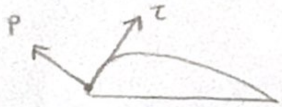
ME572 Aerodynamic Design

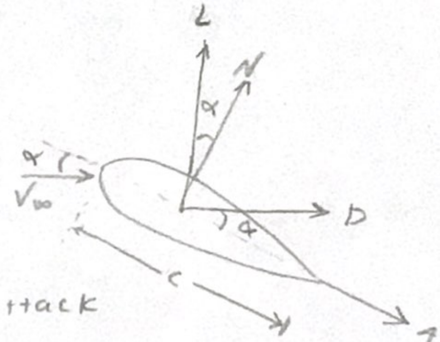
Aerodynamic forces - all forces will depend on Pressure & shear stress



· Lift & Drag forces

$L = \perp$ to V_∞ $\quad$ N = Normal force $\perp$ to c

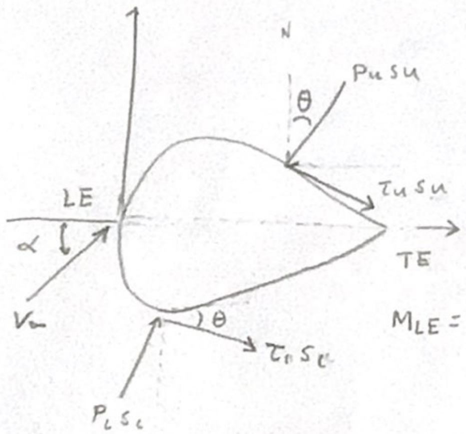
$D = //$ to V_∞ $\quad$ A = Axial force // to c



α = angle of attack

$L = N\cos\alpha - A\sin\alpha$

$D = N\sin\alpha + A\cos\alpha$



$$N' = -\int_{LE}^{TE}(P_u\cos\theta + \tau_u\sin\theta)\,ds_u + \int_{LE}^{TE}(P_l\cos\theta - \tau_l\sin\theta)\,ds_l$$

$$A' = \int_{LE}^{TE}(-P_u\sin\theta + \tau_u\cos\theta)\,ds_u + \int_{LE}^{TE}(P_l\sin\theta + \tau_l\cos\theta)\,ds_l$$

$$M_{LE} = \int_{LE}^{TE}\left[(P_u\cos\theta + \tau_u\sin\theta)\,x + (-P_u\sin\theta + \tau_u\cos\theta)\,y\right]ds_u$$
$$+$$
$$\int_{LE}^{TE}\left[(-P_l\cos\theta + \tau_l\sin\theta)\,x + (P_l\sin\theta + \tau_l\cos\theta)\,y\right]ds_l$$

Aerodynamic coefficient

Dynamic pressure $q_\infty = \frac{1}{2}\rho_\infty V_\infty^2$ $\quad C_D = \frac{D}{q_\infty S}$ $\quad C_A = \frac{A}{q_\infty S}$

(C2D uses lowercase) $\quad C_L = \frac{L}{q_\infty S}$ $\quad C_N = \frac{N}{q_\infty S}$ $\quad C_M = \frac{M}{q_\infty S c}$

$C_p = \frac{P - P_\infty}{q_\infty}$ $\quad C_f = \frac{\tau}{q_\infty}$

$D = N\overset{1}{\cancel{\sin\alpha}} + A\overset{1}{\cancel{\cos\alpha}}$

e.g.: given $\alpha = 0°$

$P_\infty = 1.01E5\ N/m^2$

$\rho_\infty = 1.23\ kg/m^3$

Ma : 2.0

$P_u = P_l = 1.31E5\ N/m^2$

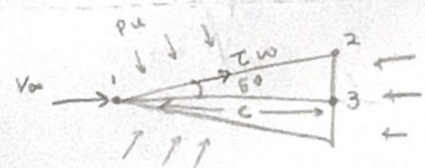
$\tau_w = 431 s^{-0.2}$

$c = 2\ m$

$C_D = ??$

$(s/2)\cos 5 = 2$

$(s/2)\sin 5 = \underline{523}$



Symmetry so U = L

P_∞

$D = 2\int_{LE}^{TE}(-P_u\sin\theta + \tau_u\cos\theta)\,ds_u$

pressure comp $= 2\left[\int_1^2 P_u\sin(-5)\,ds + \int_2^3 P_\infty\sin(90)\,ds\right]$

shear comp $= 2\left[\int_1^2 \tau_w\cos(-5)\,ds_u\right]$

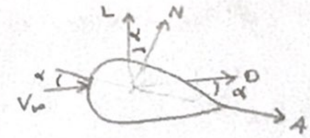
$Ma = \frac{V_\infty}{c}$ (speed of sound)

