Chapter 2

Polynomial Equations and Inequalities

Many real-life problems can be modelled by equations or inequalities. For instance, a manufacturer of electronic games models the profit on its latest device using a polynomial function in one variable. How many devices must be sold to break even, or make a profit? Solving a polynomial equation enables such questions to be answered.

By the end of this chapter, you will

- recognize that there may be more than one polynomial function that can satisfy a given set of conditions (C1.7)
- determine the equation of the family of polynomial functions with a given set of zeros and of the member of the family that passes through another given point (C1.8)
- make connections, through investigation using technology, between the polynomial function $f(x)$, the divisor $x - a$, the remainder from the division $\frac{f(x)}{x-a}$, and $f(a)$ to verify the remainder theorem and factor theorem (C3.1)
- factor polynomial expressions in one variable, of degree no higher than four, by selecting and applying strategies (C3.2)
- determine, through investigation using technology, the connection between the real roots of a polynomial equation and the x-intercepts of the graph of the corresponding polynomial function, and describe this connection (C3.3)
- solve polynomial equations in one variable, of degree no higher than four, by selecting and applying strategies, and verify solutions using technology (C3.4)
- solve problems involving applications of polynomial and equations (C3.7)
- explain, for polynomial functions, the difference between the solution to an equation in one variable and the solution to an inequality in one variable, and demonstrate that given solutions satisfy an inequality (C4.1)
- determine solutions to polynomial inequalities in one variable by graphing the corresponding functions, using graphing technology, and identifying intervals for which x satisfies the inequalities (C4.2)
- solve linear inequalities and factorable polynomial inequalities in a variety of ways, and represent the solutions on a number line or algebraically (C4.3)

Prerequisite Skills

Use Long Division

1. Use long division to find each quotient. Write the remainder.
   a) $3476 \div 28$
   b) $5973 \div 37$
   c) $2508 \div 17$
   d) $6815 \div 19$

Evaluate Functions

2. Given $P(x) = x^3 - 5x^2 + 7x - 9$, evaluate.
   a) $P(-1)$
   b) $P(3)$
   c) $P(-2)$
   d) $P\left(-\frac{1}{2}\right)$
   e) $P\left(\frac{2}{3}\right)$

Simplify Expressions

3. Expand and simplify.
   a) $(x^3 + 3x^2 - x + 1)(x - 2) + 5$
   b) $(2x^3 - 4x^2 + x - 3)(x + 4) - 7$
   c) $(x^3 + 4x^2 - x + 8)(3x - 1) + 6$
   d) $(x - \sqrt{2})(x + \sqrt{2})$
   e) $(x - 3\sqrt{5})(x + 3\sqrt{5})$
   f) $(x - 1 + \sqrt{3})(x - 1 - \sqrt{3})$

Factor Expressions

4. Factor each difference of squares. Look for common factors first.
   a) $x^2 - 4$
   b) $25m^2 - 49$
   c) $16y^2 - 9$
   d) $12c^2 - 27$
   e) $2x^4 - 32$
   f) $3n^4 - 12$

5. Factor each trinomial.
   a) $x^2 + 5x + 6$
   b) $x^2 - 9x + 20$
   c) $b^2 + 5b - 14$
   d) $2x^2 - 7x - 15$
   e) $4x^2 - 12x + 9$
   f) $6a^2 - 7a + 2$
   g) $9m^2 - 24m + 16$
   h) $3m^2 - 10m + 3$

Solve Quadratic Equations

6. Solve by factoring.
   a) $x^2 - 2x - 15 = 0$
   b) $4x^2 + x - 3 = 0$
   c) $16x^2 - 36 = 0$
   d) $9x^2 = -15 + 48x$
   e) $20 - 12x = 8x^2$
   f) $21x^2 + 1 = 10x$

7. Use the quadratic formula to solve. Round answers to one decimal place.
   a) $5x^2 + 6x - 1 = 0$
   b) $2x^2 - 7x + 4 = 0$
   c) $4x^2 = -2x + 3$
   d) $7x + 20 = 6x^2$

Determine Equations of Quadratic Functions

8. Determine an equation for the quadratic function, with the given zeros, and that passes through the given point.
   a) zeros: -4 and 1; point: $(-1, 2)$
   b) zeros: 0 and 3; point: $(2, 6)$
   c) zeros: -3 and 4; point: $(3, 24)$
   d) zeros: 5 and -1; point: $(4, -10)$
   e) zeros: $\frac{3}{2}$ and $-\frac{1}{2}$; point: $(0, 9)$

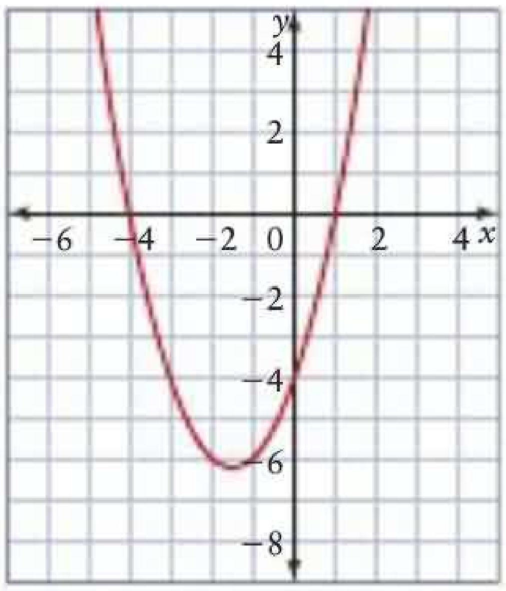
Determine Intervals From Graphs

9. For the graph of each polynomial function,

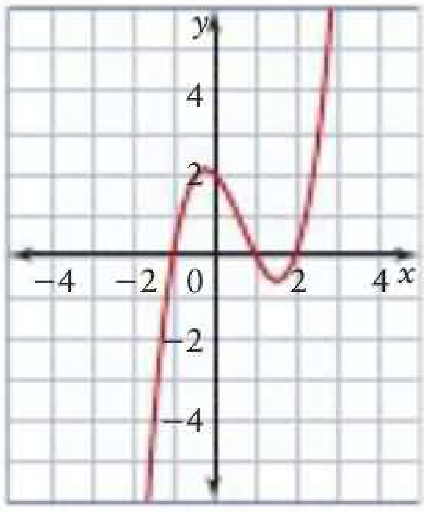
i) identify the x-intercepts

ii) write the intervals for which the graph is above the x-axis and the intervals for which the graph is below the x-axis

a)



b)



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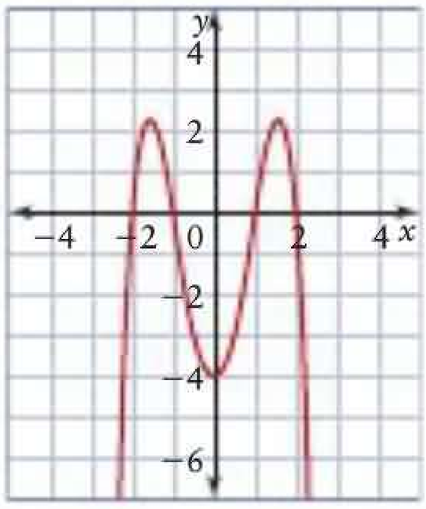


CHAPTER PROBLEM

Best of U is a company that manufactures personal care products. Much of the company's recent success is due to the hard work of three key teams. The package design team is responsible for creating attractive, practical, and low-cost containers. The marketing team keeps in close touch with up-to-date trends and consumer demands for various products. Finally, the finance team analyses production costs, revenue, and profits to ensure that the company achieves its financial goals.

Throughout this chapter, you will discover how polynomial functions may be used to model and solve problems related to some of the aspects of running this company.

2.1 The Remainder Theorem

A manufacturer of cardboard boxes receives an order for gift boxes. Based on cost calculations, the volume, V, of each box to be constructed can be modelled by the polynomial function $V(x) = x^3 + 7x^2 + 14x + 8$, where x is a positive integer such that $10 \leq x \leq 20$. The height, h, of each box is a linear function of x such that $h(x) = x + 2$. How can this information be used to determine the dimensions of the boxes in terms of polynomials?

In this section, you will apply the method of long division to divide a polynomial by a binomial. You will also learn to use the remainder theorem to determine the remainder of a division without dividing.

Investigate 1 How do you divide using long division?

1. Examine these two long divisions.

a)
$$\begin{array}{r} 34 \\ 22\overline{)753} \\ \underline{66} \\ 93 \\ \underline{88} \\ 5 \end{array}$$

b)
$$\begin{array}{r} x + 3 \\ x + 2\overline{)x^2 + 5x + 7} \\ \underline{x^2 + 2x\phantom{{}+7}} \\ 3x + 7 \\ \underline{3x + 6} \\ 1 \end{array}$$

For each division, identify the expression or value that corresponds to

i) the dividend **ii)** the divisor

iii) the quotient **iv)** the remainder

2. Reflect

a) Describe how long division is used to divide the numbers in step 1 a).

b) Describe how long division is used to divide the trinomial in step 1 b).

c) Describe similarities between the use of long division with numbers and with trinomials.

3. a) How can you check that the result of a long division is correct?

b) Write the corresponding statement that can be used to check each division.

Example 1 Divide a Polynomial by a Binomial

a) Divide $-3x^2 + 2x^3 + 8x - 12$ by $x - 1$. Express the result in quotient form.

b) Identify any restrictions on the variable.

c) Write the corresponding statement that can be used to check the division.

d) Verify your answer.

Solution

Write the polynomial in order of descending powers, just as numbers are written in order of place value: $2x^3 - 3x^2 + 8x - 12$.

a) **Method 1:** ***Use Long Division***

$$
\begin{array}{rll}
 & 2x^2 - x + 7 & \\
x - 1 \,\big) & 2x^3 - 3x^2 + 8x - 12 & \text{Divide } 2x^3 \text{ by } x \text{ to get } 2x^2. \\
 & \underline{2x^3 - 2x^2} & \text{Multiply } x - 1 \text{ by } 2x^2 \text{ to get } 2x^3 - 2x^2. \\
 & -x^2 + 8x & \text{Subtract. Bring down the next term, } 8x\text{. Then, divide } -x^2 \text{ by } x \text{ to get } -x. \\
 & \underline{-x^2 + x} & \text{Multiply } x - 1 \text{ by } -x \text{ to get } -x^2 + x. \\
 & 7x - 12 & \text{Subtract. Bring down the next term, } -12\text{. Then, divide } 7x \text{ by } x \text{ to get } 7. \\
 & \underline{7x - 7} & \text{Multiply } x - 1 \text{ by } 7 \text{ to get } 7x - 7. \\
 & -5 & \text{Subtract. The remainder is } -5.
\end{array}
$$

$$\frac{-3x^2 + 2x^3 + 8x - 12}{x - 1} = 2x^2 - x + 7 + \frac{-5}{x - 1}$$

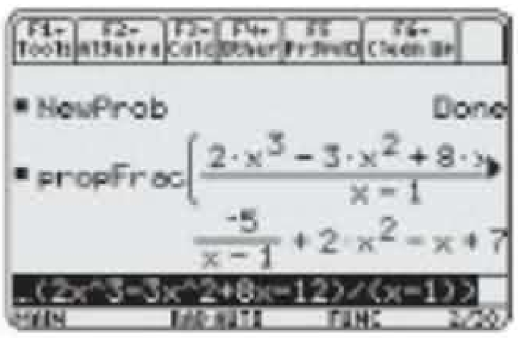
Method 2: ***Use a Computer Algebra System (CAS)***

Press (HOME) to display the CAS home screen and clear its memory using the **Clean Up** menu.

- From the **F6** menu, select **2:NewProb**.
- Press (ENTER).
- From the **F2** menu, select **7:propFrac**.

Enter the division expression.

- Press (ENTER).



So, $\dfrac{-3x^2 + 2x^3 + 8x - 12}{x - 1} = 2x^2 - x + 7 + \dfrac{-5}{x - 1}$

b) Since division by zero is not defined, the divisor cannot be zero: $x - 1 \neq 0$, or $x \neq 1$.

c) The corresponding statement is $-3x^2 + 2x^3 + 8x - 12 = (x - 1)(2x^2 - x + 7) - 5$.

d) To check: Multiply the divisor by the quotient and add the remainder.

$$
\begin{aligned}
(x - 1)(2x^2 - x + 7) - 5 &= 2x^3 - x^2 + 7x - 2x^2 + x - 7 - 5 \\
&= 2x^3 - 3x^2 + 8x - 12 \\
&= -3x^2 + 2x^3 + 8x - 12
\end{aligned}
$$

The result of the division of a polynomial $P(x)$ by a binomial of the form $x - b$ is $\dfrac{P(x)}{x - b} = Q(x) + \dfrac{R}{x - b}$, where $Q(x)$ is the quotient and R is the remainder. The corresponding statement, that can be used to check the division, is $P(x) = (x - b)Q(x) + R$.

Example 2 Divide a Polynomial by a Binomial of the Form $ax - b$

a) Divide $4x^3 + 9x - 12$ by $2x + 1$. Identify any restrictions on the variable.

b) Write the corresponding statement that can be used to check the division.

Solution

a) The polynomial does not have an x^2 term, so insert $0x^2$ as a placeholder.

$$
\begin{array}{r@{}l}
 & 2x^2 - x + 5 \\
2x + 1 & \overline{)4x^3 + 0x^2 + 9x - 12} \\
 & \underline{4x^3 + 2x^2} \\
 & -2x^2 + 9x \\
 & \underline{-2x^2 - x} \\
 & 10x - 12 \\
 & \underline{10x + 5} \\
 & -17
\end{array}
$$

Divide $4x^3$ by $2x$ to get $2x^2$.
Multiply $2x + 1$ by $2x^2$ to get $4x^3 + 2x^2$.
Subtract. Bring down the next term, $9x$.

$$\frac{4x^3 + 9x - 12}{2x + 1} = 2x^2 - x + 5 + \frac{-17}{2x + 1}$$

Restriction: $2x + 1 \neq 0$, or $x \neq -\dfrac{1}{2}$

b) The corresponding statement is $4x^3 + 9x - 12 = (2x + 1)(2x^2 - x + 5) - 17$.

Example 3 Apply Long Division to Solve for Dimensions

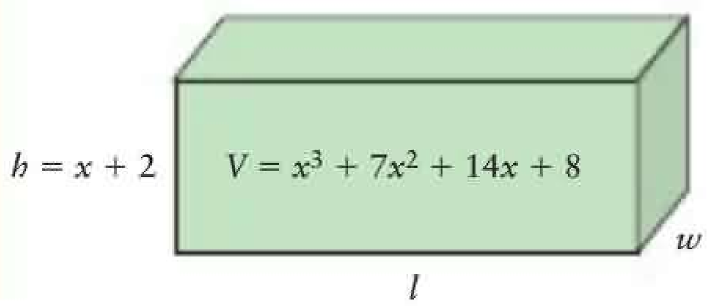
The volume, V, in cubic centimetres, of a rectangular box is given by $V(x) = x^3 + 7x^2 + 14x + 8$.

Determine expressions for possible dimensions of the box if the height, h, in centimetres, is given by $x + 2$.

Solution

CONNECTIONS

The formula for the volume of a rectangular box is $V = lwh$.



Divide the volume by the height to obtain an expression for the area of the base of the box. That is, $\frac{V}{h} = lw$, where lw is the area of the base.

$$
\begin{array}{r}
x^2 + 5x + 4 \\
x + 2\overline{)x^3 + 7x^2 + 14x + 8} \\
\underline{x^3 + 2x^2} \qquad\qquad \\
5x^2 + 14x \quad \\
\underline{5x^2 + 10x} \quad \\
4x + 8 \\
\underline{4x + 8} \\
0
\end{array}
$$

Since the remainder is 0, the volume $x^3 + 7x^2 + 14x + 8$ can be expressed as $(x + 2)(x^2 + 5x + 4)$.

The quotient $x^2 + 5x + 4$ represents the area of the base of the box. This expression can be factored as $(x + 1)(x + 4)$. The factors represent the possible width and length of the base of the box.

Expressions for the possible dimensions of the box, in centimetres, are $x + 1$, $x + 2$, and $x + 4$.

Investigate 2 How can you determine a remainder without dividing?

Method 1: *Use Pencil and Paper*

1. a) Compare each binomial to $x - b$. Write the value of b.

i) $x - 1$ **ii)** $x + 1$ **iii)** $x - 2$ **iv)** $x + 2$

b) Given a polynomial $P(x)$, what is the value of x in each?

i) $P(1)$ **ii)** $P(-1)$ **iii)** $P(2)$ **iv)** $P(-2)$

Compare these values with those found in part a). What do you notice?

2. a) Evaluate parts i) to iv) of step 1b) for the polynomial $P(x) = x^3 + 6x^2 + 2x - 4$. What do the results represent?

b) Use long division to divide $P(x) = x^3 + 6x^2 + 2x - 4$ by each binomial. Write the remainder.

i) $x - 1$ **ii)** $x + 1$ **iii)** $x - 2$ **iv)** $x + 2$

c) Compare the remainders in part b) to the values found in part a). What do you notice?

d) **Reflect** Make a conjecture about how a remainder can be found without using division.

3. a) Use your conjecture from step 2 to predict the remainder when $P(x) = x^3 + 4x^2 - 3x + 1$ is divided by each binomial.

i) $x + 1$ **ii)** $x + 2$ **iii)** $x - 3$ **iv)** $x + 3$

b) Verify your predictions using long division.

4. **Reflect** Describe the relationship between the remainder when a polynomial $P(x)$ is divided by $x - b$ and the value of $P(b)$. Why is it appropriate to call this relationship the **remainder theorem**?

Tools

- calculator with a computer algebra system (CAS)

Method 2: *Use a CAS*

1. Define $P(x) = x^3 + 6x^2 + 2x - 4$.
   - From the **F4** menu, select **1:Define**. Enter the polynomial expression.
2. Determine $P(1)$.
   - Type **P(1)** and then press ENTER.
3. Calculate $\dfrac{x^3 + 6x^2 + 2x - 4}{x - 1}$, or $\dfrac{P(x)}{x - 1}$.
   - From the **F2** menu, select **7:propFrac**.
   - Type **P(X) ÷ (X − 1))** and then press ENTER. Write the remainder.
4. a) **Reflect** Is there a relationship between $P(1)$, the linear factor $x - 1$, and the polynomial division?
   b) Investigate for other values of x. Let $x = -1$, $x = 2$, and $x = -2$, and find $P(x)$.
   c) Calculate $\dfrac{P(x)}{x + 1}$, $\dfrac{P(x)}{x + 2}$, and $\dfrac{P(x)}{x - 2}$.
   d) Compare the remainders in part c) to the values found in part b). What do you notice?
   e) **Reflect** Make a conjecture about how a remainder can be found without using division.
5. a) Use your conjecture in step 4 to predict the remainder when $P(x) = x^3 + 4x^2 - 3x + 1$ is divided by each binomial.
   i) $x + 1$ ii) $x + 2$ iii) $x - 3$ iv) $x + 3$
   b) Verify your predictions using a CAS.
6. **Reflect** Describe the relationship between the remainder when a polynomial $P(x)$ is divided by $x - b$ and the value of $P(b)$. Why is it appropriate to call this relationship the **remainder theorem**?

Remainder Theorem

When a polynomial function $P(x)$ is divided by $x - b$, the remainder is $P(b)$; and when it is divided by $ax - b$, the remainder is $P\left(\dfrac{b}{a}\right)$, where a and b are integers, and $a \neq 0$.

Example 4 Apply and Verify the Remainder Theorem

a) Use the remainder theorem to determine the remainder when $P(x) = 2x^3 + x^2 - 3x - 6$ is divided by $x + 1$.

b) Verify your answer using long division.

c) Use the remainder theorem to determine the remainder when $P(x) = 2x^3 + x^2 - 3x - 6$ is divided by $2x - 3$.

Solution

a) Since $x + 1 = x - (-1)$, the remainder is $P(-1)$.

$$\begin{aligned} P(-1) &= 2(-1)^3 + (-1)^2 - 3(-1) - 6 \\ &= -2 + 1 + 3 - 6 \\ &= -4 \end{aligned}$$

When $P(x) = 2x^3 + x^2 - 3x - 6$ is divided by $x + 1$, the remainder is -4.

b)

$$\begin{array}{r} 2x^2 - x - 2 \\ x + 1 \overline{)\,2x^3 + x^2 - 3x - 6} \\ \underline{2x^3 + 2x^2} \qquad\qquad \\ -x^2 - 3x \qquad \\ \underline{-x^2 - x} \qquad \\ -2x - 6 \\ \underline{-2x - 2} \\ -4 \end{array}$$

Using long division, the remainder is -4. This verifies the answer in part a).

c) Comparing $2x - 3$ to $ax - b$, gives $a = 2$ and $b = 3$.

The remainder $P\left(\frac{b}{a}\right)$ is $P\left(\frac{3}{2}\right)$.

$$\begin{aligned} P\left(\frac{3}{2}\right) &= 2\left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) - 6 \\ &= 2\left(\frac{27}{8}\right) + \left(\frac{9}{4}\right) - 3\left(\frac{3}{2}\right) - 6 \\ &= \frac{27}{4} + \frac{9}{4} - \frac{18}{4} - \frac{24}{4} \\ &= -\frac{6}{4} \\ &= -\frac{3}{2} \end{aligned}$$

When $P(x) = 2x^3 + x^2 - 3x - 6$ is divided by $2x - 3$, the remainder is $-\frac{3}{2}$.

Example 5 Solve for an Unknown Coefficient

Determine the value of k such that when $3x^4 + kx^3 - 7x - 10$ is divided by $x - 2$, the remainder is 8.

Solution

Let $P(x) = 3x^4 + kx^3 - 7x - 10$.

By the remainder theorem, when $P(x)$ is divided by $x - 2$, the remainder is $P(2)$.
Solve $P(2) = 8$.

$$\begin{aligned} 3(2)^4 + k(2)^3 - 7(2) - 10 &= 8 \\ 48 + 8k - 14 - 10 &= 8 \\ 24 + 8k &= 8 \\ 8k &= -16 \\ k &= -2 \end{aligned}$$

The value of k is -2.

KEY CONCEPTS

- Long division can be used to divide a polynomial by a binomial.
- The result of the division of a polynomial function $P(x)$ by a binomial of the form $x - b$ can be written as $P(x) = (x - b)Q(x) + R$ or $\dfrac{P(x)}{x - b} = Q(x) + \dfrac{R}{x - b}$, where $Q(x)$ is the quotient and R is the remainder.
- To check the result of a division, use
divisor × quotient + remainder = dividend.
- The remainder theorem states that when a polynomial function $P(x)$ is divided by $x - b$, the remainder is $P(b)$, and when it is divided by $ax - b$, the remainder is $P\left(\dfrac{b}{a}\right)$, where a and b are integers and $a \neq 0$.

Communicate Your Understanding

C1 Explain why there is a restriction on the divisor of a polynomial function. How is the restriction determined?

C2 When and why might it be necessary to use a placeholder when dividing a polynomial by a binomial?

C3 Describe the error in this statement:
$$\frac{x^3 + 3x^2 - 2x - 1}{x - 2} = (x^2 + 5x + 8) + 5$$

C4 Given a polynomial function $P(x)$ such that $P(-3) = 0$, what are the divisor and the remainder? What is the relationship between the divisor and $P(x)$?

C5 Identify the dividend, divisor, quotient, and remainder in each statement.

a) $\dfrac{6x^2 + 5x - 7}{3x + 1} = 2x + 1 - \dfrac{8}{3x + 1}$

b) $12x^3 + 2x^2 + 11x + 14 = (3x + 2)(4x^2 - 2x + 5) + 4$

c) $\dfrac{5x^3 - 7x^2 - x + 6}{x - 1} = 5x^2 - 2x - 3 + \dfrac{3}{x - 1}$

A Practise

For help with questions 1 and 2, refer to Example 1.

1. a) Divide $x^3 + 3x^2 - 2x + 5$ by $x + 1$. Express the result in quotient form.

b) Identify any restrictions on the variable.

c) Write the corresponding statement that can be used to check the division.

d) Verify your answer.

2. a) Divide $3x^4 - 4x^3 - 6x^2 + 17x - 8$ by $3x - 4$. Express the result in quotient form.

b) Identify any restrictions on the variable.

c) Write the corresponding statement that can be used to check the division.

d) Verify your answer.

For help with question 3, refer to Example 2.

3. Perform each division. Express the result in quotient form. Identify any restrictions on the variable.

a) $x^3 + 7x^2 - 3x + 4$ divided by $x + 2$

b) $6x^3 + x^2 - 14x - 6$ divided by $3x + 2$

c) $10x^3 + 11 - 9x^2 - 8x$ divided by $5x - 2$

d) $11x - 4x^4 - 7$ divided by $x - 3$

e) $3 + x^2 + 7x + 6x^3$ divided by $3x + 2$

f) $8x^3 + 4x^2 - 31$ divided by $2x - 3$

g) $6x^2 - 6 + 8x^3$ divided by $4x - 3$

4. Determine the remainder R so that each statement is true.

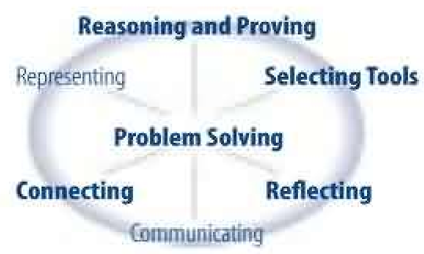
a) $(2x - 3)(3x + 4) + R = 6x^2 - x + 15$

b) $(x + 2)(x^2 - 3x + 4) + R = x^3 - x^2 - 2x - 1$

c) $(x - 4)(2x^2 + 3x - 1) + R$
$= 2x^3 - 5x^2 - 13x + 2$

For help with questions 5 and 6, refer to Example 3.

