

# Appendix A

## Orbits

As discussed in the Introduction, a good first approximation for satellite motion is obtained by assuming the spacecraft is a point mass or spherical body moving in the gravitational field of a spherical planet. This leads to the classical *two-body problem*. Since we use the term *body* to refer to a spacecraft of finite size (as in *rigid body*), it may be more appropriate to call this the *two-particle* problem, but I will use the term two-body problem in its classical sense.

The basic elements of orbital dynamics are captured in Kepler's three laws which he published in the 17th century. His laws were for the orbital motion of the planets about the Sun, but are also applicable to the motion of satellites about planets. The three laws are:

1. The orbit of each planet is an ellipse with the Sun at one focus.
2. The line joining the planet to the Sun sweeps out equal areas in equal times.
3. The square of the period of a planet is proportional to the cube of its mean distance to the sun.

The first law applies to most spacecraft, but it is also possible for spacecraft to travel in parabolic and hyperbolic orbits, in which case the period is infinite and the 3rd law does not apply. However, the 2nd law applies to all two-body motion. Newton's 2nd law and his law of universal gravitation provide the tools for generalizing Kepler's laws to non-elliptical orbits, as well as for proving Kepler's laws.

### A.1 Equations of Motion and Their Solution

In the two-body problem, one assumes a system of only two bodies that are both spherically symmetric or are both point masses. In this case the equations of motion may be developed as

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = \vec{0} \quad (\text{A.1})$$

where  $\mu = G(m_1 + m_2)$  and  $\vec{\mathbf{r}}$  is the relative position vector as shown in Fig. A.1. The parameter  $G$  is Newton's universal gravitational constant, which has the value  $G = 6.67259 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}$ . The scalar  $r$  is the magnitude of  $\vec{\mathbf{r}}$ , *i.e.*,  $r = \|\vec{\mathbf{r}}\|$ . Since normally  $m_1 \gg m_2$ ,  $\mu \approx Gm_1$ , and this is the value usually given in astrodynamics tables.

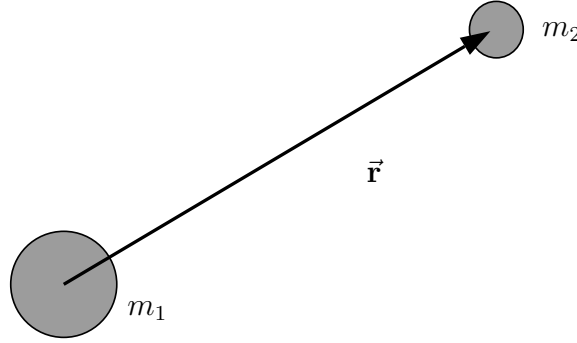


Figure A.1: Relative position vector in the two-body problem.

There are several ways to derive the conserved quantities associated with two-body motion. These constants of the motion are also called first integrals or conservation laws. Here I simply state these conservation laws and interpret them physically and graphically. First is conservation of energy,

$$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r} = \text{constant} \quad (\text{A.2})$$

where  $v = \|\vec{\mathbf{v}}\| = \|\dot{\vec{\mathbf{r}}}\|$ . Note that  $\mathcal{E}$  is the energy per unit mass:  $v^2/2$  is the kinetic energy and  $-\mu/r$  is the potential energy. As I discuss more below, energy is closely related to the size of the orbit.

The second constant of motion is the angular momentum vector

$$\vec{\mathbf{h}} = \vec{\mathbf{r}} \times \vec{\mathbf{v}} \quad (\text{A.3})$$

Again, this is the angular momentum per unit mass. The fact that  $\vec{\mathbf{h}}$  is constant in direction means that  $\vec{\mathbf{r}}$  and  $\vec{\mathbf{v}}$  always lie in the plane  $\perp$  to  $\vec{\mathbf{h}}$ . This plane is called the *orbital plane* and it is fixed in inertial space in the two-body problem. It is relatively straightforward to show that conservation of angular momentum is equivalent to

Kepler's second law, and from it can be derived the mathematical version of Kepler's third law for elliptical orbits:

$$\mathcal{P} = 2\pi\sqrt{\frac{a^3}{\mu}} \quad (\text{A.4})$$

where  $\mathcal{P}$  is the orbital period and  $a$  is the semimajor axis of the ellipse.

