Chapter 1

Polynomial Functions

Linear and quadratic functions are members of a larger group of functions known as polynomial functions. In business, the revenue, profit, and demand can be modelled by polynomial functions. An architect may design bridges or other structures using polynomial curves, while a demographer may predict population trends using polynomial functions.

This chapter focuses on the properties and key features of graphs of polynomial functions and their transformations. You will also be introduced to the concepts of average and instantaneous rate of change.

By the end of this chapter, you will

- recognize a polynomial expression and the equation of a polynomial function, and identify linear and quadratic functions as examples of polynomial functions (C1.1)
- compare, through investigation, the numeric, graphical, and algebraic representations of polynomial functions (C1.2)
- describe key features of the graphs of polynomial functions (C1.3)
- distinguish polynomial functions from sinusoidal and exponential functions (C1.4)
- investigate connections between a polynomial function given in factored form and the x-intercepts of its graph, and sketch the graph of a polynomial function given in factored form using its key features (C1.5)
- investigate the roles of the parameters a, k, d, and c in functions of the form $y = af[k(x - d)] + c$ and describe these roles in terms of transformations on the functions $f(x) = x^3$ and $f(x) = x^4$ (C1.6)
- determine an equation of a polynomial function that satisfies a given set of conditions (C1.7)
- investigate properties of even and odd polynomial functions, and determine whether a given polynomial function is even, odd, or neither (C1.9)
- investigate and recognize examples of a variety of representations of average rate of change and instantaneous rate of change (D1.1, D1.2, D1.3, D1.6)
- calculate and interpret average rates of change of functions, given various representations of the functions (D1.4)
- make connections between average rate of change and the slope of a secant, and instantaneous rate of change and the slope of a tangent (D1.7)
- recognize examples of instantaneous rates of change arising from real-world situations, and make connections between instantaneous rates of change and average rates of change (D1.5)
- solve real-world problems involving average and instantaneous rate of change (D1.9)

Prerequisite Skills

Function Notation

1. Determine each value for the function $f(x) = -4x + 7$.

a) $f(0)$ **b)** $f(3)$ **c)** $f(-1)$

d) $f\left(\frac{1}{2}\right)$ **e)** $f(-2x)$ **f)** $f(3x)$

2. Determine each value for the function $f(x) = 2x^2 - 3x + 1$.

a) $f(0)$ **b)** $f(3)$ **c)** $f(-1)$

d) $f\left(\frac{1}{2}\right)$ **e)** $3f(2x)$ **f)** $f(3x)$

Slope and y-intercept of a Line

3. State the slope and the y-intercept of each line.

a) $y = 3x + 2$ **b)** $4y = 6 - 2x$

c) $5x - y + 7 = 0$ **d)** $y + 6 = -5(x + 1)$

e) $-(x + 4) = 2(y - 3)$

Equation of a Line

4. Determine an equation for the line that satisfies each set of conditions.

a) The slope is 3 and the y-intercept is 5.

b) The x-intercept is -1 and the y-intercept is 4.

c) The slope is -4 and the line passes through the point $(7, 3)$.

d) The line passes through the points $(2, -2)$ and $(1, 5)$.

Finite Differences

5. Use finite differences to determine if each function is linear, quadratic, or neither.

a)

x	y
−2	−7
−1	−5
0	−3
1	−1
2	1
3	3
4	5

b)

x	y
−1	−8
0	−2
1	−1
2	5
3	7
4	13
5	20

c)

x	y
−4	−12
−3	−5
−2	0
−1	3
0	4
1	3
2	0

Domain and Range

6. State the domain and range of each function. Justify your answer.

a) $y = 2(x - 3)^2 + 1$

b) $y = \dfrac{1}{x + 5}$

c) $y = \sqrt{1 - 2x}$

Quadratic Functions

7. Determine the equation of a quadratic function that satisfies each set of conditions.

a) x-intercepts 1 and -1, y-intercept 3

b) x-intercept 3, and passing through the point $(1, -2)$

c) x-intercepts $-\frac{1}{2}$ and 2, y-intercept -4

8. Determine the x-intercepts, the vertex, the direction of opening, and the domain and range of each quadratic function. Then, graph the function.
   a) $y = (x + 6)(2x - 5)$
   b) $y = -2(x - 4)^2 + 1$
   c) $y = -\frac{1}{4}(x - 3)^2 + 5$
   d) $y = 5x^2 + 7x - 6$
   e) $y = -3x^2 + 5x - 2$

Transformations

9. Identify each transformation of the function $y = f(x)$ as a vertical or horizontal translation, a stretch or compression, or a reflection in the x-axis or y-axis, or any combination of these.
   a) $y = -4f(x)$
   b) $y = \frac{1}{3}f(x)$
   c) $y = f(2x)$
   d) $y = f\left(-\frac{1}{3}x\right)$
   e) $y = f(-x)$

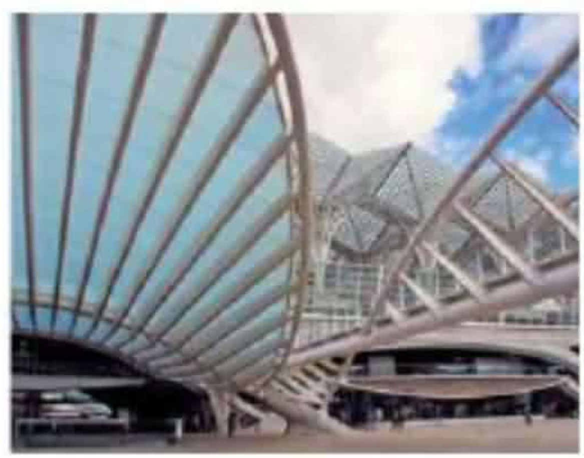
10. i) Write an equation for the transformed function of each base function.
    ii) Sketch a graph of each function.
    iii) State the domain and range.
    a) $f(x) = x$ is translated 2 units to the left and 3 units up.
    b) $f(x) = x^2$ is stretched vertically by a factor of 5, reflected in the x-axis, and translated 2 units down and 1 unit to the left.
    c) $f(x) = x$ is compressed horizontally by a factor of $\frac{1}{2}$, stretched vertically by a factor of 3, reflected in the x-axis, and translated 4 units to the left and 6 units up.

11. i) Describe the transformations that must be applied to the graph of each base function, $f(x)$, to obtain the given transformed function.
    ii) Write an equation for the transformed function.
    a) $f(x) = x$, $y = -2f(x + 3) + 1$
    b) $f(x) = x^2$, $y = \frac{1}{3}f(x) - 2$

12. Describe the transformations that must be applied to the base function $y = x^2$ to obtain the function $y = 3\left[-\frac{1}{2}(x - 1)\right]^2 + 2$.

CHAPTER PROBLEM

Mathematical shapes and curves surround us. They are found in the designs of buildings, bridges, vehicles, furniture, containers, jewellery, games, cake decorations, fabrics, amusement parks, golf courses, art, and almost everywhere else! Some careers that involve working with mathematical designs are civil engineering, architectural design, computer graphics design, interior design, and landscape architecture.



Throughout this chapter, you will explore how the curves represented by polynomial functions are applied in various design-related fields.

1.1 Power Functions

A rock that is tossed into the water of a calm lake creates ripples that move outward in a circular pattern. The area, A, spanned by the ripples can be modelled by the function $A(r) = \pi r^2$, where r is the radius. The volume, V, of helium in a spherical balloon can be modelled by the function $V(r) = \frac{4}{3}\pi r^3$, where r is again the radius. The functions that represent each situation are called **power functions**. A power function is the simplest type of **polynomial function** and has the form $f(x) = ax^n$, where x is a variable, a is a real number, and n is a whole number.

CONNECTIONS

Polynomials are the building blocks of algebra. Polynomial functions can be used to create a variety of other types of functions and are important in many areas of mathematics, including calculus and numerical analysis. Outside mathematics, the basic equations in economics and many physical sciences are polynomial equations.

A polynomial expression is an expression of the form

$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0$,

where

- n is a whole number
- x is a variable
- the **coefficients** $a_0, a_1, \ldots, a_n$ are real numbers
- the **degree** of the function is n, the exponent of the greatest power of x
- a_n, the coefficient of the greatest power of x, is the **leading coefficient**
- a_0, the term without a variable, is the **constant term**

A **polynomial function** has the form

$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0$

Polynomial functions are typically written in descending order of powers of x. The exponents in a polynomial do not need to decrease consecutively; that is, some terms may have zero as a coefficient. For example, $f(x) = 4x^3 + 2x - 1$ is still a polynomial function even though there is no x^2-term. A constant function, of the form $f(x) = a_0$, is also a type of polynomial function (where $n = 0$), as you can write the constant term a_0 in the form a_0x^0.

Investigate — What are the key features of the graphs of power functions?

Tools

- graphing calculator

1. Match each graph with the corresponding function. Justify your choices. Use a graphing calculator if necessary.

a) $y = x$

b) $y = x^2$

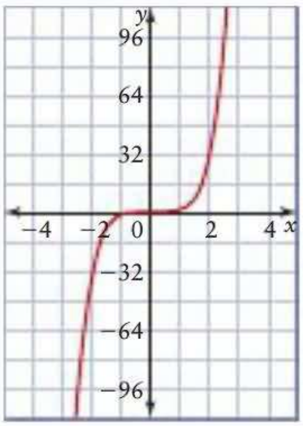
c) $y = x^3$

d) $y = x^4$

e) $y = x^5$

f) $y = x^6$

i)



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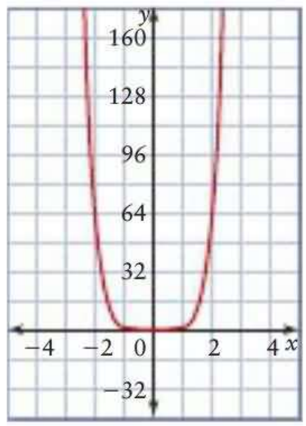

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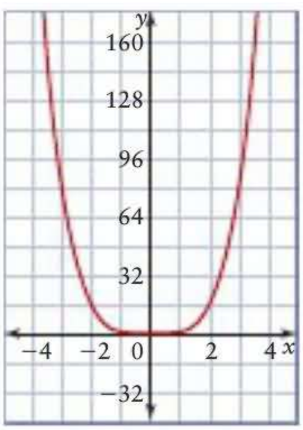

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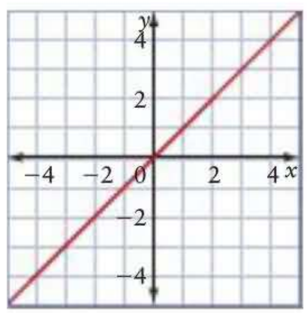

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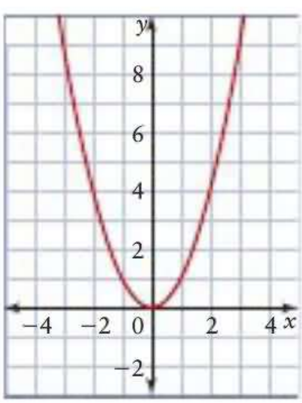

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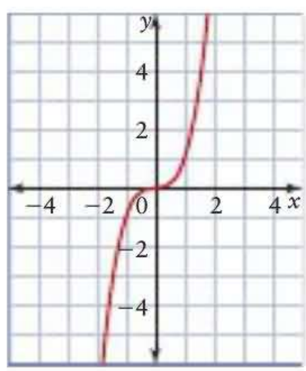


CONNECTIONS

Some power functions have special names that are associated with their degree.

Power Function	Degree	Name
$y = a$	0	constant
$y = ax$	1	linear
$y = ax^2$	2	quadratic
$y = ax^3$	3	cubic
$y = ax^4$	4	quartic
$y = ax^5$	5	quintic

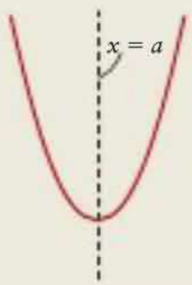
CONNECTIONS

Recall that a relation is a function if for every x-value there is only one y-value. The graph of a relation represents a function if it passes the vertical line test, that is, if a vertical line drawn anywhere along the graph intersects that graph at no more than one point.

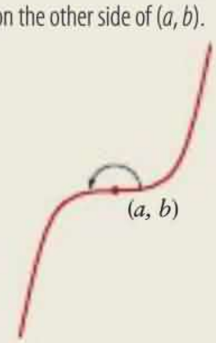
The **end behaviour** of the graph of a function is the behaviour of the y-values as x increases (that is, as x approaches positive infinity, written as $x \to \infty$) and as x decreases (that is, as x approaches negative infinity, written as $x \to -\infty$).

CONNECTIONS

- A graph has **line symmetry** if there is a line $x = a$ that divides the graph into two parts such that each part is a reflection of the other in the line $x = a$.



- A graph has **point symmetry** about a point (a, b) if each part of the graph on one side of (a, b) can be rotated 180° to coincide with part of the graph on the other side of (a, b).



2. a) **Reflect** Decide whether each graph in step 1 represent a linear, a quadratic, a cubic, a quartic, or a quintic function. Justify your answer.

b) **Reflect** Explain why each graph in step 1 represents a function.

3. a) State these key features for each graph:

i) the domain

ii) the range

iii) the intercepts

b) Describe the end behaviour of each graph as

i) $x \to \infty$

ii) $x \to -\infty$

4. a) Which graphs have both ends extending upward in quadrants 1 and 2 (that is, start high and end high)?

b) Decide whether each graph has line symmetry or point symmetry. Explain.

c) **Reflect** Describe how the graphs are similar. How are the equations similar?

5. a) Which graphs have one end extending downward in quadrant 3 (start low) and the other end extending upward in quadrant 1 (end high)?

b) Decide whether each graph has line symmetry or point symmetry. Explain.

c) **Reflect** Describe how the graphs are similar. How are the equations similar?

6. **Reflect** Summarize your findings for each group of power functions in a table like this one.

Key Features of the Graph	$y = x^n$, n is odd	$y = x^n$, n is even
Domain		
Range		
Symmetry		
End Behaviour		

7. a) Graph the function $y = x^n$ for $n = 2$, 4, and 6.

b) Describe the similarities and differences between the graphs.

c) **Reflect** Predict what will happen to the graph of $y = x^n$ for larger even values of n.

d) Check your prediction in part c) by graphing two other functions of this form.

8. a) Graph the function $y = x^n$ for $n = 1$, 3, and 5.

b) Describe the similarities and differences between the graphs.

c) **Reflect** Predict what will happen to the graph of $y = x^n$ for larger odd values of n.

d) Check your prediction in part c) by graphing two more functions of this form.

Example 1 Recognize Polynomial Functions

Determine which functions are polynomials. Justify your answer. State the degree and the leading coefficient of each polynomial function.

a) $g(x) = \sin x$

b) $f(x) = 2x^4$

c) $y = x^3 - 5x^2 + 6x - 8$

d) $g(x) = 3^x$

Solution

a) $g(x) = \sin x$

This a trigonometric function, not a polynomial function.

b) $f(x) = -2x^4$

This is a polynomial function of degree 4. The leading coefficient is -2.

c) $y = x^3 - 5x^2 + 6x - 8$

This is a polynomial function of degree 3. The leading coefficient is 1.

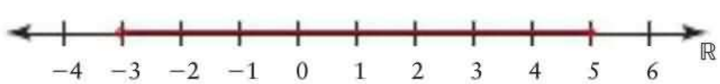
d) $g(x) = 3^x$

This is not a polynomial function but an exponential function, since the base is a number and the exponent is a variable.

Interval Notation

In this course, you will often describe the features of the graphs of a variety of types of functions in relation to real-number values. Sets of real numbers may be described in a variety of ways:

- as an inequality, $-3 < x \leq 5$
- in interval (or bracket) notation $(-3, 5]$



- graphically, on a number line

Intervals that are infinite are expressed using the symbol ∞ (infinity) or $-\infty$ (negative infinity).

Square brackets indicate that the end value is included in the interval, and round brackets indicate that the end value is not included.

A round bracket is used at infinity since the symbol ∞ means "without bound."

Below is a summary of all possible intervals for real numbers a and b, where $a < b$.

Bracket Interval	Inequality	Number Line	In Words
			The set of all real numbers x such that
(a, b)	$a < x < b$	a b $\mathbb{R}$	x is greater than a and less than b
$(a, b]$	$a < x \leq b$	a b $\mathbb{R}$	x is greater than a and less than or equal to b
$[a, b)$	$a \leq x < b$	a b $\mathbb{R}$	x is greater than or equal to a and less than b
$[a, b]$	$a \leq x \leq b$	a b $\mathbb{R}$	x is greater than or equal to a and less than or equal to b
$[a, \infty)$	$x \geq a$	a $\mathbb{R}$	x is greater than or equal to a
$(-\infty, a]$	$x \leq a$	a $\mathbb{R}$	x is less than or equal to a
(a, ∞)	$x > a$	a $\mathbb{R}$	x is greater than a
$(-\infty, a)$	$x < a$	a $\mathbb{R}$	x is less than a
$(-\infty, \infty)$	$-\infty < x < \infty$	$\mathbb{R}$	x is an element of the real numbers

Example 2 Connect the Equations and Features of the Graphs of Power Functions

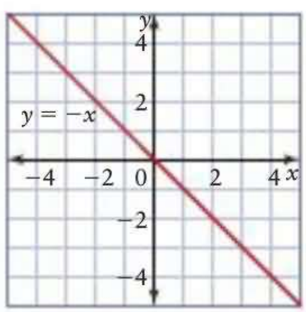
For each function

i) state the domain and range

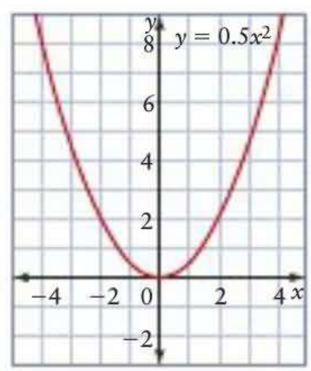
ii) describe the end behaviour

iii) identify any symmetry

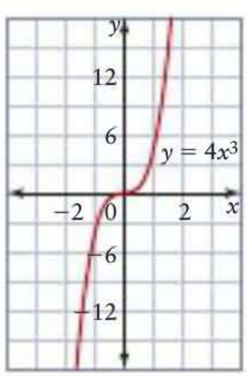
a)



b)



c)



Solution

a) $y = -x$

i) domain $\{x \in \mathbb{R}\}$ or $(-\infty, \infty)$; range $\{y \in \mathbb{R}\}$ or $(-\infty, \infty)$

ii) The graph extends from quadrant 4 to quadrant 2.
Thus, as $x \to -\infty$, $y \to \infty$, and as $x \to \infty$, $y \to -\infty$.

iii) The graph has point symmetry about the origin $(0, 0)$.

b) $y = 0.5x^2$

i) domain $\{x \in \mathbb{R}\}$ or $(-\infty, \infty)$; range $\{y \in \mathbb{R}, y \geq 0\}$ or $[0, \infty)$

ii) The graph extends from quadrant 2 to quadrant 1.
Thus, as $x \to -\infty$, $y \to \infty$, and as $x \to \infty$, $y \to \infty$.

iii) The graph has line symmetry in the y-axis.

c) $y = 4x^3$

i) domain $\{x \in \mathbb{R}\}$ or $(-\infty, \infty)$; range $\{y \in \mathbb{R}\}$ or $(-\infty, \infty)$

ii) The graph extends from quadrant 3 to quadrant 1.
Thus, as $x \to -\infty$, $y \to -\infty$, and as $x \to \infty$, $y \to \infty$.

iii) The graph has point symmetry about the origin.

Example 3 Describe the End Behaviour of Power Functions

Write each function in the appropriate row of the second column of the table. Give reasons for your choices.

$y = 2x$ $\quad$ $y = 5x^6$ $\quad$ $y = -3x^2$ $\quad$ $y = x^7$

$y = -\frac{2}{5}x^9$ $\quad$ $y = -4x^5$ $\quad$ $y = x^{10}$ $\quad$ $y = -0.5x^8$

End Behaviour	Function	Reasons
Extends from quadrant 3 to quadrant 1		
Extends from quadrant 2 to quadrant 4		
Extends from quadrant 2 to quadrant 1		
Extends from quadrant 3 to quadrant 4		

Solution

End Behaviour	Function	Reasons
Extends from quadrant 3 to quadrant 1	$y = 2x, y = x^7$	odd exponent, positive coefficient
Extends from quadrant 2 to quadrant 4	$y = -\frac{2}{5}x^9, y = -4x^5$	odd exponent, negative coefficient
Extends from quadrant 2 to quadrant 1	$y = 5x^6, y = x^{10}$	even exponent, positive coefficient
Extends from quadrant 3 to quadrant 4	$y = -3x^2, y = -0.5x^8$	even exponent, negative coefficient

Example 4 Connecting Power Functions and Volume

Helium is pumped into a large spherical balloon designed to advertise a new product. The volume, V, in cubic metres, of helium in the balloon is given by the function $V(r) = \frac{4}{3}\pi r^3$, where r is the radius of the balloon, in metres, and $r \in [0, 5]$.

a) Graph $V(r)$.

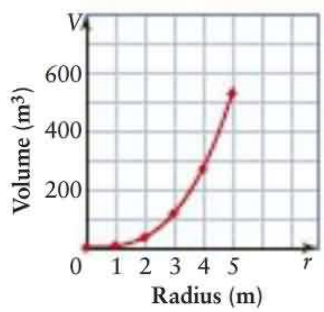
b) State the domain and range in this situation.

c) Describe the similarities and differences between the graph of $V(r)$ and the graph of $y = x^3$.

Solution

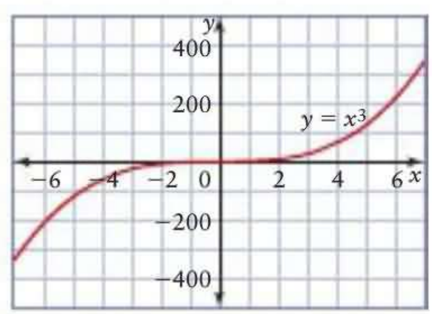
a) Make a table of values, plot the points, and connect them using a smooth curve.

r (m)	$V(r) = \frac{4}{3}\pi r^3$ (m^3)
0	$\frac{4}{3}\pi(0)^3 = 0$
1	$\frac{4}{3}\pi(1)^3 \doteq 4.2$
2	$\frac{4}{3}\pi(2)^3 \doteq 33.5$
3	$\frac{4}{3}\pi(3)^3 \doteq 113.1$
4	$\frac{4}{3}\pi(4)^3 \doteq 268.1$
5	$\frac{4}{3}\pi(5)^3 \doteq 523.6$



b) The domain is $r \in [0, 5]$. The range is approximately $V \in [0, 523.6]$.

c) The graph of $y = x^3$ is shown.



Similarities: The functions $V(r) = \frac{4}{3}\pi r^3$ and $y = x^3$ are both cubic, with positive leading coefficients. Both graphs pass through the origin (0, 0) and have one end that extends upward in quadrant 1.

Differences: the graph of $V(r)$ has a restricted domain. Since the two functions are both cubic power functions that have different leading coefficients, all points on each graph, other than (0, 0), are different.

