# Taking Linear Logic Apart

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$$
\begin{array}{rcllcl}
P, Q, R & := & x \leftrightarrow y & & \mid & (\nu x)(P \mid Q) \\
& \mid & x[y].(P \mid Q) & & \mid & x(y).P \\
& \mid & x[].0 & & \mid & x().P \\
& \mid & x \triangleleft \mathtt{inl}.P & & \mid & x \triangleleft \mathtt{inr}.P \\
& \mid & x \triangleright \{\mathtt{inl} : P; \mathtt{inr} : Q\} & & \mid & x \triangleright \{\}
\end{array}
$$

$$
\begin{aligned}
x \leftrightarrow y &\equiv y \leftrightarrow x \\
(\nu x)(P \mid Q) &\equiv (\nu x)(Q \mid P) \\
(\nu x)(P \mid (\nu y)(Q \mid R)) &\equiv (\nu y)((\nu x)(P \mid Q) \mid R)
\end{aligned}
$$

$$
\text{if } x \notin \mathrm{fv}(R) \text{ and } y \notin \mathrm{fv}(P)
$$

$$
\begin{aligned}
(\nu x)(w \leftrightarrow x \mid P) &\Longrightarrow P\{w/x\} \\
(\nu x)(x[y].(P \mid Q) \mid x(y).R) &\Longrightarrow (\nu y)(P \mid (\nu x)(Q \mid R)) \\
(\nu x)(x[].0 \mid x().P) &\Longrightarrow P \\
(\nu x)(x \triangleleft \mathtt{inl}.P \mid x \triangleright \{\mathtt{inl} : Q; \mathtt{inr} : R\}) &\Longrightarrow (\nu x)(P \mid Q) \\
(\nu x)(x \triangleleft \mathtt{inr}.P \mid x \triangleright \{\mathtt{inl} : Q; \mathtt{inr} : R\}) &\Longrightarrow (\nu x)(P \mid R)
\end{aligned}
$$

$$
\begin{aligned}
A, B, C := \;& A \otimes B \mid 1 \\
\mid\; & A \parr B \mid \bot \\
\mid\; & A \oplus B \mid \mathbf{0} \\
\mid\; & A \,\&\, B \mid \top
\end{aligned}
$$

$$\frac{}{x \leftrightarrow y \vdash x : A, y : A^{\perp}}\ \textsc{Ax} \qquad \frac{P \vdash \Gamma, x : A \qquad Q \vdash \Delta, x : A^{\perp}}{(\nu x)(P \mid Q) \vdash \Gamma, \Delta}\ \textsc{Cut}$$

$$\frac{P \vdash \Gamma, y : A \qquad Q \vdash \Delta, x : B}{x[y].(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B}\ (\otimes) \qquad \frac{P \vdash \Gamma, y : A, x : B}{x(y).P \vdash \Gamma, x : A \parr B}\ (\parr)$$

$$\frac{P \vdash \Gamma}{x().P \vdash \Gamma, x : \perp}\ (\perp) \qquad \frac{}{x[].0 \vdash x : 1}\ (1)$$

$$\frac{P \vdash \Gamma, x : A}{x \triangleleft \mathtt{inl}.P \vdash \Gamma, x : A \oplus B}\ (\oplus_1) \qquad \frac{P \vdash \Gamma, x : B}{x \triangleleft \mathtt{inr}.P \vdash \Gamma, x : A \oplus B}\ (\oplus_2)$$

$$\frac{P \vdash \Gamma, x : A \qquad Q \vdash \Gamma, x : B}{x \triangleright \{\mathtt{inl} : P; \mathtt{inr} : Q\} \vdash \Gamma, x : A \,\&\, B}\ (\&)$$

(no rule for **0**) $$\frac{}{x \triangleright \{\} \vdash \Gamma, x : \top}\ (\top)$$

