



数学分析期末试题(A)参考解答 (2007.1)

一. 1. $\lim_{x \rightarrow \infty} \left(\frac{x+a}{x-a} \right)^x = \lim_{x \rightarrow \infty} \left(1 + \frac{2a}{x-a} \right)^{\frac{x-a}{2a} \cdot \frac{2ax}{x-a}} \dots\dots\dots(2 \text{ 分})$

$$= e^{\lim_{x \rightarrow \infty} \frac{2ax}{x-a}} = e^{2a} = 9, \dots\dots\dots(5 \text{ 分})$$

$$2a = \ln 9, \quad a = \ln 3. \dots\dots\dots(6 \text{ 分})$$

2. $t = \frac{\pi}{3}$ 时, $x = -\ln 2, \quad y = \frac{\sqrt{3}}{2} - \frac{\pi}{6}, \dots\dots\dots(1 \text{ 分})$

$$\frac{dy}{dx} = \frac{\cos t - \cos t + t \sin t}{-\frac{\sin t}{\cos t}} = -t \cos t, \dots\dots\dots(4 \text{ 分})$$

$$\left. \frac{dy}{dx} \right|_{t=\frac{\pi}{3}} = -\frac{\pi}{6}, \dots\dots\dots(5 \text{ 分})$$

切线方程 $y - \frac{\sqrt{3}}{2} + \frac{\pi}{6} = -\frac{\pi}{6}(x + \ln 2). \dots\dots\dots(6 \text{ 分})$

3. $\lim_{x \rightarrow 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) = \lim_{x \rightarrow 1} \frac{x \ln x - x + 1}{(x-1) \ln x} \dots\dots\dots(2 \text{ 分})$

$$= \lim_{x \rightarrow 1} \frac{\ln x}{\ln x + \frac{x-1}{x}} = \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2}. \dots\dots\dots(6 \text{ 分})$$

4. 解 1 令 $t = \arcsin x$, 原式 $= 2 \int_0^{\frac{\pi}{6}} t \sin t dt \dots\dots\dots(2 \text{ 分})$

$$= -2 \int_0^{\frac{\pi}{6}} t d \cos t = -2(t \cos t \Big|_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \cos t dt) \dots\dots\dots(5 \text{ 分})$$

$$= -2 \left(\frac{\pi}{6} \cdot \frac{\sqrt{3}}{2} - \sin t \Big|_0^{\frac{\pi}{6}} \right) = 1 - \frac{\sqrt{3}}{6} \pi \dots\dots\dots(6 \text{ 分})$$

解 2 原式 $= 2 \int_0^{\frac{1}{2}} \frac{x \arcsin x}{\sqrt{1-x^2}} dx = -2 \int_0^{\frac{1}{2}} \arcsin x d \sqrt{1-x^2} \dots\dots\dots(2 \text{ 分})$

$$= -2 \left(\arcsin x \cdot \sqrt{1-x^2} \Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} dx \right) \dots\dots\dots(5 \text{ 分})$$

$$= -2 \left(\frac{\pi}{6} \cdot \frac{\sqrt{3}}{2} - x \Big|_0^{\frac{1}{2}} \right) = 1 - \frac{\sqrt{3}}{6} \pi. \dots\dots\dots(6 \text{ 分})$$

二. 1. $y' = 2xe^y + x^2e^y y'$,(2 分)

令 $y' = 0$, 得 $x = 0$,(3 分)

代入已知方程得 $y = 1$,(4 分)

$y'' = 2e^y + 2xe^y y' + 2xe^y y' + x^2e^y (y')^2 + x^2e^y y''$,(2 分)

$\because y''|_{x=0} = 2e > 0$, 故 $y|_{x=0} = 1$ 是极小值.(7 分)

2. $\int x \arctan x dx = \frac{1}{2} \int \arctan x d(x^2)$ (1 分)

$= \frac{1}{2} (x^2 \arctan x - \int \frac{x^2}{1+x^2} dx)$ (4 分)

$= \frac{1}{2} (x^2 \arctan x - \int (1 - \frac{1}{1+x^2}) dx)$ (6 分)

$= \frac{1}{2} (x^2 \arctan x - x + \arctan x) + C$(7 分)

3. $y_2 - y_1 = 3e^x - x$ 与 $y_3 - y_1 = -e^x + x$ (2 分)

通解为 $y = C_1 e^x + C_2 x - x^2$(7 分)

4. 由于 $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{e^{x^2} - a}{x^2} \exists$, 得 $a = 1$,(1 分)

故 $\lim_{x \rightarrow 0} \frac{e^{x^2} - a}{x^2} = \lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{x^2} = 1$,

所以 $b = 1$,(2 分)

当 $x \neq 0$, $f'(x) = (\frac{e^{x^2} - 1}{x^2})'$

$= \frac{2xe^{x^2} x^2 - (e^{x^2} - 1) \cdot 2x}{x^4} = \frac{2x^2 e^{x^2} - 2e^{x^2} + 2}{x^3}$,(5 分)

$f'(0) = \lim_{x \rightarrow 0} \frac{\frac{e^{x^2} - 1}{x^2} - 1}{x} = \lim_{x \rightarrow 0} \frac{e^{x^2} - 1 - x^2}{x^3}$

