Chapter 3

Rational Functions

In most of your studies of functions to date, the graphs have been fairly predictable once you have found the x- and y-intercepts, calculated the slopes, and found any maximum or minimum points. Rational functions take the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are both polynomial functions and $Q(x) \neq 0$. When $P(x)$ is divided by $Q(x)$, the results are not simple lines or parabolas. Special restrictions apply and some interesting and very complex graphs can result. In this chapter, you will investigate these functions, their graphs, and their rates of change and apply them to such fields as business, medicine, environmental studies, and electronics.

By the end of this chapter, you will

- determine key features of the graphs of rational functions that are the reciprocals of linear and quadratic functions, and make connections between their algebraic and graphical representations (C2.1)
- determine key features of the graphs of rational functions that have linear expressions in the numerator and denominator (C2.2)
- sketch the graph of a simple rational function using its key features, given the algebraic representation of the function (C2.3)
- determine and describe the connection between the real roots of a rational equation and the zeros of the corresponding rational function (C3.5)
- solve simple rational equations in one variable algebraically, and verify solutions using technology (C3.6)
- solve problems involving applications of polynomial and simple rational functions and equations (C3.7)
- explain the difference between the solution to a simple rational equation in one variable and the solution to an inequality, and demonstrate that given solutions satisfy an inequality (C4.1)
- solve graphically and numerically equations and inequalities whose solutions are not accessible by standard algebraic techniques (D3.2)
- recognize examples of rates of change, make connections between instantaneous and average rates of change, calculate approximate rates of change, sketch a graph that represents a rate of change, and solve related problems (D1.3, D1.4, D1.5, D1.6, D1.9)
- connect the slope of a secant with the slope of a tangent and the instantaneous rate of change, and determine the approximate slope of a tangent at a given point (D1.7, D1.8)
- recognize real-world applications of combinations of functions, and solve related problems graphically (D2.2)
- compare the characteristics of various functions, and apply concepts and procedures to solve problems (D3.1, D3.2)

Prerequisite Skills

Reciprocal Functions

1. Define the term *asymptote* and give an equation for the vertical asymptote of $f(x) = \frac{1}{x}$.
2. For each reciprocal function, write equations for the vertical and horizontal asymptotes. Use transformations to sketch each graph relative to the base function $f(x) = \frac{1}{x}$.
   a) $f(x) = \frac{1}{x - 3}$
   b) $f(x) = -\frac{1}{x + 4}$
   c) $f(x) = -\frac{3}{x - 8}$
   d) $f(x) = \frac{1}{2x - 10}$

Domain and Range

3. Write the domain and the range of each function.
   a) $f(x) = 4x - 2$
   b) $f(x) = 2(x - 1)^2 + 4$
   c) $f(x) = x^3$
   d) $f(x) = \frac{1}{x}$
   e) $f(x) = \frac{1}{x - 4}$
   f) $f(x) = -\frac{3}{x}$

Slope

4. Find the slope of the line passing through the points in each pair. Leave your answers in fraction form.
   a) $(3, -5)$ and $(2, 8)$
   b) $(-4, 0)$ and $(5, 3)$
   c) $(2, 2)$ and $(-7, 4)$
   d) $(0, 9)$ and $(1, 8)$
   e) $(1, 7)$ and $(2, -6)$
   f) $(3, 3)$ and $(-7, -9)$
5. Find the slope of the line passing through the points in each pair. Round your answers to two decimal places.
   a) $(7, 10)$ and $(-1, 7)$
   b) $(0, 6)$ and $(7, 11)$
   c) $(-4, 2)$ and $(7, 4)$
   d) $(-2, -1)$ and $(11, 4)$
   e) $(-6.6, -5.2)$ and $(1.5, -0.9)$
   f) $(5.8, -3.2)$ and $(10.1, -1.7)$

Factor Polynomials

6. Factor fully for $x \in \mathbb{R}$.
   a) $x^2 + 7x + 12$
   b) $5x^2 - 17x + 6$
   c) $6x^2 + 13x - 8$
   d) $x^3 + 2x^2 - 5x - 6$
   e) $12x^3 + 4x^2 - 5x - 2$
   f) $27x^3 - 64$

Solve Quadratic Equations

7. Find the roots of each quadratic equation.
   a) $x^2 - 4x - 32 = 0$
   b) $x^2 + 6x + 5 = 0$
   c) $2x^2 - 9x + 9 = 0$
   d) $6x^2 + 31x + 5 = 0$
   e) $2x^2 + 13x - 7 = 0$
   f) $3x^2 - 13x - 30 = 0$
8. Determine the x-intercepts. Express your answers in exact form.
   a) $y = x^2 - 4x + 2$
   b) $y = 2x^2 + 8x + 1$
   c) $y = -3x^2 + 5x + 4$
   d) $y = 5x^2 + 2x + 8$
   e) $y = 3x^2 + 8x + 2$
   f) $y = -x^2 + 2x + 7$

Solve Inequalities

9. Solve each inequality. Show your answers on a number line.
   a) $2x - 5 > 7$
   b) $4x + 9 \geq 6x - 2$
   c) $4x < 8x - 2$
   d) $2x + 1 > x - 4$
   e) $3x + 4 > x - 1$
   f) $x - 7 < 2x + 2$

10. Solve each inequality. Show your answers on a number line.
    a) $x^2 - 4 \leq 0$
    b) $x^2 - 3x - 18 > 0$
    c) $2x^2 + 4 < 30$
    d) $3x^2 + 5x - 12 > 2x^2 + 2x - 2$
    e) $2x^2 - x + 4 < x^2 - 9x - 3$
    f) $x^2 + 2x + 2 > -x^2 - 9x + 8$

CHAPTER PROBLEM

As we continue to discover the interconnectedness of our ecosystem with our environment and the world around us, care for the environment and knowledge of the effects of our actions have become increasingly important. Pollution can come in the form of chemical spills, carbon monoxide emissions, excess light in the night sky, loud sounds, and pesticides sprayed on our lawns. In this chapter, you will analyse some mathematical models of various types of pollution, including the costs of cleaning them up.

3.1 Reciprocal of a Linear Function

When you add, subtract, or multiply two polynomial functions, the result is another polynomial function. When you divide polynomial functions, the result is a **rational function**. Because division by zero is undefined, rational functions have special properties that polynomial functions do not have. These types of functions occur, for example, when expressing velocity, v, in terms of distance, d, and time, t, $v = \frac{d}{t}$, or with levers, where force is inversely proportional to the distance from the pivot point, Force $= \frac{\text{work}}{\text{distance}}$.

Investigate What is the nature of the graph of a reciprocal function?

Tools
- grid paper
- graphing calculator

or
- computer with *The Geometer's Sketchpad*®

Technology Tip

If you are using a graphing calculator, use a friendly window to avoid strange joining lines. Refer to the Extension on page 156.

CONNECTIONS

You learned how to estimate the slope of a curve at a chosen point in Chapter 1. See Section 1.5.

It is recommended that technology be used as a tool in this investigation, if available.

1. Consider the function $f(x) = \frac{1}{x}$.
   a) State the restriction on the domain. Explain your reasoning.
   b) Make a table of values and sketch a graph of $f(x) = \frac{1}{x}$.
   c) Describe what happens to the function as x approaches 0
      i) from the left
      ii) from the right
   d) Describe what happens to the function as x approaches
      i) negative infinity $(-\infty)$
      ii) positive infinity $(+\infty)$
   e) Estimate the slope of the curve at $x = 0.1$, at $x = 1$, and at $x = 10$. Repeat for $x = -0.1$, $x = -1$, and $x = -10$.
   f) **Reflect** Describe the graph in terms of asymptotes and slope.
2. Consider the function $g(x) = \frac{1}{2x - 3}$.
   a) State the restriction on the domain. Explain your reasoning.

b) Sketch a graph of $g(x) = \frac{1}{2x - 3}$.

c) Describe what happens to the function as x approaches the restricted value

i) from the left

ii) from the right

d) Describe what happens to the function

i) as $x \to +\infty$

ii) as $x \to -\infty$

e) Estimate the slope of the curve at $x = 1.6$, at $x = 2$, and at $x = 10$. Repeat for $x = 1.4$, $x = 0$, and $x = -10$.

f) **Reflect** Describe the graph in terms of asymptotes and slope.

3. **Reflect** Generalize your observations to describe the function $f(x) = \frac{1}{kx - c}$. Include the domain, range, and asymptotes in your description.

CONNECTIONS

"As $x \to +\infty$" is symbolic for "as x approaches positive infinity."

"As $x \to -\infty$" is symbolic for "as x approaches negative infinity."

Example 1 Domain, Range, and Asymptotes

Consider the function $f(x) = \frac{1}{2x - 1}$.

a) State the domain.

b) Describe the behaviour of the function near the vertical asymptote.

c) Describe the end behaviour.

d) Sketch a graph of the function.

e) State the range.

Solution

a) Since division by zero is not defined, the denominator gives a restriction:

$$2x - 1 \neq 0$$

$$x \neq \frac{1}{2}$$

Domain: $\left\{x \in \mathbb{R}, x \neq \frac{1}{2}\right\}$

CONNECTIONS

The domain can also be written using interval notation as $x \in \left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, +\infty\right)$.

b) The tables show the behaviour of the function as $x \to \frac{1}{2}$ from the left and from the right.

As $x \to \frac{1}{2}^-$:

x	$f(x)$
0	−1
0.4	−5
0.45	−10
0.49	−50

As $x \to \frac{1}{2}^+$:

x	$f(x)$
1	1
0.6	5
0.55	10
0.51	50

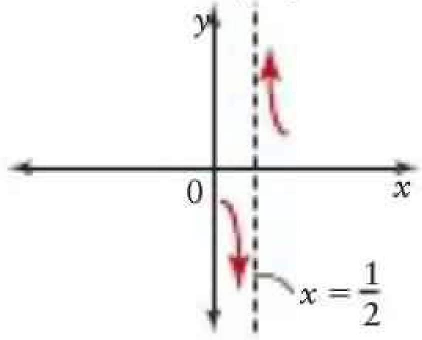
CONNECTIONS

$x \to a^+$ means as x approaches a from the right.

$x \to a^-$ means as x approaches a from the left.

As $x \to \frac{1}{2}^-$, $f(x) \to -\infty$, and as $x \to \frac{1}{2}^+$, $f(x) \to +\infty$. The function approaches a vertical line at $x = \frac{1}{2}$, but does not cross it. So, the curve is discontinuous at this line. This line is called a vertical asymptote. An equation for the vertical asymptote is $x = \frac{1}{2}$.

Behaviour of $y = f(x)$ near the vertical asymptote $x = \frac{1}{2}$



c) The tables show the end behaviour as $x \to -\infty$ and $x \to +\infty$.

As $x \to -\infty$:

x	$f(x)$
-10	$-\frac{1}{21}$
-100	$-\frac{1}{201}$
$-1\,000$	$-\frac{1}{2\,001}$
$-10\,000$	$-\frac{1}{20\,001}$

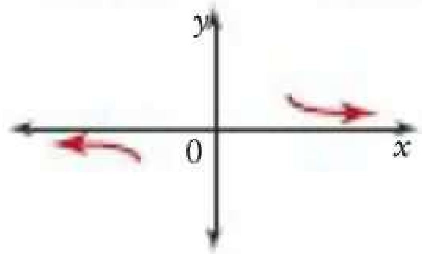
As $x \to +\infty$:

x	$f(x)$
10	$\frac{1}{19}$
100	$\frac{1}{199}$
1 000	$\frac{1}{1\,999}$
10 000	$\frac{1}{19\,999}$

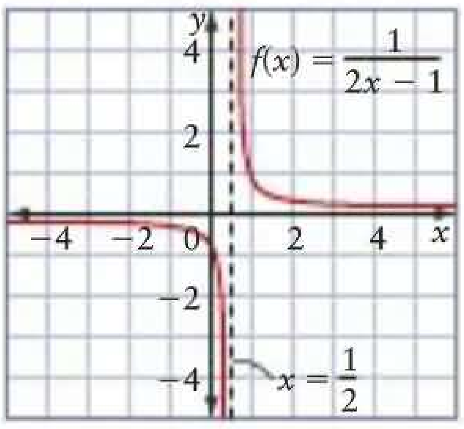
As $x \to +\infty$, $f(x) \to 0$ from above, since all values of $f(x)$ are positive. The graph approaches a horizontal line at the x-axis, but does not cross it. The horizontal asymptote at $+\infty$ has equation $y = 0$.

As $x \to -\infty$, $f(x) \to 0$ from below, since all values of $f(x)$ are negative. The horizontal asymptote at $-\infty$ has equation $y = 0$.

Behaviour of $y = f(x)$ as $x \to -\infty$ and as $x \to +\infty$



d)



e) The graph of the function shows that $f(x)$ gets close to the line $y = 0$ but never actually touches that line. Therefore, the only restriction on the range of $f(x)$ is that $y \neq 0$.

Range: $\{y \in \mathbb{R}, y \neq 0\}$

Example 2 Find Intercepts

Determine the x- and y-intercepts of the function $g(x) = \frac{2}{x + 5}$.

Solution

For the x-intercept, let $g(x) = 0$.

$$\frac{2}{x + 5} = 0$$

There is no value of x that makes this equation true. Therefore, there is no x-intercept.

For the y-intercept, let $x = 0$.

$$\begin{aligned} g(0) &= \frac{2}{0 + 5} \\ &= \frac{2}{5} \end{aligned}$$

The y-intercept is $\frac{2}{5}$.

CONNECTIONS

The x-values that result in a zero in the numerator are the x-intercepts of the function, provided that the denominator is not zero.

Example 3 Rate of Change

Describe the intervals where the slope is increasing and the intervals where it is decreasing in the two branches of the rational function $h(x) = -\frac{1}{5x + 2}$.

Solution

The two branches of the function are on either side of the vertical asymptote. For the vertical asymptote, let the denominator equal zero.

$$\begin{aligned} 5x + 2 &= 0 \\ x &= -0.4 \end{aligned}$$

The vertical asymptote has equation $x = -0.4$.

Select a few points to the left and to the right of the asymptote and analyse the slope.

Select two points to the left of $x = -0.4$. Choosing consecutive x-values gives a denominator of 1. So, the slope is the difference in y-values.

At $x = -10$, $f(x) \doteq 0.021$.
At $x = -9$, $f(x) \doteq 0.023$.

$$\begin{aligned} \text{Slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &\doteq \frac{(0.023) - (0.021)}{(-9) - (-10)} \\ &= 0.002 \end{aligned}$$

CONNECTIONS

In Chapter 1, you investigated rates of change. You can use the skills you learned in that chapter with rational functions.

Now, select two points closer to $x = -0.4$:

At $x = -4$, $f(x) \doteq 0.056$.
At $x = -3$, $f(x) \doteq 0.077$.

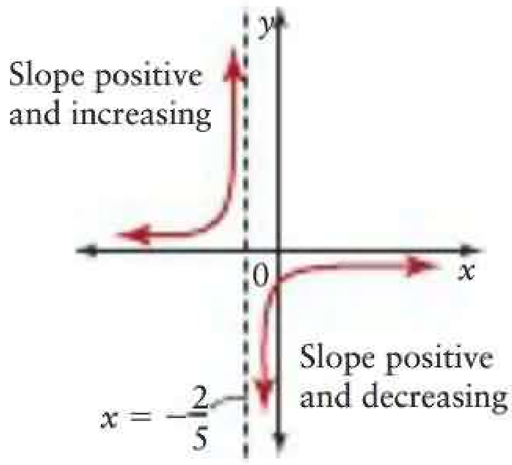
$$\begin{aligned}\text{Slope} &= f(-3) - f(-4)\\ &\doteq 0.021\end{aligned}$$

Because $0.021 > 0.002$ (the slope between the previous two points), the slope is positive and increasing within the interval $x < -0.4$.

Select two points to the right of $x = -0.4$:

At $x = 2$, $f(x) \doteq -0.083$.
At $x = 3$, $f(x) \doteq -0.059$.

$$\begin{aligned}\text{Slope} &= f(3) - f(2)\\ &\doteq 0.024\end{aligned}$$



Now, select two points closer to $x = -0.4$:

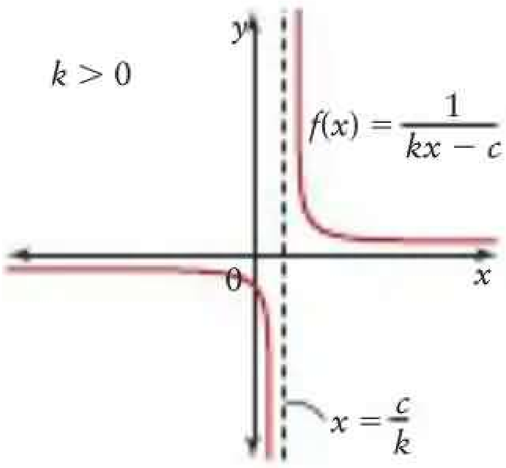
At $x = 0$, $f(x) = -0.5$.
At $x = 1$, $f(x) \doteq -0.143$.

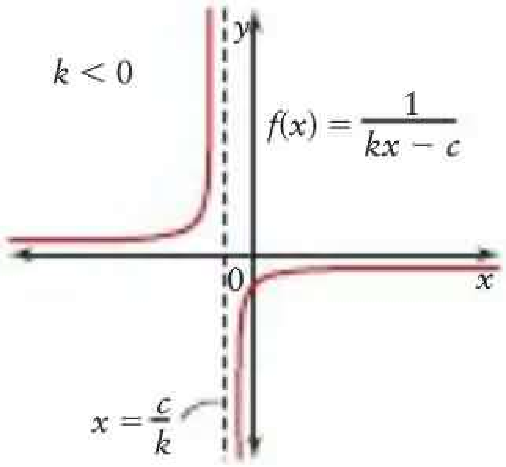
$$\begin{aligned}\text{Slope} &= f(1) - f(0)\\ &\doteq 0.357\end{aligned}$$

Because $0.357 > 0.024$, the slope is positive and decreasing within the interval $x > -0.4$.

KEY CONCEPTS

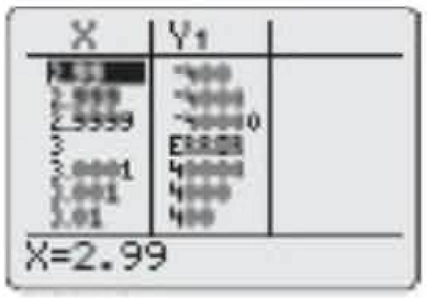
- The reciprocal of a linear function has the form $f(x) = \dfrac{1}{kx - c}$.
- The restriction on the domain of a reciprocal linear function can be determined by finding the value of x that makes the denominator equal to zero, that is, $x = \dfrac{c}{k}$.
- The vertical asymptote of a reciprocal linear function has an equation of the form $x = \dfrac{c}{k}$.
- The horizontal asymptote of a reciprocal linear function has equation $y = 0$.
  - If $k > 0$, the left branch of a reciprocal linear function has a negative, decreasing slope, and the right branch has a negative, increasing slope.
  - If $k < 0$, the left branch of a reciprocal function has positive, increasing slope, and the right branch has positive, decreasing slope.





Communicate Your Understanding

C1 The calculator screen gives a table of values for the function $f(x) = \frac{4}{x-3}$. Explain why there is an error statement.



C2 **a)** For the reciprocal of any linear function, as the denominator increases, what happens to the function value?

b) For the reciprocal of any linear function, as the denominator approaches zero, what happens to the function value?

C3 Can you find an example of a linear function whose reciprocal has no restrictions on either the domain or range? If yes, give an example. If no, explain.

A Practise

For questions 1 and 2, refer to Example 1.