KEY CONCEPTS

- A polynomial expression has the form
  $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0$
  where
  - n is a whole number
  - x is a variable
  - the coefficients $a_0, a_1, \ldots, a_n$ are real numbers
  - the degree of the function is n, the exponent of the greatest power of x
  - a_n, the coefficient of the greatest power of x, is the leading coefficient
  - a_0, the term without a variable, is the constant term
- A polynomial function has the form
  $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$
- A power function is a polynomial of the form $y = ax^n$, where n is a whole number.
- Power functions have similar characteristics depending on whether their degree is even or odd.
- Even-degree power functions have line symmetry in the y-axis, $x = 0$.
- Odd-degree power functions have point symmetry about the origin, (0, 0).

Communicate Your Understanding

C1 Explain why the function $y = 3$ is a polynomial function.

C2 How can you use a graph to tell whether the leading coefficient of a power function is positive or negative?

C3 How can you use a graph to tell whether the degree of a power function is even or odd?

C4 State a possible equation for a power function whose graph extends

a) from quadrant 3 to quadrant 1

b) from quadrant 2 to quadrant 1

c) from quadrant 2 to quadrant 4

d) from quadrant 3 to quadrant 4

A Practise

For help with questions 1 and 2, refer to Example 1.

1. Identify whether each is a polynomial function. Justify your answer.

a) $p(x) = \cos x$ **b)** $h(x) = -7x$

c) $f(x) = 2x^4$ **d)** $y = 3x^5 - 2x^3 + x^2 - 1$

e) $k(x) = 8^x$ **f)** $y = x^{-3}$

2. State the degree and the leading coefficient of each polynomial.

a) $y = 5x^4 - 3x^3 + 4$ **b)** $y = -x + 2$

c) $y = 8x^2$ **d)** $y = -\dfrac{x^3}{4} + 4x - 3$

e) $y = -5$ **f)** $y = x^2 - 3x$

For help with question 3, refer to Example 2.

3. Consider each graph.

i) Does it represent a power function of even degree? odd degree? Explain.

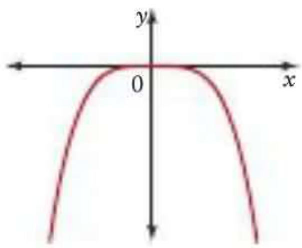
ii) State the sign of the leading coefficient. Justify your answer.

iii) State the domain and range.

iv) Identify any symmetry.

v) Describe the end behaviour.

a)



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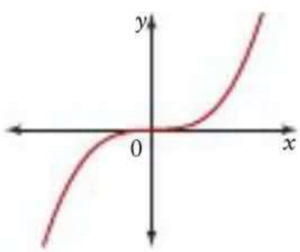
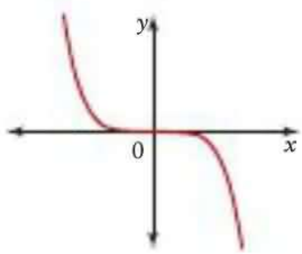


c)



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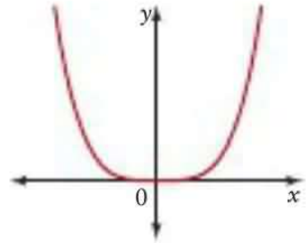

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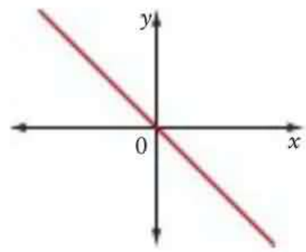


For help with question 4, refer to Example 3.

4. Copy and complete the table for the following functions.

$y = -x^3$ $\quad y = \frac{3}{7}x^2$ $\quad y = 5x$

$y = 4x^5$ $\quad y = -x^6$ $\quad y = -0.1x^{11}$

$y = 2x^4$ $\quad y = -9x^{10}$

End Behaviour	Function	Reasons
Extends from quadrant 3 to quadrant 1		
Extends from quadrant 2 to quadrant 4		
Extends from quadrant 2 to quadrant 1		
Extends from quadrant 3 to quadrant 4		

B Connect and Apply

For help with questions 5 and 6, refer to Example 4.

5. As a tropical storm intensifies and reaches hurricane status, it takes on a circular shape that expands outward from the eye of the storm. The area, A, in square kilometres, spanned by a storm with radius, r, in kilometres, can be modelled by the function $A(r) = \pi r^2$.

a) Graph $A(r)$ for $r \in [0, 10]$.

b) State the domain and range.

c) Describe the similarities and differences between the graph of $A(r)$ and the graph of $y = x^2$.

6. The circumference, C, in kilometres, of the tropical storm in question 5 can be modelled by the function $C(r) = 2\pi r$.

a) Graph $C(r)$ for $r \in [0, 10]$.

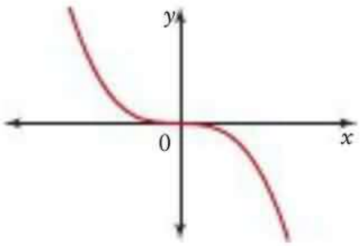
b) State the domain and range.

c) Describe the similarities and differences between the graph of $C(r)$ and the graph of $y = x$.

7. Determine whether each graph represents a power function, an exponential function, a periodic function, or none of these. Justify your choice.

> **CONNECTIONS**
>
> You worked with periodic functions when you studied trigonometric functions in grade 11. Periodic functions repeat at regular intervals.

a)



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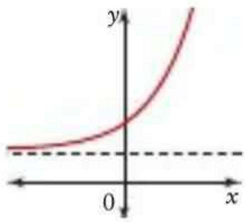

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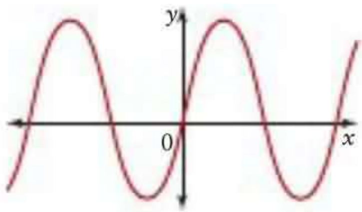

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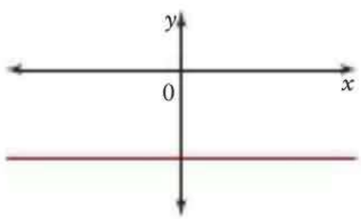

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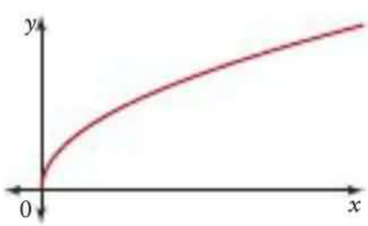

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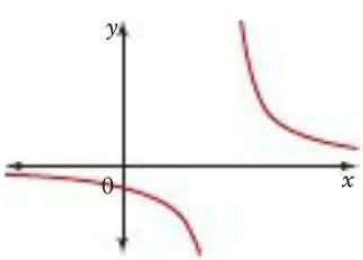
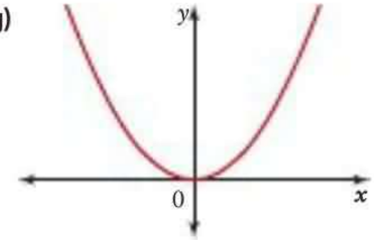


g)



8. Use Technology

a) Graph $f(x) = x^3 + x^2$, $g(x) = x^3 - x$, and $h(x) = x^3$ on the same set of axes.

b) Compare and describe the key features of the graphs of these functions.

9. Use Technology

a) Graph $f(x) = x^4 + x$, $g(x) = x^4 - x^2$, and $h(x) = x^4$ on the same set of axes.

b) Compare and describe the key features of the graphs of these functions.

10. Describe the similarities and differences between the line $y = x$ and power functions with odd degree greater than one. Use graphs to support your answer.

11. Describe the similarities and differences between the parabola $y = x^2$ and power functions with even degree greater than two. Use graphs to support your answer.

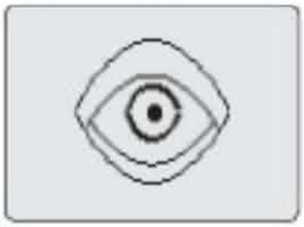
12. a) Graph the functions $y = x^3$, $y = x^3 - 2$, and $y = x^3 + 2$ on the same set of axes. Compare the graphs. Describe how the graphs are related.

b) Repeat part a) for the functions $y = x^4$, $y = x^4 - 2$, and $y = x^4 + 2$.

c) Make a conjecture about the relationship between the graphs of $y = x^n$ and $y = x^n + c$, where $c \in \mathbb{R}$ and n is a whole number.

d) Test the accuracy of your conjecture for different values of n and c.

13. Chapter Problem Part of a computer graphic designer's job may be to create and manipulate electronic images found in video games. Power functions define curves that are useful in the design of characters, objects, and background scenery. Domain restrictions allow two or more curves to be combined to create a particular form. For example, a character's eye could be created using parabolas with restricted domains.



Describe the type(s) of power function(s) that could be used to design two of the following in a new video game. Provide equations and sketches of your functions. Include the domain and range of the functions you use.

- the path of a river that extends from the southwest to the northeast part of a large forest
- the cross-section of a valley that lies between two mountain ranges
- a deep canyon where the river flows
- characters' facial expressions
- a lightning bolt
- horseshoe tracks in the sand

C Extend and Challenge

14. Use Technology

a) Graph each pair of functions. What do you notice? Provide an algebraic explanation for what you observe.

i) $y = (-x)^2$ and $y = x^2$

ii) $y = (-x)^4$ and $y = x^4$

iii) $y = (-x)^6$ and $y = x^6$

b) Repeat part a) for each of the following pairs of functions.

i) $y = (-x)^3$ and $y = -x^3$

ii) $y = (-x)^5$ and $y = -x^5$

iii) $y = (-x)^7$ and $y = -x^7$

c) Describe what you have learned about functions of the form $y = (-x)^n$, where n is a non-negative integer. Support your answer with examples.

15. a) Make a conjecture about the relationship between the graphs of $y = x^n$ and $y = ax^n$ for $a \in \mathbb{R}$.

b) Test your conjecture using different values of n and a.

16. a) Describe the relationship between the graph of $y = x^2$ and the graph of $y = 2(x - 3)^2 + 1$.

b) Predict the relationship between the graph of $y = x^4$ and the graph of $y = 2(x - 3)^4 + 1$.

c) Verify the accuracy of your prediction by sketching the graphs in part b).

17. a) Use the results of question 16 to predict a relationship between the graph of $y = x^3$ and the graph of $y = a(x - h)^3 + k$.

b) Verify the accuracy of your prediction in part a) by sketching two functions of the form $y = a(x - h)^3 + k$.

18. Math Contest Determine the number of digits in the expansion of $(2^{120})(5^{125})$ without using a calculator or computer.

19. Math Contest Find the coordinates of the two points that trisect the line segment with endpoints A(2, 3) and B(8, 2).

1.2

Characteristics of Polynomial Functions

In Section 1.1, you explored the features of power functions, which are single-term polynomial functions. Many polynomial functions that arise from real-world applications such as profit, volume, and population growth are made up of two or more terms. For example, the function $r = -0.7d^3 + d^2$, which relates a patient's reaction time, $r(d)$, in seconds, to a dose, d, in millilitres, of a particular drug is polynomial.

In this section, you will explore and identify the characteristics of the graphs and equations of general polynomial functions and establish the relationship between finite differences and the equations of polynomial functions.

Investigate 1 What are the key features of the graphs of polynomial functions?

Tools

- graphing calculator

Optional

- computer with *The Geometer's Sketchpad®*

A: *Polynomial Functions of Odd Degree*

1. a) Graph each cubic function on a different set of axes using a graphing calculator. Sketch the results.

Group A	Group B
i) $y = x^3$	**i)** $y = -x^3$
ii) $y = x^3 + x^2 - 4x - 4$	**ii)** $y = -x^3 - x^2 + 4x + 4$
iii) $y = x^3 + 5x^2 + 3x - 9$	**iii)** $y = -x^3 - 5x^2 - 3x + 9$

b) Compare the graphs in each group. Describe their similarities and differences in terms of

i) end behaviour

ii) number of minimum points and number of maximum points

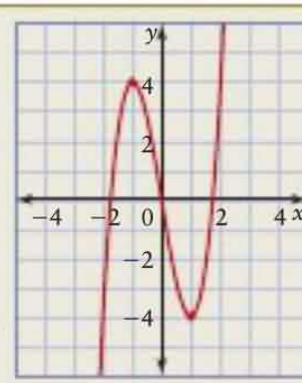
iii) number of **local minimum** and number of **local maximum** points, that is, points that are minimum or maximum points on some interval around that point

iv) number of x-intercepts

c) Compare the two groups of graphs. Describe their similarities and differences as in part b).

CONNECTIONS

In this graph, the point $(-1, 4)$ is a local maximum, and the point $(1, -4)$ is a local minimum. Notice that the point $(-1, 4)$ is not a maximum point of the function, since other points on the graph of the function are greater. Maximum or minimum points of a function are sometimes called absolute, or global, to distinguish them from local maximum and minimum points.



d) **Reflect** Which group of graphs is similar to the graph of

i) $y = x$?

ii) $y = -x$?

Explain how they are similar.

2. a) Graph each quintic function on a different set of axes using a graphing calculator. Sketch the results.

i) $y = x^5$

ii) $y = x^5 + 3x^4 - x^3 - 7x^2 + 4$

iii) $y = -x^5 + x^4 + 9x^3 - 13x^2 - 8x + 12$

b) Compare the graphs. Describe their similarities and differences.

3. **Reflect** Use the results from steps 1 and 2 to answer each question.

a) What are the similarities and differences between the graphs of linear, cubic, and quintic functions?

b) What are the minimum and the maximum numbers of x-intercepts of graphs of cubic polynomial functions?

c) Describe the relationship between the number of minimum and maximum points, the number of local minimum and local maximum points, and the degree of a polynomial function.

d) What is the relationship between the sign of the leading coefficient and the end behaviour of graphs of polynomial functions with odd degree?

e) Do you think the results in part d) are true for all polynomial functions with odd degree? Justify your answer.

B: ***Polynomial Functions of Even Degree***

1. a) Graph each quartic function.

Group A	Group B
i) $y = x^4$	**i)** $y = -x^4$
ii) $y = x^4 - x^3 - 6x^2 + 4x + 8$	**ii)** $y = -x^4 - 5x^3 + 5x + 10$
iii) $y = x^4 - 3x^3 - 3x^2 + 11x - 4$	**iii)** $y = -x^4 + 3x^3 + 3x^2 - 11x + 4$

b) Compare the graphs in each group. Describe their similarities and differences in terms of

i) end behaviour

ii) number of maximum and number of minimum points

iii) number of local minimum and number of local maximum points

iv) number of x-intercepts

c) Compare the two groups of graphs. Describe their similarities and differences as in part b).

d) **Reflect** Explain which group has graphs that are similar to the graph of

i) $y = x^2$

ii) $y = -x^2$

2. **Reflect** Use the results from step 1 to answer each question.

a) Describe the similarities and differences between the graphs of quadratic and quartic polynomial functions.

b) What are the minimum and the maximum numbers of x-intercepts of graphs of quartic polynomials?

c) Describe the relationship between the number of minimum and maximum points, the number of local maximum and local minimum points, and the degree of a polynomial function.

d) What is the relationship between the sign of the leading coefficient and the end behaviour of the graphs of polynomial functions with even degree?

e) Do you think the above results are true for all polynomial functions with even degree? Justify your answer.

Investigate 2 What is the relationship between finite differences and the equation of a polynomial function?

Tools

- graphing calculator

1. Construct a finite difference table for $y = x^3$.

Method 1: *Use Pencil and Paper*

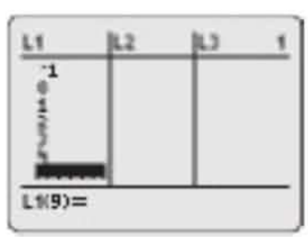
		Differences		
x	y	First	Second	...
−3				
−2				
−1				
0				
1				
2				
3				
4				

a) Create a finite difference table like this one.

b) Use the equation to determine the y-values in column 2.

c) Complete each column and extend the table as needed, until you get a constant difference.

Method 2: *Use a Graphing Calculator*



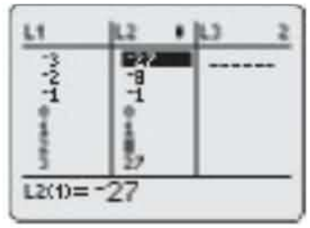
To construct a table of finite differences on a graphing calculator, press STAT **1**.

Enter the x-values in **L1**.

To enter the y-values for the function $y = x^3$ in **L2**, scroll up and right to select **L2**.

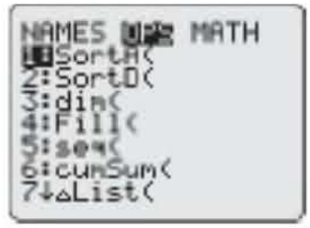
Enter the function, using **L1** to represent the variable x, by pressing ALPHA ["] 2nd [L1] ^ 3 ALPHA ["].

- Press ENTER.



To determine the first differences, scroll up and right to select **L3**.

- Press ALPHA [''] and then 2nd [LIST].
- Cursor right to select **OPS**.

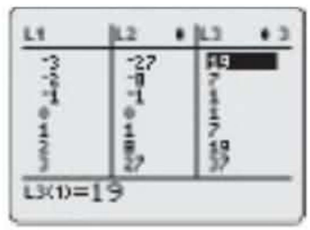


- Press **7** to select **ΔList(**.
- Press 2nd [L2]) and then ALPHA [''].

- Press ENTER.

The first differences are displayed in **L3**.

Determine the second differences and the third differences in a similar way.



2. Construct a finite difference table for

a) $y = -2x^3$

b) $y = x^4$

c) $y = -2x^4$

3. a) What is true about the third differences of a cubic function? the fourth differences of a quartic function?

b) How is the sign of the leading coefficient related to the sign of the constant value of the finite differences?

c) How is the value of the leading coefficient related to the constant value of the finite differences?

d) **Reflect** Make a conjecture about the relationship between constant finite differences and

i) the degree of a polynomial function

ii) the sign of the leading coefficient

iii) the value of the leading coefficient

e) Verify your conjectures by constructing finite difference tables for the polynomial functions $f(x) = 3x^3 - 4x^2 + 1$ and $g(x) = -2x^4 + x^3 + 3x - 1$.

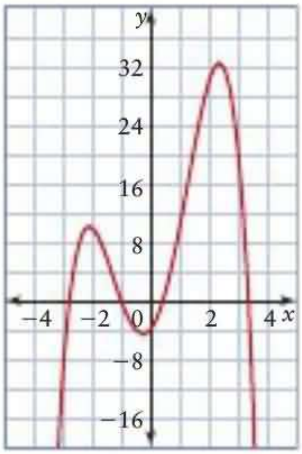
Example 1 Match a Polynomial Function With Its Graph

Determine the key features of the graph of each polynomial function. Use these features to match each function with its graph. State the number of x-intercepts, the number of maximum and minimum points, and the number of local maximum and local minimum points for the graph of each function. How are these features related to the degree of the function?

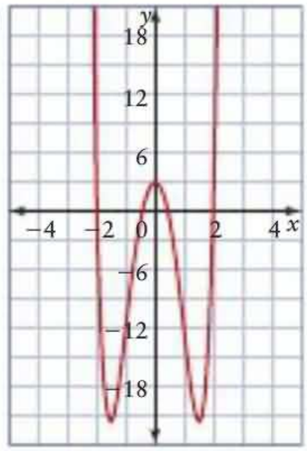
a) $f(x) = 2x^3 - 4x^2 + x + 1$ **b)** $g(x) = -x^4 + 10x^2 + 5x - 4$

c) $h(x) = -2x^5 + 5x^3 - x$ **d)** $p(x) = x^6 - 16x^2 + 3$

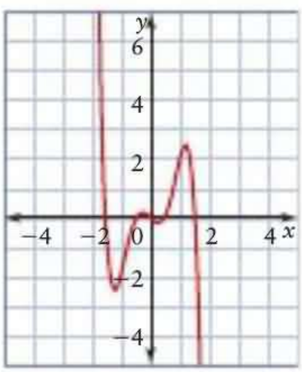
i)



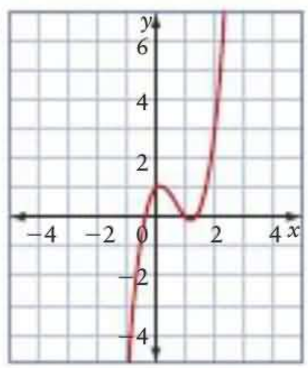
ii)



iii)



iv)



Solution

a) The function $f(x) = 2x^3 - 4x^2 + x + 1$ is cubic, with a positive leading coefficient. The graph extends from quadrant 3 to quadrant 1. The y-intercept is 1. Graph iv) corresponds to this equation.

There are three x-intercepts and the degree is three. The function has one local maximum point and one local minimum point, a total of two, which is one less than the degree. There is no maximum point and no minimum point.

b) The function $g(x) = -x^4 + 10x^2 + 5x - 4$ is quartic, with a negative leading coefficient. The graph extends from quadrant 3 to quadrant 4. The y-intercept is -4. Graph i) corresponds to this equation.

There are four x-intercepts and the degree is four. There is one maximum point and no minimum point. The graph has two local maximum points and one local minimum point, for a total of three, which is one less than the degree.

c) The function $h(x) = -2x^5 + 5x^3 - x$ is quintic, with a negative leading coefficient. The graph extends from quadrant 2 to quadrant 4. The y-intercept is 0. Graph iii) corresponds to this equation.

There are five x-intercepts and the degree is five. There is no maximum point and no minimum point. The graph has two local maximum points and two local minimum points, for a total of four, which is one less than the degree.

d) The function $p(x) = x^6 - 16x^2 + 3$ is a function of degree five with a positive leading coefficient. The graph extends from quadrant 2 to quadrant 1. The y-intercept is 3. Graph ii) corresponds to this equation.

There are four x-intercepts and the degree is six. The graph has two minimum points and no maximum point. The graph has one local maximum point and two local minimum points, for a total of three (three less than the degree).

CONNECTIONS

For any positive integer n, the product $n \times (n - 1) \times \ldots \times 2 \times 1$ may be expressed in a shorter form as $n!$, read "***n* factorial**."

