Chapter 5

Rigid Body Dynamics

In Chapter 3 we developed the equations of motion for attitude kinematics. The main results of that chapter involve the description of attitude motion using attitude variables, such as rotation matrices, Euler angles, Euler axis/angle sets, or quaternions. Knowing the values of any of these variables at a particular instant of time allows the attitude analyst to visualize the orientation of one reference frame with respect to another. Examples of mission-oriented analysis include determining the location of a target on the Earth, the direction to the Sun, and the local direction of the Earth's magnetic field.

We also developed in Chapter 3 the differential equations describing how these variables depend on the angular velocity of the reference frame. We did not investigate how the angular velocity, $\vec{\omega}$, is determined, and that is the subject of this chapter.

In orbital dynamics, we typically model a spacecraft as a point mass or as a solid, rigid sphere, ignoring the rotational motion of the spacecraft about its mass center. This model is an excellent simplification for studying the motion of a satellite in its orbit — it works well for describing the motion of Europa about Jupiter, of Jupiter about the Sun, and of artificial satellites about the Earth. For artificial satellites, however, we usually need to know the orientation of the satellite with some degree of accuracy, so we need a more detailed model than the point mass model. A real spacecraft, of course, is a complicated piece of machinery with moving parts, flexible appendages, and partially filled fluid containers. However, even with all this complexity, a useful model is the *rigid body*, and that is the model we develop in this chapter.

We begin with a rigorous development of the equations of motion for a rigid body, both translational and rotational. Because the torques applied to spacecraft are typically quite small, we find it useful to consider the special case of torque-free motion. As noted above, all spacecraft include non-rigid elements. The most notable effect of non-rigidity is the effect of energy dissipation, which has important implications for spacecraft design. We also present analysis of two important examples of rigid body dynamics: the Lagrange top, and the unbalanced rotor.

5.1 Newton's Second Law

Newton's laws are applicable to a particle, or point mass, m. As noted previously, Newton's Second Law may be expressed as one second-order vector differential equation

$$m\ddot{\vec{\mathbf{r}}} = \vec{\mathbf{f}} \tag{5.1}$$

or as two first-order vector differential equations

$$\begin{aligned} \dot{\vec{\mathbf{p}}} &= \vec{\mathbf{f}} && (5.2)\\ \dot{\vec{\mathbf{r}}} &= \vec{\mathbf{p}}/m && (5.3) \end{aligned}$$

Here $\vec{\mathbf{r}}$ is the position vector of the particle with respect to an inertially fixed point, $\vec{\mathbf{p}}$ is the linear momentum of the particle, and $\vec{\mathbf{f}}$ is the total applied force acting on the particle. Either of these equations describe the dynamics of a particle. Integration of these equations requires selection of specific coordinates or position variables. For example, with respect to an inertial reference frame, we might represent $\vec{\mathbf{r}}$ and $\vec{\mathbf{f}}$ by

$$\begin{aligned} \vec{\mathbf{r}} &= r_1\hat{\mathbf{i}}_1 + r_2\hat{\mathbf{i}}_2 + r_3\hat{\mathbf{i}}_3 = \mathbf{r}^{\mathsf{T}}\left\{\hat{\mathbf{i}}\right\} && (5.4)\\ \vec{\mathbf{f}} &= f_1\hat{\mathbf{i}}_1 + f_2\hat{\mathbf{i}}_2 + f_3\hat{\mathbf{i}}_3 = \mathbf{f}^{\mathsf{T}}\left\{\hat{\mathbf{i}}\right\} && (5.5) \end{aligned}$$

In this case, the scalar second-order equations of motion are

$$m\ddot{\mathbf{r}} = \mathbf{f} \quad \Leftrightarrow \quad \begin{aligned} m\ddot{r}_1 &= f_1 && (5.6)\\ m\ddot{r}_2 &= f_2 && (5.7)\\ m\ddot{r}_3 &= f_3 && (5.8) \end{aligned}$$

and the scalar first-order equations of motion are

$$\dot{\mathbf{p}} = \mathbf{f} \quad \Leftrightarrow \quad \begin{aligned} \dot{p}_1 &= f_1 && (5.9)\\ \dot{p}_2 &= f_2 && (5.10)\\ \dot{p}_3 &= f_3 && (5.11) \end{aligned}$$

$$\dot{\mathbf{r}} = \mathbf{p}/m \quad \Leftrightarrow \quad \begin{aligned} \dot{r}_1 &= p_1/m && (5.12)\\ \dot{r}_2 &= p_2/m && (5.13)\\ \dot{r}_3 &= p_3/m && (5.14) \end{aligned}$$

One approach to developing the equations of motion of a spacecraft is to model the spacecraft as a system of point masses, m_i, $i = 1, ..., n$ and then develop the set of $3n$ second-order equations of motion or the set of $6n$ first-order equations of motion. This approach involves summations of the mass particles. For example, the total mass is $m = \sum_{i=1}^{n} m_i$, and the total applied force is $\vec{\mathbf{f}} = \sum_{i=1}^{n} \vec{\mathbf{f}}_i$. However, if we are to model a complicated spacecraft as a system of particles, the number n will be large and the system of equations will be quite unwieldy. In the following section, we develop the rigid body as an approximate mathematical model for spacecraft attitude motion.

5.2 The Rigid Body Model

Here we introduce the concept of a continuum of differential mass elements, dm. A differential mass element is quite a different concept from a mass particle. Whereas a mass particle has zero volume but finite mass, a differential mass element has infinitesimal volume and infinitesimal mass. The mathematical approach to collecting mass particles is summation, whereas the approach to collecting differential mass elements is integration.

A common problem in studying rigid body motion is the problem of integrating over the volume of the body. This operation may have a scalar integrand, a vector integrand, or a tensor integrand, and all three of these types of integrands are important in rigid body motion. Integrating over the volume of the body is a triple integral operation, but we use the following shorthand notation:

$$\int_{\mathcal{B}} f(\vec{\mathbf{r}})\, dV = \int_{r_{1min}}^{r_{1max}} \int_{r_{2min}(r_1)}^{r_{2max}(r_1)} \int_{r_{3min}(r_1,r_2)}^{r_{3max}(r_1,r_2)} f(\vec{\mathbf{r}})\, dr_3\, dr_2\, dr_1 \tag{5.15}$$

Let us consider two examples, using a rectangular prism shape with a reference frame coinciding with one corner of the prism as shown in Fig. 5.1. The first example is the scalar integral corresponding to the total mass, and the second is the vector integral corresponding to the first moment of inertia.

