Problem Statement

A company operates a production facility that manufactures various types of products. The production process comprises several sequential steps, and each step has distinct time and resource requirements. The company aims to optimize the production process by minimizing the overall production time and cost.

Mathematical Optimization Problem Statement

Let P denote the set of products to be produced, and S denote the set of production steps. For each product $p \in P$ and step $s \in S$, let $T(p, s)$ represent the time required to complete step s for product p, and $C(p, s)$ represent the cost of resources required to perform step s for product p. We introduce a binary variable $X(p, s)$, which takes the value 1 if step s is performed for product p, and 0 otherwise. Additionally, let $Y(p)$ denote the completion time of product p.

The objective is to minimize the total production time and cost, which can be formulated as follows:

$$\min \sum_{p \in P} \sum_{s \in S} X(p, s) \cdot T(p, s) \cdot C(p, s)$$

subject to the following constraints:

1. **Completeness Constraint:** Each product must complete all the required steps:

   $$\forall p \in P, \sum_{s \in S} X(p, s) = 1$$

   This constraint ensures that every product undergoes all the necessary production steps.

2. **Non-Simultaneity Constraint:** Production steps cannot be performed simultaneously for a single product:

   $$\forall p \in P, \forall s_1 \in S, \forall s_2 \in S \text{ with } s_1 < s_2, X(p, s_1) + X(p, s_2) \leq 1$$

   This constraint guarantees that at most one step is conducted at any given time for a specific product.

3. **Production Capacity Constraint:** Production capacity limitations must be respected for each step:

   $$\forall s \in S, \sum_{p \in P} X(p, s) \cdot C(p, s) \leq \text{available\_capacity}(s)$$

This constraint ensures that the sum of resource requirements for all products undergoing a particular step does not exceed the available capacity for that step.

4. **Completion Time Constraint:** Completion time for each product must account for the time spent on individual steps:

$$\forall p \in P, \forall s \in S, Y(p) \geq Y(p) + T(p, s) \cdot X(p, s)$$

This constraint guarantees that the completion time of a product is updated accordingly based on the time spent on each step.

The above constraints collectively ensure that each product completes all the required steps, simultaneous execution of steps for a single product is prohibited, production capacity limits are adhered to, and the completion time for each product is correctly calculated.