**Theorem** (Preservation)

*If $P \vdash \Gamma$ and $P \implies Q$, then $Q \vdash \Gamma$.*

**Theorem** (Progress)
*If $P \vdash \Gamma$, then there exists a $Q$ such that $P \Longrightarrow^{\star} Q$ and $Q$ is not a cut.*

$(\nu x)($ $x[y].($✉️ $\mid$ 👩🏻‍💻$)$ $\mid$ $x(y).$🧙🏽‍♂️ $)$

$(\nu x)($ $x[y].($✉️ $\mid$ 🧗‍♀️$)$ $\mid$ $x(y).$🧙🏽‍♂️ $)$

$(\nu x)($ $x[y].($✉️ $|$ 🤷🏻‍♀️$)$ $|$ $x(y).$🧙🏽‍♂️ $)$

$$\underline{(\nu x)(} \quad x[y].(\text{✉️} \mid \text{🤷‍♀️}) \quad \underline{\mid} \quad x(y).\text{🧙‍♂️} \quad \underline{)}$$

$(\nu x)($ $x[y].($✉️ $|$ 🤷🏻‍♀️$)$ $\mid$ $x(y).$🧙🏽‍♂️ $)$

$$
\begin{array}{rclcl}
P, Q, R & := & x \leftrightarrow y & \mid & \underline{\boldsymbol{(\nu x)(}}P\underline{|}Q\underline{\boldsymbol{)}} \\
& \mid & x(y).P & \mid & \underline{\boldsymbol{x[y].(}}P\underline{|}Q\underline{\boldsymbol{)}} \\
& \mid & x().P & \mid & \underline{\boldsymbol{x[].0}} \\
& \mid & \ldots & &
\end{array}
$$

$$
\begin{array}{rcllll}
P, Q, R & := & x \leftrightarrow y & \mid \underline{(\nu x)} P & \mid \underline{(}\, P \,\underline{\mid}\, Q \,\underline{)} \\
& \mid & x(y).P & \mid \underline{x[y]}.P & \\
& \mid & x().P & \mid \underline{x[]}.P & \mid \underline{0} \\
& \mid & \ldots & & 
\end{array}
$$

$(\nu x)($ $x[y].($✉️ $\mid$ 🤷🏻‍♀️$)$ $\mid$ $x(y).$🧙🏼‍♂️ $)$

$(\nu x)\,($ $x[y].\,($ ✉️ $|$ 🤷🏻‍♀️ $)$ $|$ $x(y).$ 🧙🏽‍♂️ $)$

$$(\nu x)(P \mid (\nu y)(Q \mid R))$$

$$\equiv$$

$$(\nu y)((\nu x)(P \mid Q) \mid R)$$

if $x \notin R$ and $y \notin P$

$$(P \mid (Q \mid R)) \equiv ((P \mid Q) \mid R)$$

$$(\nu x)(P \mid Q) \equiv (P \mid (\nu x)Q) \quad \text{if } x \notin P$$

$$(\nu x)(\nu y)P \equiv (\nu y)(\nu x)P$$

$$(\nu x)(x[y].(P \mid Q) \mid x(y).R)$$

$$\Longrightarrow$$

$$(\nu y)(P \mid (\nu x)(Q \mid R))$$

$$(\nu x)(x[y].P \mid x(y).R)$$

$$\Longrightarrow$$

$$(\nu y)(\nu x)(P \mid R)$$

$$
\begin{aligned}
A, B, C \; &\Coloneqq A \otimes B \quad \mid \; 1 \\
&\mid A \parr B \quad \mid \; \bot \\
&\mid \ldots \\
\Gamma, \Delta \; &\Coloneqq \Gamma, x : A \mid \cdot
\end{aligned}
$$

$$(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp} \qquad 1^{\perp} = \perp$$

$$(A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp} \qquad \perp^{\perp} = 1$$