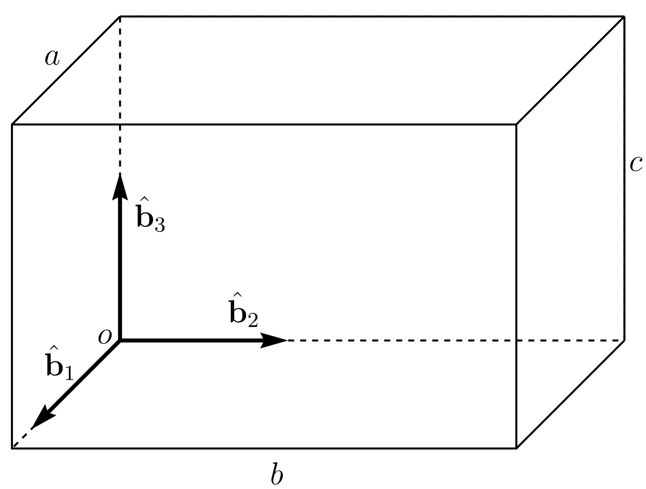

Figure 5.1: A rectangular prism for computing mass and first moment of inertia

Total mass. To compute the total mass of a rigid body, we need to know the density of the body, which we denote by μ. Units for μ are mass/length^3; for example, aluminum has density $\mu = 2.85 \times 10^3$ kg/m^3. In general, the density could vary from point to point within the body, or it could vary as a function of time. Thus, generally

$\mu = \mu(\vec{\mathbf{r}}, t)$. However, we generally assume that μ is constant, both spatially and temporally. Also, we usually use $dm = \mu\, dV$ as the variable of integration, even though m is not an independent variable. The meaning should always be clear. The total mass may then be expressed as

$$m = \int_{\mathcal{B}} \mu\, dV = \int_{\mathcal{B}} dm \tag{5.16}$$

Carrying out the indicated operations for the prism in Fig. 5.1, we obtain the expected result:

$$m = \int_0^a \int_0^b \int_0^c \mu\, dr_3\, dr_2\, dr_1 = \mu abc \tag{5.17}$$

Note that sometimes the total mass is called the *zeroth moment of inertia*, because it is a generalization of the first and second moments of inertia.

First moment of inertia. This vector quantity is defined as

$$\vec{\mathbf{c}}^o = \int_{\mathcal{B}} {}_o\vec{\mathbf{r}}\, dm \tag{5.18}$$

Clearly this quantity depends on the origin from which $\vec{\mathbf{r}}$ is measured, which motivates use of the *left* subscript o to denote that $\vec{\mathbf{r}}$ is measured *from* o, and the superscript o to denote that $\vec{\mathbf{c}}$ is the first moment *about* o. The matrix version of this equation is

$$\mathbf{c}^o = \int_{\mathcal{B}} {}_o\mathbf{r}\, dm \tag{5.19}$$

To compute the first moment, we must choose a reference frame, and the components depend on that choice of frame. For this example, we use the body frame and corner origin of Fig. 5.1. Thus, $\vec{\mathbf{c}} = \mathbf{c}_b{}^{\mathsf{T}}\{\hat{\mathbf{b}}\}$, $\vec{\mathbf{r}} = \mathbf{r}_b{}^{\mathsf{T}}\{\hat{\mathbf{b}}\}$, and

$$\mathbf{c}_b^o = \int_0^a \int_0^b \int_0^c \mu \left[r_1 \;\; r_2 \;\; r_3\right]^{\mathsf{T}} dr_3\, dr_2\, dr_1 = \frac{\mu}{2}\left[a^2bc \;\; ab^2c \;\; abc^2\right]^{\mathsf{T}} \tag{5.20}$$

Center of mass. We can use this result to determine the center of mass, c, which is defined as the point about which the first moment of inertia is zero; *i.e.* $\vec{\mathbf{c}}^c = \vec{\mathbf{0}}$. Mathematically, this definition leads to

$$\vec{\mathbf{c}}^c = \int_{\mathcal{B}} {}_c\vec{\mathbf{r}}\, dm = \vec{\mathbf{0}} \tag{5.21}$$

The position vector from o to any point may be written as

$${}_o\vec{\mathbf{r}} = {}^c_o\vec{\mathbf{r}} + {}_c\vec{\mathbf{r}} \tag{5.22}$$

where ${}^c_o\vec{\mathbf{r}}$ is the position vector of c with respect to o, which is what we want to find, and ${}_c\vec{\mathbf{r}}$ is the position vector from c to a point in the body. The matrix version of this expression is

$${}_o\mathbf{r} = {}^c_o\mathbf{r} + {}_c\mathbf{r} \tag{5.23}$$

where we omit the subscript b since the reference frame is clear. Substituting this expression into Eq. (5.19), we find

$$\mathbf{c}^o = \int_{\mathcal{B}} \left({}^c_o\mathbf{r} + {}_c\mathbf{r}\right) dm = \int_{\mathcal{B}} {}^c_o\mathbf{r}\, dm = {}^c_o\mathbf{r} \int_{\mathcal{B}} dm = m{}^c_o\mathbf{r} \tag{5.24}$$

Thus,

$$ {}^c_o\mathbf{r} = \frac{1}{m}\mathbf{c}^o = \frac{\mu}{2m}\left[a^2bc \;\; ab^2c \;\; abc^2\right]^\mathsf{T} \tag{5.25}$$

Using the expression for the mass m developed in Eq. (5.17), we obtain

$$ {}^c_o\mathbf{r} = \frac{1}{2}\left[a \;\; b \;\; c\right]^\mathsf{T} \tag{5.26}$$

which is obvious from the figure. However, this same procedure is applicable to more complicated geometric figures and to cases where μ is not constant.

We now continue with the development of equations of motion for a rigid body, beginning with the linear momentum equations, and then developing the angular momentum equations.

Linear momentum. The linear momentum of a rigid body is defined as

$$\vec{\mathbf{p}} = \int_{\mathcal{B}} \vec{\mathbf{v}}\, dm \tag{5.27}$$

where $\vec{\mathbf{v}}$ is the time derivative of the position vector $\vec{\mathbf{r}}$ of a differential mass element $dm = \mu\, dV$. This position vector must be measured from an inertial origin, and the derivative is taken with respect to inertial space. The position vector can be written as ${}^o_O\vec{\mathbf{r}} + {}_o\vec{\mathbf{r}}$, where ${}^o_O\vec{\mathbf{r}}^o$ is the position vector from the inertial origin *to* o and ${}_o\vec{\mathbf{r}}$ is the vector *from* o to a point in the body.

