Welcome to Math for Machine Learning!
Linear Algebra, Session 6

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Today's Agenda

Abstract Algebra by David Dummit and Richard Foote

Quantum Computation and Quantum Information by Michael A. Nielsen and Isaac L. Chuang

The Tensor Product

Understand the difference between "the tensor product" and "Cartesian product" of vector spaces, as well as the dimensions of these spaces.

Linear Algebra for Quantum Computation

Intro to quantum computing geared for physics grad students

Today's Agenda

Abstract Algebra by David Dummit and Richard Foote

Quantum Computing: An Applied Approach by Jack Hidary

The Tensor Product

Understand the difference between "the tensor product" and "Cartesian product" of vector spaces, as well as the dimensions of these spaces.

Quantum Computing with code

3 chapters on the math of quantum computing.

The Tensor Product

Given finite-dimensional vector spaces V and W, what is their **tensor product**, $V \otimes W$? And what is the **dimension** of this space?

Last time

tensor product of vector spaces

$V \otimes V$

a basis for that space

basis

the dimension of that space

dimension

(today)

We'll recap these today.

What is the

vector space $V \otimes W$?

Tensor Product of Vector Spaces

Let V and W be vector spaces over a field F with bases $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$, respectively.

(last time)

The **tensor product** $V \otimes W$ is the vector space with bases $\mathbf{v}_i \otimes \mathbf{w}_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

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(last time)

The **tensor product** $V \otimes W$ is the vector space with bases $\mathbf{v}_i \otimes \mathbf{w}_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

$$\dim(V \otimes W) = \dim(V)\dim(W)$$

(new)

Example

Suppose $V=\mathbb{R}^3$ and $W=\mathbb{R}^2$.

Suppose V and W are Euclidean spaces of dimensions 3 and 2, respectively.

Let's choose the **standard basis vectors** on each. In the context of machine learning, these are sometimes called "one-hot encodings."

What is the basis for the vector space V⊗W?

$$V = \mathbb{R}^3$$

with bases:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$W = \mathbb{R}^2$$

with bases:

$$w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The **basis** for the tensor product $V \otimes W$ is the set of vectors of the form $\mathbf{v}_i \otimes \mathbf{w}_j$ where $i = 1,2,3$ and $j = 1,2$.

The **basis** for the tensor product $V \otimes W$ is the set of vectors of the form $\mathbf{v}_i \otimes \mathbf{w}_j$ where $i = 1,2,3$ and $j = 1,2$.

Imagine a "multiplication table" between the bases for V and W.

There are $3 \times 2 = 6$ possibilities.

$$\dim(V \otimes W) = \dim(V) \times \dim(W) = 6$$

	w_1	w_2
v_1	$v_1 \otimes w_1$	$v_1 \otimes w_2$
v_2	$v_2 \otimes w_1$	$v_2 \otimes w_2$
v_3	$v_3 \otimes w_1$	$v_3 \otimes w_2$

basis for $V \otimes W$

This may feel abstract, but we know what each $\mathbf{v}_i \otimes \mathbf{w}_j$ looks like.

They're "just" arrays.

	w_1	w_2
v_1	$v_1 \otimes w_1$	$v_1 \otimes w_2$
v_2	$v_2 \otimes w_1$	$v_2 \otimes w_2$
v_3	$v_3 \otimes w_1$	$v_3 \otimes w_2$

basis for $V \otimes W$

Example

To form the tensor product of two coordinate vectors, simply “multiply their entries in every way possible.”

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$v \otimes w$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 \\ 1 \cdot 5 \\ \\ \\ \\ \\ \end{bmatrix}$$

$$v \otimes w$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 \\ 1 \cdot 5 \\ 2 \cdot 4 \\ 2 \cdot 5 \\ \\ \end{bmatrix}$$

$$v \otimes w$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 \\ 1 \cdot 5 \\ 2 \cdot 4 \\ 2 \cdot 5 \\ 3 \cdot 4 \\ 3 \cdot 5 \end{bmatrix}$$

$$v \otimes w$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \otimes \begin{bmatrix}4\\5\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}4\\5\end{bmatrix} = \underbrace{\begin{bmatrix}1\cdot 4\\1\cdot 5\\2\cdot 4\\2\cdot 5\\3\cdot 4\\3\cdot 5\end{bmatrix} = \begin{bmatrix}4\\5\\8\\10\\12\\15\end{bmatrix}}_{\text{in } \mathbb{R}^3 \otimes \mathbb{R}^2 \cong \mathbb{R}^6}$$

$$v \otimes w$$

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -9 \\ -3 \\ 1 \\ 9 \\ 3 \\ -1 \\ -9 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 20 \\ 1 \\ 0 \\ 4 \end{bmatrix}$$

The bases for the vector space $V \otimes W$ can be understood in the same way.

	w_1	w_2
v_1	$v_1 \otimes w_1$	$v_1 \otimes w_2$
v_2	$v_2 \otimes w_1$	$v_2 \otimes w_2$
v_3	$v_3 \otimes w_1$	$v_3 \otimes w_2$

basis for $V \otimes W$

Suppose $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

<u>Then</u> ...