1. Copy and complete each table to describe the behaviour of the function as x approaches each key value.

a) $f(x) = \frac{1}{x-2}$

As $x \rightarrow$	$f(x) \rightarrow$
2^+	
2^-	
$+\infty$	
$-\infty$	

b) $f(x) = \frac{1}{x+5}$

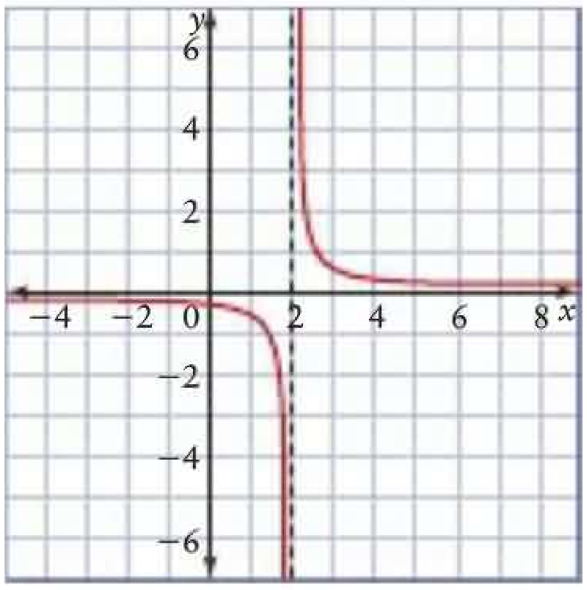
As $x \rightarrow$	$f(x) \rightarrow$
-5^+	
-5^-	
$+\infty$	
$-\infty$	

c) $f(x) = \frac{1}{x+8}$

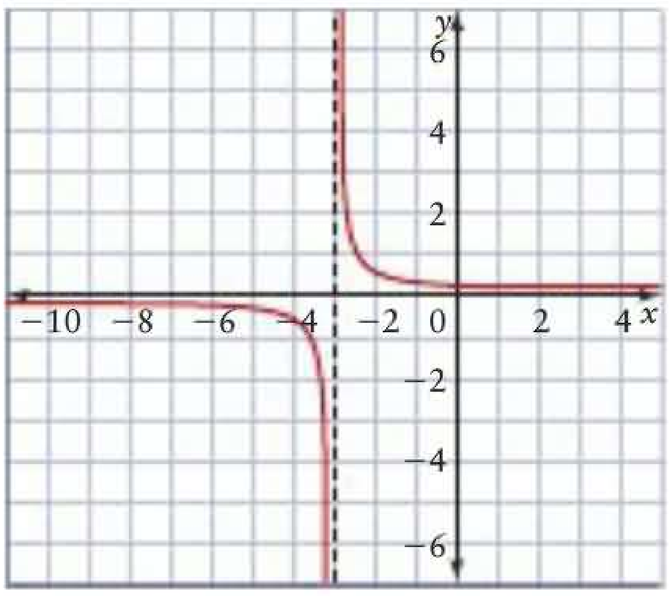
As $x \rightarrow$	$f(x) \rightarrow$
8^+	
8^-	
$+\infty$	
$-\infty$	

2. a) Write equations to represent the horizontal and vertical asymptotes of each rational function.

i)



ii)



b) Write a possible equation for each function in part a).

For help with questions 3 to 5, refer to Example 2.

3. For each reciprocal function,

i) write an equation to represent the vertical asymptote

ii) write an equation to represent the horizontal asymptote

iii) determine the y-intercept

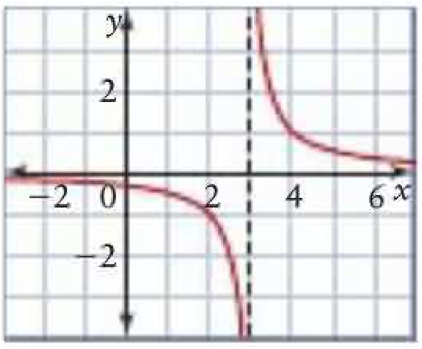
a) $f(x) = \dfrac{1}{x - 5}$ **b)** $g(x) = \dfrac{2}{x + 6}$

c) $h(x) = \dfrac{5}{1 - x}$ **d)** $k(x) = -\dfrac{1}{x + 7}$

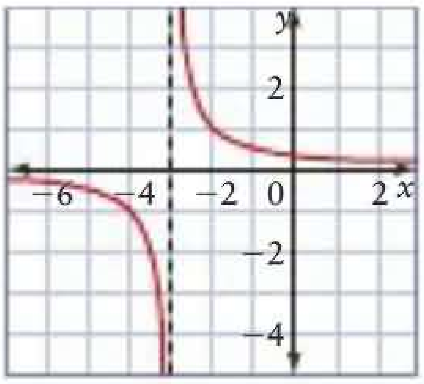
4. Use Technology Verify the vertical asymptotes in question 3 using technology.

5. Determine a possible equation to represent each function shown.

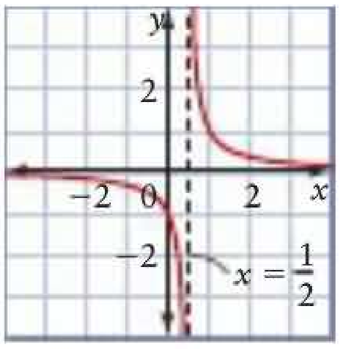
a)



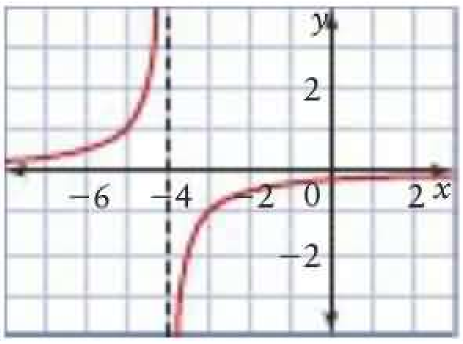
b)



c)



d)



For help with question 6, refer to Example 3.

6. Sketch each function and then describe the intervals where the slope is increasing and the intervals where it is decreasing.

a) $f(x) = \dfrac{1}{x - 3}$

b) $k(x) = \dfrac{3}{2x + 7}$

c) $m(x) = -\dfrac{2}{x + 4}$

d) $p(x) = \dfrac{5}{3 - 2x}$

B Connect and Apply

7. Sketch a graph of each function. Label the y-intercept. State the domain, range, and equations of the asymptotes.

a) $f(x) = \dfrac{1}{x - 1}$ **b)** $g(x) = \dfrac{1}{x + 4}$

c) $h(x) = \dfrac{1}{2x + 1}$ **d)** $k(x) = -\dfrac{1}{x + 4}$

e) $m(x) = -\dfrac{3}{2x - 5}$ **f)** $n(x) = \dfrac{4}{5 - x}$

g) $p(x) = \dfrac{1}{\left(x - \frac{1}{4}\right)}$ **h)** $q(x) = -\dfrac{3}{\left(x + \frac{1}{2}\right)}$

8. Determine the equation in the form $f(x) = \dfrac{1}{kx - c}$ for the function with a vertical asymptote at $x = 1$ and a y-intercept at -1.

9. Determine the equation in the form $f(x) = \dfrac{1}{kx - c}$ for the function with a vertical asymptote at $x = -1$ and a y-intercept at -0.25.

10. The time required to fly from one location to another is inversely proportional to the average speed. When the average speed to fly from Québec City to Vancouver is 350 km/h, the flying time is 11 h.

a) Write a function to represent the time as a function of the speed.

b) Sketch a graph of this function.

c) How long would the trip from Québec to Vancouver take at an average speed of 500 km/h?

d) Describe the rate of change of the time as the average speed increases.

> CONNECTIONS
>
> If two variables, x and y, are inversely proportional, then $y = \frac{k}{x}$, where k is a constant.

11. a) Investigate a variety of functions of the form $f(x) = \frac{1}{bx + 2}$.

b) What is the effect on the graph when the value of b is varied?

12. Use the results from question 11 to sketch a graph of each function.

a) $f(x) = \frac{1}{x - 5}$

b) $g(x) = \frac{1}{2x - 5}$

c) $h(x) = \frac{1}{3x - 5}$

13. The force required to lift an object is inversely proportional to the distance of the force from the fulcrum of a lever. A force of 200 N is required at a point 3 m from the fulcrum to lift a certain object.

a) Determine a function to represent the force as a function of the distance.

b) Sketch a graph of this function.

c) How much force is required to lift this object at a point 2 m from the fulcrum?

d) What is the effect on the force needed as the distance from the fulcrum is doubled?

> CONNECTIONS
>
> A fulcrum is the pivot on which a lever turns.

C Extend and Challenge

14. Analyse the key features (domain, range, vertical asymptotes, and horizontal asymptotes) of each function, and then sketch the function.

a) $f(x) = \frac{1}{\sqrt{x}}$

b) $g(x) = \frac{1}{|x|}$

c) $f(x) = \frac{3}{x - 2} + 4$

15. Graph the line $y = 2x - 5$ and find the x-intercept. Analyse the reciprocals of the y-coordinates on either side of the x-intercept. How do these numbers relate to the key features of the function $f(x) = \frac{1}{2x - 5}$?

16. Math Contest Solve for x in terms of y and z: $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$

17. Math Contest Given that $3a = 75b$, find the value of $\frac{3a - 5b}{5b}$.

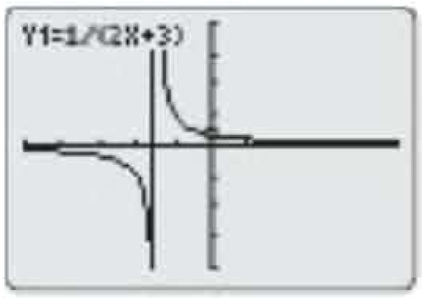
18. Math Contest Two points are chosen on the unit circle with centre at the origin. What is the probability that the distance between these two points is at least 1?

A $\frac{1}{4}$ **B** $\frac{1}{2}$ **C** $\frac{3}{4}$ **D** $\frac{1}{3}$ **E** $\frac{2}{3}$

Extension

Asymptotes and the TI-83 Plus or TI-84 Plus Graphing Calculator

A: *Friendly Windows*



The viewing window on the TI-83 Plus or TI-84 Plus graphing calculator is 94 pixels by 62 pixels. If the graph has a vertical asymptote that falls between two consecutive plot points, the calculator draws an almost vertical drag line, or "joining line." This occurs because the calculator connects the two points that span the asymptote, one with a positive y-coordinate and the other with a negative y-coordinate.

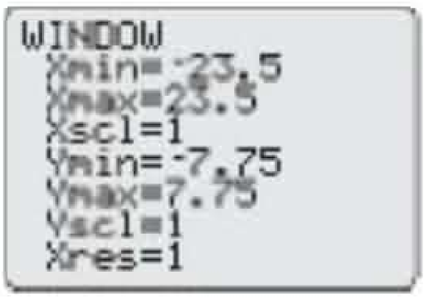
You can avoid this by using a "friendly window." The x-axis is 94 pixels across. When the calculator divides the pixels evenly into negative and positive integers, each pixel represents one tick mark. Therefore, x-values can go from -47 to 47. The y-axis has 62 pixels vertically and stretches from -31 to 31. When you press (ZOOM) and select **4:ZDecimal**, you get a small friendly window.

You can obtain other friendly windows by multiplying 94 and 62 by the same constant.

1. Graph the function shown in the first screen in the margin using the friendly window shown in the second screen. Compare the two graphs.

2. Is each a friendly window? Explain why or why not.

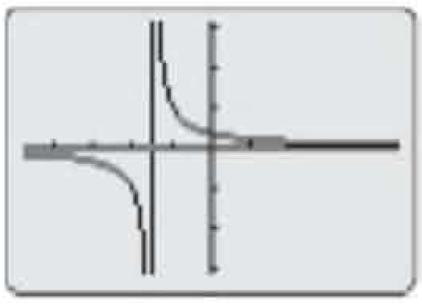
a) 

b) 

3. How can you obtain a friendly window that has Xmin = 0?

B: *Drawing in Asymptotes*

In some situations it may be helpful to actually draw in the asymptote(s) properly. This can be done using the **DRAW** feature.



- Press [2nd] (PRGM) to obtain the **DRAW** menu.
- Select **3:Horizontal** for a horizontal asymptote.
- Select **4:Vertical** for a vertical asymptote.
- Press (ENTER).
- Move the cursor to the coordinate on the axis that the asymptote will pass though.
- Press (ENTER).

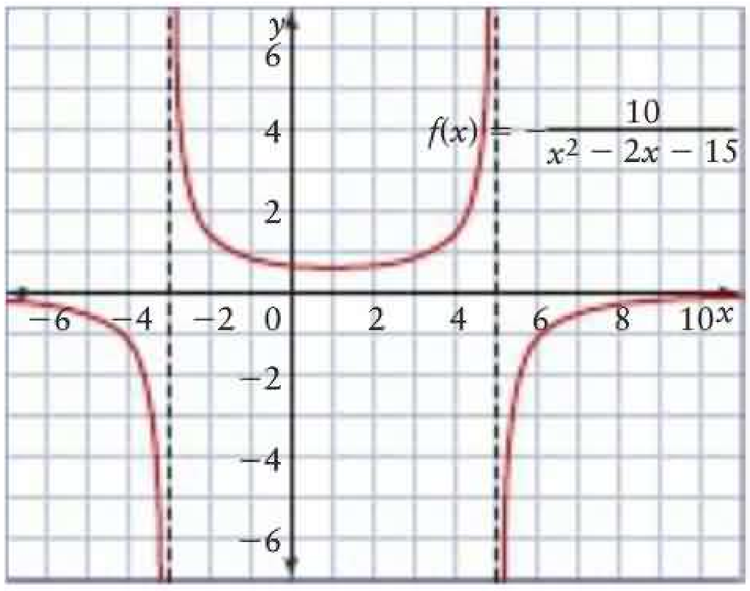
You can clear the line by selecting **1:ClrDraw** from the **DRAW** menu.

For example, an asymptote for the function $f(x) = -\dfrac{1}{2x - 3}$ is shown.

1. Use a graphing calculator to check your answers for question 3 or 7 in Section 3.1.

3.2 Reciprocal of a Quadratic Function

In Section 3.1, the rational functions you analysed had linear denominators. However, rational functions can have polynomials of any degree, such as a quadratic (or second-degree polynomial), in the numerator and denominator. Because quadratics have possibly zero, one, or two x-intercepts, a parabolic shape, and a maximum or minimum point, plotting their reciprocals becomes fairly complex. In this section, you will analyse and graph functions such as $f(x) = -\frac{10}{x^2 - 2x - 15}$.



Investigate What is the nature of the graph of the reciprocal of a quadratic function?

It is recommended that technology be used as a tool in this investigation, if available.

Tools
- grid paper
- graphing calculator

or
- computer with *The Geometer's Sketchpad*®

1. Create a graph of the function $f(x) = \frac{1}{x^2}$. Describe the key features of the graph:

a) horizontal asymptote

b) vertical asymptote

c) intercepts

d) domain and range

e) end behaviour

f) positive and negative intervals

g) increasing and decreasing intervals

2. Consider the function $g(x) = \frac{1}{(x - 2)(x + 1)}$.

a) Determine the restrictions on x and state the domain of $g(x)$.

b) **Use Technology** Graph the function using technology.

c) Describe the behaviour of the graph to the left and to the right of each asymptote.

d) Investigate the slope, and the change in slope, for the following intervals:

i) to the left of the left-most vertical asymptote

ii) to the right of the right-most vertical asymptote

e) Devise a plan for determining the maximum point in the interval between the asymptotes.

f) Use your plan from part e) to determine the coordinates of the maximum point between the asymptotes.

g) State the range of the function.

h) Explain how the graph of $g(x)$ is different from the graph of $f(x)$ in step 1.

3. Consider the function $h(x) = -\frac{2}{(x-1)^2}$.
   a) Determine the restrictions on x and state the domain of $h(x)$.
   b) **Use Technology** Graph the function using technology.
   c) Describe the behaviour of the graph to the left and to the right of the vertical asymptote.
   d) Investigate the slope, and the change in slope, for the following intervals:
      i) to the left of the vertical asymptote
      ii) to the right of the vertical asymptote
   e) State the range of the function.
   f) Explain how the graph of $h(x)$ is different from the graphs in steps 1 and 2.
4. a) Without using technology, analyse the graph of the function $k(x) = \frac{2}{x^2 + 3x + 2}$ under the following headings:
      i) Horizontal and vertical asymptotes
      ii) Domain and range
      iii) Behaviour of the slope
         - positive or negative
         - increasing or decreasing
      iv) End behaviour
      v) Intercepts
   b) Sketch a graph of the function.
   c) **Use Technology** Verify that your analysis and graph are correct using technology.
5. **Reflect** Describe how the graphs of reciprocals of quadratic functions behave differently from the graphs of reciprocals of linear functions.

Example 1 Find the Domain, Range, and Asymptotes

Consider the function $f(x) = \frac{2}{x^2 - 4}$.

a) State the domain.
b) Find the equations of the asymptotes. Describe the behaviour of the function near the asymptotes.
c) Determine the x- and y-intercepts.
d) Sketch a graph of the function.
e) State the range.

Solution

a) The function can be rewritten by factoring the difference of squares in the denominator.

$$f(x) = \frac{2}{(x+2)(x-2)}$$

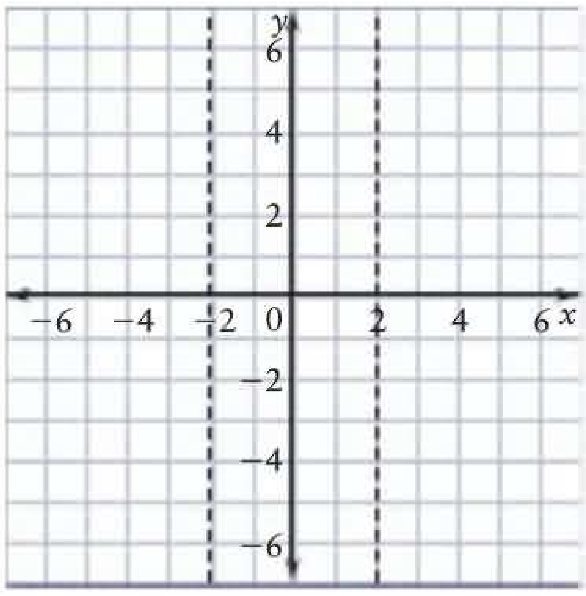
Because the denominator cannot equal zero, there are restrictions at $x = 2$ and $x = -2$.

Domain: $\{x \in \mathbb{R}, x \neq -2, x \neq 2\}$

b) The vertical asymptotes have equations $x = -2$ and $x = 2$.

As $x \to \pm\infty$, the denominator approaches $+\infty$, so $f(x)$ approaches 0. Thus, $f(x)$ approaches a horizontal line at $y = 0$, but does not cross it.

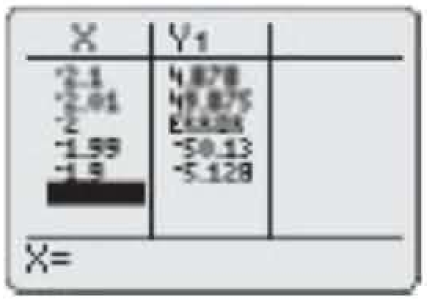
The horizontal asymptote has equation $y = 0$.



Check selected points near the vertical asymptotes.

As $x \to -2^-$, the function is positive and increasing toward $+\infty$.

As $x \to -2^+$, the function is negative and decreasing toward $-\infty$.



As $x \to 2^-$, the function is negative and decreasing toward $-\infty$ (i.e., negative slope).

As $x \to 2^+$, the function is positive and increasing toward $+\infty$ (i.e., negative slope).

A table can be used to summarize the above observations.

