The analysis of Runge Phenomenon

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Outline

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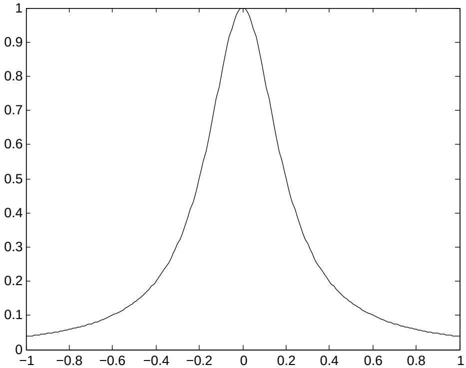
1. Introduction to Runge phenomenon
2. The analysis of Runge phenomenon
   - Theorem
   - The analysis
   - Nodes distributing analysis
   - Stability

Outline

Runge phenomenon

What is Runge phenomenon?
for example:

$$f(x) = \frac{1}{1 + 25x^2}$$



Runge phenomenon

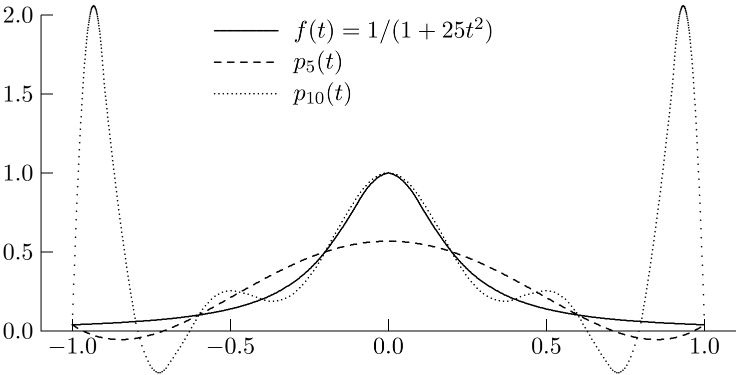


Figure: Interpolation of Runge's function at equally spaced nodes

Interpolation error analysis

Theorem

set f(x) in [a,b] include n+1 mutual different nodes:$x_0, x_1, x_2, \ldots, x_n$ *The interval*x_n *at* $[a, b]$ *has N order continuous derivative,and in (a,b) exists within n+1 order derivative. Then for any* $x \in [a, b]$*,we have*

$$R_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} W_{n+1}(x)$$

where

$$\xi \in (a, b) \quad W_{n+1}(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$$

The analysis of Runge phenomenon

- The property of the high order derivative(The high order derivative of the function must rise very fast)
  - If $\max\limits_{a \le x \le b} |f^{(n+1)}(x)| \le M_{n+1}$,we have

$$|R_n(x)| \le \frac{M_{n+1}}{(n+1)!} |W_{n+1}(x)|$$

  - If $\max\limits_{a \le x \le b} |f^{(n)}(x)| \le M$ for all n>0. Then

$$\max_{a \le x \le b} |R_n(x)| \le \frac{M}{(n+1)!}(b-a)^{n+1} \longrightarrow 0(\text{ as } n \to \infty)$$

The analysis of Runge phenomenon

For the function.

$$f(x) = \frac{1}{1 + 25x^2}$$

under the calculation of mathematics. we have

$$\max_{-1 \leq x \leq 1} f^{(11)}(x) = 1.77219 \times 10^{15}$$

The value of high order derivative increased rapidly as the order goes higher

The analysis of Runge phenomenon

So far, we know that the rapidly increased value of high order derivative has significant influence on runge phenomenon .
Carefully observed,we can find that runge phenomenon has another feature.
$\rightarrow$ The nodes are equally spaced.

Runge phenomenon

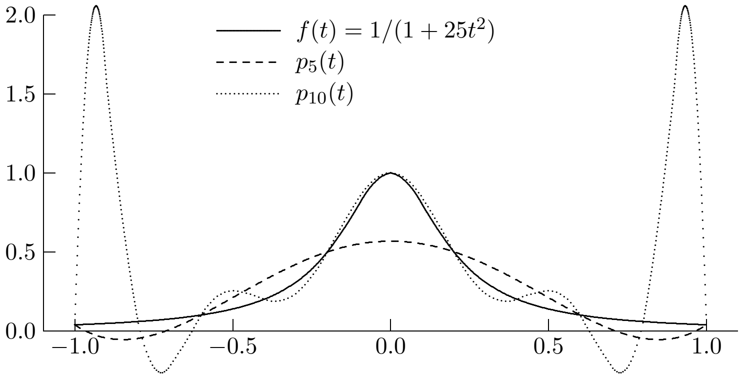


Figure: Interpolation of Runge's function at equally spaced nodes

The analysis of Runge phenomenon

Is that an important reason? What if we take the unequally spaced nodes?
There is an example of Chebyshev nodes

For the carefully selected nodes:

