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# An automatic design procedure of IIR digital filters from an analog low-pass filter

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**Abstract**

IIR digital filters are generally designed from an analog low-pass filter by any combination of the bilinear transformation, frequency transformation for analog filters and that for IIR digital filters. But the direct application of these transformations yields a complicated formula of the target transfer function, which has to be reduced by hand computation into the form of a rational polynomial. This hand computation will be replaced by an automatic procedure if the relation between the coefficients of the transfer functions is formulated. This paper derives the explicit formulae which connect those coefficients by matrices. © 1997 Elsevier Science B.V.

**Zusammenfassung**

Rekursive digitale Filter werden gewöhnlich von analogen Tiefpassfiltern abgeleitet. Dabei benutzt man sowohl bilineare Transformationen als auch Frequenztransformationen für analoge und rekursive digitale Filter. Diese Transformationen führen aber auf eine komplizierte Darstellung der gewünschten Übertragungsfunktion, die von Hand in eine rationale Funktion umgeformt werden muß. Diese Umformung wird hier durch ein automatisches Verfahren ersetzt, falls die Verhältnisse zwischen den Koeffizienten der Übertragungsfunktion bekannt sind. Dieser Artikel leitet explizite Formeln her, die diese Koeffizienten durch Matrizen miteinander verbinden. © 1997 Elsevier Science B.V.

**Résumé**

Les filtres RII sont en général conçus à partir d'un filtre analogique passe-bas en combinant la transformée bilinéaire, la transformée en fréquence des filtres analogiques et celle des filtres RII numériques. Toutefois, l'application directe de ces transformations résulte en une formule compliquée pour la fonction de transfert recherchée, qui doit alors être réduite à la main à un polynome rationnel. Ce travail à la main peut être remplacé par une procédure automatique si la relation entre les coefficients des fonctions de transfert est formulée. Cet article dérive les formules explicites qui lient ces coefficients par des relations matricielles. © 1997 Elsevier Science B.V.

*Keywords:* IIR digital filters; Analog filters; Frequency transformation; Bilinear transformation

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## 1. Introduction

Frequency transformation derives filters of various types from a filter of low-pass type. The transformation formulae have been established for analog filters in the $s$-domain [4, p. 258], and for IIR digital filters in the $z$-domain [2; 4, p. 260]. Further, bilinear transformation [4, pp. 219–224] derives IIR digital filters from analog filters, and also enables the inverse derivation. The transformation formulae are well-known and widely used. But the direct application of these transformations yields a complicated formula of the target transfer function, which has to be reduced by hand computation into the form of a rational polynomial. The yielded formula becomes more complicated for a transfer function with a higher order that can be used in practice to realize minute characteristics.

To dissolve this complexity, a transfer matrix has been formulated [1] which transforms coefficients of a polynomial in the $s$-domain to those in the $z$-domain. This matrix can replace the hand computation of the bilinear transformation for polynomials by an automatic procedure. Similarly, the hand computation of those transformations will be replaced by an automatic procedure if the relation between the coefficients of the transfer functions is formulated. This paper aims at deriving the explicit formulae which connect those coefficients by matrices. Sections 2–4 describe formulae of the transfer matrices which represent the frequency transformation for analog filters, that for IIR digital filters, and the bilinear transformation for rational polynomials, respectively.

Generally, IIR digital filters are designed from an original analog low-pass filter by any combination of the bilinear transformation, frequency transformation for analog filters and that for IIR digital filters. The proposed matrices enable an automatic design of IIR digital filters from an analog low-pass filter. Section 5 investigates conditions to design IIR digital filters directly from an analog low-pass filter.

Section 6 shows some design examples of IIR digital filters which are designed directly from an analog low-pass filter. The effectiveness of the proposed matrices can be evaluated how the process to design IIR digital filters is simplified by the proposed automatic procedure. Finally, Section 7 will give some concluding remarks.

## 2. Matrices representation of the frequency transformation for analog filters

This section derives the explicit formulae of matrices which represent the frequency transformation for analog filters. In the following discussion, frequency means angular frequency, and $\left\langle {a \atop b} \right\rangle$ denotes

$$\left\langle {a \atop b} \right\rangle := \begin{cases} 0, & b>a \text{ or } b<0 \text{ or } \\ & b \text{ is not an integer,} \\ \dfrac{a!}{b!(a-b)!}, & \text{otherwise.} \end{cases}$$