5. The volume, in cubic centimetres, of a rectangular box can be modelled by the polynomial expression $2x^3 + 17x^2 + 38x + 15$. Determine possible dimensions of the box if the height, in centimetres, is given by $x + 5$.



6. The volume, in cubic centimetres, of a square-based box is given by $9x^3 + 24x^2 - 44x + 16$. Determine possible dimensions of the box if the area of the base, in square centimetres, is $9x^2 - 12x + 4$.

For help with questions 7 to 9, refer to Example 4.

7. Use the remainder theorem to determine the remainder when $2x^3 + 7x^2 - 8x + 3$ is divided by each binomial. Verify your answer using long division.

a) $x + 1$ **b)** $x - 2$ **c)** $x + 3$
d) $x - 4$ **e)** $x - 1$

8. Determine the remainder when each polynomial is divided by $x + 2$.

a) $x^3 + 3x^2 - 5x + 2$

b) $2x^3 - x^2 - 3x + 1$

c) $x^4 + x^3 - 5x^2 + 2x - 7$

9. Use the remainder theorem to determine the remainder for each division.

a) $x^3 + 2x^2 - 3x + 9$ divided by $x + 3$

b) $2x^3 + 7x^2 - x + 1$ divided by $x + 2$

c) $x^3 + 2x^2 - 3x + 5$ divided by $x - 3$

d) $x^4 - 3x^2 - 5x + 2$ divided by $x - 2$

B Connect and Apply

For help with questions 10 to 12, refer to Example 5.

10. a) Determine the value of k such that when $P(x) = kx^3 + 5x^2 - 2x + 3$ is divided by $x + 1$, the remainder is 7.

b) Determine the remainder when $P(x)$ is divided by $x - 3$.

11. a) Determine the value of c such that when $f(x) = x^4 - cx^3 + 7x - 6$ is divided by $x - 2$, the remainder is -8.

b) Determine the remainder when $f(x)$ is divided by $x + 2$.

c) **Use Technology** Verify your answer in part b) using a CAS.

12. For what value of b will the polynomial $P(x) = -2x^3 + bx^2 - 5x + 2$ have the same remainder when it is divided by $x - 2$ and by $x + 1$?

13. For what value of k will the polynomial $f(x) = x^3 + 6x^2 + kx - 4$ have the same remainder when it is divided by $x - 1$ and by $x + 2$?

14. a) Use the remainder theorem to determine the remainder when $2x^3 + 5x^2 - 6x + 4$ is divided by $2x + 1$.

b) Verify your answer in part a) using long division.

c) **Use Technology** Verify your answer in part a) using technology.

15. a) Use the remainder theorem to determine the remainder when $10x^4 - 11x^3 - 8x^2 + 7x + 9$ is divided by $2x - 3$.

b) **Use Technology** Verify your answer in part a) using long division or using a CAS.

16. a) Determine the remainder when $6x^3 + 23x^2 - 6x - 8$ is divided by $3x - 2$.

b) What information does the remainder provide about $3x - 2$? Explain.

c) Express $6x^3 + 23x^2 - 6x - 8$ in factored form.

17. Chapter Problem The packaging design team at Best of U has determined that a cost-efficient way of manufacturing cylindrical containers for their products is to have the volume, V, in cubic centimetres, modelled by $V(x) = 9\pi x^3 + 51\pi x^2 + 88\pi x + 48\pi$, where x is an integer such that $2 \le x \le 8$. The height, h, in centimetres, of each cylinder is a linear function given by $h(x) = x + 3$.

a) Determine the quotient $\dfrac{V(x)}{h(x)}$. Interpret this result.

b) Use your answer in part a) to express the volume of a container in the form $\pi r^2 h$.

c) What are the possible dimensions and volumes of the containers for the given values of x?

CONNECTIONS

The formula for the volume of a cylinder is $V = \pi r^2 h$, where r is the radius of the circular base and h is the height.

18. Jessica Zelinka, a Canadian heptathlete, won a gold medal in javelin throw at the Pan American games in 2007. Suppose $h(t) = -5t^2 + 15t + 1$ represents the approximate height, in metres, of a javelin t seconds after it is thrown.

a) Write a statement that corresponds to the quotient $\dfrac{h(t)}{t - b}$, where b is a positive integer.

b) Show that the statement in part a) may be written as $Q(t) = \dfrac{h(t) - h(b)}{t - b}$.

c) What is the geometric interpretation of $\dfrac{h(t) - h(b)}{t - b}$? Support your answer with a diagram.

d) Use the result of part c) to explain the physical meaning of $Q(t)$ for this situation.

e) Determine the remainder when $h(t)$ is divided by $t - 3$. Interpret the remainder for this situation.

19. The shot-put is another event in a heptathlon. Suppose $h(t) = -5t^2 + 8.3t + 1.2$ represents the approximate height, h, in metres, of a shot-put t seconds after it is thrown.

a) Determine the remainder when $h(t)$ is divided by $t - 1.5$.

b) Use the results of question 18 to interpret your answer in part a) for this situation.

C Extend and Challenge

20. When the polynomial $mx^3 - 3x^2 + nx + 2$ is divided by $x + 3$, the remainder is -1. When it is divided by $x - 2$, the remainder is -4. Determine the values of m and n.

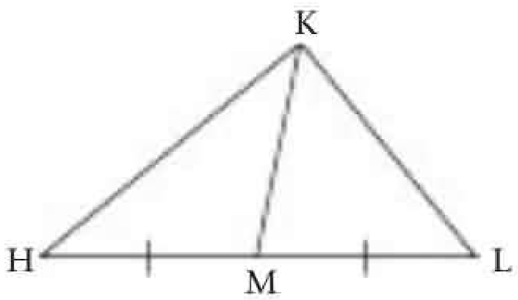
21. When the polynomial $3x^3 + ax^2 + bx - 9$ is divided by $x - 2$, the remainder is -5. When it is divided by $x + 1$, the remainder is -16. Determine the values of a and b.

22. When $3x^2 + 10x - 3$ is divided by $x + k$, the remainder is 5. Determine the values of k.

23. **Math Contest** When a number, x, is divided by 4, the remainder is 3. Determine the remainder when $5x$ is divided by 4.

24. **Math Contest** Determine the area, A, of a triangle with vertices A(4, 6), B(2, 3), and C(8, 4) by applying Heron's formula, $A = \sqrt{s(s - a)(s - b)(s - c)}$, where a, b, and c are the side lengths and $s = \frac{1}{2}(a + b + c)$.

25. **Math Contest** In $\triangle$HKL, $\angle$HKL = 90°. Prove that HM = MK.



CAREER CONNECTION

Since graduating from a 4-year environmental science program at the University of Ottawa, Chantal has been working to become a licensed environmental engineer. She works in water resources management and ensures that social, economic, environmental, and technical concerns are taken into account when water resources, such as reservoirs, are built and maintained. Chantal creates mathematical models of the water resource she is studying and tests them for various factors. For example, she may test the maximum storage capacity of a new reservoir or optimize the amount of water an existing reservoir should release.

2.2

The Factor Theorem

Ice carvers from across Canada and around the world come to Ottawa every year to take part in the ice-carving competition at the Winterlude Festival. Some artists create gigantic ice sculptures from cubic blocks of ice with sides measuring as long as 3.7 m.

Suppose the forms used to make large rectangular blocks of ice come in different dimensions such that the volume of each block can be modelled by $V(x) = 3x^3 + 2x^2 - 7x + 2$. What dimensions, in terms of x, can result in this volume? You will see that the dimensions can be found by factoring $V(x)$.

Investigate How can you determine a factor of a polynomial?

Tools

- calculator with a computer algebra system (optional)

1. a) Use the remainder theorem to determine the remainder when $x^3 + 2x^2 - x - 2$ is divided by $x - 1$.

b) Determine the quotient $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$. Write the corresonding statement that can be used to check the division.

c) Use your answer from part b) to write the factors of $x^3 + 2x^2 - x - 2$.

d) **Reflect** What is the connection between the remainder and the factors of a polynomial function?

2. a) Which of the following are factors of $P(x) = x^3 + 4x^2 + x - 6$? Justify your reasoning.

A $x + 1$

B $x - 1$

C $x + 2$

D $x - 2$

E $x + 3$

b) **Reflect** Write a statement that describes the condition when a divisor $x - b$ is a factor of a polynomial $P(x)$. Why is it appropriate to call this the **factor theorem**? How is this related to the remainder theorem?

3. a) **Reflect** Describe a method you can use to determine the factors of a polynomial $f(x)$.

b) Use your method to determine the factors of $f(x) = x^3 - 2x^2 - x + 2$.

c) Verify your answer in part b).

Factor Theorem

$x - b$ is a factor of a polynomial $P(x)$ if and only if $P(b) = 0$.

Similarly, $ax - b$ is a factor of $P(x)$ if and only if $P\left(\frac{b}{a}\right) = 0$.

CONNECTIONS

"If and only if" is a term used in logic to say that the result works both ways. Here, both of the following are true:

- If $x - b$ is a factor, then $P(b) = 0$.
- If $P(b) = 0$, then $x - b$ is a factor of $P(x)$.

With the factor theorem, you can determine the factors of a polynomial without having to divide. For instance, to determine if $x - 3$ and $x + 2$ are factors of $P(x) = x^3 - x^2 - 14x + 24$, calculate $P(3)$ and $P(-2)$.

$$\begin{aligned} P(3) &= (3)^3 - (3)^2 - 14(3) + 24 \\ &= 27 - 9 - 42 + 24 \\ &= 0 \end{aligned}$$

Since the remainder is zero, $P(x)$ is divisible by $x - 3$; that is, $x - 3$ divides evenly into $P(x)$, and $x - 3$ is a factor of $P(x)$.

$$\begin{aligned} P(-2) &= (-2)^3 - (-2)^2 - 14(-2) + 24 \\ &= -8 - 4 + 28 + 24 \\ &= 40 \end{aligned}$$

Since the remainder is not zero, $P(x)$ is not divisible by $x + 2$. So, $x + 2$ is not a factor of $P(x)$.

In general, if $P(b) = 0$, then $x - b$ is a factor of $P(x)$, and, conversely, if $x - b$ is a factor of $P(x)$, then $P(b) = 0$. This statement leads to the factor theorem, which is an extension of the remainder theorem.

Example 1 Use the Factor Theorem to Find Factors of a Polynomial

a) Which binomials are factors of the polynomial $P(x) = 2x^3 + 3x^2 - 3x - 2$? Justify your answers.

i) $x - 2$ **ii)** $x + 2$ **iii)** $x + 1$ **iv)** $x - 1$ **v)** $2x + 1$

b) Use your results in part a) to write $P(x) = 2x^3 + 3x^2 - 3x - 2$ in factored form.

Solution

a) Use the factor theorem to evaluate $P(b)$ or $P\left(\frac{b}{a}\right)$.

Method 1: *Use Pencil and Paper*

i) For $x - 2$, substitute $x = 2$ into the polynomial expression.

$$\begin{aligned} P(2) &= 2(2)^3 + 3(2)^2 - 3(2) - 2 \\ &= 16 + 12 - 6 - 2 \\ &= 20 \end{aligned}$$

Since the remainder is not zero, $x - 2$ is not a factor of $P(x)$.

ii) For $x + 2$, substitute $x = -2$ into the polynomial expression.

$$\begin{aligned} P(-2) &= 2(-2)^3 + 3(-2)^2 - 3(-2) - 2 \\ &= -16 + 12 + 6 - 2 \\ &= 0 \end{aligned}$$

Since the remainder is zero, $x + 2$ is a factor of $P(x)$.

iii) For $x + 1$, substitute $x = -1$ into the polynomial expression.

$$\begin{aligned} P(-1) &= 2(-1)^3 + 3(-1)^2 - 3(-1) - 2 \\ &= -2 + 3 + 3 - 2 \\ &= 2 \end{aligned}$$

Since the remainder is not zero, $x + 1$ is not a factor of $P(x)$.

iv) For $x - 1$, substitute $x = 1$ into the polynomial expression.

$$\begin{aligned} P(1) &= 2(1)^3 + 3(1)^2 - 3(1) - 2 \\ &= 2 + 3 - 3 - 2 \\ &= 0 \end{aligned}$$

Since the remainder is zero, $x - 1$ is a factor of $P(x)$.

v) For $2x + 1$, substitute $x = -\frac{1}{2}$ into the polynomial expression.

$$\begin{aligned} P\left(-\frac{1}{2}\right) &= 2\left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right)^2 - 3\left(-\frac{1}{2}\right) - 2 \\ &= -\frac{1}{4} + \frac{3}{4} + \frac{3}{2} - 2 \\ &= 0 \end{aligned}$$

Since the remainder is zero, $2x + 1$ is a factor of $P(x)$.

Technology Tip

Another method of finding the y-values for specific x-values is to graph **Y1**. Then, press [TRACE], input a value, and press [ENTER].

Method 2: *Use a Graphing Calculator*

Enter the function $y = 2x^3 + 3x^2 - 3x - 2$ in **Y1**.

- Press [2nd] [MODE] to return to the main screen.

i) For $x - 2$, substitute $x = 2$ and calculate Y1(2).

- Press [VARS] [▶]. Select **1:Function**, and press [ENTER].
- Enter $Y_1(2)$ by pressing [(] **2** [)].
- Press [ENTER].

ii) For $x + 2$, substitute $x = -2$ and calculate $Y_1(-2)$.

Repeat the steps of part i). Enter $Y_1(-2)$ by pressing [(] [(-)] **2** [)].

iii) For $x + 1$, calculate $Y_1(-1)$.

iv) For $x - 1$, calculate $Y_1(1)$.

v) For $2x + 1 = 0$, substitute $x = -\frac{1}{2}$ and calculate $Y_1\left(-\frac{1}{2}\right)$.

b) The factors of $P(x) = 2x^3 + 3x^2 - 3x - 2$ are $x + 2$, $x - 1$, and $2x + 1$. In factored form, $2x^3 + 3x^2 - 3x - 2 = (x + 2)(x - 1)(2x + 1)$.

CONNECTIONS

$P(x)$ is a cubic function, so it has at most three linear factors.

Consider the polynomial $P(x) = x^3 + 2x^2 - 5x - 6$.

A value $x = b$ that satisfies $P(b) = 0$ also satisfies $b^3 + 2b^2 - 5b - 6 = 0$, or $b^3 + 2b^2 - 5b = 6$. Factoring out the common factor b gives the product $b(b^2 + 2b - 5) = 6$.

For integer values of b, the value of $b^2 + 2b - 5$ is also an integer. Since the product $b(b^2 + 2b - 5)$ is 6, the possible integer values for the factors in the product are the factors of 6. They are ±1, ±2, ±3, and ±6.

The relationship between the factors of a polynomial and the constant term in the polynomial expression is stated in the **integral zero theorem**.

CONNECTIONS

The word *integral* refers to integer values of b in a factor $x - b$. The word *zero* indicates the value of b being a zero of the polynomial function $P(x)$, that is, $P(b) = 0$.

Integral Zero Theorem

If $x - b$ is a factor of a polynomial function $P(x)$ with leading coefficient 1 and remaining coefficients that are integers, then b is a factor of the constant term of $P(x)$.

Once one factor of a polynomial is found, division is used to determine the other factors. Synthetic division is an abbreviated form of long division for dividing a polynomial by a binomial of the form $x - b$. This method eliminates the use of the variable x and is illustrated in Example 2.

Example 2 Two Division Strategies to Factor a Polynomial

Factor $x^3 + 2x^2 - 5x - 6$ fully.

Solution

Let $P(x) = x^3 + 2x^2 - 5x - 6$.

Find a value $x = b$ such that $P(b) = 0$.

By the integral zero theorem, test factors of -6, that is, ±1, ±2, ±3, and ±6.

Substitute $x = 1$ to test.

$$\begin{aligned} P(1) &= (1)^3 + 2(1)^2 - 5(1) - 6 \\ &= 1 + 2 - 5 - 6 \\ &= -8 \end{aligned}$$

So, $x = 1$ is not a zero of $P(x)$ and $x - 1$ is not a factor.

Substitute $x = 2$ to test.

$$\begin{aligned} P(2) &= (2)^3 + 2(2)^2 - 5(2) - 6 \\ &= 8 + 8 - 10 - 6 \\ &= 0 \end{aligned}$$

So, $x = 2$ is a zero of $P(x)$ and $x - 2$ is a factor.

Once one factor is determined, use one of the following methods to determine the other factors.

Method 1: ***Use Long Division***

$$
\begin{array}{r}
x^2 + 4x + 3 \\
x - 2\,\overline{)\,x^3 + 2x^2 - 5x - 6} \\
\underline{x^3 - 2x^2} \qquad\qquad\quad \\
4x^2 - 5x \qquad\; \\
\underline{4x^2 - 8x} \qquad\; \\
3x - 6 \\
\underline{3x - 6} \\
0
\end{array}
$$

$x^3 + 2x^2 - 5x - 6 = (x - 2)(x^2 + 4x + 3)$

$x^2 + 4x + 3$ can be factored further to give $x^2 + 4x + 3 = (x + 3)(x + 1)$.

So, $x^3 + 2x^2 - 5x - 6 = (x - 2)(x + 3)(x + 1)$.

Method 2: ***Use Synthetic Division***

Set up a division chart for the synthetic division of $P(x) = x^3 + 2x^2 - 5x - 6$ by $x - 2$ as shown.

List the coefficients of the dividend, $x^3 + 2x^2 - 5x - 6$, in the first row. To the left, write the value of -2 from the factor $x - 2$. Below -2, place a $-$ sign to represent subtraction. Use the $\times$ sign below the horizontal line to indicate multiplication of the divisor and the terms of the quotient.

$$
\begin{array}{c|cccc}
-2 & 1 & 2 & -5 & -6 \\
- & & & & \\
\hline
\times & & & &
\end{array}
$$

Perform the synthetic division.

Bring down the first coefficient, 1, to the right of the $\times$ sign.

$$
\begin{array}{c|cccc}
-2 & 1 & 2 & -5 & -6 \\
- & & -2 & -8 & -6 \\
\hline
\times & 1 & 4 & 3 & 0
\end{array}
$$

Multiply -2 (top left) by 1 (right of $\times$ sign) to get -2.
Write -2 below 2 in the second column.
Subtract -2 from 2 to get 4.
Multiply -2 by 4 to get -8. Continue with $-5 - (-8) = 3$, $-2 \times 3 = -6$, and $-6 - (-6) = 0$.
1, 4, and 3 are the coefficients of the quotient, $x^2 + 4x + 3$.
0 is the remainder.

$x^3 + 2x^2 - 5x - 6 = (x - 2)(x^2 + 4x + 3)$

$x^2 + 4x + 3$ can be factored further to give $x^2 + 4x + 3 = (x + 3)(x + 1)$.

So, $x^3 + 2x^2 - 5x - 6 = (x - 2)(x + 3)(x + 1)$.

Example 3 Combine the Factor Theorem and Factoring by Grouping

Factor $x^4 + 3x^3 - 7x^2 - 27x - 18$.

Solution

Let $P(x) = x^4 + 3x^3 - 7x^2 - 27x - 18$.

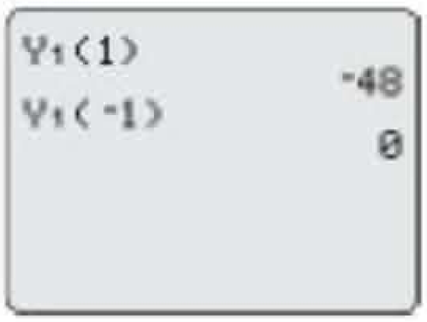
Find a value for x such that $P(x) = 0$.

By the integral zero theorem, test factors of -18, that is, ±1, ±2, ±3, ±6, ±9, and ±18.

Testing all 12 values for x can be time-consuming. Using a calculator will be more efficient.

Enter the function $y = x^4 + 3x^3 - 7x^2 - 27x - 18$ in **Y1**.

Test the factors ±1, ±2, ... by calculating $Y_1(1)$, $Y_1(-1)$, $Y_1(2)$, $Y_1(-2)$, ... until a zero is found.



Since $x = -1$ is a zero of $P(x)$, $x + 1$ is a factor.

Use division to determine the other factor.

$x^4 + 3x^3 - 7x^2 - 27x - 18 = (x + 1)(x^3 + 2x^2 - 9x - 18)$

To factor $P(x)$ further, factor $x^3 + 2x^2 - 9x - 18$ using one of the following methods.

Method 1: ***Apply the Factor Theorem and Division a Second Time***

Let $f(x) = x^3 + 2x^2 - 9x - 18$.

Test possible factors of -18 by calculating $Y_1(1)$, $Y_1(-1)$, $Y_1(2)$, $Y_1(-2)$, ... until a zero is found.

Since $Y_1(-2) = 0$, $x = -2$ is a zero of $f(x)$ and $x + 2$ is a factor.

Use division to determine the other factor.

$$\begin{aligned} f(x) &= x^3 + 2x^2 - 9x - 18 \\ &= (x + 2)(x^2 - 9) \\ &= (x + 2)(x + 3)(x - 3) \end{aligned}$$

So, $P(x) = x^4 + 3x^3 - 7x^2 - 27x - 18 = (x + 1)(x + 2)(x + 3)(x - 3)$.

Method 2: ***Factor by Grouping***

$f(x) = x^3 + 2x^2 - 9x - 18$

$= x^2(x + 2) - 9(x + 2)$ — Group the first two terms and factor out x^2. Then, group the second two terms and factor out -9.

$= (x + 2)(x^2 - 9)$ — Factor out $x + 2$.

$= (x + 2)(x + 3)(x - 3)$ — Factor the difference of squares $(x^2 - 9)$.

So, $P(x) = x^4 + 3x^3 - 7x^2 - 27x - 18 = (x + 1)(x + 2)(x + 3)(x - 3)$.

Technology Tip

Another method of finding a zero is to graph the polynomial function and use the **Zero** operation.

CONNECTIONS

The method of factoring by grouping applies when pairs of terms of a polynomial can be grouped to factor out a common factor so that the resulting binomial factors are the same.

Consider a factorable polynomial such as $P(x) = 3x^3 + 2x^2 - 7x + 2$. Since the leading coefficient is 3, one of the factors must be of the form $3x - b$, where b is a factor of the constant term 2 and $P\left(\frac{b}{3}\right) = 0$.

To determine the values of x that should be tested to find b, the integral zero theorem is extended to include polynomials with leading coefficients that are not one. This extension is known as the **rational zero theorem**.

CONNECTIONS

A rational number is any number that can be expressed as a fraction.

Rational Zero Theorem

Suppose $P(x)$ is a polynomial function with integer coefficients and $x = \frac{b}{a}$ is a zero of $P(x)$, where a and b are integers and $a \neq 0$. Then,

- b is a factor of the constant term of $P(x)$
- a is a factor of the leading coefficient of $P(x)$
- $ax - b$ is a factor of $P(x)$

Example 4 Solve a Problem Using the Rational Zero Theorem

The forms used to make large rectangular blocks of ice come in different dimensions such that the volume, V, in cubic centimetres, of each block can be modelled by $V(x) = 3x^3 + 2x^2 - 7x + 2$.

a) Determine possible dimensions in terms of x, in metres, that result in this volume.

b) What are the dimensions of blocks of ice when $x = 1.5$?

Solution

a) Determine possible dimensions of the rectangular blocks of ice by factoring $V(x) = 3x^3 + 2x^2 - 7x + 2$.

Let b represent the factors of the constant term 2, which are ± 1 and ± 2.

Let a represent the factors of the leading coefficient 3, which are ± 1 and ± 3.

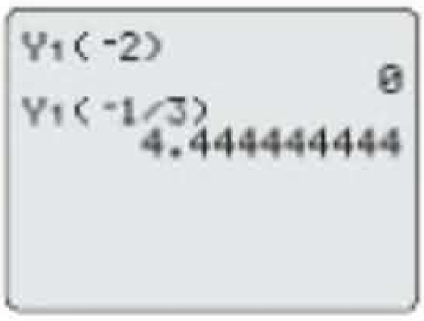
The possible values of $\frac{b}{a}$ are $\pm\frac{1}{1}$, $\pm\frac{1}{3}$ $\pm\frac{2}{1}$, and $\pm\frac{2}{3}$ or ± 1, ± 2, $\pm\frac{1}{3}$, and $\pm\frac{2}{3}$.

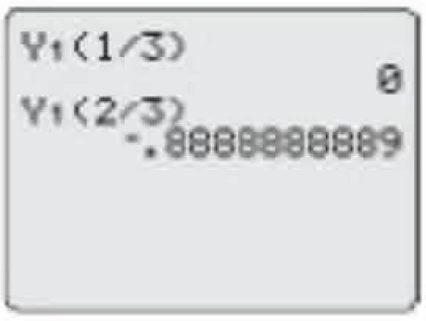
Test the values of $\frac{b}{a}$ for x to find the zeros. Use a graphing calculator.