$c_D = \frac{D}{q_\infty S} = \frac{D}{\frac{1}{2}\rho_\infty V_\infty^2 \cdot c}$

e.g

given $\alpha = 12^\circ$ $C_N = 1.2$ $C_D = 0.03$ $C_L = C_n \cos\alpha - C_A \sin\alpha$

$C_D = C_n \sin\alpha + C_A \cos\alpha$



e.g Infinity thinplate $\theta = 0^\circ$

given $c = 1m$ $P_u = 4E4(x-1)^2 + 5.4E4$ $\tau_u = 288x^{-0.2}$

$\alpha = 10^\circ$ $P_L = 2E4(x-1)^2 + 1.73E6$ $\tau_L = 779x^{-0.2}$

$$N = -\int_{LE}^{TE} (P_u \cancel{\cos}\theta + \tau_u \cancel{\sin}\theta)\,ds_u + \int_{LE}^{TE} (P_L \cancel{\cos}\theta - \tau_u \cancel{\sin}\theta)\,ds_L$$

$$N = \int_0^1 (P_L - P_u)\,dx$$

$$A = \int_{LE}^{TE} (-P_u \cancel{\sin}\theta + \tau_u \cos\theta)\,ds_u + \int_{LE}^{TE} (P_L \cancel{\sin}\theta + \tau_L \cos\theta)\,ds_L$$

$$A = \int_0^1 (\tau_u + \tau_l)\,dx$$

Lift $= N\cos\alpha - A\sin\alpha$ Drag $= N\sin\alpha + A\cos\alpha$

$$M = \int_0^1 (P_u - P_L)\,x\,dx$$

Dimentional Analysis

Buckingham π - Theorem

Basic units - M L T

$\rho_\infty = ML^{-3}$

$V_\infty = LT^{-1}$

$c_p, c_v = L^2T^{-2}K^{-1}$ (specific heat)

$D_w = MLT^{-2}$

$P = ML^{-1}T^{-2}$

e.g. $C_{D,w} = f(M_\infty, \gamma)$ $\quad \gamma = \frac{c_v}{c_p}$ $\quad C_{D,w} = \frac{D_w}{q_\infty S}$ $\quad q = f(\rho, V)$

$S \rightarrow L$

$D_w = MLT^{-2}$ $\quad r = 4$ basic variables

$a_\infty = LT^{-1}$ $\quad k = 7$ variables $\rightarrow 3\pi$ groups

$\rho_\infty = ML^{-3}$

$C_v = L^2T^{-2}K^{-1}$

$c_p = L^2T^{-2}K^{-1}$

$V_\infty = LT^{-1}$

$L = L$

$$\pi_1 = D_w \rho_\infty^a C_v^b V_\infty^c L^d$$
$$= (MLT^{-2})(ML^{-3})^a(L^2T^{-2}K^{-1})^b(LT^{-1})^c(L)^d = 0$$

$M: 1 + a = 0$

$L: 1 - 3a + 2b + d + c = 0$

$T: -2 - 2b - c = 0$

$K \quad -b = 0$

$b = 0$, $c = -2$, $a = -1$, $d = -2$

$$\pi_1 = \frac{D_w}{\rho_\infty V_\infty^2 L} \approx \frac{D_w}{q_\infty S}$$

$$\pi_2 = a_\infty \rho_\infty^a C_v^b V_\infty^c L^d = (LT^{-1})(ML^{-3})^a(L^2T^{-2}K^{-1})^b(LT^{-1})^c(L)^d = 0$$

$M = \quad a = 0$

$L = -3a + 2b + c + d + 1 = 0$

$T = -1 - 2b - c = 0$

$K = \quad b = 0$

$a = 0$, $b = 0$, $c = -1$, $d = 0$

$$\pi_2 = \frac{a_\infty}{V_\infty} = M_\infty$$

$$\pi_3 = c_p \rho_\infty^a C_v^b V_\infty^c L^d = (L^2T^{-2}K^{-1})(ML^{-3})^a(L^2T^{-2}K^{-1})^b(LT^{-1})^c(L)^d$$