Suppose $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

<u>Then</u> ...

$$v_1 \otimes w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_1 \otimes w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 \otimes w_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 \otimes w_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_3 \otimes w_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 \otimes w_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So, what is the vector space $V \otimes W$?

$V = \mathbb{R}^3$ $W = \mathbb{R}^2$

Guess

If V and W are vector spaces, then their **tensor product** $V \otimes W$ consists of all vectors of the form

$$\mathbf{v} \otimes \mathbf{w}$$

where **v** is a vector in V, and **w** is a vector in W.

nope.

If V and W are vector spaces, then their **tensor product** $V \otimes W$ consists of all vectors of the form

$$\mathbf{v} \otimes \mathbf{w}$$

where $\mathbf{v}$ is a vector in V, and $\mathbf{w}$ is a vector in W.

The vector space $V \otimes W$ consists of all linear combinations of the 6 basis vectors $\mathbf{v}_i \otimes \mathbf{w}_j$.

$$7 v_1 \otimes w_1 + 3 v_3 \otimes w_2 = \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

$$-v_1 \otimes w_2 + 5 v_2 \otimes w_2 - 4 v_3 \otimes w_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 5 \\ -4 \\ 0 \end{bmatrix}$$

More generally, if V and W have bases $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$, respectively, then an arbitrary vector in $V \otimes W$ is a linear combination of the *nm* basis vectors $\mathbf{v}_i \otimes \mathbf{w}_j$.

$$g = \sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} g_{ij} \, v_i \otimes w_j \in V \otimes W$$

"tensor" from Mesch's session

Well, actually...

Technically speaking, the tensor product of coordinate vectors $\mathbf{v}_i \otimes \mathbf{w}_j$ is called their **outer product,** which is a matrix representing a linear map from W to V.

$$v \otimes w := vw^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 & 1 \cdot 5 \\ 2 \cdot 4 & 2 \cdot 5 \\ 3 \cdot 4 & 3 \cdot 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{bmatrix}$$

Well, actually...

But that matrix can be reshaped into an array, which is an easy way to think of $\mathbf{v}_i \otimes \mathbf{w}_j$ when first introduced to tensor products.

$$\begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{bmatrix} \xrightarrow{\text{reshape}} \begin{bmatrix} 4 \\ 5 \\ 8 \\ 10 \\ 12 \\ 15 \end{bmatrix}$$

a 3x2 matrix

a 6x1 vector

What about $V \otimes V \otimes V \otimes \cdots \otimes V$?

$$\dim(V \otimes V \otimes V)$$

$$\dim(V \otimes V \otimes V) = \dim(V \otimes V \otimes V)$$

$$\dim(V \otimes V \otimes V) = \dim(V \otimes V \otimes V)$$

$$= \dim(V \otimes V) \cdot \dim(V)$$

$$\dim(V \otimes V \otimes V) = \dim(V \otimes V \otimes V)$$

$$= \dim(V \otimes V) \cdot \dim(V)$$

$$= \dim(V) \cdot \dim(V) \cdot \dim(V)$$

$$= \dim(V)^3$$

$$\dim(V \otimes V \otimes V \otimes V)$$

$$\dim(V \otimes V \otimes V \otimes V) = \dim(V \otimes V \otimes V) \cdot \dim(V)$$

$$\dim(V \otimes V \otimes V \otimes V) = \dim(V \otimes V \otimes V) \cdot \dim(V)$$

$$= (\dim V)^3 \cdot \dim V$$

$$\dim(V \otimes V \otimes V \otimes V) = \dim(V \otimes V \otimes V) \cdot \dim(V)$$

$$= (\dim V)^3 \cdot \dim V$$

$$= (\dim V)^4$$

$$\dim\Big(\underbrace{V \otimes V \otimes \cdots \otimes V}_{N \text{ factors}}\Big) = \dim\left(V^{\otimes N}\right)$$

$$= \left(\dim V\right)^N$$

<u>example</u>: $V = \mathbb{C}^2 \Rightarrow \dim\left(V^{\otimes N}\right) = 2^N$

Tensor product vs. Cartesian product

Besides the tensor product, there’s another way to combine two vector spaces - the **Cartesian product**.