Enter the function $y = 3x^3 + 2x^2 - 7x + 2$ in **Y1** and calculate $Y_1(1)$, $Y_1(-1)$, $Y_1(2)$, $Y_1(-2)$, ... to find the zeros.

Technology Tip

As a short cut, after one value has been found, press 2nd ENTER. The calculator will duplicate the previous calculation. Change the value in the brackets and press ENTER.





The zeros are 1, −2, and $\frac{1}{3}$. The corresponding factors are $x - 1$, $x + 2$, and $3x - 1$.

So, $3x^3 + 2x^2 - 7x + 2 = (x - 1)(x + 2)(3x - 1)$.

Possible dimensions of the rectangular block of ice, in metres, are $x - 1$, $x + 2$, and $3x - 1$.

b) For $x = 1.5$,

$$\begin{aligned} x - 1 &= 1.5 - 1 \\ &= 0.5 \end{aligned} \qquad \begin{aligned} x + 2 &= 1.5 + 2 \\ &= 3.5 \end{aligned} \qquad \begin{aligned} 3x - 1 &= 3(1.5) - 1 \\ &= 4.5 - 1 \\ &= 3.5 \end{aligned}$$

When $x = 1.5$, the dimensions are 0.5 m by 3.5 m by 3.5 m.

In Example 4, once one factor is determined for a polynomial whose leading coefficient is not 1, you can use division to determine the other factors.

KEY CONCEPTS

For integer values of a and b with $a \neq 0$,

- The factor theorem states that $x - b$ is a factor of a polynomial $P(x)$ if and only if $P(b) = 0$.
  Similarly, $ax - b$ is a factor of $P(x)$ if and only if $P\left(\frac{b}{a}\right) = 0$.
- The integral zero theorem states that if $x - b$ is a factor of a polynomial function $P(x)$ with leading coefficient 1 and remaining coefficients that are integers, then b is a factor of the constant term of $P(x)$.
- The rational zero theorem states that if $P(x)$ is a polynomial function with integer coefficients and $x = \frac{b}{a}$ is a rational zero of $P(x)$, then
  - b is a factor of the constant term of $P(x)$
  - a is a factor of the leading coefficient of $P(x)$
  - $ax - b$ is a factor of $P(x)$

Communicate Your Understanding

C1 **a)** Which of the following binomials are factors of the polynomial $P(x) = 2x^3 + x^2 - 7x - 6$? Justify your answers.

A $x - 1$ **B** $x + 1$ **C** $x + 2$ **D** $x - 2$ **E** $2x + 1$ **F** $2x + 3$

b) Use the results of part a) to write $P(x) = 2x^3 + x^2 - 7x - 6$ in factored form.

C2 When factoring a trinomial $ax^2 + bx + c$, you consider the product ac. How does this relate to the rational zero theorem?

C3 Describe the steps required to factor the polynomial $2x^3 - 3x^2 + 5x - 4$.

C4 Identify the possible factors of the expression $x^3 + 2x^2 - 5x - 4$. Explain your reasoning.

A Practise

For help with questions 1 and 2, refer to Example 1.

1. Write the binomial factor that corresponds to the polynomial $P(x)$.

a) $P(4) = 0$ **b)** $P(-3) = 0$

c) $P\left(\frac{2}{3}\right) = 0$ **d)** $P\left(-\frac{1}{4}\right) = 0$

2. Determine if $x + 3$ is a factor of each polynomial.

a) $x^3 + x^2 - x + 6$

b) $2x^3 + 9x^2 + 10x + 3$

c) $x^3 + 27$

For help with question 3, refer to Example 2.

3. List the values that could be zeros of each polynomial. Then, factor the polynomial.

a) $x^3 + 3x^2 - 6x - 8$

b) $x^3 + 4x^2 - 15x - 18$

c) $x^3 - 3x^2 - 10x + 24$

For help with question 4, refer to Example 3.

4. Factor each polynomial by grouping terms.

a) $x^3 + x^2 - 9x - 9$

b) $x^3 - x^2 - 16x + 16$

c) $2x^3 - x^2 - 72x + 36$

d) $x^3 - 7x^2 - 4x + 28$

e) $3x^3 + 2x^2 - 75x - 50$

f) $2x^4 + 3x^3 - 32x^2 - 48x$

For help with question 5, refer to Example 4.

5. Determine the values that could be zeros of each polynomial. Then, factor the polynomial.

a) $3x^3 + x^2 - 22x - 24$

b) $2x^3 - 9x^2 + 10x - 3$

c) $6x^3 - 11x^2 - 26x + 15$

d) $4x^3 + 3x^2 - 4x - 3$

B Connect and Apply

6. Factor each polynomial.

a) $x^3 + 2x^2 - x - 2$

b) $x^3 + 4x^2 - 7x - 10$

c) $x^3 - 5x^2 - 4x + 20$

d) $x^3 + 5x^2 + 3x - 4$

e) $x^3 - 4x^2 - 11x + 30$

f) $x^4 - 4x^3 - x^2 + 16x - 12$

g) $x^4 - 2x^3 - 13x^2 + 14x + 24$

7. **Use Technology** Factor each polynomial.

a) $8x^3 + 4x^2 - 2x - 1$

b) $2x^3 + 5x^2 - x - 6$

c) $5x^3 + 3x^2 - 12x + 4$

d) $6x^4 + x^3 - 8x^2 - x + 2$

e) $5x^4 + x^3 - 22x^2 - 4x + 8$

f) $3x^3 + 4x^2 - 35x - 12$

g) $6x^3 - 17x^2 + 11x - 2$

8. An artist creates a carving from a rectangular block of soapstone whose volume, V, in cubic metres, can be modelled by $V(x) = 6x^3 + 25x^2 + 2x - 8$. Determine possible dimensions of the block, in metres, in terms of binomials of x.

9. Determine the value of k so that $x + 2$ is a factor of $x^3 - 2kx^2 + 6x - 4$.

10. Determine the value of k so that $3x - 2$ is a factor of $3x^3 - 5x^2 + kx + 2$.

11. Factor each polynomial.

a) $2x^3 + 5x^2 - x - 6$

b) $4x^3 - 7x - 3$

c) $6x^3 + 5x^2 - 21x + 10$

d) $4x^3 - 8x^2 + 3x - 6$

e) $2x^3 + x^2 + x - 1$

f) $x^4 - 15x^2 - 10x + 24$

12. a) Factor each difference of cubes.

i) $x^3 - 1$

ii) $x^3 - 8$

iii) $x^3 - 27$

iv) $x^3 - 64$

b) Use the results of part a) to predict a pattern for factoring $x^3 - a^3$.

c) Use your pattern from part b) to factor $x^3 - 125$. Verify your answer by expanding.

d) Factor each polynomial.

i) $8x^3 - 1$ **ii)** $125x^6 - 8$

iii) $64x^{12} - 27$ **iv)** $\frac{8}{125}x^3 - 64y^6$

13. a) Factor each sum of cubes.

i) $x^3 + 1$ **ii)** $x^3 + 8$

iii) $x^3 + 27$ **iv)** $x^3 + 64$

b) Use the results of part a) to predict a pattern for factoring $x^3 + a^3$.

c) Use your pattern from part b) to factor $x^3 + 125$. Verify your answer by expanding.

d) Factor each polynomial.

i) $8x^3 + 1$ **ii)** $125x^6 + 8$

iii) $64x^{12} + 27$ **iv)** $\frac{8}{125}x^3 + 64y^6$

14. Show that $x^4 + x^2 + 1$ is non-factorable over the integers.

15. Factor by letting $m = x^2$.

a) $4x^4 - 37x^2 + 9$

b) $9x^4 - 148x^2 + 64$

✓ Achievement Check

16. Chapter Problem Best of U has produced a new body wash. The profit, P, in dollars, can be modelled by the function $P(x) = x^3 - 6x^2 + 9x$, where x is the number of bottles sold, in thousands.

a) Use the factor theorem to determine if $x - 1$ is a factor of $P(x)$.

b) Use the rational zero theorem to write the possible values of $\frac{b}{a}$ for the factored form: $P(x) = x(x^2 - 6x + 9)$

c) Use long division to check that $x - 3$ is a factor.

d) The company is happy with the profit and manufactured a similar body spray. The profit of this product can be modelled by the function $P(x) = 4x^3 + 12x^2 - 16x$. Find the factors of $P(x)$.

C Extend and Challenge

17. Factor each polynomial.

a) $2x^5 + 3x^4 - 10x^3 - 15x^2 + 8x + 12$

b) $4x^6 + 12x^5 - 9x^4 - 51x^3 - 30x^2 + 12x + 8$

18. Determine the values of m and n so that the polynomials $2x^3 + mx^2 + nx - 3$ and $x^3 - 3mx^2 + 2nx + 4$ are both divisible by $x - 2$.

19. Determine a polynomial function $P(x)$ that satisfies each set of conditions.

a) $P(-4) = P\left(-\frac{3}{4}\right) = P\left(\frac{1}{2}\right) = 0$ and $P(-2) = 50$

b) $P(3) = P(-1) = P\left(\frac{2}{3}\right) = P\left(-\frac{3}{2}\right) = 0$ and $P(1) = -18$

20. a) Factor each expression.

i) $x^4 - 1$ **ii)** $x^4 - 16$

iii) $x^5 - 1$ **iv)** $x^5 - 32$

b) Use the results of part a) to predict a pattern for factoring $x^n - a^n$.

c) Use your pattern from part b) to factor $x^6 - 1$. Verify your answer by expanding.

d) Factor each expression.

i) $x^4 - 625$ **ii)** $x^5 - 243$

21. Is there a pattern for factoring $x^n + a^n$? Justify your answer.

22. Math Contest When a polynomial is divided by $(x + 2)$, the remainder is -19. When the same polynomial is divided by $(x - 1)$, the remainder is 2. Determine the remainder when the polynomial is divided by $(x - 1)(x + 2)$.

2.3 Polynomial Equations

Suppose the volume, V, in cubic centimetres, of a block of ice that a sculptor uses to carve the wings of a dragon can be modelled by $V(x) = 9x^3 + 60x^2 + 249x$, where x represents the thickness of the block, in centimetres. What maximum thickness of wings can be carved from a block of ice with volume 2532 cm^3? The solution to this problem can be determined by solving the cubic equation $9x^3 + 60x^2 + 249x = 2532$.

In this section, you will learn methods of solving polynomial equations of degree higher than two by factoring (using the factor theorem) and by using technology. You will also identify the relationship between the roots of polynomial equations, the x-intercepts of the graph of a polynomial function, and the zeros of the function.

CONNECTIONS

The Arabic mathematician Al-Khwarizmi (c. 780-850 A.D.) developed an algorithm for determining the roots of a quadratic equation in about 830 A.D. Methods for solving cubic and quartic equations were not discovered until about 700 years later. The Italian mathematician Scipione del Ferro (1465–1526) developed a method for solving cubic equations of the form $x^3 + mx = n$. In 1539, Niccolo Tartaglia (1499–1557) used an algorithm for solving cubic equations to win a challenge.

Investigate — How are roots, x-intercepts, and zeros related?

Tools

- graphing calculator

1. a) Graph the function $f(x) = x^4 - 13x^2 + 36$.
   b) Determine the x-intercepts from the graph.
   c) Factor $f(x)$. Then, use the factors to determine the zeros of $f(x)$.
   d) **Reflect** What is the relationship between the zeros of the function and the x-intercepts of the corresponding graph?
2. a) Set the polynomial function $f(x) = x^4 - 13x^2 + 36$ equal to 0. Solve the equation $x^4 - 13x^2 + 36 = 0$ to determine the roots.
   b) Compare the roots to the x-intercepts of the corresponding graph. What do you notice?
   c) **Reflect** What is the relationship between the zeros of the function, the x-intercepts of the corresponding graph, and the roots of the polynomial equation?

In Chapter 1, you found that when a polynomial function is given in factored form, you can identify the zeros of the function and the x-intercepts of the corresponding graph. For a polynomial function $y = P(x)$, the roots are determined by letting $y = 0$ and solving the polynomial equation $P(x) = 0$. If the polynomial equation is factorable, then the values of the roots can be determined algebraically by solving each linear or quadratic factor. Polynomial equations of the form $P(x) = 0$ may also be solved graphically by examining the x-intercepts.

Example 1 Solve Polynomial Equations by Factoring

Solve.

a) $x^3 - x^2 - 2x = 0$

b) $3x^3 + x^2 - 12x - 4 = 0$

Solution

a) $x^3 - x^2 - 2x = 0$

$x(x^2 - x - 2) = 0$ — Factor out the common factor x.

$x(x - 2)(x + 1) = 0$ — Factor the trinomial.

$x = 0$ or $x - 2 = 0$ or $x + 1 = 0$

$x = 0$ or $x = 2$ or $x = -1$

b) $3x^3 + x^2 - 12x - 4 = 0$ — Factor by grouping.

$x^2(3x + 1) - 4(3x + 1) = 0$ — Factor out x^2 from the first two terms and -4 from the last two terms.

$(3x + 1)(x^2 - 4) = 0$

$(3x + 1)(x + 2)(x - 2) = 0$ — Factor the difference of squares $x^2 - 4$.

$3x + 1 = 0$ or $x + 2 = 0$ or $x - 2 = 0$

$x = -\frac{1}{3}$ or $x = -2$ or $x = 2$

Example 2 Use the Factor Theorem to Solve a Polynomial Equation

a) Solve $2x^3 + 3x^2 - 11x - 6 = 0$.

b) What do the values of x in part a) represent in terms of the related polynomial function?

Solution

a) Factor the polynomial $2x^3 + 3x^2 - 11x - 6$.

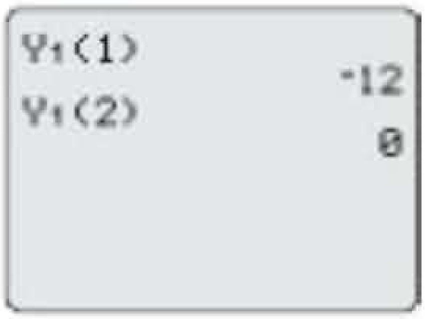
Use the rational zero theorem to determine the values that should be tested.

Let b represent the factors of the constant term -6, which are ± 1, ± 2, ± 3, and ± 6.

Let a represent the factors of the leading coefficient 2, which are ± 1 and ± 2.

The possible values of $\frac{b}{a}$ are $\pm\frac{1}{1}$, $\pm\frac{1}{2}$, $\pm\frac{2}{1}$, $\pm\frac{2}{2}$, $\pm\frac{3}{1}$, $\pm\frac{3}{2}$, $\pm\frac{6}{1}$, and $\pm\frac{6}{2}$, or ± 1, ± 2, ± 3, ± 6, $\pm\frac{1}{2}$, and $\pm\frac{3}{2}$.

Test the values of $\frac{b}{a}$ for x to find the zeros.



Since 2 is a zero of the function, $x - 2$ is a factor.

It is possible to begin with a different factor depending on which values are tested first.

Divide to determine the other factor.

$$\begin{array}{r|rrrr} -2 & 2 & 3 & -11 & -6 \\ - & \downarrow & -4 & -14 & -6 \\ \hline \times & 2 & 7 & 3 & 0 \end{array}$$

$$\begin{aligned} 2x^3 + 3x^2 - 11x - 6 &= (x - 2)(2x^2 + 7x + 3) \\ &= (x - 2)(2x + 1)(x + 3) \end{aligned}$$

Solve $2x^3 + 3x^2 - 11x - 6 = 0$.

$(x - 2)(2x + 1)(x + 3) = 0$

$x - 2 = 0$ or $2x + 1 = 0$ or $x + 3 = 0$

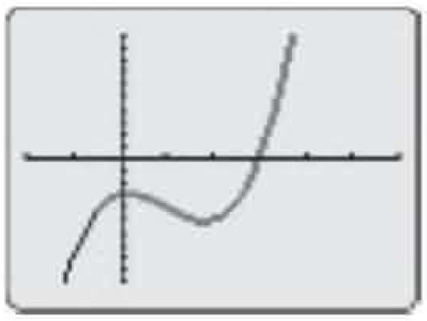
$x = 2$ or $x = -\frac{1}{2}$ or $x = -3$

b) The values 2, $-\frac{1}{2}$, and -3 are the roots of the equation $2x^3 + 3x^2 - 11x - 6 = 0$ and are the x-intercepts of the graph of the related function $y = 2x^3 + 3x^2 - 11x - 6$.

A polynomial equation may have real and non-real roots.

Consider the solution to the polynomial equation $(x - 3)(x^2 + 1) = 0$.

$(x - 3)(x^2 + 1) = 0$
$x - 3 = 0$ or $x^2 + 1 = 0$
$x = 3$ or $x^2 = -1$
$x = 3$ or $x = \pm\sqrt{-1}$



Since the square root of a negative number is not a real number, the only real root is $x = 3$.

The function $y = (x - 3)(x^2 + 1)$ has only one real zero, so the equation $(x - 3)(x^2 + 1) = 0$ has one real root. The x-intercept of the graph is 3.

The x-intercepts of the graph of a polynomial function correspond to the real roots of the related polynomial equation.

Example 3 Solve a Problem by Determining the Roots of a Polynomial Equation

The volume, V, in cubic centimetres, of a block of ice that a sculptor uses to carve the wings of a dragon can be modelled by $V(x) = 9x^3 + 60x^2 + 249x$, where x represents the thickness of the block, in centimetres. What maximum thickness of wings can be carved from a block of ice with volume 2532 cm^3?

Solution

Determine the value of x that satisfies $V(x) = 2532$.

That is, solve the equation $9x^3 + 60x^2 + 249x = 2532$.

$9x^3 + 60x^2 + 249x - 2532 = 0$
$3(3x^3 + 20x^2 + 83x - 844) = 0$ Factor out the common factor 3.
$3x^3 + 20x^2 + 83x - 844 = 0$

Use the rational zero theorem to determine the values that should be tested.

Let b represent the factors of the constant term 844, which are ±1, ±2, ±4, ±211, ±422, and ±844.

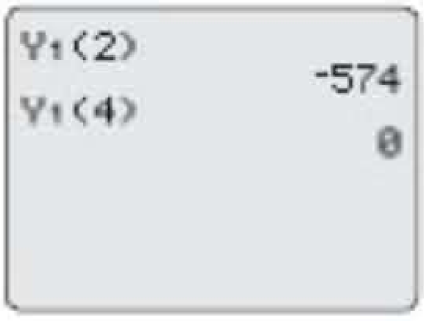
Let a represent the factors of the leading coefficient 3, which are ±1 and ±3.

The possible values of $\frac{b}{a}$ are $\pm\frac{1}{1}$, $\pm\frac{1}{3}$, $\pm\frac{2}{1}$, $\pm\frac{2}{3}$, $\pm\frac{4}{1}$, $\pm\frac{4}{3}$, $\pm\frac{211}{1}$, $\pm\frac{211}{3}$, $\pm\frac{422}{1}$, $\pm\frac{422}{3}$, $\pm\frac{844}{1}$, and $\pm\frac{844}{3}$, or ±1, ±2, ±4, ±211, ±422, ±844, $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, $\pm\frac{4}{3}$, $\pm\frac{211}{3}$, $\pm\frac{422}{3}$, and $\pm\frac{844}{3}$.

CONNECTIONS

It is not necessary to list all the possible factors, unless a question asks for it. Do use a systematic method of checking possible factors starting with the simplest, $+/-1$.

Test only positive values of $\frac{b}{a}$ for x since x represents thickness.



Since 4 is a zero of the function, $x - 4$ is a factor.

Divide to determine the other factor of $3x^3 + 20x^2 + 83x - 844$.

$3x^3 + 20x^2 + 83x - 844 = (x - 4)(3x^2 + 32x + 211)$

Solve $(x - 4)(3x^2 + 32x + 211) = 0$.

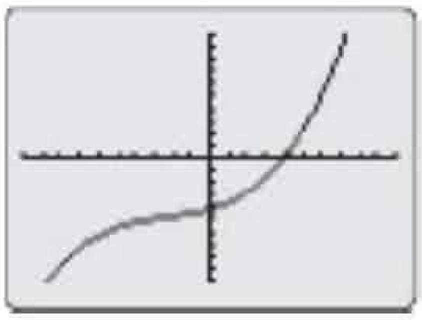
$x - 4 = 0$ or $3x^2 + 32x + 211 = 0$ — The trinomial $3x^2 + 32x + 211$ cannot be factored.

$x = 4$ or $x = \dfrac{-32 \pm \sqrt{32^2 - 4(3)(211)}}{2(3)}$ — Use the quadratic formula.

$x = \dfrac{-32 \pm \sqrt{-1508}}{6}$ — These roots are not real.

Since the only positive real root is $x = 4$, the thickness of the wings is 4 cm.

A graph of the function verifies this solution.



Example 4 Determine the Roots of a Non-Factorable Polynomial Equation

Solve $x^3 - 3x = -1$. Round the roots to one decimal place.

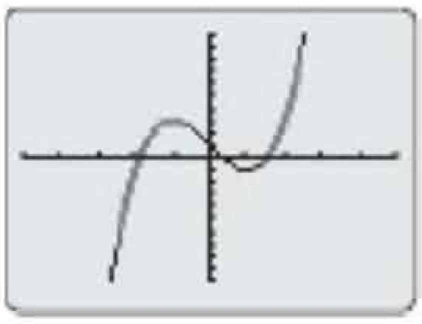
Solution

Write the equation as $x^3 - 3x + 1 = 0$.

The only factors of 1 are ± 1, neither of which makes the left side of the equation equal to 0 when tested.

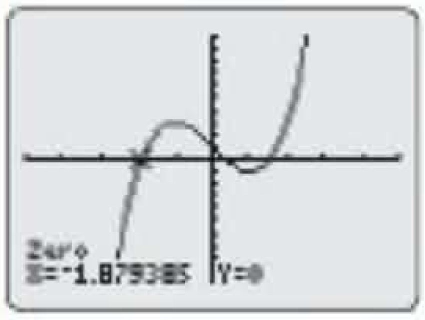
Since the polynomial cannot be factored, determine the roots graphically using a graphing calculator.

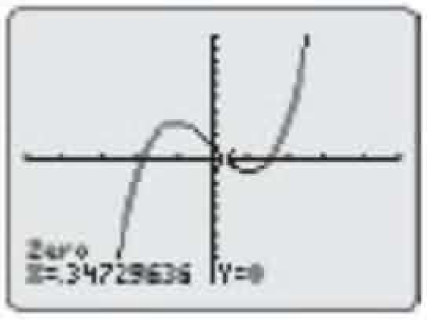
Graph $y = x^3 - 3x + 1$. Use the window settings shown.

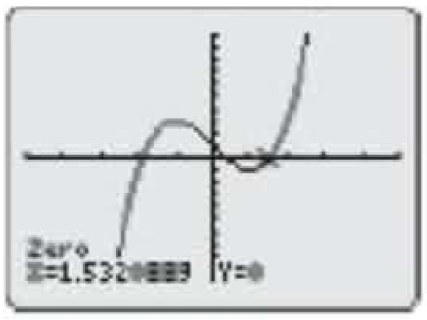


From the graph, there are three x-intercepts, one near -2, another near 0, and a third near 2.

Use the **Zero** operation.







The three roots of the equation are -1.9, 0.3, and 1.5, to one decimal place.

CONNECTIONS

Another method of solving the equation with a graphing calculator is to find the points of intersection of the graphs of the two functions $y = x^3 - 3x$ and $y = -1$.

KEY CONCEPTS

- The real roots of a polynomial equation $P(x) = 0$ correspond to the x-intercepts of the graph of the polynomial function $P(x)$.
- The x-intercepts of the graph of a polynomial function correspond to the real roots of the related polynomial equation.
- If a polynomial equation is factorable, the roots are determined by factoring the polynomial, setting its factors equal to zero, and solving each factor.
- If a polynomial equation is not factorable, the roots can be determined from the graph using technology.

Communicate Your Understanding

C1 Describe what is meant by a root, a zero, and an x-intercept. How are they related?

C2 Without solving, describe two ways to show that 2, -1, 3, and -2 are the roots of the polynomial equation $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$.

C3 A polynomial equation of degree four has exactly two distinct real roots. How many x-intercepts does the graph of the polynomial function have?

C4 Describe the different methods that can be used to factor a polynomial function.

C5 Suppose the degree of a polynomial function is n. What is the maximum number of real roots of the corresponding equation? Will the number of x-intercepts of the graph of the function be the same as the number of roots? Explain.

A Practise

For help with question 1, refer to Example 1.

