the recurrence $P_3 = (1,2)_T \cdot P_2 - P_1$.

$$P_3 = (1,2)_T \cdot [t^{-4}(3,4)_T + t^4(1,4)_T + (1,2)_T \eta] - P_1 \tag{23}$$

The intermediate products $|det| = 2$ between simple curves, generating new η terms.

$$P_3 = t^{-4}[t^{-2}(4,6)_T + t^2(2,2)_T + \eta] + t^4[t^2(2,6)_T + t^{-2}(0,2)_T + \eta] \tag{24}$$
$$+ (1,2)_T^2\eta - P_1 \tag{25}$$
$$= t^{-6}(4,6)_T + t^{-2}(2,2)_T + t^{-4}\eta + t^6(2,6)_T + t^2(0,2)_T + t^4\eta \tag{26}$$
$$+ ((2,4)_T + 2)\eta - P_1 \tag{27}$$

We substitute P_1 back in. The main terms $t^{-2}(2,2)_T + t^2(0,2)_T$ cancel with the main terms of $-P_1$, and one η cancels.

$$P_3 = (3,6)_T \cdot (1,0)_T = t^{-6}(4,6)_T + t^6(2,6)_T \tag{28}$$
$$+ [t^4 + t^{-4} + 1 + (2,4)_T]\eta \tag{29}$$

Calculation for n=4

The product is $P_4 = (4,8)_T \cdot (1,0)_T$. We use the recurrence $P_4 = (1,2)_T \cdot P_3 - P_2$.

$$P_4 = (1,2)_T \cdot [t^{-6}(4,6)_T + t^6(2,6)_T + (t^4 + t^{-4} + 1 + (2,4))\eta] - P_2 \tag{30}$$

The intermediate products involve composite curves, so no new η terms are generated.

$$P_4 = t^{-6}[t^{-2}(5,8)_T + t^2(3,4)] + t^6[t^2(3,8)_T + t^{-2}(1,4)_T] \tag{31}$$
$$+ (1,2)_T(t^4 + t^{-4} + 1)\eta + (1,2)_T \cdot (2,4)_T\eta - P_2 \tag{32}$$
$$= t^{-8}(5,8)_T + t^{-4}(3,4)_T + t^8(3,8) + t^4(1,4)_T \tag{33}$$
$$+ (t^4 + t^{-4} + 1)(1,2)_T\eta + ((3,6)_T + (1,2)_T)\eta - P_2 \tag{34}$$

We substitute P_2 back in. The main terms $t^{-4}(3,4)_T$ and $t^4(1,4)_T$ cancel, and one $(1,2)\eta$ cancels.

$$P_4 = (4,8)_T \cdot (1,0)_T = t^{-8}(5,8)_T + t^8(3,8)_T \tag{35}$$
$$+ [(t^4 + t^{-4} + 1)(1,2)_T + (3,6)_T]\eta \tag{36}$$

Calculation for n=5

The product is $P_5 = (5,10)_T \cdot (1,0)_T$. We use the recurrence $P_5 = (1,2)_T \cdot P_4 - P_3$. We start by expanding the term $(1,2)_T \cdot P_4$:

$$(1,2)_T \cdot P_4 = (1,2)_T \cdot \left[t^{-8}(5,8)_T + t^8(3,8)_T + \epsilon_4\right]$$

where $\epsilon_4 = [(t^4 + t^{-4} + 1)(1,2)_T + (3,6)_T]\eta$.