$= \lim_{x \rightarrow 0} \frac{2xe^{x^2} - 2x}{3x^2} = \lim_{x \rightarrow 0} \frac{2e^{x^2} - 2}{3x} = 0$(7 分)

三. $y' = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$,(2 分)

当 $x \in (0,3)$, 令 $y' = 0$, 得 $x = 2$,(3 分)

函数在 $(0,2)$ 与 $(2,3)$ 内单调,(4 分)

又 $y(0) = 10 > 0$, $y(2) = -22 < 0$, $y(3) = 37 > 0$ (6 分)

函数在 $(0,2)$ 与 $(2,3)$ 内各有一个零点, 故在 $(0,3)$ 内有两个零点.....(7 分)

四. $f''(x) = g'(x) = 2e^x - f(x)$,

$f''(x) + f(x) = 2e^x$,(1 分)

$f(0) = 0$, $f'(0) = g(0) = 2$,(2 分)

$r^2 + 1 = 0$, $r = \pm i$,

$\bar{f}(x) = C_1 \cos x + C_2 \sin x$,(4 分)

设 $f^*(x) = Ae^x$, 代入方程得 $A = 1$, $f^*(x) = e^x$,.....(6 分)

通解 $f(x) = C_1 \cos x + C_2 \sin x + e^x$,(7 分)

由初始条件得 $C_1 = -1$, $C_2 = 1$,

$f(x) = -\cos x + \sin x + e^x$(8 分)

五. 令 $F(x) = \int_0^{\sin^2 x} \arcsin \sqrt{t} dt + \int_0^{\cos^2 x} \arccos \sqrt{t} dt$,

$F'(x) = \arcsin \sin x \cdot 2 \sin x \cos x + \arccos \cos x \cdot (-2) \cos x \sin x$

$= 2x \sin x \cos x - 2x \cos x \sin x = 0$,(3 分)

故 $F(x) = C$,(4 分)

又 $F(0) = \int_0^1 \arccos \sqrt{t} dt$ (令 $\arccos \sqrt{t} = u$)(5 分)

$= -\int_0^{\frac{\pi}{2}} u d \cos^2 u$

$= -u \cos^2 u \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos^2 u du = \frac{\pi}{4}$,

$\therefore C = \frac{\pi}{4}$(7 分)

六. $\int_1^t \pi f^2(x) dx = \pi[t^2 f^2(t) - f^2(1)], \dots\dots\dots(3 \text{ 分})$

对 t 求导得 $f^2(t) = 2tf^2(t) + 2t^2 f(t)f'(t), \dots\dots\dots(5 \text{ 分})$

$$f'(t) = \frac{1-2t}{2t^2} f(t), \quad \frac{df(t)}{f(t)} = \left(\frac{1}{2t^2} - \frac{1}{t}\right) dt, \dots\dots\dots(8 \text{ 分})$$

$$\ln|f(t)| = -\frac{1}{2t} - \ln t + C_1,$$

$$f(x) = Ce^{-\frac{1}{2x} - \ln x} = \frac{C}{x} e^{-\frac{1}{2x}}. \dots\dots\dots(10 \text{ 分})$$

七. 设 t 时刻含盐量为 $m(t)$ 克, 则

$$dm = 4 \times 5 dt - \frac{m}{100} \cdot 5 dt, \dots\dots\dots(3 \text{ 分})$$

$$\begin{cases} \frac{dm}{dt} + \frac{m}{20} = 20, \\ m(0) = 0 \end{cases}, \dots\dots\dots(4 \text{ 分})$$

通解为 $m(t) = Ce^{-\frac{t}{20}} + 400, \dots\dots\dots(7 \text{ 分})$

由初始条件得 $C = -400,$

$$\therefore m(t) = 400(1 - e^{-\frac{t}{20}}). \dots\dots\dots(8 \text{ 分})$$

八. (1) 由于 $\lim_{x \rightarrow 0^+} \frac{\ln(1 + \frac{f(x)}{x})}{\sin x} = 3$, 所以 $\lim_{x \rightarrow 0^+} \frac{f(x)}{x} = 0, \dots\dots\dots(1 \text{ 分})$

故 $f(0) = 0, \quad f'(0) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x} = 0; \dots\dots\dots(3 \text{ 分})$

(2) 令 $F(x) = f'(x)e^x, \dots\dots\dots(4 \text{ 分})$

根据积分中值定理, $\exists c \in [1, 2]$, 使

$$f(c) = \int_1^2 f(x) dx = 0, \dots\dots\dots(5 \text{ 分})$$

由洛尔定理, $\exists c_1 \in (0, c)$, 使 $f'(c_1) = 0, \dots\dots\dots(6 \text{ 分})$

$$\therefore F(0) = F(c_1),$$

由洛尔定理, $\exists \xi \in (0, c_1) \subset (0, 2)$, 使 $F'(\xi) = 0,$

即 $f''(\xi)e^\xi + f'(\xi)e^\xi = 0,$

$$f'(\xi) + f''(\xi) = 0. \dots\dots\dots(8 \text{ 分})$$