Let the cut-off frequency of the original analog low-pass filter be $\Omega_0 = 1$, which is fixed throughout this investigation. The transfer function of this filter can be written as

$$H_1(s) := \frac{a_0 + a_1 s + \cdots + a_m s^m}{b_0 + b_1 s + \cdots + b_n s^n} \quad (m \leqslant n).$$

The transformation formulae [4, p. 258] of the frequency transformation for analog filters are shown in Table 1. The transfer functions of the derived analog low-pass and high-pass filters can be written as

$$H_2(s) = \frac{c_0 + c_1 s + \cdots + c_n s^n}{d_0 + d_1 s + \cdots + d_n s^n}.$$

Those of the derived digital band-pass and band-stop filters can be written as

$$\tilde{H}_2(s) = \frac{c_0 + c_1 s + \cdots + c_n s^n + \cdots + c_{2n} s^{2n}}{d_0 + d_1 s + \cdots + d_n s^n + \cdots + d_{2n} s^{2n}}.$$

Then, matrices which transform the transfer function of the original low-pass filter to transfer functions of the derived filters are formulated as follows.

### 2.1. Design of analog low-pass filter

The analog low-pass filter with the cut-off frequency $\Omega_1$ is designed from the original analog low-pass filter. From the transformation formula in Table 1, the relation of coefficients from $\{a_i\}_{i=0}^m$ and $\{b_i\}_{i=0}^n$ into $\{c_i\}_{i=0}^n$ and $\{d_i\}_{i=0}^n$ can be easily written as

$$c_i = \begin{cases} (\Omega_1)^{-i} a_i, & i = 0, 1, \ldots, m, \\ 0, & i = m+1, m+2, \ldots, n, \end{cases}$$

$$d_i = (\Omega_1)^{-i} b_i.$$

Table 1
Transformation formulae of the frequency transformation for analog filters

| Filter type | Transformation formulae |
| --- | --- |
| cut-off frequency | Parameters |
| Low-pass | $s = \dfrac{s}{\Omega_1}$ |
| $\Omega_1$ | |
| High-pass | $s = \dfrac{\Omega_1}{s}$ |
| $\Omega_1$ | |
| Band-pass | $s = \dfrac{s^2 + \Omega_a^2}{\Omega_b s}$ |
| $\Omega_1, \Omega_2$ $(\Omega_1 < \Omega_2)$ | $\Omega_a = \sqrt{\Omega_1 \Omega_2}$, $\Omega_b = \Omega_2 - \Omega_1$ |
| Band-stop | $s = \dfrac{\Omega_b s}{s^2 + \Omega_a^2}$ |
| $\Omega_1, \Omega_2$ $(\Omega_1 < \Omega_2)$ | $\Omega_a = \sqrt{\Omega_1 \Omega_2}$, $\Omega_b = \Omega_2 - \Omega_1$ |

On the other hand, we define the following two matrices:

$$\boldsymbol{X}_{nm}^{\mathrm{LP}} := \begin{bmatrix} \boldsymbol{I}_{m+1} \\ \boldsymbol{O}_{n-m,m+1} \end{bmatrix} \in \mathbb{R}^{(n+1)\times(m+1)},$$

$$\boldsymbol{S}_{n}^{\mathrm{LP}} := \begin{bmatrix} 1 & & & 0 \\ & (\Omega_1)^{-1} & & \\ & & \ddots & \\ 0 & & & (\Omega_1)^{-n} \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},$$

where $\boldsymbol{I}_{m+1}$ is an identity matrix of $(m+1)\times(m+1)$, and $\boldsymbol{O}_{n-m,m+1}$ is a zero matrix of $(n-m)\times(m+1)$. Then, the relation of coefficients can be rewritten as

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \boldsymbol{X}_{nm}^{\mathrm{LP}} \boldsymbol{S}_{m}^{\mathrm{LP}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{bmatrix}, \qquad \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_n \end{bmatrix} = \boldsymbol{S}_{n}^{\mathrm{LP}} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

It remarks that the matrix $\boldsymbol{X}_{nm}^{\mathrm{LP}}$ equals to $\boldsymbol{I}_{m+1}$ in the case $m = n$.