The intermediate products $(1,2)_T \cdot (5,8)_T$ and $(1,2)_T \cdot (3,8)_T$ are between simple curves with $|det| = \pm 2$, and thus they generate new η terms:

$$\begin{aligned}(1,2)_T \cdot P_4 &= t^{-8}\left[t^{-2}(6,10)_T + t^2(4,6)_T + \eta\right] + t^8\left[t^2(4,10)_T + t^{-2}(2,6)_T + \eta\right] \\ &\quad + (1,2)_T \cdot \epsilon_4 \\ &= t^{-10}(6,10)_T + t^{-6}(4,6)_T + t^{-8}\eta + t^{10}(4,10)_T + t^6(2,6)_T + t^8\eta \\ &\quad + (1,2)_T \cdot \left[(t^4 + t^{-4} + 1)(1,2)_T + (3,6)_T\right]\eta\end{aligned}$$

We expand the coefficient of η:

$$\begin{aligned}(1,2)_T \cdot \epsilon_4 &= \left[(t^4 + t^{-4} + 1)(1,2)_T^2 + (1,2)_T \cdot (3,6)_T\right]\eta \\ &= \left[(t^4 + t^{-4} + 1)((2,4)_T + 2) + ((4,8)_T + (2,4)_T)\right]\eta \\ &= \left[(t^4 + t^{-4} + 2)(2,4)_T + 2(t^4 + t^{-4} + 1) + (4,8)_T\right]\eta\end{aligned}$$

Now we assemble the full expression for $P_5 = (1,2)_T \cdot P_4 - P_3$ and substitute the known formulas for P_4 and P_3. The main terms $t^{-6}(4,6)_T$ and $t^6(2,6)_T$ from the expansion cancel perfectly with the main terms from $-P_3$. The remaining terms form the new correction term ϵ_5:

$$\begin{aligned}\epsilon_5 &= (t^8 + t^{-8})\eta + (1,2)_T\epsilon_4 - \epsilon_3 \\ &= (t^8 + t^{-8})\eta + \left[(t^4 + t^{-4} + 2)(2,4)_T + 2(t^4 + t^{-4} + 1) + (4,8)_T\right]\eta \\ &\quad - \left[t^4 + t^{-4} + 1 + (2,4)_T\right]\eta\end{aligned}$$

Collecting the coefficients of the skeins inside the η term:

- Scalar part: $(t^8 + t^{-8}) + 2(t^4 + t^{-4} + 1) - (t^4 + t^{-4} + 1) = t^8 + t^{-8} + t^4 + t^{-4} + 1$
- $(2,4)_T$ part: $(t^4 + t^{-4} + 2)(2,4)_T - (2,4)_T = (t^4 + t^{-4} + 1)(2,4)_T$
- $(4,8)_T$ part: $(4,8)_T$

This gives the final expression for the product:

$$P_5 = (5,10)_T \cdot (1,0)_T = t^{-10}(6,10)_T + t^{10}(4,10)_T + \epsilon_5$$

where

$$\epsilon_5 = \eta\left[(t^8 + t^{-8} + t^4 + t^{-4} + 1) + (t^4 + t^{-4} + 1)(2,4)_T + (4,8)_T\right]$$

B Normalizations and comparison with Wang–Wong

Remark B.1 (Normalization: T_0 vs. T_0'). *After developing our formulas, we compared them with the Wang-Wong fast algorithm [WWo, p.9] and noticed a notational normalization that is worth recording. We package the terminal peel contribution using*

$$(T_0 - \delta_{0,0}) \qquad \textit{with } T_0 = 2,$$

so that the last term in the cascade contributes $S_j(t^2 + t^{-2}) \cdot 1$. Wang-Wong adopt the commonly used Chebyshev variation $T'_0 = 1$ and $T'_k = T_k$ for $k \geq 1$. Equivalently,

$$S_j(t^2 + t^{-2})\,(T_0 - \delta_{0,0}) \;=\; S_j(t^2 + t^{-2})\,T'_0$$

and their printed $T(0,0)$ should be read as the unit 1. This dictionary eliminates the apparent factor-of-two ambiguity in the terminal η-term and makes our peel-cascade coefficients coincide term-by-term with the Wang-Wong output.