$5! = 5 \times 4 \times 3 \times 2 \times 1$
$= 120$

Finite Differences

For a polynomial function of degree n, where n is a positive integer, the nth differences

- are equal (or constant)
- have the same sign as the leading coefficient
- are equal to $a[n \times (n - 1) \times \ldots \times 2 \times 1]$, where a is the leading coefficient

Example 2 Identify Types of Polynomial Functions From Finite Differences

Each table of values represents a polynomial function. Use finite differences to determine

i) the degree of the polynomial function

ii) the sign of the leading coefficient

iii) the value of the leading coefficient

a)

x	y
−3	−36
−2	−12
−1	−2
0	0
1	0
2	4
3	18
4	48

b)

x	y
−2	−54
−1	−8
0	0
1	6
2	22
3	36
4	12
5	−110

Solution

Construct a finite difference table. Determine finite differences until they are constant.

a) i)

x	y	First Differences	Second Differences	Third Differences
-3	-36			
-2	-12	$-12 - (-36) = 24$		
-1	-2	$-2 - (-12) = 10$	$10 - 24 = -14$	
0	0	$0 - (-2) = 2$	$2 - 10 = -8$	$-8 - (-14) = 6$
1	0	$0 - 0 = 0$	$0 - 2 = -2$	$-2 - (-8) = 6$
2	4	$4 - 0 = 4$	$4 - 0 = 4$	$4 - (-2) = 6$
3	18	$18 - 4 = 14$	$14 - 4 = 10$	$10 - 4 = 6$
4	48	$48 - 18 = 30$	$30 - 14 = 16$	$16 - 10 = 6$

The third differences are constant. So, the table of values represents a cubic function. The degree of the function is three.

ii) The leading coefficient is positive, since 6 is positive.

iii) The value of the leading coefficient is the value of a such that $6 = a[n \times (n - 1) \times \ldots \times 2 \times 1]$.

Substitute $n = 3$:

$$6 = a(3 \times 2 \times 1)$$
$$6 = 6a$$
$$a = 1$$

b) i)

x	y	First Differences	Second Differences	Third Differences	Fourth Differences
-2	-54				
-1	-8	46			
0	0	8	-38		
1	6	6	-2	36	
2	22	16	10	12	-24
3	36	14	-2	-12	-24
4	12	-24	-38	-36	-24
5	-110	-122	-98	-60	-24

Since the fourth differences are equal and negative, the table of values represents a quartic function. The degree of the function is four.

ii) This polynomial has a negative leading coefficient.

iii) The value of the leading coefficient is the value of a such that $-24 = a[n \times (n - 1) \times \ldots \times 2 \times 1]$.

Substitute $n = 4$:

$$-24 = a(4 \times 3 \times 2 \times 1)$$
$$-24 = 24a$$
$$a = -1$$

CONNECTIONS

You will learn why the constant finite differences of a degree-n polynomial with leading coefficient a are equal to $a \times n!$ if you study calculus.

Example 3 Application of a Polynomial Function

A new antibacterial spray is tested on a bacterial culture. The table shows the population, P, of the bacterial culture t minutes after the spray is applied.

t (min)	P
0	800
1	799
2	782
3	737
4	652
5	515
6	314
7	37

a) Use finite differences to

i) identify the type of polynomial function that models the population growth

ii) determine the value of the leading coefficient

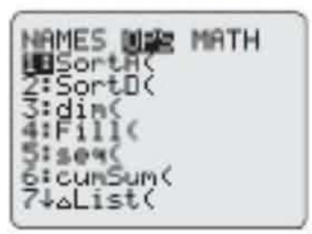
b) Use the regression feature of a graphing calculator to determine an equation for the function that models this situation.

Solution

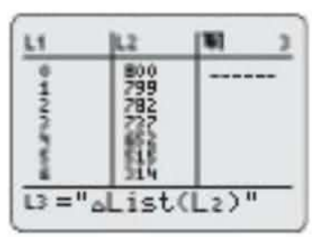
- Press STAT **1**. Enter the t-values in **L1** and the P-values in **L2**.

Use the calculator to calculate the first differences in **L3** as follows.

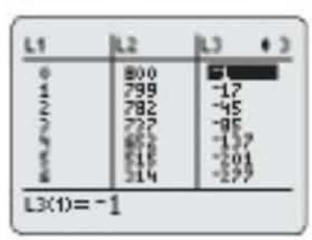
- Press ALPHA ["] and then 2nd [LIST].
- Scroll right to select **OPS**.



- Press **7** to select **ΔList(**.
- Press 2nd [L2]) and then ALPHA ["].

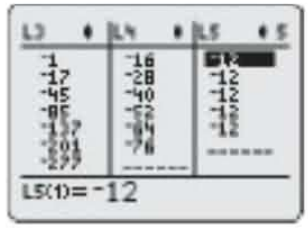


- Press ENTER.



The first differences are displayed in **L3**.

Repeat the steps to determine the second differences in **L4** and the third differences in **L5**.



a) **i)** Since the third differences are constant, the population models a cubic function.

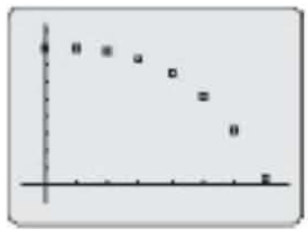
ii) The value of the leading coefficient is the value of a such that $-12 = a[n \times (n - 1) \times \ldots \times 2 \times 1]$.
Substitute $n = 3$:

$$-12 = a(3 \times 2 \times 1)$$
$$-12 = 6a$$
$$a = -2$$

b) Create a scatter plot as follows.

- Press 2nd [STAT PLOT] **1** ENTER.

Turn on **Plot 1** by making sure **On** is highlighted, **Type** is set to the graph type you prefer, and **L1** and **L2** appear after **Xlist** and **Ylist**.



Display the graph as follows.

- Press ZOOM **9** for **ZoomStat**.

Determine the equation of the curve of best fit as follows.

- Press STAT.

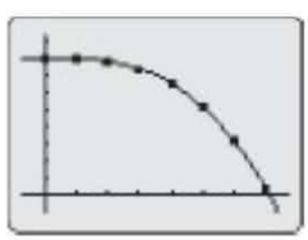
Move the cursor to **CALC**, and then press **6** to select a cubic regression function, since we know $n = 3$.

- Press 2nd [L1], 2nd [L2] , VARS.
- Cursor over to **Y-VARS**.
- Press **1** twice to store the equation of the curve of best fit in **Y1** of the equation editor.
- Press ENTER to display the results.

Substitute the values of a, b, c, and d into the general cubic equation as shown on the calculator screen.

An equation that models the data is $y = -2x^3 - 2x^2 + 3x + 800$.

- Press GRAPH to plot the curve.

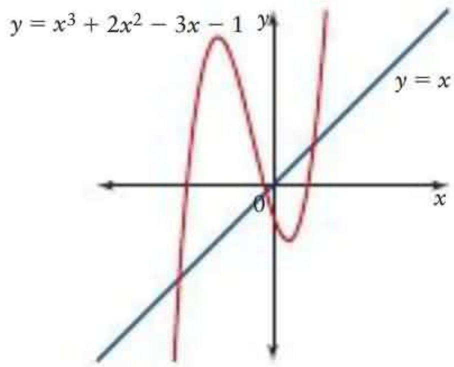


KEY CONCEPTS

Key Features of Graphs of Polynomial Functions With Odd Degree

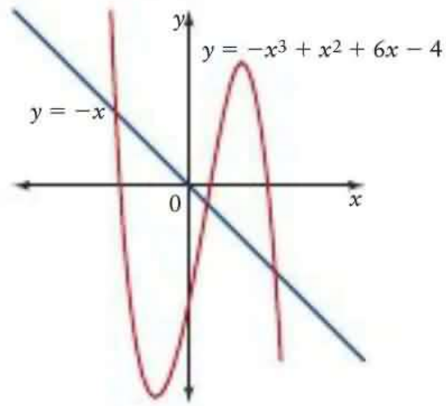
Positive Leading Coefficient

- the graph extends from quadrant 3 to quadrant 1 (similar to the graph of $y = x$)



Negative Leading Coefficient

- the graph extends from quadrant 2 to quadrant 4 (similar to the graph of $y = -x$)

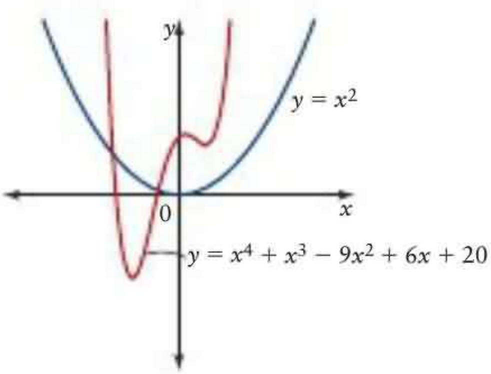


- Odd-degree polynomials have at least one x-intercept, up to a maximum of n x-intercepts, where n is the degree of the function.
- The domain of all odd-degree polynomials is $\{x \in \mathbb{R}\}$ and the range is $\{y \in \mathbb{R}\}$. Odd-degree functions have no maximum point and no minimum point.
- Odd-degree polynomials may have point symmetry.

Key Features of Graphs of Polynomial Functions With Even Degree

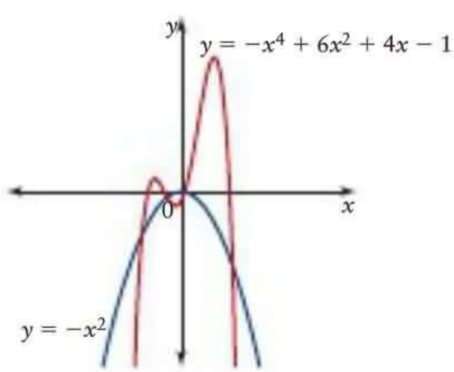
Positive Leading Coefficient

- the graph extends from quadrant 2 to quadrant 1 (similar to the graph of $y = x^2$)
- the range is $\{y \in \mathbb{R}, y \geq a\}$, where a is the minimum value of the function
- an even-degree polynomial with a positive leading coefficient will have at least one minimum point



Negative Leading Coefficient

- the graph extends from quadrant 3 to quadrant 4 (similar to the graph of $y = -x^2$)
- the range is $\{y \in \mathbb{R}, y \leq a\}$, where a is the maximum value of the function
- an even-degree polynomial with a negative leading coefficient will have at least one maximum point



- Even-degree polynomials may have from zero to a maximum of n x-intercepts, where n is the degree of the function.
- The domain of all even-degree polynomials is $\{x \in \mathbb{R}\}$.
- Even-degree polynomials may have line symmetry.

Key Features of Graphs of Polynomial Functions

- A polynomial function of degree n, where n is a whole number greater than 1, may have at most $n - 1$ local minimum and local maximum points.
- For any polynomial function of degree n, the nth differences
  - are equal (or constant)
  - have the same sign as the leading coefficient
  - are equal to $a[n \times (n - 1) \times \ldots \times 2 \times 1]$, where a is the leading coefficient

Communicate Your Understanding

C1 Describe the similarities between

a) the lines $y = x$ and $y = -x$ and the graphs of other odd-degree polynomial functions

b) the parabolas $y = x^2$ and $y = -x^2$ and the graphs of other even-degree polynomial functions

C2 Discuss the relationship between the degree of a polynomial function and the following features of the corresponding graph:

a) the number of x-intercepts

b) the number of maximum and minimum points

c) the number of local maximum and local minimum points

C3 Sketch the graph of a quartic function that

a) has line symmetry

b) does not have line symmetry

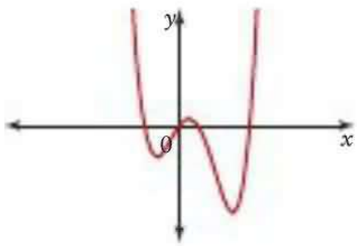
C4 Explain why even-degree polynomials have a restricted range. What does this tell you about the number of maximum or minimum points?

A Practise

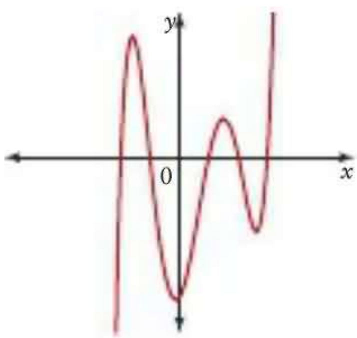
For help with questions 1 to 3, refer to Example 1.

1. Each graph represents a polynomial function of degree 3, 4, 5, or 6. Determine the least possible degree of the function corresponding to each graph. Justify your answer.

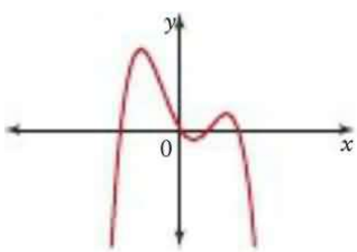
a)



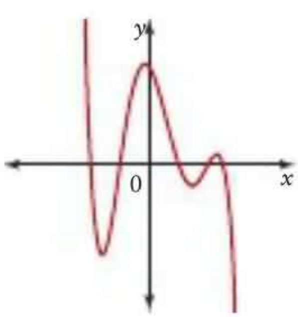
b)



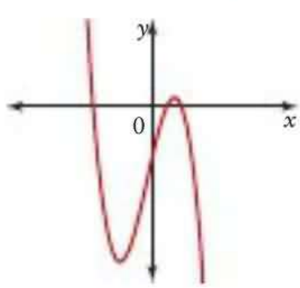
c)



d)

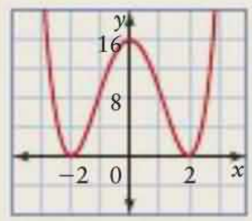


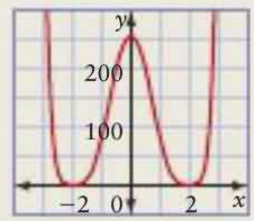
e)



CONNECTIONS

The least possible degree refers to the fact that it is possible for the graphs of two polynomial functions with either odd degree or even degree to *appear* to be similar, even though one may have a higher degree than the other. For instance, the graphs of $y = (x - 2)^2(x + 2)^2$ and $y = (x - 2)^4(x + 2)^4$ have the same shape and the same x-intercepts, -2 and 2, but one function has a double root at each of these values, while the other has a quadruple root at each of these values.





2. Refer to question 1. For each graph, do the following.

a) State the sign of the leading coefficient. Justify your answer.

b) Describe the end behaviour.

c) Identify any symmetry.

d) State the number of minimum and maximum points and local minimum and local maximum points. How are these related to the degree of the function?

3. Use the degree and the sign of the leading coefficient to

i) describe the end behaviour of each polynomial function

ii) state which finite differences will be constant

iii) determine the value of the constant finite differences

a) $f(x) = x^2 + 3x - 1$

b) $g(x) = -4x^3 + 2x^2 - x + 5$

c) $h(x) = -7x^4 + 2x^3 - 3x^2 + 4$

d) $p(x) = 0.6x^5 - 2x^4 + 8x$

e) $f(x) = 3 - x$

f) $h(x) = -x^6 + 8x^3$

For help with question 4, refer to Example 2.

4. State the degree of the polynomial function that corresponds to each constant finite difference. Determine the value of the leading coefficient for each polynomial function.

a) second differences $= -8$

b) fourth differences $= -48$

c) third differences $= -12$

d) fourth differences $= 24$

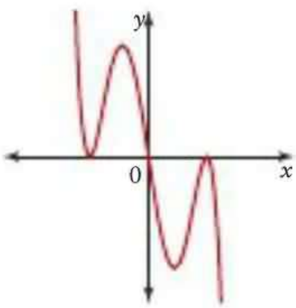
e) third differences $= 36$

f) fifth differences $= 60$

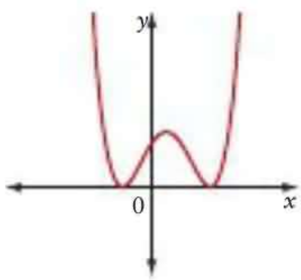
B Connect and Apply

5. Determine whether each graph represents an even-degree or an odd-degree polynomial function. Explain your reasoning.

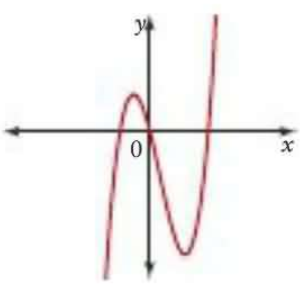
a)



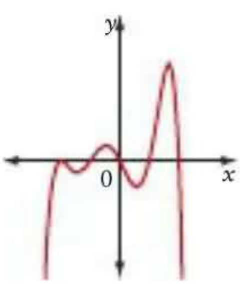
b)



c)



d)



6. Refer to question 5. For each graph, do the following.

a) State the least possible degree.

b) State the sign of the leading coefficient.

c) Describe the end behaviour of the graph.

d) Identify the type of symmetry, if it exists.

For help with question 7, refer to Example 2.

7. Each table represents a polynomial function. Use finite differences to determine the following for each polynomial function.

i) the degree

ii) the sign of the leading coefficient

iii) the value of the leading coefficient

a)

x	y
−3	−45
−2	−16
−1	−3
0	0
1	−1
2	0
3	9
4	32

b)

x	y
−2	−40
−1	12
0	20
1	26
2	48
3	80
4	92
5	30

For help with questions 8 and 9, refer to Example 3.

8. A snowboard manufacturer determines that its profit, P, in thousands of dollars, can be modelled by the function $P(x) = x + 0.001\,25x^4 - 3$, where x represents the number, in hundreds, of snowboards sold.

a) What type of function is $P(x)$?

b) Without calculating, determine which finite differences are constant for this polynomial function. What is the value of the constant finite differences? Explain how you know.

c) Describe the end behaviour of this function, assuming that there are no restrictions on the domain.

d) State the restrictions on the domain in this situation.

e) What do the x-intercepts of the graph represent for this situation?

f) What is the profit from the sale of 3000 snowboards?

9. **Use Technology** The table shows the displacement, s, in metres, of an inner tube moving along a waterslide after time, t, in seconds.

t (s)	s (m)
0	10
1	34
2	42
3	46
4	58
5	90
6	154
7	262

a) Use finite differences to

i) identify the type of polynomial function that models s

ii) determine the value of the leading coefficient

b) Graph the data in the table using a graphing calculator. Use the regression feature of the graphing calculator to determine an equation for the function that models this situation.

10. a) Sketch graphs of $y = \sin x$ and $y = \cos x$.

b) Compare the graph of a periodic function to the graph of a polynomial function. Describe any similarities and differences. Refer to the end behaviour, local maximum and local minimum points, and maximum and minimum points.

11. The volume, V, in cubic centimetres, of a collection of open-topped boxes can be modelled by $V(x) = 4x^3 - 220x^2 + 2800x$, where x is the height of each box, in centimetres.

a) Graph $V(x)$. State the restrictions.

b) Fully factor $V(x)$. State the relationship between the factored form of the equation and the graph.

c) State the value of the constant finite differences for this function.

12. A medical researcher establishes that a patient's reaction time, r, in minutes, to a dose of a particular drug is $r(d) = -0.7d^3 + d^2$, where d is the amount of the drug, in millilitres, that is absorbed into the patient's blood.

a) What type of function is $r(d)$?

b) Without calculating the finite differences, state which finite differences are constant for this function. How do you know? What is the value of the constant differences?

c) Describe the end behaviour of this function if no restrictions are considered.

d) State the restrictions for this situation.

13. By analysing the impact of growing economic conditions, a demographer establishes that the predicted population, P, of a town t years from now can be modelled by the function $p(t) = 6t^4 - 5t^3 + 200t + 12\,000$.

a) Describe the key features of the graph represented by this function if no restrictions are considered.

b) What is the value of the constant finite differences?

c) What is the current population of the town?

d) What will the population of the town be 10 years from now?

e) When will the population of the town be approximately 175 000?

✓ Achievement Check

14. Consider the function $f(x) = x^3 + 2x^2 - 5x - 6$.
   a) How do the degree and the sign of the leading coefficient correspond to the end behaviour of the polynomial function?
   b) Sketch a graph of the polynomial function.
   c) What can you tell about the value of the third differences for this function?

C Extend and Challenge

15. Graph a polynomial function that satisfies each description.
   a) a quartic function with a negative leading coefficient and three x-intercepts
   b) a cubic function with a positive leading coefficient and two x-intercepts
   c) a quadratic function with a positive leading coefficient and no x-intercepts
   d) a quintic function with a negative leading coefficient and five x-intercepts

16. a) What possible numbers of x-intercepts can a quintic function have?
   b) Sketch an example of a graph of a quintic function for each possibility in part a).

17. **Use Technology**
   a) What type of polynomial function is each of the following? Justify your answer.
      i) $f(x) = (x + 4)(x - 1)(2x + 5)$
      ii) $f(x) = (x + 4)^2(x - 1)$
      iii) $f(x) = (x + 4)^3$
   b) Graph each function.
   c) Describe the relationship between the x-intercepts and the equation of the function.

18. A storage tank is to be constructed in the shape of a cylinder such that the ratio of the radius, r, to the height of the tank is 1 : 3.
   a) Write a polynomial function to represent
      i) the surface area of the tank in terms of r
      ii) the volume of the tank in terms of r
   b) Describe the key features of the graph that corresponds to each of the above functions.

19. **Math Contest**
   a) Given the function $f(x) = x^3 - 2x$, sketch $y = f(|x|)$.
   b) Sketch $g(x) = |x^2 - 1| - |x^2 - 4|$.
   c) Sketch the region in the plane to show all points (x, y) such that $|x| + |y| \leq 2$.

CAREER CONNECTION

Davinder completed a 2-year course in mining engineering technology at a Canadian college. He works with an engineering and construction crew to blast openings in rock faces for road construction. In his job as the explosives specialist, he examines the structure of the rock in the blast area and determines the amounts and kinds of explosives needed to ensure that the blast is not only effective but also safe. He also considers environmental concerns such as vibration and noise. Davinder uses mathematical reasoning and a knowledge of physical principles to choose the correct formulas to solve problems. Davinder then creates a blast design and initiation sequence.

1.3

Equations and Graphs of Polynomial Functions

A rollercoaster is designed so that the shape of a section of the ride can be modelled by the function $f(x) = -0.000\,004x(x - 15)(x - 25)(x - 45)^2(x - 60) + 9$, $x \in [0, 60]$, where x is the time, in seconds, and $f(x)$ is the height of the ride above the ground, in metres, at time x.

What are the similarities and differences between this polynomial function and the ones studied in previous sections? What useful information does this form of the equation provide that can be used to sketch a graph to represent the path of the rollercoaster?

In this section, you will investigate the relationship between the factored form of a polynomial function and the x-intercepts of the corresponding graph. You will examine the effect of repeated factors on the graph of a polynomial function, and you will establish an algebraic method for determining whether a polynomial function has line symmetry or point symmetry.

Investigate 1 What is the connection between the factored form of a polynomial function and its graph?

Tools

- graphing calculator

Optional

- computer with *The Geometer's Sketchpad®*

1. a) Graph the function $y = x(x - 3)(x + 2)(x + 1)$.

b) From the graph, determine

i) the degree of the polynomial function

ii) the sign of the leading coefficient

iii) the x-intercepts

iv) the y-intercept

c) Refer to the graph in part a). The x-intercepts divide the x-axis into five intervals. Copy the table below and write the intervals in the first row. In the second row, indicate whether the function is positive (above the x-axis) or negative (below the x-axis) for the corresponding interval.

Interval					
Sign of $f(x)$					

d) **Reflect**

i) How can you determine the degree and the sign of the leading coefficient from the equation?

ii) What is the relationship between the x-intercepts and the equation of the function? the y-intercept and the equation of the function?

iii) What happens to the sign of $f(x)$ near each x-intercept?

2. a) Graph each function on a separate set of axes. Sketch each graph in your notebook.

i) $y = (x + 1)(x - 2)(x + 3)$

ii) $y = (x - 2)^2(x + 3)$

iii) $y = (x - 2)^3$

b) State the x-intercepts and the y-intercept of each graph in part a).

c) **Reflect** What is the relationship between the number of x-intercepts, the repeated factors in the equation of the function, and the sign of $f(x)$?