Tensor product vs. Cartesian product

Besides the tensor product, there's another way to combine two vector spaces - the **Cartesian product**.

Let V and W be vector spaces over a field F with bases $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$, respectively. The **Cartesian product** $V \times W$ is the vector space with basis given by:

$$\left\{ (v_i, 0), (0, w_j) \;\middle|\; 1 \leq i \leq n \text{ and } 1 \leq j \leq m \right\}$$

Tensor product vs. Cartesian product

Besides the tensor product, there's another way to combine two vector spaces - the **Cartesian product***.

So the **dimension** of the Cartesian product V×W is the *sum* of the dimensions of V and W.

$$\dim(V \times W) = \dim(V) + \dim(W)$$

*This is also called the **direct product** of V and W. If V and W are finite-dimensional, then V×W is the *same* as their **direct sum** V⊕W. But these two constructions are not the same if V and W are infinite dimensional.

Example

Suppose $V=\mathbb{R}^3$ and $W=\mathbb{R}^2$.

$$V = \mathbb{R}^3$$

with bases:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$W = \mathbb{R}^2$$

with bases:

$$w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$V = \mathbb{R}^3$$

with bases:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$W = \mathbb{R}^2$$

with bases:

$$w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$V \times W \cong \mathbb{R}^5$$

with bases

the 0 vector in $\mathbb{R}^2$

$$(v_1, 0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (v_2, 0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (v_3, 0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(0, w_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad (0, w_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

the 0 vector in $\mathbb{R}^3$

tensor product

$$v \otimes w = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 8 \\ 10 \\ 12 \\ 15 \end{bmatrix}$$

$$\mathbb{R}^3 \otimes \mathbb{R}^2 \cong \mathbb{R}^{3 \times 2}$$

dimensions multiply

Cartesian product

$$(v, w) = \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix} \right) \rightsquigarrow \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

$$\mathbb{R}^3 \times \mathbb{R}^2 \cong \mathbb{R}^{3+2}$$

dimensions add

$$\dim(\underbrace{V \times V \times \cdots \times V}_{N \text{ factors}}) = \dim(V^N) = N \cdot \dim V$$

very different!

$$\dim(\overbrace{V \otimes V \otimes \cdots \otimes V}) = \dim(V^{\otimes N}) = (\dim V)^N$$

$$\dim(\underbrace{V \times V \times \cdots \times V}_{N \text{ factors}}) = \dim(V^N) = N \cdot \dim V$$

very different!

$$\dim(\overbrace{V \otimes V \otimes \cdots \otimes V}^{N \text{ factors}}) = \dim(V^{\otimes N}) = (\dim V)^N$$

<u>Special example</u>: $V = \mathbb{C}^2$

$$\dim(V \times V) = 2 + 2 = 2^2 = \dim(V \otimes V)$$

But wait! There’s more:

The tensor product of linear maps

V V'

W W'

$$V \xrightarrow{f} V'$$

$$W \xrightarrow[g]{} W'$$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, phantom, "\otimes"] & V' \arrow[d, phantom, "\otimes"] \\ W \arrow[r, "g"'] & W' \end{tikzcd}$$

$$\begin{array}{ccc} V & & V' \\ \otimes & \longrightarrow & \otimes \\ W & & W' \end{array}$$

familiar thing

familiar thing

$$V \otimes W \xrightarrow{f \otimes g} V' \otimes W'$$

What's this?

The Kronecker product

Suppose we've chosen bases for our vector spaces, so we may consider the matrix representations of the linear maps f and g.

The **tensor product** $f \otimes g$ is the linear map whose matrix representation is the **Kronecker product** of the individual matrices.

$$V \otimes W \xrightarrow{f \otimes g} V' \otimes W'$$

$$f = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}$$

$$g = \begin{bmatrix} 0 & 2 \\ -1 & 4 \\ 5 & 0 \end{bmatrix}$$

$$f = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}$$

$$g = \begin{bmatrix} 0 & 2 \\ -1 & 4 \\ 5 & 0 \end{bmatrix}$$

$$f \otimes g \text{ "=" } \begin{bmatrix} 1 \cdot g & 2 \cdot g \\ -3 \cdot g & 0 \cdot g \end{bmatrix}$$