### 2.2. Analog high-pass filter

The analog high-pass filter with the cut-off frequency $\Omega_1$ is designed from the original analog low-pass filter. From the transformation formula in Table 1, the relation of coefficients from $\{a_i\}_{i=0}^{m}$ and $\{b_i\}_{i=0}^{n}$ into $\{c_i\}_{i=0}^{n}$ and $\{d_i\}_{i=0}^{n}$ can be easily written as

$$c_i = \begin{cases} 0, & i = 0, 1, \ldots, n-m-1, \\ (\Omega_1)^{n-i} a_{n-i}, & i = n-m, n-m+1, \ldots, n, \end{cases}$$

$$d_i = (\Omega_1)^{n-i} b_{n-i}.$$

Then, the matrices which transform those coefficients can be easily obtained in the same manner as (2.1), i.e.,

$$\boldsymbol{X}_{nm}^{\mathrm{HP}} := \begin{bmatrix} \boldsymbol{O}_{n-m,m+1} \\ \boldsymbol{I}_{m+1} \end{bmatrix} \in \mathbb{R}^{(n+1)\times(m+1)},$$

$$\boldsymbol{S}_{n}^{\mathrm{HP}} := \begin{bmatrix} 0 & & & (\Omega_1)^n \\ & & \cdot^{\cdot^{\cdot}} & \\ & \Omega_1 & & \\ 1 & & & 0 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}.$$

The matrix $\boldsymbol{X}_{nm}^{\mathrm{HP}}$ equals to $\boldsymbol{I}_{m+1}$ in the case $m = n$.

### 2.3. Analog band-pass filter

The analog band-pass filter with the cut-off frequencies $\Omega_1, \Omega_2$ $(\Omega_1 < \Omega_2)$ is designed from the original analog low-pass filter. The matrices

$$\boldsymbol{X}_{nm}^{\mathrm{BP}} := (\Omega_b)^{n-m} \begin{bmatrix} \boldsymbol{O}_{n-m,2m+1} \\ \boldsymbol{I}_{2m+1} \\ \boldsymbol{O}_{n-m,2m+1} \end{bmatrix} \in \mathbb{R}^{(2n+1)\times(2m+1)},$$

$$\boldsymbol{S}_{n}^{\mathrm{BP}} := \left[ s_{ij}^{\mathrm{BP}} \right] \in \mathbb{R}^{(2n+1)\times(n+1)},$$

where

$$s_{ij}^{\mathrm{BP}} = \left\langle \begin{matrix} i \\ (i+j-n)/2 \end{matrix} \right\rangle (\Omega_a)^{j-i+n} (\Omega_b)^{n-j},$$

transform $\{a_i\}_{i=0}^{m}$ and $\{b_i\}_{i=0}^{n}$ into $\{c_i\}_{i=0}^{2n}$ and $\{d_i\}_{i=0}^{2n}$ as

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \\ \vdots \\ c_{2n} \end{bmatrix} = \boldsymbol{X}_{nm}^{\mathrm{BP}} \boldsymbol{S}_{m}^{\mathrm{BP}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{bmatrix}, \qquad \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_n \\ \vdots \\ d_{2n} \end{bmatrix} = \boldsymbol{S}_{n}^{\mathrm{BP}} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

The matrix $\boldsymbol{X}_{nm}^{\mathrm{BP}}$ equals to $\boldsymbol{I}_{2m+1}$ in the case $m = n$.

### 2.4. Analog band-stop filter

The analog band-stop filter with the cut-off frequencies $\Omega_1, \Omega_2$ ($\Omega_1 < \Omega_2$) is designed from the original analog low-pass filter. The matrices which transform $\{a_i\}_{i=0}^{m}$ and $\{b_i\}_{i=0}^{n}$ into $\{c_i\}_{i=0}^{2n}$ and $\{d_i\}_{i=0}^{2n}$ can be obtained in the same manner as (2.3), i.e.,

$$X_{nm}^{\mathrm{BS}} := \left[x_{ij}^{\mathrm{BS}}\right] \in \mathbb{R}^{(2n+1)\times(2m+1)},$$
$$S_{n}^{\mathrm{BS}} := \left[s_{ij}^{\mathrm{BS}}\right] \in \mathbb{R}^{(2n+1)\times(n+1)},$$

where

$$x_{ij}^{\mathrm{BS}} = \left\langle \begin{matrix} n-m \\ (i-j)/2 \end{matrix} \right\rangle (\Omega_a)^{2n-2m+j-i},$$
$$s_{ij}^{\mathrm{BS}} = s_{i,n-j}^{\mathrm{BP}} = \left\langle \begin{matrix} n-j \\ (i-j)/2 \end{matrix} \right\rangle (\Omega_a)^{2n-i+j}(\Omega_b)^{j}.$$

## 3. Matrices representation of the frequency transformation for IIR digital filters

This section derives the explicit formulae of matrices which represent the frequency transformation for IIR digital filters.