1. Solve.

a) $x(x + 2)(x - 5) = 0$

b) $(x - 1)(x - 4)(x + 3) = 0$

c) $(3x + 2)(x + 9)(x - 2) = 0$

d) $(x - 7)(3x + 2)(x + 1) = 0$

e) $(4x - 1)(2x - 3)(x + 8) = 0$

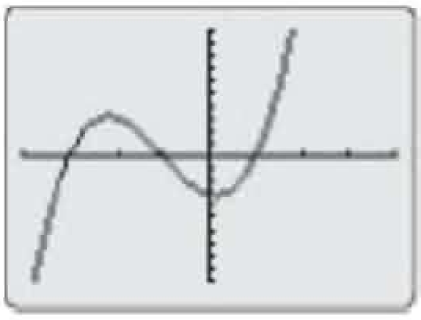
f) $(2x - 5)(2x + 5)(x - 7) = 0$

g) $(5x - 8)(x + 3)(2x - 1) = 0$

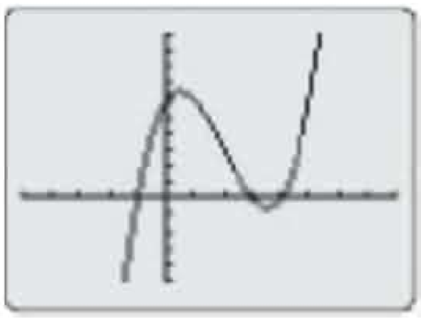
For help with question 2, refer to Example 2.

2. Use the graph to determine the roots of the corresponding polynomial equation. The roots are all integral values.

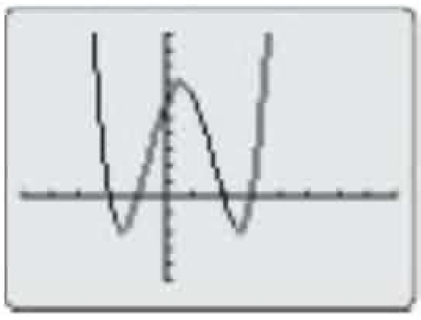
a) Window variables: $x \in [-4, 4]$, $y \in [-10, 10]$



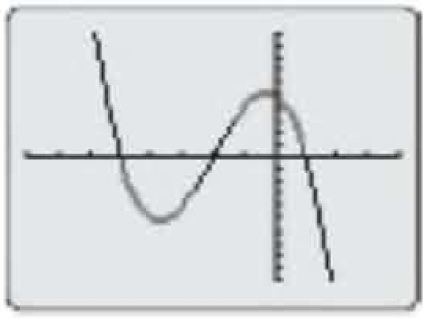
b) Window variables: $x \in [-5, 8]$, $y \in [-10, 20]$, Yscl = 2



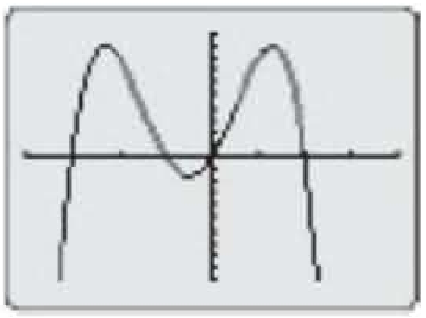
c) Window variables: $x \in [-5, 8]$, $y \in [-10, 20]$, Yscl = 2



d) Window variables: $x \in [-8, 4]$, $y \in [-20, 20]$, Yscl = 2



e) Window variables: $x \in [-4, 4]$, $y \in [-10, 10]$



For help with question 3, refer to Example 3.

3. Determine the real roots of each polynomial equation.

a) $(x^2 + 1)(x - 4) = 0$

b) $(x^2 - 1)(x^2 + 4) = 0$

c) $(3x^2 + 27)(x^2 - 16) = 0$

d) $(x^4 - 1)(x^2 - 25) = 0$

e) $(4x^2 - 9)(x^2 + 16) = 0$

f) $(x^2 + 7x + 12)(x^2 - 49) = 0$

g) $(2x^2 + 5x - 3)(4x^2 - 100) = 0$

4. Determine the x-intercepts of the graph of each polynomial function.

a) $y = x^3 - 4x^2 - 45x$

b) $f(x) = x^4 - 81x^2$

c) $P(x) = 6x^3 - 5x^2 - 4x$

d) $h(x) = x^3 + x^2 - 4x - 4$

e) $g(x) = x^4 - 16$

f) $k(x) = x^4 - 2x^3 - x^2 + 2x$

g) $t(x) = x^4 - 29x^2 + 100$

B Connect and Apply

5. Is each statement true or false? If the statement is false, reword it to make it true.

a) If the graph of a quartic function has two x-intercepts, then the corresponding quartic equation has four real roots.

b) All the roots of a polynomial equation correspond to the x-intercepts of the graph of the corresponding polynomial function.

c) A polynomial equation of degree three must have at least one real root.

d) All polynomial equations can be solved algebraically.

e) All polynomial equations can be solved graphically.

6. Solve by factoring.

a) $x^3 - 4x^2 - 3x + 18 = 0$

b) $x^3 - 4x^2 - 7x + 10 = 0$

c) $x^3 - 5x^2 + 7x - 3 = 0$

d) $x^3 + x^2 - 8x - 12 = 0$

e) $x^3 - 3x^2 - 4x + 12 = 0$

f) $x^3 + 2x^2 - 7x + 4 = 0$

g) $x^3 - 3x^2 + x + 5 = 0$

7. Solve by factoring.

a) $2x^3 + 3x^2 - 5x - 6 = 0$

b) $2x^3 - 11x^2 + 12x + 9 = 0$

c) $9x^3 + 18x^2 - 4x - 8 = 0$

d) $5x^3 - 8x^2 - 27x + 18 = 0$

e) $8x^4 - 64x = 0$

f) $4x^4 - 2x^3 - 16x^2 + 8x = 0$

g) $x^4 - x^3 - 11x^2 + 9x + 18 = 0$

8. Solve by factoring.

a) $x^3 - 5x^2 + 8 = -2x$

b) $x^3 - x^2 = 4x + 6$

c) $2x^3 - 7x^2 + 10x - 5 = 0$

d) $x^4 - x^3 = 2x + 4$

e) $x^4 + 13x^2 = -36$

For help with question 9, refer to Example 4.

9. **Use Technology** Solve. Round answers to one decimal place.

a) $x^3 - 4x + 2 = 0$

b) $2x^3 + 9x^2 = x + 3$

c) $x^4 = 2$

d) $3x^3 + 6 = x$

e) $x^4 = x^3 + 7$

f) $4x^3 - 3x^2 - 5x + 2 = 0$

g) $x^4 + x^2 - x + 4 = 0$

10. The width of a square-based storage tank is 3 m less than its height. The tank has a capacity of 20 m^3. If the dimensions are integer values in metres, what are they?

11. The passenger section of a train has width $2x - 7$, length $2x + 3$, and height $x - 2$, with all dimensions in metres. Solve a polynomial equation to determine the dimensions of the section of the train if the volume is 117 m^3.

12. Is it possible for a polynomial equation to have exactly one irrational root? Use an example to justify your answer.

13. Is it possible for a polynomial equation to have exactly one non-real root? Use an example to justify your answer.

14. The distance, d, in kilometres, travelled by a plane after t hours can be represented by $d(t) = -4t^3 + 40t^2 + 500t$, where $0 \leq t \leq 10$. How long does the plane take to fly 4088 km?

15. A steel beam is supported by two vertical walls. When a 1000-kg weight is placed on the beam, x metres from one end, the vertical deflection, d, in metres, can be calculated using the formula $d(x) = 0.0005(x^4 - 16x^3 + 512x)$. How far from the end of the beam should the weight be placed for a deflection of 0 m?

16. Chapter Problem Based on research, the marketing team at Best of U predicts that when the price of a bottle of a new SPF 50 sunscreen is x dollars, the number, D, in hundreds, of bottles sold per month can be modelled by the function $D(x) = -x^3 + 8x^2 + 9x + 100$.

a) Graph the function $D(x)$. Write the domain for this situation.

b) How many bottles are sold per month when the price of each bottle is \$5?

c) Determine the value(s) of x that will result in sales of 17 200 bottles of sunscreen per month. Interpret this answer.

17. Solve. Round answers to one decimal place if necessary.

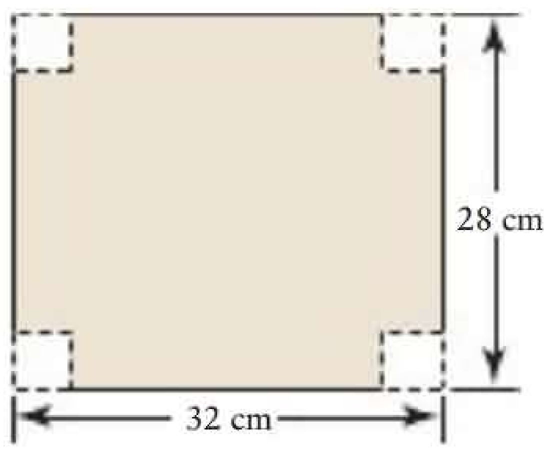
a) $2(x - 1)^3 = 16$

b) $2(x^2 - 4x)^2 - 5(x^2 - 4x) = 3$

18. a) Determine the value of k such that -2 is one root of the equation $2x^3 + (k + 1)x^2 = 4 - x^2$.

b) Determine the other roots of the equation. Justify your answer.

19. Open-top boxes are constructed by cutting equal squares from the corners of cardboard sheets that measure 32 cm by 28 cm. Determine possible dimensions of the boxes if each has a volume of 1920 cm^3.



C Extend and Challenge

20. A complex number is a number that can be written in the form $a + ib$, where a and b are real numbers and $i = \sqrt{-1}$. When the quadratic formula is used and the discriminant is negative, complex numbers result. See Example 3 of this section. The non-real roots $x = \dfrac{-32 \pm \sqrt{-1508}}{6}$ are complex roots. They may be written in terms of i, as shown below.

$$x = \frac{-32 \pm \sqrt{(-1)(1508)}}{6}$$

$$x = \frac{-32 \pm i\sqrt{1508}}{6}$$

a) Find all the real and complex solutions to $x^3 - 27 = 0$.

b) Determine a polynomial equation of degree three with roots $x = 3 \pm i$ and $x = -4$. Is this equation unique? Explain.

21. The dimensions of a gift box are consecutive positive integers such that the height is the least integer and the length is the greatest integer. If the height is increased by 1 cm, the width is increased by 2 cm, and the length is increased by 3 cm, then a larger box is constructed such that the volume is increased by 456 cm^3. Determine the dimensions of each box.

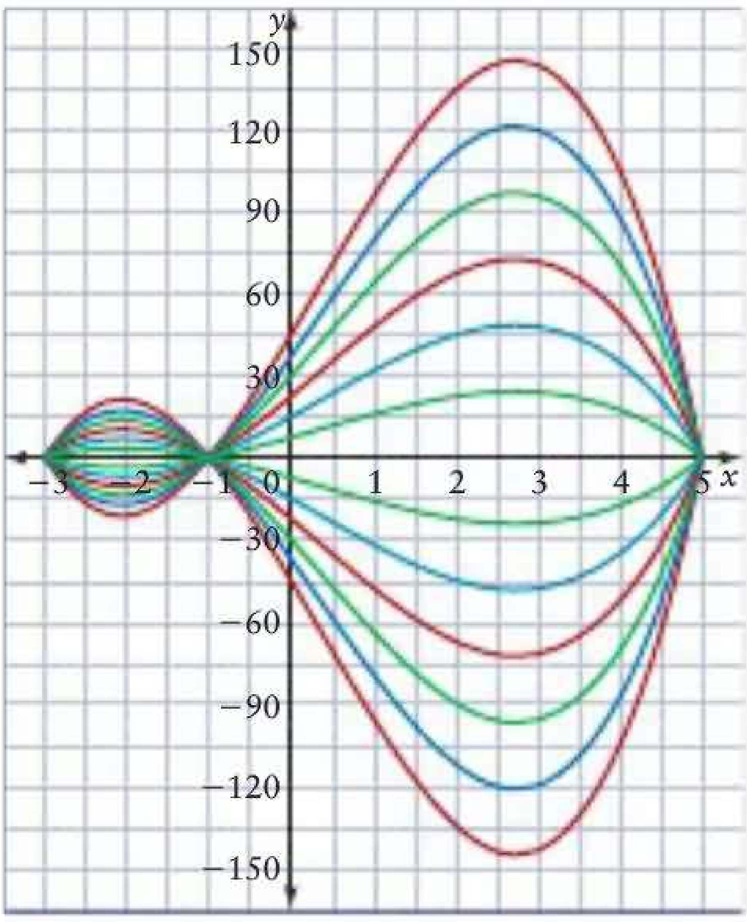
22. The roots of the equation $6x^3 + 17x^2 - 5x - 6 = 0$ are represented by a, b, and c (from least to greatest). Determine an equation whose roots are $a + b$, $\frac{a}{b}$, and ab.

23. Math Contest AB is the diameter of a circle with centre O. P is a point on the circle, and AP is extended to C such that PC = OP. If $\angle$COB = 45°, what is the measure of $\angle$POC?

24. Math Contest Determine the product of all values of k for which the polynomial equation $2x^3 - 9x^2 + 12x - k = 0$ has a double root.

2.4 Families of Polynomial Functions

Crystal pieces for a large chandelier are to be cut according to the design shown. The graph shows how the design is created using polynomial functions. What do all the functions have in common? How are they different? How can you determine the equations that are used to create the design?



In this section, you will determine equations for a family of polynomial functions from a set of zeros. Given additional information, you will determine an equation for a particular member of the family.

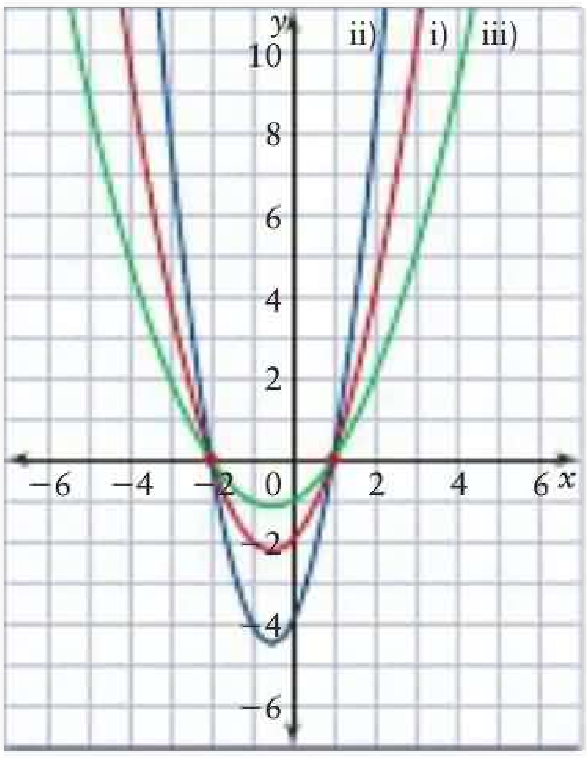
Investigate How are polynomial functions with the same zeros related?

Tools

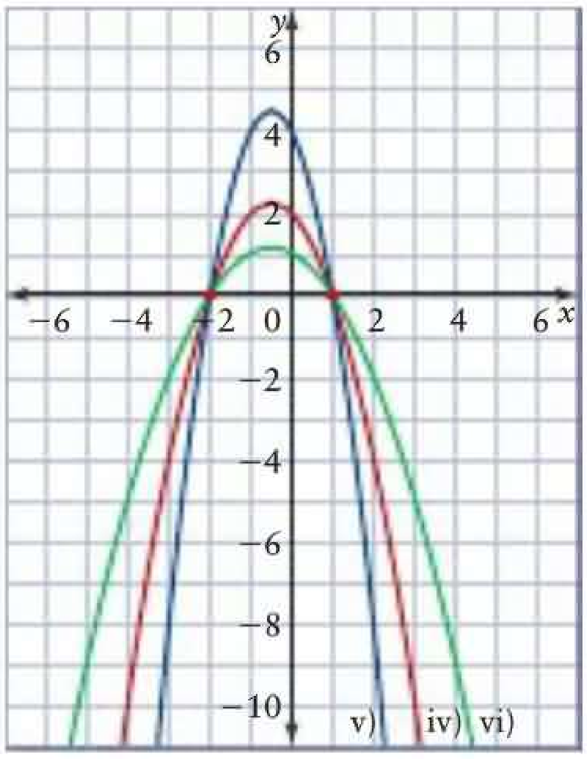
- graphing calculator

1. a) Examine each set of parabolas and the corresponding functions.

Set A



Set B



i) $y = (x - 1)(x + 2)$

ii) $y = 2(x - 1)(x + 2)$

iii) $y = \frac{1}{2}(x - 1)(x + 2)$

iv) $y = -(x - 1)(x + 2)$

v) $y = -2(x - 1)(x + 2)$

vi) $y = -\frac{1}{2}(x - 1)(x + 2)$

b) **Reflect** How are the graphs of the functions in part a) similar and how are they different?

2. **Reflect** Describe the relationship between the graphs of functions of the form $y = k(x - 1)(x + 2)$, where $k \in \mathbb{R}$. Why do you think this is called a **family of functions**?

3. a) Examine the following functions. How are they similar? How are they different?

i) $y = -2(x - 1)(x + 3)(x - 2)$

ii) $y = -(x - 1)(x + 3)(x - 2)$

iii) $y = (x - 1)(x + 3)(x - 2)$

iv) $y = 2(x - 1)(x + 3)(x - 2)$

b) **Reflect** Predict how the graphs of the functions in part a) will be similar and how they will be different.

4. a) Use a graphing calculator to graph the functions in step 3 on the same set of axes.

b) Examine the graphs. Was your prediction accurate? If not, explain how it should be changed.

5. **Reflect** Describe the relationship between the graphs of polynomial functions of the form $y = k(x - r)(x - s)(x - t)$, where $k \in \mathbb{R}$. Why is it appropriate to call this a **family of polynomial functions**?

A family of functions is a set of functions that have the same characteristics. Polynomial functions with the same zeros are said to belong to the same family. The graphs of polynomial functions that belong to the same family have the same x-intercepts but have different y-intercepts (unless zero is one of the x-intercepts).

An equation for the family of polynomial functions with zeros $a_1, a_2, a_3, \ldots, a_n$ is $y = k(x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n)$, where $k \in \mathbb{R}$, $k \neq 0$.

Example 1 Represent a Family of Functions Algebraically

The zeros of a family of quadratic functions are 2 and -3.

a) Determine an equation for this family of functions.

b) Write equations for two functions that belong to this family.

c) Determine an equation for the member of the family that passes through the point (1, 4).

Solution

a) The factor associated with 2 is $x - 2$ and the factor associated with -3 is $x + 3$.

An equation for this family is $y = k(x - 2)(x + 3)$, where $k \in \mathbb{R}$.

b) Use any two values for k to write two members of the family.

For $k = 8$, $y = 8(x - 2)(x + 3)$.
For $k = -3$, $y = -3(x - 2)(x + 3)$.

c) To find the member whose graph passes through (1, 4), substitute $x = 1$ and $y = 4$ into the equation and solve for k.

$$4 = k(1 - 2)(1 + 3)$$
$$4 = k(-1)(4)$$
$$4 = -4k$$
$$k = -1$$

The equation is $y = -(x - 2)(x + 3)$.

Example 2 Determine an Equation for a Family of Cubic Functions Given Integral Zeros

The zeros of a family of cubic functions are -2, 1, and 3.

a) Determine an equation for this family.

b) Write equations for two functions that belong to this family.

c) Determine an equation for the member of the family whose graph has a y-intercept of -15.

d) Sketch graphs of the functions in parts b) and c).

Solution

a) Since the zeros are -2, 1, and 3, then $x + 2$, $x - 1$, and $x - 3$ are factors of the family of cubic functions. An equation for this family is $y = k(x + 2)\ (x - 1)(x - 3)$, where $k \in \mathbb{R}$.

b) Use any two values for k to write two members of the family.
For $k = 2$, $y = 2(x + 2)(x - 1)(x - 3)$.
For $k = -1$, $y = -(x + 2)(x - 1)(x - 3)$.

c) Since the y-intercept is -15, substitute $x = 0$ and $y = -15$ into $y = k(x + 2)(x - 1)(x - 3)$.

$$-15 = k(0 + 2)(0 - 1)(0 - 3)$$
$$-15 = 6k$$
$$k = -2.5$$

The equation is $y = -2.5(x + 2)(x - 1)(x - 3)$.

d) From part a), the three functions have zeros, or x-intercepts, -2, 1, and 3.
From part c), the y-intercept of $y = -2.5(x + 2)(x - 1)(x - 3)$ is -15.
Substitute $x = 0$ to determine the y-intercepts of the functions from part b).

$$\begin{aligned} y &= 2(x + 2)(x - 1)(x - 3) \\ &= 2(0 + 2)(0 - 1)(0 - 3) \\ &= 12 \end{aligned}$$

The y-intercept of $y = 2(x + 2)(x - 1)(x - 3)$ is 12.

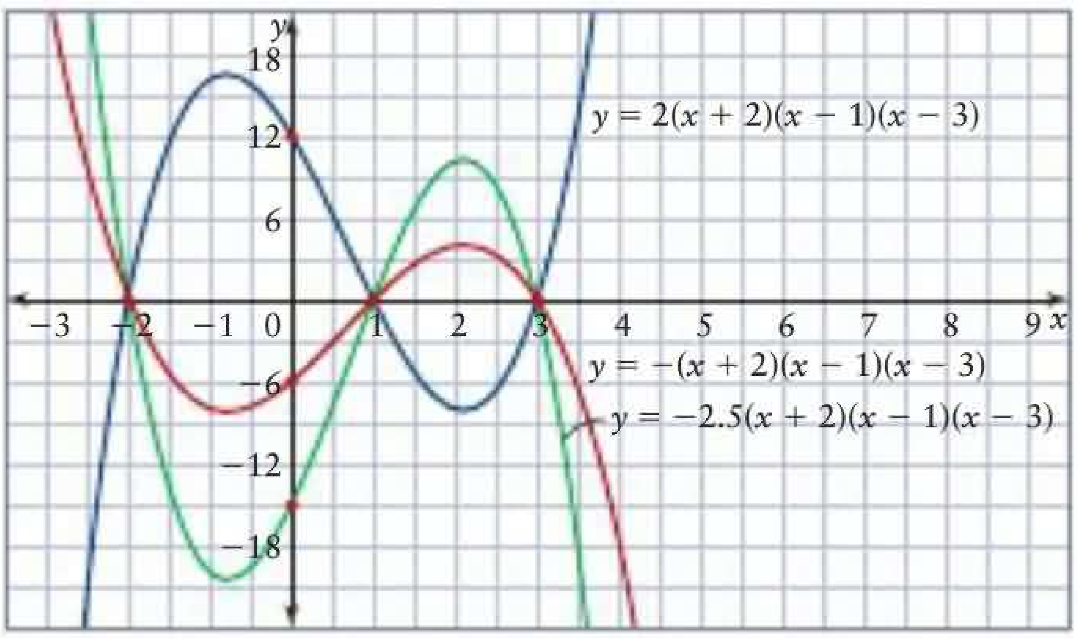
$$\begin{aligned} y &= -(x + 2)(x - 1)(x - 3) \\ &= -(0 + 2)(0 - 1)(0 - 3) \\ &= -6 \end{aligned}$$

The y-intercept of $y = -(x + 2)(x - 1)(x - 3)$ is -6.

To sketch a graph of the functions, plot the common x-intercepts. Plot the y-intercept for each function.

The cubic function $y = 2(x + 2)(x - 1)(x - 3)$ has a positive leading coefficient, so its graph will extend from quadrant 3 to quadrant 1.

The cubic functions $y = -(x + 2)(x - 1)(x - 3)$ and $y = -2.5(x + 2)(x - 1)(x - 3)$ have negative leading coefficients, so their graphs will extend from quadrant 2 to quadrant 4.



Example 3 Determine an Equation for a Family of Quartic Functions Given Irrational Zeros

a) Determine a simplified equation for the family of quartic functions with zeros ± 1 and $2 \pm \sqrt{3}$.

b) Determine an equation for the member of the family whose graph passes through the point (2, 18).

Solution

a) The zeros are 1, -1, $2 + \sqrt{3}$, and $2 - \sqrt{3}$.

So, $(x - 1)$, $(x + 1)$, $(x - 2 - \sqrt{3})$, and $(x - 2 + \sqrt{3})$ are factors of the family of quartic functions. An equation for this family is

$$\begin{aligned} y &= k(x - 1)(x + 1)(x - 2 - \sqrt{3})(x - 2 + \sqrt{3}) \\ &= k(x - 1)(x + 1)[(x - 2) - \sqrt{3}][(x - 2) + \sqrt{3}] \\ &= k(x^2 - 1)[(x - 2)^2 - (\sqrt{3})^2] \\ &= k(x^2 - 1)(x^2 - 4x + 4 - 3) \\ &= k(x^2 - 1)(x^2 - 4x + 1) \\ &= k(x^4 - 4x^3 + x^2 - x^2 + 4x - 1) \\ &= k(x^4 - 4x^3 + 4x - 1) \end{aligned}$$

CONNECTIONS

Each pair of factors has the difference of squares pattern: $(a - b)(a + b) = a^2 - b^2$.