$$f = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}$$

$$g = \begin{bmatrix} 0 & 2 \\ -1 & 4 \\ 5 & 0 \end{bmatrix}$$

$$f \otimes g \text{ "=" } \begin{bmatrix} 1 \begin{bmatrix} 0 & 2 \\ -1 & 4 \\ 5 & 0 \end{bmatrix} & 2 \begin{bmatrix} 0 & 2 \\ -1 & 4 \\ 5 & 0 \end{bmatrix} \\ -3 \begin{bmatrix} 0 & 2 \\ -1 & 4 \\ 5 & 0 \end{bmatrix} & 0 \begin{bmatrix} 0 & 2 \\ -1 & 4 \\ 5 & 0 \end{bmatrix} \end{bmatrix}$$

$$f = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}$$

$$g = \begin{bmatrix} 0 & 2 \\ -1 & 4 \\ 5 & 0 \end{bmatrix}$$

$$f \otimes g = \begin{bmatrix} 0 & 2 & 0 & 4 \\ -1 & 4 & -2 & 8 \\ 5 & 0 & 10 & 0 \\ 0 & -6 & 0 & 0 \\ 3 & -12 & 0 & 0 \\ -15 & 0 & 0 & 0 \end{bmatrix}$$

Disclaimer:

This was a basis-dependent discussion.

Basis-*independent* statements are better.

Tip

A good mathematical construction should not depend on a choice of "perspective," e.g. a basis. Instead, **it's better to be characterized by some property that holds true from *all* perspectives**.

To do identify such a property, it's often helpful to look at the wider landscape.

Better Version:

Universal Property

Better Version:

Universal Property

The tensor product $V \otimes W$ is characterized by linear maps *out* of it:

$V \otimes W$ is the special vector space with the property that linear maps out of $V \otimes W$ are in one-to-one correspondence with multilinear maps out of the Cartesian product $V \times W$.

Takeaways

The dimension of the **Cartesian product** of vector spaces is the *sum* of their dimensions.

$$\dim(V \times W) = \dim V + \dim W$$

The dimension of the **tensor product** of vector spaces is the *product* of their dimensions.

$$\dim(V \otimes W) = \dim V \cdot \dim W$$

Okay. So what?

Quantum computing

A pure **quantum state** of a quantum many body system is a unit vector in a **tensor product** of spaces.

$$|\psi\rangle \in V \otimes V$$

A **quantum gate** is a **linear transformation** satisfying a special property.

$$\mathcal{U} : V \otimes V \longrightarrow V \otimes V$$

A **quantum circuit** is a composition of **tensor products** of quantum gates.

$$\mathcal{U}' \circ (\mathcal{U} \otimes \mathrm{id}_V) : V^{\otimes 3} \longrightarrow V^{\otimes 4}$$

But first, bra-ket notation.

Previously, we've seen different ways to denote vectors: **bold font** and superscript arrows. When inner products are involved, there's a much better notation: **bra-ket (Dirac) notation**.

$$\mathbf{v}$$

bold

$$\vec{v}$$

arrow

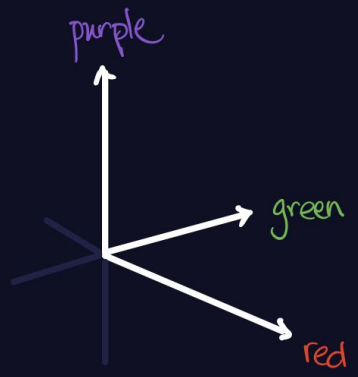
$$|v\rangle$$

ket

Bra-ket notation

Example: Given any finite set X, consider the complex vector space whose basis corresponds to elements of X. Denote this space by $\mathbb{C}^X$.

$$X = \{\text{red}, \text{green}, \text{purple}\} \rightsquigarrow$$



$$\mathbb{C}^X \cong \mathbb{C}^3$$

Bra-ket notation

For each element x in X, let $|x\rangle$ denote the basis vector corresponding to x.

$$X = \{\text{red}, \text{green}, \text{purple}\}$$

$$|\text{red}\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad |\text{green}\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad |\text{purple}\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Bra-ket notation

For each element x in X, let $|x\rangle$ denote the basis vector corresponding to x.

An arbitrary vector $|\psi\rangle$ in $\mathbb{C}^X$ is a linear combination of elements of X.

