American Options With Discrete Dividends Solved by Highly Accurate Discretizations

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Summary. We present an accurate numerical solution for the discrete Black-Scholes equation with only a few grid points. European and American option problems with deterministic discrete dividend modelled by a jump condition at the ex-dividend date are solved. Fourth order finite differences are employed, as well as a grid stretching in space and a Lagrange interpolation at the ex-dividend date.

Key words: American options, Black-Scholes, 4th order discretization, stretched grid, discrete dividend.

1 Black-Scholes Equation, Discretization

Research in option pricing theory concerns, among other issues, the computation of the fair price of an option. We aim for accurate American option values by 4th order finite differences and grid stretching in space. The Black-Scholes equation represents a simple model for pricing a put or a call option. Option value u depends on asset price s and is influenced by exercise price K, expiration time T $(0 \leq t \leq T)$, interest rate r, and volatility σ. The equation reads

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} + rs\frac{\partial u}{\partial s} - ru = 0, \quad 0 \leq s < \infty, \quad 0 \leq t < T \ . \tag{1}$$

It is valid under the assumption of a geometric Brownian motion for the asset price process $\{S_t\}$ and comes with boundary and final conditions distinguishing a put from a call and a European from an American option. We adopt the technique of modelling discrete dividends D by a jump condition at the ex-dividend date t_d [9]:

$$u(s, t_d^-) = u(s - D, t_d^+), \tag{2}$$

where t_d^-, t_d^+ represent the times just before and after the ex-dividend date.

Boundary conditions arise naturally from financial arguments [9].

$$\text{For a call: } u(0,t) = 0, \text{ and } u(s_{max}, t) = s_{max} - Ke^{-r(T-t)}.$$

It is shown in [5] that the boundary condition at $s = s_{max}$ for a call after dividend payment changes into $u(s_{max}, t) = s_{max} - Ke^{-r(T-t)} - De^{-r(t_d-t)}$. The final condition at $t = T$ for the European call reads $u(s,T) = \max(s - K, 0)$.

The exercise of American-style options is permitted at any time during the lifetime of an option, $0 < t \leq T$. When early exercise is permitted, a constraint "$u(s,t) \geq$payoff" plus a smooth pasting condition at the payoff must be imposed to the solution of (1), giving rise to a free boundary problem. A special s-value exists, the optimal exercise price $s_f(t)$, which is not known in advance and needs to be computed. For the American call, early exercise may be optimal just before the dividend payment *only if* the dividend payment is large enough, *i.e.*, $D > K\left(1 - e^{-r(T-t_d)}\right)$. The boundary condition at $s = s_{max}$ for the American call before the dividend payment, $t < t_d$ reads

$$u(s_{max}, t) = \max\{s_{max} - Ke^{-r(T-t)} - De^{-r(t_d-t)},\ s_{max} - Ke^{-r(t_d-t)}\}, \quad (3)$$

and the final condition before the dividend payment reads:

$$u(s, t_d) = \max\{s - Ke^{-r(T-t_d)} - D,\ s - K\}, \quad (4)$$

which is also the condition whether the option should be exercised at t_d^-.

For the American put option the boundary condition at $s = 0$ changes after the dividend payment from $u(0,t) = Ke^{-r(T-t)}$ into

$$u(0,t) = \max\{Ke^{-r(T-t)} + De^{-r(t_d-t)},\ K\},$$

and we have $u(s_{max}, t) = 0$. The payoff for a put after a dividend has been paid remains $u(s,T) = \max(K - s,\ 0)$. Early exercise of a put option may be optimal at any time within the option's lifetime.

1.1 Grid Transformation and Discretization

The implicit 4th order accurate backward differentiation formula, BDF4 [2], with time discretization is employed on an equidistant grid with time step k. It is preceded by two Crank-Nicholson and one BDF3 step. Crank-Nicholson is unconditionally stable, whereas BDF3 and BDF4 have a stability region. For our applications these stability constraints are not problematic.

In space, we use an analytic coordinate transformation, which results in an a-priori stretching of the grid around $s = K$, *i.e.*, at the non-differentiability in the final condition. An equidistant grid discretization can be used after the analytic transformation, as only the coefficients in front of the derivatives in (1) change. The spatial transformation used for Black-Scholes originates from [1, 8]:

$$y = \psi(s) = \sinh^{-1}(\mu(s-K)) + \sinh^{-1}(\mu K)\,. \tag{5}$$

The parameter μ determines the rate of stretching; $\mu K = 15$ is satisfactory in many cases.

A fourth order "long stencil" finite difference discretization in space on the equidistant y-grid based on Taylor's expansion is employed. First-order derivatives are discretized by central differences. For the fourth order approximation, near-boundary points $y_1,\ y_{N-1}$ need special treatment by means of one-sided differences. Important for our future applications is a small discretization error with only a few grid points. This has been achieved in [7] with the techniques proposed: for some reference Black-Scholes option pricing problems with known exact solutions only $20-40$ space- and time-steps were necessary to get an accuracy of less than one cent (€ 0.01).