First we must ask, *What is the velocity* $\vec{\mathbf{v}}$ *of a differential mass element?* We describe the motion of the body frame by the velocity of the origin, ${}^o_O\vec{\mathbf{v}}$ and the angular velocity of the body frame with respect to the inertial frame, $\vec{\omega}^{bi}$. Then the velocity of a mass element is $\vec{\mathbf{v}} = {}^o_O\vec{\mathbf{v}} + \vec{\omega}^{bi} \times {}_o\vec{\mathbf{r}}$. Thus the linear momentum is

$$\vec{\mathbf{p}} = \int_{\mathcal{B}} \left({}^o_O\vec{\mathbf{v}} + \vec{\omega}^{bi} \times {}_o\vec{\mathbf{r}}\right) dm \tag{5.28}$$

The velocity of the origin and the angular velocity of the frame do not depend on the particular differential mass element, so they are constant with respect to the integration, so that the momentum becomes

$$\vec{\mathbf{p}} = m{}^o_O\vec{\mathbf{v}} + \vec{\omega}^{bi} \times \int_{\mathcal{B}} {}_o\vec{\mathbf{r}}\, dm \tag{5.29}$$

The integral on the right can be recognized as the first moment of inertia about the origin, so that

$$\vec{\mathbf{p}} = m{}^O_o\vec{\mathbf{v}} + \vec{\omega}^{bi} \times \vec{\mathbf{c}}^o \tag{5.30}$$

Thus, if o is the mass center c, then the linear momentum is simply $\vec{\mathbf{p}} = m{}^{c}_{O}\vec{\mathbf{v}}$.

Now, we can extend Newton's second law, $\dot{\vec{\mathbf{p}}} = \vec{\mathbf{f}}$ to the rigid body case as follows. The total force acting on a differential element of mass is

$$d\vec{\mathbf{f}} = \dot{\vec{\mathbf{v}}}\,dm \tag{5.31}$$

Integrating both sides of this expression over the body, we get

$$\int_{\mathcal{B}} d\vec{\mathbf{f}} = \int_{\mathcal{B}} \dot{\vec{\mathbf{v}}}\,dm = \dot{\vec{\mathbf{p}}} \tag{5.32}$$

Before integrating the differential forces, we first distinguish between the external applied forces, $d\vec{\mathbf{f}}_{ext}$, and the internal forces due to the other differential mass elements, $d\vec{\mathbf{f}}_{int}$. Thus

$$\int_{\mathcal{B}} \left(d\vec{\mathbf{f}}_{ext} + d\vec{\mathbf{f}}_{int}\right) = \dot{\vec{\mathbf{p}}} \tag{5.33}$$

By Newton's third law, the internal forces are equal and opposite, so that the integral $\int_{\mathcal{B}} d\vec{\mathbf{f}}_{int} = \vec{\mathbf{0}}$. Thus

$$\int_{\mathcal{B}} d\vec{\mathbf{f}}_{ext} = \dot{\vec{\mathbf{p}}} \tag{5.34}$$

We define the integral of the external forces by

$$\vec{\mathbf{f}} = \int_{\mathcal{B}} d\vec{\mathbf{f}}_{ext} \tag{5.35}$$

so that the translational equations of motion for a rigid body are

$$\vec{\mathbf{p}} = m{}^{o}_{O}\vec{\mathbf{v}} + \vec{\omega}^{bi} \times \vec{\mathbf{c}}^{o} \tag{5.36}$$

$$\vec{\mathbf{f}} = \dot{\vec{\mathbf{p}}} \tag{5.37}$$

These are vector equations of motion. To integrate them, we need to choose a reference frame and work with the components of the appropriate matrices in that frame. For example, we usually work with a body-fixed reference frame, $\mathcal{F}_b$, whose angular velocity with respect to inertial space is $\vec{\omega}^{bi}$. To differentiate a vector expressed in a rotating reference frame, we must recall the rule:

$$\frac{d}{dt}\left[\left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}}\mathbf{a}\right] = \left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}}\left[\dot{\mathbf{a}} + \omega^{\times}\mathbf{a}\right] \tag{5.38}$$

Applying this rule to Eqs. (5.36) and (5.37), we get

$$\mathbf{p} = m{}^{o}_{O}\mathbf{v} + \omega^{bi\times}\mathbf{c}^{o} \tag{5.39}$$

$$\mathbf{f} = \dot{\mathbf{p}} + \omega^{bi\times}\mathbf{p} \tag{5.40}$$

Note that the velocity term ${}^{o}_{O}\mathbf{v}$ may also involve using Eq. (5.38).

5.3 Euler's Law for Moment of Momentum

Now we develop the rotational equations of motion for a rigid body. One approach to developing these equations is to develop an expression for the angular momentum, and then apply Newton's second law. Another approach is to apply Euler's Law

$$\dot{\vec{\mathbf{h}}} = \vec{\mathbf{g}} \tag{5.41}$$

where $\vec{\mathbf{h}}$ is the angular momentum about the mass center, and $\vec{\mathbf{g}}$ is the net applied moment about the mass center. One advantage of the approach based on Newton's second law is that it also yields valid expressions when the moments are taken about some point other than the mass center. This generality is frequently useful in spacecraft dynamics problems where there are flexible components resulting in a movable mass center.

As we develop in the remainder of this section, the angular momentum can be expressed in $\mathcal{F}_b$ as $\mathbf{h} = \mathbf{I}\omega$, where $\mathbf{I}$ is the moment of inertia matrix, and ω is the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$ (*i.e.*, $\omega = \omega^{bi}$). Since $\mathbf{I}$ is constant in the body frame, we want to express the equations of motion in that frame (so that we do not have to deal with $\dot{\mathbf{I}}$ terms), and the body frame is rotation. Recalling Eq. (5.38), then, Eq. (5.41) in body-frame components is written:

$$\dot{\mathbf{h}} + \omega^{\times}\mathbf{h} = \mathbf{g} \tag{5.42}$$

Since $\mathbf{h} = \mathbf{I}\omega$, and since $\dot{\mathbf{I}} = \mathbf{0}$ *in the body frame*, we can rewrite this equation as

$$\dot{\omega} = -\mathbf{I}^{-1}\omega^{\times}\mathbf{I}\omega + \mathbf{I}^{-1}\mathbf{g} \tag{5.43}$$

If a *principal* reference frame is used, then $\mathbf{I}$ is diagonal; *i.e.*, $\mathbf{I} = \mathrm{diag}(I_1, I_2, I_3)$. This matrix equation can then be expanded to obtain the "standard" version of Euler's equations for the rotational motion of a rigid body:

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \frac{g_1}{I_1} \tag{5.44}$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_1\omega_3 + \frac{g_2}{I_2} \tag{5.45}$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \frac{g_3}{I_3} \tag{5.46}$$

Therefore, to determine the motion of a rigid body, we need to simultaneously integrate Eqs. (5.43) and one of the sets of kinematics differential equations developed in Chapter 3 and summarized in § 3.2.2.