$M \quad a = 0$

$L \quad 2 - 3a + 2b + c + d = 0$

$T \quad -2 - 2b - c = 0$

$K \quad -1 - b = 0$

$b = -1$, $c = 0$, $a = 0$, $d = 0$

$$\pi_3 = \frac{c_p}{c_v} \approx \gamma$$

$\pi_1 = f(\pi_2, \pi_3)$

$C_{Dw} = f(M, \gamma)$

Stokes' theorem $\oint_C \vec{V}\, ds = \oiint_S (\nabla \times \vec{V})\, d\vec{s}$

Divergence theorem $\oiint_S \vec{V}\, d\vec{s} = \iiint_{\forall} (\nabla \cdot \vec{V})\, d\forall$

Gradient theorem $\oiint_S P\, ds = \iiint_{\forall} (\nabla P)\, d\forall$

Continuity Eq: $\frac{dm}{dt} + \dot{m}_o - \dot{m}_i = 0$ $\quad dm = \rho V ds$ $\quad$ Steady $t = 0$

$\nabla(\rho V) = 0$

$\oiint_S \rho V ds = 0$

$= \frac{\partial}{\partial t} \iiint_V \rho\, d\forall + \oiint_S \rho \vec{V}\, d\vec{s} = 0$

↳ divergence theorem

$= \iiint_{\forall} \left(\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{V}) \right) d\forall = 0 \longrightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$

momentum eq: $\frac{\partial}{\partial t} \iiint_{\forall} \vec{V} \rho\, d\forall + \oiint_S (\rho \vec{V})\, d\vec{s}\, \vec{V} = -\oiint_S P\, ds + \iiint_{\forall} \rho \vec{f}\, d\forall + F_{vis}$

time rate of momentum $\quad$ Net rate of Momentum $\quad$ P. force $\quad$ Body force $\quad$ v.s. force

Navier stokes x comp. $\quad u$

$\iiint_{\forall} \frac{\partial \rho u}{\partial t} d\forall + \iiint_{\forall} \nabla(\rho u \vec{V})\, d\forall = -\iiint_{\forall} \nabla P\, d\forall + \iiint_{\forall} \rho \vec{f}\, d\forall + F_{vis}$

$\frac{\partial \rho u}{\partial t} + \nabla(\rho u \vec{V}) = -\frac{\partial P}{\partial x} + \rho f_x + F_{vis}$

Euler equations - no body, viscouse, steady momentum equation

$\nabla(\rho u \vec{V}) = -\frac{\partial P}{\partial x}$

Energy Eq: $\delta q + \delta w = \delta e$

$\iiint_{\forall} q \rho\, d\forall + \dot{Q}_{vis} + -\oiint_S (P d\vec{s})\, V + \iiint_{\forall} \rho (\vec{f} \cdot \vec{V})\, d\forall + \dot{W}_{vis} = \oiint_S (\rho \vec{V}\, d\vec{s})(e + \frac{V^2}{2}) + \frac{\partial}{\partial t} \iiint_V \rho (e + \frac{V^2}{2})\, d\forall$

$-\iiint_{\forall} \nabla \cdot P\, d\forall$

$\iiint_{\forall} \nabla(\rho(e + \frac{V^2}{2})\vec{V})\, d\forall$

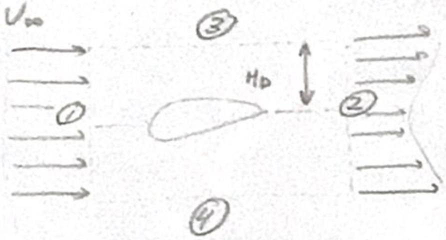
$= \iiint_{\forall} \left(q\rho - \nabla P + \rho \vec{f} \vec{V} - \nabla(\rho(e + \frac{V^2}{2}) \cdot V - \frac{\partial}{\partial t}[\rho(e + \frac{V^2}{2})\right) + \dot{Q}_{vis} + \dot{W}_{vis} = 0$