Let $T$ denotes the sampling interval of digital filters, and the cut-off frequency of the original digital low-pass filter be $\omega_0$. Assume that IIR digital filters are derived from analog filters by the bilinear transformation. Then the transfer functions of the original digital low-pass and high-pass filter can be easily obtained from [3] as

$$H_3(z) := \frac{e_0 + e_1 z^{-1} + \cdots + e_n z^{-n}}{f_0 + f_1 z^{-1} + \cdots + f_n z^{-n}}.$$

The transformation formulae [2; 4, p. 260], of the frequency transformation for IIR digital filters are shown in Table 2. The transfer functions of the derived digital low-pass and high-pass filters can be reduced to

$$H_4(z) = \frac{g_0 + g_1 z^{-1} + \cdots + g_n z^{-n}}{h_0 + h_1 z^{-1} + \cdots + h_n z^{-n}}.$$

Those of the derived digital band-pass and band-stop filters can be reduced to

$$\tilde{H}_4(z) = \frac{g_0 + g_1 z^{-1} + \cdots + g_n z^{-n} + \cdots + g_{2n} z^{-2n}}{h_0 + h_1 z^{-1} + \cdots + h_n z^{-n} + \cdots + h_{2n} z^{-2n}}.$$

Table 2
Transformation formulae of frequency transformation for IIR digital filters

| Filter type / cut-off frequency | Transformation formulae / Parameters |
|---|---|
| Low-pass | $z^{-1} = \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$ |
| $\omega_1$ | $\alpha = \dfrac{\sin\left(\frac{\omega_0 - \omega_1}{2}\right)T}{\sin\left(\frac{\omega_0 + \omega_1}{2}\right)T}$ |
| High-pass | $z^{-1} = -\dfrac{z^{-1} + \tilde{\alpha}}{1 + \tilde{\alpha} z^{-1}},$ |
| $\omega_1$ | $\tilde{\alpha} = -\dfrac{\cos\left(\frac{\omega_0 + \omega_1}{2}\right)T}{\cos\left(\frac{\omega_0 - \omega_1}{2}\right)T}$ |
| Band-pass | $z^{-1} = -\dfrac{z^{-2} + v z^{-1} + u}{u z^{-2} + v z^{-1} + 1}$ <br> $u = \dfrac{k-1}{k+1}, \quad v = -\dfrac{2\alpha k}{k+1},$ |
| $\omega_1, \omega_2$ ($\omega_1 < \omega_2$) | $k = \cot\left(\dfrac{\omega_2 - \omega_1}{2}\right) T \tan\dfrac{\omega_0}{2} T,$ <br> $\alpha = \dfrac{\cos\left(\frac{\omega_2 + \omega_1}{2}\right)T}{\cos\left(\frac{\omega_2 - \omega_1}{2}\right)T}$ |
| Band-stop | $z^{-1} = \dfrac{z^{-2} + \tilde{v} z^{-1} + \tilde{u}}{\tilde{u} z^{-2} + \tilde{v} z^{-1} + 1}$ <br> $\tilde{u} = -\dfrac{\tilde{k}-1}{\tilde{k}+1}, \quad \tilde{v} = -\dfrac{2\tilde{\alpha}}{\tilde{k}+1},$ |
| $\omega_1, \omega_2$ ($\omega_1 < \omega_2$) | $\tilde{k} = \tan\left(\dfrac{\omega_2 - \omega_1}{2}\right) T \tan\dfrac{\omega_0}{2} T,$ <br> $\tilde{\alpha} = \dfrac{\cos\left(\frac{\omega_2 + \omega_1}{2}\right)T}{\cos\left(\frac{\omega_2 - \omega_1}{2}\right)T}$ |

Then, matrices which transform the transfer function of the original low-pass filter to transfer functions of the derived filters are formulated.

### 3.1. Design of digital low-pass filter

The digital low-pass filter with the cut-off frequency $\omega_1$ is designed from the original digital low-pass filter.

The matrix

$$T_n^{\mathrm{LP}} := \left[t_{ij}^{\mathrm{LP}}\right] \in \mathbb{R}^{(n+1)\times(n+1)},$$

where

$$t_{ij}^{\mathrm{LP}} = \sum_{k=0}^{i} (-1)^{j-i} \left\langle \begin{matrix} j \\ i-k \end{matrix} \right\rangle \left\langle \begin{matrix} n-j \\ k \end{matrix} \right\rangle \alpha^{j-i+2k},$$

transforms $\{e_i\}_{i=0}^{n}$ and $\{f_i\}_{i=0}^{n}$ into $\{g_i\}_{i=0}^{n}$ and $\{h_i\}_{i=0}^{n}$ as

$$\begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_n \end{bmatrix} = T_n^{\mathrm{LP}} \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_n \end{bmatrix}, \qquad \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_n \end{bmatrix} = T_n^{\mathrm{LP}} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