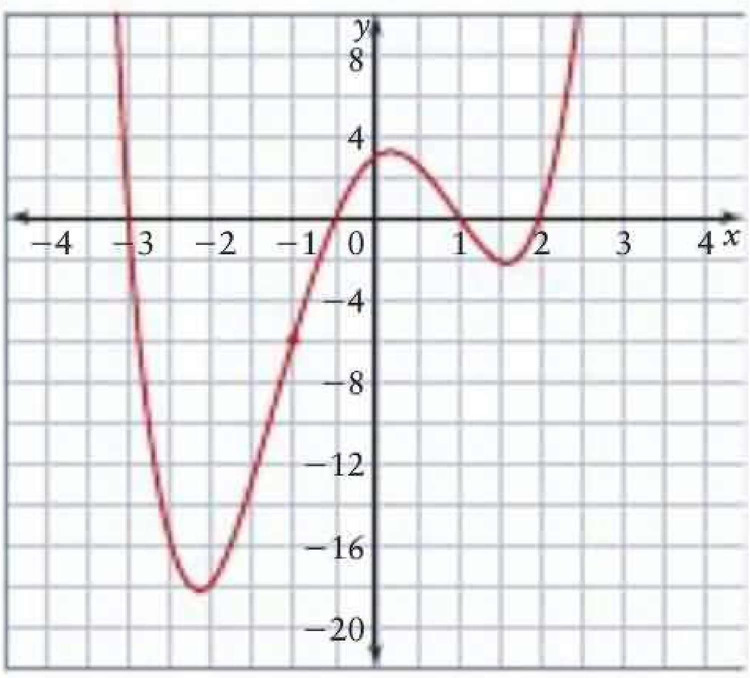
b) To find the member whose graph passes through (2, 18), substitute $x = 2$ and $y = 18$ into the equation and solve for k.

$$\begin{aligned} 18 &= k[(2)^4 - 4(2)^3 + 4(2) - 1] \\ 18 &= -9k \\ k &= -2 \end{aligned}$$

The equation is $y = -2(x^4 - 4x^3 + 4x - 1)$, or $y = -2x^4 + 8x^3 - 8x + 2$.

Example 4 Determine an Equation for a Quartic Function From a Graph

Determine an equation for the quartic function represented by this graph.



Solution

From the graph, the x-intercepts are -3, $-\frac{1}{2}$, 1, and 2.

The corresponding factors are $x + 3$, $2x + 1$, $x - 1$, and $x - 2$.

An equation for the family of polynomial functions with these zeros is $y = k(x + 3)(2x + 1)(x - 1)(x - 2)$.

Select a point that the graph passes through, such as $(-1, -6)$.

Substitute $x = -1$ and $y = -6$ into the equation to solve for k.

$$
\begin{aligned}
-6 &= k[(-1) + 3][2(-1) + 1][(-1) - 1][(-1) - 2] \\
-6 &= k(2)(-1)(-2)(-3) \\
-6 &= -12k \\
k &= 0.5
\end{aligned}
$$

The equation is $y = 0.5(x + 3)(2x + 1)(x - 1)(x - 2)$.

KEY CONCEPTS

- A family of functions is a set of functions with the same characteristics.
- Polynomial functions with graphs that have the same x-intercepts belong to the same family.
- A family of polynomial functions with zeros $a_1, a_2, a_3, \ldots, a_n$ can be represented by an equation of the form $y = k(x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n)$, where $k \in \mathbb{R}$, $k \neq 0$.
- An equation for a particular member of a family of polynomial functions can be determined if a point on the graph is known.

Communicate Your Understanding

C1 How many polynomial functions can have the same x-intercepts? Explain.

C2 What information is required to determine an equation for a family of polynomial functions?

C3 What information is required to determine an equation for a particular member of a family of polynomial functions?

C4 Describe how the graphs of the members of a family of polynomial functions are the same and how they are different.

A Practise

For help with question 1, refer to Example 1.

1. The zeros of a quadratic function are −7 and −3.

a) Determine an equation for the family of quadratic functions with these zeros.

b) Write equations for two functions that belong to this family.

c) Determine an equation for the member of the family that passes through the point (2, 18).

For help with questions 2 to 4, refer to Example 2.

2. Examine the following functions. Which function does not belong to the same family? Explain.

A $y = 1.5(x + 4)(x - 5)(x - 2)$

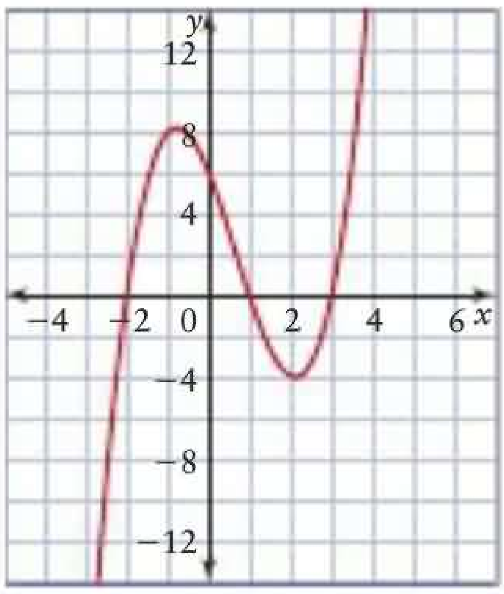
B $y = -1.5(x - 2)(x - 5)(x + 4)$

C $y = 1.5(x - 2)(x + 4)(x - 2)$

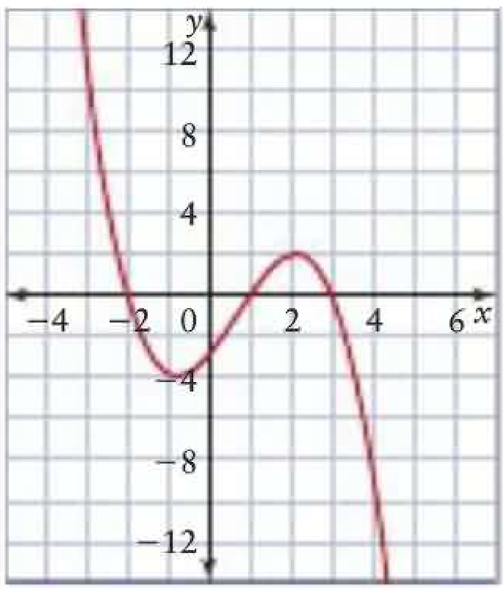
D $y = 3(x - 5)(x - 2)(x + 4)$

3. The graphs of four polynomial functions are given. Which graphs represent functions that belong to the same family? Explain.

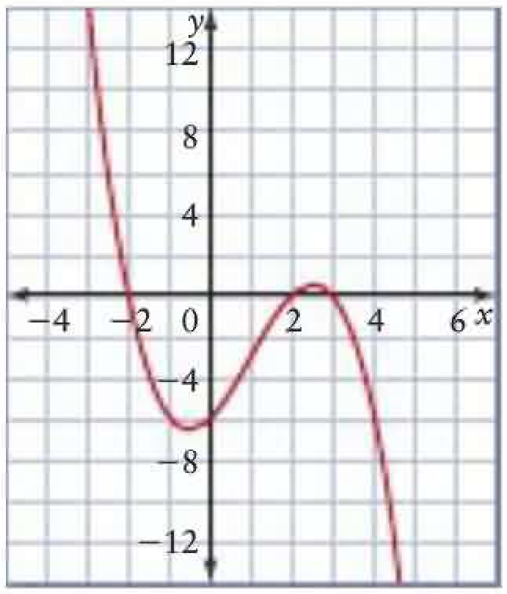
A



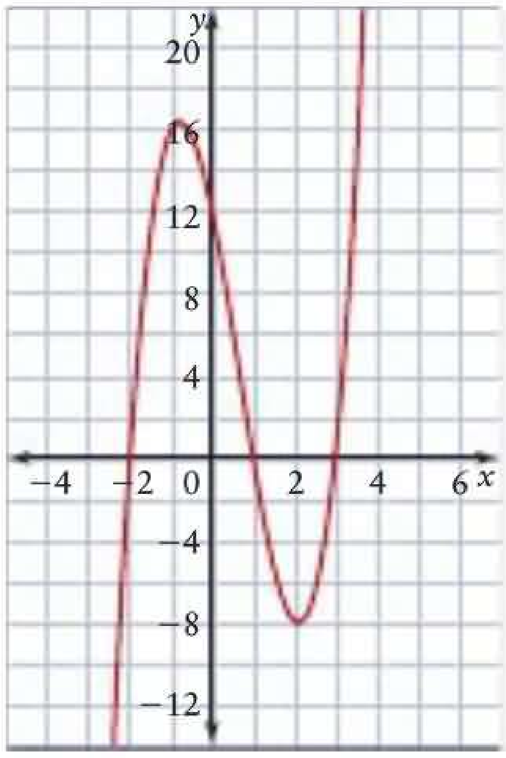
B



C



D



4. Which of the following polynomial functions belong to the same families? Explain. Sketch a graph of the functions in each family to verify your answer.

A $f(x) = (x + 2)(x - 1)(x + 3)$

B $h(x) = -(x - 2)(x + 1)(x - 3)$

C $g(x) = 3(x + 2)(x - 1)(x + 3)$

D $p(x) = 0.4(x - 3)(x + 1)(x - 2)$

E $r(x) = -\frac{2}{5}(x - 1)(x + 2)(x + 3)$

F $q(x) = -\sqrt{3}\,(x + 1)(x - 3)(x - 2)$

For help with question 5, refer to Example 3.

5. Write an equation for a family of polynomial functions with each set of zeros.

   a) $-5, 2, 3$ b) $1, 6, -3$

   c) $-4, -1, 9$ d) $-7, 0, 2, 5$

For help with question 6, refer to Example 4.

6. Determine an equation for the function that corresponds to each graph in question 3.

B Connect and Apply

7. a) Determine an equation for the family of cubic functions with zeros -4, 0, and 2.

   b) Write equations for two functions that belong to this family.

   c) Determine an equation for the member of the family whose graph passes through the point $(-2, 4)$.

   d) Sketch a graph of the functions in parts b) and c).

8. a) Determine an equation for the family of cubic functions with zeros -2, -1, and $\frac{1}{2}$.

   b) Write equations for two functions that belong to this family.

   c) Determine an equation for the member of the family whose graph has a y-intercept of 6.

   d) Sketch a graph of the functions in parts b) and c).

9. a) Determine an equation for the family of quartic functions with zeros -4, -1, 2, and 3.

   b) Write equations for two functions that belong to this family.

   c) Determine an equation for the member of the family whose graph has a y-intercept of -4.

   d) Sketch a graph of the functions in parts b) and c).

10. a) Determine an equation for the family of quartic functions with zeros $-\frac{5}{2}$, -1, $\frac{7}{2}$, and 3.

    b) Write equations for two functions that belong to this family.

    c) Determine an equation for the member of the family whose graph passes through the point $(-2, 25)$.

    d) Sketch a graph of the functions in parts b) and c).

11 a) Determine an equation, in simplified form, for the family of cubic functions with zeros $1 \pm \sqrt{2}$ and $-\frac{1}{2}$.

   b) Determine an equation for the member of the family whose graph passes through the point $(3, 35)$.

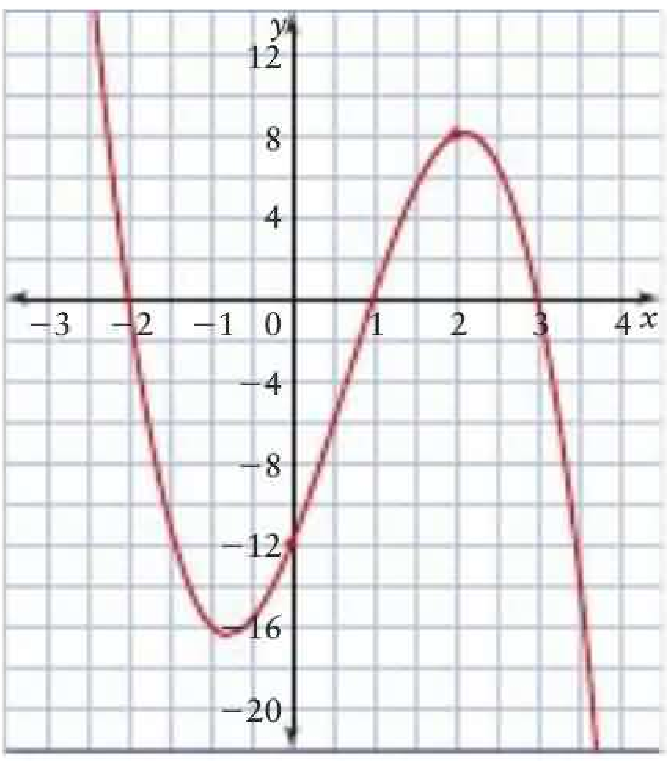
12. a) Determine an equation, in simplified form, for the family of quartic functions with zeros 3 (order 2) and $-4 \pm \sqrt{3}$.

    b) Determine an equation for the member of the family whose graph passes through the point $(1, -22)$.

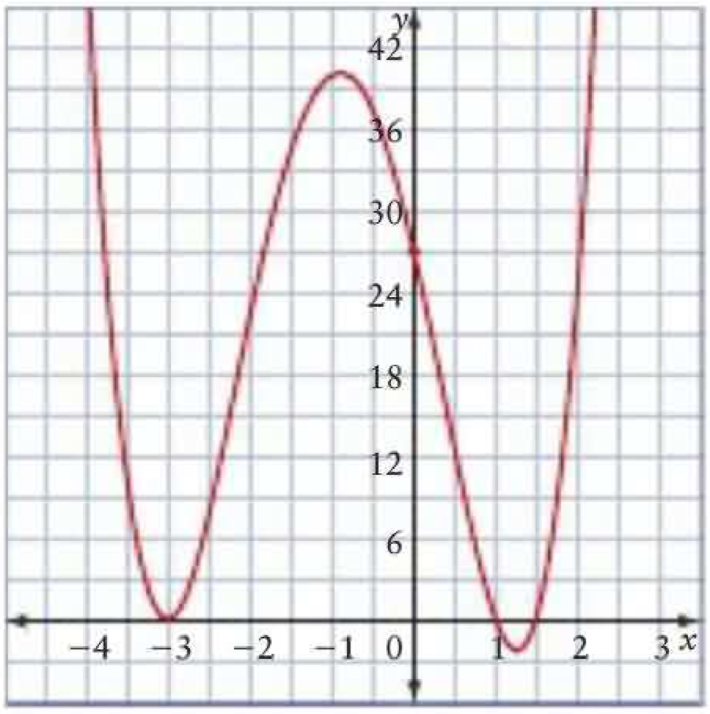
13. a) Determine an equation, in simplified form, for the family of quartic functions with zeros $-1 \pm \sqrt{5}$ and $2 \pm \sqrt{2}$.

    b) Determine an equation for the member of the family whose graph has a y-intercept of -32.

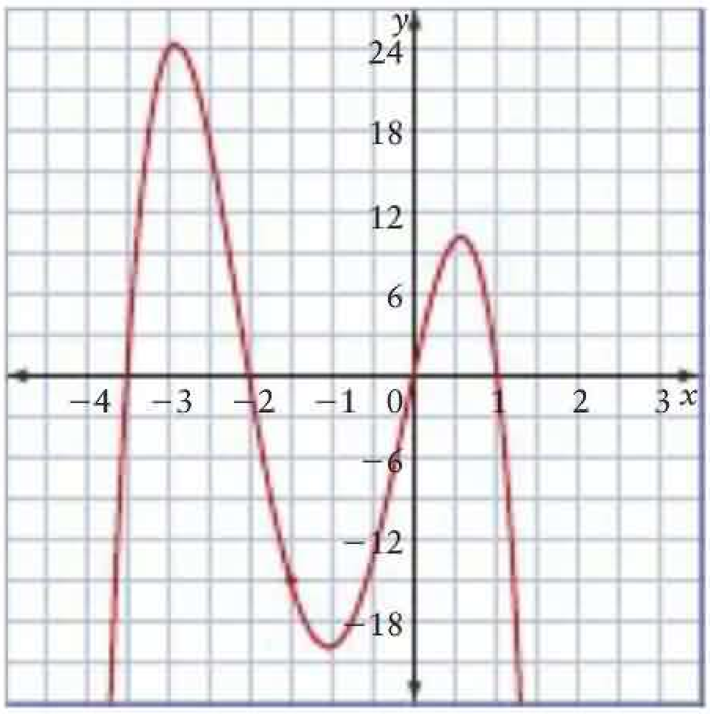
14. Determine an equation for the cubic function represented by this graph.



15. Determine an equation for the quartic function represented by this graph.



16. Determine an equation for the quartic function represented by this graph.



17. Use Technology Are the functions in each set a family? Justify your answer.

Set A

$y = (3x + 1)(2x - 1)(x + 3)(x - 2)$

$y = 2(3x + 1)(2x - 1)(x + 3)(x - 2) + 1$

$y = 3(3x + 1)(2x - 1)(x + 3)(x - 2) + 2$

$y = 4(3x + 1)(2x - 1)(x + 3)(x - 2) + 3$

Set B

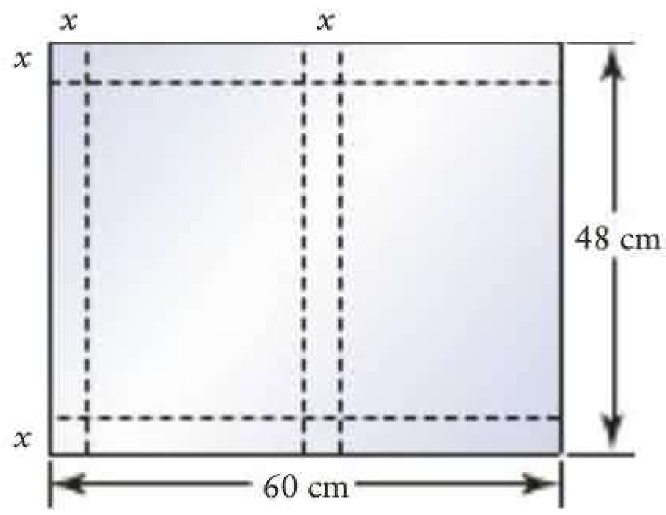
$y = (3x + 1)(2x - 1)(x + 3)(x - 2)$

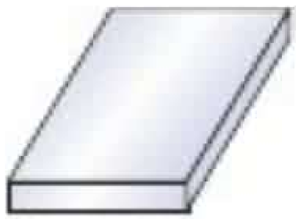
$y = (3x + 1)(4x - 2)(x + 3)(x - 2)$

$y = 3(3x + 1)(1 - 2x)(x + 3)(x - 2)$

$y = 4(3x + 1)(2x - 1)(x + 3)(6 - 3x)$

18. Chapter Problem Clear plastic sheets that measure 48 cm by 60 cm are to be used to construct gift boxes for Best of U personal care products. The boxes are formed by folding the sheets along the dotted lines, as shown in the diagram.





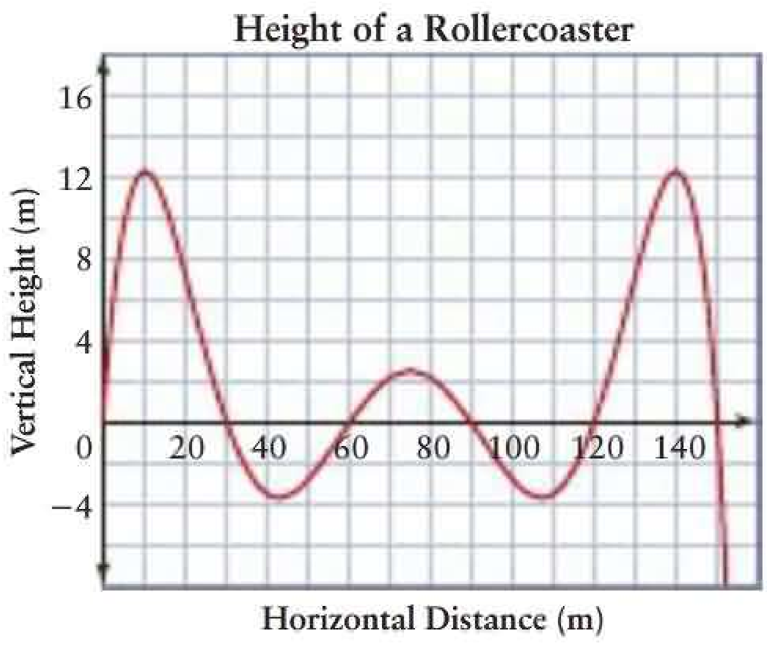
a) Express the volume of one of the boxes as a function of x.

b) Determine possible dimensions of the box if the volume of each box is to be 2300 cm^3.

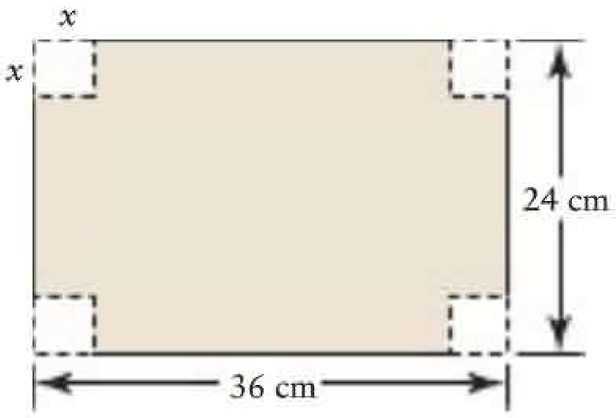
c) How does the volume function in part a) change if the height of the box is doubled? tripled? Describe the family of functions formed by multiplying the height by a constant.

d) Sketch graphs of two members of this family on the same coordinate grid.

19. The graph represents a section of the track of a rollercoaster. Write an equation for the family of functions that models the section of the track.



20. An open-top box is to be constructed from a piece of cardboard by cutting congruent squares from the corners and then folding up the sides. The dimensions of the cardboard are shown.



a) Express the volume of the box as a function of x.

b) Write an equation to represent a box with volume that is

i) twice the volume of the box represented by the function in part a)

ii) three times the volume of the box represented by the function in part a)

c) How are the equations in part b) related to the one in part a)?

d) Sketch graphs of all three functions on the same coordinate grid.

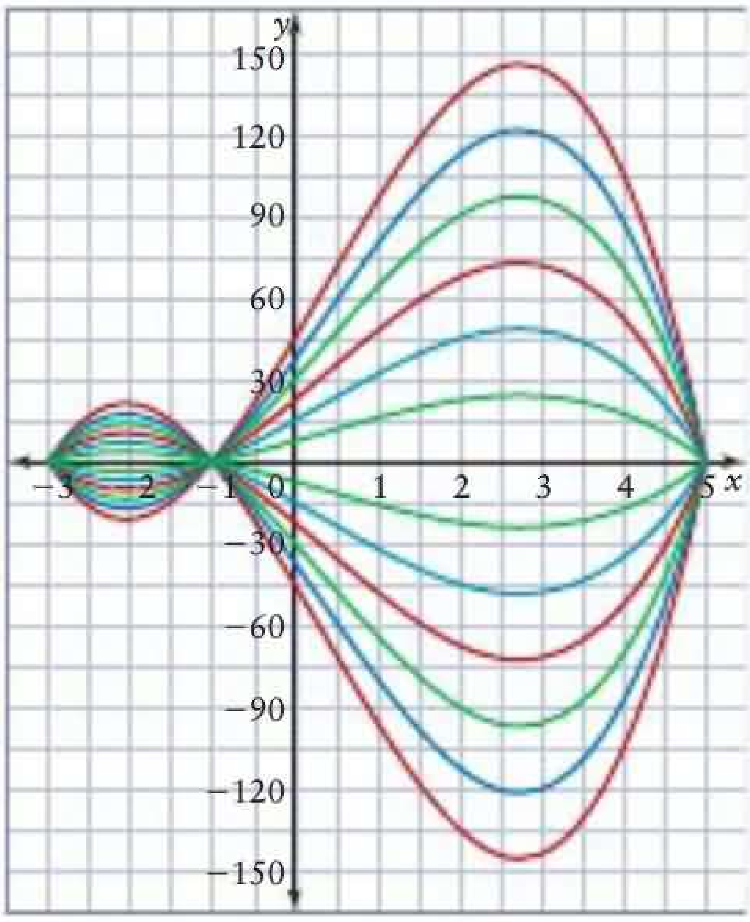
e) Determine possible dimensions of a box with volume 1820 cm^3.

✓ Achievement Check

21. a) The design for the crystal pieces of the chandelier at the beginning of this section is shown below. Determine an equation for the family of functions used to create the design.

b) Find equations for the members of the family that make up the design.

c) Create a design of your own. Write equations for the family of functions and the members used in your design.



C Extend and Challenge

22. a) Write an equation for a family of even functions with four x-intercepts, two of which are $\frac{2}{3}$ and 5.

b) What is the least degree this family of functions can have?

c) Determine an equation for the member of this family that passes through the point $(-1, -96)$.

d) Determine an equation for the member of this family that is a reflection in the x-axis of the function in part c).

23. Refer to question 19. Design your own rollercoaster track using a polynomial function of degree six or higher. Sketch a graph of your rollercoaster.

24. Math Contest Two concentric circles have radii 9 cm and 15 cm. Determine the length of a chord of the larger circle that is tangent to the smaller circle.

25. Math Contest Given a function $g(x)$ such that $g(x^2 + 2) = x^4 + 5x^2 + 3$, determine $g(x^2 - 1)$.

2.5

Solve Inequalities Using Technology

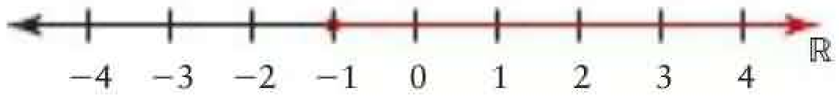
An electronics manufacturer determines that the revenue, R, in millions of dollars, from yearly sales of MP3 players can be modelled by the function $R(t) = t^3 + 0.8t^2 - 2t + 1$, where t is the time, in years, since 2003. How can this model be used to determine when the yearly sales will be \$100 million or more, that is, when will $t^3 + 0.8t^2 - 2t + 1 \geq 100$?

