A generalization of a reconciliation mechanism

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Here we focus on the even modulus, for simplicity we will assume that the modulus is a power of two (in our instantiation/implementation $q = 2^{32}$).

In the Section 1 we describe the mechanism for extracting 2 bits ([Pei14] describes the mechamism for extracting 1 bit) from a single ring element. In Section 2 we generalize the mechanism to extracting an arbitrary number of bits (as long as the noise is not too big). In Section 3 we show how these changes affect the correctness of key exchange by upper bounding the probability for two parties to get different keys.

1 Extracting 2 bits from a single element

Following the original idea of Peikert [Pei14] for element $v \in \mathbb{Z}_q$ we define functions

$$\lfloor \cdot \rceil_4 : \mathbb{Z}_q \to \{0,1,2,3\} \text{ s.t. } \lfloor v \rceil_4 = \left\lfloor \frac{4}{q} x \right\rceil$$

$$\langle \cdot \rangle_4 : \mathbb{Z}_q \to \{0,1\} \text{ s.t. } \langle v \rangle_4 = \left\lfloor \frac{8}{q} \right\rfloor \mod 2$$

We define disjoint intervals $I_0 := \{0, 1, \ldots, \frac{q}{4} - 1\}; I_k := I_0 + k \cdot \frac{q}{4}$, where $k \in \{1,2,3\}$.

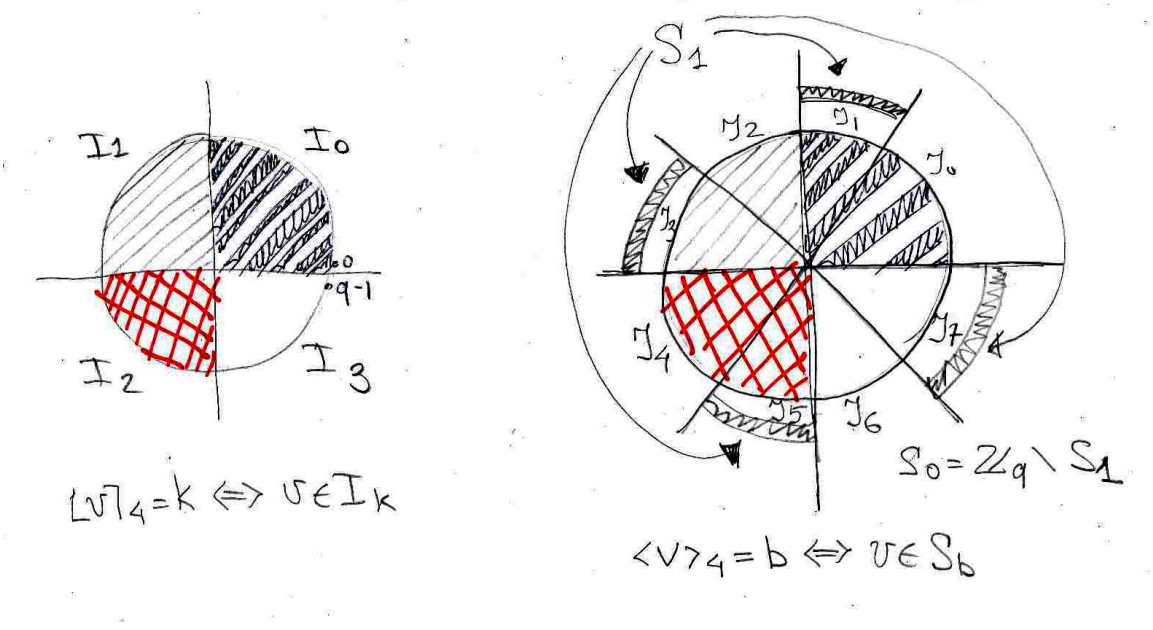
$$\lfloor v \rceil_4 = k \text{ iff } v \in I_k.$$

We define disjoint intervals which are twice smaller: $J_0 := \{0, 1, \ldots, \frac{q}{8} - 1\}; J_k := J_0 + k \cdot \frac{q}{8}$, where $k \in \{1, \ldots, 7\}$.