Instead of "vector," we sometimes say "**ket**."

$$X = \{\text{red}, \text{green}, \text{purple}\}$$

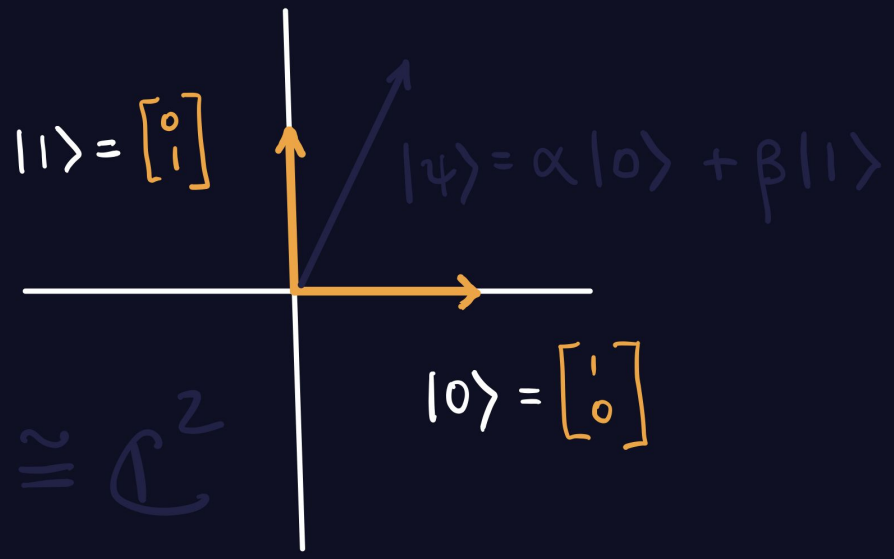
$$|\text{red}\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad |\text{green}\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad |\text{purple}\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

linear combination:

$$|\psi\rangle = 7|\text{red}\rangle - 3i\,|\text{green}\rangle + 5\,|\text{purple}\rangle$$

$$X = \{0, 1\} = \{\text{"spin up"}, \text{"spin down"}\}$$

Better example:
qubits



$$\mathbb{C}^X \cong \mathbb{C}^2$$

Bra-ket notation

Every vector space V gives rise to another vector space V* called the **dual vector space**, or **co-vector space**, associated to V.

Bra-ket notation

Every vector space V gives rise to another vector space V* called the **dual vector space**, or **co-vector space**, associated to V.

Elements of V* are **linear functions** from V to the ground field: $\mathbb{R}$, or $\mathbb{C}$, or…. (Such functions are also called **functionals.**)

$$V^* = \left\{ f : V \to \mathbb{C} \;\middle|\; f \text{ is linear} \right\}$$

Bra-ket notation

For each vector $|v\rangle$ in V, there is a special vector in V^* defined by the inner product.

Bra-ket notation

For each vector $|v\rangle$ in V, there is a special vector in V^* defined by the inner product.

Pick $|v\rangle \in V$.

Define $v^* : V \to \mathbb{C}$ by

$$V \longrightarrow \mathbb{C}$$

$$|w\rangle \longmapsto \langle v, w\rangle = v^*(|w\rangle)$$

Bra-ket notation

For each vector |v⟩ in V, there is a special vector in V* defined by the inner product.

Pick $|v\rangle \in V$.

Define $\cancel{v^*}: V \to \mathbb{C}$ by

$$V \longrightarrow \mathbb{C}$$

$$|w\rangle \longmapsto \langle v, w\rangle$$

$$\cancel{v^*(|w\rangle)}$$

Be smarter.

Bra-ket notation

For each vector $|v\rangle$ in V, there is a special vector in V^* defined by the inner product.

Denote that special vector by a "**bra**" $\langle v|$.

Pick $|v\rangle \in V$.

Define $\langle v| : V \to \mathbb{C}$ by

$$V \longrightarrow \mathbb{C}$$

$$|w\rangle \longmapsto \langle v|w\rangle$$

	Usual	Bra-Ket	Arrays
vector	v	$\lvert v\rangle$	$\begin{bmatrix} \vdots \end{bmatrix}$
co-vector	v^*	$\langle v\rvert$	$\begin{bmatrix} \cdots \end{bmatrix}$
tensor product	$v \otimes w$	$\lvert v\rangle \otimes \lvert w\rangle = \lvert v\rangle\lvert w\rangle = \lvert vw\rangle$	$\begin{bmatrix} \vdots \end{bmatrix} \otimes \begin{bmatrix} \vdots \end{bmatrix}$
inner product	$\langle v, w\rangle$	$\langle v \rvert w\rangle$	$\underset{v}{\begin{bmatrix} \cdots \end{bmatrix}} \underset{w}{\begin{bmatrix} \vdots \end{bmatrix}}$
linear map V → W	$w \otimes v^*$	$\lvert w\rangle\langle v\rvert$	$\underset{w}{\begin{bmatrix} \vdots \end{bmatrix}} \underset{v}{\begin{bmatrix} \cdots \end{bmatrix}}$