Interval of x	$x < -2$	$-2 < x < 2$	$x > 2$
Sign of $f(x)$	+	−	+
Sign of Slope	+	+ to −	−

c) For the x-intercepts, let $f(x) = 0$:

$$\frac{2}{(x + 2)(x - 2)} = 0$$

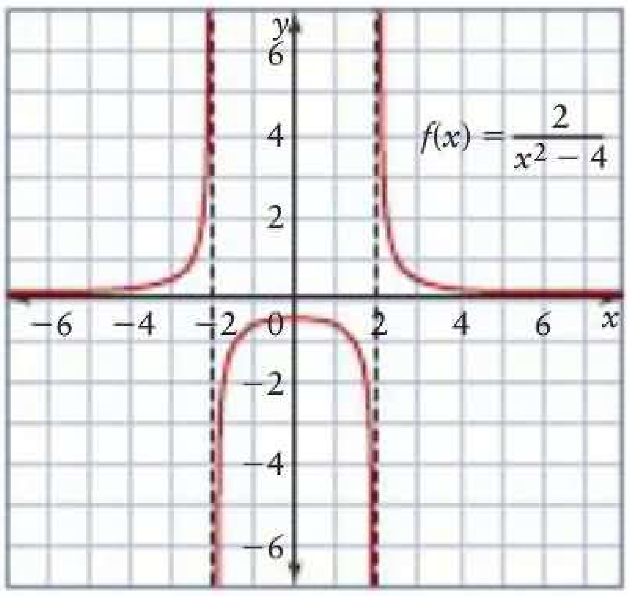
Since there is no value of x that makes this statement true, there is no x-intercept.

For the y-intercept, let $x = 0$:

$$f(0) = \frac{2}{(0 + 2)(0 - 2)}$$
$$= -0.5$$

Because of the symmetry of the function, since this point lies exactly halfway between the two vertical asymptotes, $(0, -0.5)$ is the maximum point for this interval.

d)



e) Range: $y > 0$ or $y \le -0.5$

Example 2 Rate of Change

Describe the increasing and decreasing intervals in the branches of the function $f(x) = -\frac{1}{2x^2 - x - 6}$. Then, graph the function.

Solution

The function can be rewritten as $f(x) = -\frac{1}{(2x + 3)(x - 2)}$ by factoring the denominator.

The restrictions on the domain occur at $x = -1.5$ and $x = 2$.

The intervals to be analysed are (i) $x < -1.5$, (ii) $-1.5 < x < 2$, and (iii) $x > 2$.

Select points in each interval and approximate the slope at each point.

i) Consider the interval $x < -1.5$.
At $x = -4$, $f(x) \doteq -0.033\,33$.

x	y	Slope of secant with $(-4, -0.033\,33)$
−3.9	−0.035 31	$\frac{(-0.035\,31) - (-0.033\,33)}{(-3.9) - (-4)}$ $= -0.0198$
−3.99	−0.033 52	−0.019
−3.999	−0.033 35	−0.02

The slope is approximately -0.02 at the point $(-4, -0.033\,33)$.

At $x = -2$, $f(x) = -0.25$.

x	y	Slope of secant with $(-2, -0.25)$
−2.1	−0.203 25	−0.467 5
−2.01	−0.244 49	−0.551
−2.001	−0.249 44	−0.56

The slope is approximately −0.56 at the point (−2, −0.25).

The slope is negative and decreasing on the interval $x < -1.5$.

ii) Consider the interval $x > 2$. At $x = 3$, $f(x) \doteq -0.111\ 11$.

x	y	Slope of secant with $(3, -0.111\ 11)$
3.1	−0.098 81	0.123
3.01	−0.109 77	0.134
3.001	−0.110 98	0.13

The slope is approximately 0.13 at the point (3, −0.111 11).

At $x = 4$, $f(x) \doteq -0.045\ 45$.

x	y	Slope of secant with $(4, -0.045\ 45)$
4.1	−0.042 52	0.029 3
4.01	−0.045 15	0.03
4.001	−0.045 42	0.03

The slope is approximately 0.03 at the point (4, −0.045 45).

The slope is positive and decreasing on the interval $x > 2$.

iii) Consider the interval $-1.5 < x < 2$. As can be seen in the Investigate and Example 1, there will be a maximum or a minimum point between the asymptotes. Just as a quadratic has the x-coordinate of its maximum or minimum point exactly halfway between the x-intercepts, the reciprocal of a quadratic has the x-coordinate of its maximum or minimum point exactly halfway between the vertical asymptotes.

For the midpoint:

$$x = \frac{-1.5 + 2}{2}$$
$$= 0.25$$

$$f(0.25) = \frac{8}{49}$$
$$\doteq 0.163\,27$$

Consider points just to the left of (0.25, 0.163 27).

x	y	Slope of secant with $(0.25, 0.163\ 27)$
0.23	0.163 29	−0.001
0.24	0.163 27	0

Consider points just to the right of (0.25, 0.163 27).

x	y	Slope of secant with (0.25, 0.163 27)
0.26	0.163 27	0
0.27	0.163 29	0.001

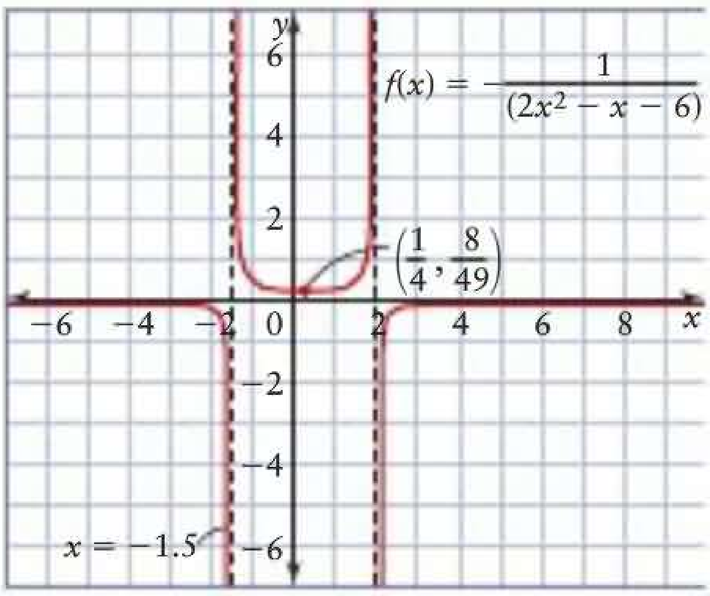
For the interval $-1.5 < x < 0.25$, the slope is negative and increasing.

For the interval $0.25 < x < 2$, the slope is positive and increasing.

At $x = 0.25$, the slope is 0.

This can all be summarized in a table.

Interval	$x < -1.5$	$-1.5 < x < 0.25$	$x = 0.25$	$0.25 < x < 2$	$x > 2$
Sign of $f(x)$	−	+	+	+	−
Sign of slope	−	−	0	+	+
Change in slope	− decreasing	+ increasing		+ increasing	− decreasing



Example 3 Describe the Key Features of a Function

Analyse the key features of the function $f(x) = \frac{1}{x^2 + 4}$ and sketch its graph.

Solution

For the domain and vertical asymptotes:

Consider the denominator $x^2 + 4$.

Since $x^2 \geq 0$, $x^2 + 4 > 0$ for all values of x.

Therefore, there is no restriction on the domain and there is no vertical asymptote.

Domain: $\mathbb{R}$

For the x-intercepts:

Let $f(x) = 0$.

$$\frac{1}{x^2 + 4} = 0$$

Because there are no values of x that make this statement true, there is no x-intercept.

For the horizontal asymptote:

As $x \to -\infty$, $f(x) \to 0$ and $f(x) > 0$.

As $x \to +\infty$, $f(x) \to 0$ and $f(x) > 0$.

The horizontal asymptote has equation $y = 0$ and the curve lies entirely above the asymptote.

For the y-intercept:

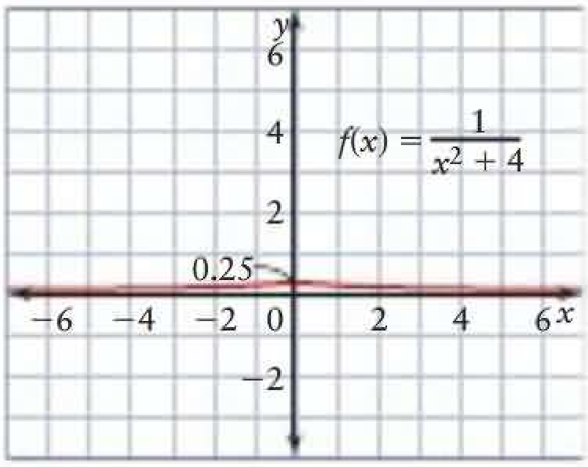
Let $x = 0$.

$$f(x) = \frac{1}{x^2 + 4}$$

$$f(0) = \frac{1}{(0)^2 + 4}$$

$$= 0.25$$

Range: $\{y \in \mathbb{R}, 0 < y \leq 0.25\}$



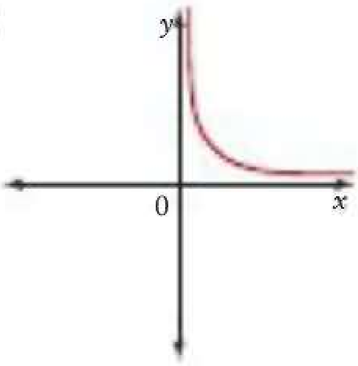
KEY CONCEPTS

- Rational functions can be analysed using key features: asymptotes, intercepts, slope (positive or negative, increasing or decreasing), domain, range, and positive and negative intervals.
- Reciprocals of quadratic functions with two zeros have three parts, with the middle one reaching a maximum or minimum point. This point is equidistant from the two vertical asymptotes.
- The behaviour near asymptotes is similar to that of reciprocals of linear functions.
- All of the behaviours listed above can be predicted by analysing the roots of the quadratic relation in the denominator.

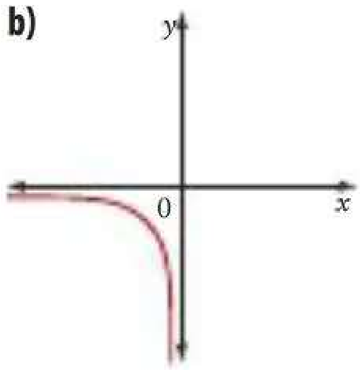
Communicate Your Understanding

C1 Describe the slope and change in slope for each graph.

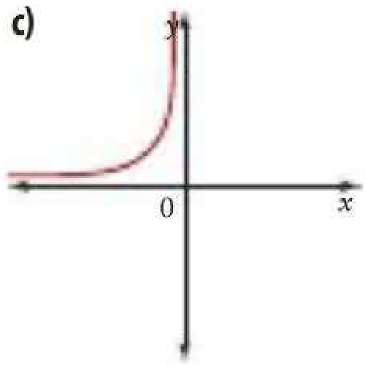
a)



b)



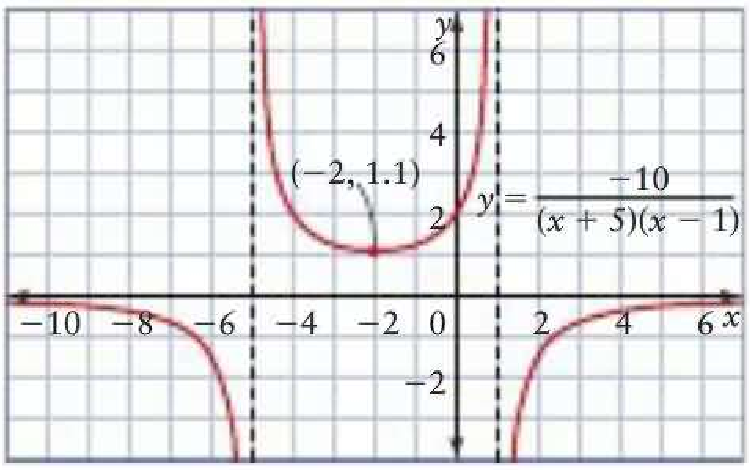
c)



C2 A reciprocal of a quadratic function has the following summary table. Sketch a possible graph of the function.

Interval	$x < -3$	$-3 < x < 1$	$x = 1$	$1 < x < 5$	$x > 5$
Sign of $f(x)$	+	−	−	−	+
Sign of Slope	+	+	0	−	−
Change in Slope	+	−	−	−	+

C3 Describe the key features of the function shown.



A Practise

For help with questions 1 and 2, refer to Example 1.

1. Copy and complete each table to describe the behaviour of the function as x approaches each key value.

a) $f(x) = \dfrac{1}{(x-3)(x-1)}$

As $x \rightarrow$	$f(x) \rightarrow$
3^-	
3^+	
1^-	
1^+	
$-\infty$	
$+\infty$	

b) $f(x) = \dfrac{1}{(x-5)(x+4)}$

As $x \rightarrow$	$f(x) \rightarrow$
-4^-	
-4^+	
5^-	
5^+	
$-\infty$	
$+\infty$	

c) $f(x) = -\dfrac{1}{(x+6)^2}$

As $x \rightarrow$	$f(x) \rightarrow$
-6^-	
-6^+	
$-\infty$	
$+\infty$	

2. Determine equations for the vertical asymptotes, if they exist, for each function. Then, state the domain.

   a) $g(x) = \dfrac{1}{(x - 4)^2}$

   b) $f(x) = \dfrac{1}{(x - 2)(x + 7)}$

   c) $v(x) = \dfrac{1}{x^2 + 1}$

   d) $m(x) = \dfrac{3}{x^2 - 25}$

   e) $h(x) = \dfrac{1}{x^2 - 4x + 3}$

   f) $k(x) = -\dfrac{2}{x^2 + 7x + 12}$

   g) $n(x) = -\dfrac{2}{3x^2 + 2x - 8}$

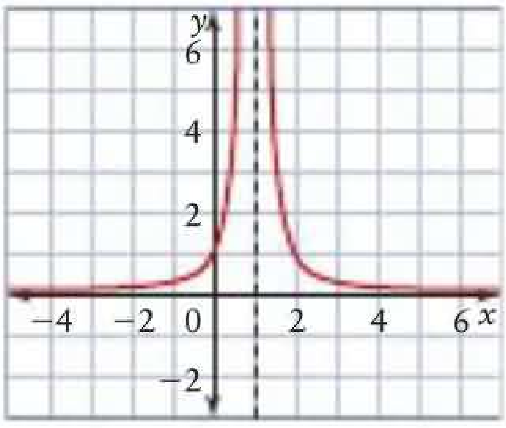
   h) $u(x) = -\dfrac{2}{2x^2 + 3x + 8}$

For help with questions 3 and 4, refer to Example 2.

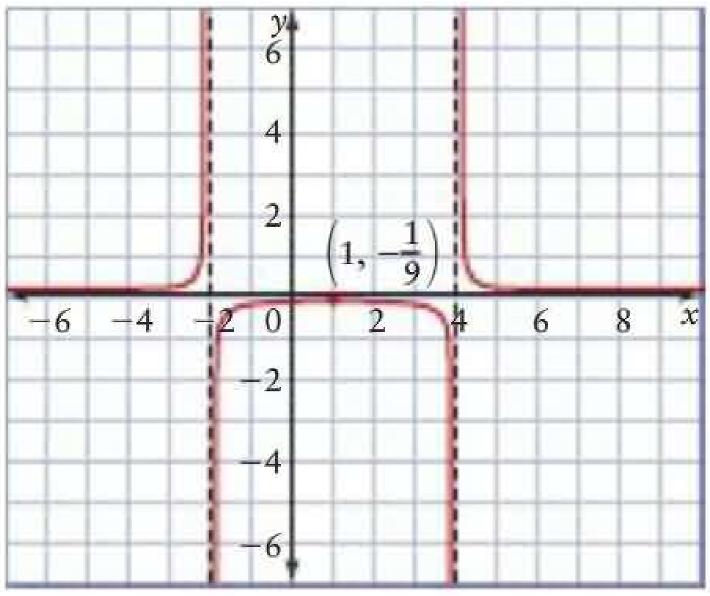
3. Make a summary table with the headings shown for each graph.

Interval	
Sign of Function	
Sign of Slope	
Change in Slope	

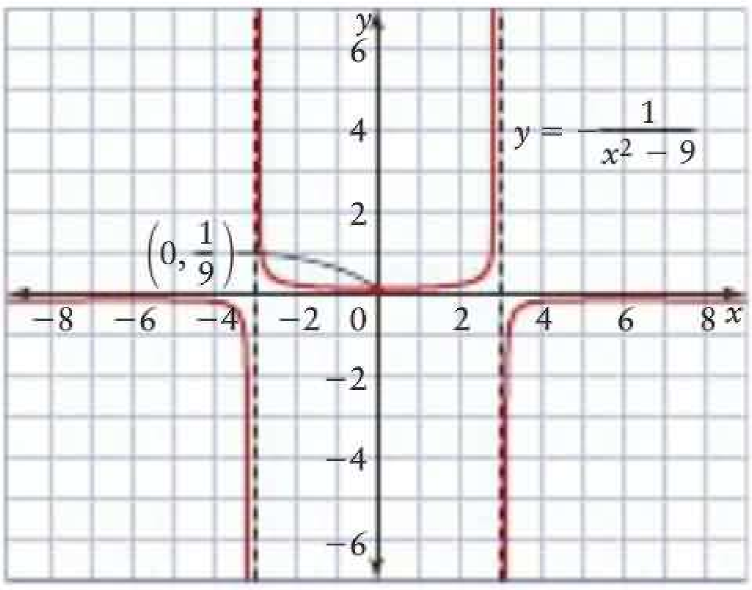
   a)



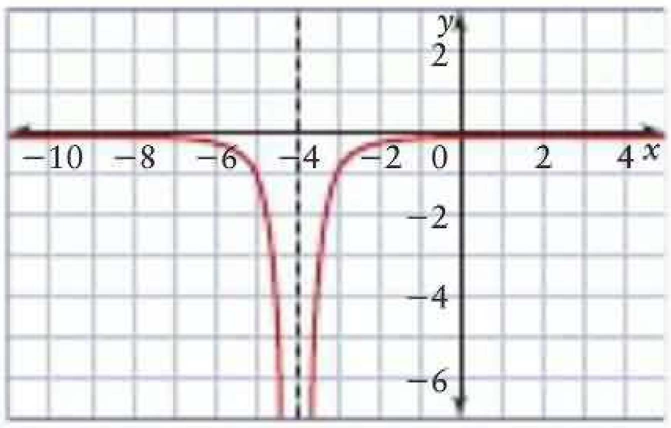
   b)



   c)



   d)



4. Determine a possible equation for each function in question 3.

B Connect and Apply

For help with question 5, refer to Example 3.