3. Repeat step 2 for the following polynomial functions:

i) $y = -(x + 1)(x - 2)(x - 4)(x + 3)$

ii) $y = -(x - 2)^2(x - 4)(x + 3)$

iii) $y = -(x - 2)^4$

iv) $y = -(x - 2)^2(x + 3)^2$

4. **Reflect**

a) Describe the effect on the graph of a polynomial function when a factor is repeated

i) an even number of times

ii) an odd number of times

b) What is the relationship between

i) an even number of repeated factors and the sign of $f(x)$?

ii) an odd number of repeated factors and the sign of $f(x)$?

5. a) **Reflect** Describe how a graph of a polynomial function can be sketched using the x-intercepts, the y-intercept, the sign of the leading coefficient, and the degree of the function.

b) Sketch a graph of $y = -2(x - 1)^2(x + 2)(x - 4)$. Use technology to verify your sketch.

c) Sketch a graph of each function. Use technology to verify your graph.

i) $y = -(x + 4)(x - 1)^2$

ii) $y = -(x - 1)^3$

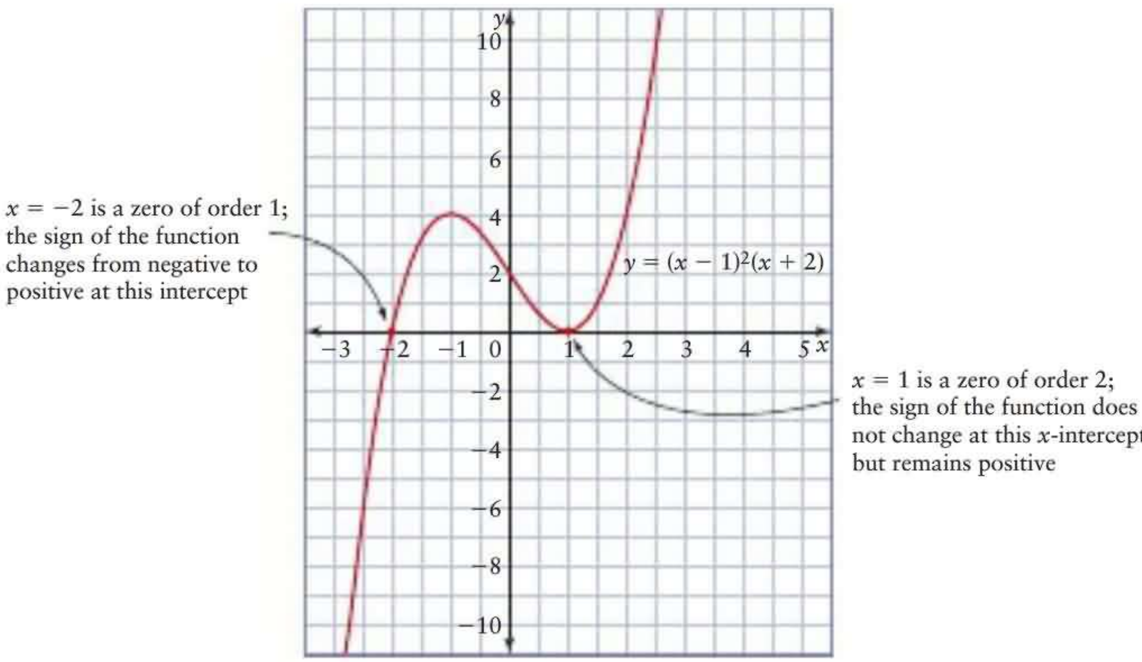
iii) $y = (x + 4)^2(x - 1)^2$

iv) $y = (x + 4)^4$

The zeros of a polynomial function $y = f(x)$ correspond to the x-intercepts of the graph and to the roots of the corresponding equation $f(x) = 0$. For example, the function $f(x) = (x - 2)(x + 1)$ has zeros 2 and -1. These are the roots of the equation $(x - 2)(x + 1) = 0$.

If a polynomial function has a factor $(x - a)$ that is repeated n times, then $x = a$ is a zero of **order** n. The function $f(x) = (x - 2)(x + 1)^2$ has a zero of order 2 at $x = -1$ and the equation $(x - 2)(x + 1)^2 = 0$ has a double root at $x = -1$.

The graph of a polynomial function changes sign (from positive to negative or negative to positive) at zeros that are of odd order but does not change sign at zeros that are of even order.



Test values close to either side of an x-intercept to check if the function is positive or negative.

Consider the function $y = (x - 1)^2(x + 2)$ shown in the graph above.

To the left of -2, choose $x = -3$.

$$\begin{aligned} f(-3) &= (-3 - 1)^2(-3 + 2) \\ &= 16(-1) \\ &= -16 \end{aligned}$$

Thus, $f(x) < 0$ to the left of -2.

To the right of -2, choose $x = -1$.

$$\begin{aligned} f(-1) &= (-1 - 1)^2(-1 + 2) \\ &= 4(1) \\ &= 4 \end{aligned}$$

Thus, $f(x) > 0$ to the right of -2.

To the left of 1, choose $x = 0$.

$$\begin{aligned} f(0) &= (0 - 1)^2(0 + 2) \\ &= 1(2) \\ &= 2 \end{aligned}$$

Thus, $f(x) > 0$ to the left of 1.

To the right of 1, choose $x = 2$.

$$\begin{aligned} f(2) &= (2 - 1)^2(2 + 2) \\ &= 1(4) \\ &= 4 \end{aligned}$$

Thus, $f(x) > 0$ to the right of 1.

Example 1 Analysing Graphs of Polynomial Functions

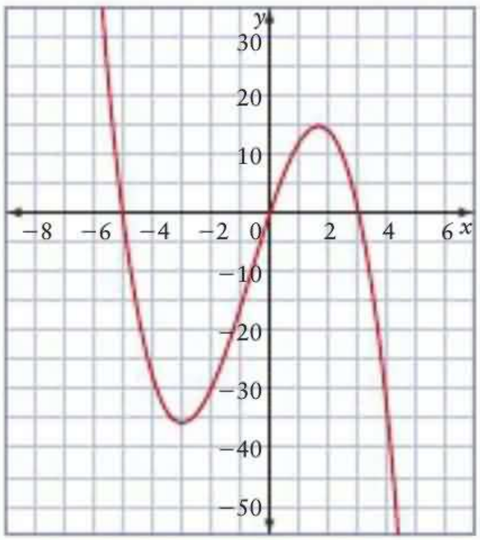
For each graph of a polynomial function, determine

i) the least possible degree and the sign of the leading coefficient

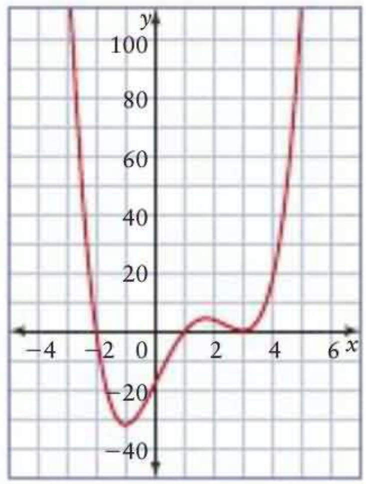
ii) the x-intercepts and the factors of the function

iii) the intervals where the function is positive and the intervals where it is negative

a)



b)



Solution

a) i) The three x-intercepts are of order one, so the least possible degree is 3. The graph extends from quadrant 2 to quadrant 4, so the leading coefficient is negative.

ii) The x-intercepts are -5, 0, and 3. The factors are $x + 5$, x, and $x - 3$.

iii)

Interval	$(-\infty, -5)$	$(-5, 0)$	$(0, 3)$	$(3, \infty)$
Sign of $f(x)$	+	−	+	−

The function is positive for $x \in (-\infty, -5)$ and $x \in (0, 3)$.

The function is negative for $x \in (-5, 0)$ and $x \in (3, \infty)$.

b) i) Two x-intercepts are of order one, and the third is of order two, so the least possible degree is four. The graph extends from quadrant 2 to quadrant 1, so the leading coefficient is positive.

ii) The x-intercepts are -2, 1, and 3 (order 2). The factors are $x + 2$, $x - 1$, and $(x - 3)^2$.

iii)

Interval	$(-\infty, -2)$	$(-2, 1)$	$(1, 3)$	$(3, \infty)$
Sign of $f(x)$	+	−	+	+

The function is positive for $x \in (-\infty, -2)$, $x \in (1, 3)$, and $x \in (3, \infty)$ and negative for $x \in (-2, 1)$.

Example 2 Analysing Equations to Sketch Graphs of Polynomial Functions

Sketch a graph of each polynomial function.

a) $y = (x - 1)(x + 2)(x + 3)$

b) $y = -2(x - 1)^2(x + 2)$

c) $y = -(2x + 1)^3(x - 3)$

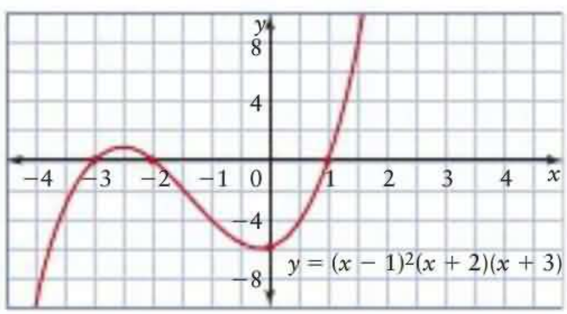
Solution

Use a table to organize information about each function. Then, use the information to sketch a graph.

a) $y = (x - 1)(x + 2)(x + 3)$

Degree	Leading Coefficient	End Behaviour	Zeros and x-intercepts	y-intercept
Each factor has one x. Their product is x^3. The function is cubic (degree 3).	The product of all the x-coefficients is 1.	A cubic with a positive leading coefficient extends from quadrant 3 to quadrant 1.	The zeros are 1, -2, and -3. These are the x-intercepts.	The y-intercept is $(0 - 2)(0 + 2)(0 + 3)$, or -6.

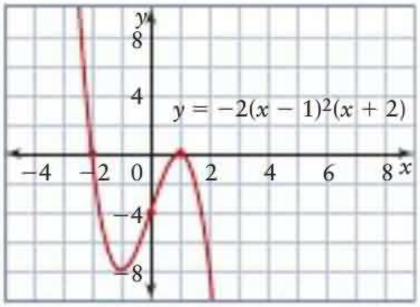
Mark the intercepts. Since the order of each zero is 1, the graph changes sign at each x-intercept. Beginning in quadrant 3, sketch the graph so that it passes up through $x = -3$ to the positive side of the x-axis, back down through $x = -2$ to the negative side of the x-axis, through the y-intercept -6, up through $x = 1$, and upward in quadrant 1.



b) $y = -2(x - 1)^2(x + 2)$

Degree	Leading Coefficient	End Behaviour	Zeros and x-intercepts	y-intercept
The product of all factors will give a multiple of x^3. The function is cubic (degree 3).	The product of all the x-coefficients is $-2 \times 1^2 \times 1$ or -2.	A cubic with a negative leading coefficient extends from quadrant 2 to quadrant 4.	The zeros are 1 (order 2) and -2. These are the x-intercepts.	The y-intercept is $-2(0 - 1)^2(0 + 2)$, or -4.

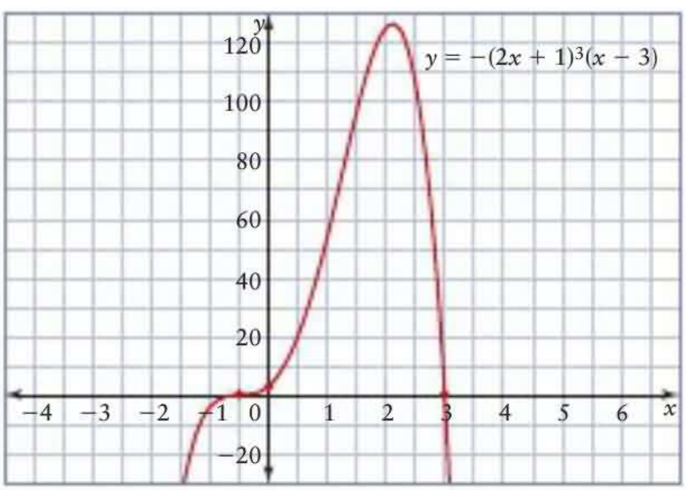
Mark the intercepts. The graph changes sign at $x = -2$ (order 1) but not at $x = 1$ (order 2). Begin in quadrant 2. Sketch the graph so that it passes through $x = -2$ to the negative side of the x-axis, through $y = -4$ and up to touch the x-axis at $x = 1$, and then down again, extending downward in quadrant 4.



c) $y = -(2x + 1)^3(x - 3)$

Degree	Leading Coefficient	End Behaviour	Zeros and x-intercepts	y-intercept
The product of the x's in the factors is x^4. The function is quartic (degree 4).	The product of all the x-coefficients is $-1 \times 2^3 \times 1$, or -8.	A quartic with a negative leading coefficient extends from quadrant 3 to quadrant 4.	The zeros are $-\frac{1}{2}$ (order 3) and 3. These are the x-intercepts.	The y-intercept is $-(2(0) + 1)^3(0 - 3)$, or 4.

Mark the intercepts. Since the order of each zero is odd, the graph changes sign at both intercepts. Beginning in quadrant 3, sketch the graph so that it passes up through $x = -\frac{1}{2}$ (flatter near the x-axis, since this zero is of order 3) to the positive side of the x-axis, through $y = 3$ continuing upward, and then back down to pass through $x = 3$, extending downward in quadrant 4.



Investigate 2 How is symmetry represented in the equation of a polynomial function?

Tools

- graphing calculator

Optional

- computer with *The Geometer's Sketchpad®*

A: ***Even-Degree Polynomial Functions***

1. a) Graph each function on a separate set of axes. Sketch each graph in your notebook.

i) $f(x) = x^4$

ii) $f(x) = x^4 - 8x^2$

iii) $f(x) = -x^4 + 9x^2$

iv) $f(x) = -x^6 + 7x^4 + 3x^2$

b) Reflect What type of symmetry does each graph have?

c) Reflect How can the exponents of the terms be used to identify these as even functions?

2. a) For each function in step 1a), determine $f(-x)$ by substituting $-x$ for x.

b) Reflect Compare $f(x)$ and $f(-x)$. What do you notice?

B: ***Odd-Degree Polynomial Functions***

1. a) Graph each function on a separate set of axes. Sketch each graph in your notebook.

i) $f(x) = x^3$

ii) $f(x) = x^3 - 4x$

iii) $f(x) = -x^5 + 16x^3$

iv) $f(x) = -x^5 + 5x^3 + 6x$

b) Reflect What type of symmetry does each graph have?

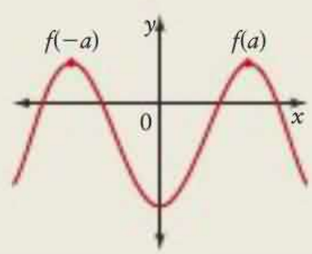
c) Reflect How can the exponents of the terms be used to identify these as odd functions?

2. a) For each function in step 1a), determine $f(-x)$ and $-f(x)$.

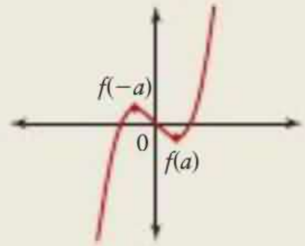
b) Reflect Compare $-f(x)$ and $f(-x)$. What do you notice?

3. Reflect Use the results of parts A and B to describe two ways that the equation of a polynomial function can be used to determine the type of symmetry exhibited by the graph of that function.

An even-degree polynomial function is an **even function** if the exponent of each term of the equation is even. An even function satisfies the property $f(-x) = f(x)$ for all x in the domain of $f(x)$. An even function is symmetric about the y-axis.



An odd-degree polynomial function is an **odd function** if each term of the equation has an odd exponent. An odd function satisfies the property $f(-x) = -f(x)$ for all x in the domain of $f(x)$. An odd function is rotationally symmetric about the origin.



Example 3 Identify Symmetry

Without graphing, determine if each polynomial function has line symmetry about the y-axis, point symmetry about the origin, or neither. Verify your response.

a) $f(x) = 2x^4 - 5x^2 + 4$
b) $f(x) = -3x^5 + 9x^3 + 2x$
c) $f(x) = 2x(x + 1)(x - 2)$
d) $f(x) = x^6 - 4x^3 + 6x^2 - 4$

Solution

a) Since the exponent of each term is even, $f(x) = 2x^4 - 5x^2 + 4$ is an even function and has line symmetry about the y-axis.
Verify that $f(-x) = f(x)$.

$$\begin{aligned} f(-x) &= 2(-x)^4 - 5(-x)^2 + 4 && \text{Substitute } -x \text{ in the equation.} \\ &= 2x^4 - 5x^2 + 4 \\ &= f(x) \end{aligned}$$

b) Since the exponent of each term is odd, $f(x) = -3x^5 + 9x^3 + 2x$ is an odd function and has point symmetry about the origin. Verify that $f(-x) = -f(x)$.

$$\begin{aligned} f(-x) &= -3(-x)^5 + 9(-x)^3 + 2(-x) && \text{Substitute } -x. \\ &= 3x^5 - 9x^3 - 2x \end{aligned}$$

$$\begin{aligned} -f(x) &= -(-3x^5 + 9x^3 + 2x) && \text{Multiply } f(x) \text{ by } -1. \\ &= 3x^5 - 9x^3 - 2x \end{aligned}$$

The resulting expressions are equal, so $f(-x) = -f(x)$.

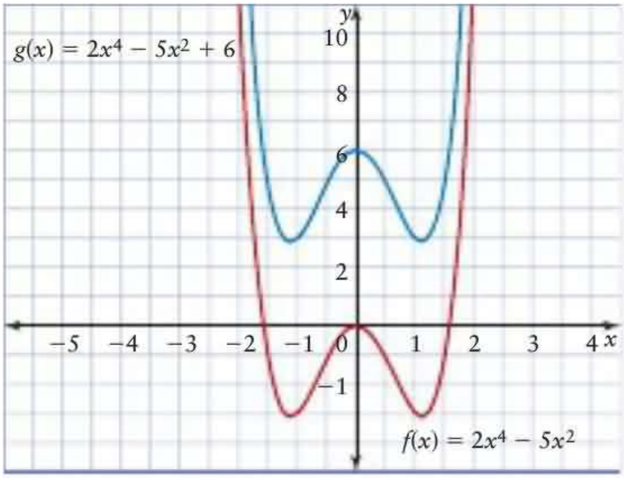
c) Since $f(x) = 2x(x + 1)(x - 2)$ is a cubic function, it may be odd and thus have point symmetry about the origin.

$$\begin{aligned} f(-x) &= 2(-x)(-x + 1)(-x - 2) && \text{Substitute } -x. \\ &= -2x(-1)(x - 1)(-1)(x + 2) && \text{Factor } -1 \text{ from each factor.} \\ &= -2x(x - 1)(x + 2) \end{aligned}$$

$$-f(x) = -2x(x + 1)(x - 2) \qquad \text{Multiply } f(x) \text{ by } -1.$$

The resulting expressions are not equal, so the function is not odd and does not have point symmetry about the origin.

d) Some exponents in $f(x) = x^6 - 4x^3 + 6x^2 - 4$ are even and some are odd, so the function is neither even nor odd and does not have line symmetry about the y-axis or point symmetry about the origin.



When a constant term is added to an even function, the function remains even. For example, the graph of $g(x) = 2x^4 - 5x^2 + 6$ represents a vertical translation of 6 units up of the graph of $f(x) = 2x^4 - 5x^2$. Thus, since $f(x) = 2x^4 - 5x^2$ is even and has line symmetry, the same is true for $g(x) = 2x^4 - 5x^2 + 6$.

CONNECTIONS

Recall that constant terms can be thought of as coefficients of x^0.

KEY CONCEPTS

- The graph of a polynomial function can be sketched using the x-intercepts, the degree of the function, and the sign of the leading coefficient.
- The x-intercepts of the graph of a polynomial function are the roots of the corresponding polynomial equation.
- When a polynomial function is in factored form, the zeros can be easily determined from the factors. When a factor is repeated n times, the corresponding zero has order n.
- The graph of a polynomial function changes sign only at x-intercepts that correspond to zeros of odd order. At x-intercepts that correspond to zeros of even order, the graph touches but does not cross the x-axis.
- An even function satisfies the property $f(-x) = f(x)$ for all x in its domain and is symmetric about the y-axis. An even-degree polynomial function is an even function if the exponent of each term is even.
- An odd function satisfies the property $f(-x) = -f(x)$ for all x in its domain and is rotationally symmetric about the origin. An odd-degree polynomial function is an odd function if the exponent of each term is odd.

Communicate Your Understanding

C1 Are all even-degree polynomial functions even? Are all odd-degree polynomial functions odd? Explain.

C2 Why is it useful to express a polynomial function in factored form?

C3 **a)** What is the connection between the order of the zeros of a polynomial function and the graph?

b) How can you tell from a graph if the order of a zero is 1, 2, or 3?

C4 How can symmetry be used to sketch a graph of a polynomial function?

A Practise

For help with questions 1 and 2, refer to Example 1.

1. For each polynomial function:

i) state the degree and the sign of the leading coefficient

ii) describe the end behaviour of the graph of the function

iii) determine the x-intercepts

a) $f(x) = (x - 4)(x + 3)(2x - 1)$

b) $g(x) = -2(x + 2)(x - 2)(1 + x)(x - 1)$

c) $h(x) = (3x + 2)^2(x - 4)(x + 1)(2x - 3)$

d) $p(x) = -(x + 5)^3(x - 5)^3$

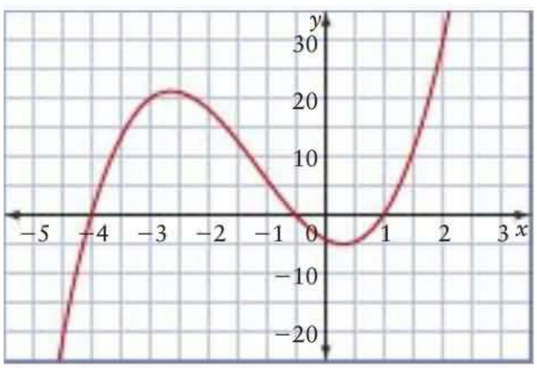
2. For each graph, do the following.

i) State the x-intercepts.

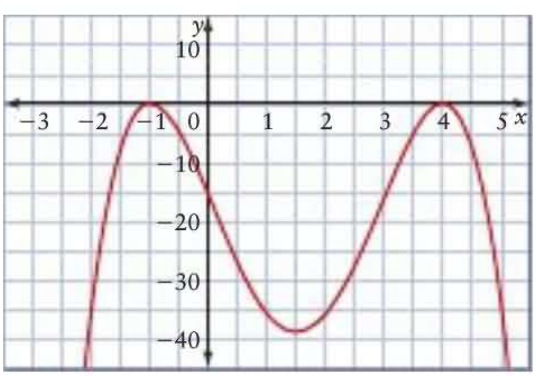
ii) State the intervals where the function is positive and the intervals where it is negative.

iii) Explain whether the graph might represent a polynomial function that has zeros of order 2 or of order 3.