In the remainder of this chapter we further develop these equations for the kinetics of a rigid body.

Angular Momentum and Moment of Momentum[*]*.* There are actually at least three quantities that are commonly called *angular momentum* in the literature. They are all related, but have slightly different definitions. We define two different vector quantities: *angular momentum* and *momentum of momentum.*

The *moment of momentum*, $\vec{\mathbf{H}}^o$, of a rigid body about a point o is defined as

$$\vec{\mathbf{H}}^o = \int_{\mathcal{B}} {}_o\vec{\mathbf{r}} \times \vec{\mathbf{v}}\, dm \tag{5.47}$$

where ${}_o\vec{\mathbf{r}}$ is the position vector of a differential mass element from point o, and $\vec{\mathbf{v}}$ is the velocity of the mass element with respect to inertial space. Note that the quantity $\vec{\mathbf{v}}\, dm$ is in fact the momentum of the differential mass element, and by taking the cross product with the moment arm ${}_o\vec{\mathbf{r}}$, we are forming the moment of momentum about o.

The *angular momentum*, $\vec{\mathbf{h}}^o$, of a rigid body about a point o is defined as

$$\vec{\mathbf{h}}^o = \int_{\mathcal{B}} {}_o\vec{\mathbf{r}} \times {}_o\dot{\vec{\mathbf{r}}}\, dm \tag{5.48}$$

where the velocity $\vec{\mathbf{v}}$ appearing in $\vec{\mathbf{H}}^o$ has been replaced with the time derivative of the moment arm vector itself. Note that ${}_o\dot{\vec{\mathbf{r}}}\, dm$ is not the momentum of the differential mass element, but it is "momentum-like;" *i.e.*, it has the correct units.

We subsequently derive expressions for $\dot{\vec{\mathbf{H}}}^o$ and $\dot{\vec{\mathbf{h}}}^o$, but first we develop expressions for $\vec{\mathbf{H}}^o$ and $\vec{\mathbf{h}}^o$ involving the angular velocity of the body reference frame and the moment of inertia tensor.

Beginning with Eq. (5.47), we use the fact that the velocity of the differential mass element dm may be written as

$$\vec{\mathbf{v}} = {}^o_O\vec{\mathbf{v}} + \vec{\omega} \times {}_o\vec{\mathbf{r}} \tag{5.49}$$

where ${}^o_O\vec{\mathbf{v}}$ is the velocity of the point o, $\vec{\omega}$ is the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$, and ${}_o\vec{\mathbf{r}}$ is the position vector from o to the mass element. With this expression for $\vec{\mathbf{v}}$, Eq. (5.47) may be written as

$$\vec{\mathbf{H}}^o = \int_{\mathcal{B}} {}_o\vec{\mathbf{r}} \times \left[{}^o_O\vec{\mathbf{v}} + \vec{\omega} \times {}_o\vec{\mathbf{r}}\right]\, dm \tag{5.50}$$

The integrand may be expanded so that

$$\vec{\mathbf{H}}^o = \int_{\mathcal{B}} \left[{}_o\vec{\mathbf{r}} \times {}^o_O\vec{\mathbf{v}}\right]\, dm + \int_{\mathcal{B}} \left[{}_o\vec{\mathbf{r}} \times (\vec{\omega} \times {}_o\vec{\mathbf{r}})\right]\, dm \tag{5.51}$$

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[*]An excellent development of this material can be found in the out-of-print text by Peter Likins.[1] Professor Likins studied with the well-known dynamicist Thomas Kane at Stanford, and subsequently taught space mechanics at UCLA with D. Lewis Mingori and Robert Roberson. While at UCLA he also coined the term "dual-spin spacecraft." He went on to be Dean of Engineering at Columbia University and was President of Lehigh University for several years before becoming President of the University of Arizona.

The triple vector product in the second integral can be expanded using the vector identity

$$\vec{\mathbf{a}} \times \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) = \vec{\mathbf{b}}\left(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}\right) - \vec{\mathbf{c}}\left(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\right) \tag{5.52}$$

In this case $\vec{\mathbf{a}} = \vec{\mathbf{c}} = {}_o\vec{\mathbf{r}}$, and $\vec{\mathbf{b}} = \vec{\omega}$, so that the expression becomes

$$\int_{\mathcal{B}} \left[{}_o\vec{\mathbf{r}} \times \left(\vec{\omega} \times {}_o\vec{\mathbf{r}}\right)\right] dm = \int_{\mathcal{B}} \left[\vec{\omega}\left({}_o\vec{\mathbf{r}} \cdot {}_o\vec{\mathbf{r}}\right) - {}_o\vec{\mathbf{r}}\left(\vec{\omega} \cdot {}_o\vec{\mathbf{r}}\right)\right] dm \tag{5.53}$$

We can reverse the order of the second dot product, and *we can omit the parentheses in the integrand*. That is, we write

$$\vec{\omega}\left({}_o\vec{\mathbf{r}} \cdot {}^o\vec{\mathbf{r}}\right) = {}_o\vec{\mathbf{r}} \cdot {}_o\vec{\mathbf{r}}\,\vec{\omega} \tag{5.54}$$

and

$${}_o\vec{\mathbf{r}}\left(\vec{\omega} \cdot {}_o\vec{\mathbf{r}}\right) = {}_o\vec{\mathbf{r}}{}_o\vec{\mathbf{r}} \cdot \vec{\omega} \tag{5.55}$$

Whenever we write three vectors in the form $\vec{\mathbf{a}}\vec{\mathbf{b}}\cdot\vec{\mathbf{c}}$, parentheses are implied around the two vectors with the $\cdot$ between them. Thus, $\vec{\mathbf{a}}\vec{\mathbf{b}} \cdot \vec{\mathbf{c}} = \vec{\mathbf{a}}\left(\vec{\mathbf{b}} \cdot \vec{\mathbf{c}}\right)$, and $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\vec{\mathbf{c}} = \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\right)\vec{\mathbf{c}}$. There is no ambiguity in this notation, and we discuss it in more detail later.