### 3.2. Digital high-pass filter

The digital high-pass filter with the cut-off frequency $\omega_1$ is designed from the original digital low-pass filter. The matrix which transforms $\{e_i\}_{i=0}^{n}$ and $\{f_i\}_{i=0}^{n}$ into $\{g_i\}_{i=0}^{n}$ and $\{h_i\}_{i=0}^{n}$ can be obtained in the same manner as (3.1), i.e.,

$$T_n^{\mathrm{HP}} := [t_{ij}^{\mathrm{HP}}] \in \mathbb{R}^{(n+1)\times(n+1)}$$

$$= \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & (-1)^n \end{bmatrix} T_n^{\mathrm{LP}}\Big|_{\alpha=\tilde{\alpha}},$$

where

$$t_{ij}^{\mathrm{HP}} = (-1)^i t_{ij}^{\mathrm{LP}}\Big|_{\alpha=\tilde{\alpha}}$$

$$= \sum_{k=0}^{i} (-1)^{j} \left\langle \begin{matrix} j \\ i-k \end{matrix} \right\rangle \left\langle \begin{matrix} n-j \\ k \end{matrix} \right\rangle \tilde{\alpha}^{j-i+2k}.$$

### 3.3. Digital band-pass filter

The digital band-pass filter with the cut-off frequencies $\omega_1, \omega_2$ $(\omega_1 < \omega_2)$ is designed from the original digital low-pass filter. The matrix

$$T_n^{\mathrm{BP}} := \left[t_{ij}^{\mathrm{BP}}\right] \in \mathbb{R}^{(2n+1)\times(n+1)},$$

where

$$t_{ij}^{\mathrm{BP}} = \sum_{\ell=0}^{i} (-1)^j \beta_{i-\ell,j}\gamma_{\ell,j},$$

$$\beta_{\ell,j} = \sum_{k=0}^{\lfloor \ell/2 \rfloor} \left\langle \begin{matrix} j \\ \ell-k \end{matrix} \right\rangle \left\langle \begin{matrix} \ell-k \\ k \end{matrix} \right\rangle u^{j-\ell+k} v^{\ell-2k},$$

$$\gamma_{\ell,j} = \sum_{k=0}^{\lfloor \ell/2 \rfloor} \left\langle \begin{matrix} n-j \\ \ell-k \end{matrix} \right\rangle \left\langle \begin{matrix} \ell-k \\ k \end{matrix} \right\rangle u^{k} v^{\ell-2k},$$

transforms $\{e_i\}_{i=0}^{n}$ and $\{f_i\}_{i=0}^{n}$ into $\{g_i\}_{i=0}^{2n}$ and $\{h_i\}_{i=0}^{2n}$ as

$$\begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_n \\ \vdots \\ g_{2n} \end{bmatrix} = T_n^{\mathrm{BP}} \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_n \end{bmatrix}, \qquad \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_n \\ \vdots \\ h_{2n} \end{bmatrix} = T_n^{\mathrm{BP}} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

In the above formulation of $\beta_{\ell,j}$ and $\gamma_{\ell,j}$, $\lfloor \ell/2 \rfloor$ means the maximum integer not exceeding $\ell/2$.

### 3.4. Digital band-stop filter

The digital band-stop filter with the cut-off frequencies $\omega_1, \omega_2$ $(\omega_1 < \omega_2)$ is designed from the original digital low-pass filter. The matrix which transforms $\{e_i\}_{i=0}^{n}$ and $\{f_i\}_{i=0}^{n}$ into $\{g_i\}_{i=0}^{2n}$ and $\{h_i\}_{i=0}^{2n}$ can be obtained in the same manner as (3.3), i.e.,

$$T_n^{\mathrm{BS}} := \left[t_{ij}^{\mathrm{BS}}\right] \in \mathbb{R}^{(2n+1)\times(n+1)}$$

$$= T_n^{\mathrm{BP}}\Big|_{u=\tilde{u},v=\tilde{v}} \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & (-1)^n \end{bmatrix},$$

where

$$t_{ij}^{\mathrm{BS}} = (-1)^j t_{ij}^{\mathrm{BP}}\Big|_{u=\tilde{u},v=\tilde{v}}$$

$$= \sum_{\ell=0,}^{i} \beta_{i-\ell,j}\gamma_{\ell,j}\Big|_{u=\tilde{u},v=\tilde{v}}.$$

## 4. Matrices representation of the bilinear transformation for rational polynomials

The transfer matrices are derived which represent the bilinear transformation for rational polynomials.