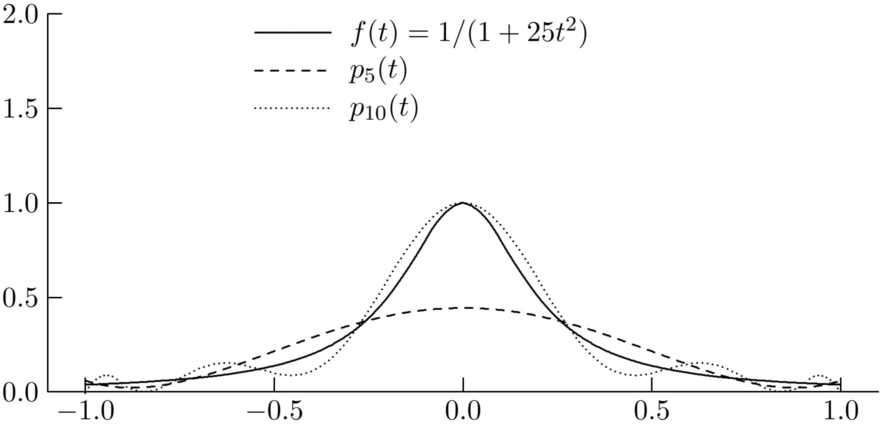


Figure: Interpolation of Runge's function at the Chebyshev points.

Nodes distributing analysis

We denote by $P_n g$,the polynomial of order n that agrees with a given g at the given n sides of the sequence $\{x_i\}$,we assume that the interpolation site sequence lies in some interval [a,b],and we measure the size of a continuous function f on [a,b] by

$$\|f\| = \max_{a \le x \le b} |f(x)|$$

Nodes distributing analysis

Further,we recall from the Lagrange Polynomials

$$l_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_j - x_j}$$

In terms of which

$$|(P_n g)(x)| = |\sum_i g(x_i) l_i| \leq \sum_i |g(x_i)||l_i(x)| \leq (\max_i (g(x_i)) \sum_i |l_i(x)|)$$

Nodes distributing analysis

We introduce the so-called lebesgue functiom

$$\lambda_n(x) = \sum_{i=1}^{n} |l_i(x)|$$

since $\max_i |g(x_i)| \leq \|g\|$,we obtain the estimate

$$\|P_n g\| \leq \|\lambda_n\| \|g\|$$

Actually,there exist functions g(other than zero function) for which

$$\|p_n g\| = \|\lambda_n\| \|g\|$$

Nodes distributing analysis

On the other hand, for uniformly spaced interpolation sites,we have

$$\|\lambda_n\| \sim \frac{2^n}{en \ln n}$$

(as first proved by A.N.Turetskii,1940)

Nodes distributing analysis

Fortunately,this situation can be be improved if we carefully select the nodes.
There is a proof for Chebyshev nodes

Chebyshev sites are good

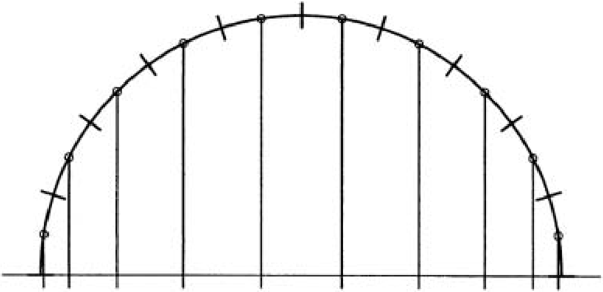


FIGURE. The Chebyshev sites for the interval $[a \,..\, b]$ are obtained by subdividing the semicircle over it into n equal arcs and then projecting the midpoint of each arc onto the interval.

Chebyshev sites are good

Computed by M.J.D.Powell, with λ_n^c the corresponding Lebesgue function ,we have

$$\|\lambda_n^c\| \sim \frac{2}{\Pi} \ln n$$

Stability

Let us consider a set of function values $\{\tilde{f}(x_i)\}$ which is a perturbation of the data $f(x_i)$ relative to the nodes x_i,within $i = 0, \ldots, n$, in an interval [a,b]. The perturbation may be due,for instance,to the effect of rounding errors,or may be cause by an error in the experimental measure of the data.
Denoting by $\Pi_n \tilde{f}$ the interpolating polynomial on the set of values $\tilde{f}(x_i)$,we have

$$\|\Pi_n f - \Pi_n \tilde{f}\|_\infty = \max_{a \leq x \leq b} \left| \sum_{j=0}^{n} (f(x_i) - \tilde{f}(x_j)) l_j(x) \right| \tag{1}$$

$$\leq \Lambda_n(X) \max_{i=0,\ldots,n} |f(x_i) - \tilde{f}(x_i)| \tag{2}$$

As previous noticed,$\Lambda_n(X)$ grow as n$\rightarrow\infty$ and in particular, in the case of Lagrange interpolation on equally spaced nodes, it can be proved that(see[Nat65])

$$\Lambda_n(X) \simeq \frac{2^{n+1}}{en\log n}$$

For the carefully selected nodes, it can be proved that

$$\Lambda_n(X) \sim \frac{2}{\Pi}\log(n+1), n = 0, 1 \ldots$$

The reason of Runge phenomenon

- The high order derivative increased so fast
- The nodes are equally spaced

The disadvantage of high order interpolation

The disadvantage of high order interpolation

- The Nature of Non uniform convergence

The disadvantage of high order interpolation

- The Nature of Non uniform convergence
- Computational Complexity

The disadvantage of high order interpolation

- The Nature of Non uniform convergence
- Computational Complexity
- Too much Swing and Oscillation
- For these reasons,in many cases,high order interpolation is not practical

How to avoid

- Spline interpolation

How to avoid

- Spline interpolation
- Unequally spaced interpolation