5. For each function,
   i) give the domain
   ii) determine equations for the asymptotes
   iii) determine the y-intercepts
   iv) sketch a graph of the function
   v) include a summary table of the slopes
   vi) give the range

   a) $f(x) = \dfrac{1}{x^2 - 9}$

   b) $t(x) = \dfrac{1}{x^2 - 2x - 15}$

   c) $p(x) = -\dfrac{1}{x^2 + 5x - 21}$

   d) $w(x) = \dfrac{1}{3x^2 - 5x - 2}$

   e) $q(x) = \dfrac{1}{x^2 + 2}$

6. For each function in question 5, approximate the instantaneous rate of change at each y-intercept.

7. Recall that a quadratic function with a double zero is tangent to the x-axis at its vertex. Describe the key features of the reciprocal of a perfect square quadratic function, after investigating the graphs of the following functions.

   a) $f(x) = \dfrac{1}{x^2}$

   b) $g(x) = \dfrac{1}{(x-1)^2}$

   c) $h(x) = \dfrac{1}{(x+2)^2}$

8. Sketch each function. Then, determine the intervals where the function is increasing and those where it is decreasing.

   a) $f(x) = \dfrac{1}{x^2 - 1}$

   b) $c(x) = \dfrac{1}{x^2 + 8x + 15}$

   c) $q(x) = \dfrac{4}{x^2 + x - 6}$

   d) $h(x) = -\dfrac{1}{4x^2 - 4x - 3}$

   e) $w(x) = \dfrac{8}{x^2 + 1}$

   f) $g(x) = -\dfrac{1}{(x-6)^2}$

   g) $k(x) = -\dfrac{1}{x^2 + 3}$

   h) $m(x) = \dfrac{1}{9x^2 - 6x + 1}$

9. **a)** Describe how to find the vertex of the parabola defined by the function $f(x) = x^2 + 6x + 11$.

   b) Explain how to use your method in part a) to find the maximum point of the function $g(x) = \dfrac{1}{x^2 + 6x + 11}$.

   c) Use your technique to sketch a graph of each function.

   i) $h(x) = \dfrac{4}{2x^2 - 8x + 9}$

   ii) $k(x) = -\dfrac{5}{x^2 + 5x + 8}$

10. Without graphing, describe the similarities and differences between the graphs of the functions in each pair. Check your answers by graphing.

    a) $f(x) = \dfrac{1}{x^2 - 7x + 12}$, $g(x) = -\dfrac{1}{x^2 - 7x + 12}$

    b) $h(x) = \dfrac{1}{x^2 - 9}$, $k(x) = \dfrac{2}{x^2 - 9}$

    c) $m(x) = \dfrac{1}{x^2 - 4}$, $n(x) = \dfrac{1}{x^2 - 25}$

11. Each function described below is the reciprocal of a quadratic function. Write an equation to represent each function.

    a) The horizontal asymptote is $y = 0$.
    The vertical asymptotes are $x = 2$ and $x = -3$.
    For the intervals $x < -3$ and $x > 2$, $y > 0$.

    b) The horizontal asymptote is $y = 0$.
    There is no vertical asymptote.
    The maximum point is $(0, 0.5)$.
    Domain: $\mathbb{R}$

    c) The horizontal asymptote is $y = 0$.
    The vertical asymptote is $x = -3$.
    Domain: $\{x \in \mathbb{R}, x \neq -3\}$

12. **Chapter Problem** Radiation from the Sun keeps us all alive, but with the thinning of the ozone layer, it is important to limit exposure. The intensity of radiation is inversely proportional to the square of the distance that the Sun's rays travel. The formula $I = \dfrac{k}{d^2}$ models the relationship between intensity, I, in watts per square metre (W/m^2), and distance, d, in astronomical units (AU). The intensity of radiation from the Sun is 9140 W/m^2 on Mercury, which is 0.387 AU away.

    a) Determine an equation relating the intensity of radiation and the distance from the Sun.

    b) Sketch a graph of this relationship.

    c) Determine the intensity of radiation and its rate of change on Earth, which is 1 AU from the Sun.

13. When astronauts go into space, they feel lighter. This is because weight decreases as a person rises above Earth's gravitational pull according to the formula $W(h) = \dfrac{W_e}{\left(1 + \dfrac{h}{6400}\right)^2}$ where W_e is the person's weight, in newtons, at sea level on Earth, and $W(h)$ is the weight at h kilometres above sea level.

a) Sketch a graph of this function for an astronaut whose weight is 820 N at sea level.

b) What is this astronaut's weight at each altitude?

i) 10 km **ii)** 4000 km

c) At what range of altitudes will this astronaut have a weight of less than 10 N?

14. Use your knowledge of transformations to sketch each function.

a) $f(x) = \dfrac{1}{x^2} + 3$

b) $g(x) = \dfrac{1}{x^2 - 9} - 4$

✓Achievement Check

15. Consider the function $f(x) = \dfrac{3}{x^2 - 25}$.

a) Determine any restrictions on x.

b) State the domain and range.

c) State equation(s) for the asymptote(s).

d) Determine any x- and y-intercepts.

e) Sketch a graph of the function.

f) Describe the behaviour of the function as x approaches -5 and 5.

C Extend and Challenge

16. One method of graphing rational functions that are reciprocals of polynomial functions is to sketch the polynomial function and then plot the reciprocals of the y-coordinates of key ordered pairs.

Use this technique to sketch $y = \dfrac{1}{f(x)}$ for each function.

a) $f(x) = 2x$

b) $f(x) = 4x^2$

c) $f(x) = x^2 - 1$

d) $f(x) = x^3$

17. Determine whether the graph of each function is symmetric about the x-axis, the y-axis, the origin, the line $y = x$, the line $y = -x$, or none of these.

a) $y = \dfrac{1}{x^3}$ **b)** $y = \dfrac{1}{x^4}$

18. Sketch each function. Compare your results to those in question 12 and explain the connection.

a) $f(x) = \dfrac{3x^2 + 1}{x^2}$

b) $g(x) = \dfrac{-4x^2 + 37}{x^2 - 9}$

19. **Math Contest** Write a rational equation that cannot have a or b as a root, given that $a, b \in \mathbb{R}$.

20. **Math Contest** The ratio of $a + 5$ to $2a - 1$ is greater than 40%. Solve for a.

21. **Math Contest** Consider the equation $g(z + 1) = \dfrac{g(z - 2)g(z - 1) + 1}{g(z)}$. Find $g(5)$, given that $g(1) = 1$, $g(2) = 2$, and $g(3) = 3$.

22. **Math Contest** A circle with centre O intersects another circle with centre P at points C and D. If $\angle COD = 30°$ and $\angle CPD = 60°$, what is the ratio of the area of the circle with centre O to the area of the circle with centre P?

A $9:1$

B $(\sqrt{6} + \sqrt{2}):1$

C $1:(2 - \sqrt{3})$

D $(6 + 2\sqrt{3}):1$

3.3 Rational Functions of the Form $f(x) = \dfrac{ax + b}{cx + d}$

In Sections 3.1 and 3.2, you investigated rational functions that are reciprocals of polynomial functions. In this section, you will look at polynomial functions in which both the numerator and the denominator are linear expressions. Because there is a variable in both the numerator and the denominator, there are effects on both the vertical and horizontal asymptotes and, as a result, the domain and range. You will see how these concepts are applied to pollutants flowing into a pond in question 10.

Investigate

How can you determine the asymptotes of a rational function of the form $f(x) = \dfrac{ax + b}{cx + d}$?

Tools

- grid paper
- graphing calculator

or

- computer with *The Geometer's Sketchpad®*

1. Investigate the key features of the family of functions of the form $f(x) = \dfrac{x}{cx - 3}$ for $c \in \mathbb{Z}$. Include both negative and positive values of c. Make a table using the headings shown.

c	Equation(s) of Vertical Asymptote(s)	Equation of Horizontal Asymptote	Comparison to $f(x) = \dfrac{x}{x - 3}$

2. a) Describe the nature of the vertical asymptote(s).

b) Describe the nature of the horizontal asymptote.

c) Develop a method for determining an equation for the horizontal asymptote.

3. Reflect Summarize the behaviour of the function $f(x) = \dfrac{x}{cx - 3}$ using the headings shown.

Interval	
Sign of Function	
Sign of Slope	
Change in Slope	

4. Sketch a graph of each function listed in your table.

Example 1 Determine Key Features

Determine the key features of $f(x) = \dfrac{x}{x - 2}$. Use the key features to graph the function.

Solution

The vertical asymptote has equation $x = 2$.

For the horizontal asymptote:

At $x \to \infty$, the numerator and denominator both approach infinity.

Divide each term by x.

$$f(x) = \dfrac{\frac{x}{x}}{\frac{x}{x} - \frac{2}{x}}$$

$$= \dfrac{1}{1 - \frac{2}{x}}$$

As $x \to +\infty$, $\dfrac{2}{x}$ gets very close to 0.

$$f(x) \to \dfrac{1}{1 - 0}$$

$$f(x) \to 1$$

As $x \to -\infty$, $\dfrac{2}{x}$ gets very close to 0.

$$f(x) \to \dfrac{1}{1 - 0}$$

$$f(x) \to 1$$

The function approaches the line $y = 1$ for both $x \to +\infty$ and $x \to -\infty$. So, $y = 1$ is the equation of the horizontal asymptote.

For the x-intercept, let $f(x) = 0$.

$$\dfrac{x}{x - 2} = 0$$

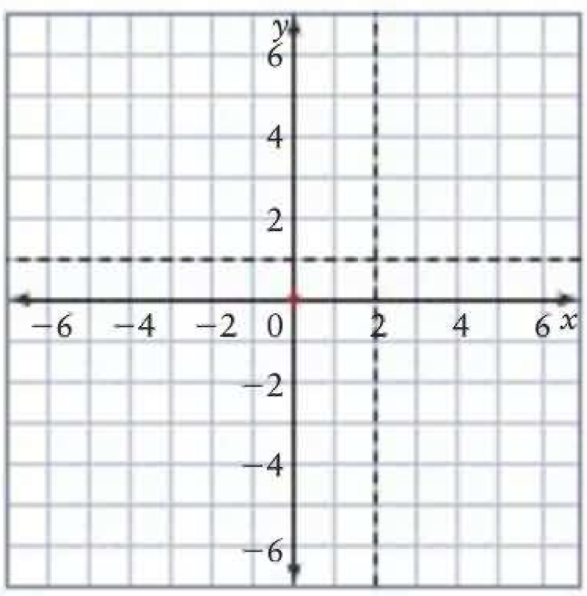
$$x = 0$$

For the y-intercept, let $x = 0$.

$$f(0) = \dfrac{0}{0 - 2}$$

$$= 0$$

The y-intercept is 0.



To determine where the slope is negative and where it is positive, select points in the interval to the left of the vertical asymptote and in the interval to the right of the vertical asymptote.

Select two points in the interval $x < 2$, say $x = -8$ and $x = 1$:

$f(-8) = 0.8$ and $f(1) = -1$.

$f(x)$ is decreasing, so the slope is negative.

Select two points in the interval $x > 2$, say $x = 3$ and $x = 10$:

$f(3) = 3$ and $f(10) = 1.25$.

$f(x)$ is decreasing, so the slope is negative.

There is a vertical asymptote at $x = 2$, but what is the sign of $f(x)$ on either side of this asymptote?

$f(1.99)$ will give a positive numerator and a negative denominator, so $f(1.99)$ is negative.

$f(2.01)$ will give a positive numerator and a positive denominator, so $f(2.01)$ is positive.

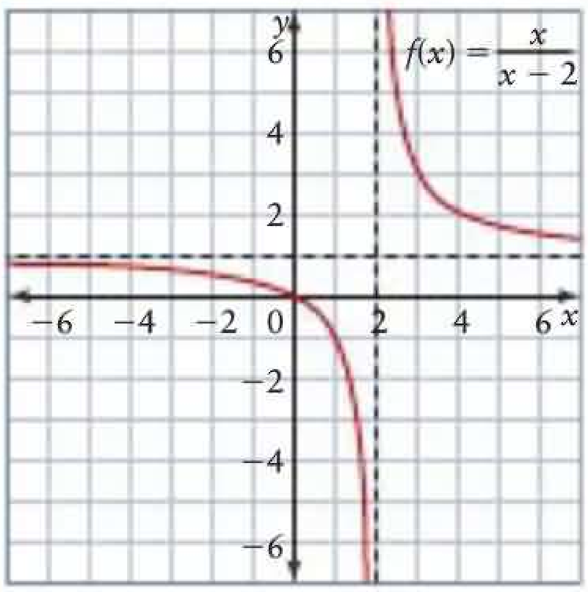
What is the end behaviour of this function?
Choose large positive and negative values of x.

$f(10\ 000) > 1$, so the function is above the asymptote.

$f(-10\ 000) < 1$, so the function is below the asymptote.

Summarize the intervals. At $x = 0$, $y = 0$, so the curve crosses the x-axis at the origin:

Interval	$x < 0$	$x = 0$	$0 < x < 2$	$x > 2$
f(x)	+	0	−	+
Slope	−	−	−	−



Domain: $\{x \in \mathbb{R}, x \neq 2\}$

Range: $\{y \in \mathbb{R}, y \neq 1\}$

Example 2 Compare Rational Functions

a) Compare the graphs of the functions $f(x) = \dfrac{x - 1}{2x + 5}$, $g(x) = \dfrac{x - 5}{2x + 5}$, and $h(x) = \dfrac{x - 10}{2x + 5}$.

b) Explain the effects of the coefficient b in $f(x) = \dfrac{ax + b}{cx + d}$.

Solution

a) The vertical asymptote for all three functions has equation $x = -\dfrac{5}{2}$.

For the horizontal asymptote, divide each term in the numerator and the denominator by x:

$$f(x) = \frac{\frac{x}{x} - \frac{1}{x}}{\frac{2x}{x} + \frac{5}{x}}$$

$$= \frac{1 - \frac{1}{x}}{2 + \frac{5}{x}}$$

As $x \to +\infty$, $\dfrac{1}{x} \to 0$ and $\dfrac{5}{x} \to 0$, so

$$f(x) \to \frac{1 - 0}{2 + 0}$$

$$f(x) \to \frac{1}{2}$$

As $x \to -\infty$, $\dfrac{1}{x} \to 0$ and $\dfrac{5}{x} \to 0$, so

$$f(x) \to \frac{1 - 0}{2 + 0}$$

$$f(x) \to \frac{1}{2}$$

The horizontal asymptote for $f(x)$ has equation $y = \dfrac{1}{2}$. By similar reasoning, the functions $g(x)$ and $h(x)$ have the same horizontal asymptote as $f(x)$.

Determine the intercepts.

Function	$f(x) = \dfrac{x-1}{2x+5}$	$g(x) = \dfrac{x-5}{2x+5}$	$h(x) = \dfrac{x-10}{2x+5}$
x-intercept	1	5	10
y-intercept	$-\dfrac{1}{5}$	-1	-2

To determine where the slope is negative and where it is positive, select points in each interval.

For the interval $x < -\dfrac{5}{2}$, $f(-10) = 0.7\overline{3}$ and $f(-5) = 1.2$.

The function f is increasing, so the slope is positive.

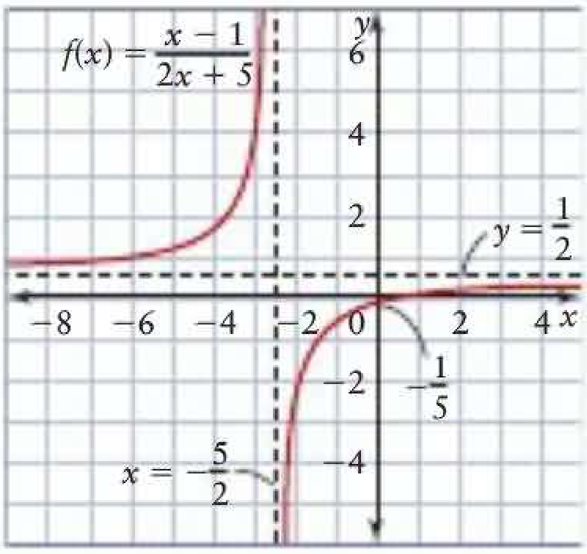
For the interval $x > -\dfrac{5}{2}$, $f(-2) = -3$ and $f(10) = 0.36$.

The function f is increasing, so the slope is positive.

Use the same reasoning to find where g and h are increasing and decreasing. Build an interval summary table.

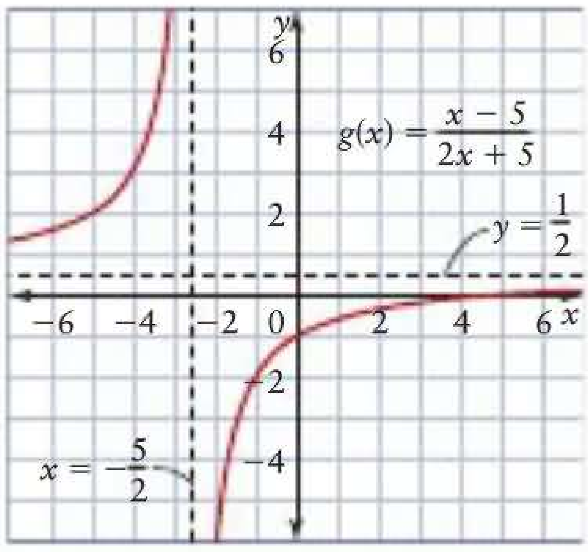
	Interval	$x < -\frac{5}{2}$	$-\frac{5}{2} < x < x$-intercept	x-intercept	$x > x$-intercept
$f(x) = \frac{x-1}{2x+5}$	Sign of $f(x)$	$+$	$-$	0	$+$
	Sign of Slope	$+$	$+$	$+$	$+$
$g(x) = \frac{x-5}{2x+5}$	Sign of $g(x)$	$+$	$-$	0	$+$
	Sign of Slope	$+$	$+$	$+$	$+$
$h(x) = \frac{x-10}{2x+5}$	Sign of $h(x)$	$+$	$-$	0	$+$
	Sign of Slope	$+$	$+$	$+$	$+$

Graph the functions:



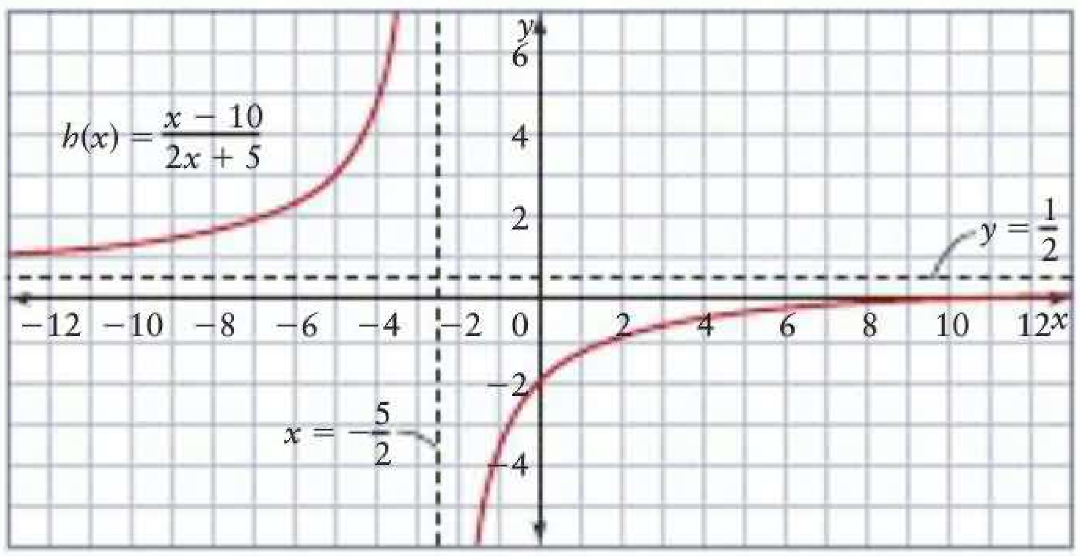
Domain: $\left\{x \in \mathbb{R}, x \neq -\frac{5}{2}\right\}$

Range: $\left\{y \in \mathbb{R}, y \neq \frac{1}{2}\right\}$



Domain: $\left\{x \in \mathbb{R}, x \neq -\frac{5}{2}\right\}$

Range: $\left\{y \in \mathbb{R}, y \neq \frac{1}{2}\right\}$



Domain: $\left\{x \in \mathbb{R}, x \neq \frac{1}{2}\right\}$

Range: $\left\{y \in \mathbb{R}, y \neq \frac{1}{2}\right\}$

The graphs have a similar shape, with the same horizontal and vertical asymptotes. The intervals have the same sign and slope analyses.
The domain and range are the same. The graphs have different x- and y-intercepts.

b) The constant b in $f(x) = \frac{ax + b}{cx + d}$ has an effect that is similar to both a vertical and a horizontal stretch factor but does not affect the asymptotes, domain, or range.