The transfer function of an analog filter was written as

$$H_1(s) := \frac{a_0 + a_1 s + \cdots + a_m s^m}{b_0 + b_1 s + \cdots + b_n s^n}, \quad m \leqslant n.$$

The transformation formula [4, pp. 219–224] of the bilinear transformation is given as

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}.$$

In designing filters, it is equivalent [3] to

$$s = \frac{1 - z^{-1}}{1 + z^{-1}}.$$

Then the transfer function of the derived digital filter can be reduced to

$$H_3(z) = \frac{e_0 + e_1 z^{-1} + \cdots + e_n z^{-n}}{f_0 + f_1 z^{-1} + \cdots + f_n z^{-n}}.$$

Then the matrices

$$\boldsymbol{Y}_{nm} := [y_{ij}] \in \mathbb{R}^{(n+1)\times(m+1)},$$

$$\boldsymbol{Q}_n := [q_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)},$$

where

$$y_{ij} = \left\langle \begin{matrix} n-m \\ i-j \end{matrix} \right\rangle,$$

$$q_{ij} = \sum_{k=0}^{i} (-1)^{i-k} \left\langle \begin{matrix} j \\ i-k \end{matrix} \right\rangle \left\langle \begin{matrix} n-j \\ k \end{matrix} \right\rangle$$

transform $\{a_i\}_{i=0}^m$ and $\{b_i\}_{i=0}^n$ into $\{e_i\}_{i=0}^n$ and $\{f_i\}_{i=0}^n$ as

$$\begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_n \end{bmatrix} = \boldsymbol{Y}_{nm}\boldsymbol{Q}_m \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{bmatrix}, \qquad \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix} = \boldsymbol{Q}_n \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

The matrix $\boldsymbol{Y}_{nm}$ equals to $\boldsymbol{I}_{m+1}$ in the case $m = n$. Besides, the recursive formula to obtain $\boldsymbol{Q}_n$ is shown in [1].

## 5. Conditions for the direct design of IIR digital filters

In this section, conditions to design IIR digital filters directly from an analog low-pass filter are investigated.

In designing an IIR digital filter, two different ways can be considered as shown in Fig. 1, i.e.,

(i) Frequency transformation for analog filters first, and then bilinear transformation.

(ii) Bilinear transformation first, and then frequency transformation for IIR digital filters.

Though they look quite different from each other at a glance, the transformation formulae of them had been totally same so that the same digital filter is designed [4, p. 261]. Hence, we can choose simpler one.

Since the frequency transformation formulae to design analog low-pass and high-pass filters can be simply defined, it looks more simple to design digital low-pass and high-pass filters by the way (i) than by the way (ii). It is unknown which way is simpler to design digital band-pass and band-stop filters.

To design digital low-pass and high-pass filter with cut-off frequency $\omega_1$ by the way (i), the cut-off frequency $\Omega_1$ of analog filters must be

$$\Omega_1 = \tan\frac{\omega_1}{2}T.$$

To design band-pass and band-stop digital filter with cut-off frequencies $\omega_1, \omega_2$ ($\omega_1 < \omega_2$) by the way (i), the cut-off frequencies $\Omega_1, \Omega_2$ of analog filters must be

$$\Omega_1 = \tan\frac{\omega_1}{2}T, \qquad \Omega_2 = \tan\frac{\omega_2}{2}T.$$

To design any type of IIR digital filters by the way (ii), the cut-off frequency $\omega_0$ of the original digital low-pass filter must be

$$\omega_0 = \frac{\pi}{2T}.$$

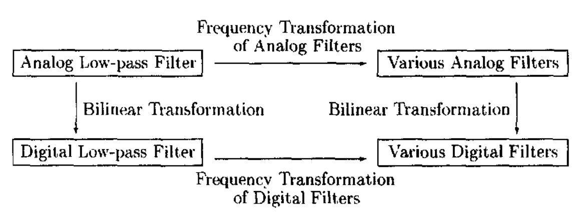


Fig. 1. The design process of IIR digital filters.

## 6. Design examples of IIR digital filters

Some simple digital filters are designed to evaluate the effectiveness of the proposed matrices. By using the matrices, transfer functions of IIR digital filters of various types are derived directly and automatically from a transfer function of an analog low-pass filter.

### 6.1. Original analog filter of low-pass type

Let the original analog low-pass filter be the second-order normalized Butterworth low-pass filter. The transfer function is given as

$$H_1(s) := \frac{1}{1+\sqrt{2}s+s^2}.$$

Coefficient vectors are

$$[a_0]=[1], \qquad \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}.$$

The amplitude characteristics of this filter is shown in Fig. 2.