The third constant is also a vector quantity, sometimes called the *eccentricity vector*

$$\vec{e} = \frac{1}{\mu} \left[ \left( v^2 - \frac{\mu}{r} \right) \vec{r} - (\vec{r} \cdot \vec{v}) \vec{v} \right] \quad (\text{A.5})$$

Note that  $\vec{e}$  is in the orbital plane, since it is a linear combination of  $\vec{r}$  and  $\vec{v}$ . In fact,  $\vec{e}$  points in the direction of periapsis, the closest point of the orbit to the central body. Also,  $e = \|\vec{e}\|$  is the eccentricity of the orbit.

Using these constants, it is possible to develop a solution to Eq. (A.1) in the form

$$r = \frac{p}{1 + e \cos \nu} \quad (\text{A.6})$$

where  $\nu$  is an angle called the *true anomaly*, and  $p = h^2/\mu$  is the *semilatus rectum* of the conic section. Equation (A.6) is the mathematical expression of Kepler's second law. This relation describes how the radius changes as the satellite moves but it does not tell how  $\nu$  varies with time. To determine  $\nu(t)$ , we must solve Kepler's equation

$$M = n(t - t_0) = E - e \sin E \quad (\text{A.7})$$

where  $E$  is the *eccentric anomaly*,  $M$  is the *mean anomaly*,  $n = \sqrt{\mu/a^3}$  is the *mean motion*, and  $t_0$  is the time of periapsis passage.

A satellite's orbit is normally described in terms of its *orbital elements*. The "classical" orbital elements are semimajor axis,  $a$ , eccentricity,  $e$ , inclination,  $i$ , right ascension of the ascending node,  $\Omega$ , argument of periapsis,  $\omega$ , and true anomaly of epoch,  $\nu_0$ . The last of these six orbital elements is sometimes replaced by mean anomaly at epoch,  $M_0$ , or by the time of periapsis passage,  $T_p$ .

## A.2 Orbit Determination and Prediction

We often need to be able to determine the orbit of a spacecraft or other space object. The primary motivation for orbit determination is to be able to predict where the object will be at some later time. This problem was posed and solved in 1801 by Carl Friedrich Gauss in order to predict the reappearance of the newly discovered minor planet, Ceres. He also invented the method of least squares which is described below.

In practice, orbit determination is accomplished by processing a sequence of observations in an orbit determination algorithm. Since it takes six numbers (elements) to describe an orbit, the algorithm is deterministic if and only if there are six independent numbers in the observation data. If there are fewer than six numbers, then the

problem is underdetermined and there is an infinite number of solutions. If there are more than six numbers in the observations, then the problem is overdetermined, and in general no solution exists. In the overdetermined case, a statistical method such as least squares is used to find an orbit that is as close as possible to fitting all the observations. In this section, we first present some deterministic orbit determination algorithms, then a statistical algorithm, and finally we discuss the orbit prediction problem.

### A.2.1 Solving Kepler's Equation

```
% function E = EofMe(M,e,tol)
%
% This function computes E as a function of M and e.
function E = EofMe(M,e,tol)
    if ( nargin<3 ), tol=1e-11; end
    En = M;
    En1 = En - (En-e*sin(En)-M)/(1-e*cos(En));
    while ( abs(En1-En) > tol )
        En = En1;
        En1 = En - (En-e*sin(En)-M)/(1-e*cos(En));
    end;
    E = En1;
```