Lets define $S_0 = J_0 \cup J_2 \cup J_4 \cup J_6$, and $S_1 = J_1 \cup J_3 \cup J_5 \cup J_7 = \mathbb{Z}_q \backslash S_0$.

$$\langle v \rangle_4 = b \text{ iff } v \in S_b,$$

The partition is shown on the picture below:



Claim 1. *If $v \in \mathbb{Z}_q$ is uniformly random, then $\lfloor v \rceil_4$ is uniformly ramdom given $\langle v \rangle_4$.*

Proof. For any $b \in \{0,1\}$, if we condition on $\langle v \rangle_4 = b$, then v is uniform over $J_b \cup J_{2+b} \cup J_{4+b} \cup J_{6+b}$. If $v \in J_{b+2k}$, then $\lfloor v \rceil_4 = k$, so $\lfloor v \rceil_4$ is uniformly random given $\langle v \rangle_4$. □

Define the reconciliation function $rec : \mathbb{Z}_q \times \{0,1\} \to \mathbb{Z}_4$ as

$$rec(w, b) := k \text{ if } w \in J_{2k+b} + E$$

Claim 2. *If $w = v + e \mod q$ for $v \in \mathbb{Z}_q$ and $e \in E$, where $E = \left[-\frac{q}{16}, \frac{q}{16}\right) \cap \mathbb{Z}$, then $rec(w, \langle v \rangle_4) = \lfloor v \rceil_4$*

Proof. Let $b = \langle v \rangle_4 \in \{0,1\}$, so $v \in J_b \cup J_{2+b} \cup J_{4+b} \cup J_{6+b}$. Then $\lfloor v \rceil_4 = k$ if and only if $v \in I_k$. This holds if and only if $w \in J_{2k+b} + E$, because $J_{2k+b} + E$ are disjoint for different k's. □

2 Extracting B bits from a single element

The approach described above can be generalized to extracting B bits. We again define functions

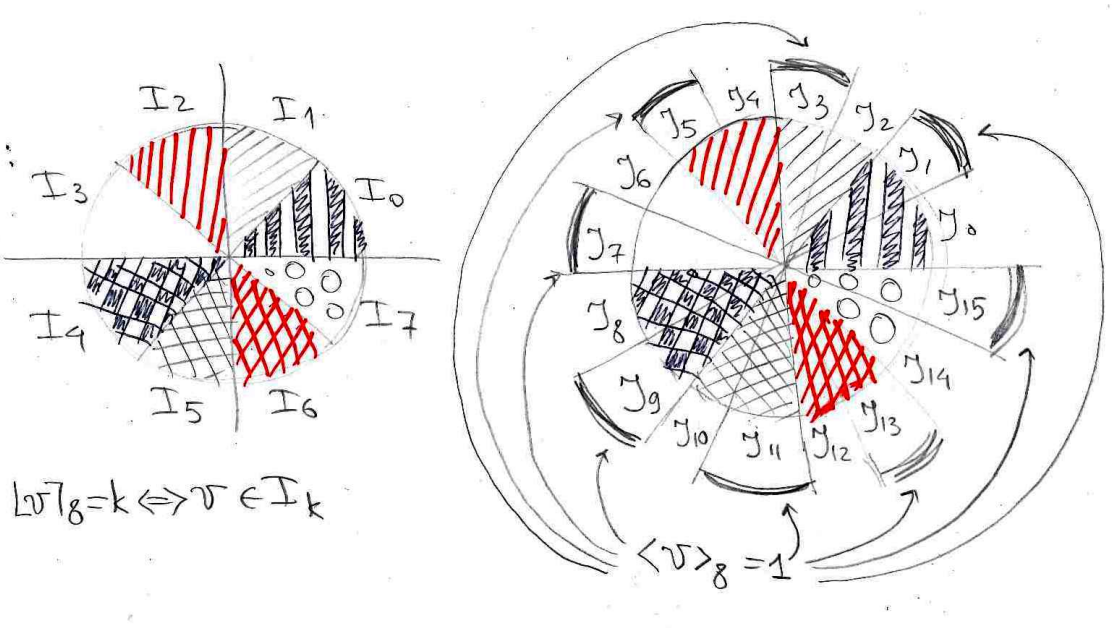
$$\lfloor \cdot \rceil_{2^B} : \mathbb{Z}_q \to \{0, \ldots, 2^B - 1\} \text{ s.t. } \lfloor v \rceil_{2^B} = \left\lfloor \frac{2^B}{q} x \right\rceil$$

$$\langle \cdot \rangle_{2^B} : \mathbb{Z}_q \to \{0,1\} \text{ s.t. } \langle v \rangle_{2^B} = \left\lfloor \frac{2^{B+1}}{q} \right\rfloor \mod 2$$