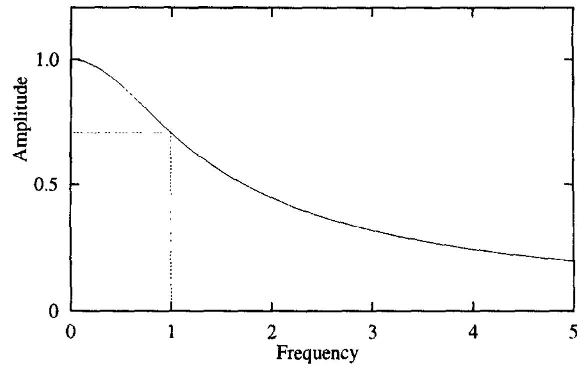


Fig. 2. Amplitude characteristics of the original analog low-pass filter.

### 6.2. Design of IIR digital filter of high-pass type

The digital high-pass filter with the cut-off frequency $\omega_1 = 0.2\pi/T$ is designed from the original analog low-pass filter. This filter is designed by the way (i), which is simpler one.

The cut-off frequency of the analog high-pass filter, which is derived by the frequency transformation for analog filters, is given as

$$\Omega_1 = \tan\frac{\omega_1}{2}T = \tan 0.1\pi \simeq 0.3249.$$

The transfer matrices $\boldsymbol{Q}_2, \boldsymbol{X}_{2,0}^{\mathrm{HP}}, \boldsymbol{S}_0^{\mathrm{HP}}$ and $\boldsymbol{S}_2^{\mathrm{HP}}$ are given as

$$\boldsymbol{Q}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}, \qquad \boldsymbol{X}_{2,0}^{\mathrm{HP}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\boldsymbol{S}_0^{\mathrm{HP}} = [1], \qquad \boldsymbol{S}_2^{\mathrm{HP}} = \begin{bmatrix} 0 & 0 & 0.1056 \\ 0 & 0.3249 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

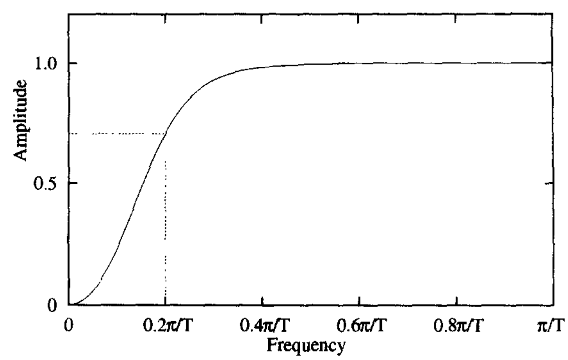


Fig. 3. Amplitude characteristics of the derived digital high-pass filter.

Then the coefficient vectors of the derived digital high-pass filter are obtained as

$$\begin{bmatrix} g_0 \\ g_1 \\ g_2 \end{bmatrix} = \boldsymbol{Q}_2\boldsymbol{X}_{2,0}^{\mathrm{HP}}\boldsymbol{S}_0^{\mathrm{HP}}[a_0] = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \end{bmatrix} = \boldsymbol{Q}_2\boldsymbol{S}_2^{\mathrm{HP}} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1.5651 \\ -1.7888 \\ 0.6461 \end{bmatrix}.$$

The derived transfer function of this filter is written as

$$H_4(z) = \frac{1-2z^{-1}+z^{-2}}{1.5651-1.7888z^{-1}+0.6461z^{-2}}.$$

The amplitude characteristics of this filter is shown in Fig. 3. The designed digital filter certainly realizes high-pass characteristics.

### 6.3. IIR digital filter of band-pass type

The digital band-pass filter with the cut-off frequencies $\omega_1=0.2\pi/T, \omega_2=0.6\pi/T$ is designed from the original analog low-pass filter. This filter is designed by both the ways (i) and (ii).