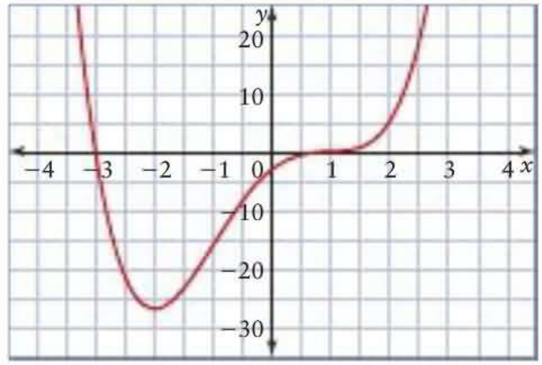
a)



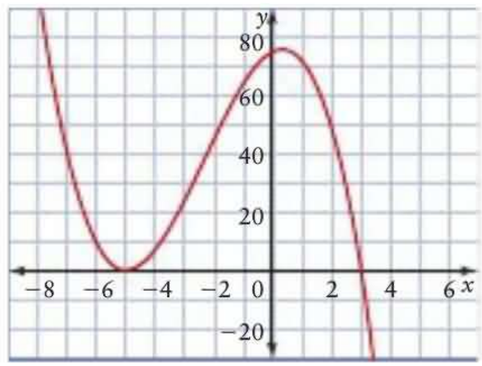
b)



c)



d)



For help with question 3, refer to Example 2.

3. a) Determine the zeros of each polynomial function. Indicate whether they are of order 1, 2, or 3.

i) $f(x) = -2(x - 3)(x + 2)(4x - 3)$

ii) $g(x) = (x - 1)(x + 3)(1 + x)(3x - 9)$

iii) $h(x) = -(x + 4)^2(x - 1)^2(x + 2)(2x - 3)$

iv) $p(x) = 3(x + 6)(x - 5)^2(3x - 2)^3$

b) Determine algebraically if each function is even or odd.

c) Sketch a graph of each function in part a).

For help with questions 4 and 5, refer to Example 3.

4. Determine, algebraically, whether each function in question 1 has point symmetry about the origin or line symmetry about the y-axis. State whether each function is even, odd, or neither. Give reasons for your answer.

5. i) Determine whether each function even, odd, or neither. Explain.

ii) Without graphing, determine if each polynomial function has line symmetry about the y-axis, point symmetry about the origin, or neither. Explain.

a) $y = x^4 - x^2$

b) $y = -2x^3 + 5x$

c) $y = -4x^5 + 2x^2$

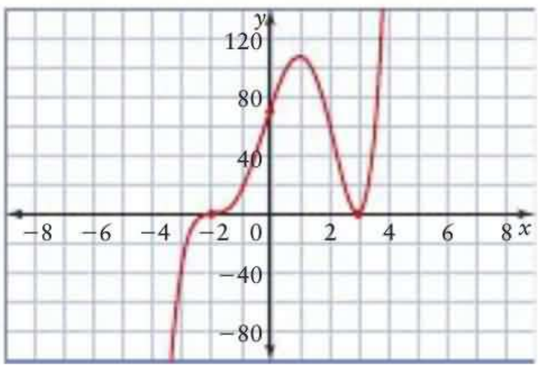
d) $y = x(2x + 1)^2(x - 4)$

e) $y = -2x^6 + x^4 + 8$

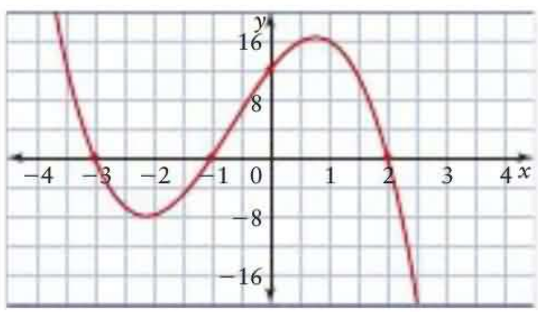
B Connect and Apply

6. Determine an equation for the polynomial function that corresponds to each graph.

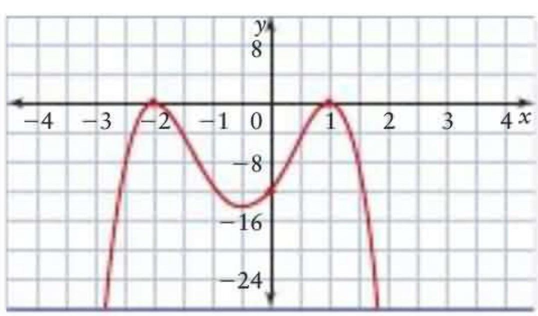
a)



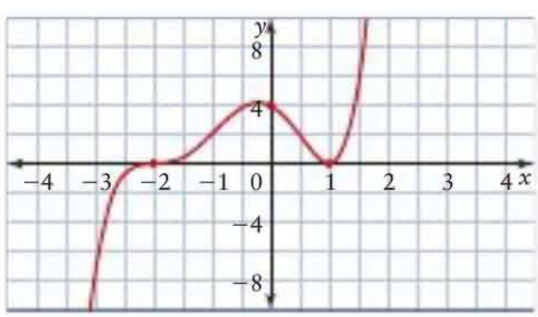
b)



c)



d)



7. Determine an equation for each polynomial function. State whether the function is even, odd, or neither. Sketch a graph of each.

a) a cubic function with zeros -2 (order 2) and 3 and y-intercept 9

b) a quartic function with zeros -1 (order 3) and 1 and y-intercept -2

c) a quintic function with zeros -1 (order 3) and 3 (order 2) that passes through the point $(-2, 50)$

d) a quintic function with zeros -3, -2 (order 2), and 2 (order 2) that passes through the point $(1, -18)$

8. Without graphing, determine if each polynomial function has line symmetry, point symmetry, or neither. Verify your response using technology.

a) $f(x) = -6x^5 + 2x$

b) $g(x) = -7x^6 + 3x^4 + 6x^2$

c) $h(x) = x^3 - 3x^2 + 5x$

d) $p(x) = -5x^3 + 2x$

9. Each polynomial function has zeros at -3, -1, 2. Write an equation for each function. Then, sketch a graph of the function.

a) a cubic function with a positive leading coefficient

b) a quartic function that touches the x-axis at -1

c) a quartic function that extends from quadrant 3 to quadrant 4

d) a quintic function that extends from quadrant 3 to quadrant 1

10. Chapter Problem An engineer designs a rollercoaster so that a section of the ride can be modelled by the function $h(x) = -0.000\,000\,4x(x - 15)(x - 25)(x - 45)^2(x - 60)$, where x is the horizontal distance from the boarding platform, in metres; $x \in [0, 60]$; and h is the height, in metres, above or below the boarding platform.

a) What are the similarities and differences between this polynomial function and those studied in Sections 1.1 and 1.2?

b) What useful information does this form of the equation provide that can be used to sketch a graph of the path of the rollercoaster?

c) Use the information from part b) to sketch a graph of this section of the rollercoaster.

d) Estimate the maximum and the minimum height of the rollercoaster relative to the boarding platform.

11. a) Determine the zeros of $f(x) = (2x^2 - x - 1)(x^2 - 3x - 4)$.

b) Use graphing technology to verify your answer.

12. a) Determine the zeros of each polynomial function.

i) $f(x) = x^4 - 13x^2 + 36$

ii) $g(x) = 6x^5 - 7x^3 - 3x$

b) State whether each function is even, odd, or neither. Verify your answers algebraically.

c) Sketch a graph of each function.

C Extend and Challenge

13. Use Technology Consider the polynomial function $f(x) = (x - 3)(x - 1)(x + 2)^2 + c$, where c is a constant. Determine a value of c such that the graph of the function has each number of x-intercepts. Justify your answer graphically.

a) four

b) three

c) two

d) one

e) zero

14. a) Write equations for two even functions with four x-intercepts, two of which are $\frac{2}{3}$ and 5.

b) Determine an equation for a function with x-intercepts at $\frac{2}{3}$ and 5, passing through the point $(-1, -96)$.

c) Determine an equation for a function with x-intercepts at $\frac{2}{3}$ and 5 that is a reflection in the x-axis of the function in part b).

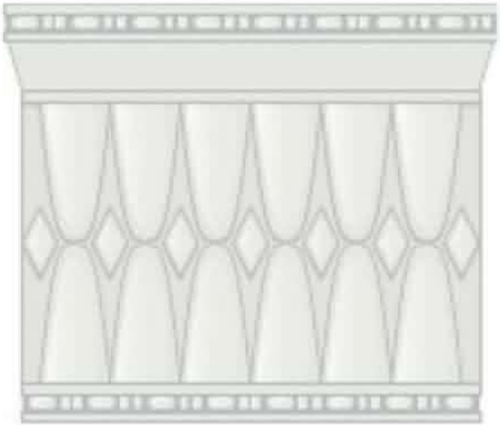
15. Explain algebraically why a polynomial that is an odd function is no longer an odd function when a nonzero constant is added.

16. Math Contest If the value of a continuous function $f(x)$ changes sign in an interval, there is a root of the equation $f(x) = 0$ in that interval. For example, $f(x) = x^3 - 4x - 2$ has a zero between 2 and 3. Evaluate the function at the endpoints and at the midpoint of the interval. This gives $f(2) = -2$, $f(2.5) = 3.625$, and $f(3) = 13$. The function changes sign between $x = 2$ and $x = 2.5$, so a root lies in this interval. Since $f(2.25) \doteq 0.39$, there is a root between $x = 2$ and $x = 2.25$. Continuing in this way gives increasingly better approximations of that root.

a) Determine, correct to two decimal places, the root of $x^3 - 3x + 1 = 0$ that lies between 0 and 1.

b) Calculate the greatest root of $2x^3 - 4x^2 - 3x + 1 = 0$, to three decimal places.

1.4 Transformations



In the architectural design of a new hotel, a pattern is to be carved in the exterior crown moulding. What power function forms the basis of the pattern? What transformations are applied to the power function to create the pattern?

In this section, you will investigate the roles of the parameters a, k, d, and c in polynomial functions of the form $y = a[k(x - d)]^n + c$. You will determine equations to model given transformations. You will also apply transformations to the graphs of basic power functions of the form $y = x^n$ to sketch the graphs of functions of the form $y = a[k(x - d)]^n + c$.

Investigate — What are the roles of a, k, d, and c in polynomial functions of the form $y = a[k(x - d)]^n + c$, where $n \in \mathbb{N}$?

Tools

- graphing calculator

Optional

- computer with *The Geometer's Sketchpad®*

Apply your prior knowledge of transformations to predict the roles of the parameters a, k, d, and c in polynomial functions of the form $y = a[k(x - d)]^n + c$. Complete each part to verify the accuracy of your prediction.

A: ***Describe the Roles of d and c in Polynomial Functions of the Form $y = a[k(x - d)]^n + c$***

1. a) Graph each group of functions on one set of axes. Sketch the graphs in your notebook.

Group A	Group B
i) $y = x^3$	**i)** $y = x^4$
ii) $y = x^3 + 2$	**ii)** $y = (x + 2)^4$
iii) $y = x^3 - 2$	**iii)** $y = (x - 2)^4$

b) Compare the graphs in group A. For any constant c, describe the relationship between the graphs of $y = x^3$ and $y = x^3 + c$.

c) Compare the graphs in group B. For any constant d, describe the relationship between the graphs of $y = x^4$ and $y = (x - d)^4$.

2. Reflect Describe the roles of the parameters c and d in functions of the form $y = a[k(x - d)]^n + c$. Summarize your results in tables like these.

Value of c in $y = a[k(x - d)]^n + c$	Effect on the Graph of $y = x^n$
$c > 0$	
$c < 0$	

Value of d in $y = a[k(x - d)]^n + c$	Effect on the Graph of $y = x^n$
$d > 0$	
$d < 0$	

B: *Describe the Roles of a and k in Polynomial Functions of the Form $y = a[k(x - d)]^n + c$*

1. a) Graph each group of functions on one set of axes. Sketch the graphs in your notebook.

Group A	Group B
i) $y = x^3$	**i)** $y = x^4$
ii) $y = 3x^3$	**ii)** $y = \frac{1}{3}x^4$
iii) $y = -3x^3$	**iii)** $y = -\frac{1}{3}x^4$

b) Compare the graphs in group A. For any integer value a, describe the relationship between the graphs of $y = x^3$ and $y = ax^3$.

c) Compare the graphs in group B. For any rational value a such that $a \in (-1, 0)$ or $a \in (0, 1)$, describe the relationship between the graphs of $y = x^4$ and $y = ax^4$.

d) **Reflect** Describe the role of the parameter a in functions of the form $y = a[k(x - d)]^n + c$.

2. a) Graph each group of functions on one set of axes. Sketch the graphs in your notebook.

Group A	Group B
i) $y = x^3$	**i)** $y = x^4$
ii) $y = (3x)^3$	**ii)** $y = \left(\frac{1}{3}x\right)^4$
iii) $y = (-3x)^3$	**iii)** $y = \left(-\frac{1}{3}x\right)^4$

b) Compare the graphs in group A. For any integer value k, describe the relationship between the graphs of $y = x^3$ and $y = (kx)^3$.

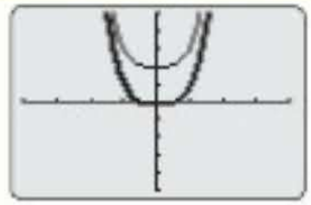
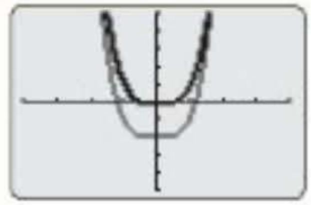
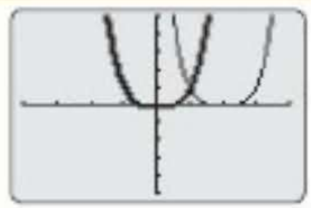
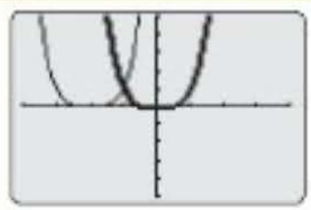
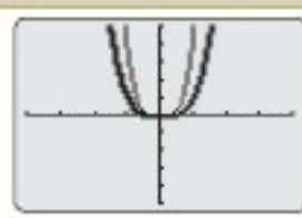
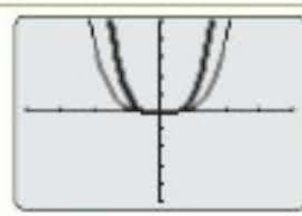
c) Compare the graphs in group B. For any value $k \in (-1, 0)$ or $k \in (0, 1)$, describe the relationship between the graphs of $y = x^4$ and $y = (kx)^4$.

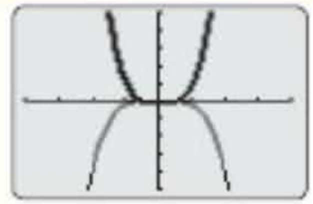
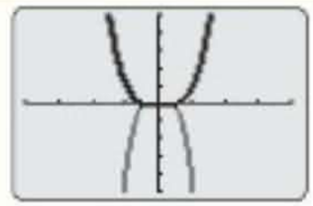
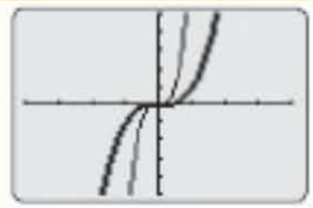
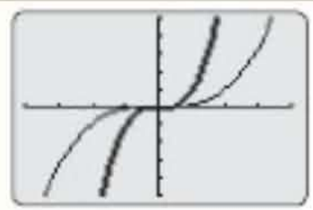
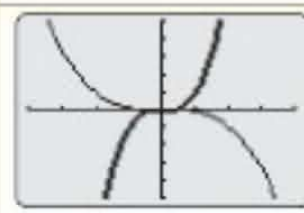
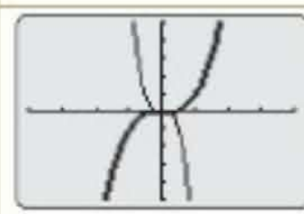
d) **Reflect** Describe the role of the parameter k in functions of the form $y = a[k(x - d)]^n + c$.

3. Summarize your results in tables like these.

Value of a in $y = a[k(x - d)]^n + c$	Effect on the Graph of $y = x^n$
$a > 1$	
$0 < a < 1$	
$-1 < a < 0$	
$a < 1$	

Value of k in $y = a[k(x - d)]^n + c$	Effect on the Graph of $y = x^n$
$k > 1$	
$0 < k < 1$	
$-1 < k < 0$	
$k < -1$	

The Roles of the Parameters a, k, d, and c in Polynomial Functions of the Form $y = a[k(x - d)]^n + c$, where $n \in \mathbb{N}$		
Value of c in $y = a[k(x - d)]^n + c$	**Transformation of the Graph of $y = x^n$**	**Example Using the Graph of $y = x^4$**
$c > 0$	Translation c units up	 $c = 2$ $y = x^4 + 2$
$c < 0$	Translation c units down	 $c = -2$ $y = x^4 - 2$
Value of d in $y = a[k(x - d)]^n + c$		
$d > 0$	Translation d units right	 $d = 2$ $y = (x - 2)^4$
$d < 0$	Translation d units left	 $d = -2$ $y = (x + 2)^4$
Value of a in $y = a[k(x - d)]^n + c$		
$a > 1$	Vertical stretch by a factor of a	 $a = 4$ $y = 4x^4$
$0 < a < 1$	Vertical compression by a factor of a	 $a = 0.25$ $y = 0.25x^4$

The Roles of the Parameters a, k, d, and c in Polynomial Functions of the Form $y = a[k(x - d)]^n + c$, where $n \in \mathbb{N}$		
Value of a in $y = a[k(x - d)]^n + c$	**Transformation of the Graph of $y = x^n$**	**Example Using the Graph of $y = x^4$**
$-1 < a < 0$	Vertical compression by a factor of $\lvert a \rvert$ and a reflection in the x-axis	 $a = -0.25$ $y = -0.25x^4$
$a < -1$	Vertical stretch by a factor of $\lvert a \rvert$ and a reflection in the x-axis	 $a = -4$ $y = -4x^4$
Value of k in $y = a[k(x - d)]^n + c$	**Transformation of the Graph of $y = x^n$**	**Example Using the Graph of $y = x^3$**
$k > 0$	Horizontal compression by a factor of $\frac{1}{k}$	 $k = 2$ $y = (2x)^3$
$0 < k < 1$	Horizontal stretch by a factor of $\frac{1}{k}$	 $k = 0.5$ $y = (0.5x)^3$
$-1 < k < 0$	Horizontal stretch by a factor of $\left\lvert \frac{1}{k} \right\rvert$ and a reflection in the y-axis	 $k = -0.5$ $y = (-0.5x)^3$
$k < -1$	Horizontal compression by a factor of $\left\lvert \frac{1}{k} \right\rvert$ and a reflection in the y-axis	 $k = -2$ $y = (-2x)^3$

The graph of a function of the form $y = a[k(x - d)]^n + c$ is obtained by applying transformations to the graph of the power function $y = x^n$. An accurate sketch of the transformed graph is obtained by applying the transformations represented by a and k *before* the transformations represented by c and d.

CONNECTIONS

The order of transformations coincides with the order of operations on numerical expressions. Multiplication and division (represented by reflections, stretches, and compressions) are applied before addition and subtraction (translations). Transformations represented by *a* and *k* may be applied at the same time, followed by *c* and *d* together.

Example 1 Applying Transformations to Sketch a Graph

The graph of $y = x^3$ is transformed to obtain the graph of $y = -3[2(x + 1)]^3 + 5$.

a) State the parameters and describe the corresponding transformations.

b) Complete the table.

$y = x^3$	$y = (2x)^3$	$y = -3(2x)^3$	$y = -3[2(x + 1)]^3 + 5$
(−2, −8)			
(−1, −1)			
(0, 0)			
(1, 1)			
(2, 8)			

c) Sketch a graph of $y = -3[2(x + 1)]^3 + 5$.

d) State the domain and range.

Solution

a) The base power function is $f(x) = x^3$.

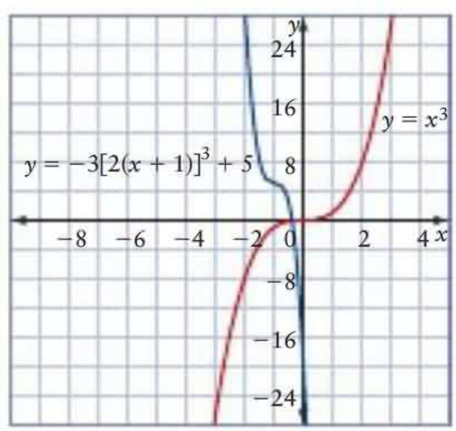
Compare $y = -3[2(x + 1)]^3 + 5$ to $y = a[k(x - d)]^n + c$.

- $k = 2$ corresponds to a horizontal compression of factor $\frac{1}{2}$. Divide the x-coordinates of the points in column 1 by 2.
- $a = -3$ corresponds to a vertical stretch of factor 3 and a reflection in the x-axis. Multiply the y-coordinates of the points in column 2 by -3.
- $d = -1$ corresponds to a translation of 1 unit to the left and $c = 5$ corresponds to a translation of 5 units up.

b)

$y = x^3$	$y = (2x)^3$	$y = -3(2x)^3$	$y = -3[2(x + 1)]^3 + 5$
(−2, −8)	(−1, −8)	(−1, 24)	(−2, 29)
(−1, −1)	(−0.5, −1)	(−0.5, 3)	(−1.5, 8)
(0, 0)	(0, 0)	(0, 0)	(−1, 5)
(1, 1)	(0.5, 1)	(0.5, −3)	(−0.5, 2)
(2, 8)	(1, 8)	(1, −24)	(0, −19)

c) To sketch the graph, plot the points from column 4 and draw a smooth curve through them.



d) There are no restrictions on the domain or range.

The domain is $\{x \in \mathbb{R}\}$ and the range is $\{y \in \mathbb{R}\}$.

When n is even, the graphs of polynomial functions of the form $y = a[k(x - d)]^n + c$ are even functions and have a vertex at (d, c). The axis of symmetry is $x = d$.

For $a > 0$, the graph opens upward. The vertex is the minimum point on the graph and c is the minimum value. The range of the function is $\{y \in \mathbb{R}, y \geq c\}$.

For $a < 0$, the graph opens downward. The vertex is the maximum point on the graph and c is the maximum value. The range of the function is $\{y \in \mathbb{R}, y \leq c\}$.