### A.2.2 Deterministic Orbit Determination

In the best of all possible worlds, we might be able to measure  $\vec{r}$  and  $\vec{v}$  directly, in which case we would have a set of obs comprising six numbers, three in each vector. With this set of observations, it is straightforward to develop an algorithm that computes the orbital elements as a function of  $\vec{r}$  and  $\vec{v}$ . This algorithm is developed in numerous orbit dynamics texts, and here we simply present the mathematical algorithm and a MatLab function that implements the algorithm.

```
%
% rv2oe.m  r,v,mu -> orbital elements
%
% oe = rv2oe(rv,vv,mu)
%      rv = [X Y Z]'      vv = [v1 v2 v3]'      in IJK frame
%      oe = [a e i Omega omega nu0]
%
function oe = rv2oe(rv,vv,mu)
%
% first do some preliminary calculations
%
    K = [0;0;1];
```

```

    hv = cross(rv,vv);
    nv = cross(K,hv);
    n = sqrt(nv'*nv);
    h2 = (hv'*hv);
    v2 = (vv'*vv);
    r = sqrt(rv'*rv);
    ev = 1/mu * ( (v2-mu/r)*rv - (rv'*vv)*vv );
    p = h2/mu;
%
% now compute the oe's
%
    e = sqrt(ev'*ev);          % eccentricity
    a = p/(1-e*e);             % semimajor axis
    i = acos(hv(3)/sqrt(h2));   % inclination
    Om = acos(nv(1)/n);        % RAAN
    if ( nv(2) < 0-eps )        % fix quadrant
        Om = 2*pi-Om
    end;
    om = acos(nv'*ev/n/e);      % arg of periapsis
    if ( ev(3) < 0 )            % fix quadrant
        om = 2*pi-om;
    end;
    nu = acos(ev'*rv/e/r);      % true anomaly
    if ( rv'*vv < 0 )           % fix quadrant
        nu = 2*pi-nu;
    end;
    oe = [a e i Om om nu];      % assemble "vector"

```

## A.3 Two-Line Element Sets

Once an orbit has been determined for a particular space object, we might want to communicate that information to another person who is interested in tracking the object. The standard format for communicating the orbit of an Earth-orbiting object is the two-line element set, or TLE. An example TLE for a Russian spy satellite is

```

COSMOS 2278 1 23087U 94023A
98011.59348139 .00000348 00000-0 21464-3 0 5260 2 23087
71.0176 58.4285 0007185 172.8790 187.2435 14.12274429191907

```

Note that the TLE actually has three lines.

Documentation for the TLE format is available at the website <http://celestrak.com> and is reproduced here for your convenience. This website also includes current TLEs for many satellites.

Data for each satellite consists of three lines in the following format:

AAAAAAAAAAAAAAAAAAAAAAAAAAAA

```
1 NNNNNU NNNNAAA NNNN.NNNNNNNN +.NNNNNNNN +NNNNN-N +NNNNN-N N NNNNN
2 NNNNN NNN.NNNN NNN.NNNN NNNNNNNN NNN.NNNN NNN.NNNN NN.NNNNNNNNNNNNNNN
```

Line 0 is a twenty-four character name (to be consistent with the name length in the NORAD SATCAT).

Lines 1 and 2 are the standard Two-Line Orbital Element Set Format identical to that used by NORAD and NASA. The format description is:

Line 1

Column	Description
01	Line Number of Element Data
03-07	Satellite Number
08	Classification (U=Unclassified)
10-11	International Designator (Last two digits of launch year)
12-14	International Designator (Launch number of the year)
15-17	International Designator (Piece of the launch)
19-20	Epoch Year (Last two digits of year)
21-32	Epoch (Day of the year and fractional portion of the day)
34-43	First Time Derivative of the Mean Motion
45-52	Second Time Derivative of Mean Motion (decimal point assumed)
54-61	BSTAR drag term (decimal point assumed)
63	Ephemeris type
65-68	Element number
69	Checksum (Modulo 10)
	(Letters, blanks, periods, plus signs = 0; minus signs = 1)

Line 2

Column	Description
01	Line Number of Element Data
03-07	Satellite Number
09-16	Inclination [Degrees]
18-25	Right Ascension of the Ascending Node [Degrees]
27-33	Eccentricity (decimal point assumed)
35-42	Argument of Perigee [Degrees]
44-51	Mean Anomaly [Degrees]
53-63	Mean Motion [Revs per day]
64-68	Revolution number at epoch [Revs]
69	Checksum (Modulo 10)

All other columns are blank or fixed.

Example:

NOAA 14

```
1 23455U 94089A 97320.90946019 .00000140 00000-0 10191-3 0 2621
2 23455 99.0090 272.6745 0008546 223.1686 136.8816 14.11711747148495
```

For further information, see "Frequently Asked Questions: Two-Line Element Set Format" in the Computers & Satellites column of Satellite Times, Volume 4 Number 3.