Steady, invisicid, adiabatic, no bf

$\nabla[\rho(e + \frac{V^2}{2})\vec{V}] = -\nabla(\rho \vec{V})$

e.g.: consider a steady, incompressible, 2D flow $P_{gage} = 0$

$v = V_\infty$ $u = V_\infty(1 - 0.5\cos(\pi y/2H_D))$



$H_D = 0.025c$

a) Volume flow Rate across 3 & 4 $\int \cancel{\frac{\partial}{\partial t}\rho\, d\forall} + \int \rho v\, ds = 0$

$$\int_2 u\,dx - \int_1 u\,dx - \int_{3:4} u\,dx = 0$$

$$\int_{34} u\,dx = \int_2 u\,dx - \int_1 u\,dx = 2\int_0^{HD} V_\infty(1 - 0.5\cos(\pi y/2H_D)) - 2\int_0^{HD} V_\infty\,dx$$

b) $C_D = \frac{D}{q_\infty S_{(c)}}$ conservation of momentum

$$\int \cancel{\frac{\partial}{\partial t}\rho}\, d\forall + \int \rho \hat{v}\, ds \cdot \hat{v} = \Sigma F^{\leftarrow \text{Drag}}$$

$$-D = -\int_{①} \rho u u\,dy + \int_{②} \rho u u\,dy - \int_{34} \rho u u\,dy$$

from part(a) $\int \rho u\,dx = \dots \times u$

Lecture #5 Basic Equations in Aerodynamics 2/8/24

- Continuity Equation $\frac{dm}{dt}_{sys} = \sum_{in} \dot{m}_i - \sum_{out} \dot{m}_e$

- momentum Equation $\frac{dP}{dt}_{sys} = F_{ext} + \sum \dot{m}_i V_{in} - \sum \dot{m}_e V_{out}$

- Energy Equation $\frac{dE}{dt}_{sys} = \dot{Q}_{net in} + \dot{W}_{out} + \left(\sum_{in} \dot{m}_i (u + \frac{1}{2}V^2 + gz)_i\right) - \sum_{out} \dot{m}_e (u + \frac{1}{2}V^2 + gz)_e$

- gradient scalar field (max ROC) $\nabla \cdot P = \frac{\partial P}{\partial x}\hat{i} + \frac{\partial P}{\partial y}\hat{j} + \frac{\partial P}{\partial z}\hat{k}$ VECTOR

  $(\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k})$

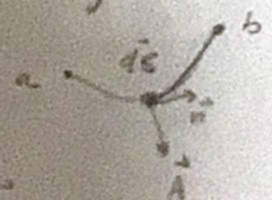
- divergence vector field (ROC of volume small cv element) $\nabla \cdot V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$ SCALAR

- Curl of vector field $\nabla \times V = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$

$$\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}$$

- Line integral: $\vec{A} = \vec{A}(x, y, z) = \vec{A}(r, \theta, z) = \vec{A} = (r, \theta, \phi)$

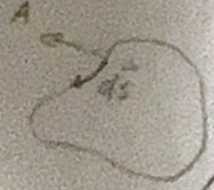
"C"



$d\vec{s} = \vec{n} \cdot ds$ (vector) ds: line element

line integral of $\vec{A}$ along curve "C"

$$\int_a^b \vec{A}\, d\vec{s}$$



$$\oint_C \vec{A}\, d\vec{s}$$

"C" closed (+↺)

• Surface integral:

"S" - open surface

"C" bound

$\vec{ds}$ now surface element

$\vec{ds} = \vec{n}\, ds$ (vector)

$\iint_S \vec{A}\, \vec{ds}$ surface integral vector $\vec{A}$

$\iint_S P\, \vec{ds}$ surface integral of scalar P

"S" = closed surface

$\oiint \vec{A}\, \vec{ds}$

$\oiint P\, \vec{ds}$

• Volume integral

"$\forall$" in space

$\iiint_{\forall} \vec{A}\, d\forall$ - volume integral of vector $\vec{A}$

$\iiint_{\forall} \rho\, d\forall$ - volume integral of scalar

• Stoke's theorem - relates line & surface integral

$$\oint_c \vec{A}\, \vec{ds} = \iint_S \underbrace{(\vec{\nabla} \times \vec{A})}_{\text{(curl of vector)}} \vec{ds}$$

• Divergence theorem - relates surface & volume integral

$$\oiint_S \vec{A}\, ds = \iiint_{\forall} \underbrace{(\nabla \cdot A)}_{\text{(divergence of vector)}}\, d\forall$$

• gradient theorem - relates surface & volume (scalar):

$$\oiint_S P\, \vec{ds} = \iiint_{\forall} \underbrace{(\nabla P)}_{\text{(gradient of scalar)}}\, d\forall$$