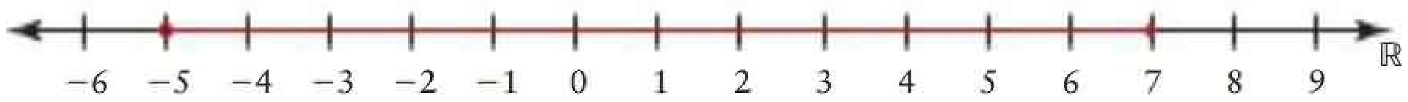
You have solved polynomial equations by determining the roots. In some problems, such as the one for MP3 players, the solution is a range of values. The equal sign in the equation is replaced with an inequality symbol. In this section, you will learn the meaning of a polynomial inequality and examine methods for solving polynomial inequalities using technology.

Investigate How are a polynomial inequality and the graph of a polynomial function related?

1. Write an inequality that corresponds to the values of x shown on each number line.

a) 

b) 

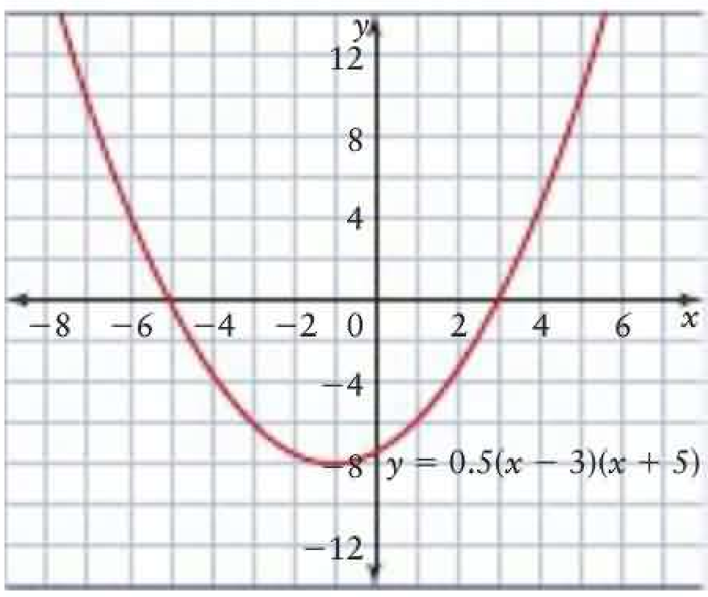
c) 

2. Graphs of two parabolas with their equations are given.

a) For graph A, write inequalities for the values of x for which the graph lies above the x-axis. What is the sign of the y-values of the graph for these values of x?

b) **Reflect** What is the relationship between the inequalities in part a) and $0.5(x - 3)(x + 5) > 0$?

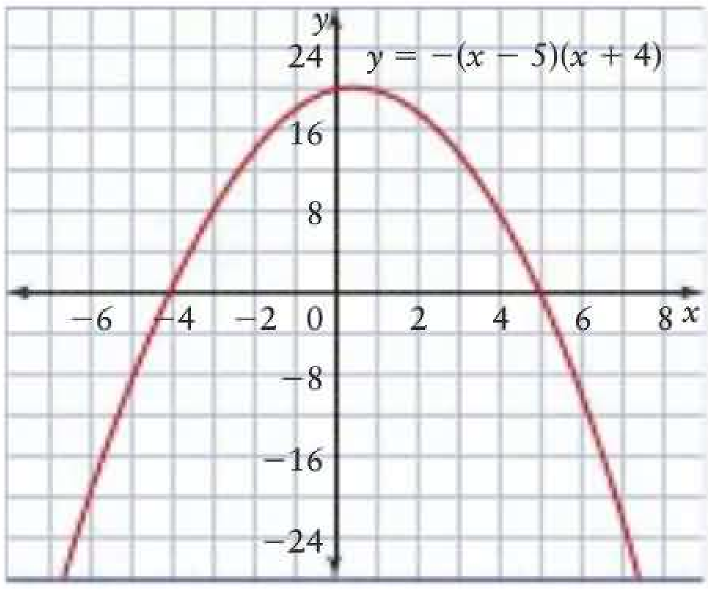
Graph A



c) For graph B, write inequalities for the values of x for which the graph is below the x-axis. What is the sign of the y-values of the graph for these values of x?

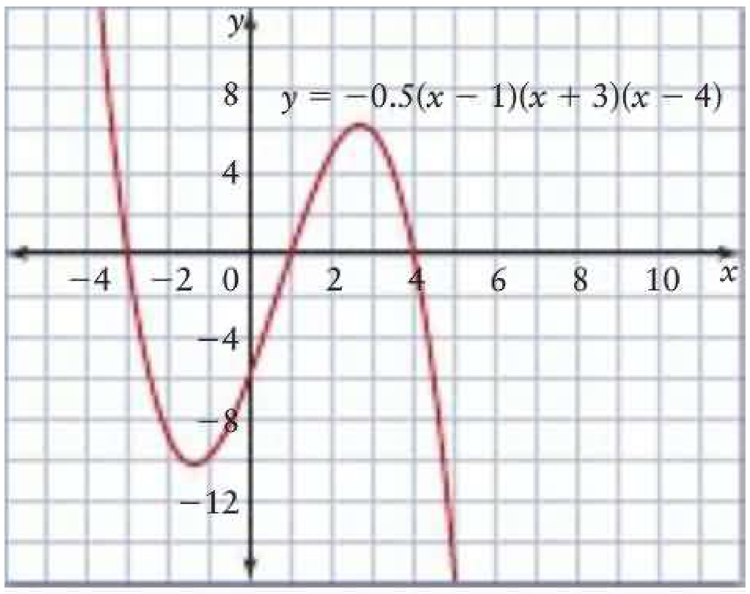
d) **Reflect** What is the relationship between the inequalities in part c) and $-(x - 5)(x + 4) < 0$?

Graph B



3. **Reflect** What is the relationship between the x-intercepts of the graphs and the inequalities determined in step 2?

4. Consider the cubic graph shown.



a) Write inequalities for the values of x such that $y \leq 0$.

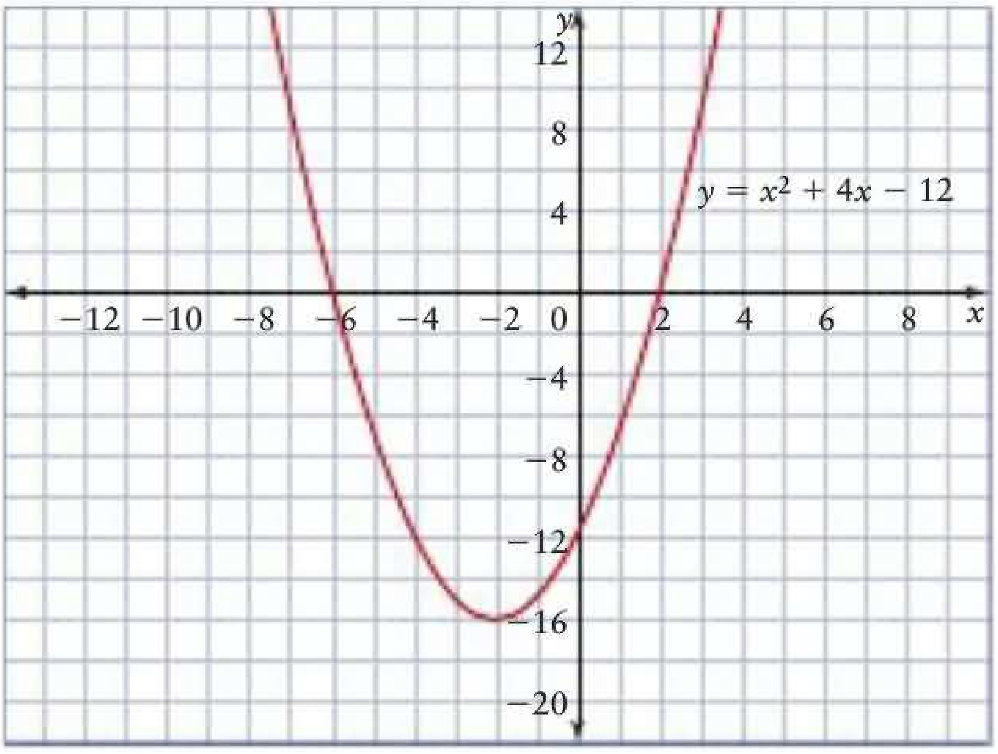
b) Write inequalities for the values of x such that $y > 0$.

c) **Reflect** Explain how a graph can be used to solve each polynomial inequality.

i) $-0.5(x - 1)(x + 3)(x - 4) \leq 0$

ii) $-0.5(x - 1)(x + 3)(x - 4) > 0$

Examine the graph of $y = x^2 + 4x - 12$.



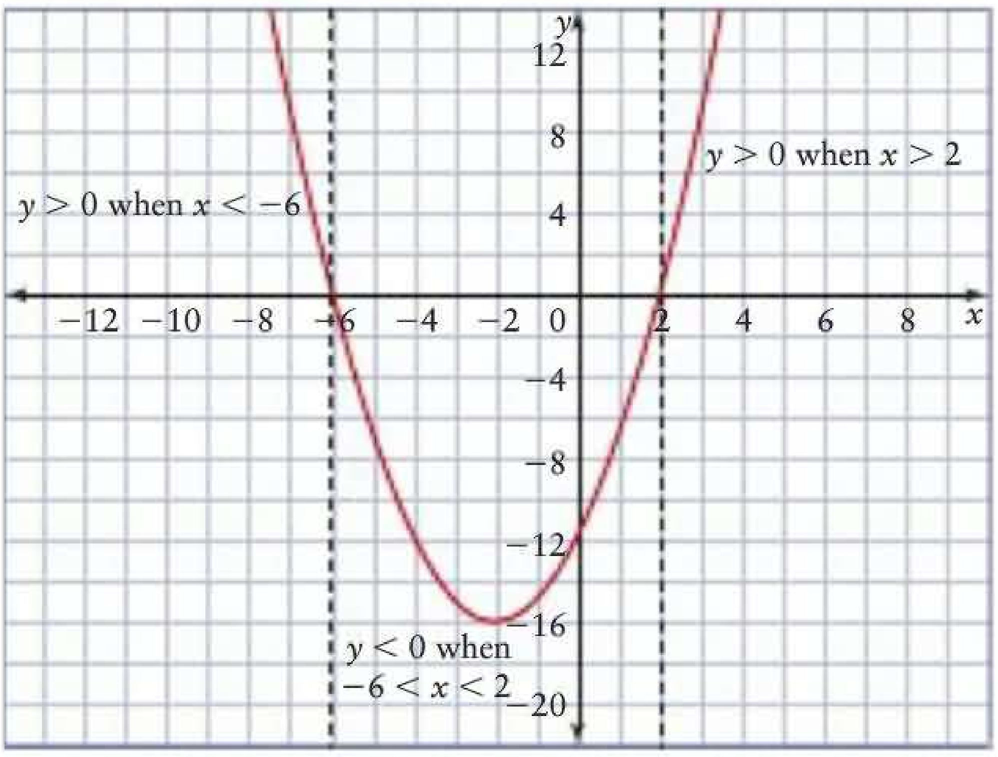
The x-intercepts are -6 and 2. These correspond to the zeros of the function $y = x^2 + 4x - 12$. By moving from left to right along the x-axis, we can make the following observations.

- The function is positive when $x < -6$ since the y-values are positive.
- The function is negative when $-6 < x < 2$ since the y-values are negative.
- The function is positive when $x > 2$ since the y-values are positive.

The zeros -6 and 2 divide the x-axis into three intervals: $x < -6$, $-6 < x < 2$, and $x > 2$. In each interval, the function is either positive or negative.

The information can be summarized in a table and is shown on the graph below.

Interval	$x < -6$	$-6 < x < 2$	$x > 2$
Sign of Function	+	−	+



Example 1 Solve a Polynomial Inequality Graphically

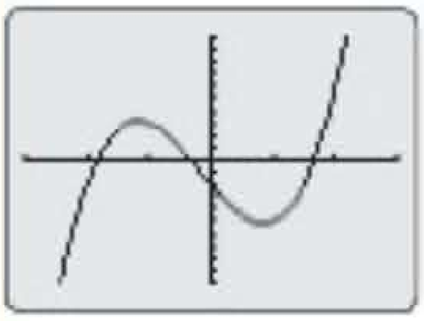
Solve the polynomial inequality. Round answers to one decimal place.

$2x^3 + x^2 - 6x - 2 \geq 0$

Solution

Use a graphing calculator to graph the corresponding polynomial function.

Graph the function $y = 2x^3 + x^2 - 6x - 2$.



Use the **Zero** operation.

The three zeros are -1.8, -0.3, and 1.7, to one decimal place.

The values that satisfy the inequality $2x^3 + x^2 - 6x - 2 \geq 0$ are the values of x for which the graph is zero or positive (on or above the x-axis). From the graph, this occurs when $-1.8 \leq x \leq -0.3$ or $x \geq 1.7$.

A polynomial inequality may also be solved numerically by using the roots of the polynomial equation to determine the possible intervals on which the corresponding function changes from positive to negative and vice versa. Once the intervals are established, a value from each interval is used to test if the function is positive or negative in that interval.

Example 2 Solve Polynomial Inequalities Numerically Using a CAS

Solve $x^4 - 5x^2 + 4 < 0$. Verify your answer graphically.

Solution

Use the CAS on a graphing calculator.

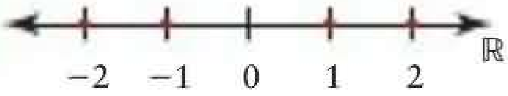
Determine the intervals using the roots of the polynomial equation and then numerically verify if the corresponding function is positive or negative within the intervals.

First, solve the equation $x^4 - 5x^2 + 4 = 0$.

- From the F2 menu, select **1:solve**.
  Enter the equation to solve for x.

The roots are -2, -1, 1, and 2.

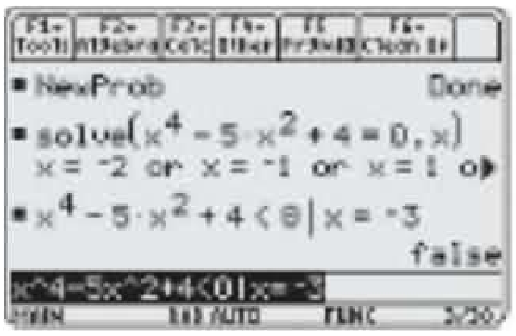
Arrange the roots in order from least to greatest on a number line.



The intervals are $x < -2$, $-2 < x < -1$, $-1 < x < 1$, $1 < x < 2$, and $x > 2$.

To solve the inequality $x^4 - 5x^2 + 4 < 0$, use numerical values in each interval to test if the function is negative to determine if the values in that interval make the inequality true.

For $x < -2$, test $x = -3$.



- Press the key sequence
  [X] [^] 4 [−] 5 [X] [^] 2 [+] 4
  [2nd] 0 0 [|] [X] [=] [(−)] 3.
- Press [ENTER].

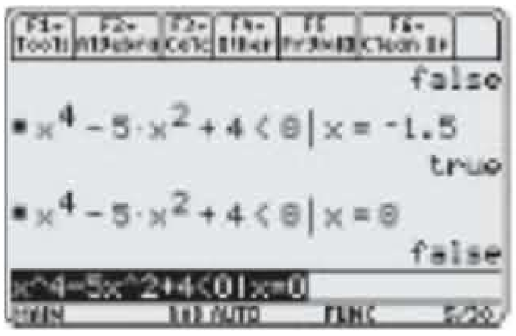
The inequality statement is false.

For $-2 < x < -1$, test $x = -1.5$.

The inequality statement is true.

For $-1 < x < 1$, test $x = 0$.

The inequality statement is false.

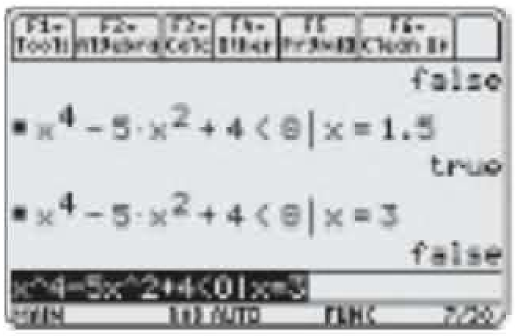


For $1 < x < 2$, test $x = 1.5$.

The inequality statement is true.

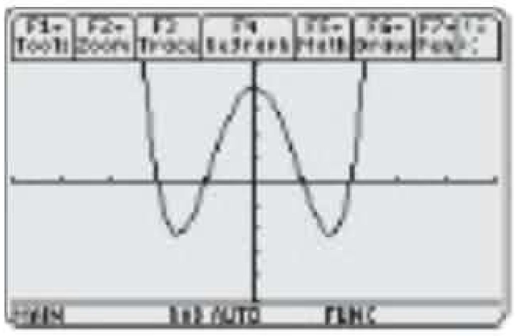
For $x > 2$, test $x = 3$.

The inequality statement is false.



Since the inequality statement is true for the values tested in the two intervals $-2 < x < -1$ and $1 < x < 2$, the solutions to $x^4 - 5x^2 + 4 < 0$ are values of x such that $-2 < x < -1$ or $1 < x < 2$.

The graph of $y = x^4 - 5x^2 + 4$ verifies that $x^4 - 5x^2 + 4 < 0$ when $-2 < x < -1$ or $1 < x < 2$.



Example 3 Solve a Problem Involving an Inequality

An electronics manufacturer determines that the revenue, R, in millions of dollars, from yearly sales of MP3 players can be modelled by the function $R(t) = t^3 + 0.8t^2 - 2t + 1$, where t is the time, in years, since 2003. Use this model to determine when the yearly sales will be \$100 million or more, that is, when $t^3 + 0.8t^2 - 2t + 1 \geq 100$ will hold.

Solution

Method 1: *Graph a Single Function*

Write the inequality as $t^3 + 0.8t^2 - 2t + 1 - 100 \geq 0$, or $t^3 + 0.8t^2 - 2t - 99 \geq 0$.

Graph the function $y = x^3 + 0.8x^2 - 2x - 99$.

Window variables: $x \in [-10, 10]$, $y \in [-200, 200]$, Yscl = 20

There is one x-intercept, which cannot be read easily from the graph. It is located between 4 and 5. Use the **Zero** operation to find its value.

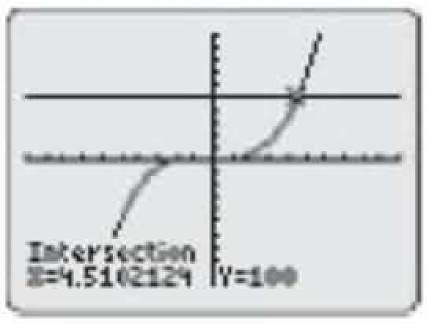
The yearly sales will be \$100 million or more approximately 4.5 years from 2003, or halfway through 2007.

Method 2: *Find the Intersection Point of Two Functions*

Consider the graphs of the two functions $y = t^3 + 0.8t^2 - 2t + 1$ and $y = 100$. The two functions are equal at their point(s) of intersection. Once this point is identified, determine where the y-values of $y = t^3 + 0.8t^2 - 2t + 1$ are greater than $y = 100$.

Graph $y = x^3 + 0.8x^2 - 2x + 1$ in **Y1** and $y = 100$ in **Y2**.

Window variables: $x \in [-10, 10]$, $y \in [-200, 200]$, Yscl = 20



Use the **Intersect** operation to find the coordinates of the point of intersection.

The point of intersection is approximately (4.5, 100).

From the graphs, the value of $y = x^3 + 0.8x^2 - 2x + 1$ is greater than or equal to $y = 100$ when $x \geq 4.5$. So, $t^3 + 0.8t^2 - 2t + 1 \geq 100$ when $t \geq 4.5$.

The yearly sales will be \$100 million or more approximately 4.5 years from 2003, or halfway through 2007.

KEY CONCEPTS

- A polynomial inequality results when the equal sign in a polynomial equation is replaced with an inequality symbol.
- The real zeros of a polynomial function, or x-intercepts of the corresponding graph, divide the x-axis into intervals that can be used to solve a polynomial inequality.
- Polynomial inequalities may be solved graphically by determining the x-intercepts and then using the graph to determine the intervals that satisfy the inequality.
- A CAS may be used to solve a polynomial inequality numerically by determining the roots of the polynomial equation and then testing values in each interval to see if they make the inequality true.

Communicate Your Understanding

C1 Explain the difference between a polynomial equation and a polynomial inequality. Support your answer with examples.

C2 Describe the connection between the solution to a polynomial inequality and the graph of the corresponding function.

C3 Describe the role of the real roots of a polynomial equation when solving the related inequality.

C4 Describe how technology can be used to solve the inequality $1.2x^3 - 5x^2 + 3.5x + 2 \leq 0$.

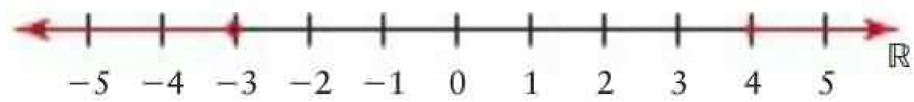
A Practise

1. Write inequalities for the values of x shown.

a)

b)

c)



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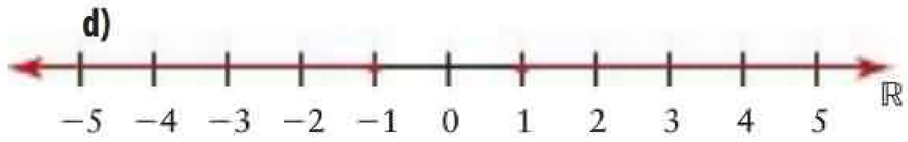


2. Write the intervals into which the x-axis is divided by each set of x-intercepts of a polynomial function.

a) $-1, 5$ **b)** $-7, 2, 0$

c) $-6, 0, 1$ **d)** $-4, -2, \frac{2}{5}, 4.3$

For help with questions 3 to 5, refer to Example 1.

3. Sketch a graph of a cubic polynomial function $y = f(x)$ such that $f(x) < 0$ when $-4 < x < 3$ or $x > 7$ and $f(x) > 0$ when $x < -4$ or $3 < x < 7$.

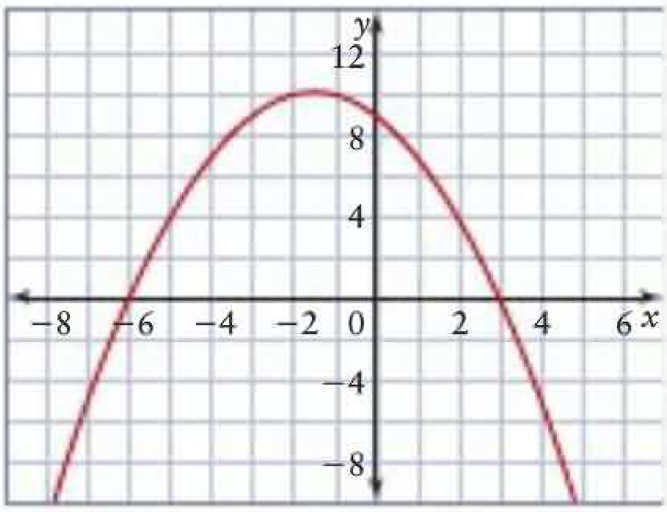
4. Describe what the solution to each inequality indicates about the graph of $y = f(x)$.

a) $f(x) < 0$ when $-2 < x < 1$ or $x > 6$

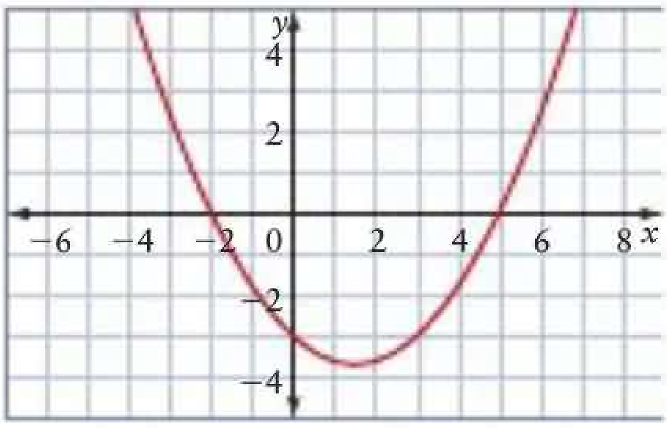
b) $f(x) \geq 0$ when $x \leq -3.6$ or $0 \leq x \leq 4.7$ or $x \geq 7.2$

5. For each graph, write
   i) the x-intercepts
   ii) the intervals of x for which the graph is positive
   iii) the intervals of x for which the graph is negative

a)



b)



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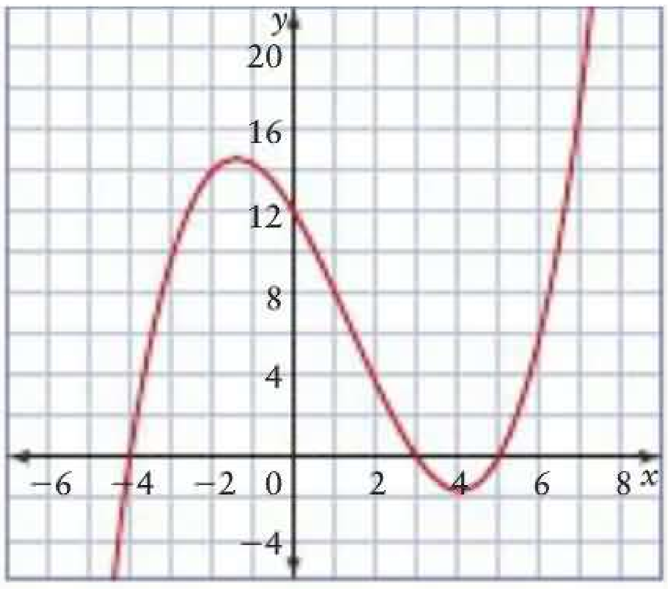

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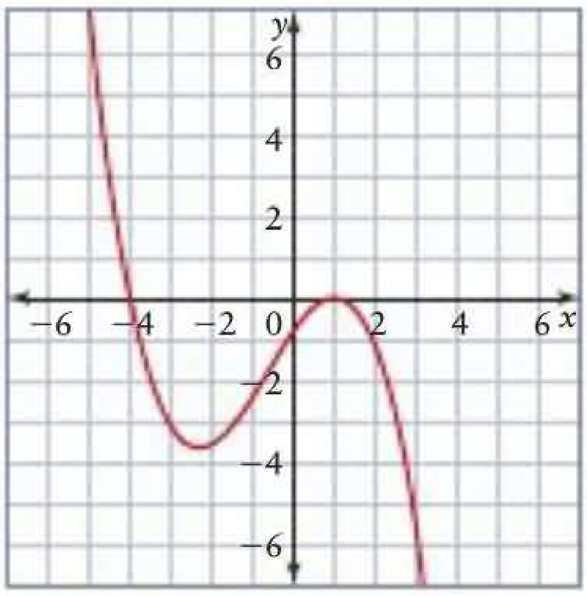


B Connect and Apply

6. Solve each polynomial inequality by graphing the polynomial function.
   a) $x^2 - x - 12 < 0$
   b) $x^2 + 8x + 15 \leq 0$
   c) $x^3 - 6x^2 + 11x - 6 > 0$
   d) $x^3 + 8x^2 + 19x + 12 \geq 0$
   e) $x^3 - 2x^2 - 9x + 18 < 0$
   f) $x^3 + x^2 - 16x - 16 \leq 0$

For help with questions 7 to 9, refer to Example 2.