Appendix

Classical to Quantum Probability

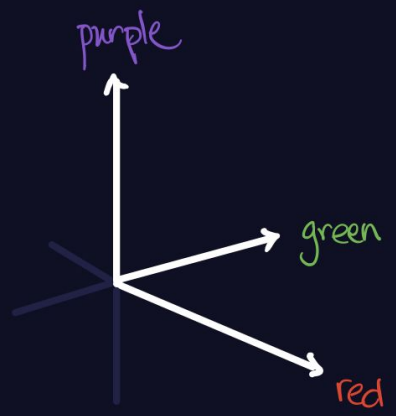
Key Analogies

Classical	Quantum
set X	Hilbert space $\mathbb{C}^X$
Cartesian product $X \times Y$	tensor product $\mathbb{C}^X \otimes \mathbb{C}^Y$
probability distribution $\pi : X \to [0,1]$	quantum state $\rho : \mathbb{C}^X \to \mathbb{C}^X$
joint distribution $\pi : X \times Y \to [0,1]$	quantum state on tensor product $\rho : \mathbb{C}^X \otimes \mathbb{C}^Y \to \mathbb{C}^X \otimes \mathbb{C}^Y$

The quantum analogue of a set is a Hilbert space.

Given any finite set X, consider the complex vector space whose basis corresponds to elements of X. Denote this space by $\mathbb{C}^X$.

$$X = \{\text{red}, \text{green}, \text{purple}\} \rightsquigarrow$$



$$\mathbb{C}^X \cong \mathbb{C}^3$$

If we fix an ordering on X, then we can associate elements of X with standard basis vectors in $\mathbb{C}^X$.

Instead of using **bold font** to denote the vector $\mathbf{x} \in \mathbb{C}^X$ associated to an element $x \in X$, let's use **Dirac notation**:

> Let $|x\rangle$ denote the vector associated to $x \in X$.
> This is called a **ket**.

Arbitrary vectors $|\psi\rangle$ in $\mathbb{C}^X$ are linear combinations of elements of X.

$$X = \{\text{red}, \text{green}, \text{purple}\}$$

$$|\text{red}\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad |\text{green}\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad |\text{purple}\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

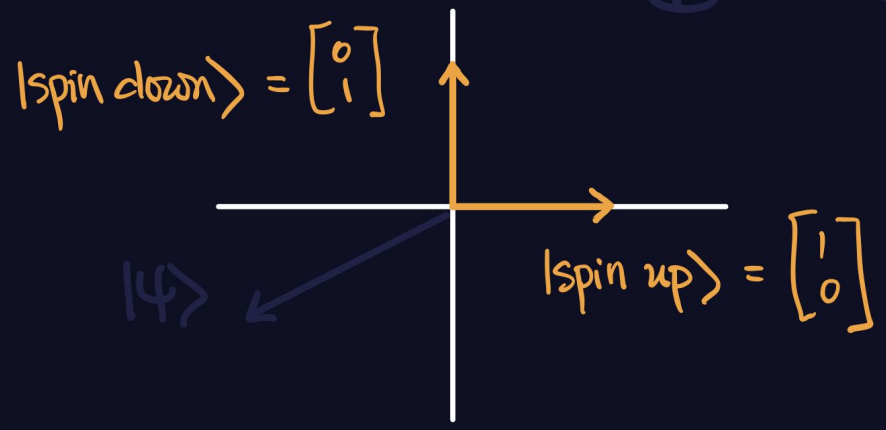
linear combination:

$$|\psi\rangle = 7|\text{red}\rangle - 3i\,|\text{green}\rangle + 5\,|\text{purple}\rangle$$

$X = \{\text{spin up}, \text{spin down}\}$

$\mathbb{C}^X \cong \mathbb{C}^2$

Better example:
qubits



$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha|\text{spin up}\rangle + \beta|\text{spin down}\rangle$

The quantum analogue of a set is a Hilbert space.

There is a natural **inner product** on $\mathbb{C}^X$. For any two elements x,x' in X, define the following:

$$\langle x, x' \rangle = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases}$$

This defines an inner product on all of $\mathbb{C}^X$. A finite-dimensional vector space equipped with an inner product is called a **Hilbert space**.*

$$\langle \text{up}, \text{up} \rangle = 1$$

$$\langle \text{up}, \text{down} \rangle = 0$$

*Actually, a vector space must satisfy a more technical requirement to be a "Hilbert space," but it turns out this requirement is automatically satisfied in finite dimensions.

The quantum analogue of the
Cartesian product is the tensor product.

Given two sets X and Y, consider the Hilbert space $\mathbb{C}^{X\times Y}$ whose basis corresponds to the Cartesian product $X\times Y$.

This space is isomorphic to the tensor product of $\mathbb{C}^X$ and $\mathbb{C}^Y$.