Lecture#6 Basic Equation in Aerodynamics 2/15/24

$$\nabla(\rho u \vec{V}) = -\frac{\partial P}{\partial x}$$
$$\nabla(\rho v \vec{V}) = -\frac{\partial P}{\partial y}$$
$$\nabla(\rho w \vec{V}) = -\frac{\partial P}{\partial z}$$

Euler's equation
steady, inviscous, no body force momentum equations

Energy Equation;

$$\delta q + \delta w = de$$

$$\delta q : \iiint_{V} q\rho \, dV + \dot{Q}_{\text{viscous}}$$

$$\delta w : \underbrace{-\oiint_S (P\, d\vec{s})\, \vec{V}}_{\text{pressure}} + \underbrace{\iiint_V (\rho \vec{f}\, \vec{V})\, dV}_{\text{Body}} + \underbrace{\dot{W}_{\text{viscous}}}_{\text{viscous}}$$

$$de : \oiint_S \underbrace{(\rho \vec{V} d\vec{s})}_{\dot{m}} \left(e + \frac{V^2}{2}\right) \quad \text{Net energy flow out}$$

$$\frac{\partial}{\partial t} \iiint_V \rho \left(e + \frac{V^2}{2}\right) dV \quad \text{Net energy rate}$$

$$\iiint_V q\rho \, dV + \dot{Q}_{vis} - \oiint_S (P\vec{V})\, d\vec{s} + \iiint_V \rho \vec{f}\, \vec{V}\, dV + \dot{w}_{vis} =$$
$$\oiint_S (\rho \vec{V} ds)\left(e + \frac{V^2}{2}\right) + \frac{\partial}{\partial t} \iiint_V \rho\left(e + \frac{V^2}{2}\right) dV$$

$$\left[-\oiint_S (P\vec{V})\, d\vec{s} = -\iiint_V \nabla (P\vec{V})\, dV\right] \quad \text{divergence of P. in work}$$

$$\left[\oiint_S \rho\left(e + \frac{V^2}{2}\right)\vec{V}\, d\vec{s} = \iiint_V \nabla \rho\left(e + \frac{V^2}{2}\right)\vec{V}\, dV\right) \quad \text{div. of Net energy}$$

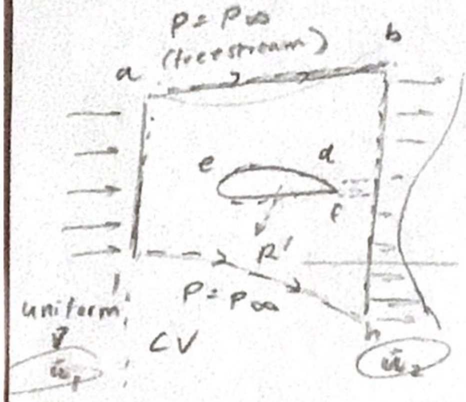
$$\Rightarrow \iiint_V \left(\frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right] + \vec{\nabla}\rho\left(e + \frac{V^2}{2}\right)\vec{V} - \rho \dot{q} - \dot{Q}_{vis} + \nabla P\vec{V} - \rho \vec{f}\, \vec{V}\right] = 0$$

$$\frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right] + \vec{\nabla}\left[\rho\left(e + \frac{V^2}{2}\right)\vec{V}\right] = \rho \dot{q} - \nabla(P\vec{V}) + \rho(\vec{f} \cdot \vec{V}) + \dot{Q}_{vis} + \dot{W}_{vis}$$

steady-state, inviscous, adiabatic, (no heat transfer) no body

$$\vec{\nabla}\left[\rho\left(e + \frac{V^2}{2}\right)\vec{V}\right] = -\nabla(P\vec{V})$$

e.g. momentum equation allow to see the pressure distribution, vel.
account for mounting in testing airfoil $\longrightarrow$ solves for drag force



Surface force of CV: pressure over "abhi"

$$-\iint_{abhi} p\,d\vec{s}$$

(CV opposing force of airfoil)

force on "def"

$$-R'$$

Total force: $-\iint_{abhi} p\,d\vec{s} - R'$

P + viscous $\rightarrow$ resultant = L + D (or N + A)

momentum equation:

$$\frac{\partial}{\partial t}\iiint_{V} \rho\vec{V}\,dV + \iint_{S}(\rho\vec{V}\cdot d\vec{s}\,\vec{V}) = -\iint_{abhi} p\,d\vec{s} - R'$$