We again define 2^B intervals $\{I_k\}$ and 2^{B+1} intervals $\{J_{2k+b}\}$ such that

$$\lfloor v \rceil_4 = k \text{ iff } v \in I_k,$$
$$\langle v \rangle_4 = b \text{ iff } v \in J_{b+2k} \text{ for some } k.$$

The case of $B = 3$ is shown on the picture below:



Similar claims can be proven for $E = \left[-\frac{q}{2^{B+2}}, \frac{q}{2^{B+2}}\right) \cap \mathbb{Z}$.

3 Correctness for key exchange mechanism

For correctness we require that for all pairs of indices (i, j), $|(E'S + S'E + E'')_{ij}| < \frac{q}{2^{2+B}} - \frac{1}{2}$. Here $S, E \in \mathbb{Z}_q^{n \times \overline{n}}$, $S', E' \in \mathbb{Z}_q^{\overline{m} \times n}$, $E'' \in \mathbb{Z}_q^{\overline{m} \times \overline{n}}$ and they all are draw from a Gaussian distribution with standard deviation σ. For a fixed pair (i, j), we bound the probability of $|(E'S + S'E + E'')_{ij}| > \frac{q}{2^{2+B}} - \frac{1}{2}$ as follows. There are $2n + 1$ terms in the sum, if the (ij) element is greater than $\frac{q}{2^{2+B}} - \frac{1}{2}$, then at least one of the elements of the gaussian matrix must exceed $z = \sqrt{\frac{q}{2^{2+B}(2n+1)}}$ in absolute value. The probability of individual gaussian coefficient exceeding z in absolute value can be bounded by $e^{-(z/(\sqrt{2}\sigma))^2}$. The probability that one out of (4n + 1) exceeds z is bounded above by the sum $(4n + 1)e^{-(z/(\sqrt{2}\sigma))^2}$. Similarly, the probability that at least one coefficient of k_A and k_B disagree is clearly bounded above by the sum of all the p_{ij}, so we get

$$\Pr(k_A \neq k_B) \leq \sum_{i=0}^{\overline{n}} \sum_{j=0}^{\overline{n}} p_{ij} \leq \overline{n} \cdot \overline{m}(4n + 1)e^{-q/(2 \cdot 2^{2+B}(2n+1)\sigma^2)} \tag{1}$$

For our choice of parameters: $q = 2^{32}$, $n = 1024$, $\overline{n} \cdot \overline{m} = 128$, $\sigma = 3.2$, for different numbers of extracted bits the decay in this probability is shown in the table below:

$$\begin{aligned}(\overline{n} \cdot \overline{m})(4n + 1)e^{-q/(2 \cdot 2^{2+B}(2n+1)\sigma^2)} = \\ 2^7 \cdot 2^{12} \exp\left(-\frac{2^{32}}{2^{3+B} \cdot 2^{11} \cdot \sigma^2}\right) = \\ 2^{19} \exp(-2^{18-B}/\sigma^2) = \\ 10^{\log_{10} 2 \cdot 19 - \log_{10} e \cdot 2^{18-B}/\sigma^2}\end{aligned}$$

Bits extracted from one ring element (i)	Probability of failure
1	1e-5553
2	1e-2773
3	1e-1384
4	1e-689
5	1e-341
6	1e-167
7	1e-81
8	1e-37
9	1e-15
10	1e-5
11	1e1

Note: 1e-x stands for 10^{-x}.
The parameters of the scheme scale as $\frac{1}{\sqrt{B}}$.

References

[Pei14] Chris Peikert. Lattice cryptography for the internet. In *Post-Quantum Cryptography*, pages 197–219. Springer, 2014.