KEY CONCEPTS

- A rational function of the form $f(x) = \frac{ax + b}{cx + d}$ has the following key features:
  - The equation of the vertical asymptote can be found by setting the denominator equal to zero and solving for x, provided the numerator does not have the same zero.
  - The equation of the horizontal asymptote can be found by dividing each term in both the numerator and the denominator by x and investigating the behaviour of the function as $x \to \pm\infty$.
  - The coefficient b acts to stretch the curve but has no effect on the asymptotes, domain, or range.
  - The coefficient d shifts the vertical asymptote.
  - The two branches of the graph of the function are equidistant from the point of intersection of the vertical and horizontal asymptotes.

Communicate Your Understanding

C1 Describe the end behaviour (that is, as $x \to \pm\infty$) of each function.

a) $f(x) = \frac{x - 1}{x + 3}$

b) $g(x) = \frac{2x}{4x - 7}$

c) $h(x) = \frac{3 - x}{2 + x}$

C2 Describe the roles of the numerator and the denominator in determining the key features of the graph of a rational function of the form $f(x) = \frac{ax + b}{cx + d}$.

A Practise

For help with questions 1 to 3, refer to Example 1.

1. Determine an equation for the vertical asymptote of each function. Then, state the domain.

a) $f(x) = \frac{x}{x - 7}$ **b)** $g(x) = \frac{2x}{x + 5}$

c) $h(x) = -\frac{x}{x + 8}$ **d)** $k(x) = \frac{x}{3x - 1}$

e) $m(x) = \frac{5x - 3}{4x + 9}$ **f)** $n(x) = \frac{6 - x}{5 - x}$

2. Determine an equation for the horizontal asymptote of each function. Then, state the range.

a) $p(x) = \frac{x}{x - 6}$ **b)** $q(x) = \frac{3x}{x + 4}$

c) $r(x) = \frac{x - 1}{x + 1}$ **d)** $s(x) = \frac{5x - 2}{2x + 3}$

e) $t(x) = \frac{x - 6}{4 - x}$ **f)** $u(x) = \frac{3 - 4x}{1 - 2x}$

3. Sketch each function and then summarize the increasing and decreasing intervals.

a) $f(x) = \frac{x}{x - 5}$ **b)** $c(x) = \frac{4x}{x + 8}$

c) $k(x) = \frac{x + 1}{4 - x}$ **d)** $w(x) = \frac{x + 2}{4x - 5}$

e) $d(x) = \frac{-2x - 3}{x + 5}$ **f)** $m(x) = \frac{3x + 1}{2x + 1}$

For help with questions 4 to 6, refer to Example 2.

4. a) For the function $f(x) = \frac{2x}{x - 3}$, compare the slopes of the tangents

i) at the points where $x = 3.5$ and $x = 20$

ii) at the points where $x = 2.5$ and $x = -20$

b) What do these results indicate about the key features of the graph?

B Connect and Apply

5. a) Determine an equation for the horizontal asymptote of each function.

i) $f(x) = \frac{x - 5}{2x + 1}$ **ii)** $g(x) = \frac{3 - 5x}{2x + 1}$

b) How do the equations of the horizontal asymptotes relate to the coefficients in each function?

c) Summarize your findings to describe how to determine an equation for the horizontal asymptote of the function $f(x) = \frac{ax + b}{cx + d}$.

6. Use your results from question 5 to determine an equation for the horizontal asymptote of each function. Then, determine an equation for the vertical asymptote and graph the function. State the domain and range.

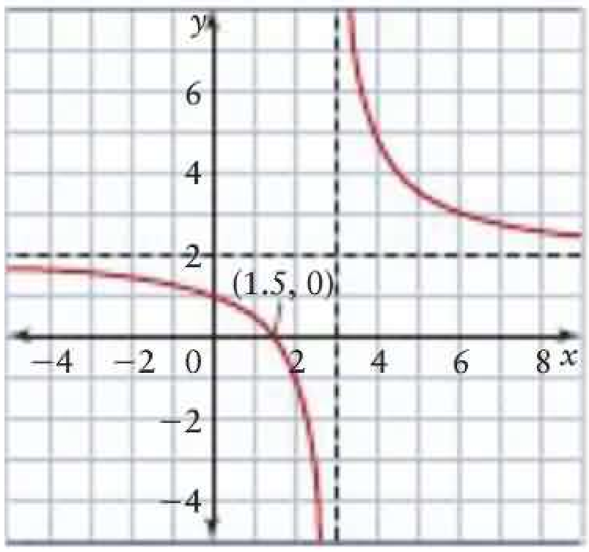
a) $f(x) = \frac{x}{x - 9}$ **b)** $g(x) = \frac{3x}{x + 2}$

c) $h(x) = \frac{4x - 3}{2x + 1}$ **d)** $k(x) = \frac{x - 3}{2x - 5}$

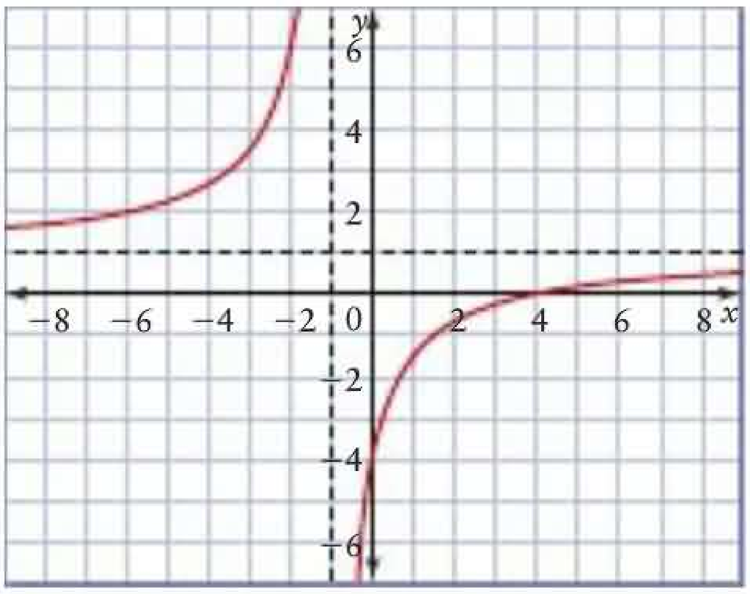
e) $m(x) = \frac{4 - x}{5 + x}$ **f)** $p(x) = \frac{6 - 8x}{3x - 4}$

7. Write an equation for the rational function shown on each graph.

a)



b)



8. Write an equation for a rational function whose graph has all of the indicated features.
- x-intercept of -4
- y-intercept of -2
- vertical asymptote with equation $x = 2$
- horizontal asymptote with equation $y = 1$

9. Write an equation for a rational function whose graph has all of the indicated features.
- x-intercept of $\frac{3}{5}$
- y-intercept of -3
- vertical asymptote with equation $x = -\frac{1}{2}$
- horizontal asymptote with equation $y = \frac{5}{2}$

10. **Chapter Problem** After a train derailment in Northern Ontario, the concentration, C, in grams per litre, of a pollutant after t minutes in a 5 000 000-L pond can be modelled by the function $C(t) = \frac{30t}{200\,000 + t}$, when a pollutant concentration of 30 g/L flows into the pond at a rate of 25 L/min.

a) Sketch a graph showing the concentration of the pollutant after t minutes.

b) What happens as t becomes very large?

c) When a concentration level of 0.05 g/L in the pond is reached, the fish stock will be irreversibly damaged. When will this occur?

11. a) Use long division to rewrite the function $f(x) = \frac{4x + 5}{2x - 1}$ as the sum of a constant and a rational function.

b) Explain how this method could be used to graph rational functions.

c) Use this method to sketch a graph of $f(x)$.

12. Use your method from question 11 to graph each function.

a) $p(x) = \frac{2x + 3}{x + 1}$

b) $t(x) = \frac{5x - 4}{2x + 5}$

✓Achievement Check

13. Consider the function $g(x) = \frac{x}{x - 7}$.

a) Determine an equation for the vertical asymptote.

b) State the domain.

c) Determine an equation for the horizontal asymptote.

d) State the range.

e) Sketch the function.

f) Summarize the increasing and decreasing intervals.

g) Compare the slopes of the tangents at the points where

i) $x = 7.5$ and $x = 20$

ii) $x = 6.5$ and $x = -20$

C Extend and Challenge

14. A golf ball of mass 4.6 g is struck by a golf club at a speed of 50 m/s. The ball has initial velocity, v, in metres per second, of $v(m) = \frac{83m}{m + 0.046}$, where m is the mass, in grams, of the golf club. Describe the rate of change of the initial velocity as the mass of the club increases.

15. Analyse the key features of the function $f(x) = \frac{\sqrt{x}}{\sqrt{x} - 1}$ and sketch its graph. How does it compare to the graph of $f(x) = \frac{x}{x - 1}$?

16. Rational functions can have any polynomial in the numerator and denominator. Analyse the key features of each function and sketch its graph. Describe the common features of the graphs.

a) $f(x) = \dfrac{x}{x^2 - 1}$

b) $g(x) = \dfrac{x - 2}{x^2 + 3x + 2}$

c) $h(x) = \dfrac{x + 5}{x^2 - x - 12}$

Many rational functions have an asymptote that is not vertical or horizontal but on an angle or slant. These asymptotes are called slant or oblique asymptotes. Use this information to answer questions 17 to 19.

17. Math Contest Investigate when you can expect to have an oblique asymptote. Graph each function on a graphing calculator and determine what factor leads to an oblique asymptote.

a) $f(x) = \dfrac{x^3 + 1}{x^3}$ **b)** $f(x) = \dfrac{x^3 + 1}{x^2}$

c) $f(x) = \dfrac{x^2 + 1}{x}$ **d)** $f(x) = \dfrac{x^2 + 1}{x^3}$

18. Refer to question 17. Given the general function $f(x) = \dfrac{x^a + k}{x^b + m}$, a linear oblique asymptote will occur when

A $a > b$ **B** $b > a$

C $a - b = 1$ **D** $|a - b| = 1$

19. Math Contest Given the function $f(x) = \dfrac{x^2 - 2}{x}$:

a) Perform long division on $f(x)$ and state the quotient and the remainder.

b) Write $f(x)$ in terms of its quotient, remainder, and divisor in the form $f(x) = q(x) + \dfrac{r(x)}{d(x)}$.

c) Determine equations for all asymptotes of each function and graph the function without using technology.

i) $g(x) = \dfrac{x^2 - 3x - 4}{x - 2}$

ii) $f(x) = \dfrac{x^2 + x - 2}{2x - 4}$

iii) $z(x) = \dfrac{x^2 - 9}{x + 3}$

CAREER CONNECTION

Marissa graduated from the University of Ontario Institute of Technology after studying applied and industrial mathematics for 5 years. She now works in the field of mathematical modelling, helping an aircraft manufacturer to design faster, safer, and environmentally cleaner airplanes. Marissa uses her knowledge of fluid mechanics and software programs that can, for example, model a wind tunnel, to run experiments. Data from these tests help her to translate physical phenomena into equations. She then analyses and solves these complex equations and interprets the solutions. As computers become more powerful, Marissa will be able to tackle more complex problems and get answers in less time, thereby reducing research and development costs.

3.4

Solve Rational Equations and Inequalities

Proper lighting is of critical importance in surgical procedures. If a surgeon requires more illumination from a light source of fixed power, the light needs to be moved closer to the patient. By how much is the intensity increased if the distance between the light and the patient is halved? The intensity of illumination is inversely proportional to the square of the distance to the light source. The function $I = \frac{k}{d^2}$ relates the illumination, I, to the distance, d, from the light source. If a specific intensity is needed, the distance needs to be solved for in order to determine the placement of the light source.

Example 1 Solve Rational Equations Algebraically

Solve.

a) $\frac{4}{3x - 5} = 4$

b) $\frac{x - 5}{x^2 - 3x - 4} = \frac{3x + 2}{x^2 - 1}$

Solution

a)

$$\frac{4}{3x - 5} = 4$$

$$4 = 4(3x - 5), \quad x \neq \frac{5}{3}$$

Multiply both sides by $(3x - 5)$.

$$4 = 12x - 20$$

$$12x = 24$$

$$x = 2$$

b)

$$\frac{x - 5}{x^2 - 3x - 4} = \frac{3x + 2}{x^2 - 1}$$

$$\frac{x - 5}{(x - 4)(x + 1)} = \frac{3x + 2}{(x - 1)(x + 1)}, \quad x \neq -1, x \neq 1, x \neq 4$$

Factor the expressions in the denominators.

$$(x - 5)(x - 1) = (3x + 2)(x - 4)$$

Multiply both sides by $(x - 4)(x + 1)(x - 1)$.

$$x^2 - 6x + 5 = 3x^2 - 10x - 8$$

$$2x^2 - 4x - 13 = 0$$

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(2)(-13)}}{2(2)}$$
$$= \frac{4 \pm \sqrt{120}}{4}$$
$$= \frac{4 \pm 2\sqrt{30}}{4}$$
$$= \frac{2 \pm \sqrt{30}}{2}$$

$2x^2 - 4x - 13$ cannot be factored. Use the quadratic formula.

Example 2 Solve a Rational Equation Using Technology

Solve $\dfrac{x}{x-2} = \dfrac{2x^2 - 3x + 5}{x^2 + 6}$.

Solution

$$\frac{x}{x-2} = \frac{2x^2 - 3x + 5}{x^2 + 6}$$

$$x(x^2 + 6) = (x - 2)(2x^2 - 3x + 5),\ x \neq 2$$

Multiply both sides by $(x - 2)(x^2 + 6)$.

$$x^3 + 6x = 2x^3 - 7x^2 + 11x - 10$$
$$x^3 - 7x^2 + 5x - 10 = 0$$

Use technology to solve the equation.

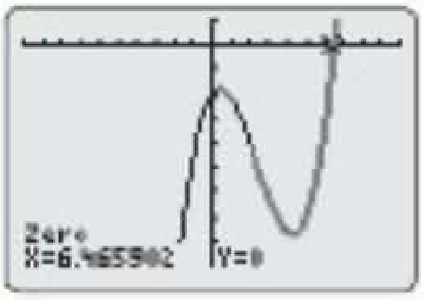
CONNECTIONS

You can solve using technology directly from the initial equation. With a graphing calculator, you can graph both sides separately and find the point of intersection. With a CAS, you can use the **solve** operation directly. Other methods are possible.

Method 1: ***Use a Graphing Calculator***

Graph the function $y = x^3 - 7x^2 + 5x - 10$.

Then, use the **Zero** operation.



$x \doteq 6.47$ is the only solution.

Method 2: ***Use a Computer Algebra System (CAS)***

Use the **solve** function or the **zero** function.

Example 3 Solve a Simple Rational Inequality

Solve $\frac{2}{x - 5} < 10$.

Solution

Method 1: ***Consider the Key Features of the Graph of a Related Rational Function***

$$\frac{2}{x - 5} < 10$$

$$\frac{2}{x - 5} - 10 < 0$$

$$\frac{2 - 10x + 50}{x - 5} < 0 \quad \text{Combine the terms using a common denominator of } x - 5.$$

$$\frac{-10x + 52}{x - 5} < 0$$

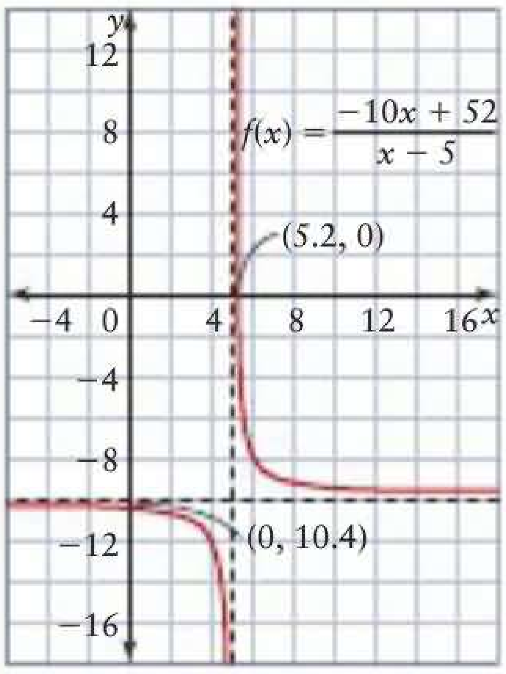
Consider the function $f(x) = \frac{-10x + 52}{x - 5}$.

The vertical asymptote has equation $x = 5$.

The horizontal asymptote has equation $y = -10$.

The x-intercept is 5.2.

The y-intercept is -10.4.



From the graph, the coordinates of all of the points below the x-axis satisfy the inequality $f(x) < 0$.

So, $\frac{-10x + 52}{x - 5} < 0$, or $\frac{2}{x - 5} < 10$, for $x < 5$ or $x > 5.2$.

In interval notation, $x \in (-\infty, 5) \cup (5.2, +\infty)$.

Method 2: ***Solve Algebraically***

$$\frac{2}{x-5} < 10$$

Because $x - 5 \neq 0$, either $x > 5$ or $x < 5$.

Case 1:

$$\begin{aligned} x &> 5 \\ 2 &< 10(x-5) && \text{Multiply both sides by } (x-5)\text{, which is positive if } x > 5. \\ 2 &< 10x - 50 \\ 52 &< 10x \\ 5.2 &< x \\ x &> 5.2 \end{aligned}$$

$x > 5.2$ is within the inequality $x > 5$, so the solution is $x > 5.2$.

Case 2:

$$\begin{aligned} x &< 5 \\ \frac{2}{x-5} &< 10 \\ 2 &> 10(x-5) && \text{Change the inequality when multiplying by a negative.} \\ 2 &> 10x - 50 \\ 52 &> 10x \\ 5.2 &> x \\ x &< 5.2 \end{aligned}$$

$x < 5$ is within the inequality $x < 5.2$, so the solution is $x < 5$.

Combining the two cases, the solution to the inequality is $x < 5$ or $x > 5.2$.

Example 4 Solve a Quadratic Over a Quadratic Rational Inequality

Solve $\frac{x^2 - x - 2}{x^2 + x - 12} \geq 0$.

Solution

Method 1: ***Solve Using an Interval Table***

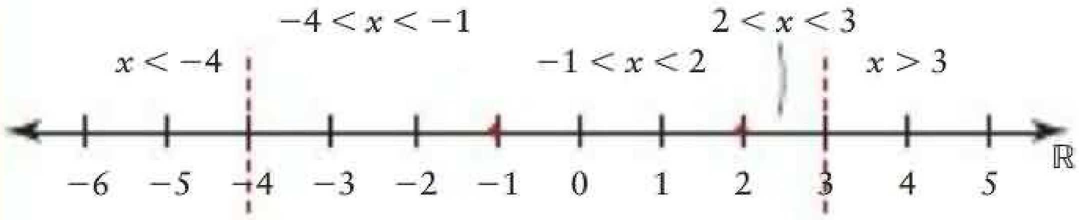
$$\frac{x^2 - x - 2}{x^2 + x - 12} \geq 0$$

$$\frac{(x-2)(x+1)}{(x-3)(x+4)} \geq 0$$

From the numerator, the zeros occur at $x = 2$ and $x = -1$, so solutions occur at these values of x.

From the denominator, the restrictions occur at $x = 3$ and $x = -4$.

A number line can be used to consider intervals.



Use a table to consider the signs of the factors on each interval. Pick an arbitrary number in each interval. The number line is broken into five intervals, as shown. Select test values of x for each interval. Check whether the value of $\frac{(x - 2)(x + 1)}{(x - 3)(x + 4)}$ in each interval is greater than or equal to 0.

For example, for $x < -4$, test $x = -5$.