Now we introduce a mathematical object, $\vec{\mathbf{1}}$, defined by the following properties:

$$\vec{\mathbf{1}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{1}} = \vec{\mathbf{v}} \tag{5.56}$$

for any vector $\vec{\mathbf{v}}$. Like the $\vec{\mathbf{a}}\vec{\mathbf{b}} \cdot \vec{\mathbf{c}}$ notation, this notation is simple, unambiguous, and convenient for working with angular momentum and moment of momentum. We use this operator in the expression in Eq. (5.54):

$${}_o\vec{\mathbf{r}} \cdot {}_o\vec{\mathbf{r}}\,\vec{\omega} = {}_o\vec{\mathbf{r}} \cdot {}_o\vec{\mathbf{r}}\,\vec{\mathbf{1}} \cdot \vec{\omega} = {}_o r^2\vec{\mathbf{1}} \cdot \vec{\omega} \tag{5.57}$$

We can now write the moment of momentum vector as

$$\vec{\mathbf{H}}^o = \int_{\mathcal{B}} \left[{}_o\vec{\mathbf{r}} \times {}^o_O\vec{\mathbf{v}}\right] dm + \int_{\mathcal{B}} \left[({}_o r)^2\vec{\mathbf{1}} \cdot \vec{\omega} - {}_o\vec{\mathbf{r}}{}_o\vec{\mathbf{r}} \cdot \vec{\omega}\right] dm \tag{5.58}$$

With respect to the variables of integration, the two vectors ${}^o_O\vec{\mathbf{v}}^o$ and $\vec{\omega}$ are both constants (note that they are not constants with respect to time, though). Thus these two terms can be brought outside of the integral sign to get

$$\vec{\mathbf{H}}^o = \int_{\mathcal{B}} {}_o\vec{\mathbf{r}}\, dm \times {}^o_O\vec{\mathbf{v}} + \int_{\mathcal{B}} \left[({}_o r)^2\vec{\mathbf{1}} - {}_o\vec{\mathbf{r}}{}_o\vec{\mathbf{r}}\right] dm \cdot \vec{\omega} \tag{5.59}$$

The first integral is the first moment of inertia about o, and the second integral is a mathematical object called the second moment of inertia about o. We denote this object by

$$\vec{\mathbf{I}}^o = \int_{\mathcal{B}} \left[{}_o r^2\vec{\mathbf{1}} - {}_o\vec{\mathbf{r}}{}_o\vec{\mathbf{r}}\right] dm \tag{5.60}$$

We develop the moment of inertia in more detail below. For now, we simply note that it is a constant property of a rigid body.

Now we can write the moment of momentum as

$$\vec{\mathbf{H}}^o = \vec{\mathbf{c}}^o \times {}^o_O\vec{\mathbf{v}} + \vec{\mathbf{I}}^o \cdot \vec{\omega} \tag{5.61}$$

Note that if we choose the mass center of the rigid body as the origin, *i.e.*, $o \equiv c$, then $\vec{\mathbf{c}}^c = \vec{\mathbf{0}}$, and $\vec{\mathbf{H}}^c = \vec{\mathbf{I}}^c \cdot \vec{\omega}$.

Similar operations lead to the following expression for the angular momentum:

$$\vec{\mathbf{h}}^o = \vec{\mathbf{I}}^o \cdot \vec{\omega} \tag{5.62}$$

Note that when $o \equiv c$, the distinction between angular momentum and moment of momentum vanishes. This equivalence is part of the reason that these two expressions are sometimes confused in the literature. The careful student of spacecraft dynamics always clearly states which vector is being used.

In what follows, we work with the special case where $o \equiv c$, and we use the angular momentum.

Second Moment of Inertia. The integral in Eq. (5.60) is an important quantity called the *second moment of inertia tensor.* If we express the vector ${}_c\vec{\mathbf{r}}$ in a body-fixed frame, say $\mathcal{F}_b$, then we can develop a matrix version of $\vec{\mathbf{I}}$. That is, we write ${}_c\vec{\mathbf{r}} = {}_c\mathbf{r}_b^T\{\vec{\mathbf{b}}\} = \{\vec{\mathbf{b}}\}^\mathsf{T}{}_c\mathbf{r}_b$, and substitute one or the other of these expressions into Eq. (5.60). This substitution leads to

$$\vec{\mathbf{I}}^c = \left\{\vec{\mathbf{b}}\right\}^\mathsf{T} \int_{\mathcal{B}} \left[{}_c\mathbf{r}_b^T\,{}_c\mathbf{r}_b\mathbf{1} - {}_c\mathbf{r}_b\,{}_c\mathbf{r}_b^T\right] dm \left\{\vec{\mathbf{b}}\right\} \tag{5.63}$$

The integral is the *moment of inertia matrix*, denoted

$$\mathbf{I}_b^c = \int_{\mathcal{B}} \left[{}_c\mathbf{r}_b^T\,{}_c\mathbf{r}_b\mathbf{1} - {}_c\mathbf{r}_b\,{}_c\mathbf{r}_b^T\right] dm \tag{5.64}$$

so that

$$\vec{\mathbf{I}}^c = \left\{\vec{\mathbf{b}}\right\}^\mathsf{T}\mathbf{I}_b^c\left\{\vec{\mathbf{b}}\right\} \tag{5.65}$$

or

$$\mathbf{I}_b^c = \left\{\vec{\mathbf{b}}\right\} \cdot \vec{\mathbf{I}}^c \cdot \left\{\vec{\mathbf{b}}\right\}^\mathsf{T} \tag{5.66}$$

Note that the moment of inertia matrix depends explicitly on the choice of point about which moments are taken (c here) and on the choice of body-fixed reference frame in which the inertia tensor is expressed. This frame-dependence is equivalent to the fact that the components of a vector depend on the choice of reference frame.

We frequently need to be able to change the reference point or to change the reference frame. The two tools we need to be able to do make these changes are the *parallel axis theorem* and the *change of vector basis theorem.*

Parallel Axis Theorem. Suppose we know the moment of inertia matrix about the mass center, c, expressed in a particular reference frame, and we want the moment of inertia matrix about a different point o, defined by $\mathbf{r}^{o/c}$. The former is denoted $\mathbf{I}^c$ and the latter is denoted $\mathbf{I}^o$ (we omit the subscript, since the reference frame is the same for both).