Remark B.2 (Agreement with Wang-Wong [WWo]). *Write $x = t^2 + t^{-2}$. The Chebyshev identity*

$$S_j(x) = \frac{t^{2(j+1)} - t^{-2(j+1)}}{t^2 - t^{-2}}$$

shows that each peel-cascade coefficient in Proposition 5.1

$$t^{\varepsilon\,(n-2-2j)}\,S_j(x)$$

coincides with the coefficient produced by the Wang-Wong fast algorithm for the same term, namely

$$t^{\varepsilon\,(n-2-2j)}\;\frac{t^{2(j+1)} - t^{-2(j+1)}}{t^2 - t^{-2}}.$$

Hence the maximal-thread expansion matches the Wang-Wong output term-by-term. See Appendix C for a worked example when $(\alpha,\beta) = (2,1), (3,4)$ and several others.

C Maximal Thread Results and Comparison

Example: $(4,3)\cdot(0,1)$ in the maximal-thread regime

Let $\alpha = (4,3)$ and $\beta = (0,1)$. Then

$$n = \det\begin{pmatrix} 4 & 0 \\ 3 & 1 \end{pmatrix} = 4, \tag{37}$$

$$C_+ = \alpha + \beta = (4,4), \quad d_+ = \gcd(4,4) = 4 = n, \quad \mu_+ = (1,1), \tag{38}$$

$$C_- = \alpha - \beta = (4,2), \quad d_- = \gcd(4,2) = 2. \tag{39}$$

Thus the "+" Frohman–Gelca summand is maximally threaded ($d_+ = n$), while the "−" summand has thread degree 2. The closed-torus part is

$$t^4\,T_4(1,1) + t^{-4}\,T_2(2,1) = t^4\,T(4,4) + t^{-4}\,T(4,2).$$

By Proposition 5.1, the once-punctured torus product adds a peel-cascade from the threaded summand. With $\lfloor (n-2)/2 \rfloor = 1$ we have two levels $j = 0,1$ and $\varepsilon = +1$:

$$\eta \sum_{j=0}^{1} t^{\,4-2-2j}\Big(T_{4-2-2j}(1,1) - \delta_{\,4-2-2j,0}\Big)\,S_j\big(t^2 + t^{-2}\big).$$

Using $S_0 = 1$ and $S_1(x) = x$ with $x = t^2 + t^{-2}$, this equals

$$\eta \left[t^2 \, T_2(1,1) + (T_0 - 1) \, S_1(x) \right] .$$

Altogether,

$$\begin{aligned} (4,3) \cdot (0,1) = {} & t^{-4} \, T(4,2) + t^4 \, T(4,4) \\ & + \eta \left[t^2 \, T_2(1,1) + (T_0 - 1) \, S_1(x) \right] , \quad x = t^2 + t^{-2}. \end{aligned} \tag{40}$$

Since $T_0 = 2$, the factor $(T_0 - 1)$ equals the unit, so the terminal term is simply $S_1(x)$.

Wang–Wong's Fast Algorithm output and simplification

The Wang–Wong fast algorithm for the same input returns

$$\begin{aligned} & t^{-4} \, T(4,2) + t^4 \, T(4,4) \\ & + \eta \left[\frac{-t^4 + t^{-4}}{-t^2 + t^{-2}} \, T'(0,0) + t^2 \, T(2,2) \quad \right] . \end{aligned} \tag{41}$$

Using the paper's T'-normalization, $T'(0,0) = 1$ (the unit), and

$$\frac{-t^4 + t^{-4}}{-t^2 + t^{-2}} = \frac{t^4 - t^{-4}}{t^2 - t^{-2}} = S_1(x), \qquad T(2,2) = T_2(1,1),$$

this simplifies to

$$t^{-4} \, T(4,2) + t^4 \, T(4,4) + \eta \left[t^2 \, T_2(1,1) + S_1(x) \right] ,$$

which matches the maximal-thread peel-cascade expression above.

Reconciliation with Cho's $|\det| = 4$ calculation

Cho writes

$$(4,3) \cdot (0,1) = t^{-4} \, T(4,2) + t^4 \, T(4,4) + \eta \big[t^2 \, (2,2)_T + t^{-2} \, (0,0)_S \big] .$$

Changing basis and using $S_1(x) = \dfrac{t^4 - t^{-4}}{t^2 - t^{-2}} = t^2 + t^{-2}$ and $T_2(1,1) = (1,1)^2 - 2$, we obtain

$$\eta \left[t^2 \, T_2(1,1) + S_1(x) \cdot 1 \right] ,$$

which is exactly the same η-term as in the maximal-thread and Wang–Wong expressions (recall the unit convention $T'(0,0) = 1$). Hence all three presentations agree term-by-term.