Example 2 Describing Transformations From an Equation

i) Describe the transformations that must be applied to the graph of each power function, $f(x)$, to obtain the transformed function. Then, write the corresponding equation.

ii) State the domain and range. State the vertex and the equation of the axis of symmetry for functions that are even.

a) $f(x) = x^4$, $y = 2f\left[\frac{1}{3}(x - 5)\right]$ **b)** $f(x) = x^5$, $y = \frac{1}{4}f[-2x + 6] + 4$

Solution

Compare the transformed equation with $y = af[k(x - d)]^n + c$ to determine the values of a, k, d, and c.

a) **i)** For $y = 2f\left[\frac{1}{3}(x - 5)\right]$, the parameters are $a = 2$, $k = \frac{1}{3}$, $d = 5$, and $c = 0$.
The function $f(x) = x^4$ is stretched vertically by a factor of 2, stretched horizontally by a factor of 3, and translated 5 units to the right, so the equation of the transformed function is $y = 2\left[\frac{1}{3}(x - 5)\right]^4$.

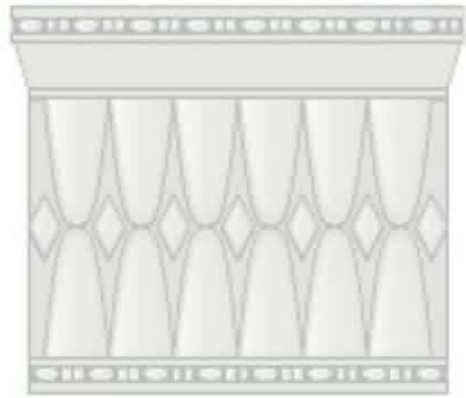
ii) This is a quartic function with $a = 2$, so it opens upward.
vertex $(5, 0)$; axis of symmetry $x = 5$; domain $\{x \in \mathbb{R}\}$; range $\{y \in \mathbb{R}, y \geq 0\}$

b) **i)** $y = \frac{1}{4}f(-2x + 6) + 4$ is not in the form $y = af[k(x - d)]^n + c$ since $-2x + 6$ is not expressed in the form $k(x - d)$. To determine the value of k, factor -2 from the expression $-2x + 6$:
$y = \frac{1}{4}f[-2(x - 3)] + 4$
This is now in the desired form. The parameters are $a = \frac{1}{4}$, $k = -2$, $d = 3$, and $c = 4$. The function $f(x) = x^5$ is compressed vertically by a factor of $\frac{1}{4}$, compressed horizontally by a factor of $\frac{1}{2}$, reflected in the y-axis, translated 3 units to the right, and translated 4 units up, so the equation of the transformed function is $y = \frac{1}{4}[-2(x - 3)]^5 + 4$.

ii) This is a quintic function.
domain $\{x \in \mathbb{R}\}$; range $\{y \in \mathbb{R}\}$

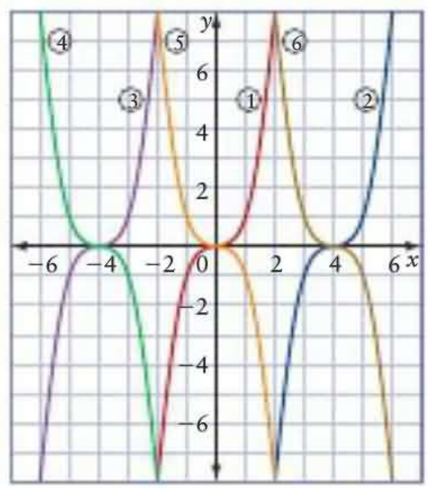
Example 3 Determine an Equation Given the Graph of a Transformed Function

Recall the crown moulding pattern introduced at the beginning of this section. Determine equations that could make up this pattern.



Solution

Overlay the pattern on a grid and identify features of the graphs. The pattern is created by transforming a single polynomial function. Use points to identify the main power function. Then, determine the transformations that need to be applied to create the other graphs, and hence the entire pattern.



Examine graph ①. Since the graph extends from quadrant 3 to quadrant 1, it represents an odd-degree function with a positive leading coefficient. Some points on this graph are $(-2, -8)$, $(-1, -1)$, $(0, 0)$, $(1, 1)$, and $(2, 8)$. These points satisfy $y = x^3$, so an equation for this graph is $y = x^3$.

Consider graphs ② and ③. Each of these is a horizontal translation of graph ①. The point $(0, 0)$ on graph ① corresponds to the point $(4, 0)$ on graph ②. Thus, to obtain graph ②, translate the graph of $y = x^3$ to the right 4 units. An equation for graph ② is $y = (x - 4)^3$. To obtain graph ③, translate the graph of $y = x^3$ to the left 4 units. An equation for graph ③ is $y = (x + 4)^3$.

To obtain graph ⑤, reflect the graph of $y = x^3$ in the x-axis. An equation for graph ⑤ is $y = -x^3$.

To obtain graph ⑥, translate graph ⑤ to the right 4 units. Its equation is $y = -(x - 4)^3$.

To obtain graph ④, translate graph ⑤ to the left 4 units. Its equation is $y = -(x + 4)^3$.

Thus, the pattern is created by graphing the functions $y = x^3$, $y = (x - 4)^3$, $y = (x + 4)^3$, $y = -x^3$, $y = -(x - 4)^3$, and $y = -(x + 4)^3$.

KEY CONCEPTS

- The graph of a polynomial function of the form $y = a[k(x - d)]^n + c$ can be sketched by applying transformations to the graph of $y = x^n$, where $n \in \mathbb{N}$. The transformations represented by a and k must be applied before the transformations represented by c and d.
- The parameters a, k, d, and c in polynomial functions of the form $y = a[k(x - d)]^n + c$, where n is a non-negative integer, correspond to the following transformations:
  - a corresponds to a vertical stretch or compression and, if $a < 0$, a reflection in the x-axis
  - k corresponds to a horizontal stretch or compression and, if $k < 0$, a reflection in the y-axis
  - c corresponds to a vertical translation up or down
  - d corresponds to a horizontal translation to the left or right

Communicate Your Understanding

C1 **a)** Which parameters cause the graph of a power function to become wider or narrower?

b) Describe what values of the parameters in part a) make a graph

i) wider **ii)** narrower

C2 Which parameters do not change the shape of a power function? Provide an example.

C3 Which parameters can cause the graph of a power function to be reflected? Describe the type of reflections.

C4 **a)** Describe the order in which the transformations should be applied to obtain an accurate graph.

b) What sequences of transformations produce the same result?

A Practise

For help with question 1, refer to Example 1.

1. **a)** The graph of $y = x^4$ is transformed to obtain the graph of $y = 4[3(x + 2)]^4 - 6$. State the parameters and describe the corresponding transformations.

b) Copy and complete the table.

$y = x^4$	$y = (3x)^4$	$y = 4(3x)^4$	$y = 4[3(x + 2)]^4 - 6$
$(-2, 16)$			
$(-1, 1)$			
$(0, 0)$			
$(1, 1)$			
$(2, 16)$			

c) Sketch a graph of $y = 4[3(x + 2)]^4 - 6$.

d) State the domain and range, the vertex, and the equation of the axis of symmetry.

For help with questions 2 to 4, refer to Example 2.

2. Match each function with the corresponding transformation of $y = x^n$.

a) $y = -x^n$

b) $y = (-x)^n + 2$

c) $y = -(-x)^n$

d) $y = x^n$

i) no reflection

ii) reflection in the x-axis

iii) reflection in the x-axis and the y-axis

iv) reflection in the y-axis

3. Match each function with the corresponding transformation of $y = x^n$.

a) $y = 2x^n$

b) $y = (2x)^n$

c) $y = \frac{1}{2}x^n$

d) $y = \left(\frac{1}{2}x\right)^n$

i) horizontally stretched by a factor of 2

ii) vertically compressed by a factor of $\frac{1}{2}$

iii) vertically stretched by a factor of 2

iv) horizontally compressed by a factor of $\frac{1}{2}$

4. Compare each polynomial function with the equation $y = a[k(x - d)]^n + c$. State the values of the parameters a, k, d, and c and the degree n, assuming that the base function is a power function. Describe the transformation that corresponds to each parameter.

a) $y = (3x)^3 - 1$

b) $y = 0.4(x + 2)^2$

c) $y = x^3 + 5$

d) $y = \frac{3}{4}[-(x - 4)]^3 + 1$

e) $y = 2\left(\frac{1}{3}x\right)^4 - 5$

f) $y = 8[(2x)^3 + 3]$

For help with question 5, refer to Example 3.

5. Match each graph with the corresponding function. Justify your choice.

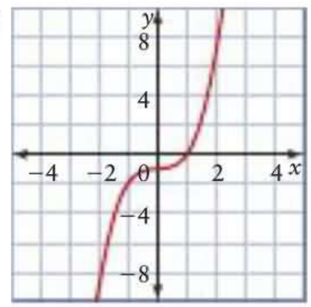
i) $y = -\frac{1}{4}x^3$

ii) $y = x^3 - 1$

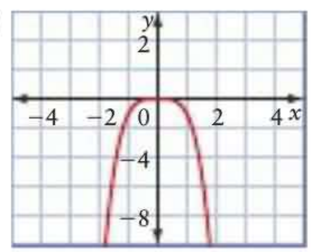
iii) $y = \left(-\frac{1}{4}x\right)^5$

iv) $y = -x^4$

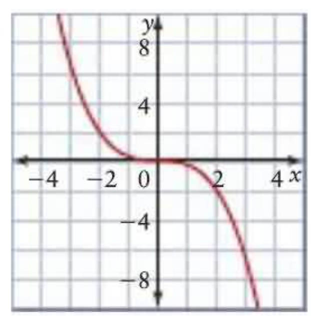
a)



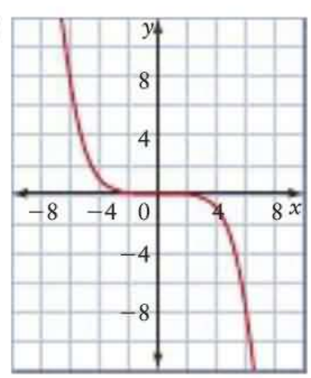
b)



c)



d)



B Connect and Apply

For help with questions 6 to 8, refer to Example 2.

6 Describe the transformations that must be

ii) State the domain and range. For even functions, give the vertex and the equation

12. i) Write an equation for the function that results from the given transformations.

ii) State the domain and range. For even functions, give the vertex and the equation of the axis of symmetry.

a) The function $f(x) = x^4$ is translated 2 units to the left and 3 units up.

b) The function $f(x) = x^5$ is stretched horizontally by a factor of 5 and translated 12 units to the left.

c) The function $f(x) = x^4$ is stretched vertically by a factor of 3, reflected in the x-axis, and translated 6 units down and 1 unit to the left.

d) The function $f(x) = x^6$ is reflected in the x-axis, stretched horizontally by a factor of 5, reflected in the y-axis, and translated 3 units down and 1 unit to the right.

e) The function $f(x) = x^6$ is compressed horizontally by a factor of $\frac{4}{5}$, stretched vertically by a factor of 7, reflected in the x-axis, and translated 1 unit to the left and 9 units up.

✓ Achievement Check

13. a) The graph of $y = x^4$ is transformed to obtain the graph of $y = \frac{1}{4}[-2(x - 1)]^4 + 2$. List the parameters and describe the corresponding transformations.

b) Sketch a graph of $y = \frac{1}{4}[-2(x - 1)]^4 + 2$.

C Extend and Challenge

14. a) Predict the relationship between the graph of $y = x^3 - x^2$ and the graph of $y = (x - 2)^3 - (x - 2)^2$.

b) **Use Technology** Graph each function using technology to verify the accuracy of your prediction.

c) Factor each function in part a) to determine the x-intercepts.

15. **Use Technology**

a) Describe the transformations that must be applied to the graph of $y = x^4 - x^3 + x^2$ to obtain the graph of $y = -3\left(\left[\frac{1}{2}(x + 4)\right]^4 - \left[\frac{1}{2}(x + 4)\right]^3 + \left[\frac{1}{2}(x + 4)\right]^2\right)$.

b) Sketch each graph using technology.

c) Factor each function in part a) to determine the x-intercepts.

16. a) The function $h(x) = 3(x - 3)(x + 2)(x - 5)$ is translated 4 units to the left and 5 units down. Write an equation for the transformed function.

b) Suppose the transformed function is then reflected in the x-axis and vertically compressed by a factor of $\frac{2}{5}$. Write an equation for the new transformed function.

17. **Math Contest** A farm has a sale on eggs, selling 13 eggs for the usual price of a dozen eggs. As a result, the price of eggs is reduced by 24 cents a dozen. What was the original price for a dozen eggs?

18. **Math Contest** Given $f_0(x) = x^2$ and $f_{n+1} = f_0(f_n(x))$, where n is any natural number

a) determine $f_1(x)$, $f_2(x)$, and $f_3(x)$

b) determine a formula for $f_n(x)$ in terms of n

1.5 Slopes of Secants and Average Rate of Change

Change occurs in many aspects of everyday life. A person's height and mass change from birth to adulthood. The distance that a car travels in a period of time changes according to its speed. Temperature fluctuations occur daily and with each season. The value of a house or antique may increase, or appreciate, over a period of time. The cost of a train ticket varies with the distance travelled. Some things change over a long period of time, while others change in an instant. Being able to understand and predict the rate at which a change occurs can provide useful information.

A **rate of change** is a measure of the change in one quantity (the dependent variable) with respect to a change in another quantity (the independent variable). There are two types of rates of change, average and instantaneous. An **average rate of change** is a change that takes place over an interval, while an instantaneous rate of change is a change that takes place in an instant. Instantaneous rates of change will be examined more closely in the next section of this chapter.

Consider the following scenario.

A car leaves Ottawa at 12 p.m. and arrives in Toronto at 4 p.m. after travelling a distance of 400 km. The average rate of change of distance with respect to time—the average velocity—is determined as shown:

$$\begin{aligned}\text{Average velocity} &= \frac{\text{change in distance}}{\text{change in time}}\\ &= \frac{\Delta d}{\Delta t}\\ &= \frac{400}{4}\\ &= 100\end{aligned}$$

Therefore, the average velocity of the car is 100 km/h.

Investigate How can you connect average rate of change and slope?

1. Seismic activity at a certain point on the ocean floor creates a wave that spreads in a circular pattern over the calm surface of the ocean. The table shows the radius of the circular pattern during the first 10 s as the wave moves outward.

Time, t (s)	Radius, r (m)
0	0
1	2
2	4
3	6
4	8
5	10
6	12
7	14
8	16
9	18
10	20

a) Identify the independent variable and the dependent variable. Justify your choice.

b) Determine $\frac{\Delta r}{\Delta t} = \frac{\text{change in radius}}{\text{change in time}}$ for each time interval.

i) [0, 10] **ii)** [0, 1] **iii)** [9, 10]

c) **Reflect** Interpret the values found in part b). State the units for these values.

d) Graph the data. What type of polynomial function does the graph represent? Explain.

e) A **secant** is a line that connects two points on a curve. Draw a secant on the graph to connect the pair of points associated with each time interval in part b). What is the slope of each secant line?

f) **Reflect** What is the relationship between the values found in part b) and the graph? Explain.

2. This table shows the total area covered by the wave during the first 10 s.

Radius, r (m)	Area, A (m^2)
0	0
2	12.57
4	50.27
6	113.10
8	201.06
10	314.16
12	452.39
14	615.75
16	804.25
18	1017.88
20	1256.64

a) Identify the independent variable and the dependent variable. Justify your choice.

b) Determine $\frac{\Delta A}{\Delta r} = \frac{\text{change in area}}{\text{change in radius}}$ for each radius interval.

i) [0, 20]

ii) [0, 4]

iii) [6, 12]

iv) [0, 2]

v) [14, 16]

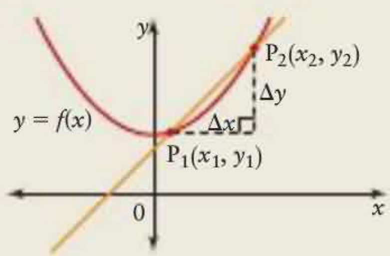
c) **Reflect** Interpret the values found in part b). State the units for these values.

d) Graph the data. What type of polynomial function does the graph represent? Explain.

e) On the graph, draw a secant to connect the pair of points associated with each radius interval in part b). What is the slope of each secant line?

f) **Reflect** What is the relationship between the values found in part b) and the secant lines? How are these related to the shape of the graph? Explain.

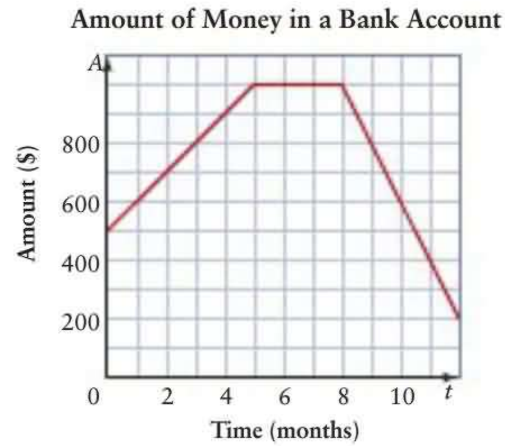
The average rate of change between two points corresponds to the slope of the secant between the points. For example, the average rate of change of y with respect to x between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is determined as follows:



$$\begin{aligned}\text{Average rate of change} &= \frac{\Delta y}{\Delta x}\\ &= \frac{\text{change in } y}{\text{change in } x}\\ &= \frac{y_2 - y_1}{x_2 - x_1}\end{aligned}$$

Example 1 Calculate and Interpret Average Rates of Change From a Graph

The graph represents the amount of money in a bank account over a 1-year period.



a) How much money was in the account

i) at the start of the year?

ii) at the end of the year?

b) What does the graph tell you about the average rate of change of the amount of money in the account in the following intervals:

i) month 0 to month 5

ii) month 5 to month 8

iii) month 8 to month 12

c) Determine the average rate of change for the time periods in part b). Interpret these values for this situation.

Solution

a) The first point on the graph is (0, 500) and the last point is (12, 200), so

i) the initial amount of money in the account is $500

ii) the amount at the end of the year is $200

b) **i)** Between month 0 and month 5, the graph is a line with positive slope. The average rate of change is constant and positive. The amount of money in the account is increasing by the same amount each month.

ii) Between month 5 and month 8, the graph is a horizontal line with zero slope. The average rate of change is 0. The amount of money in the account does not change.

iii) Between month 8 and month 12, the graph is a line with negative slope. The average rate of change is constant and negative. The amount of money in the account is decreasing by the same amount each month.

c) **i)** The points that correspond to month 0 and month 5 are (0, 500) and (5, 1000).

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{change in amount}}{\text{change in time}}\\ &= \frac{1000 - 500}{5 - 0}\\ &= \frac{500}{5}\\ &= 100\end{aligned}$$

The amount of money in the account is increasing on average by \$100 per month.

ii) The points that correspond to month 5 and month 8 are (5, 1000) and (8, 1000).

$$\begin{aligned}\text{Average rate of change} &= \frac{1000 - 1000}{8 - 5}\\ &= \frac{0}{3}\\ &= 0\end{aligned}$$

No change occurs. The amount of money in the account remains the same.

iii) The points that correspond to month 8 and month 12 are (8, 1000) and (12, 200).

$$\begin{aligned}\text{Average rate of change} &= \frac{200 - 1000}{12 - 8}\\ &= \frac{-800}{4}\\ &= -200\end{aligned}$$

The negative value indicates that the amount of money is decreasing.

The amount of money in the account is decreasing on average by \$200 per month.

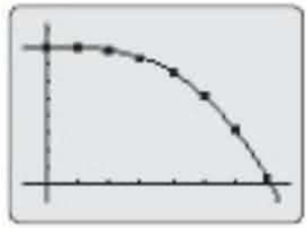
Notice that the average rate of change and the slope are the same for each time period.

Example 2 Calculate and Interpret the Average Rate of Change From a Table of Values

Recall the problem from Example 3 in Section 1.2, page 22.

A new antibacterial spray is tested on a bacterial culture. The table shows the population, P, of the bacterial culture t minutes after the spray is applied.

t (min)	P
0	800
1	799
2	782
3	737
4	652
5	515
6	314
7	37



a) How can you tell that the average rate of change is negative by examining

i) the table of values? **ii)** the graph?

b) Determine the average rate of change of the number of bacteria over the entire time period shown in the table. Interpret this value for this situation.

c) Compare the average rate of change of the number of bacteria in the first 3 min and in the last 3 min. Explain any similarities and differences.

d) How can you tell that this situation involves a non-constant rate of change by examining

i) the table of values? **ii)** the graph? **iii)** the average rate of change?

Solution

a) **i)** From the table, the value of the dependent variable, P, is decreasing as the value of the independent variable, t, is increasing. So, the average rate of change will be negative.

ii) The graph decreases from left to right, so the slope of any secant will be negative.

b) From the table, the points are (0, 800) and (7, 37).

$$\begin{aligned}\text{Average rate of change} &= \frac{\Delta P}{\Delta t}\\ &= \frac{37 - 800}{7 - 0}\\ &= \frac{-763}{7}\\ &= -109\end{aligned}$$

During the entire 7 min, the number of bacteria decreases on average by 109 bacteria per minute.

c) From the table, the endpoints of the first interval are (0, 800) and (3, 737).

$$\begin{aligned}\text{Average rate of change} &= \frac{\Delta P}{\Delta t}\\ &= \frac{737 - 800}{3 - 0}\\ &= \frac{-63}{3}\\ &= -21\end{aligned}$$

During the first 3 min, the number of bacteria decreases on average by 21 bacteria per minute.

From the table, the endpoints of the last interval are (4, 652) and (7, 37).

$$\begin{aligned}\text{Average rate of change} &= \frac{\Delta P}{\Delta t}\\ &= \frac{37 - 652}{7 - 4}\\ &= \frac{-615}{3}\\ &= -205\end{aligned}$$

During the last 3 min, the number of bacteria decreases on average by 205 bacteria per minute.

Similarity: The average rates of change in the number of bacteria are decreasing.

Difference: The average rate of decrease is greater in the last 3 min than in the first 3 min. This may be because more bacteria are exposed to the spray as time passes.

d) **i)** From the table, the number of bacteria is decreasing at a different rate. The change in the dependent variable varies for each 1-min increase in the independent variable.

ii) The graph is a curve rather than a straight line, so the rate of change is not constant.

iii) The value of the average rate of change varies, which indicates that the bacteria population decreases at a non-constant rate. In the first 3 min, the average rate of decrease in the bacteria population is 21 bacteria per minute, while in the last 3 min it is 205 bacteria per minute.

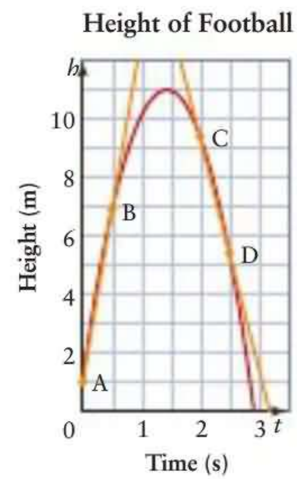
Example 3 Calculate and Interpret an Average Rate of Change From an Equation

A football is kicked into the air such that its height, h, in metres, after t seconds can be modelled by the function $h(t) = -4.9t^2 + 14t + 1$.

a) Determine the average rate of change of the height of the ball for each time interval.

i) $[0, 0.5]$ **ii)** $[2, 2.5]$

b) Consider the graph of $h(t) = -4.9t^2 + 14t + 1$ with secant lines AB and CD. Describe the relationship between the values in part a), the secant lines, and the graph.