The strategy to solve the Black-Scholes equation with discrete deterministic dividends numerically is as follows (see also [8]):

- Start solving from $t = T$ to $t = t_d$ with the usual pay-off.
- Apply an interpolation to calculate the new asset and option price on the s-grid discounted with D.
- Restart the numerical process with the PDE from the interpolated price as final condition from t_d tot $t = 0$.

In our computations we place t_d exactly on a time line, t_d^- and t_d^+ are assumed to lie on the same line.

2 Numerical Results with Discrete Dividend

2.1 European Call

We present European call results for multiple discrete dividends, as in [3], with problem parameters $s_0 = K = 100,\ r = 0.06, \sigma = 0.25$ and multiple dividends of 4 (ex-dividend date is each half year). Table 1 presents numerical results for $T = 1,\ T = 2$ and $T = 3$, with one, two and three dividend payments, respectively. It compares the numerical approximation to the exact solution from [3] (HHL in the table). The value $s_{max} = 3K$ is chosen according to a well-known formula [4] and the stretching parameter is set to $\mu = 0.15$. For larger values of T the number of points in time increases proportionally. The numerical results obtained with only a few grid points agree very well with the reference.

Other discrete dividend results from [3] can also be confirmed with our discretization techniques, whereas binomial tree approaches may need special techniques based on financial arguments to get the correct price.

2.2 American Put

Next, we consider an American put with parameters from [6]: $K = 100,\ \sigma = 0.4,\ r = 0.08,\ D = 2,\ t_d = 0.3,\ T = 0.5$ and $\mu = 0.15,\ s_{max} = 3K$. Results

Table 1. Multiple discrete dividends payments, $K = 100,\ d = 4,\ \mu = 0.15$.

	$T = 1$		$T = 2$		$T = 3$
Grid	$u\ (s_0, t = 0)$	Grid	$u\ (s_0, t = 0)$	Grid	$u\ (s_0, t = 0)$
10×10	10.612	10×20	15.177	10×30	18.698
20×20	10.660	20×40	15.202	20×60	16.607
40×40	10.661	40×80	15.201	40×120	18.600
HHL	10.661		15.199		18.598

for $s = 80,\ 100,\ 120$ at $t = 0$ on a 20^2 – and 40^2– grid are compared to those in [6] in Table 2. With only 20 points in space and time the results from [6] are obtained.

Table 2. American put reference problem from [6], $K = 100,\ D = 2,\ \mu = 0.15$.

Grid	$u_h\ (80, t = 0)$	$u_h\ (100, t = 0)$	$u_h\ (120, t = 0)$
20×20	0.223	0.105	0.043
40×40	0.223	0.105	0.043
Meyer (J. C. Fin. 2001) [6]:	0.223	0.105	0.043

Finally, we consider an American put with two ex-dividend dates, $t_{d_1} = 0.3,\ t_{d_2} = 0.8\ (T = 1)$ and visualize the free boundary as a function of time for different dividend payment strategies (as in [6]). Figure 1 shows the free boundary for an American put with problem parameters: $K = 100,\ \sigma = 0.4,\ r = 0.08,\ T = 1$. In the figure the free boundary functions presented are: one without any dividend payment ($D = 0$, solid line), with a fixed dividend payment $D = 2$ at $t_{d_1},\ t_{d_2}$ (dashed line) and with a payment proportional to the asset price $D = 0.98s$ (dotted line) at the ex-dividend dates. It can be seen that after the discrete dividend payment the free boundary may disappear, and reappear, indicating that early exercise is not always favorable after an ex-dividend date.

3 Conclusion

In this paper we have solved the Black-Scholes equation for a European and American option with discrete dividend with only a few grid points. Fourth order accurate space and time discretizations have been employed, using spatial grid stretching by means of an analytical coordinate transformation. The discrete dividend payment is handled very satisfactorily by the stretched grid discretization and a 4th order Lagrange interpolation at the ex-dividend date. Reference results for a European call with multiple dividends and American puts from the literature are retained with only $20 - 40$ grid points in space and time.

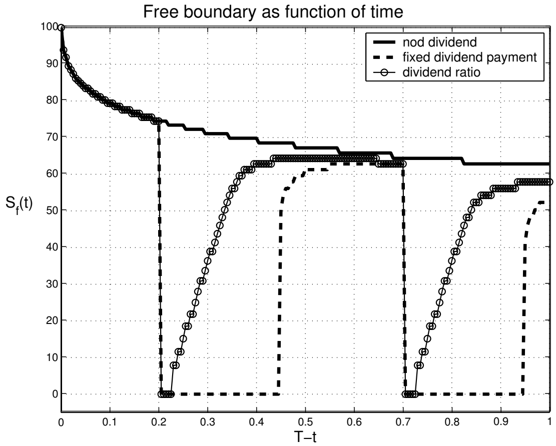

Fig. 1. Free boundary as function of time with two ex-dividend dates and different forms of dividend payment: $D = 0$ (solid), $D = 2$ (dashed) vs. $D = 0.98$s (dotted).

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