A Matlab function to compute the orbital elements given a TLE is provided below.

```
% function [oe,epoch,yr,M,E,satname] = TLE2oe(fname);
%     fname is a filename string for a file containing
%         a two-line element set (TLE)
%     oe is a 1/6 matrix containing the orbital elements
%         [a e i Om om nu]
%     yr is the two-digit year
%     M is the mean anomaly at epoch
%     E is the eccentric anomaly at epoch
%     satname is the satellite name
%
% Calls Newton iteration function file EofMe.m

function [oe,epoch,yr,M,E,satname] = TLE2oe(fname);

% Open the file up and scan in the elements

fid = fopen(fname, 'rb');
A = fscanf(fid,'%13c*s',1);
B = fscanf(fid,'%d%6d%*c%5d%*3c%2d%f%f%5d%*c%*d%5d%*c%*d%d%5d',[1,10]);
C = fscanf(fid,'%d%6d%f%f%f%f%f', [1,8]);
fclose(fid);
satname=A;

% The value of mu is for the earth
mu = 3.986004415e5;

% Calculate 2-digit year (Oh no!, look out for Y2K bug!)

yr = B(1,4);

% Calculate epoch in julian days
epoch = B(1,5);
%ndot = B(1,6);
% n2dot = B(1,7);
```

```

% Assign variables to the orbital elements
i = C(1,3)*pi/180;          % inclination
Om = C(1,4)*pi/180;         % Right Ascension of the Ascending Node
e = C(1,5)/1e7;             % Eccentricity
om = C(1,6)*pi/180;         % Argument of periapsis
M = C(1,7)*pi/180;          % Mean anomaly
n = C(1,8)*2*pi/(24*3600);  % Mean motion

% Calculate the semi-major axis
a = (mu/n^2)^(1/3);

% Calculate the eccentric anomaly using mean anomaly
E = EofMe(M,e,1e-10);

% Calculate true anomaly from eccentric anomaly
cosnu = (e*cos(E)) / (e*cos(E)-1);
sinnu = ((a*sqrt(1-e*e)) / (a*(1-e*cos(E))))*sin(E);
nu = atan2(sinnu,cosnu);
if (nu<0), nu=nu+2*pi; end

% Return the orbital elements in a 1x6 matrix
oe = [a e i Om om nu];

```

## A.4 Summary

This appendix is intended to give a brief overview of orbital mechanics so that the problems involving two-line element sets can be completed. Here is a summary of some basic formulas and constants.

### Two-body problem relationships

$$\begin{array}{ll}
 r &= a(1 - e^2)/(1 + e \cos \nu) & p &= a(1 - e^2) = h^2/\mu \\
 \vec{h} &= \vec{r} \times \vec{v} & h &= rv \cos \phi \\
 TP &= 2\pi\sqrt{a^3/\mu} & \mathcal{E} &= v^2/2 - \mu/r \\
 v_c &= \sqrt{\mu/r} & v_{\text{esc}} &= \sqrt{2\mu/r} \\
 \vec{e} &= \frac{1}{\mu}[(v^2 - \mu/r)\vec{r} - (\vec{r} \cdot \vec{v})\vec{v}] & e &= (r_a - r_p)/(r_a + r_p) \\
 \vec{n} &= \hat{\mathbf{K}} \times \vec{h}
 \end{array}$$



**Orbital elements**

$a$	semimajor axis	$b = \sqrt{ap}$	semiminor axis
$e$	eccentricity		
$i$	inclination	$h \cos i = \vec{h} \cdot \hat{K}$	
$\Omega$	RAAN	$n \cos \Omega = \vec{n} \cdot \hat{I}$	check $n_j$
$\omega$	argument of periapsis	$ne \cos \omega = \vec{n} \cdot \vec{e}$	check $e_k$
$\nu$	true anomaly	$er \cos \nu = \vec{e} \cdot \vec{r}$	check $\vec{r} \cdot \vec{v}$

**Perifocal frame**

$$\vec{v} = \sqrt{\frac{\mu}{p}} \left[ -\sin \nu \hat{P} + (e + \cos \nu) \hat{Q} \right]$$