Example: $(2,1)\cdot(3,4)$ in the maximal-thread regime

Let $\alpha=(2,1)$ and $\beta=(3,4)$. Then

$$n=\det\begin{pmatrix}2&3\\1&4\end{pmatrix}=5,\qquad C_+=\alpha+\beta=(5,5),\ d_+=\gcd(5,5)=5,\ \mu_+=(1,1),$$

$$C_-=\alpha-\beta=(-1,-3),\ d_-=\gcd(1,3)=1.$$

Thus the "+" Frohman-Gelca summand is maximally threaded ($d_+=n$), while the "−" summand is simple. The closed-torus part is

$$t^5\,T_5(1,1)\ +\ t^{-5}\,T_1(-1,-3)\ =\ t^5\,T_5(1,1)\ +\ t^{-5}\,(1,3),$$

since T_1 is the identity and $(-1,-3)$ represents the same simple curve as $(1,3)$.

By Proposition 5.1, the once-punctured torus product adds a peel-cascade from the threaded summand. With $\lfloor(n-2)/2\rfloor=1$ we have two levels $j=0,1$:

$$\eta\sum_{j=0}^{1}t^{\varepsilon\,(n-2-2j)}\Big(T_{n-2-2j}(1,1)-\delta_{n-2-2j,0}\Big)\,S_j(t^2+t^{-2}),\quad \varepsilon=+1.$$

Using $S_0=1$ and $S_1(x)=x$, this equals

$$\eta\Big[t^3\,T_3(1,1)\ +\ t^1\,T_1(1,1)\,(t^2+t^{-2})\Big]=\eta\Big[t^3\,T_3(1,1)\ +\ (t^3+t^{-1})\,(1,1)\Big].$$

Altogether,

$$(2,1)\cdot(3,4)=t^5\,T_5(1,1)\ +\ t^{-5}\,(1,3)\ +\ \eta\Big[t^3\,T_3(1,1)\ +\ (t^3+t^{-1})\,(1,1)\Big].$$

Wang–Wong's Fast Algorithm output and simplification

The Wang–Wong fast algorithm for the same input returns

$$t^5\,T(5,5)\ +\ t^{-5}\,T(-1,-3)\ +\ \eta\Big[t^3\,T(3,3)\ +\ t\,\frac{t^4-t^{-4}}{t^2-t^{-2}}\,T(1,1)\Big].$$

Identifying $T(5,5)=T_5(1,1)$ and $T(3,3)=T_3(1,1)$, it remains to simplify the rational coefficient:

$$t\,\frac{t^4-t^{-4}}{t^2-t^{-2}}=t\,\frac{(t^2)^2-(t^2)^{-2}}{t^2-t^{-2}}=t\,(t^2+t^{-2})=t^3+t^{-1}.$$

This matches the maximal-thread peel-cascade term above exactly.

Coefficient dictionary: S_j vs. Wang–Wong

Throughout the peel cascade, the Chebyshev–S factor admits the standard closed form

$$S_j\big(t^2+t^{-2}\big)=\frac{t^{2(j+1)}-t^{-2(j+1)}}{t^2-t^{-2}},$$

so a generic peel-cascade term has coefficient

$$t^{\varepsilon\,(n-2-2j)}\,S_j\big(t^2+t^{-2}\big)\ =\ t^{\varepsilon\,(n-2-2j)}\,\frac{t^{2(j+1)}-t^{-2(j+1)}}{t^2-t^{-2}},$$

which is precisely the coefficient appearing in the Wang–Wong formula.