#### 6.3.1. The way (i)

The cut-off frequencies of the analog band-pass filter, which are derived by the frequency transformation for analog filters, are given as

$$\Omega_1 = \tan\frac{\omega_1}{2}T = \tan 0.1\pi \simeq 0.3249,$$

$$\Omega_2 = \tan\frac{\omega_2}{2}T = \tan 0.3\pi \simeq 1.3764.$$

The transfer matrices $\boldsymbol{Q}_4, \boldsymbol{X}_{2,0}^{\mathrm{BP}}, \boldsymbol{S}_0^{\mathrm{BP}}$ and $\boldsymbol{S}_2^{\mathrm{BP}}$ are given as

$$\boldsymbol{Q}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & -2 & 0 & 2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix},$$

$$\boldsymbol{X}_{2,0}^{\mathrm{BP}} = \begin{bmatrix} 0 \\ 0 \\ 1.1056 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{S}_0^{\mathrm{BP}} = [1],$$

$$\boldsymbol{S}_2^{\mathrm{BP}} = \begin{bmatrix} 0 & 0 & 0.2000 \\ 0 & 0.4702 & 0 \\ 1.1056 & 0 & 0.8944 \\ 0 & 1.0515 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, the coefficient vectors of the derived digital high-pass filter are obtained as

$$\begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \boldsymbol{Q}_4 \boldsymbol{X}_{2,0}^{\mathrm{BP}} \boldsymbol{S}_0^{\mathrm{BP}} [a_0] = \begin{bmatrix} 1.1056 \\ 0 \\ -2.2112 \\ 0 \\ 1.1056 \end{bmatrix},$$

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \boldsymbol{Q}_4 \boldsymbol{S}_2^{\mathrm{BP}} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 5.3520 \\ -4.8442 \\ 3.2000 \\ -1.5558 \\ 1.0480 \end{bmatrix}.$$

The derived transfer function of this filter is written as

$$\tilde{H}_4(z) = \frac{1.1056 - 2.2112z^{-2} + 1.1056z^{-4}}{5.3520 - 4.8442z^{-1} + 3.2000z^{-2} - 1.5558z^{-3} + 1.0480z^{-4}}.$$

The amplitude characteristics of this filter is shown in Fig. 4. The designed digital filter certainly realizes band-pass characteristics.

#### 6.3.2. The way (ii)

The transfer matrices $\boldsymbol{T}_2^{\mathrm{BP}}, \boldsymbol{Y}_{2,0}$ and $\boldsymbol{Q}_0$ are given as

$$\boldsymbol{T}_2^{\mathrm{BP}} = \begin{bmatrix} 1 & -0.1584 & 0.0251 \\ -0.8850 & 0.5126 & -0.1402 \\ 0.5126 & -1.2209 & 0.5126 \\ -0.1402 & 0.5126 & -0.8850 \\ 0.0251 & -0.1584 & 1 \end{bmatrix},$$

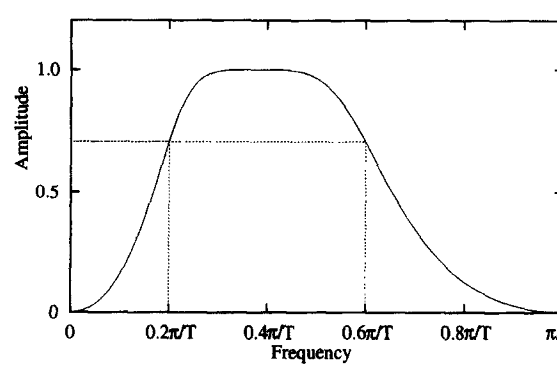


Fig. 4. Amplitude characteristics of the derived digital band-pass filter.

$$Y_{2,0}=\begin{bmatrix}1\\2\\1\end{bmatrix},\qquad Q_0=[1].$$

Then, the coefficient vectors of the derived digital high-pass filter are obtained as

$$\begin{bmatrix}g_0\\g_1\\g_2\\g_3\\g_4\end{bmatrix}=T_2^{\mathrm{BP}}Y_{2,0}Q_0[a_0]=\begin{bmatrix}0.7083\\0\\-1.4166\\0\\0.7083\end{bmatrix},$$

$$\begin{bmatrix}h_0\\h_1\\h_2\\h_3\\h_4\end{bmatrix}=T_2^{\mathrm{BP}}Q_2\begin{bmatrix}b_0\\b_1\\b_2\end{bmatrix}=\begin{bmatrix}3.4289\\-3.1037\\2.0503\\-0.9970\\0.6713\end{bmatrix}.$$

These coefficients are different from those obtained by the way (i), but the derived transfer functions become same. Therefore, the same digital filter was certainly designed.

## 7. Conclusions

This paper proposed the matrices which represent the relation of the coefficients of the transfer functions in the frequency transformation for analog filters, that for IIR digital filters and the bilinear transformation for rational polynomials. The proposed matrices could replace hand computation in these transformations by automatic procedures. Using the proposed matrices, an automatic procedure was established to design various types of IIR digital filters from an analog low-pass filter.

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