Centroidal angular momentum matrix

February 22, 2024

1 Derivation of $\boldsymbol{H}_A$

The angular momentum of a robot (or a group of rigid bodies) about its centre of mass (COM) denoted by $\boldsymbol{h}_R$ is defined as:

$$\boldsymbol{h}_R = \sum \boldsymbol{h}_i + \sum (\boldsymbol{\rho}_i \times (m_i \boldsymbol{v}_i)), \tag{1}$$

where $\boldsymbol{h}_i$ is the angular momentum of a rigid body about its own COM, $\boldsymbol{\rho}_i$ denotes the location of the body-COM w.r.t. the robot-COM, and $\boldsymbol{v}_i$ is the linear velocity of the body-COM. All the quantities in the above equation are expressed in the global frame of reference.

In the following manipulations, we show that it does not matter if we use the absolute linear velocities of individual body-COMs ($\boldsymbol{v}_i$) or relative velocities of body-COMs w.r.t. the robot-COM ($\boldsymbol{v}_{iG}$) for the computation of $\boldsymbol{h}_R$:

$$\boldsymbol{h}_R = \sum \boldsymbol{h}_i + \sum (\boldsymbol{\rho}_i \times (m_i(\boldsymbol{v}_G + \boldsymbol{v}_{iG}))), \tag{2}$$

$$\boldsymbol{h}_R = \sum \boldsymbol{h}_i + \boldsymbol{v}_G(\sum (m_i \boldsymbol{\rho}_i)) + \sum (\boldsymbol{\rho}_i \times (m_i \boldsymbol{v}_{iG})), \tag{3}$$

$$\boldsymbol{h}_R = \sum \boldsymbol{h}_i + \boldsymbol{0} + \sum (\boldsymbol{\rho}_i \times (m_i \boldsymbol{v}_{iG})). \tag{4}$$

The central term in Eq. (3) vanishes due to the very definition of the robot-COM. Hence, Eq. (1) and Eq. (4) are equivalent. We will use Eq. (1) for the computations. The first term on the RHS of Eq. (1) can be expanded as:

$$\boldsymbol{h}_i = {}^0_i\boldsymbol{R}({}^i\boldsymbol{I}\,{}^i\omega_i), \tag{5}$$

$$\boldsymbol{h}_i = {}^0_i\boldsymbol{R}({}^i\boldsymbol{I}\,{}^0_i\boldsymbol{R}^\top\,{}^0\omega_i), \tag{6}$$

$$\boldsymbol{h}_i = ({}^0_i\boldsymbol{R}\,{}^i\boldsymbol{I}\,{}^0_i\boldsymbol{R}^\top)\,{}^0\omega_i, \tag{7}$$

$$\boldsymbol{h}_i = ({}^0_i\boldsymbol{R}\,{}^i\boldsymbol{I}\,{}^0_i\boldsymbol{R}^\top)\boldsymbol{J}_{\omega_i}\dot{\boldsymbol{q}}, \tag{8}$$

where ${}^0_i\boldsymbol{R}$ transforms a vector from body frame to global frame. In the context of MuJoCo, ${}^0_i\boldsymbol{R} =$ `mjData.ximat`. The pre-superscript denotes the frame in which a quantity is expressed, i.e., ${}^i\omega_i$ is the angular velocity of a body in its own frame, while ${}^0\omega_i$ denotes the same quantity in the global frame. If the pre-superscript is not mentioned, it is assumed to be 0 (ground frame). The inertia matrix of a body in its own frame is denoted by ${}^i\boldsymbol{I}$. In the context of MuJoCo, it is the diagonal body inertia given by `mjModel.body_inertia`.

The second term on the RHS of Eq. (1) can be rewritten as:

$$\boldsymbol{\rho}_i \times (m_i \boldsymbol{v}_i) = m_i[\boldsymbol{\rho}_i \times]\boldsymbol{J}_{v_i}\dot{\boldsymbol{q}} \tag{9}$$

The angular velocity *Jacobian* $\boldsymbol{J}_{\omega_i}$ and linear velocity Jacobian $\boldsymbol{J}_{v_i}$ are given in MuJoCo by the `mj_jacBodyCom()` function. Finally $\boldsymbol{h}_R$ can be written as:

$$\boldsymbol{h}_R = \boldsymbol{H}_A \dot{\boldsymbol{q}}, \tag{10}$$

$$\boldsymbol{H}_A = \sum (({}^0_i\boldsymbol{R}\, {}^i\boldsymbol{I}\, {}^0_i\boldsymbol{R}^\top)\boldsymbol{J}_{\omega_i} + m_i[\boldsymbol{\rho}_i \times]\boldsymbol{J}_{v_i}) \tag{11}$$