(Check!)

$$\mathbb{C}^{X\times Y} \cong \mathbb{C}^X \otimes \mathbb{C}^Y$$

$$(x, y)$$

$$\downarrow \uparrow$$

$$|x\rangle \otimes |y\rangle$$

The quantum analogue of a
probability distribution is a quantum state.

A **probability distribution** on a finite set X is a function:

$$\pi : X \to \mathbb{R}$$

real-valued function

$$\pi(x) \geq 0$$

values are nonnegative

$$\sum_{x \in X} \pi(x) = 1$$

values sum to 1

Example

Let $X = \{H, T\}$ denote the outcomes of a fair coin toss.

$$\pi(H) = \frac{1}{2} \qquad \pi(T) = \frac{1}{2}$$

The quantum analogue of a probability distribution on X is a particular **linear operator** on $\mathbb{C}^X$.

$$\pi : X \to \mathbb{R} \quad \rightsquigarrow \quad \rho : \mathbb{C}^X \to \mathbb{C}^X$$

must satisfy conditions analogous to probabilities

The quantum analogue of a probability distribution on a finite set X is a **quantum state** on $\mathbb{C}^X$. *What is a "quantum state"?*

<u>Answer (v.0):</u>

A quantum state is a **unit vector** in $\mathbb{C}^X$.

$$|\psi\rangle \in \mathbb{C}^X$$

<u>Answer (v.1):</u>

A quantum state is a **density operator** on $\mathbb{C}^X$.

$$\rho : \mathbb{C}^X \longrightarrow \mathbb{C}^X$$

These linear operators are called **density operators** or **quantum states**.

$\rho : \mathbb{C}^{\times} \to \mathbb{C}^{\times}$ is a <u>density operator</u> if:

It's **positive semidefinite.**

$\langle v|\rho|v\rangle \geq 0 \quad \forall |v\rangle$

It's **self-adjoint.**

$\rho = \rho^{\dagger}$

Its **trace is 1.**

$\mathrm{tr}(\rho) = 1$

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It's **positive semidefinite.**

$\langle v|\rho|v\rangle \geq 0 \ \forall |v\rangle$

This guarantees its eigenvalues are positive.

It's **self-adjoint.**

$\rho = \rho^\dagger$

This guarantees its eigenvalues are real.

Its **trace is 1.**

$\mathrm{tr}(\rho) = 1$

This guarantees its eigenvalues sum to 1.

Example

Every probability distribution defines a density operator. (And in more than one way!)

1. A Diagonal Operator

Suppose $X = \{x,y,z\}$ and consider the probability distribution π on X defined by $\pi(x)=1/7$ and $\pi(y)=2/7$ and $\pi(z)=4/7$. For simplicity, let's write this as a tuple $\pi = (1/7, 2/7, 4/7)$.

1. A Diagonal Operator

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$$\rho = \begin{bmatrix} 1/7 & 0 & 0 \\ 0 & 2/7 & 0 \\ 0 & 0 & 4/7 \end{bmatrix} : \mathbb{C}^3 \to \mathbb{C}^3$$

positive semidefinite
(entries are nonnegative)
✓

self-adjoint
(matrix is symmetric)
✓

unit trace
(sum of diagonals is 1)
✓

2. A Rank 1 Projection Operator

Again, suppose $\pi = (1/7, 3/7, 4/7)$ and consider the orthogonal projection onto the following unit vector:

2. A Rank 1 Projection Operator

Again, suppose π = (1/7, 3/7, 4/7) and consider the orthogonal projection onto the following unit vector:

$$|\psi\rangle = \sqrt{\tfrac{1}{7}}\,|x\rangle + \sqrt{\tfrac{2}{7}}\,|y\rangle + \sqrt{\tfrac{4}{7}}\,|z\rangle \in \mathbb{C}^X$$

$$= \begin{bmatrix} \sqrt{1/7} \\ \sqrt{2/7} \\ \sqrt{4/7} \end{bmatrix}$$

sum of elements in X weighted by the square roots of their probabilities.

$$\rho = \mathrm{Proj}_{|\psi\rangle}$$

$$= |\psi\rangle\langle\psi|$$

the outer product of $|\psi\rangle$ with itself

$$= \begin{bmatrix} \sqrt{1/7} \\ \sqrt{2/7} \\ \sqrt{4/7} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{1}{7}} & \sqrt{\frac{2}{7}} & \sqrt{\frac{4}{7}} \end{bmatrix}$$

a rank 1 operator

$$= \begin{bmatrix} \frac{1}{7} & \frac{\sqrt{2}}{7} & \frac{2}{7} \\ \frac{\sqrt{2}}{7} & \frac{2}{7} & \frac{\sqrt{8}}{7} \\ \frac{2}{7} & \frac{\sqrt{8}}{7} & \frac{4}{7} \end{bmatrix} : \mathbb{C}^3 \to \mathbb{C}^3$$

✓ **positive semidefinite**
(inner product with any vector $|v\rangle$ is $|\langle v|\psi\rangle|^2 \geq 0$.)