**Rotation matrices**

PQW to IJK

$$\mathbf{R} = \begin{bmatrix} \cos \Omega \cos \omega - \sin \Omega \sin \omega \cos i & -\cos \Omega \sin \omega - \sin \Omega \cos \omega \cos i & \sin \Omega \sin i \\ \sin \Omega \cos \omega + \cos \Omega \sin \omega \cos i & -\sin \Omega \sin \omega + \cos \Omega \cos \omega \cos i & -\cos \Omega \sin i \\ \sin \omega \sin i & \cos \omega \sin i & \cos i \end{bmatrix}$$

**Constants and other parameters**

Heliocentric	Geocentric
1 AU (or 1 DU) = $1.4959965 \times 10^8$ km	1 DU ( $R_{\oplus}$ ) = 6378.145 km
1 TU = $5.0226757 \times 10^6$ sec	1 TU = 806.8118744 sec
$\mu_{\odot} = 1.3271544 \times 10^{11}$ km <sup>3</sup> /s <sup>2</sup>	$\mu_{\oplus} = 3.986012 \times 10^5$ km <sup>3</sup> /s <sup>2</sup>

## A.5 References and further reading

One good reference for this material also has the advantage of being the least expensive. Bate, Mueller, and White<sup>1</sup> is an affordable Dover paperback that includes most of the basic information about orbits that you'll need to know. Another inexpensive book is Thomson's *Introduction to Space Dynamics*,<sup>2</sup> which covers orbital dynamics as well as attitude dynamics, rocket dynamics, and some advanced topics from analytical dynamics. Other recent texts include Prussing and Conway,<sup>3</sup> Vallado,<sup>4</sup> and Wiesel.<sup>5</sup>

## Bibliography

- [1] Roger R. Bate, Donald D. Mueller, and Jerry E. White. *Fundamentals of Astrodynamics*. Dover, New York, 1971.
- [2] W. T. Thomson. *Introduction to Space Dynamics*. Dover, New York, 1986.
- [3] John E. Prussing and Bruce A. Conway. *Orbital Mechanics*. Oxford University Press, Oxford, 1993.

- [4] David A. Vallado. *Fundamentals of Astrodynamics and Applications*. McGraw-Hill, New York, 1997.
- [5] William E. Wiesel. *Spaceflight Dynamics*. McGraw-Hill, New York, second edition, 1997.

## A.6 Exercises

1. The Earth rotates about its axis with an angular velocity of about  $2\pi$  radians per day (or  $360^\circ/\text{day}$ ). Approximately how fast does the Sun rotate about its axis?
2. Approximately how much of the Solar System's mass does the Earth contribute?
3. What are the approximate minimum and maximum communication delays between the Earth and Mars?
4. What are the Apollo asteroids?
5. What are the latitude and longitude of Blacksburg, Virginia. Give your answer in degrees, minutes, and seconds to the nearest second, and in radians to an equivalent accuracy. Also, what is the distance on the surface of the earth equivalent to 1 second of longitude error at the latitude of Blacksburg? Give your answer in kilometers. You may assume a spherical earth with a radius of 6378 km.
6. Describe the main features of the Earth's orbit about the Sun and its rotational motion.
7. Imagine that you are standing at the front center of our classroom (which you can assume to be the "origin"), facing the rear of the room (which you can assume to be North). What are the azimuth and elevation of a point 100 meters to the North, 50 meters to the East, and 10 meters above the floor of the classroom?
8. What time does this class meet in UT1?
9. Given two vectors,  $\vec{r}_1 = 3\hat{\mathbf{I}} + 4\hat{\mathbf{J}} + 5\hat{\mathbf{K}}$  and  $\vec{r}_2 = 4\hat{\mathbf{I}} - 5\hat{\mathbf{J}} - 2\hat{\mathbf{K}}$ , compute the following quantities. Use a reasonably accurate sketch to illustrate the geometry of the required calculations.
  - a)  $\vec{r}_1 + \vec{r}_2$                       b)  $\vec{r}_1 - \vec{r}_2$
  - c)  $\vec{r}_1 \cdot \vec{r}_2$                         d)  $\vec{r}_1 \times \vec{r}_2$
  - e) a unit vector perpendicular to both  $\vec{r}_1$  and  $\vec{r}_2$

10. Make a plot of the gravitational acceleration magnitude *vs.* radius for a satellite in Earth orbit. The abscissa should be the orbital radius in units of Earth radii (use 6378 km) from 1 to 10, and the ordinate should be acceleration in units of *gees* (1 “gee” = 9.81 m/s<sup>2</sup>). Your plot should be labeled and should include labeled markers to indicate space station altitude, GlobalStar altitude, GPS altitude and geostationary altitude.