7. Solve each polynomial inequality. Use a CAS if available.
   a) $2x^2 + 7x - 4 \geq 0$
   b) $2x^2 - 5x - 3 < 0$
   c) $x^3 + 5x^2 + 2x - 8 \leq 0$
   d) $x^3 + 2x^2 - 19x - 20 > 0$
   e) $x^3 - 39x - 70 < 0$
   f) $x^3 - 3x^2 - 24x - 28 \leq 0$

8. Solve each polynomial inequality by first finding the approximate zeros of the related polynomial function. Round answers to two decimal places.
   a) $x^2 + 4x - 3 < 0$
   b) $-3x^2 - 4x + 8 > 0$
   c) $x^3 + x^2 - 3x - 1 \leq 0$
   d) $2x^3 + 4x^2 - x - 1 \geq 0$
   e) $3x^3 + 4x^2 - 5x - 3 < 0$
   f) $-x^4 + x^3 - 2x + 3 \geq 0$

9. Solve.
   a) $5x^3 - 7x^2 - x + 4 > 0$
   b) $-x^3 + 28x + 48 \geq 0$
   c) $3x^3 + 4x^2 - 35x - 12 \leq 0$
   d) $3x^3 + 2x^2 - 11x - 10 < 0$
   e) $-2x^3 + x^2 + 13x + 6 > 0$
   f) $2x^4 + x^3 - 26x^2 - 37x - 12 > 0$

10. The height, h, in metres, of a golf ball t seconds after it is hit can be modelled by the function $h(t) = -4.9t^2 + 32t + 0.2$. When is the height of the ball greater than 15 m?

11. The number, n, in hundreds, of tent caterpillars infesting a forested area after t weeks can be modelled by the function $n(t) = -t^4 + 5t^3 + 5t^2 + 6t$.

a) When is the tent caterpillar population greater than 10 000?

b) What will happen after 6 weeks?

12. Chapter Problem The Best of U marketing team has determined that the number, c, in thousands, of customers who purchase the company's products on-line from the Best of U Web site t years after 2003 can be modelled by the function $c(t) = 0.1t^3 - 2t + 8$.

a) When will there be fewer than 8000 on-line customers?

b) When will the number of on-line customers exceed 10 000?

13. a) Create a cubic polynomial inequality such that the corresponding equation has

i) one distinct real root

ii) two distinct real roots

iii) three real roots, one of which is of order 2

b) Solve the inequalities you created in part a).

14. a) Create a quartic inequality such that the corresponding quartic equation has

i) no real roots

ii) two distinct real roots

iii) three distinct real roots

iv) four real roots, two of which are of order 2

b) Solve the inequalities you created in part a).

15. The solutions below correspond to inequalities involving a cubic function. For each solution, write two possible cubic polynomial inequalities, one with the less than symbol ($<$) and the other with the greater than symbol ($>$).

a) $-\frac{2}{3} < x < \frac{4}{5}$ or $x > 3.5$

b) $x < -1 - \sqrt{3}$ or $-1 + \sqrt{3} < x < 4$

C Extend and Challenge

16. Solve $3x^4 + 2x^2 - 4x + 6 \geq 6x^4 - 5x^3 - x^2 - 9x + 2$.

17. Write the domain and range of each function.

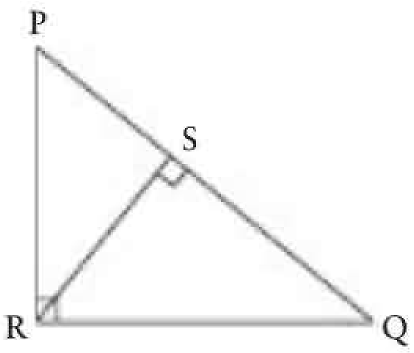
a) $f(x) = \sqrt{-x - x^2}$

b) $g(x) = \dfrac{1}{\sqrt{x^2 - 1}}$

18. Math Contest Given a circle with centre O, PQ and PR are two tangents from a point P outside the circle. Prove that $\angle POQ = \angle POR$.

19. Math Contest Given that $3x - 5$ is a factor of the polynomial function $f(x) = kx^2 - bx + k$, determine the ratio $k:b$ in simplest form.

20. Math Contest In the figure, RS is perpendicular to PQ, PS = 4, and QS = 6. Find the exact length of RS.



2.6

Solve Factorable Polynomial Inequalities Algebraically

A rectangular in-ground swimming pool is to be installed. The engineer overseeing the construction project estimates that at least 1408 m^3 of earth and rocks needs to be excavated. What are the minimum dimensions of the excavation if the depth must be 2 m more than one quarter of the width, and the length must be 12 m more than four times the width? The solution to this problem can be found by solving a cubic polynomial inequality algebraically.

Example 1 Solve Linear Inequalities

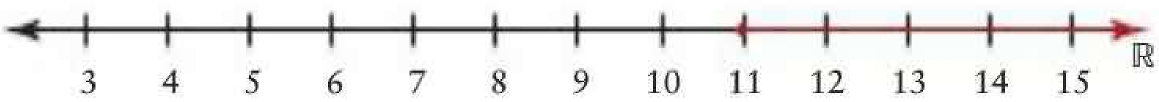
Solve each inequality. Show the solution on a number line.

a) $x - 8 \geq 3$

b) $-4 - 2x < 12$

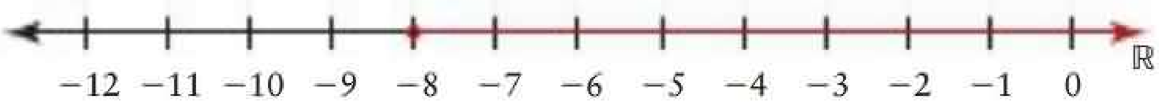
Solution

a)
$$\begin{aligned} x - 8 &\geq 3 \\ x &\geq 3 + 8 \\ x &\geq 11 \end{aligned}$$



b)
$$\begin{aligned} -4 - 2x &< 12 \\ -2x &< 12 + 4 \\ -2x &< 16 \\ x &> \frac{16}{-2} \\ x &> -8 \end{aligned}$$

Divide both sides by -2. Reverse the inequality.



CONNECTIONS

Solving linear inequalities is similar to solving linear equations; however, when both sides of an inequality are multiplied or divided by a negative number, the inequality sign must be reversed.

Example 2 Solve Polynomial Inequalities Algebraically

Solve each inequality.

a) $(x + 3)(2x - 3) > 0$

b) $-2x^3 - 6x^2 + 12x + 16 \leq 0$

Solution

a) $(x + 3)(2x - 3) > 0$

Method 1: ***Consider All Cases***

$(x + 3)(2x - 3) > 0$

A product mn is positive when m and n are

- both positive or
- both negative

Case 1

$$\begin{aligned} x + 3 &> 0 \\ x &> -3 \end{aligned} \qquad \begin{aligned} 2x - 3 &> 0 \\ 2x &> 3 \\ x &> \frac{3}{2} \end{aligned}$$

$x > \frac{3}{2}$ is included in the inequality $x > -3$. So, the solution is $x > \frac{3}{2}$.

Case 2

$$\begin{aligned} x + 3 &< 0 \\ x &< -3 \end{aligned} \qquad \begin{aligned} 2x - 3 &< 0 \\ 2x &< 3 \\ x &< \frac{3}{2} \end{aligned}$$

$x < -3$ is included in the inequality $x < \frac{3}{2}$. So, the solution is $x < -3$.

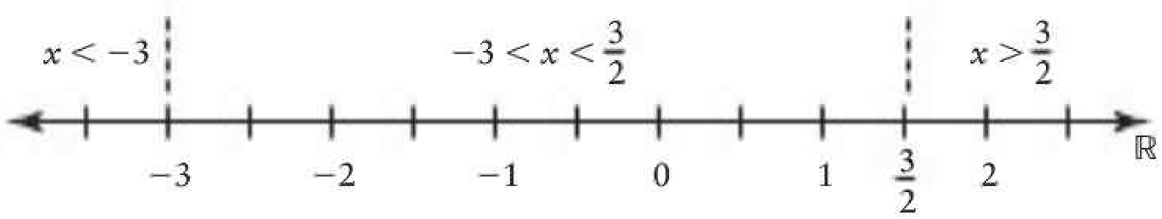
Combining the results of the two cases, the solution is $x < -3$ or $x > \frac{3}{2}$.

Method 2: ***Use Intervals***

$(x + 3)(2x - 3) > 0$

The roots of the equation $(x + 3)(2x - 3) = 0$ are $x = -3$ and $x = \frac{3}{2}$.

Use the roots to break the number line into three intervals.



Test arbitrary values of x for each interval.

For $x < -3$, test $x = -4$.

$(-4 + 3)[2(-4) - 3] = 11$

Since $11 > 0$, $x < -3$ is a solution.

For $-3 < x < \frac{3}{2}$, test $x = 0$.

$(0 + 3)[2(0) - 3] = -9$

Since $-9 < 0$, $-3 < x < \frac{3}{2}$ is not a solution.

For $x > \frac{3}{2}$, test $x = 2$.

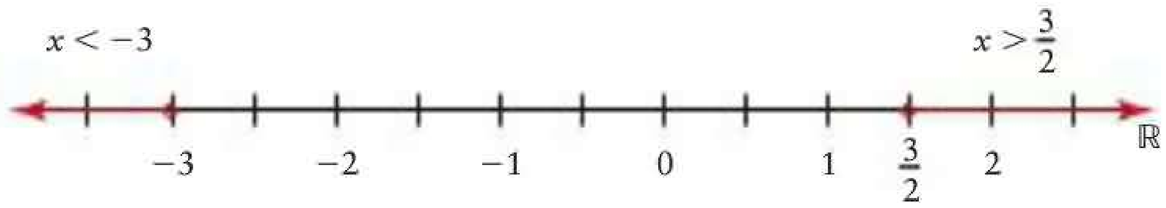
$(2 + 3)[2(2) - 3] = 5$

Since $5 > 0$, $x > \frac{3}{2}$ is a solution.

All of the information can be summarized in a table:

Interval / Factor	$x < -3$	$x = -3$	$-3 < x < \frac{3}{2}$	$x = \frac{3}{2}$	$x > \frac{3}{2}$
$(x + 3)$	$-$	0	$+$	$+$	$+$
$(2x - 3)$	$-$	$-$	$-$	0	$+$
$(x + 3)(2x - 3)$	$+$	0	$-$	0	$+$

The solution is $x < -3$ or $x > \frac{3}{2}$. This can be shown on a number line.



b) $-2x^3 - 6x^2 + 12x + 16 \le 0$

Factor $-2x^3 - 6x^2 + 12x + 16$ using the factor theorem.

$-2x^3 - 6x^2 + 12x + 16 = -2(x + 4)(x - 2)(x + 1)$

So, the inequality becomes $-2(x + 4)(x - 2)(x + 1) \le 0$.

CONNECTIONS

You will obtain the same final result if you divide the inequality by -2 and consider $(x + 4)(x - 2)(x + 1) \ge 0$.

Method 1: *Consider All Cases*

$-2(x + 4)(x - 2)(x + 1) \le 0$

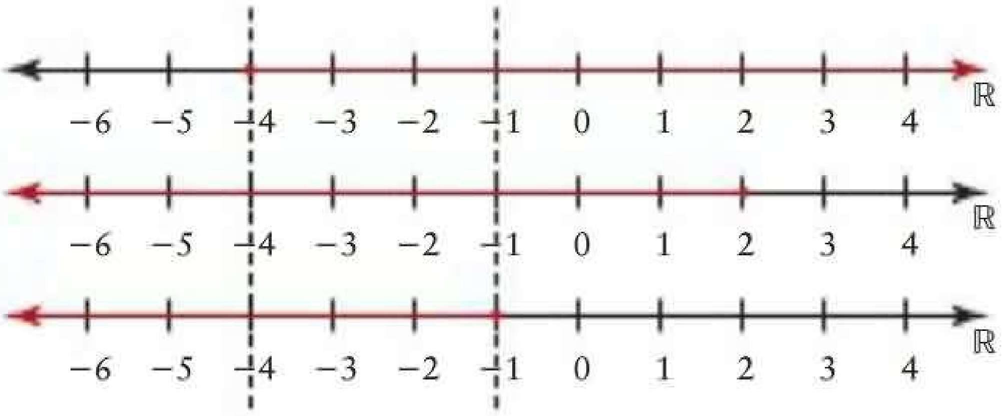
Since -2 is a constant factor, it can be combined with $(x + 4)$ to form one factor.

Thus, the three factors of $-2(x + 4)(x - 2)(x + 1)$ are $-2(x + 4)$, $x - 2$, and $x + 1$.

A product abc is negative when all three factors, a, b, and c, are negative, or when two of the factors are positive and the third one is negative. There are four cases to consider.

Case 1

$-2(x + 4) \le 0$	$x - 2 \le 0$	$x + 1 \le 0$
$x + 4 \ge 0$	$x \le 2$	$x \le -1$
$x \ge -4$		



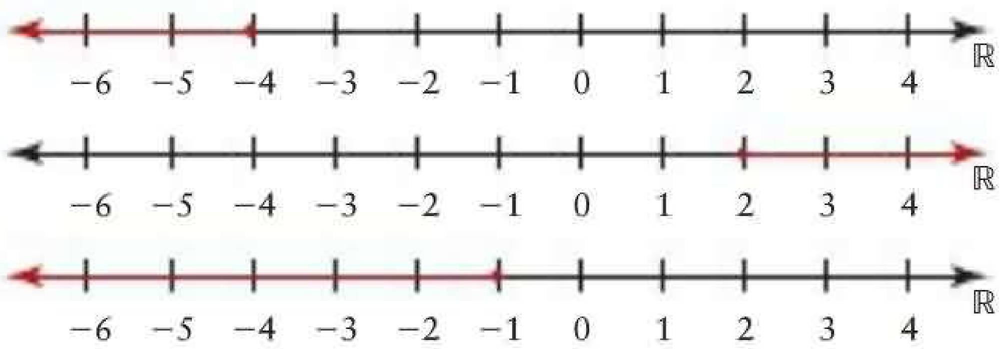
The broken lines indicate that $-4 \le x \le -1$ is common to all three intervals.

The values of x that are common to all three inequalities are $x \ge -4$ and $x \le -1$.

So, $-4 \le x \le -1$ is a solution.

Case 2

$-2(x + 4) \ge 0$	$x - 2 \ge 0$	$x + 1 \le 0$
$x + 4 \le 0$	$x \ge 2$	$x \le -1$
$x \le -4$		

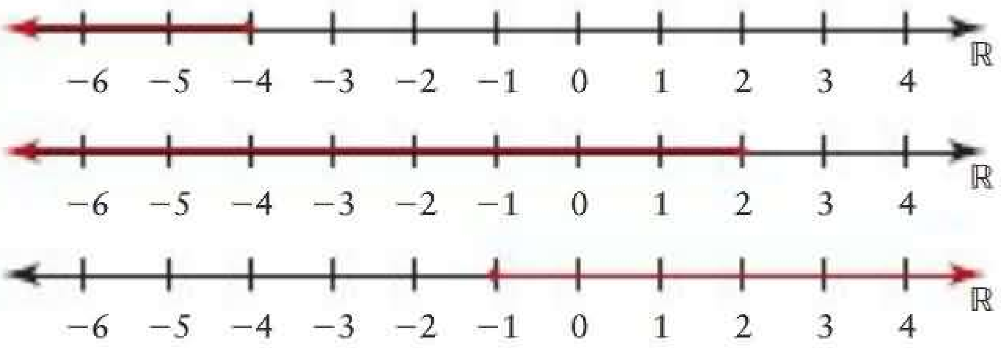


There are no x-values that are common to all three inequalities.

Case 2 has no solution.

Case 3

$-2(x + 4) \ge 0$	$x - 2 \le 0$	$x + 1 \ge 0$
$x + 4 \le 0$	$x \le 2$	$x \ge -1$
$x \le -4$		



There are no x-values that are common to all three inequalities.

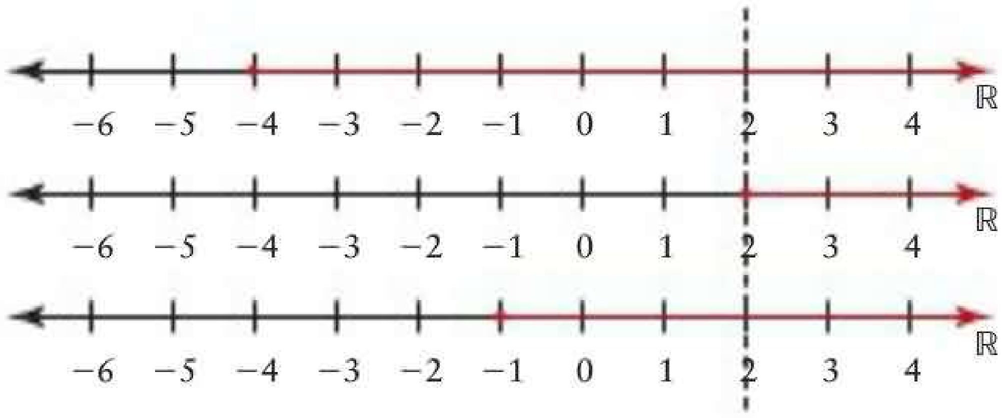
Case 3 has no solution.

Case 4

$-2(x + 4) \le 0$ $\quad$ $x - 2 \ge 0$ $\quad$ $x + 1 \ge 0$

$x + 4 \ge 0$ $\quad$ $x \ge 2$ $\quad$ $x \ge -1$

$x \ge -4$



The broken line indicates that $x \ge 2$ is common to all three intervals.

$x \ge 2$ is included the intervals $x \ge -4$ and $x \ge -1$.

So, $x \ge 2$ is a solution.

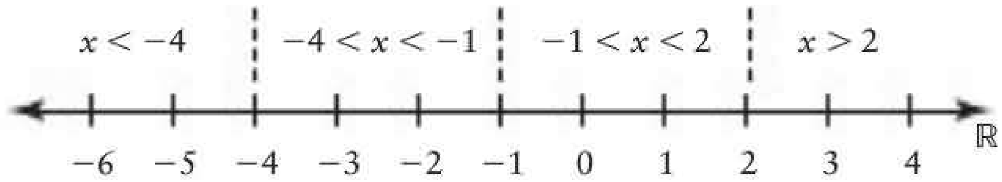
Combining the results of the four cases, the solution is $-4 \le x \le -1$ or $x \ge 2$.

Method 2: *Use Intervals*

$-2(x + 4)(x - 2)(x + 1) \le 0$

The roots of $-2(x + 4)(x - 2)(x + 1) = 0$ are $x = -4$, $x = -1$, and $x = 2$.

Use the roots to break the number line into four intervals.



Test arbitrary values of x in each interval.

For $x < -4$, test $x = -5$.

$-2(-5 + 4)(-5 - 2)(-5 + 1) = 56$

Since $56 > 0$, $x < -4$ is not a solution.

For $-4 < x < -1$, test $x = -3$.

$-2(-3 + 4)(-3 - 2)(-3 + 1) = -20$

Since $-20 < 0$, $-4 < x < -1$ is a solution.

For $-1 < x < 2$, test $x = 0$.

$-2(0 + 4)(0 - 2)(0 + 1) = 16$

Since $16 > 0$, $-1 < x < 2$ is not a solution.

For $x > 2$, test $x = 3$.

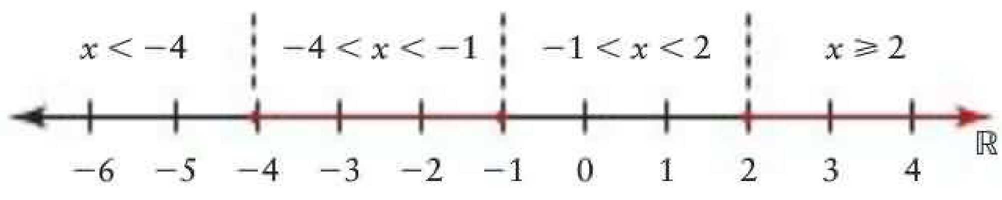
$-2(3 + 4)(3 - 2)(3 + 1) = -56$

Since $-56 < 0$, $x > 2$ is a solution.

Factor \ Interval	$x < -4$	$x = -4$	$-4 < x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$x > 2$
$-2(x + 4)$	+	0	−	−	−	−	−
$(x - 2)$	−	−	−	−	−	0	+
$(x + 1)(2x - 3)$	−	−	−	0	+	+	+
$-2(x + 4)(x - 2)(x + 1)$	+	0	−	0	+	0	−

The solution is $-4 \le x \le -1$ or $x \ge 2$.

This can be shown on a number line.



Example 3 Solve a Problem Involving a Factorable Polynomial Inequality

A rectangular in-ground pool is to be installed. The engineer overseeing the construction project estimates that at least 1408 m^3 of earth and rocks needs to be excavated. What are the minimum dimensions of the excavation if the depth must be 2 m more than one quarter of the width, and the length must be 12 m more than four times the width?

Solution

From the given information, $h = \frac{1}{4}w + 2$ and $l = 4w + 12$, with $l > 0$, $w > 0$, and $h > 0$.

$$\begin{aligned} V &= lwh \\ &= (4w + 12)(w)\left(\frac{1}{4}w + 2\right) \\ &= w^3 + 11w^2 + 24w \end{aligned}$$

CONNECTIONS

The formula for the volume, V, of a rectangular prism is $V = lwh$, where l is the length, w is the width, and h is the height.

Since the volume must be at least 1408 m^3, $V \ge 1408$; that is, $w^3 + 11w^2 + 24w \ge 1408$.

Solve $w^3 + 11w^2 + 24w - 1408 \ge 0$.

Factor the corresponding polynomial function.

$w^3 + 11w^2 + 24w - 1408 = (w - 8)(w^2 + 19w + 176)$

Then, solve $(w - 8)(w^2 + 19w + 176) \ge 0$. $w^2 + 19w + 176$ cannot be factored further.

CONNECTIONS

The discriminant can be used to test for factors. If $b^2 - 4ac$ is a perfect square then the quadratic can be factored. Here $b^2 - 4ac = -343$, so the expression has no real roots.

Case 1

Both factors are non-negative.

$w - 8 \ge 0$ $\qquad$ $w^2 + 19w + 176 \ge 0$

$w \ge 8$

$w^2 + 19w + 176 \ge 0$ is true for all values of w. These include values for $w \ge 8$.

So, $w \ge 8$ is a solution.

Case 2

Both factors are non-positive, and w is positive (because w represents the width).

$0 < w \leq 8$ $\qquad$ $w^2 + 19w + 176 \leq 0$

$w^2 + 19w + 176 \leq 0$ is not possible for any values of w. There is no solution.