✓ **self-adjoint**
(matrix is symmetric)

✓ **unit trace**
(sum of diagonals is 1)

Pure states

A density operator with **rank = 1** is called a pure quantum state.

From a mathematical perspective, pure quantum states and unit vectors are “synonymous.”

Why? The image of a rank 1 linear operator is a *line*, i.e. a 1D vector space spanned by a certain vector!

$\mathrm{rank}(\rho) = 1$

Mixed states

A density operator with **rank > 1** is called a mixed quantum state.

It can be written as a linear combination of pure states: **the spectral decomposition theorem** of linear algebra.

$\mathrm{rank}(\rho) > 1$

So what?

Recall the analogies.

Classical	Quantum
set X	Hilbert space $\mathbb{C}^X$
Cartesian product $X \times Y$	tensor product $\mathbb{C}^X \otimes \mathbb{C}^Y$
probability distribution $\pi : X \to [0,1]$	quantum state $\rho : \mathbb{C}^X \to \mathbb{C}^X$
joint distribution $\pi : X \times Y \to [0,1]$	quantum state on tensor product $\rho : \mathbb{C}^X \otimes \mathbb{C}^Y \to \mathbb{C}^X \otimes \mathbb{C}^Y$

The quantum analogue of a joint probability distribution is a quantum state on a tensor product.

Let X and Y be finite sets. The quantum version of a **joint probability distribution** on the Cartesian product $X \times Y$

$$\pi : X \times Y \to \mathbb{R} \qquad \pi(x,y) \geq 0 \qquad \sum_{x,y} \pi(x,y) = 1$$

is a density operator on the tensor product $\mathbb{C}^X \otimes \mathbb{C}^Y$.

$$\rho : \mathbb{C}^X \otimes \mathbb{C}^Y \to \mathbb{C}^X \otimes \mathbb{C}^Y$$

Example:
multi-qubit systems

Q: What is the **state** of a system consisting of one qubit?

A: A **density operator** on $\mathbb{C}^2$.

It can be represented as a 2×2 matrix, for example:

$$\rho = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^2$$

Example:
multi-qubit systems

Q: What is the **state** of a system consisting of two qubits?

A: A **density operator** on $\mathbb{C}^2 \otimes \mathbb{C}^2$.

It can be represented as a 4×4 matrix, for example:

$$\rho = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} : \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$$

4 × 4

dimension is 4

Example:
multi-qubit systems

Q: What is the **state** of a system consisting of three qubits?

A: A **density operator** on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

It can be represented as an 8×8 matrix, for example:

$$\rho = \begin{bmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix} : \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

dimension 8

8 × 8

Example:
multi-qubit systems

Q: What is the **state** of a system consisting of four qubits?

A: A **density operator** on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

It can be represented as an 16×16 matrix, for example:

$$\rho = \underbrace{\begin{bmatrix} 256 \\ \text{numbers} \end{bmatrix}}_{16 \times 16} : \underbrace{(\mathbb{C}^2)^{\otimes 4}}_{\text{dimension } 16} \longrightarrow \underbrace{(\mathbb{C}^2)^{\otimes 4}}_{\text{dimension } 16}$$

Example:
multi-qubit systems

Q: What is the **state** of a system consisting of N qubits?

A: A **density operator** on the N-fold tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$.

It can be represented as a $2^N \times 2^N$ matrix, for example:

$$\rho = \underbrace{\begin{bmatrix} 2^{2N} \\ \text{numbers} \end{bmatrix}}_{2^N \times 2^N} : \underbrace{(\mathbb{C}^2)^{\otimes N}}_{\text{dimension } 2^N} \longrightarrow \underbrace{(\mathbb{C}^2)^{\otimes N}}_{\text{dimension } 2^N}$$

Quantum computing

A pure **quantum state** of a quantum many body system is a unit vector in a **tensor product** of spaces.

$$|\psi\rangle \in V \otimes V$$

A **quantum gate** is a **linear transformation** satisfying a special property.

$$\mathcal{U} : V \otimes V \longrightarrow V \otimes V$$

A **quantum circuit** is a composition of **tensor products** of quantum gates.

$$\mathcal{U}' \circ (\mathcal{U} \otimes \mathrm{id}_V) : V^{\otimes 3} \longrightarrow V^{\otimes 4}$$