Appendix A

An overview of results from real analysis

In this Appendix we gather a number of results from real analysis which are assumed at various places in the text. Some of these will be familiar from any course on the subject, and no proofs are given; a small number may be less familiar, and we give proofs of these for completeness.

Theorem A.1 (The Intermediate Value Theorem) *Suppose that f is a real-valued function, defined and continuous on the closed interval $[a, b]$ of $\mathbb{R}$. Then, f is a bounded function on the interval $[a, b]$ and, if y is any number such that*

$$\inf_{x\in[a,b]} f(x) \le y \le \sup_{x\in[a,b]} f(x)\,,$$

then there is a number $\xi \in [a, b]$ such that $f(\xi) = y$. In particular, the infimum and the supremum of f are achieved, and can be replaced by $\min_{x\in[a,b]}$ *and* $\max_{x\in[a,b]}$*, respectively.*

The next result, known as Rolle's Theorem, was published in an obscure book in 1691 by the French mathematician Michel Rolle (1652–1719) who invented the notation $\sqrt[n]{x}$ for the nth root of x.

Theorem A.2 (Rolle's Theorem) *Suppose that f is a real-valued function, defined and continuous on the closed interval $[a, b]$ of $\mathbb{R}$, differentiable in the open interval (a, b), and such that $f(a) = f(b)$. Then, there exists a number $\xi \in (a, b)$ such that $f'(\xi) = 0$.*

It is often important in our applications that the point $\xi \in (a, b)$, *i.e.*, $a < \xi < b$. For instance it may happen that $f'(a) = f'(b) = 0$, as well as $f(a) = f(b)$; Theorem A.2 then states that, in addition to the endpoints

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of the interval $[a, b]$, there is also an interior point $\xi \in (a, b)$ at which the derivative vanishes.

Theorem A.3 (The Mean Value Theorem) *Suppose that f is a real-valued function, defined and continuous on the closed interval $[a, b]$ of $\mathbb{R}$, and f is differentiable in the open interval (a, b). Then, there exists a number $\xi \in (a, b)$ such that*

$$f(b) - f(a) = f'(\xi)\,(b - a)\,.$$

Theorem A.4 (Taylor's Theorem) *Suppose that n is a nonnegative integer, and f is a real-valued function, defined and continuous on the closed interval $[a, b]$ of $\mathbb{R}$, such that the derivatives of f of order up to and including n are defined and continuous on the closed interval $[a, b]$. Suppose further that $f^{(n)}$ is differentiable on the open interval (a, b). Then, for each value of x in $[a, b]$, there exists a number $\xi = \xi(x)$ in the open interval (a, b) such that*

$$\begin{aligned} f(x) &= f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) \\ &\quad + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)\,. \end{aligned}$$

Theorem A.5 (Taylor's Theorem with integral remainder) *Let n be a nonnegative integer and suppose that f is a real-valued function, defined and continuous on the closed interval $[a, b]$ of $\mathbb{R}$, such that the derivatives of f of order up to and including n are defined and continuous on $[a, b]$, $f^{(n)}$ is differentiable on the open interval (a, b), and $f^{(n+1)}$ is integrable on (a, b). Then, for each $x \in [a, b]$,*

$$\begin{aligned} f(x) &= f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) \\ &\quad + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\mathrm{d}t\,. \end{aligned}$$

Proof As this version of the theorem may be rather less familiar we include a proof.

The theorem is trivially true for $n = 0$. Suppose that the theorem is true for some nonnegative integer, say $n = k$. Then, provided that $f^{(k+1)}$ is differentiable on (a, b) and $f^{(k+2)}$ is integrable on (a, b), integration by parts shows that

$$\int_a^x \frac{(x-t)^{k+1}}{(k+1)!} f^{(k+2)}(t)\mathrm{d}t \;=\; -\frac{(x-a)^{k+1}}{(k+1)!} f^{(k+1)}(a) - \int_a^x -\frac{(x-t)^k}{k!} f^{(k+1)}(t)\mathrm{d}t\,;$$

use of the theorem when $n = k$ now shows that it is also true for $n = k+1$. The proof by induction is then complete. □

Theorem A.6 (The Integral Mean Value Theorem) *Suppose that f is a real-valued function, defined and continuous on a closed interval $[a,b]$ of $\mathbb{R}$, and let g be a function, defined, nonnegative and integrable on (a,b). Then, there exists a number $\xi \in (a,b)$ such that*

$$\int_a^b f(x)g(x)\,\mathrm{d}x = f(\xi)\int_a^b g(x)\,\mathrm{d}x\,.$$

Proof Since f is continuous on $[a,b]$, it is bounded on $[a,b]$, say

$$m \le f(x) \le M, \qquad x \in [a,b]\,.$$

Then, as $g(x) \ge 0$ for all $x \in (a,b)$, we have that

$$mg(x) \le f(x)g(x) \le Mg(x)\,, \qquad x \in (a,b)\,.$$

Integrating these inequalities gives

$$m\int_a^b g(x)\,\mathrm{d}x \le \int_a^b f(x)g(x)\,\mathrm{d}x \le M\int_a^b g(x)\,\mathrm{d}x\,.$$

If $\int_a^b g(x)\mathrm{d}x = 0$, then the result trivially follows. If, on the other hand, $\int_a^b g(x)\mathrm{d}x > 0$, then

$$m \le \frac{\int_a^b f(x)g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x} \le M.$$

The existence of the required value of $\xi \in (a,b)$ now follows from the Intermediate Value Theorem. □

Theorem A.6 obviously also holds provided that $g(x) \le 0$ on (a,b); it is only important that g has constant sign on (a,b). Note also that we do not require that g is continuous, only that it is integrable. For example, Theorem A.6 will hold if f is a continuous function defined on $[0,1]$ and $g(x) = x^{-1/2}$, $x \in (0,1)$.

Theorem A.7 (Taylor's Theorem for several variables) *Suppose that f is a real-valued function of n real variables, $n \geq 1$, such that f and all of its partial derivatives up to and including order $k+1$ are defined, continuous and bounded in a neighbourhood of the point $\boldsymbol{a}$ in $\mathbb{R}^n$. Let A denote an upper bound on the absolute values of all the derivatives of order $k+1$ in this neighbourhood. Then*

$$f(\boldsymbol{a}+\boldsymbol{\eta}) = f(\boldsymbol{a}) + \sum_{r=1}^{k} \frac{U_r(\boldsymbol{a})}{r!} + E_k\,,$$

where

$$U_r(\boldsymbol{a}) = \left[\left(\eta_1 \frac{\partial}{\partial x_1} + \cdots + \eta_n \frac{\partial}{\partial x_n}\right)^r f\right](\boldsymbol{a})\,, \qquad r = 1, \ldots, k\,,$$

and

$$|E_k| \leq \tfrac{1}{(k+1)!} A n^{k+1} \, \|\boldsymbol{\eta}\|_\infty^{k+1}.$$

Proof The proof involves the application of Theorem A.4, Taylor's Theorem, to the function of one variable

$$\varphi(t) = f(\boldsymbol{a} + t\boldsymbol{\eta})$$

to give a series expansion for $\varphi(1)$. Then, the expressions for the derivatives of φ in terms of the partial derivatives of $\boldsymbol{f}$, *via* the chain rule, yield the required result; n^{k+1} is the number of partial derivatives of order $k+1$ for a function of n variables. $\square$

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