So, the possible solution is $w \geq 8$.

When $w = 8$, $h = \frac{1}{4}(8) + 2 = 4$ and $l = 4(2) + 12 = 20$.

The dimensions of the excavation that give a volume of at least 1408 cm^3 are width 8 m, depth 4 m, and length 20 m.

KEY CONCEPTS

- Factorable inequalities can be solved algebraically by
  - considering all cases
  - using intervals and then testing values in each interval
- Tables and number lines can help organize intervals to provide a visual clue to solutions.

Communicate Your Understanding

C1 Why is it necessary to reverse an inequality sign when each side is multiplied or divided by a negative value? Support your answer with examples.

C2 What are the similarities between solving a linear inequality and solving a polynomial inequality?

C3 Which method is more efficient for solving factorable inequalities algebraically, using cases or using intervals? Explain.

A Practise

For help with question 1, refer to Example 1.

1. Solve each inequality. Show each solution on a number line.

a) $x + 3 \leq 5$ **b)** $2x + 1 > -4$

c) $5 - 3x \geq 6$ **d)** $7x < 4 + 3x$

e) $2 - 4x > 5x + 20$ **f)** $2(1 - x) \leq x - 8$

For help with questions 2 to 4, refer to Example 2.

2. Solve by considering all cases. Show each solution on a number line.

a) $(x + 2)(x - 4) > 0$

b) $(2x + 3)(4 - x) \leq 0$

3. Solve using intervals. Show each solution on a number line.

a) $(x + 3)(x - 2) > 0$

b) $(x - 6)(x - 9) \leq 0$

c) $(4x + 1)(2 - x) \geq 0$

4. Solve.

a) $(x + 2)(3 - x)(x + 1) < 0$

b) $(-x + 1)(3x - 1)(x + 7) \geq 0$

c) $(7x + 2)(1 - x)(2x + 5) > 0$

d) $(x + 4)(-3x + 1)(x + 2) \leq 0$

B Connect and Apply

5. Solve by considering all cases. Show each solution on a number line.

a) $x^2 - 8x + 15 \geq 0$

b) $x^2 - 2x - 15 < 0$

c) $15x^2 - 14x - 8 \leq 0$

d) $x^3 - 2x^2 - 5x + 6 < 0$

e) $2x^3 + 3x^2 - 2x - 3 \geq 0$

6. Solve using intervals.

a) $x^3 + 6x^2 + 7x + 12 \geq 0$

b) $x^3 + 9x^2 + 26x + 24 < 0$

c) $5x^3 - 12x^2 - 11x + 6 \leq 0$

d) $6x^4 - 7x^3 - 4x^2 + 8x + 12 > 0$

7. Solve.

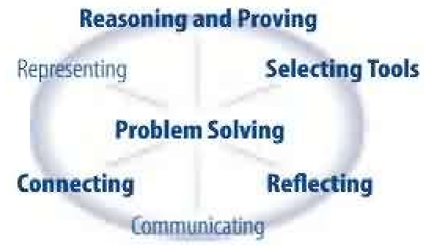
a) $x^2 + 4x - 5 \leq 0$

b) $-2x^3 + x^2 + 13x + 6 < 0$

c) $2x^3 + x^2 - 2x - 1 > 0$

d) $x^3 - 5x + 4 \geq 0$

For help with questions 8 and 9, refer to Example 3.



8. Cookies are packaged in boxes that measure 18 cm by 20 cm by 6 cm. A larger box is being designed by increasing the length, width, and height of the smaller box by the same length so that the volume is at least 5280 cm^3. What are the minimum dimensions of the larger box?

9. The price, p, in dollars, of a stock t years after 1999 can be modelled by the function $p(t) = 0.5t^3 - 5.5t^2 + 14t$. When will the price of the stock be more than \$90?

✓ Achievement Check

10. a) Solve the inequality $x^3 - 5x^2 + 2x + 8 < 0$ by

i) using intervals

ii) considering all cases

b) How are the two methods the same? How are they different?

C Extend and Challenge

11. a) How many cases must be considered when solving $(x + 4)(x - 2)(x + 1)(x - 1) \leq 0$? Justify your answer.

b) Would it be more efficient to solve this inequality using intervals? Justify your answer.

12. Solve $x^5 + 7x^3 + 6x < 5x^4 + 7x^2 + 2$.

13. A demographer develops a model for the population, P, of a small town n years from today such that $P(n) = -0.15n^5 + 3n^4 + 5560$.

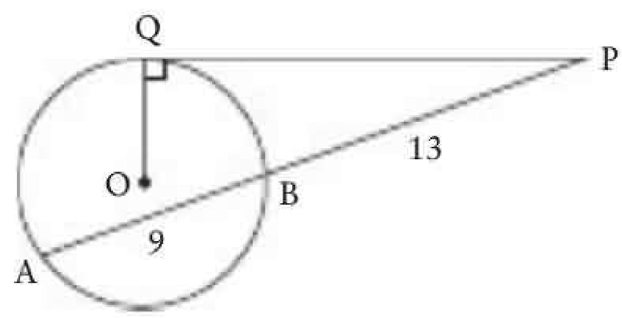
a) When will the population of the town be between 10 242 and 25 325?

b) When will the population of the town be more than 30 443?

c) Will the model be valid after 20 years? Explain.

14. Write two possible quartic inequalities, one using the less than or equal to symbol ($\leq$) and the other using the greater than or equal to symbol ($\geq$), that correspond to the following solution:
$-6 - \sqrt{2} < x < -6 + \sqrt{2}$ or
$6 - \sqrt{2} < x < 6 + \sqrt{2}$

15. Math Contest Determine the exact length of PQ in the figure.



16. Math Contest Determine an equation for the line that is tangent to the circle with equation $x^2 + y^2 - 25 = 0$ and passes through the point $(4, -3)$.

Chapter 2 REVIEW

2.1 The Remainder Theorem

1. i) Use the remainder theorem to determine the remainder for each division.

ii) Perform each division. Express the result in quotient form. Identify any restrictions on the variable.

a) $x^3 + 9x^2 - 5x + 3$ divided by $x - 2$

b) $12x^3 - 2x^2 + x - 11$ divided by $3x + 1$

c) $-8x^4 - 4x + 10x^3 - x^2 + 15$ divided by $2x - 1$

2. a) Determine the value of k such that when $f(x) = x^4 + kx^3 - 3x - 5$ is divided by $x - 3$, the remainder is -10.

b) Determine the remainder when $f(x)$ is divided by $x + 3$.

c) **Use Technology** Verify your answer in part b) using technology.

3. For what value of b will the polynomial $P(x) = 4x^3 - 3x^2 + bx + 6$ have the same remainder when it is divided by $x - 1$ and by $x + 3$?

2.2 The Factor Theorem

4. Factor each polynomial.

a) $x^3 - 4x^2 + x + 6$

b) $3x^3 - 5x^2 - 26x - 8$

c) $5x^4 + 12x^3 - 101x^2 + 48x + 36$

5. Factor.

a) $-4x^3 - 4x^2 + 16x + 16$

b) $25x^3 - 50x^2 - 9x + 18$

c) $2x^4 + 5x^3 - 8x^2 - 20x$

6. Rectangular blocks of limestone are to be cut up and used to build the front entrance of a new hotel. The volume, V, in cubic metres, of each block can be modelled by the function $V(x) = 2x^3 + 7x^2 + 2x - 3$.

a) Determine the dimensions of the blocks in terms of x.

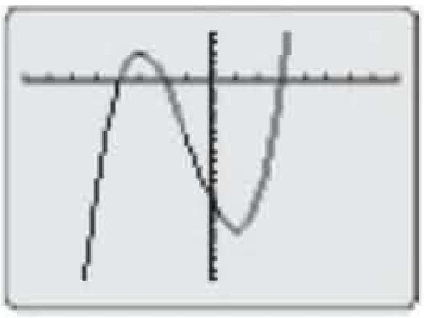
b) What are the possible dimensions of the blocks when $x = 1$?

7. Determine the value of k so that $x + 3$ is a factor of $x^3 + 4x^2 - 2kx + 3$.

2.3 Polynomial Equations

8. Use the graph to determine the roots of the corresponding polynomial equation.

Window variables: $x \in [-8, 8]$, $y \in [-40, 10]$, Yscl = 2



9. Determine the real roots of each equation.

a) $(5x^2 + 20)(3x^2 - 48) = 0$

b) $(2x^2 - x - 13)(x^2 + 1) = 0$

10. Solve. Round answers to one decimal place, if necessary.

a) $7x^3 + 5x^2 - 5x - 3 = 0$

b) $-x^3 + 9x^2 = x + 6$

11. The specifications for a cardboard box state that the width is 5 cm less than the length, and the height is 1 cm more than double the length. Write an equation for the volume of the box and find possible dimensions for a volume of 550 cm^3.

2.4 Families of Polynomial Functions

12. Examine the following functions. Which function does not belong to the same family? Explain.

A $y = 3.5(x + 2)(x - 1)(x - 3)$

B $y = -0.2(x - 3)(2x + 4)(2x - 3)$

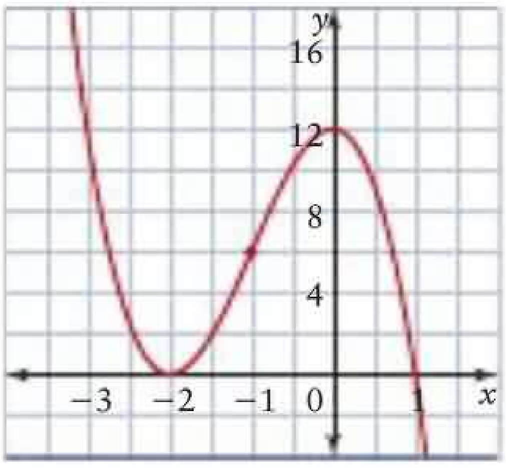
C $y = (4x - 12)(x + 2)(x - 1)$

D $y = -7(x - 1)(x - 3)(x + 2)$

13. a) Determine an equation, in simplified form, for the family of cubic functions with zeros $2 \pm \sqrt{5}$ and 0.

b) Determine an equation for the member of the family with graph passing through the point (2, 20).

14. Determine an equation for the function represented by this graph.



2.5 Solving Inequalities Using Technology

15. **Use Technology** Solve. Round the zeros to one decimal place, if necessary.

a) $x^2 + 3x - 5 \geq 0$

b) $2x^3 - 13x^2 + 17x + 12 > 0$

c) $x^3 - 2x^2 - 5x + 2 < 0$

d) $3x^3 + 4x^2 - 35x - 12 \leq 0$

e) $-x^4 - 2x^3 + 4x^2 + 10x + 5 < 0$

16. **Use Technology** A section of a water tube ride at an amusement park can be modelled by the function $h(t) = -0.002t^4 + 0.104t^3 - 1.69t^2 + 8.5t + 9$, where t is the time, in seconds, and h is the height, in metres, above the ground. When will the riders be more than 15 m above the ground?

2.6 Solving Factorable Polynomial Inequalities Algebraically

17. Solve each inequality. Show the solution on a number line.

a) $(5x + 4)(x - 4) < 0$

b) $-(2x + 3)(x - 1)(3x - 2) \leq 0$

c) $(x^2 + 4x + 4)(x^2 - 25) > 0$

18. Solve by factoring.

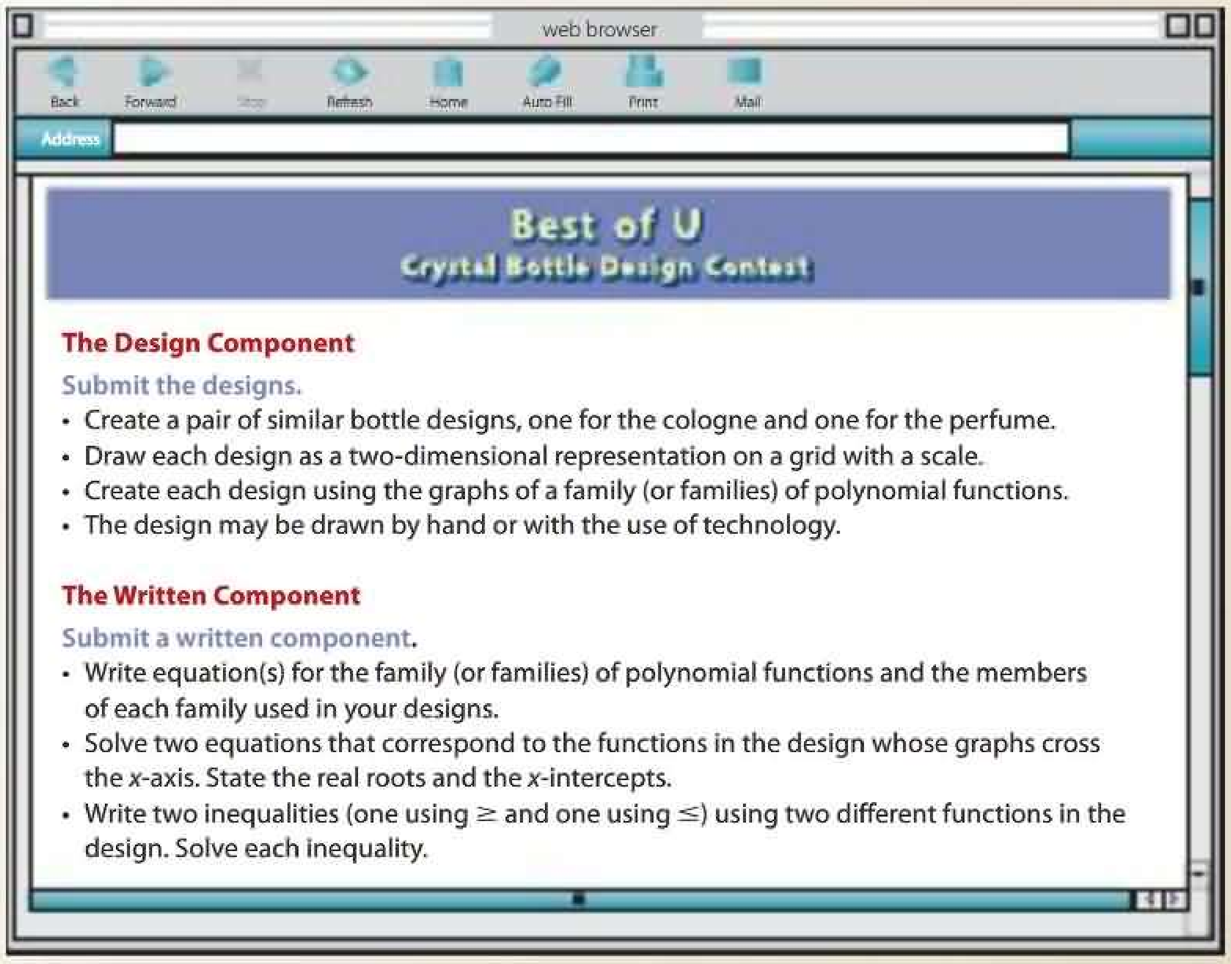
a) $12x^2 + 25x - 7 \geq 0$

b) $6x^3 + 13x^2 - 41x + 12 \leq 0$

c) $-3x^4 + 10x^3 + 20x^2 - 40x + 32 < 0$

CHAPTER PROBLEM WRAP-UP

Best of U has developed a new cologne and perfume and is now searching for artistic designs for the crystal bottles that will contain these products. The company has decided to run a promotional design contest on its Web site. Here are the specifications. Your task is to prepare an entry for the contest.



Best of U
Crystal Bottle Design Contest

The Design Component

Submit the designs.

- Create a pair of similar bottle designs, one for the cologne and one for the perfume.
- Draw each design as a two-dimensional representation on a grid with a scale.
- Create each design using the graphs of a family (or families) of polynomial functions.
- The design may be drawn by hand or with the use of technology.

The Written Component

Submit a written component.

- Write equation(s) for the family (or families) of polynomial functions and the members of each family used in your designs.
- Solve two equations that correspond to the functions in the design whose graphs cross the x-axis. State the real roots and the x-intercepts.
- Write two inequalities (one using $\geq$ and one using $\leq$) using two different functions in the design. Solve each inequality.

Chapter 2 PRACTICE TEST

For questions 1 to 3, select the best answer.

1. Which statement is true for $P(x) = 5x^3 + 4x^2 - 3x + 2$?

A When $P(x)$ is divided by $x + 1$, the remainder is 8.

B $x + 2$ is a factor of $P(x)$.

C $P(-2) = -16$

D $P(x) = (x + 1)(5x^2 - x - 2) - 4$

2. Which of the following is not a factor of $2x^3 - 5x^2 - 9x + 18$?

A $2x - 3$

B $x + 2$

C $x - 2$

D $x - 3$

3. Which set of values for x should be tested to determine the possible zeros of $4x^3 + 5x^2 - 23x - 6$?

A $\pm1, \pm2, \pm3, \pm4, \pm6$

B $\pm1, \pm2, \pm3, \pm4, \pm6, \pm\frac{1}{2}, \pm\frac{2}{3}$

C $\pm4, \pm6, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm\frac{1}{4}$

D $\pm1, \pm2, \pm3, \pm6, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm\frac{1}{4}$

4. a) Divide $x^3 - 4x^2 + 3x - 7$ by $x + 3$. Express the result in quotient form.

b) Identify any restrictions on the variable.

c) Write the corresponding statement that can be used to check the division.

d) Verify your answer.

5. a) Determine the value of k such that when $f(x) = x^4 + kx^3 - 2x^2 + 1$ is divided by $x + 2$, the remainder is 5.

b) Determine the remainder when $f(x)$ is divided by $x + 4$.

c) Verify your answer in part b) using long division.

6. Factor.

a) $x^3 - 5x^2 + 2x + 8$

b) $x^3 + 2x^2 - 9x - 18$

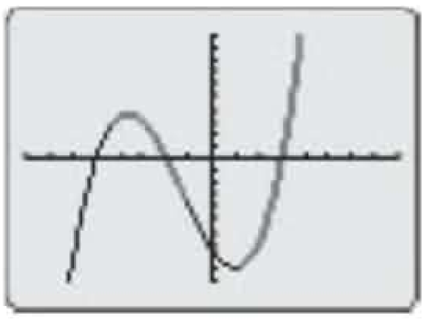
c) $x^3 + 5x^2 - 2x - 24$

d) $5x^3 + 7x^2 - 8x - 4$

e) $x^3 + 9x^2 + 26x + 24$

f) $2x^4 + 13x^3 + 28x^2 + 23x + 6$

7. Use the graph to determine the roots of the corresponding polynomial equation.
Window variables: $x \in [-8, 8]$,
$y \in [-40, 40]$, Yscl = 4



8. Determine the real roots of each equation.

a) $(x^2 + 5)(x - 2) = 0$

b) $(x^2 - 121)(x^2 + 16) = 0$

c) $(x^2 - 2x + 3)(2x^2 - 50) = 0$

d) $(3x^2 - 27)(x^2 - 3x - 10) = 0$

9. Solve by factoring.

a) $x^3 + 4x^2 + 5x + 2 = 0$

b) $x^3 - 13x + 12 = 0$

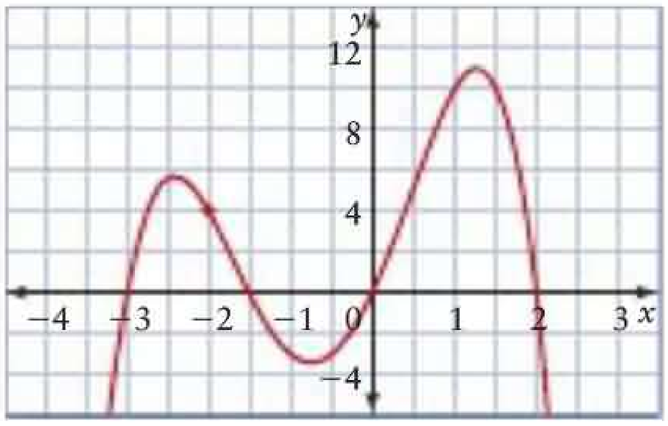
c) $32x^3 - 48x^2 - 98x + 147 = 0$

d) $45x^4 - 27x^3 - 20x^2 + 12x = 0$

10. a) Describe the similarities and differences between polynomial equations, polynomial functions, and polynomial inequalities. Support your answer with examples.

b) What is the relationship between roots, zeros, and x-intercepts? Support your answer with examples.

11. a) Determine an equation for the quartic function represented by this graph.

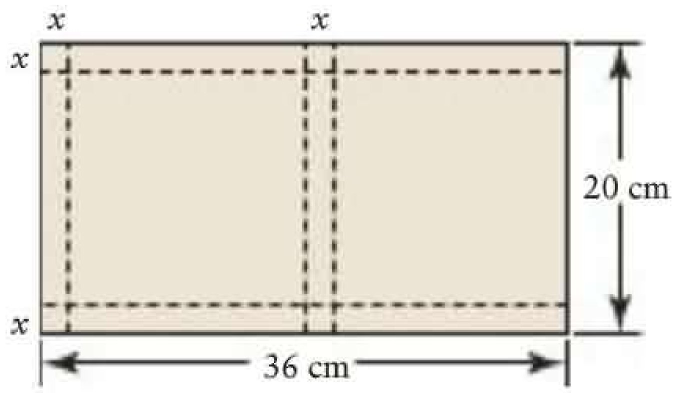


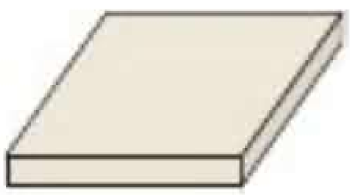
b) Use the graph to identify the intervals on which the function is below the x-axis.

12. a) Determine an equation, in simplified form, for the family of quartic functions with zeros 5 (order 2) and $-2 \pm \sqrt{6}$.

b) Determine an equation for the member of the family whose graph has a y-intercept of 20.

13. Boxes for chocolates are to be constructed from cardboard sheets that measure 36 cm by 20 cm. Each box is formed by folding a sheet along the dotted lines, as shown.





a) Express the volume of the box as a function of x.

b) Determine the possible dimensions of the box if the volume is to be 450 cm^3. Round answers to the nearest tenth of a centimetre.

c) Write an equation for the family of functions that corresponds to the function in part a).

d) Sketch graphs of two members of this family on the same coordinate grid.

14. Use Technology Solve. Round answers to one decimal place.

a) $x^3 + 3x \leq 8x^2 - 9$

b) $-x^4 + 3x^3 + 9x^2 > 5x + 5$

15. Use Technology Solve each inequality.

a) $x^3 + 3x^2 - 4x - 7 < 0$

b) $2x^4 + 5x^3 - 3x^2 - 15x - 9 \geq 0$

16. Solve by factoring.

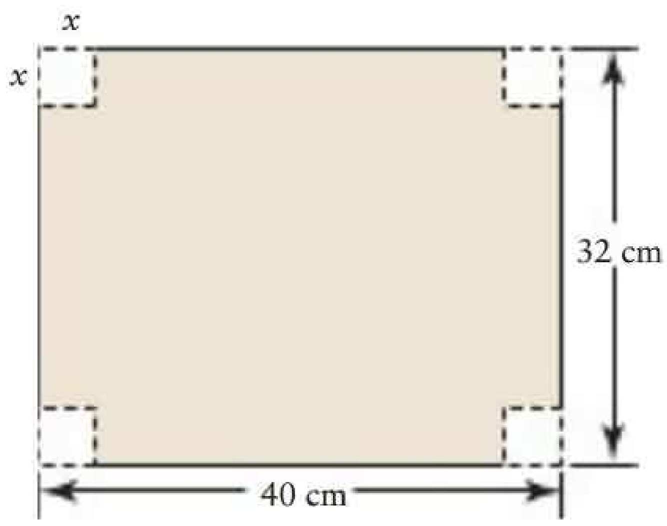
a) $9x^2 - 16 < 0$

b) $-x^3 + 6x^2 - 9x > 0$

c) $2x^3 + 5x^2 - 18x - 45 \leq 0$

d) $2x^4 + 5x^3 - 8x^2 - 17x - 6 \geq 0$

17. A open-top box is to be constructed from a piece of cardboard by cutting congruent squares from the corners and then folding up the sides. The dimensions of the cardboard are shown.



a) Express the volume of the box as a function of x.

b) Write an equation that represents the box with volume

i) twice the volume of the box represented by the function in part a)

ii) half the volume of the box represented by the function in part a)

c) How are the equations in part b) related to the one in part a)?

d) Use your function from part a) to determine the values of x that will result in boxes with a volume greater than 2016 cm^3.

TASK

Can You Tell Just by Looking?

a) Determine the roots of each equation.

i) $x^2 - 3x + 2 = 0$

ii) $x^3 - 6x^2 + 11x - 6 = 0$

iii) $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$

iv) $x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 = 0$

b) Use technology to graph the function $y = f(x)$ that corresponds to each equation in part a). Confirm that the zeros of the function are the roots that you found for the corresponding equation.

c) Determine and explain patterns in the roots.

d) Conjecture clues in the equation that may help you predict the pattern in the roots.

e) Test your conjecture on the following equations.

i) $x^3 - 7x + 6 = 0$

ii) $x^4 - x^3 - 7x^2 + 13x - 6 = 0$

iii) $x^6 - 21x^5 + 175x^4 - 735x^3 + 1624x^2 - 1764x + 720 = 0$