Discuss the significance of the term “zero-gravity” commonly used to describe the near-Earth orbital environment.

11. A 100 kg spacecraft is in a circular orbit ( $e = 0$ ) at 400 km altitude. What are the orbital period [s], total mechanical energy [W], specific mechanical energy [W/kg], and specific angular momentum [km<sup>2</sup>/s<sup>2</sup>]?

What is the instantaneous access area of the satellite [km<sup>2</sup>]?

If the spacecraft is in an equatorial orbit ( $i = 0$ ) and the position vector is  $\vec{r} = 6778 \hat{\mathbf{J}}$  [km], then what is the velocity vector [km/s]? Make a supporting sketch.

12. A spacecraft is in a geostationary orbit. Compute the acceleration magnitude [m/s<sup>2</sup>] due to the gravitational attraction of Earth, Moon, Mars, Sun, and Jupiter. You will need to make some assumptions about the relative positions of the spacecraft and the primary bodies. You should choose the closest approach (worst case) and you can assume circular orbits for all the planets.

Discuss the significance of the two-body assumptions used in developing Keplerian motion.

## A.7 Problems

1. An orbit with parameter  $p = 2$  DU connects two position vectors:

$$\vec{r}_1 = 3 \hat{\mathbf{I}} \text{ DU} \qquad \vec{r}_2 = 4 \hat{\mathbf{J}} \text{ DU}$$

Assuming a “long-way” trajectory, how long does a spacecraft take to get from  $\vec{r}_1$  to  $\vec{r}_2$ ?

$$t = \underline{\hspace{10cm}} \quad [\text{TU}]$$

2. A mission to Mars begins in a parking orbit at 500 km altitude above the Earth. Assume the parking orbit is in the plane of the ecliptic, and that Earth and Mars are in coplanar circular orbits.

**(a) 5 points.** Determine the angular separation required between Earth and Mars at the time of initiating a heliocentric Hohmann transfer.

$$\theta = \text{_____} \quad [\text{rad}]$$

**(b) 5 points.** Determine the  $\Delta v$  required to get from the same heliocentric circular orbit that the Earth is on to the heliocentric orbit of Mars, using a Hohmann transfer.

$$\Delta v = \text{_____} \quad [\text{km/s}]$$

**(c) 10 points.** Determine the actual  $\Delta v$  required to get from the Earth parking orbit to the heliocentric transfer orbit.

$$\Delta v = \text{_____} \quad [\text{km/s}]$$

**(d) 5 points.** Assuming that the maneuver takes place on the first day of spring, determine the sidereal time of the position at which the actual  $\Delta v$  takes place.

$$\theta_{\Delta v} = \text{_____} \quad [\text{degrees}]$$

3. The Mars Global Surveyor used an aerobraking maneuver to reduce the energy in the orbit until it reached a low Mars orbit suitable for its surface-mapping mission (400 km altitude circular orbit). Useful information about Mars is in your textbook on pp. 360–361.

The aerobraking maneuver is equivalent to applying a  $\Delta v$  at each periapsis to reduce the periapsis speed, thereby reducing the energy and lowering the apoapsis altitude, without appreciably affecting periapsis altitude.

**(a) 20 points.** Assume that the periapsis altitude is 110 km. If the apoapsis altitude in one orbit is 20,000 km, and the apoapsis altitude in the next orbit is 19,500 km, determine what the *effective*  $\Delta v$  at periapsis was (in km/s).

$$\Delta v = \text{_____} [\text{km/s}]$$

**(b) 5 points.** Assuming that the effective  $\Delta v$  is the same at each periapsis passage, determine the number of orbits it will take to lower the apoapsis to  $\leq 400$  km.

$$N = \text{_____} [\text{orbits}]$$