$\frac{(-5 - 2)(-5 + 1)}{(-5 - 3)(-5 + 4)} = \frac{28}{8} > 0$, so $x < -4$ is part of the solution.

Continue testing a selected point in each interval and record the results in a table.

Interval	Choice for x in the interval	Signs of Factors of $\frac{(x-2)(x+1)}{(x-3)(x+4)}$	Sign of $\frac{(x-2)(x+1)}{(x-3)(x+4)}$
$(-\infty, -4)$	$x = -5$	$\frac{(-)(-)}{(-)(-)}$	$+$
$(-4, -1)$	$x = -2$	$\frac{(-)(-)}{(-)(+)}$	$-$
-1	$x = 1$	$\frac{(-)(0)}{(-)(+)}$	0
$(-1, 2)$	$x = 0$	$\frac{(-)(+)}{(-)(+)}$	$+$
2	$x = 2$	$\frac{(0)(+)}{(-)(+)}$	0
$(2, 3)$	$x = 2.5$	$\frac{(+)(+)}{(-)(+)}$	$-$
$(3, +\infty)$	$x = 4$	$\frac{(+)(+)}{(+)(+)}$	$+$

The same data are shown in a different format below. The critical values of x are bolded, and test values are chosen within each interval.

x	-5	$\mathbf{-4}$	-2	$\mathbf{-1}$	0	**2**	2.5	**3**	4
$\frac{(x-2)(x+1)}{(x-3)(x+4)}$	$+$	∞	$-$	0	$+$	0	$-$	∞	$+$

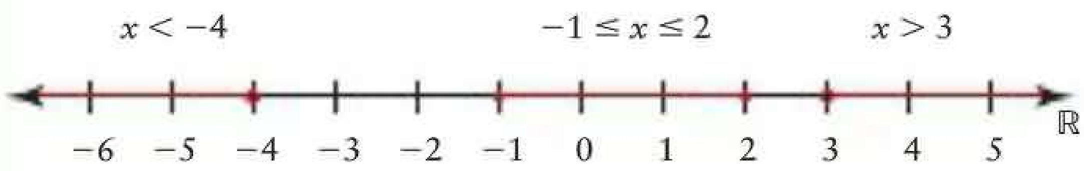
CONNECTIONS

The critical values of x are those values where there is a vertical asymptote, or where the slope of the graph of the inequality changes sign.

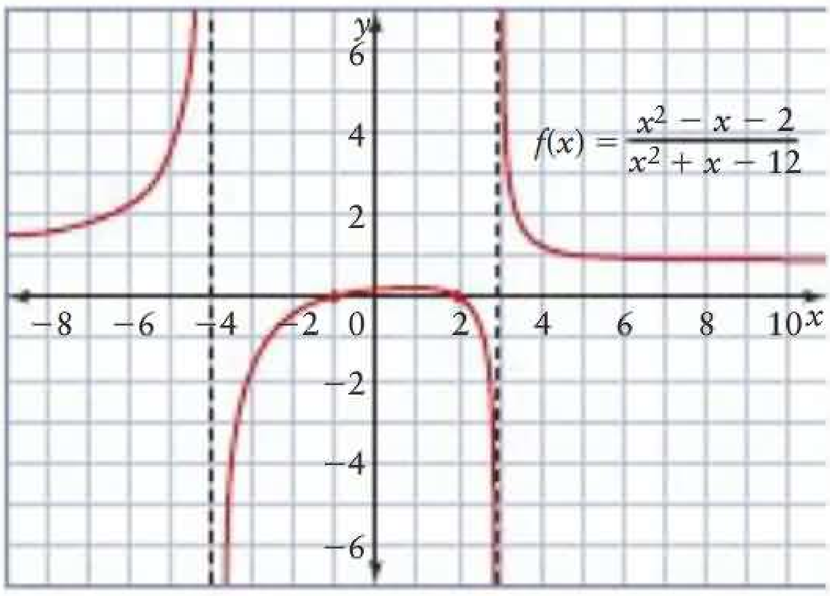
For the inequality $\frac{x^2 - x - 2}{x^2 + x - 12} \geq 0$, the solution is $x < -4$ or $-1 \leq x \leq 2$ or $x > 3$.

In interval notation, the solution set is $x \in (-\infty, -4) \cup [-1, 2] \cup (3, +\infty)$.

The solution can be shown on a number line.

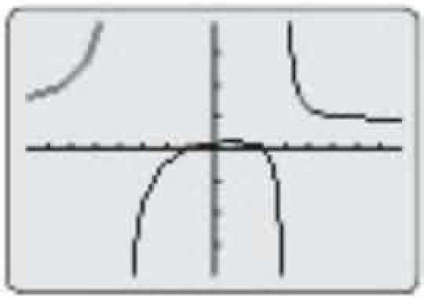


A graph of the related function, $f(x) = \frac{x^2 - x - 2}{x^2 + x - 12}$, confirms this solution.



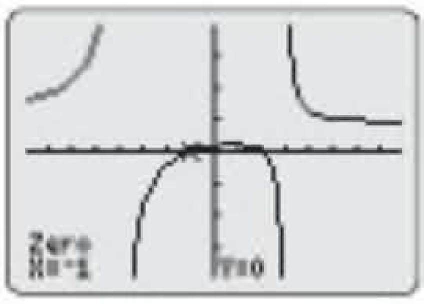
Method 2: ***Use a Graphing Calculator***

Enter the function $f(x) = \frac{x^2 - x - 2}{x^2 + x - 12}$ and view its graph.

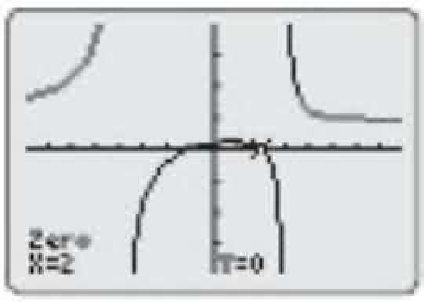


Either visually or by checking the table, you can see that there are asymptotes at $x = -4$ and $x = 3$, so there are no solutions there.

Use the **Zero** operation to find that the zeros are at $x = -1$ and $x = 2$.



Using the graph and the zeros, you can see that $f(x) \geq 0$ for $x < -4$, for $-1 \leq x \leq 2$, and for $x > 3$.



KEY CONCEPTS

- To solve rational equations algebraically, start by factoring the expressions in the numerator and denominator to find asymptotes and restrictions.
- Next, multiply both sides by the factored denominators, and simplify to obtain a polynomial equation. Then, solve using techniques from Chapter 2.
- For rational inequalities:
  - It can often help to rewrite with the right side equal to 0. Then, use test points to determine the sign of the expression in each interval.
  - If there is a restriction on the variable, you may have to consider more than one case. For example, if $\frac{a}{x-k} < b$, case 1 is $x > k$ and case 2 is $x < k$.
- Rational equations and inequalities can be solved by studying the key features of a graph with paper and pencil or with the use of technology.
- Tables and number lines can help organize intervals and provide a visual clue to solutions.

Communicate Your Understanding

C1 Describe the process you would use to solve $\frac{2}{x-1} = \frac{3}{x+5}$.

C2 Explain why $\frac{1}{x^2 + 2x + 9} < 0$ has no solution.

C3 Explain the difference between the solution to the equation $\frac{4}{x-5} = \frac{3}{x+4}$ and the solution to the inequality $\frac{4}{x-5} < \frac{3}{x+4}$.

A Practise

For help with questions 1 and 2, refer to Example 1.

1. Determine the x-intercept(s) for each function. Verify using technology.

a) $y = \frac{x+1}{x}$

b) $y = \frac{x^2 + x - 12}{x^2 - 3x + 5}$

c) $y = \frac{2x-3}{5x+1}$

d) $y = \frac{x}{x^2 - 3x + 2}$

2. Solve algebraically. Check each solution.

a) $\frac{4}{x-2} = 3$

b) $\frac{1}{x^2 - 2x - 7} = 1$

c) $\frac{2}{x-1} = \frac{5}{x+3}$

d) $x - \frac{5}{x} = 4$

e) $\frac{1}{x} = \frac{x-34}{2x^2}$

f) $\frac{x-3}{x-4} = \frac{x+2}{x+6}$

For help with question 3, refer to Example 2.

3. **Use Technology** Solve each equation using technology. Express your answers to two decimal places.

   a) $\dfrac{5x}{x-4} = \dfrac{3x}{2x+7}$

   b) $\dfrac{2x+3}{x-6} = \dfrac{5x-1}{4x+7}$

   c) $\dfrac{x}{x-2} = \dfrac{x^2-4x+1}{x-3}$

   d) $\dfrac{x^2-1}{2x^2-3} = \dfrac{2x^2+3}{x^2+1}$

For help with question 4, refer to Example 3.

4. Solve each inequality without using technology. Illustrate the solution on a number line.

   a) $\dfrac{4}{x-3} < 1$ b) $\dfrac{7}{x+1} > 7$

   c) $\dfrac{5}{x+4} \le \dfrac{2}{x+1}$ d) $\dfrac{(x-2)(x+1)^2}{(x-4)(x+5)} \ge 0$

   e) $\dfrac{x^2-16}{x^2-4x-5} > 0$ f) $\dfrac{x-2}{x} < \dfrac{x-4}{x-6}$

For help with question 5, refer to Example 4.

5. Solve each inequality using an interval table. Check using technology.

   a) $\dfrac{x^2+9x+14}{x^2-6+5} > 0$

   b) $\dfrac{2x^2+5x-3}{x^2+8x+16} < 0$

   c) $\dfrac{x^2-3x-4}{x^2+11x+30} \le 0$

   d) $\dfrac{3x^2-8x+4}{2x^2-9x-5} \ge 0$

B Connect and Apply

6. Write a rational equation that cannot have $x = 3$ or $x = -5$ as a solution. Explain your reasoning.

7. Solve $\dfrac{x}{x+1} < \dfrac{2x}{x-2}$ by graphing the functions $f(x) = \dfrac{x}{x+1}$ and $g(x) = \dfrac{2x}{x-2}$ with or without using technology. Determine the points of intersection and when $f(x) < g(x)$.

8. Use the method from question 7 to solve $\dfrac{x}{x-3} > \dfrac{3x}{x+5}$.

9. Solve and check.

   a) $\dfrac{1}{x} + 3 = \dfrac{2}{x}$ b) $\dfrac{2}{x+1} + 5 = \dfrac{1}{x}$

   c) $\dfrac{12}{x} + x = 8$ d) $\dfrac{x}{x-1} = 1 - \dfrac{1}{1-x}$

   e) $\dfrac{2x}{2x+3} - \dfrac{2x}{2x-3} = 1$

   f) $\dfrac{7}{x-2} - \dfrac{4}{x-1} + \dfrac{3}{x+1} = 0$

10. Solve. Illustrate graphically.

    a) $\dfrac{2}{x} + 3 > \dfrac{29}{x}$ b) $\dfrac{16}{x} - 5 < \dfrac{1}{x}$

    c) $\dfrac{5}{6x} + \dfrac{2}{3x} > \dfrac{3}{4}$ d) $6 + \dfrac{30}{x-1} < 7$

11. The ratio of $x + 2$ to $x - 5$ is greater than $\dfrac{3}{5}$. Solve for x.

12. Compare the solutions to $\dfrac{2x-1}{x+7} > \dfrac{x+1}{x+3}$ and $\dfrac{2x-1}{x+7} < \dfrac{x+1}{x+3}$.

13. Compare the solutions to $\dfrac{x+1}{x-4} \le \dfrac{x-3}{x+5}$ and $\dfrac{x-4}{x+1} \le \dfrac{x+5}{x-3}$.

14. A number x is the harmonic mean of two numbers a and b if $\dfrac{1}{x}$ is the mean of $\dfrac{1}{a}$ and $\dfrac{1}{b}$.

    a) Write an equation to represent the harmonic mean of a and b.

    b) Determine the harmonic mean of 12 and 15.

    c) The harmonic mean of 6 and another number is 1.2. Determine the other number.

15. **Chapter Problem** Light pollution is caused by many lights being on in a concentrated area. Think of the night sky in the city compared to the night sky in the country. Light pollution can be a problem in cities, as more and more bright lights are used in such things as advertising and office buildings. The intensity of illumination is inversely proportional to the square of the distance to the light source and is defined by the formula $I = \frac{k}{d^2}$, where I is the intensity, in lux; d is the distance from the source, in metres; and k is a constant. When the distance from a certain light source is 10 m, the intensity is 900 lux.
    a) Determine the intensity when the distance is
        i) 5 m   ii) 200 m
    b) What distance, or range of distance, results in an intensity of
        i) 4.5 lux?   ii) at least 4500 lux?

16. The relationship between the object distance, d, and image distance, I, both in centimetres, for a camera with focal length 2.0 cm is defined by the relation $d = \frac{2.0I}{I - 2.0}$. For what values of I is d greater than 10.0 cm?

✓ Achievement Check

17. Consider the functions $f(x) = \frac{1}{x} + 4$ and $g(x) = \frac{2}{x}$. Graph f and g on the same grid.
    a) Determine the points of intersection of the two functions.
    b) Show where $f(x) < g(x)$.
    c) Solve the equation $\frac{1}{x} + 4 = \frac{2}{x}$ to check your answer to part a).
    d) Solve the inequality $\frac{1}{x} + 4 < \frac{2}{x}$ to check your answer to part b).

C Extend and Challenge

18. a) A rectangle has perimeter 64 cm and area 23 cm². Solve the following system of equations to find the rectangle's dimensions.
    $l = \frac{23}{w}$
    $l + w = 32$
    b) Solve the system of equations.
    $x^2 + y^2 = 1$
    $xy = 0.5$

19. Use your knowledge of exponents to solve.
    a) $\frac{1}{2^x} = \frac{1}{x + 2}$
    b) $\frac{1}{2^x} > \frac{1}{x^2}$

20. Determine the region(s) of the Cartesian plane for which
    a) $y > \frac{1}{x^2}$
    b) $y \le x^2 + 4$ and $y \ge \frac{1}{x^2 + 4}$

21. **Math Contest** In some situations, it is convenient to express a rational expression as the sum of two or more simpler rational expressions. These simpler expressions are called *partial fractions*. Partial fractions are used when, given $\frac{P(x)}{D(x)}$, the degree of $P(x)$ is less than the degree of $D(x)$.
    Decompose each of the following into partial fractions. Check your solutions by graphing the equivalent function on a graphing calculator. Hint: Start by factoring the denominator, if necessary.
    a) $f(x) = \frac{5x + 7}{x^2 + 2x - 3}$
    b) $g(x) = \frac{7x + 6}{x^2 - x - 6}$
    c) $h(x) = \frac{6x^2 - 14x - 27}{(x + 2)(x - 3)^2}$

3.5 Making Connections With Rational Functions and Equations

Most of us know that getting closer in a concert means getting a better view, but it also means more exposure to potentially damaging sound levels. Just how much more intense can sound levels become as you move closer and closer to the source? The intensity, I, increases by what is known as the inverse square law, or the reciprocal of the square of the distance, d, from the sound source, so that $I = \frac{k}{d^2}$. This law also applies to gravitational force and light intensity.

Example 1 Intensity of Sound

The intensity of sound, in watts per square metre, varies inversely as the square of the distance, in metres, from the source of the sound. The intensity of the sound from a loudspeaker at a distance of 2 m is 0.001 W/m^2.

a) Determine a function to represent this relationship.

b) Graph this function.

c) What is the effect of halving the distance from the source of the sound?

Solution

a) Let I represent the intensity of sound. Let d represent the distance from the source of the sound. I varies inversely as the square of d.

So, $I = \frac{k}{d^2}$, where k is a constant.

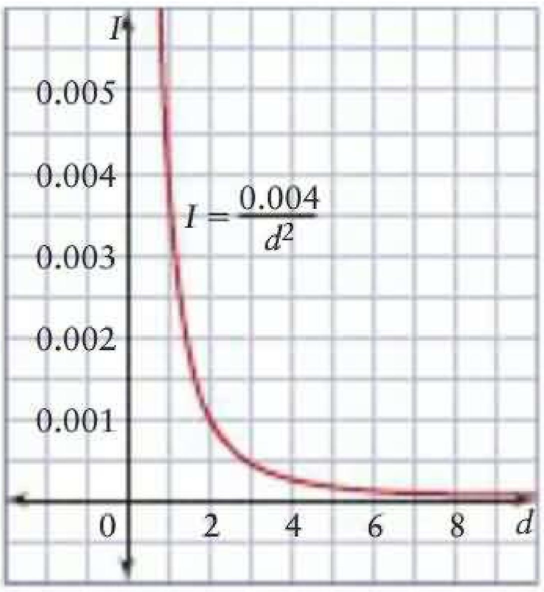
Substitute $I = 0.001$ and $d = 2$ into the equation to find k.

$$0.001 = \frac{k}{2^2}$$

$$k = 0.004$$

$$I = \frac{0.004}{d^2}, d > 0$$

b)



c) Substitute $\frac{1}{2}d$ for d.

$$
\begin{aligned}
I &= \frac{0.004}{\left(\frac{1}{2}d\right)^2} \\
&= \frac{0.004}{\left(\frac{d^2}{4}\right)} \\
&= 4 \times \frac{0.004}{d^2}
\end{aligned}
$$

If the distance is halved, the sound is four times as intense.

Example 2 Diving Time

The maximum time, T, in minutes, a scuba diver can rise without stopping for decompression on the way up to the surface is defined by the equation $T(d) = \frac{525}{d - 10}$, $d > 10$, where d is the depth of the dive, in metres. For the maximum time to be less than 30 min, how deep can the diver dive?

Solution

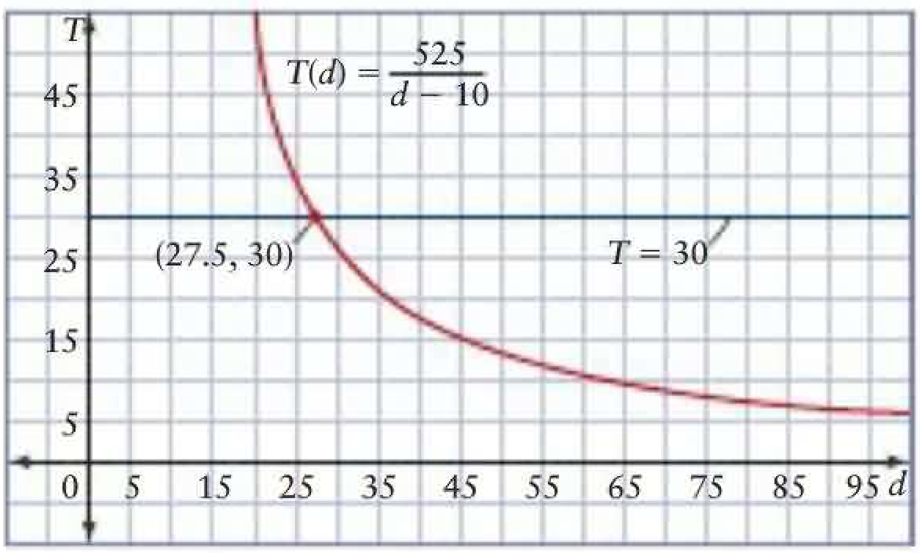
$$\frac{525}{d - 10} < 30$$

Since $d > 10$, there is only one case.

$525 < 30(d - 10)$ Multiply both sides by $d - 10$.

$$
\begin{aligned}
\frac{525}{30} &< d - 10 \\
d &> \frac{525}{30} + 10 \\
d &> 27.5
\end{aligned}
$$

This can be illustrated graphically.



If the diver dives deeper than 27.5 m, the time without decompression stops will be less than 30 min.

Example 3 Special Cases

Sketch a graph of each function. Explain how these are special cases.