Returning to the definition of the inertia matrix in Eq. (5.64), we express the term ${}_c\mathbf{r}$ as

$$ {}_c\mathbf{r} = {}^o_c\mathbf{r} + {}_o\mathbf{r} \tag{5.67} $$

Upon substituting this expression into Eq. (5.64), we obtain

$$ \mathbf{I}^c = \int_{\mathcal{B}} \left[{}_o\mathbf{r}^{\mathsf{T}} {}_o\mathbf{r}\mathbf{1} - {}_o\mathbf{r}_o\mathbf{r}^T \right] dm - m\left[{}^c_o\mathbf{r}^T {}^c_o\mathbf{r}\mathbf{1} - {}^c_o\mathbf{r}^c_o\mathbf{r}^T \right] \tag{5.68} $$

The integral in this expression is recognized as the moment of inertia matrix about the point o (expressed in $\mathcal{F}_b$). Thus, the parallel axis theorem may be written as

$$ \mathbf{I}^o = \mathbf{I}^c + m\left[{}^c_o\mathbf{r}^T {}^c_o\mathbf{r}\mathbf{1} - {}^c_o\mathbf{r}^c_o\mathbf{r}^T \right] \tag{5.69} $$

This theorem may also be written as

$$ \mathbf{I}^o = \mathbf{I}^c - m{}^c_o\mathbf{r}^{\times} {}^c_o\mathbf{r}^{\times} \tag{5.70} $$

where we have made use of the result of the identity

$$ \mathbf{a}^{\times}\mathbf{a}^{\times} = \mathbf{1} - \mathbf{a}\mathbf{a}^{\times} \tag{5.71} $$

which is left as an exercise.

Change Of Vector Basis Theorem. The parallel axis theorem is used to change the reference point about which the moment of inertia is taken. The *change of vector basis theorem* is used to express the moment of inertia matrix in a different reference frame, but about the same reference point. For this development, it makes no difference whether the reference point is the mass center or an arbitrary point o, so we omit the superscript to simplify the notation.

Suppose we know the inertia matrix with respect to a certain body-fixed reference frame, $\mathcal{F}_b$, and we want to know it with respect to a different body-fixed reference frame, $\mathcal{F}_a$. Ordinarily, we know the relative attitude between these two reference frames, usually expressed as a rotation matrix, $\mathbf{R}^{ab}$, that takes vectors from $\mathcal{F}_b$ to $\mathcal{F}_a$. That is,

$$ \mathbf{v}_a = \mathbf{R}^{ab}\mathbf{v}_b \tag{5.72} $$

We want to develop an equivalent expression involving $\mathbf{R}^{ab}$, $\mathbf{I}_a$, and $\mathbf{I}_b$.

Again, we begin with Eq. (5.64), substituting $\mathbf{R}^{ba}\mathbf{r}_a$ for $\mathbf{r}_b$. Upon multiplying through and factoring out the rotation matrices (which do not depend on the independent variable of integration), we find that

$$ \mathbf{I}_b = \mathbf{R}^{ba} \int_{\mathcal{B}} \left[\mathbf{r}_a^T\mathbf{r}_a\mathbf{1} - \mathbf{r}_a\mathbf{r}_a^T \right] dm\, \mathbf{R}^{ab} \tag{5.73} $$

The integral may be immediately recognized as $\mathbf{I}_a$, so that

$$\mathbf{I}_b = \mathbf{R}^{ba}\mathbf{I}_a\mathbf{R}^{ab} \tag{5.74}$$

Using the properties of rotation matrices, this identity may also be written as

$$\mathbf{I}_a = \mathbf{R}^{ab}\mathbf{I}_b\mathbf{R}^{ba} \tag{5.75}$$

Equations (5.74) and (5.75) are the tensor equivalents of Eq. (5.72). Note the ordering of the sub- and superscripts.

Principal Axes. One of the most important applications of the change of basis vector theorem is the use of a rigid body's *principal axes.* The principal axes comprise a body-fixed reference frame for which the moment of inertia matrix is diagonal; *i.e.*, the moment of inertia matrix takes the form

$$\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{5.76}$$

That such a reference frame always exists is a consequence of the fact that the moment of inertia tensor is *symmetric*; *i.e.*,

$$\vec{\mathbf{u}} \cdot \vec{\vec{\mathbf{I}}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\vec{\mathbf{I}}} \cdot \vec{\mathbf{u}} \tag{5.77}$$

for any vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. This fact means that *in any reference frame* the moment of inertia matrix is symmetric; *i.e.*, $\mathbf{I} = \mathbf{I}^{\mathsf{T}}$. A well-known theorem states that for any symmetric matrix, its eigenvalues are real, and its eigenvectors are orthogonal. For the moment of inertia matrix, these eigenvalues are the principal moments of inertia, and the eigenvectors provide the directions of the principal axes.

If we know $\mathbf{I}$ in a non-principal reference frame $\mathcal{F}_a$, and want to know $\mathbf{I}$ in principal reference frame, $\mathcal{F}_b$, we must find the rotation matrix $\mathbf{R}^{ba}$, which requires the establishment and solution of an eigenvalue problem. Beginning with Eq. (5.75), we post-multiply by $\mathbf{R}^{ab}$ to obtain

$$\mathbf{I}_a\mathbf{R}^{ab} = \mathbf{R}^{ab}\mathbf{I}_b \tag{5.78}$$

Recalling that the columns of $\mathbf{R}^{ab}$ are the components of the unit vectors of $\mathcal{F}_b$ expressed in $\mathcal{F}_a$, and that the principal moment of inertia matrix $\mathbf{I} = \mathrm{diag}[I_1, I_2, I_3]$, we can rewrite Eq. (5.78) as

$$\mathbf{I}_a \left[\mathbf{b}_{1a} \;\; \mathbf{b}_{2a} \;\; \mathbf{b}_{3a}\right] = \left[I_1\mathbf{b}_{1a} \;\; I_2\mathbf{b}_{2a} \;\; I_3\mathbf{b}_{3a}\right] \tag{5.79}$$

or as the three separate equations

$$\mathbf{I}_a\mathbf{b}_{1a} = I_1\mathbf{b}_{1a} \qquad \mathbf{I}_a\mathbf{b}_{2a} = I_2\mathbf{b}_{2a} \qquad \mathbf{I}_a\mathbf{b}_{3a} = I_3\mathbf{b}_{3a} \tag{5.80}$$

Each of these three equations is in the form

$$\mathbf{I}_a \mathbf{x} = \lambda \mathbf{x} \tag{5.81}$$

which is recognized as the eigenvalue problem, where solutions $(\lambda, \mathbf{x})$ are eigenvalues and eigenvectors of $\mathbf{I}_a$. In general, since $\mathbf{I}_a$ is a 3×3 matrix, there are three eigenvalues, $\lambda_i, i = 1, 2, 3$, each with an associated eigenvector, $\mathbf{x}_i$. Furthermore, since $\mathbf{I}_a$ is a real symmetric matrix, all its eigenvalues are real, and the associated eigenvectors are mutually orthogonal. The eigenvalues of $\mathbf{I}_a$ are the principal moments of inertia of the body, and the eigenvectors are the components of the principal axes expressed in $\mathcal{F}_a$.