$$
\dfrac{\dfrac{\text{✉️} \vdash \Gamma, y : \text{📩} \qquad \text{🤷‍♀️} \vdash \Delta, x : \text{❓}}{x[y].(\text{✉️} \mid \text{🤷‍♀️}) \vdash \Gamma, \Delta, x : \text{📩} \otimes \text{❓}} \qquad x(y).\text{🧙} \vdash \Theta, x : \text{📩}^{\perp} \parr \text{❓}^{\perp}}{(\nu x)(x[y].(\text{✉️} \mid \text{🤷‍♀️}) \mid x(y).\text{🧙}) \vdash \Gamma, \Delta, \Theta}
$$

$$\frac{P \vdash \Gamma, y : A \qquad Q \vdash \Delta, x : B}{x[y].(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B}\,(\otimes) \qquad \frac{P \vdash \Gamma, y : A, x : B}{x(y).P \vdash \Gamma, x : A \parr B}\,(\parr)$$

$$\frac{P \vdash \Gamma, x : A \qquad Q \vdash \Delta, x : A^{\perp}}{(\nu x)(P \mid Q) \vdash \Gamma, \Delta}\,\textsc{Cut}$$

$$\frac{P \vdash \Gamma, x : A \quad Q \vdash \Delta, x : A^{\perp}}{(\nu x)(P \mid Q) \vdash \Gamma, \Delta}\textsc{Cut}$$

$$\frac{P \vdash \Gamma \quad Q \vdash \Delta}{(P \mid Q) \vdash \Gamma, \Delta}\textsc{Mix}$$

$$\frac{P \vdash \Gamma, x : A, y : A^{\perp}}{(\nu xy)P \vdash \Gamma}\textsc{Cycle}$$

$$
\dfrac{\dfrac{\dfrac{\overline{x \leftrightarrow y \vdash x : A, y : A^{\perp}} \quad \overline{z \leftrightarrow w \vdash z : A, w : A^{\perp}}}{(x \leftrightarrow y \mid z \leftrightarrow w) \vdash x : A, y : A^{\perp}, z : A, w : A^{\perp}}}{(\nu xw)(x \leftrightarrow y \mid z \leftrightarrow w) \vdash y : A^{\perp}, z : A}}{(\nu yz)(\nu xw)(x \leftrightarrow y \mid z \leftrightarrow w) \vdash \cdot}
$$

$$\frac{P \vdash \Gamma \quad Q \vdash \Delta}{(P \mid Q) \vdash \Gamma, \Delta}\textsc{Mix}$$

$$\frac{P \vdash \Gamma, x : A, y : A^{\perp}}{(\nu xy)P \vdash \Gamma}\textsc{Cycle}$$

$$\mathcal{G}, \mathcal{H} ::= \mathcal{G} \mathbin{\underline{|}} \Gamma \mid \underline{\varnothing}$$

$$\frac{P \vdash \mathcal{G} \quad Q \vdash \mathcal{H}}{(P \mid Q) \vdash \mathcal{G} \mid \mathcal{H}} \text{H-Mix}$$

$$\frac{P \vdash \mathcal{G} \mid \Gamma, x : A \mid \Delta, x : A^{\perp}}{(\nu x)P \vdash \mathcal{G} \mid \Gamma, \Delta} \text{H-Cut}$$

$$\mathcal{G},\mathcal{H} := \mathcal{G} \mathbin{\underline{\mid}} \Gamma \mid \underline{\varnothing}$$

$$\frac{P \vdash \mathcal{G} \qquad Q \vdash \mathcal{H}}{(P \mid Q) \vdash \mathcal{G} \mid \mathcal{H}}\,\textsc{H-Mix}$$

$$\mathcal{G}, \mathcal{H} \coloneqq \mathcal{G} \mathbin{\underline{|}} \Gamma \mid \underline{\varnothing}$$

$$\frac{P \vdash \mathcal{G} \mid \Gamma, x : A \mid \Delta, x : A^{\perp}}{(\nu x)P \vdash \mathcal{G} \mid \Gamma, \Delta}\,\textsc{H-Cut}$$