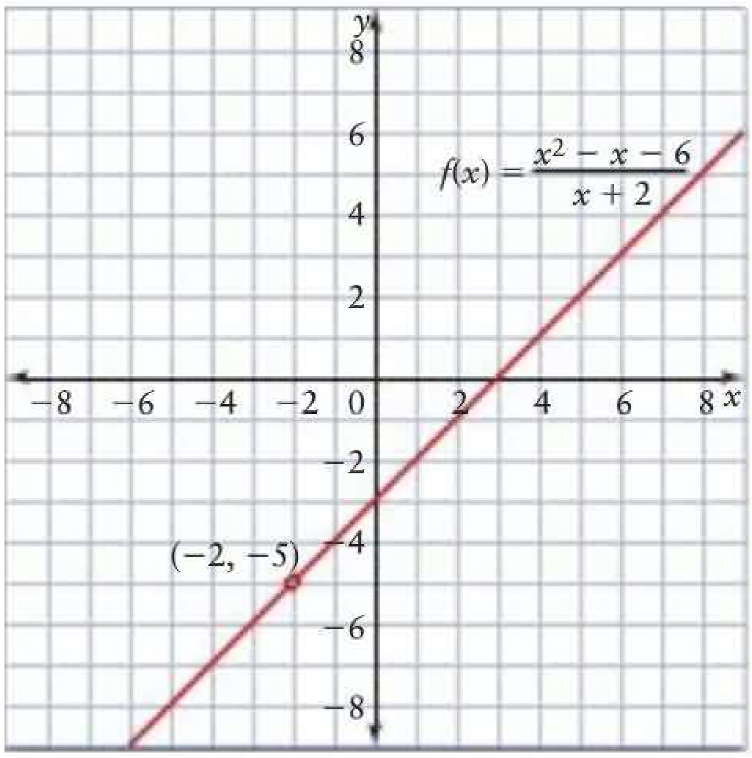
a) $f(x) = \dfrac{x^2 - x - 6}{x + 2}$

b) $g(x) = \dfrac{2x^2 - 7x - 4}{2x^2 + 5x + 2}$

Solution

a)
$$\begin{aligned} f(x) &= \frac{x^2 - x - 6}{x + 2} \\ &= \frac{(x - 3)(x + 2)}{x + 2} \\ &= x - 3, x \neq -2 \end{aligned}$$

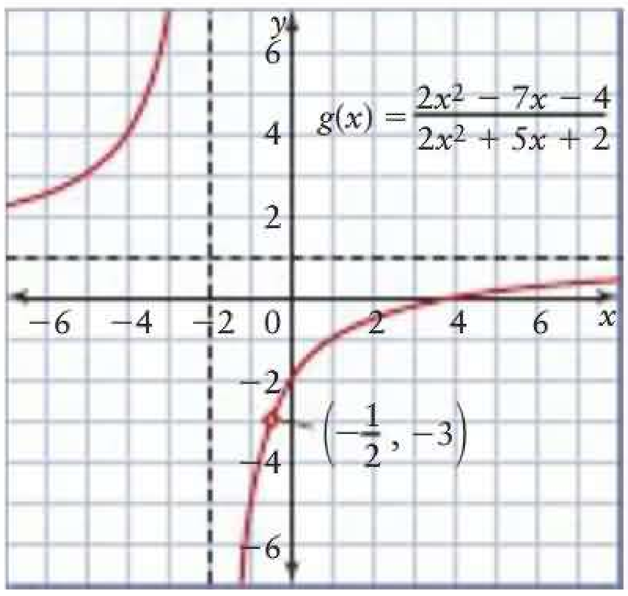
Notice that $f(x)$ simplifies to a linear relationship. So, even though the function is undefined at $x = -2$, we can find what value f approaches as $x \to -2$. From the simplified form, as $x \to -2$, $f \to -5$.



This is a special case of a line that is **discontinuous** at $x = -2$. This is a value at which the function is undefined. Here, the discontinuity is not an asymptote, but the point $(-2, -5)$. This type of discontinuity is called a hole or a gap.

b)
$$\begin{aligned} g(x) &= \frac{2x^2 - 7x - 4}{2x^2 + 5x + 2} \\ &= \frac{(2x + 1)(x - 4)}{(2x + 1)(x + 2)}, x \neq -2, x \neq -\frac{1}{2} \\ &= \frac{x - 4}{x + 2} \end{aligned}$$

This is a special case because there is a hole at the point $\left(-\dfrac{1}{2}, -3\right)$.



KEY CONCEPTS

- When solving a problem, it is important to read carefully to determine whether a function is being analysed or an equation or inequality is to be solved.
- A full analysis will involve four components:
  - numeric (tables, ordered pairs, calculations)
  - algebraic (formulas, solving equations)
  - graphical
  - verbal (descriptions)
- When investigating special cases of functions, factor and reduce where possible. Indicate the restrictions on the variables in order to identify hidden discontinuities.
- When investigating new types of rational functions, consider what is different about the coefficients and the degree of the polynomials in the numerator and denominator. These differences could affect the stretch factor of the curve and the equations of the asymptotes and they could cause other discontinuities.

Communicate Your Understanding

C1 Other than at asymptotes, describe when a discontinuity can occur in a rational function.

C2 The maximum height, h, in kilometres, of a rocket launched with initial velocity, v, in kilometres per hour, is given by the formula $h = \frac{6400v^2}{125\,440 - v^2}$, ignoring air resistance. Explain why the velocity at the vertical asymptote is considered the escape velocity of the rocket.

CONNECTIONS

Escape velocity is the speed at which the magnitude of the kinetic energy of an object is equal to the magnitude of its gravitational potential energy. It is commonly described as the speed needed to "break free" from a gravitational field.

A Practise

For help with question 1, refer to Example 1.

1. The intensity of illumination is inversely proportional to the square of the distance to the light source. It is modelled by the formula $I = \frac{k}{d^2}$, where I is the intensity, in lux; d is the distance, in metres, from the source; and k is a constant. When the distance from a certain light source is 50 m, the intensity is 6 lux.

a) Sketch a graph of this relation.

b) Describe what happens to the light intensity as the distance becomes greater.

c) Comment on the model for values of d close to 0.

For help with questions 2 and 3, refer to Example 2.

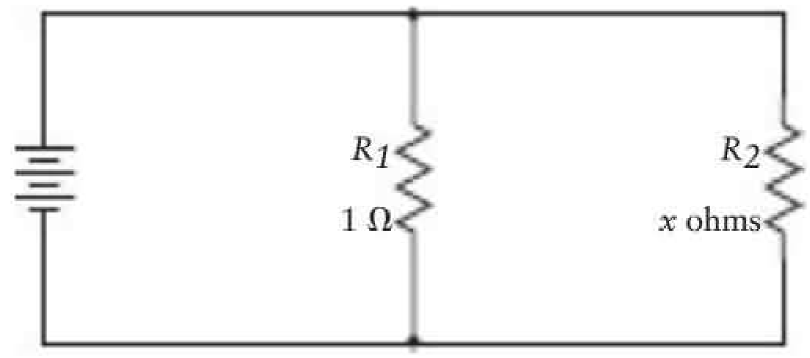
2. According to Boyle's law, under constant temperature, the volume of gas varies inversely with the pressure. A tank holds 10 L of hydrogen gas at a pressure of 500 kPa.

a) Determine a function to relate volume and pressure for this gas.

b) Sketch a graph of this relation, showing the volume of gas for different atmospheric pressures.

c) What is the effect of doubling the pressure?

3. When connected in parallel, a resistor of x ohms and a resistor of 1 Ω will have a total resistance defined by the function $R(x) = \frac{x}{1 + x}$.



a) For the total resistance to be 0.5 Ω, what does the resistance x need to be?

b) For the total resistance to be less than 0.25 Ω, what does the resistance x need to be?

For help with question 4, refer to Example 3.

4. Sketch a graph of each function. Describe each special case.

a) $f(x) = \frac{x}{x^2 - 2x}$

b) $g(x) = \frac{x - 4}{x^2 + x - 20}$

c) $h(x) = \frac{x^2 + 7x + 12}{x^2 + 2x - 3}$

d) $k(x) = \frac{3x^2 + x - 2}{2x^2 + 7x + 5}$

e) $m(x) = \frac{x}{x^3 - 4x^2 - 12x}$

f) $n(x) = \frac{x^2 - 3x + 2}{x^3 - 7x + 6}$

B Connect and Apply

5. The profit, P, in thousands of dollars, from the sale of x kilograms of coffee can be modelled by the function $P(x) = \frac{4x - 200}{x + 400}$.

a) Sketch a graph of this relation.

b) The average profit for a given sales level, x, can be found by drawing a secant from the origin to the point $(x, P(x))$. Explain how average profit is related to the slope of a secant.

c) Estimate where the average profit is the greatest. Verify using slopes.

d) Determine the rate of change of the profit at a sales level of 1000 kg.

6. The electrical resistance, R, in ohms, of a wire varies directly with its length, l, in metres, and inversely with the square of the diameter, d, in millimetres, of its cross section according to the function $R = \frac{kl}{d^2}$.

a) If 1000 m of 4-mm-diameter wire has a resistance of 40 Ω, determine an equation to model length and cross section.

b) Sketch a graph of the function for the electrical resistance of 1000 m of wire at various cross sections.

7. A bus company models its cost, C, in dollars, per person for a bus charter trip with the equation $C(x) = \frac{10\,000}{10 + x}$, where x is the number of passengers over its minimum number of 10. Describe the change in the cost model represented by each of the following, and accompany each with a graph.

a) $C(x) = \frac{10\,000}{8 + x}$

b) $C(x) = \frac{20\,000}{10 + x}$

c) $C(x) = \frac{15\,000}{12 + x}$

8. A function has equation $f(x) = \frac{x^2 - 2x - 5}{x - 1}$. In addition to a vertical asymptote, it has an oblique, or slant, asymptote, which is neither vertical nor horizontal.

a) **Use Technology** Graph the function using technology.

b) Describe what is meant by the term *oblique asymptote*.

c) Use long division to rewrite this function.

d) How can the new form of the equation be used to determine an equation for the slant asymptote?

9. Refer to question 8 to graph each function.

a) $f(x) = \dfrac{x^2 + 5x - 2}{x + 3}$

b) $g(x) = \dfrac{2x^2 - 5x + 3}{x + 2}$

10. In the event of a power failure, a computer model estimates the temperature, T, in degrees Celsius, in a food-processing plant's freezer to be $T = \dfrac{2t^2}{t + 1} - 15$, where t is the time, in hours, after the power failure.

a) Sketch a graph of this function. Use technology or the method from question 8.

b) How long would it take for the temperature to reach 0°C?

c) A generator starts up when the temperature is −5°C. How long would it take for this to happen?

11. A cylindrical tank is to have a volume of 100 000 cm³.

a) Write a formula for the height in terms of the radius.

b) Sketch a graph of the relationship between height and radius.

12. **Use Technology** As blood moves away from the heart, the systolic pressure, P, in millimetres of mercury (mmHg), after t seconds, changes according to the function $P(t) = \dfrac{25t^2 + 125}{t^2 + 1}$, $0 \leq t \leq 10$.

a) Graph this function using technology.

b) Describe what happens to the systolic pressure over the first 10 s.

c) To measure the rate of change in systolic pressure, you can use the function $R(t) = -\dfrac{200t}{(t^2 + 1)^2}$. Graph this function using technology. Describe the rate of change.

d) Compare the rate of change at $t = 5$ s, using the slope of a tangent of $P(t)$, to the rate of change function $R(t)$ at $t = 5$ s.

C Extend and Challenge

13. For $x > 0$, what value of x gives the least sum of x and its reciprocal?

14. The function $C(t) = \dfrac{0.16t}{t^2 + t + 2}$ models the concentration, C, in milligrams per cubic centimetre, of a drug in the bloodstream after time, t, in minutes.

a) Sketch a graph of the function, without using technology.

b) Explain the shape of the graph in the context of the concentration of the drug in the bloodstream.

15. A generator produces electrical power, P, in watts, according to the function $P(R) = \dfrac{120R}{(0.4 + R)^2}$, where R is the resistance, in ohms. Determine the intervals on which the power is increasing.

16. **Math Contest** Is the following statement true or false? The function $g(x) = \dfrac{x^n - n^2}{x^{n-1} - n}$, $n \in \mathbb{N}$, has a slant asymptote for all values of n. Give a reason for your answer.

17. **Math Contest** When the polynomial $5x^4 + 4x^3 + 3x^2 + Px + Q$ is divided by $x^2 - 1$, the remainder is 0. What is the value of $P + Q$?

A −12 **B** 212 **C** 26 **D** 12 **E** 6

18. **Math Contest** Solve the equation $\sqrt{3}\sin x + \cos x = 2$ for x.

Chapter 3 REVIEW

3.1 Reciprocal of a Linear Function

1. Determine equations for the vertical and horizontal asymptotes of each function.

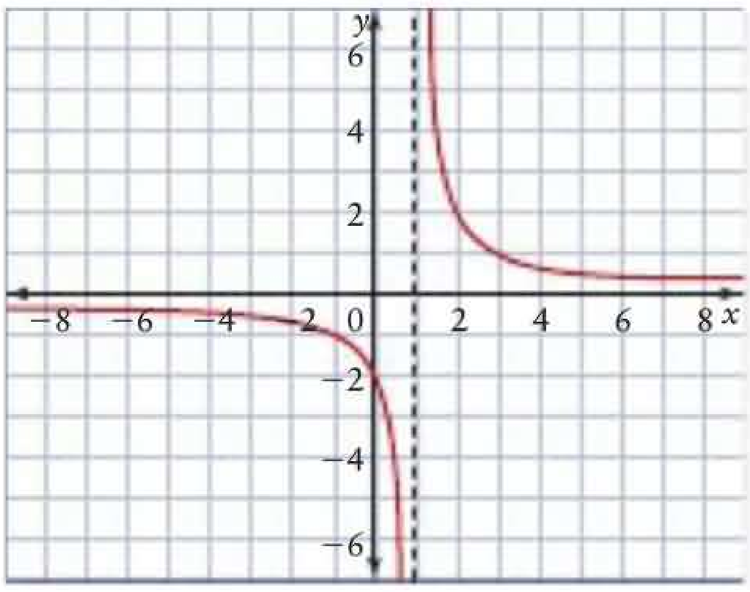
a) $f(x) = \dfrac{1}{x - 2}$

b) $g(x) = \dfrac{3}{x + 7}$

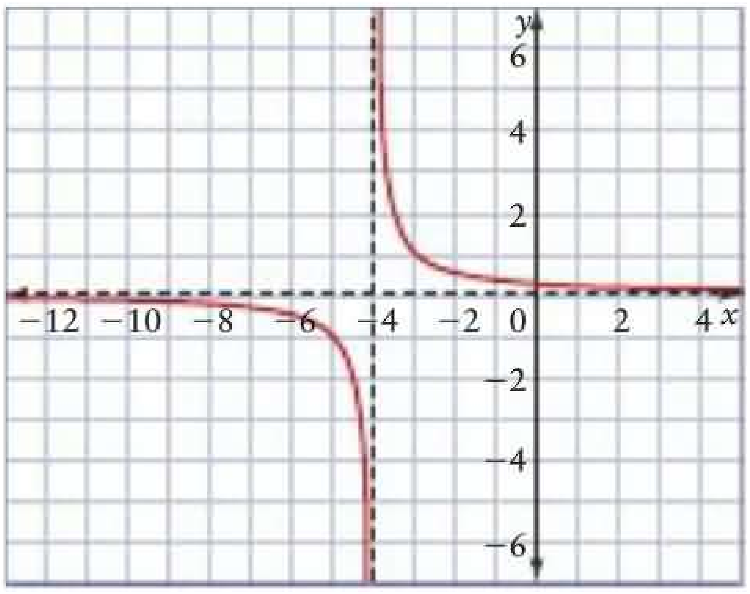
c) $h(x) = -\dfrac{4}{x - 5}$

2. Determine an equation to represent each function.

a)



b)



3. Sketch a graph of each function. State the domain, range, y-intercepts, and equations of the asymptotes.

a) $f(x) = \dfrac{5}{x - 3}$

b) $g(x) = -\dfrac{1}{x - 4}$

c) $h(x) = \dfrac{1}{2x - 3}$

d) $k(x) = -\dfrac{8}{5x + 4}$

3.2 Reciprocal of a Quadratic Function

4. Determine equations for the vertical asymptotes of each function. Then, state the domain.

a) $f(x) = \dfrac{1}{(x - 3)(x + 4)}$

b) $g(x) = -\dfrac{2}{(x + 3)^2}$

c) $h(x) = \dfrac{1}{x^2 + 8x + 12}$

5. For each function,

i) determine equations for the asymptotes

ii) determine the y-intercepts

iii) sketch a graph

iv) describe the increasing and decreasing intervals

v) state the domain and range

a) $f(x) = \dfrac{1}{x^2 + 6x + 5}$

b) $g(x) = \dfrac{1}{x^2 - 5x - 24}$

c) $h(x) = -\dfrac{1}{x^2 - 6x + 9}$

d) $k(x) = -\dfrac{2}{x^2 + 5}$

6. Analyse the slope, and change in slope, for the intervals of the function $f(x) = \dfrac{1}{2x^2 + 3x - 5}$ by sketching a graph of the function.

7. Write an equation for a function that is the reciprocal of a quadratic and has the following properties:

- The horizontal asymptote is $y = 0$.
- The vertical asymptotes are $x = -4$ and $x = 5$.
- For the intervals $x < -4$ and $x > 5$, $y < 0$.

3.3 Rational Functions of the Form $f(x) = \dfrac{ax + b}{cx + d}$

8. Determine an equation for the horizontal asymptote of each function.

a) $a(x) = \dfrac{x}{x + 5}$

b) $b(x) = -\dfrac{2x}{x - 3}$

c) $c(x) = \dfrac{x + 2}{x - 2}$

9. Summarize the key features of each function. Then, sketch a graph of the function.

a) $f(x) = \dfrac{x}{x - 2}$ b) $g(x) = -\dfrac{3x}{x + 1}$

c) $h(x) = \dfrac{x - 2}{x + 4}$ d) $k(x) = \dfrac{6x + 2}{2x - 1}$

10. Write an equation of a rational function of the form $f(x) = \dfrac{ax + b}{cx + d}$ whose graph has all of the following features:

- x-intercept of $\dfrac{1}{4}$
- y-intercept of $-\dfrac{1}{2}$
- vertical asymptote with equation $x = -\dfrac{2}{3}$
- horizontal asymptote with equation $y = \dfrac{4}{3}$

3.4 Solve Rational Equations and Inequalities

11. Solve algebraically. Check each solution.

a) $\dfrac{7}{x - 4} = 2$ b) $\dfrac{3}{x^2 + 6x - 24} = 1$

12. **Use Technology** Solve each equation using technology. If necessary, express your answers to two decimal places.

a) $\dfrac{4x}{x + 2} = \dfrac{5x}{3x + 1}$

b) $\dfrac{5x + 2}{2x - 9} = \dfrac{3x - 1}{x + 2}$

c) $\dfrac{x^2 - 3x + 1}{2 - x} = \dfrac{x^2 + 5x + 4}{x - 6}$

13. Solve each inequality without using technology. Illustrate the solution on a number line. Check your solutions using technology.

a) $\dfrac{3}{x + 5} < 2$ b) $\dfrac{3}{x + 2} \leq \dfrac{4}{x + 3}$

c) $\dfrac{x^2 - x - 20}{x^2 - 4x - 12} > 0$ d) $\dfrac{x}{x + 5} > \dfrac{x - 1}{x + 7}$

14. **Use Technology** Solve each inequality using technology.

a) $\dfrac{x^2 + 5x + 4}{x^2 - 5x + 6} < 0$

b) $\dfrac{x^2 - 6x + 9}{2x^2 + 17x + 8} > 0$

3.5 Making Connections With Rational Functions and Equations

15. A manufacturer is predicting profit, P, in thousands of dollars, on the sale of x tonnes of fertilizer according to the equation $P(x) = \dfrac{600x - 15000}{x + 100}$.

a) Sketch a graph of this relation.

b) Describe the predicted profit as sales increase.

c) Compare the rates of change of the profit at sales of 100 t and 500 t of fertilizer.