5.4 Summary of Notation

There are several subscripts and superscripts used in this and preceding chapters. This table summarizes the meanings of these symbols.

Symbol	Meaning
$\vec{\mathbf{c}}^o$	first moment of inertia *relative to* or *about* o (a vector)
$\mathbf{c}_b^o$	first moment of inertia *relative to* or *about* o expressed in $\mathcal{F}_b$ (a 3×1 matrix)
${}_o\vec{\mathbf{r}}$	position vector from o to a point
${}_O^o\vec{\mathbf{r}}$	position vector from an inertial origin (O) to point o
${}_o^c\vec{\mathbf{r}}$	position vector from o to point c
${}_O^o\vec{\mathbf{v}}$	velocity vector of point o with respect to an inertial origin
$\vec{\mathbf{H}}^o$	moment of momentum of a rigid body about point o
$\vec{\mathbf{h}}^o$	angular momentum of a rigid body about point o
$\vec{\omega}^{bi}$	angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$
$\mathbf{R}^{bi}$	rotation matrix of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$
$\vec{\mathbf{1}}$	the identity tensor
$\mathbf{1}$	the identity matrix
$\vec{\mathbf{I}}^o$	the moment of inertia of a rigid body with respect to o (a tensor)
$\mathbf{I}_b^o$	the moment of inertia of a rigid body with respect to o expressed in $\mathcal{F}_b$ (a 3×3 matrix)

5.5 References and further reading

The textbook by Likins[1] has one of the best developments of rigid body dynamics that I have seen. Unfortunately the book is out of print. If you are interested in the history of Eq. (5.41), I highly recommend Truesdell's essay *Whence the Law of Moment of Momentum?*, which appears in Ref. 2. The advanced dynamics book by Meirovitch[3] develops rigid body equations of motion, and has two chapters of space-related applications. Hughes's[4] textbook gives a rigorous development of the equations of motion for a variety of spacecraft dynamics problems, beginning with a thorough treatment of the basics of rigid body dynamics. Synge and Griffith[5] is one of the few elementary mechanics texts that treats the elliptic function solution of the torque-free rigid body equations of motion. Goldstein's[6] advanced dynamics book is primarily aimed at the particle physics community; however, he provides an excellent treatment of the basics of rigid body kinematics and kinetics. Wiesel[7] develops rigid body dynamics from the point of view of spacecraft dynamics.

Bibliography

[1] Peter W. Likins. *Elements of Engineering Mechanics*. McGraw-Hill, New York, 1973.

[2] C. Truesdell. *Essays in the History of Mechanics*. Springer-Verlag, New York, 1968.

[3] Leonard Meirovitch. *Methods of Analytical Dynamics*. McGraw-Hill, New York, 1970.

[4] Peter C. Hughes. *Spacecraft Attitude Dynamics*. John Wiley & Sons, New York, 1986.

[5] John L. Synge and Byron A. Griffith. *Principles of Mechanics*. McGraw-Hill, New York, third edition, 1959.

[6] Herbert Goldstein. *Classical Mechanics*. Addison-Wesley, Reading, MA, second edition, 1980.

[7] William E. Wiesel. *Spaceflight Dynamics*. McGraw-Hill, New York, second edition, 1997.

5.6 Exercises

1. Develop the scalar first-order differential equations of motion for the case where $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$ are expressed in terms of spherical coordinates.

2. Develop the scalar first-order differential equations of motion for the case where $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$ are expressed in terms of cylindrical coordinates.

3. Use the first-order vector differential equations for a system of n point masses to develop two first-order vector differential equations for the system's mass center.

4. Explicitly carry out the integrations required to compute the total mass, first moment, and second moment of inertia matrix for the rectangular prism used in § 5.2.

5. Show that $\vec{\mathbf{b}}_1\vec{\mathbf{b}}_1 + \vec{\mathbf{b}}_2\vec{\mathbf{b}}_2 + \vec{\mathbf{b}}_3\vec{\mathbf{b}}_3$ satisfies the definition of $\vec{\mathbf{1}}$.

6. Prove Eq. (5.57).

7. Develop the equation $\vec{\mathbf{h}}^o = \vec{\mathbf{I}}^o \cdot \omega$.

8. Show that Eq. (5.71) is true.

9. Beginning with the definition of kinetic energy,

$$T = \frac{1}{2}\int_{\mathcal{B}} \vec{\mathbf{r}} \times \vec{\mathbf{v}}\, dm$$

show that

$$T = \frac{1}{2}\vec{\omega} \cdot \vec{\mathbf{I}} \cdot \vec{\omega} = \frac{1}{2}\omega^{\mathsf{T}}\mathbf{I}\omega$$

Be sure to distinguish between vectors and tensors, and their representations in reference frames.

5.7 Problems

1. For the body shown in Fig. 5.2, determine the volume, V, mass, m, mass center position relative to o, and the first, $\mathbf{c}^o$, and second, $\mathbf{I}^o$, moments of inertia about o. Use the parallel axis theorem to determine the second moments of inertia about the mass center. Develop these expressions analytically (*i.e.*, in terms of a, b, c, *etc.*), and then use the following numbers to compute numerical values: $a = 1$ m, $b = 3$ m, $c = 2$ m, $\mu = 10$ kg/m^3, $\rho = 0.4$ m.

2. A spacecraft is comprised of 3 rigid bodies: a *cylinder*, a *rod*, and a *panel*, as shown in Fig. 5.3. The $\mathcal{F}_{b'}$ frame shown is at the center of mass of the cylinder with the $\hat{\mathbf{b}}'_3$ axis parallel to the cylinder's symmetry axis, and the $\hat{\mathbf{b}}'_2$ axis parallel to the rod axis. The cylinder has a diameter of 1 m, a height of 2 m, and a mass density of 100 kg/m^3. The rod has a length of 2 m and a mass of 1 kg. The panel has length 2 m, width 0.5 m, and mass 4 kg. The panel is rotated about the rod axis through an angle $\theta = 87°$; *i.e.*, if $\theta = 0$, then the panel would be in the $\hat{\mathbf{b}}'_2\hat{\mathbf{b}}'_3$ plane, and if $\theta = 90°$, then the panel would be parallel to the $\hat{\mathbf{b}}'_1\hat{\mathbf{b}}'_2$ plane.