Solution

a) Use the equation to determine the endpoints corresponding to each interval.

i) For $[0, 0.5]$:

Substitute $t = 0$ to find the height at 0 s.

$$h(0) = -4.9(0)^2 + 14(0) + 1 = 1$$

Substitute $t = 0.5$ to find the height at 0.5 s.

$$h(0.5) = -4.9(0.5)^2 + 14(0.5) + 1 = 6.775$$

The points that correspond to 0 s and 0.5 s are (0, 1) and (0.5, 6.775).

$$\text{Average rate of change} = \frac{\Delta h}{\Delta t} = \frac{6.775 - 1}{0.5 - 0} = 11.55$$

The average rate of change of the height of the football from 0 s to 0.5 s is 11.55 m/s.

ii) For $[2, 2.5]$:

Substitute $t = 2$ to find the height at 2 s.

$$h(2) = -4.9(2)^2 + 14(2) + 1 = 9.4$$

Substitute $t = 2.5$ to find the height at 2.5 s.

$$h(2.5) = -4.9(2.5)^2 + 14(2.5) + 1 = 5.375$$

The points that correspond to 2 s and 2.5 s are (2, 9.4) and (2.5, 5.375).

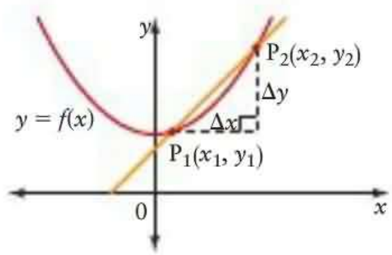
$$\begin{aligned}\text{Average rate of change} &= \frac{\Delta h}{\Delta t}\\ &= \frac{5.375 - 9.4}{2.5 - 2}\\ &= -8.05\end{aligned}$$

The average rate of change of the height of the football from 2 s to 2.5 s is −8.05 m/s.

b) The average rate of change of the height of the football for $t \in [0, 0.5]$ corresponds to the slope of secant AB. The average rate of change is positive, as is the slope of AB. The height of the football is increasing. The average rate of change of the height of the football for $t \in [2, 2.5]$ corresponds to the slope of secant CD. The average rate of change is negative, as is the slope of the secant. The height of the football is decreasing.

KEY CONCEPTS

- A rate of change is a measure of how quickly one quantity (the dependent variable) changes with respect to another quantity (the independent variable).
- Average rates of change
  - represent the rate of change over a specified interval
  - correspond to the slope of a secant between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on a curve



$$\begin{aligned}\text{Average rate of change} &= \frac{\Delta y}{\Delta x}\\ &= \frac{y_2 - y_1}{x_2 - x_1}\end{aligned}$$

- An average rate of change can be determined by calculating the slope between two points given in a table of values or by using an equation.

Communicate Your Understanding

C1 Describe a situation for which the average rate of change is

a) constant and positive **b)** constant and negative **c)** zero

C2 State the average rate of change for this situation. When the change in the independent variable is −3, the change in the dependent variable is 12.

C3 **a)** What information is provided by the sign (positive or negative) of an average rate of change?

b) How can you tell from a graph if the average rate of change over an interval is positive or negative? Is it always possible to do so? Explain.

A Practise

1. Which of the following does not represent a situation that involves an average rate of change? Justify your answer.

a) A child grows 8 cm in 6 months.

b) The temperature at a 750-m-high ski hill is 2°C at the base and −8°C at the top.

c) A speedometer shows that a vehicle is travelling at 90 km/h.

d) A jogger ran 23 km in 2 h.

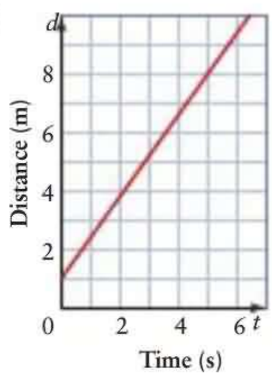
e) The laptop cost $750.

f) A plane travelled 650 km in 3 h.

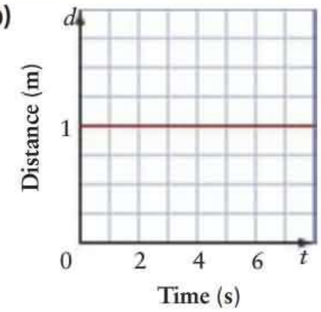
For help with questions 2 and 3, refer to Example 1.

2. Identify if the average rate of change for pairs of points along each graph is constant and positive, constant and negative, zero, or non-constant. Justify your response.

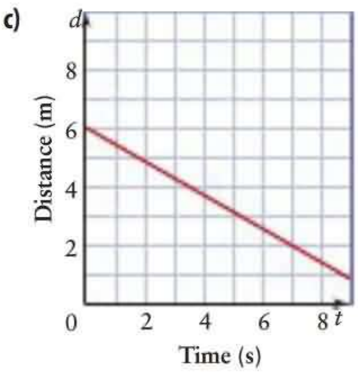
a)



b)



c)



3. Determine the average rate of change for two points on each line segment in question 2.

4. In 1990, 16.2% of households had a home computer, while 66.8% of households had a home computer in 2003. Determine the average rate of change of the percent of households that had a home computer over this time period.

Source: Statistics Canada, Canada at a Glance 2006, page 9, Household facilities.

B Connect and Apply

For help with question 5, refer to Example 2.

5. The table shows the percent of Canadian households that used e-mail from 1999 to 2003.

Year	Households (%)
1999	26.3
2000	37.4
2001	46.1
2002	48.9
2003	52.1

Source: Statistics Canada, Canada at a Glance, 2006, page 9, Household Internet use at home by Internet activity.

a) Determine the average rate of change of the percent of households using e-mail from 1999 to 2003. What are the units for this average rate of change?

b) Why might someone want to know the average rate of change found in part a)?

c) Determine the average rate of change of the percent of households using e-mail for each pair of consecutive years from 1999 to 2003.

d) Compare the values found in part c). Which value is the greatest? the least? What is the significance of these values?

e) Compare the values found in part a) with those in part c). Explain any similarities or differences.

10. As water drains out of a 2000-L hot tub, the amount of water remaining in the tub can be modelled by the function $V = 0.000\,02(100 - t)^4$, where t is the time, in minutes, $0 \le t \le 100$, and $V(t)$ is the volume of water, in litres, remaining in the tub at time t.

a) Determine the average rate of change of the volume of water during

i) the entire 100 min

ii) the first 30 min

iii) the last 30 min

b) What do the values from part a) tell you about the rate at which water drains from the tub?

c) **Use Technology** Graph the function $V(t) = 0.000\,02(100 - t)^4$. Sketch the function in your notebook.

d) On your drawing, sketch the secant lines that correspond to the time intervals in part a).

C Extend and Challenge

11. **Use Technology** The table shows the amount of water remaining in a swimming pool as it is being drained.

Reasoning and Proving
Representing
Selecting Tools
Problem Solving
Connecting
Reflecting
Communicating

Time (h)	Amount of Water (L)
0	18 750
1	17 280
2	15 870
3	14 520
4	13 230
5	12 000
6	10 830
7	9 720
8	8 670
9	7 680
10	6 750

a) Determine the average rate of change of the volume of water for consecutive hours.

b) Compare the values in part a). When is the rate of change the greatest? the least?

c) Create a table of finite differences for the data. Use the finite differences to determine the type of polynomial function that best models this situation.

d) Is there a relationship between the finite differences and the average rates of change? Explain.

e) Enter the data in a graphing calculator. Use the regression feature to obtain the equation of the curve of best fit for the data.

f) Graph the equation in part e).

g) How long will it take the pool to fully drain? Justify your answer.

12. The height, h, in metres, of a ball above the ground after t seconds can be modelled by the function $h(t) = -4.9t^2 + 20t$.

a) What does the average rate of change represent for this situation?

b) Determine the average rate of change of the height of the ball for each time interval.

i) $[1, 2]$ ii) $[1, 1.5]$ iii) $[1, 1.1]$

iv) $[1, 1.01]$ v) $[1, 1.001]$ vi) $[1, 1.0001]$

c) Compare the values in part b). Describe their relationship.

d) Explain how the values in part b) can be used to estimate the instantaneous rate of change of the ball at 1 s.

13. **Math Contest** A right triangle has an area of 25 cm^2. Express the hypotenuse, h, of this triangle as a function of its perimeter, p. Hint: Consider the square of the perimeter.

1.6 Slopes of Tangents and Instantaneous Rate of Change

When you hit or kick a ball, the height, h, in metres, of the ball can be modelled by the equation $h(t) = -4.9t^2 + v_0t + c_0$. In this equation, t is the time, in seconds; c_0 represents the initial height of the ball, in metres; and v_0 represents the initial vertical velocity of the ball, in metres per second. In Section 1.5, you learned how to calculate the average rate of change of the height of the ball, or the average velocity of the ball, over an interval of time. What if you want to know the velocity of the ball 1 s after it was hit? To determine how fast the ball is travelling at a specific time, that is, to determine the **instantaneous rate of change** of the height of the ball, other methods are required. In this section, you will investigate the connection between average and instantaneous rates of change and learn how the slope of a tangent is used to determine an instantaneous rate of change.

Investigate

What is the connection between the slopes of secants, the slope of a tangent, and the instantaneous rate of change?

1. A golf ball lying on the grass is hit so that its initial vertical velocity is 25 m/s. The height, h, in metres, of the ball after t seconds can be modelled by the quadratic function $h(t) = -4.9t^2 + 25t$. Copy and complete the table for $h(t)$.

Interval	Δh	Δt	Average rate of change, $\frac{\Delta h}{\Delta t}$
$1 \le t \le 2$	$h(2) - h(1)$ $= [-4.9(2)^2 + 25(2)] - [-4.9(1)^2 + 25(1)]$ $= 30.4 - 20.1$ $= 10.3$	$2 - 1 = 1$	
$1 \le t \le 1.5$			
$1 \le t \le 1.1$			
$1 \le t \le 1.01$			
$1 \le t \le 1.001$			

2. Explain how the time intervals in the first column are changing.

3. **Reflect** Describe how the average rate of change values in the fourth column are changing in relation to the time intervals.

4. a) Graph the function.

   b) **Reflect** On the graph, sketch the secants that correspond to the average rates of change in the table. How do the secants illustrate the relationship you described in step 3?

5. a) Sketch a tangent line, a line that touches the graph only at the point that corresponds to $t = 1$ s. Describe how the values of the average rate of change in your table could be used to estimate the slope of the tangent at $t = 1$ s.

   b) **Reflect** What does the slope of the tangent represent in this situation? Explain.

   c) What would happen if you approached $t = 1$ from below, that is, using $t = 0.9$, $t = 0.99$, $t = 0.999$, and so on? Explain.

6. **Reflect** Explain the relationship between

   a) the slopes of secants near a point on a curve and the slope of the tangent at that point

   b) the slope of a tangent at a point on a curve and the instantaneous rate of change

---

Relationship Between the Slope of Secants and the Slope of a Tangent

As a point Q becomes very close to a tangent point P, the slope of the secant line becomes closer to (approaches) the slope of the tangent line.

Often an arrow is used to denote the word "approaches." So, the above statement may be written as follows:

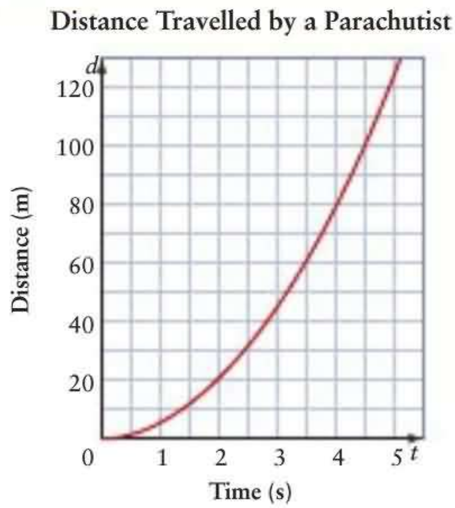
As Q $\rightarrow$ P, the slope of secant PQ $\rightarrow$ the slope of the tangent at P.

Thus, the average rate of change between P and Q becomes closer to the value of the instantaneous rate of change at P.

When the graph of a function is given, an approximate value for the instantaneous rate of change at a point on the curve may be determined using either of two methods: by calculating the slope of a line passing through the given point and another point on the curve that is very close to the given point, or by sketching the tangent line at the point and calculating the average rate of change over an interval between the tangent point and another point on the tangent line.

Example 1 Estimate an Instantaneous Rate of Change From a Graph

The graph shows the approximate distance travelled by a parachutist in the first 5 s after jumping out of a helicopter. How fast was the parachutist travelling 2 s after jumping out of the helicopter?



Solution

The parachutist's velocity at 2 s corresponds to the instantaneous rate of change at $t = 2$ s. Calculate an approximate value for the instantaneous rate of change. Let P be the point on the graph at $t = 2$ s.

Method 1: *Use the Slope of a Secant*

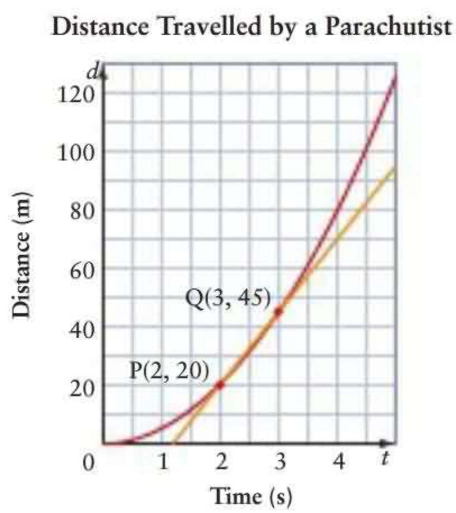
Determine the slope of a secant passing through P and another point on the curve that is very close to P.

Select the point Q(3, 45), which is close to P and easy to read from the graph.

The slope of the secant PQ is

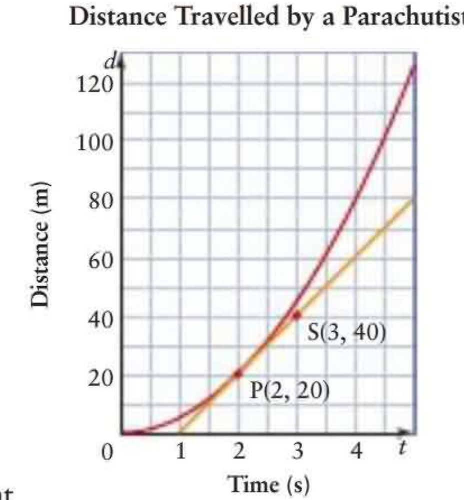
$$m_{PQ} = \frac{\Delta d}{\Delta t}$$
$$= \frac{45 - 20}{3 - 2}$$
$$= 25$$

Using a secant, the parachutist's velocity after falling for 2 s is approximately 25 m/s.



Method 2: *Use Two Points on an Approximate Tangent Line*

Another way to estimate the slope of a tangent at a point from a graph is to sketch an approximate tangent line through that point and then select a second point on that line. Select the point (3, 40). Label it S.



The slope of PS is

$$m_{PS} = \frac{\Delta d}{\Delta t}$$
$$= \frac{40 - 20}{3 - 2}$$
$$= 20$$

Using points on the approximate tangent line, the parachutist's velocity after falling for 2 s is approximately 20 m/s.

Note that both methods give an approximate value of the slope of the tangent because the calculations depend on either

- the accuracy of the coordinates of the selected point on the curve
- the accuracy of the approximate tangent line, as well as the accuracy of the coordinates of the second point on this approximate tangent line

In both methods, the closer the selected point is to P the better is the approximation of the slope of the tangent line.

Example 2 Estimate an Instantaneous Rate of Change From a Table of Values

In the table, the distance of the parachutist in Example 1 is recorded at 0.5-s intervals. Estimate the parachutist's velocity at 2 s.

t (s)	*d* (m)
0	0
0.5	1.25
1.0	5.00
1.5	11.25
2.0	20.00
2.5	31.25
3.0	45.00
3.5	61.25
4.0	80.00

Solution

To estimate an instantaneous rate of change from a table, calculate the average rate of change over a short interval by using points in the table that are closest to the tangent point.

The table shows that at 2 s the parachutist has travelled a distance of 20 m. This corresponds to the tangent point (2, 20) on the graph shown in Example 1. A point that is close to (2, 20) is the point (2.5, 31.25).

The average rate of change between (2, 20) and (2.5, 31.25) is

$$\begin{aligned}\text{Average rate of change} &= \frac{\Delta d}{\Delta t} \\ &= \frac{31.25 - 20}{2.5 - 2} \\ &= \frac{11.25}{0.5} \\ &= 22.5\end{aligned}$$

The parachutist's velocity at 2 s is approximately 22.5 m/s.

Notice that we also could have chosen the point (1.5, 11.25).

Example 3 Estimate an Instantaneous Rate of Change From an Equation

The function $d(t) = 5t^2$ may be used to model the approximate distance travelled by the parachutist in Examples 1 and 2. Use the equation to estimate the velocity of the parachutist after 2 s.

Solution

Determine the average rate of change over shorter and shorter intervals.

Interval	Δd	Δt	$\frac{\Delta d}{\Delta t}$
$2 \le t \le 3$	$d(3) - d(2) = 5(3)^2 - 5(2)^2$ $= 45 - 20$ $= 25$	$3 - 2 = 1$	$\frac{25}{1} = 25$
$2 \le t \le 2.5$	$d(2.5) - d(2) = 5(2.5)^2 - 5(2)^2$ $= 31.25 - 20$ $= 11.25$	$2.5 - 2 = 1$	$\frac{11.25}{0.5} = 22.5$
$2 \le t \le 2.1$	$d(2.1) - d(2) = 5(2.1)^2 - 5(2)^2$ $= 22.05 - 20$ $= 2.05$	$2.1 - 2 = 0.1$	$\frac{2.05}{0.1} = 20.5$
$2 \le t \le 2.01$	$d(2.01) - d(2) = 5(2.01)^2 - 5(2)^2$ $= 20.2005 - 20$ $= 0.2005$	$2.01 - 2 = 0.01$	$\frac{0.2005}{0.01} = 20.05$
$2 \le t \le 2.001$	$d(2.001) - d(2) = 5(2.001)^2 - 5(2)^2$ $= 20.020\,005 - 20$ $= 0.020\,005$	$2.001 - 2 = 0.001$	$\frac{0.020\,005}{0.001} = 20.005$

As the time intervals decrease, the average rate of change (which corresponds to the slope of a secant line), becomes closer to (or approaches) 20. Thus, the velocity of the parachutist after 2 s is approximately 20 m/s. You also could have used values of t less than 2.

KEY CONCEPTS

- An instantaneous rate of change corresponds to the slope of a tangent to a point on a curve.
- An approximate value for an instantaneous rate of change at a point may be determined using
  - a graph, either by estimating the slope of a secant passing through that point or by sketching the tangent and estimating the slope between the tangent point and a second point on the approximate tangent line
  - a table of values, by estimating the slope between the point and a nearby point in the table
  - an equation, by estimating the slope using a very short interval between the tangent point and a second point found using the equation

Communicate Your Understanding

C1 **a)** Does the speedometer of a car measure average speed or instantaneous speed? Explain.

b) Describe situations in which the instantaneous speed and the average speed would be the same.

C2 State if each situation represents average rate of change or instantaneous rate of change. Give reasons for your answer.

a) At 3 p.m., the plane was travelling at 850 km/h.

b) The average speed travelled by the train during the 10-h trip was 130 km/h.

c) The fire was spreading at a rate of 2 ha/h.

d) 5 s after an antiseptic spray is applied, the bacteria population is decreasing at a rate of 60 bacteria per second.

e) He lost 4 kg per month over a 5-month period.

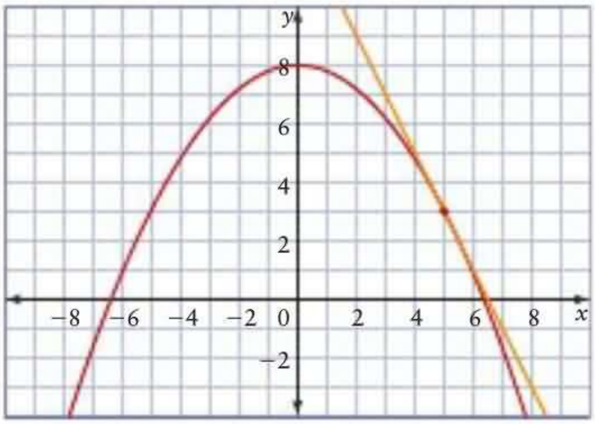
f) After being heated for 2 min, the water temperature was rising at 1°C/min.

C3 Which method from Examples 1, 2, and 3 is the most accurate for finding the instantaneous rate of change? Explain.

A Practise

For help with question 1, refer to Example 1.

1. Consider the graph shown.



a) State the coordinates of the tangent point.

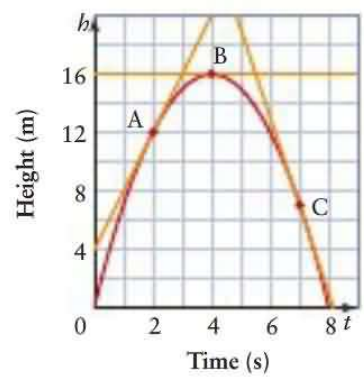
b) State the coordinates of another point on the tangent line.

c) Use the points you found in parts a) and b) to determine the slope of the tangent line.

d) What does the value you found in part c) represent?

2. a) At each of the indicated points on the graph, is the instantaneous rate of change positive, negative, or zero? Explain.

Height of a Tennis Ball



b) Estimate the instantaneous rate of change at points A and C.

c) Interpret the values in part b) for the situation represented by the graph.

B Connect and Apply

For help with question 3, refer to Example 3.

3. A firework is shot into the air such that its height, h, in metres, after t seconds can be modelled by the function $h(t) = -4.9t^2 + 27t + 2$.

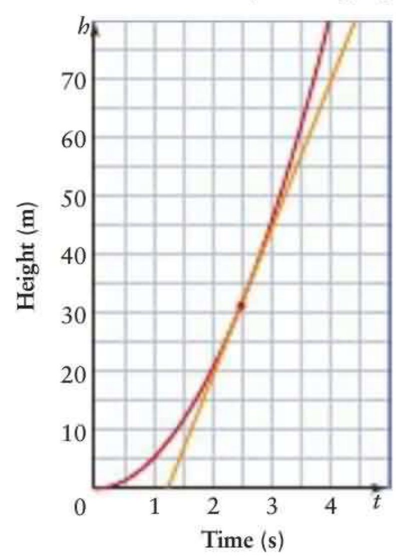
a) Copy and complete the table.

Interval	Δh	Δt	$\frac{\Delta h}{\Delta t}$
$3 \le t \le 3.1$			
$3 \le t \le 3.01$			
$3 \le t \le 3.001$			

b) Use the table to estimate the velocity of the firework after 3 s.