$$\dfrac{\dfrac{\overline{x \leftrightarrow y \vdash x : A, y : A^\perp} \qquad \overline{y \leftrightarrow x \vdash y : A, x : A^\perp}}{(x \leftrightarrow y \mid y \leftrightarrow x) \vdash x : A, y : A^\perp \mid y : A, x : A^\perp}}{(\nu x)(x \leftrightarrow y \mid y \leftrightarrow x) \vdash y : A^\perp, y : A}$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\text{✉️} \vdash \Gamma, y : \text{📩} \qquad \text{🤷‍♀️} \vdash \Delta, x : \text{❓}}{(\text{✉️} \mid \text{🤷‍♀️}) \vdash \Gamma, y : \text{📩} \mid \Delta, x : \text{❓}}}{x[y].(\text{✉️} \mid \text{🤷‍♀️}) \vdash \Gamma, \Delta, x : \text{📩} \otimes \text{❓}} \qquad x(y).\text{🧙} \vdash \Theta, x : \text{📩}^{\perp} \parr \text{❓}^{\perp}}{(x[y].(\text{✉️} \mid \text{🤷‍♀️}) \mid x(y).\text{🧙}) \vdash \Gamma, \Delta, x : \text{📩} \otimes \text{❓} \mid \Theta, x : \text{📩}^{\perp} \parr \text{❓}^{\perp}}}{(\nu x)(x[y].(\text{✉️} \mid \text{🤷‍♀️}) \mid x(y).\text{🧙}) \vdash \Gamma, \Delta, \Theta}
$$

$$\frac{P \vdash \mathcal{G} \mid \Gamma, y : A \mid \Delta, x : B}{x[y].P \vdash \mathcal{G} \mid \Gamma, \Delta, x : A \otimes B} \otimes \qquad \frac{P \vdash \mathcal{G} \mid \Gamma, y : A, x : B}{x(y).P \vdash \mathcal{G} \mid \Gamma, x : A \mathbin{⅋} B} (⅋)$$

$$\frac{P \vdash \mathcal{G} \qquad Q \vdash \mathcal{H}}{P \mid Q \vdash \mathcal{G} \mid \mathcal{H}} \text{H-MIX} \qquad \frac{P \vdash \mathcal{G} \mid \Gamma, x : A \mid \Delta, x : A^{\perp}}{(\nu x)P \vdash \mathcal{G} \mid \Gamma, \Delta} \text{H-CUT}$$

$$\frac{P \vdash \mathcal{G} \mid \Gamma, y : A \mid \Delta, x : B}{x[y].P \vdash \mathcal{G} \mid \Gamma, \Delta, x : A \otimes B} \otimes$$

$$\frac{P \vdash \mathcal{G} \mid \Gamma, y : A, x : B}{x(y).P \vdash \mathcal{G} \mid \Gamma, x : A \parr B} (\parr)$$

$$(\nu x)(x(y).\ldots)$$

$$(\nu x)(x(y).x[y].\ldots)$$

$$\frac{P \;\vdash\; \mathcal{G} \mid \Gamma, y : A \mid \Delta, x : B}{x[y].P \;\vdash\; \mathcal{G} \mid \Gamma, \Delta, x : A \otimes B}\otimes$$

$$\frac{P \;\vdash\; \mathcal{G} \mid \Gamma, y : A, x : B}{x(y).P \;\vdash\; \mathcal{G} \mid \Gamma, x : A \mathbin{⅋} B}(⅋)$$

if $x \notin \mathcal{G}$

**Theorem** (Progress)

*If $P \vdash \mathcal{G}$, then there exists a $Q$ such that $P \Longrightarrow^{\star} Q$ and $Q$ is not a cut or a mix.*

**Theorem** (Representability)
*If* $\vdash\ \Gamma_1 \mid \ldots \mid \Gamma_n$ *in our new calculus, then* $\vdash\ \bigparr \Gamma_1 \otimes \ldots \otimes \bigparr \Gamma_n$ *in CP.*