Example: $(11,67)\cdot(3,19)$ in the maximal-thread regime

Let $\alpha=(11,67)$ and $\beta=(3,19)$. Then

$$n=\det\begin{pmatrix}11 & 3\\ 67 & 19\end{pmatrix}=11\cdot 19-67\cdot 3=8, \tag{42}$$

$$C_+=\alpha+\beta=(14,86),\quad d_+=\gcd(14,86)=2,\quad \mu_+=(7,43), \tag{43}$$

$$C_-=\alpha-\beta=(8,48),\quad d_-=\gcd(8,48)=8=n,\quad \mu_-=(1,6). \tag{44}$$

Thus the "$-$" Frohman–Gelca summand is maximally threaded ($d_-=n$), while the "+" summand has thread degree 2. The closed-torus part is

$$t^8\,T_2(7,43)+t^{-8}\,T_8(1,6).$$

By Proposition 5.1, the once-punctured torus product adds a peel-cascade from the maximally threaded summand. With $\lfloor (n-2)/2\rfloor=3$ we have four levels $j=0,1,2,3$:

$$\eta\sum_{j=0}^{3} t^{\varepsilon\,(n-2-2j)}\Big(T_{n-2-2j}(1,6)-\delta_{n-2-2j,0}\Big)\,S_j(t^2+t^{-2}),$$

where $\varepsilon=-1$.

Using $S_0=1$, $S_1(x)=x$, $S_2(x)=x^2-1$, $S_3(x)=x^3-2x$ with $x=t^2+t^{-2}$, this equals

$$\eta\Big[t^{-6}\,T_6(1,6)+t^{-4}\,S_1(x)\,T_4(1,6)+t^{-2}\,S_2(x)\,T_2(1,6)+S_3(x)\,\big(T_0-1\big)\Big],$$

where $T_0=2$. Altogether,

$$(11,67)\cdot(3,19)=t^8\,T_2(7,43)+t^{-8}\,T_8(1,6) \tag{45}$$

$$+\,\eta\begin{bmatrix}t^{-6}\,T_6(1,6)+t^{-4}\,S_1(x)\,T_4(1,6)\\ +\,t^{-2}\,S_2(x)\,T_2(1,6)+S_3(x)\,(T_0-1)\end{bmatrix}, \tag{46}$$

where $x=t^2+t^{-2}$.

Wang–Wong's Fast Algorithm output and simplification

The Wang–Wong fast algorithm for the same input returns

$$\begin{aligned}&t^8\,T(14,86)+t^{-8}\,T(8,48)\\ &+\eta\begin{bmatrix}\dfrac{t^8-t^{-8}}{t^2-t^{-2}}\,T'(0,0)+\dfrac{t^6-t^{-6}}{t^2\big(t^2-t^{-2}\big)}\,T(2,12)\\ +\,\dfrac{t^4-t^{-4}}{t^4\big(t^2-t^{-2}\big)}\,T(4,24)+t^{-6}\,T(6,36)\end{bmatrix}.\end{aligned} \tag{47}$$

Identifying $T(14,86)=T_2(7,43)$ and $T(8,48)=T_8(1,6)$, and using

$$\frac{t^{2m}-t^{-2m}}{t^2-t^{-2}}=S_{m-1}(t^2+t^{-2}),$$

the rational coefficients simplify term-by-term to the peel-cascade factors:

$$\frac{t^4 - t^{-4}}{t^4(t^2 - t^{-2})} = t^{-4} S_1(x), \tag{48}$$

$$\frac{t^6 - t^{-6}}{t^2(t^2 - t^{-2})} = t^{-2} S_2(x), \tag{49}$$

$$\frac{t^8 - t^{-8}}{t^2 - t^{-2}} = S_3(x). \tag{50}$$

Moreover $T(6,36) = T_6(1,6)$, $T(4,24) = T_4(1,6)$, $T(2,12) = T_2(1,6)$, and $T(0,0) = T_0 = 2$. The only notational difference is our normalization $(T_0 - 1)$ (Kronecker-delta convention) versus $T'_0 = 1$ (and $T'_k = T_k$) in Wang–Wong; this adjusts the constant multiple of $S_3(x)$ inside the η-term and is purely conventional. Hence the two expressions agree exactly.

References

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