16. Sketch a graph of each function. Describe each special case.

a) $f(x) = \dfrac{x}{x^2 + 5x}$

b) $g(x) = \dfrac{x^2 - 2x - 35}{x^2 - 3x - 28}$

CHAPTER PROBLEM WRAP-UP

The cost, C, in millions of dollars, of cleaning up an oil spill can be modelled by the function $C(p) = \dfrac{20}{100 - p}$, where p is the percent of the oil that was spilled. The rate of change of the cost, in millions of dollars, is given by $R(p) = \dfrac{20}{(100 - p)^2}$.

a) State the domain and range of each function, and explain what they mean.

b) Calculate the rate of change at $p = 50$, using the slope of the tangent to $C(p)$.

c) Compare the rate of change from part b) to that using the rate of change function $R(p)$ for $p = 50$.

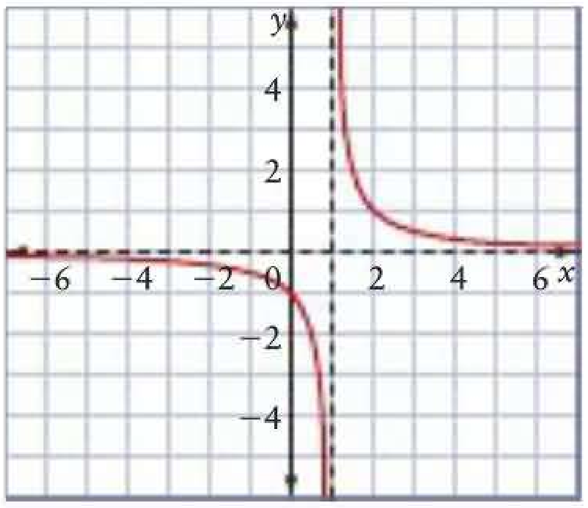
d) Write a report outlining the cost of cleaning up the oil spill. Include the following:

- graphs of $C(p)$ and $R(p)$
- the total cost to clean up 25%, 50%, and 90% of the spill
- the rate of change of the cost at 25%, 50%, and 90% of the spill

For questions 1 to 3, select the best answer.

1. Which graph represents $f(x) = \dfrac{1}{x^2 - 1}$?

A



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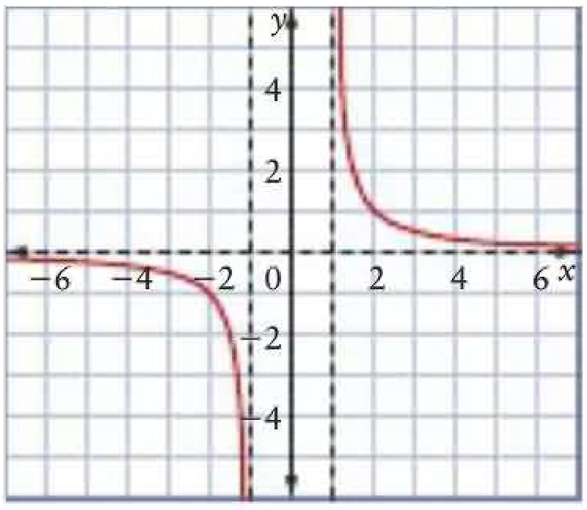

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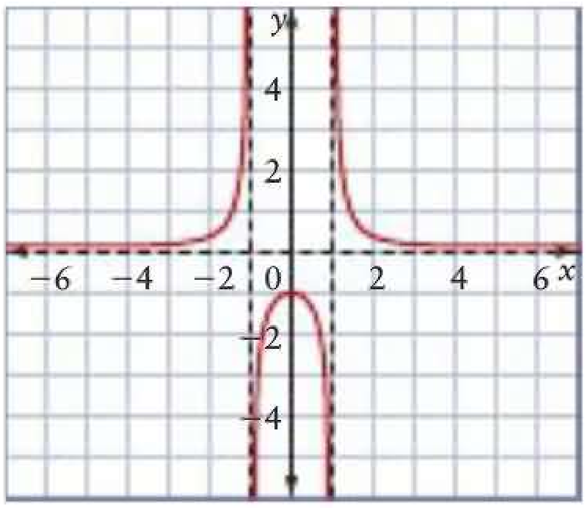

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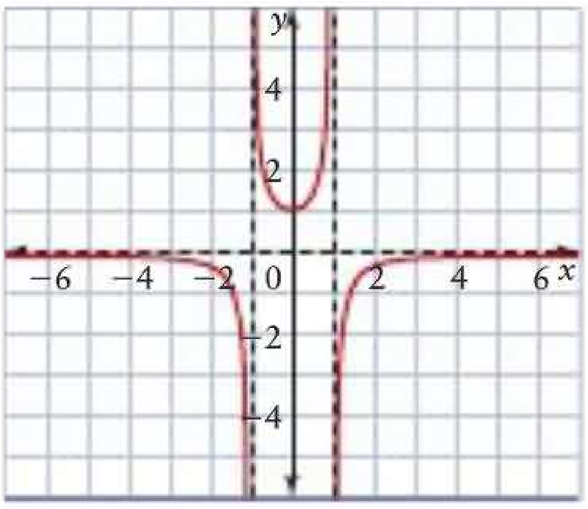


2. For the function $f(x) = \dfrac{2}{x + 5}$, which statement is correct?

A As $x \to +\infty$, $f(x) \to +\infty$.

B As $x \to +\infty$, $f(x) \to 0$.

C As $x \to 5^+$, $f(x) \to +\infty$.

D As $x \to 5^-$, $f(x) \to +\infty$.

3. For the function $f(x) = \dfrac{x + 2}{x - 5}$, which statement is true?

A Domain: $\{x \in \mathbb{R}, x \neq 5\}$,
Range: $\{y \in \mathbb{R}, y \neq 1\}$

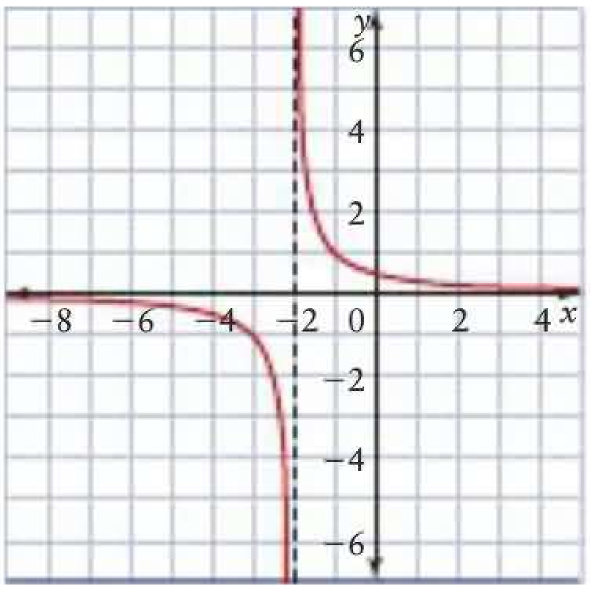
B Domain: $\{x \in \mathbb{R}, x \neq -2\}$,
Range: $\{y \in \mathbb{R}, y \neq 1\}$

C Domain: $\{x \in \mathbb{R}, x \neq 5\}$,
Range: $\{y \in \mathbb{R}, y \neq 0\}$

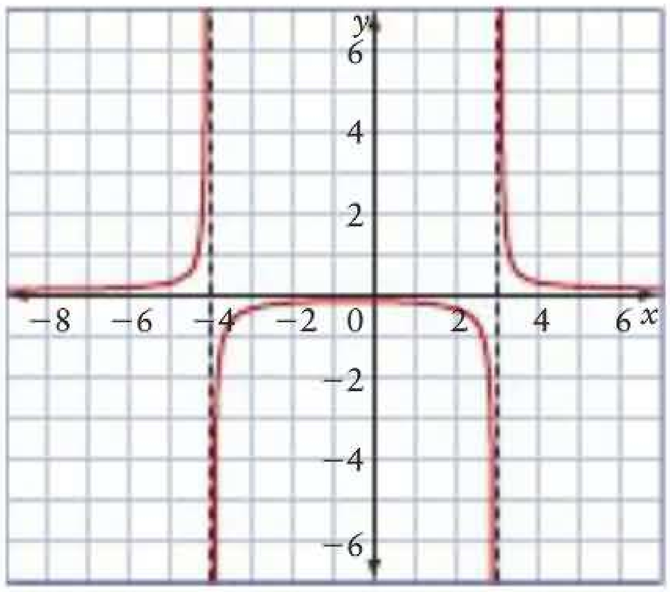
D Domain: $\{x \in \mathbb{R}, x > 5\}$,
Range: $\{y \in \mathbb{R}, y > 0\}$

4. Write a possible equation for the function in each graph.

a)



b)



5. Consider the function $f(x) = -\frac{4}{x^2 + 2}$.

a) Determine the following key features of the function:

i) domain and range

ii) intercepts

iii) asymptotes

iv) intervals where the function is increasing and intervals where it is decreasing

b) Sketch a graph of the function.

6. If $f(x)$ is a polynomial function, does $\frac{1}{f(x)}$ always have a horizontal asymptote? If yes, explain why. If no, provide a counterexample.

7. Solve each equation. Provide an exact solution.

a) $\frac{3x + 5}{x - 4} = \frac{1}{2}$

b) $\frac{20}{x^2 - 4x + 7} = x + 2$

8. Solve each inequality. Illustrate your solution on a number line.

a) $\frac{5}{2x + 3} < 4$

b) $\frac{x + 1}{x - 2} > \frac{x + 7}{x + 1}$

9. a) Determine an equation of the form $f(x) = \frac{ax + b}{cx + d}$ for the rational function with x-intercept 2, vertical asymptote at $x = -1$, and horizontal asymptote at $y = -\frac{1}{2}$.

b) Is it possible for another function to have the same key features? If not, explain why not. If so, provide an example.

10. The acceleration due to gravity is inversely proportional to the square of the distance from the centre of Earth. The acceleration due to gravity for a satellite orbiting 7000 km above the centre of Earth is 8.2 m/s^2.

a) Write a formula for this relationship.

b) Sketch a graph of the relation.

c) At what height will the acceleration due to gravity be 6.0 m/s^2?

11. When a saw is used to cut wood, a certain percent is lost as sawdust, depending on the thickness of the saw-blade. The wood lost is called the kerf. The percent lost, $P(t)$, can be modelled by the function $P(t) = \frac{100t}{t + W}$, where t is the thickness of the blade and W is the thickness of the wood, both in millimetres. Consider a saw cutting a 30-mm-thick piece of wood.

a) Sketch a graph of the function, in the context of this situation.

b) State the domain and range.

c) Explain the significance of the horizontal asymptote.

12. The electric power, P, in watts, delivered by a certain battery is given by the function $P = \frac{100R}{(2 + R)^2}$, where R is the resistance, in ohms.

a) Sketch a graph of this function.

b) Describe the power output as the resistance increases from 0 Ω to 20 Ω.

c) Show that the rate of change is 0 at $R = 2$. What does this indicate about the power?

13. Investigate the graphs of functions of the form $f(x) = \frac{1}{x^n}$, where $n \in \mathbb{N}$. Summarize what happens to the asymptotes and slopes as n increases. Consider, also, when n is even or odd.

Chapters 1 to 3 REVIEW

Chapter 1

1. Sketch a graph of each polynomial function. Determine the x- and y-intercepts.
   a) $y = x^3 - 2x^2 - x + 2$
   b) $y = x^4 - 2x^3 - 8x^2 + 12x - 16$

2. Describe the symmetry and the end behaviour of each function.
   a) $f(x) = 5x^4 - 2x^3 + x^2 - 3x + 1$
   b) $g(x) = 4x^3 - 5x^2 + 7x - 3$

3. a) Determine the average rate of change of the cubic function $f(x) = -4x^3$ for the intervals
   i) $x = 2.0$ to $x = 2.5$
   ii) $x = 1.5$ to $x = 2.0$
   b) Find the average of these two rates of change and describe what the result approximates.

4. Sketch graphs of the functions in each pair on the same set of axes. Label fully.
   a) $f(x) = x^3$ and $g(x) = 0.5(x - 1)^3 + 3$
   b) $f(x) = x^4$ and $g(x) = -(2x + 6)^4$

5. A function is represented by the table of values.

x	y
−5	−288
−4	−50
−3	0
−2	−18
−1	−32
0	−18
1	0
2	−50
3	−288

   a) Sketch a graph of the function, assuming that all zeros are given in the table and that they are each of order 2.
   b) Determine the degree of the function.
   c) Write an equation in factored form to represent the function.
   d) Explain what effect the leading coefficient has on the graph.

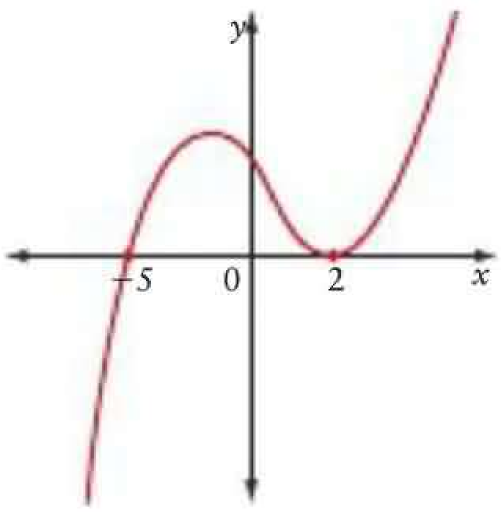
6. Consider the function $f(x) = x^3 - 5x^2 + 4x - 5$.
   a) Determine the instantaneous rate of change at the point where $x = 2$.
   b) Determine the instantaneous rate of change at the point where $x = 4$.
   c) What special kind of point occurs between the points in parts a) and b)? Explain.

7. A section of rollercoaster can be modelled by $h(x) = 0.0025x(x - 8)(x - 16)(x - 32)$, $0 \le x \le 32$, where x is the horizontal distance and h is the height relative to the platform, with all measurements in metres.
   a) Determine the maximum and minimum height of each hill and valley. Round answers to two decimal places.
   b) What is the average rate of change of the rollercoaster between each pair of maximum and minimum points?
   c) Where would the instantaneous rate of change of the rollercoaster be the greatest? Justify your answer using appropriate calculations.

Chapter 2

8. Write two possible equations for a quartic function with zeros at $x = -7$ and $x = 0$ and a zero of order two at $x = 3$.

9. Write an equation representing the family of curves defined by the graph shown. Provide two possible examples of specific functions that are members of this family and give the y-intercept for each.



10. Use long division to divide. State the restriction on the variable.
   a) $4x^3 - 5x^2 + 6x + 2$ divided by $2x + 1$
   b) $3x^4 - 5x^2 - 28$ divided by $x - 2$

11. Use the remainder theorem to determine the remainder of each division.

a) $6x^3 - 7x^2 + 5x + 8$ divided by $x - 2$

b) $3x^4 + x^3 - 2x + 1$ divided by $3x + 4$

12. Use the factor theorem to determine whether the second polynomial is a factor of the first.

a) $4x^5 - 3x^3 - 2x^2 + 5$, $x + 5$

b) $3x^3 - 15x^2 + 10x + 8$, $x - 4$

13. Determine the value of k so that, when $3x^4 - 4x^3 + kx - 3x + 6$ is divided by $x - 2$, the remainder is 3.

14. Factor fully.

a) $x^3 - 27$

b) $2x^3 + 4x^2 - 13x - 6$

15. Find the real roots of each equation.

a) $x^3 - 2x^2 - 19x + 20 = 0$

b) $5x^3 + 23x^2 = 9 - 21x$

16. Solve each inequality.

a) $x^2 - 7x + 6 \geq 0$

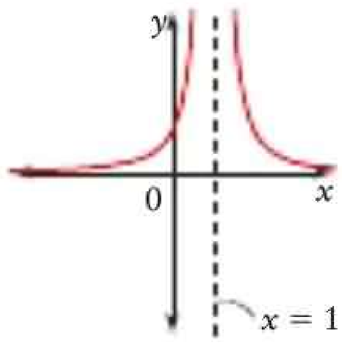
b) $x^3 + 3x^2 - 4x - 12 < 0$

17. The mass, m, in tonnes, of fuel in a rocket t minutes after it is launched is given by $m = -t^2 - 140t + 2000$. During what period of time is the mass of the fuel greater than 500 t?

Chapter 3

18. Which graph represents $f(x) = \dfrac{1}{(x - 1)^2}$?

A



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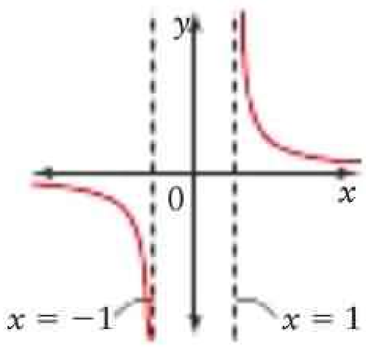

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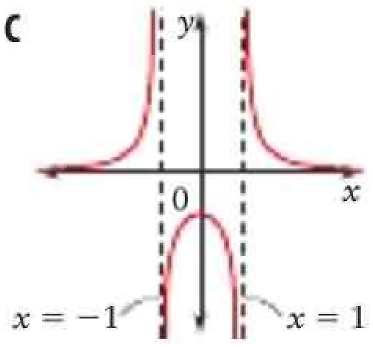

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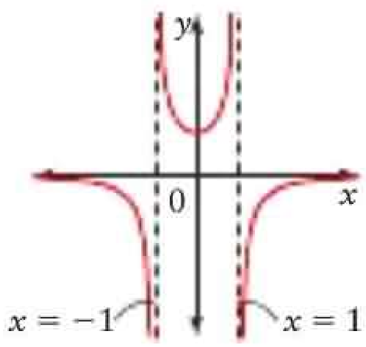

19. Describe what happens to the function $f(x) = \dfrac{4}{x - 3}$ as x approaches each value.

a) $+\infty$ b) $-\infty$ c) 3^+ d) 3^-

20. State the domain and the range of the function $f(x) = \dfrac{x - 2}{x + 1}$.

21. i) State the key features of each function:
- domain and range
- x- and y-intercepts
- equations of the asymptotes
- slope of the function in each interval
- change in slope for each interval

ii) Sketch a graph of each function.

a) $f(x) = \dfrac{6x + 1}{2x - 4}$ b) $f(x) = \dfrac{1}{x^2 - 9}$

22. Determine an equation for the rational function of the form $f(x) = \dfrac{ax + b}{cx + d}$ that has an x-intercept of -2, a vertical asymptote at $x = 1$, and a horizontal asymptote at $y = 3$.

23. Solve each equation.

a) $\dfrac{2x + 3}{x - 2} = \dfrac{1}{3}$

b) $\dfrac{10}{x^2 - 3x + 5} = x + 1$

24. Solve each inequality. Show your solution on a number line.

a) $\dfrac{3}{2x + 4} > -2$

b) $\dfrac{x + 3}{x - 1} \leq \dfrac{x + 4}{x - 2}$

25. The profit, in thousands of dollars, from the sale of x kilograms of tuna fish can be modelled by the function $P(x) = \dfrac{5x - 400}{x + 600}$.

a) Sketch a graph of this function.

b) State the domain and the range.

c) Explain the significance of the horizontal asymptote.

26. The current, I, in ohms (Ω), in an electrical circuit after t milliseconds is given by $I = \dfrac{11 - t}{10 - t}$. What is an appropriate domain for this function? Explain your reasoning.