4. Use two different methods to estimate the slope of the tangent at the point indicated on the graph.

Distance Travelled by a Bungee Jumper



5. The data show the percent of households that play games over the Internet.

Year	% Households
1999	12.3
2000	18.2
2001	24.4
2002	25.7
2003	27.9

Source: Statistics Canada, Canada at a Glance 2006, page 9, Household Internet use at home by Internet activity.

a) Determine the average rate of change, in percent, of households that played games over the Internet from 1999 to 2003.

b) Estimate the instantaneous rate of change in percent of households that played games over the Internet

i) in 2000 **ii)** in 2002

c) Compare the values found in parts a) and b). Explain any similarities and differences.

6. The table shows the consumer price index (CPI) every 5 years from 1955 to 2005.

Year	CPI
1955	16.8
1960	18.5
1965	20.0
1970	24.2
1975	34.5
1980	52.4
1985	75.0
1990	93.3
1995	104.2
2000	113.5
2005	127.3

Source: Statistics Canada, CANSIM Table 326-0002.

a) Determine the average rate of change in the CPI from 1955 to 2005.

b) Estimate the instantaneous rate of change in the CPI for

i) 1965 **ii)** 1985 **iii)** 2000

c) Compare the values found in parts a) and b). Explain any similarities and differences.

7. A soccer ball is kicked into the air such that its height, h, in metres, after t seconds can be modelled by the function $h(t) = -4.9t^2 + 12t + 0.5$.

a) Determine the average rate of change of the height of the ball from 1 s to 2 s.

b) Estimate the instantaneous rate of change of the height of the ball after 1 s.

c) Sketch the curve and the tangent.

d) Interpret the average rate of change and the instantaneous rate of change for this situation.

8. On Earth, the height, h, in metres, of a free-falling object after t seconds can be modelled by the function $h(t) = -4.9t^2 + k$, while on Venus, the height can be modelled by $h(t) = -4.45t^2 + k$, where $t \geq 0$ and k is the height, in metres, from which the object is dropped. Suppose a rock is dropped from a height of 60 m on each planet.

a) Determine the average rate of change of the height of the rock in the first 3 s after it is dropped.

b) Estimate the instantaneous rate of change of the height of the rock 3 s after it is dropped.

c) Interpret the values in parts a) and b) for this situation.

9. A manufacturer estimates that the cost, C, in dollars, of producing x MP3 players can be modelled by $C(x) = 0.000\,15x^3 + 100x$.

a) Determine the average rate of change of the cost of producing from 100 to 200 MP3 players.

b) Estimate the instantaneous rate of change of the cost of producing 200 MP3 players.

c) Interpret the values found in parts a) and b) for this situation.

d) Does the cost ever decrease? Explain.

CONNECTIONS

The CPI measures the average price of consumer goods and services purchased by households. The percent change in the CPI is one measure of inflation.

10. Suppose the revenue, R, in dollars, from the sales of x MP3 players described in question 9 is given by $R(x) = x(350 - 0.000\ 325x^2)$.

a) Determine the average rate of change of revenue from selling from 100 to 200 MP3 players.

b) Estimate the instantaneous rate of change of revenue from the sale of 200 MP3 players.

c) Interpret the values found in parts a) and b) for this situation.

d) The profit, P, in dollars, from the sale of x MP3 players is given by the profit function $P(x) = R(x) - C(x)$. Determine an equation for the profit function.

e) Determine the average rate of change of profit on the sale of from 100 to 200 MP3 players.

f) Estimate the instantaneous rate of change of profit on the sale of 200 MP3 players.

g) Interpret the values found in parts e) and f) for this situation.

11. A worldwide distributor of basketballs determines that the yearly profit, P, in thousands of dollars, earned on the sale of x thousand basketballs can be modelled by the function $P(x) = -0.09x^3 + 1.89x^2 + 9x$, where $x \in [0, 25]$.

a) Determine the average rate of change of profit earned on the sale of from

i) 2000 to 6000 basketballs

ii) 16 000 to 20 000 basketballs

b) What conclusions can you make by comparing the values in part b)? Explain your reasoning.

c) Estimate the instantaneous rate of change of profit earned on the sale of

i) 5000 basketballs

ii) 18 000 basketballs

d) What conclusions can you make from the values found in parts c)? Explain your reasoning.

e) **Use Technology** Graph the function. How does the graph support your answers in parts a) and c)?

C Extend and Challenge

12. The population, P, of a small town after t years can be modelled by the function $P(t) = 0.5t^3 + 150t + 1200$, where $t = 0$ represents the beginning of this year.

a) Write an expression for the average rate of change of the population from $t = 8$ to $t = 8 + h$.

b) Use the expression in part a) to determine the average rate of change of the population when

i) $h = 2$ **ii)** $h = 4$ **iii)** $h = 5$

c) What do the values you found in part b) represent?

d) Describe how the expression in part a) could be used to estimate the instantaneous rate of change of the population after 8 years.

e) Use the method you described in part d) to estimate the instantaneous rate of change of the population after 8 years.

13. **Math Contest** Determine the exact value of $\sqrt{10 + \sqrt{10 + \sqrt{10 + \ldots}}}$.

14. **Math Contest** Find $|m|$ given $\sqrt[3]{m + 9} = 3 + \sqrt[3]{m - 9}$.

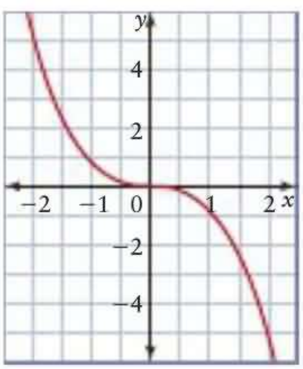
Chapter 1 REVIEW

1.1 Power Functions

1. i) Which functions are polynomial functions? Justify your answer.
   ii) State the degree and the leading coefficient of each polynomial function.
   a) $f(x) = 3x^4 - 5x + 1$
   b) $g(x) = x(4 - x)$
   c) $h(x) = 3x + 2x$
   d) $m(x) = x^{-2}$
   e) $r(x) = 5(x - 1)^3$

2. For each graph, do the following.
   i) State whether the corresponding function has even degree or odd degree.
   ii) State whether the leading coefficient is positive or negative.
   iii) State the domain and range.
   iv) Describe the end behaviour.
   v) Identify the type of symmetry.

a)



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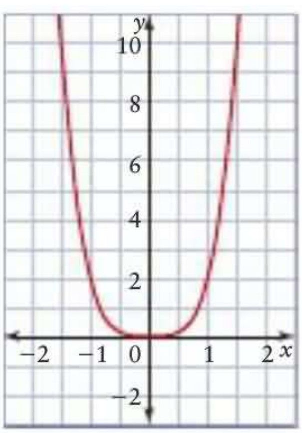


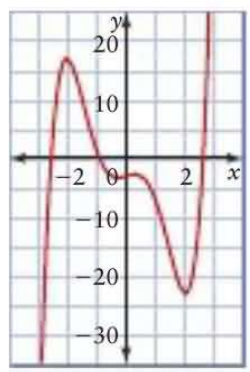
3. Set up a table as shown. Write each function in the appropriate row of column 2. Give reasons for your choices.
   $y = -x^5$, $y = \frac{2}{3}x^4$, $y = 4x^3$, $y = 0.2x^6$

End Behaviour	Function	Reasons
Extends from quadrant 3 to quadrant 1		
Extends from quadrant 2 to quadrant 4		
Extends from quadrant 2 to quadrant 1		
Extends from quadrant 3 to quadrant 4		

1.2 Characteristics of Polynomial Functions

4. Match each graph of a polynomial function with the corresponding equation. Justify your choice.
   i) $g(x) = 0.5x^4 - 3x^2 + 5x$
   ii) $h(x) = x^5 - 7x^3 + 2x - 3$
   iii) $p(x) = -x^6 + 5x^3 + 4$

a)



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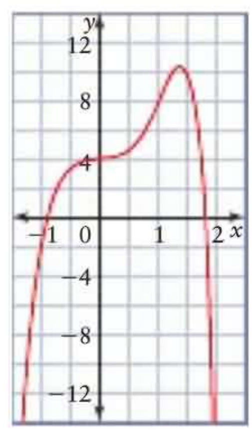
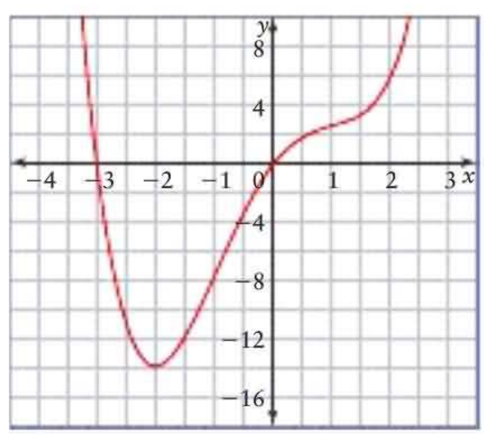


c)



5. For each polynomial function in question 4, do the following.
   a) Determine which finite differences are constant.
   b) Find the value of the constant finite differences.
   c) Identify the type of symmetry, if it exists.

6. a) State the degree of the polynomial function that corresponds to each constant finite difference.
      i) first differences $= -5$
      ii) fifth differences $= -60$
      iii) fourth differences $= 36$
      iv) second differences $= 18$
      v) third differences $= 42$
      vi) third differences $= -18$
   b) Determine the value of the leading coefficient of each polynomial function in part a).

7. Each table of values represents a polynomial function. Use finite differences to determine the following for each:
   i) the degree
   ii) the sign of the leading coefficient
   iii) the value of the leading coefficient

a)

x	y
−3	124
−2	41
−1	8
0	1
1	−4
2	−31
3	−104
4	−247

b)

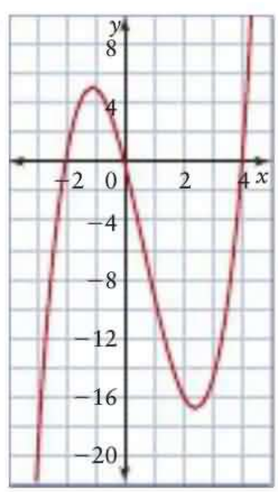
x	y
−2	−229
−1	−5
0	3
1	−7
2	−53
3	−129
4	35
5	1213

8. A parachutist jumps from a plane 3500 m above the ground. The height, h, in metres, of the parachutist above the ground t seconds after the jump can be modelled by the function $h(t) = 3500 - 4.9t^2$.
   a) What type of function is $h(t)$?
   b) Without calculating the finite differences, determine
      i) which finite differences are constant for this polynomial function
      ii) the value of the constant finite differences

      Explain how you found your answers.
   c) Describe the end behaviour of this function assuming there are no restrictions on the domain.
   d) Graph the function. State any reasonable restrictions on the domain.
   e) What do the t-intercepts of the graph represent for this situation?

1.3 Equations and Graphs of Polynomial Functions

9. Use each graph of a polynomial function to determine
   i) the least possible degree and the sign of the leading coefficient
   ii) the x-intercepts and the factors of the function
   iii) the intervals where the function is positive and the intervals where it is negative

a)


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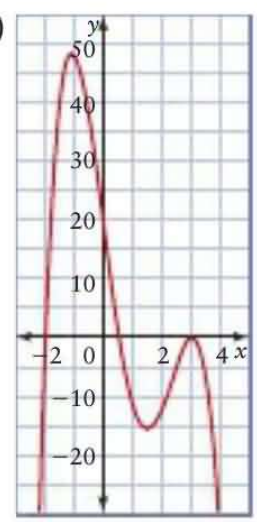


10. Sketch a graph of each polynomial function.

a) $y = (x + 1)(x - 3)(x + 2)$

b) $y = -x(x + 1)(x + 2)^2$

c) $y = (x - 4)^2(x + 3)^3$

11. The zeros of a quartic function are -3, -1, and 2 (order 2). Determine

a) equations for two functions that satisfy this condition

b) an equation for a function that satisfies this condition and passes through the point (1, 4)

12. Without graphing, determine if each polynomial function has line symmetry about the y-axis, point symmetry about the origin, or neither. Graph the functions to verify your answers.

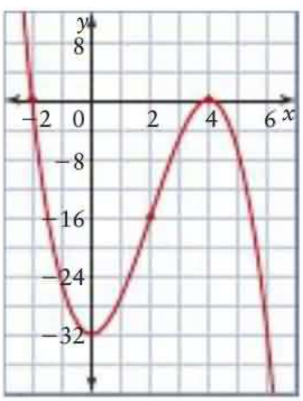
a) $f(x) = -x^5 + 7x^3 + 2x$

b) $f(x) = x^4 + 3x^2 + 1$

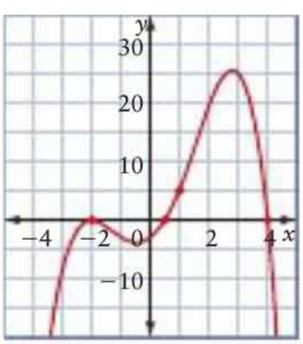
c) $f(x) = 4x^3 - 3x^2 + 8x + 1$

13. Determine an equation for the polynomial function that corresponds to each graph.

a)



b)



1.4 Transformations

14. i) Describe the transformations that must be applied to the graph of each power function, $f(x)$, to obtain the transformed function. Then, write the corresponding equation.

ii) State the domain and range of the transformed function. For even functions, state the vertex and the equation of the axis of symmetry.

a) $f(x) = x^3$, $y = -\frac{1}{4}f(x) - 2$

b) $f(x) = x^4$, $y = 5f\left[\frac{2}{5}(x - 3)\right] + 1$

15. i) Write an equation for the function that results from each set of transformations.

ii) State the domain and range. For even functions, state the vertex and the equation of the axis of symmetry.

a) $f(x) = x^4$ is compressed vertically by a factor of $\frac{3}{5}$, stretched horizontally by a factor of 2, reflected in the y-axis, and translated 1 unit up and 4 units to the left.

b) $f(x) = x^3$ is compressed horizontally by a factor of $\frac{1}{4}$, stretched vertically by a factor of 5, reflected in the x-axis, and translated 2 units to the left and 7 units up.

1.5 Slopes of Secants and Average Rate of Change

16. Which are not examples of average rates of change? Explain why.

a) The average height of the players on the basketball team is 2.1 m.

b) The temperature of water in the pool decreased by 5°C over 3 days.

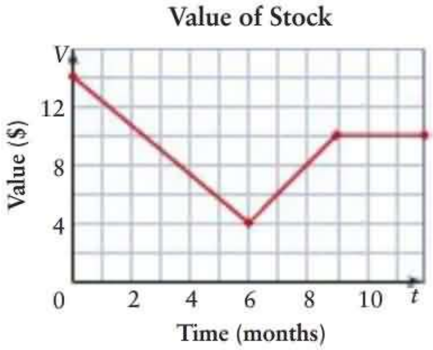
c) The snowboarder raced across the finish line at 60 km/h.

d) The class average on the last math test was 75%.

e) The value of the Canadian dollar increased from \$0.75 U.S. to \$1.01 U.S. in 8 months.

f) Approximately 30 cm of snow fell over a 5-h period.

17. The graph represents the approximate value of a stock over 1 year.



a) What was the value of the stock at the start of the year? at the end of the year?

b) What does the graph tell you about the average rate of change of the value of the stock in each interval?

i) month 0 to month 6

ii) month 6 to month 9

iii) month 9 to month 12

c) Determine the average rate of change of the value of the stock for the time periods in part b). Interpret these values for this situation.

1.6 Slopes of Tangents and Instantaneous Rate of Change

18. The table shows the percent of Canadian households that used the Internet for electronic banking.

Year	Households (%)
1999	8.0
2000	14.7
2001	21.6
2002	26.2
2003	30.8

Source: Statistics Canada, Canada at a Glance 2006, page 9, Household Internet use at home by Internet activity.

a) Determine the average rate of change, in percent, of households that used the Internet for electronic banking from 1999 to 2003.

b) Estimate the instantaneous rate of change in the percent of households that used the Internet for electronic banking in the year 2000, and also in 2002.

c) Compare the values you found in parts a) and b). Explain any similarities and differences.

CHAPTER

PROBLEM WRAP-UP

Throughout this chapter, you have seen how polynomial functions can be used in different careers involving design. Create a design that uses polynomial functions. The design could be for items such as furniture, vehicles, games, clothing, or jewellery.

Provide a report on the design you select that includes the following:

- a description of the item the design relates to and why you selected it
- a drawing of the design using appropriate colours
- an explanation of how the design integrates the graphs of a variety of polynomial functions

Include the following information about each function you used to create the design:

- the equation, domain and range, and end behaviour
- the value of the constant finite differences
- any transformations used
- any symmetry
- any connections to average and instantaneous rate of change

You may wish to do some research on the Internet for ideas and you may want to use technology to help you create your design.

Chapter 1 PRACTICE TEST

For questions 1 to 3, select the best answer.

1. Which statement is true? For those that are false, provide a counterexample.

A All odd-degree polynomial functions are odd functions.

B Even-degree polynomial functions have an even number of x-intercepts.

C Odd-degree polynomial functions have at least one x-intercept.

D All even-degree polynomial functions are even.

2. Which statement is true? For those that are false, provide a counterexample.

A A polynomial function with constant third differences has degree four.

B A polynomial function with a negative leading coefficient may extend from quadrant 3 to quadrant 4.

C A power function with even degree has point symmetry.

D A polynomial function with four x-intercepts has degree four.

3. Which statement is true? For those that are false, provide a counterexample.

A A vertical compression of factor $\frac{1}{3}$ is the same as a horizontal stretch of factor 3.

B When applying transformations, translations are applied before stretches and compressions.

C Stretches must be applied before compressions.

D A negative k-value in $y = a[k(x - d)]^n + c$ results in a reflection in the x-axis.

E The equation of a transformed polynomial function can be written in the form $y = a[k(x - d)]^n + c$.

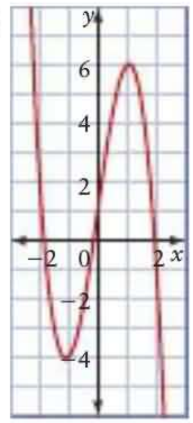
4. Match each graph of a polynomial function with the corresponding equation. Justify your choice.

i) $f(x) = -2x^3 + 7x + 1$

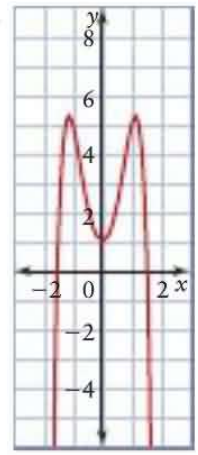
ii) $h(x) = x^5 - 7x^3 + 2x + 1$

iii) $p(x) = -x^6 + 5x^2 + 1$

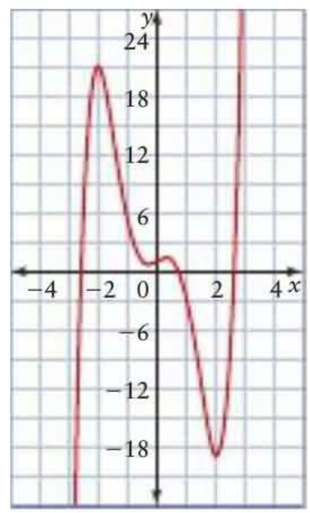
a)



b)



c)



5. For each polynomial function in question 4, answer the following.

a) Which finite differences are constant?

b) Find the values of the constant finite differences.

c) Identify any symmetry. Verify your answer algebraically.

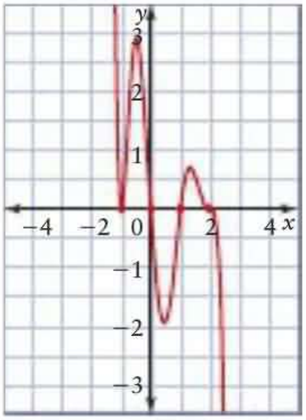
6. A quartic function has zeros −1, 0, and 3 (order 2).

a) Write equations for two distinct functions that satisfy this description.

b) Determine an equation for a function satisfying this description that passes through the point (2, −18).

c) Sketch the function you found in part b). Then, determine the intervals on which the function is positive and the intervals on which it is negative.

7. Determine an equation for the polynomial function that corresponds to this graph.



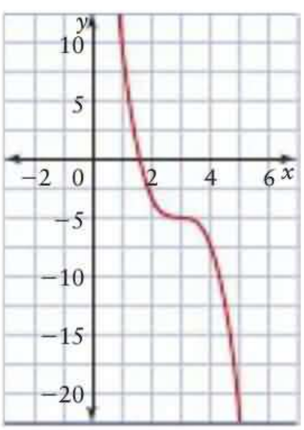
8. a) Identify the parameters a, k, d, and c in the polynomial function $y = \frac{1}{3}[-2(x + 3)]^4 - 1$. Describe how each parameter transforms the base function $y = x^4$.

   b) State the domain and range, the vertex, and the equation of the axis of symmetry of the transformed function.

   c) Describe two possible orders in which the transformations can be applied to the graph of $y = x^4$ to produce the graph of $y = \frac{1}{3}[-2(x + 3)]^4 - 1$.

   d) Sketch graphs of the base function and the transformed function on the same set of axes.

9. Transformations are applied to $y = x^3$ to obtain the graph shown. Determine its equation.



10. Describe a real-life situation that corresponds to

   a) a constant, positive average rate of change

   b) a constant, negative average rate of change

   c) a non-constant average rate of change

   d) a zero average rate of change

11. In 1990, 15.5% of households had a CD player, while 76.1% of households had a CD player in 2003. Determine the average rate of change of the percent of households that had a CD player over this time period.

Source: Statistics Canada, Canada at a Glance 2006, page 9, Household facilities.

12. An oil tank is being drained. The volume, V, in litres, of oil remaining in the tank after t minutes can be modelled by the function $V(t) = 0.2(25 - t)^3$, where $t \in [0, 25]$.

   a) How much oil is in the tank initially?

   b) Determine the average rate of change of volume during

      i) the first 10 min ii) the last 10 min

   c) Compare the values you found in part b). Explain any similarities and differences.

   d) Sketch a graph to represent the volume.

   e) What do the values found in part b) represent on the graph?

13. The distance, d, in metres travelled by a boat from the moment it leaves shore can be modelled by the function $d(t) = 0.002t^3 + 0.05t^2 + 0.3t$, where t is the time, in seconds.

   a) Determine the average rate of change of the distance travelled during the first 10 s after leaving the shore.

   b) Estimate the instantaneous rate of change 10 s after leaving the shore.

   c) Interpret each value in parts a) and b) for this situation.

TASK

Create Your Own Water Park

Apply your knowledge of polynomial functions to create a design for a water park, with a minimum of two appropriate rides per age group:

- under 6 years old
- ages 6 to 12
- over age 12

a) Sketch a graph representing the height versus horizontal distance for each ride.

b) Describe each ride and explain why it is appropriate for the age group.

c) Create a polynomial equation for each ride. Explain how each equation represents the ride.

d) Create a map, diagram, or model of your park.

e) Share your design and equations with a partner.