   (a) Find the mass center of the system, relative to the $\{\hat{\mathbf{b}}'\}$ frame shown in the sketch. Give your answer as

$$\vec{\mathbf{r}}^c = x\hat{\mathbf{b}}'_1 + y\hat{\mathbf{b}}'_2 + z\hat{\mathbf{b}}'_3$$

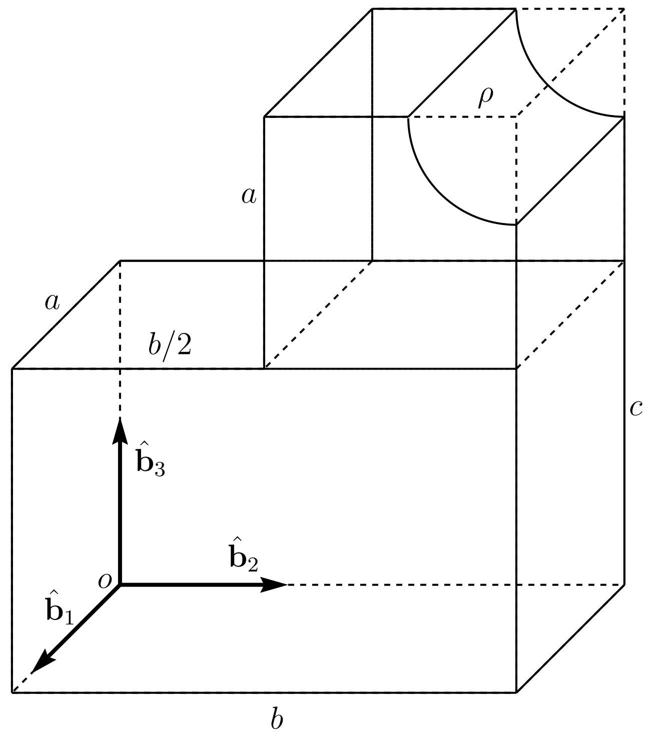

Figure 5.2: Composite rigid body

where x, y, and z are lengths given in meters with at least 5 significant digits.

(b) Find the moment of inertia matrix of the system, about the mass center of the cylinder, and relative to $\mathcal{F}_{b'}$.

(c) Find the moment of inertia matrix of the system about the mass center of the system, and relative to $\mathcal{F}_{b'}$.

(d) Find the principal axes and principal moments of inertia of the system. That is, find the rotation matrix $\mathbf{R}^{bb'}$ that takes vectors from $\mathcal{F}_{b'}$ to $\mathcal{F}_b$. You may find the Matlab function `[V,D] = eig(I)` useful in solving this problem.

(e) Suppose this spacecraft is in an orbit with the TLE given below. (What spacecraft is this really the TLE for?) Further suppose that at epoch, the attitude of $\mathcal{F}_b$ with respect to the orbital frame is given by the identity matrix (*i.e.*, the two reference frames are aligned). Determine the latitude and longitude of the subsatellite point, and of the "targets" pointed to by the $\hat{\mathbf{b}}_3'$ axis and the vector $\hat{\mathbf{n}}_p$ which is normal to the panel. How far apart are these two targets, measured in km as an arc on the surface of the Earth?

```
1 25544U 98067A   99077.49800042  .00020263  00000-0  26284-3 0  4071
```

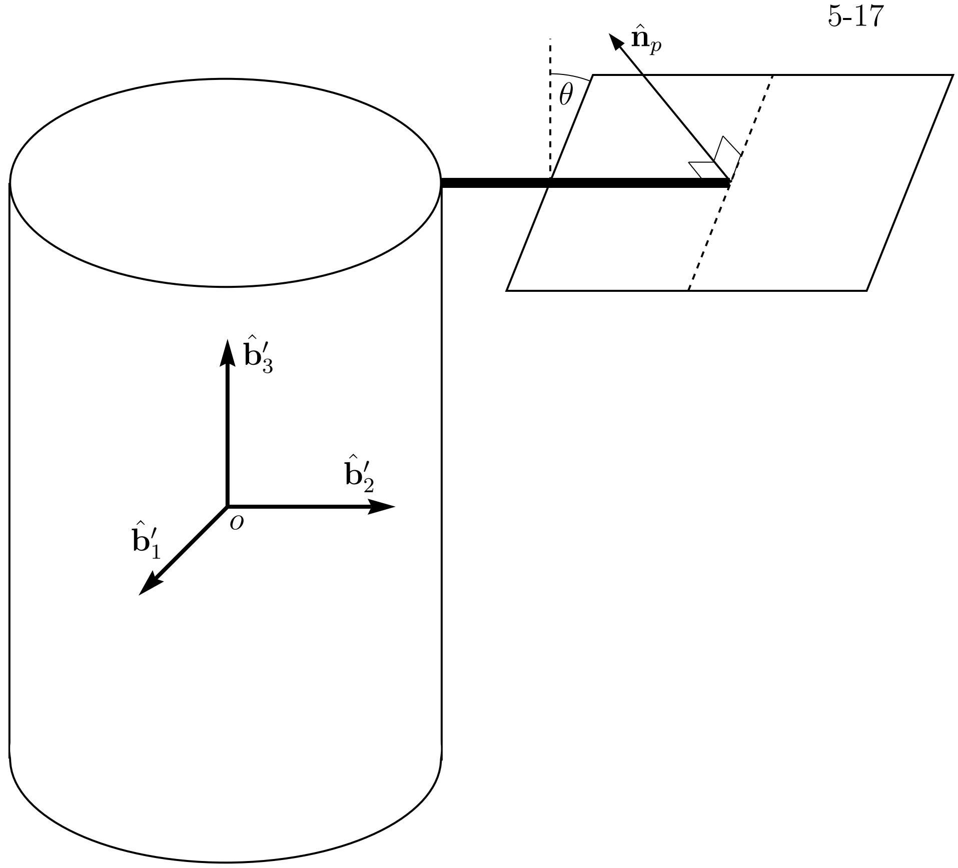

Figure 5.3: Cylinder+rod+panel

```
2 25544  51.5921 294.4693 0004426 235.8661 124.1905 15.58818692 18441
```