Condition
- steady state
- Pressure along abhi = constant

$$R' = -\iint_S(\rho\vec{V}\cdot d\vec{s})\vec{V} - \iint_{abhi} p\,d\vec{s}$$

x comp: $D' = -\iint_S(\rho\vec{V}\cdot d\vec{s})u - \iint_{abhi}(p\,d\vec{s})_x$

$$\left[-\iint_{abhi}(p\,d\vec{s})_x = 0\right] \text{ constant}$$

along "ab", "hi", "def" $\vec{V} \perp d\vec{s}$ $\quad \vec{V}\cdot d\vec{s} = 0$

$$\iint_S(\rho\vec{V}\cdot d\vec{s})u = -\int_i^a \rho_1 u_1^2\,dy + \int_h^b \rho_2 u_2^2\,dy$$

$$-\int_i^a \rho u_1\,dy + \int_h^b \rho u_2\,dy = 0$$ $\leftarrow$ continuity equation (to rewrite momentum)

$$= \int_i^a \rho u_1\,dy = \int_h^b \rho u_2 u_1\,dy$$

calc. wake vel. allows us to calc. Drag

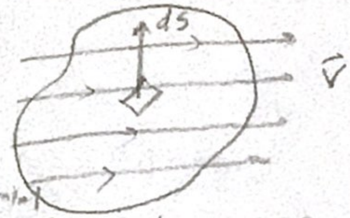
$$= \iint_S(\rho\vec{V}\cdot d\vec{s})u = -\int_h^b \rho_2 u_2 u_1\,dy + \int_h^b \rho_2 u_2^2\,dy$$

$$= -\int_h^b \rho_2 u_2(u_1 - u_2)\,dy$$

$$D' = \int_h^b \rho_2 u_2(u_1 - u_2)\,dy$$

incompressible $\downarrow$ $\quad D' = \rho\int_h^b u_2(u_1 - u_2)\,dy$

Continuity Equation (mass flow rate) $(\dot{m} = \rho V A)$ (Both comp. + incomp. flows)



$\underbrace{d\dot{m} = \rho \vec{V} d\vec{s}}_{\text{scalar}} = \rho V_n ds$

elemental mass flow rate across elemental area

mass flowrate across the surface : $\iint_S \rho \vec{V} d\vec{s}$

total mass in the volume : $\iiint_V \rho d\mathcal{V}$

net mass flow rate out of CV = rate of decrease of mass in CV

$$\oiint_S \rho \vec{V} d\vec{s} = -\frac{\partial}{\partial t} \iiint_V \rho d\mathcal{V}$$

$$\frac{\partial}{\partial t} \iiint_V \rho d\mathcal{V} + \oiint_S \rho \vec{V} d\vec{s} = 0 \quad \text{continuity Equation}$$

↓ convert to volume integral w/ divergence theorem

$$\oiint_S \rho \vec{V} d\vec{s} = \iiint_V \nabla(\rho \vec{V}) d\mathcal{V}$$

$$= \frac{\partial}{\partial t} \iiint_V \rho \, d\mathcal{V} = \iiint_V \frac{\partial \rho}{\partial t} d\mathcal{V}$$

$$= \iiint_V \frac{\partial \rho}{\partial t} d\mathcal{V} + \iiint_V \nabla \cdot (\rho \vec{V}) d\mathcal{V} = 0$$

$$= \iiint_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) \right] d\mathcal{V} = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{V}) = 0 \quad \text{continuity Equation (partial diff.)}$$

density change w/ time

unsteady flow - spacial + time function $\rho = \rho(x, y, z, t)$

steady flow - spacial function only $\rho = \rho(x, y, z)$

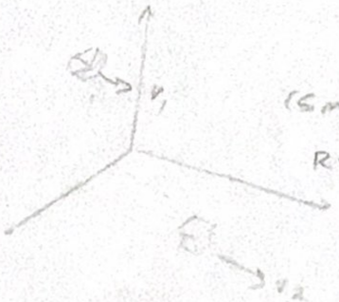
$$\frac{\partial}{\partial t} = 0$$

steady continuity : $\oiint \rho \vec{V} ds = 0$; $\nabla(\rho \vec{V}) = 0$

incompressible steady $\nabla \vec{V} = 0$

Lecture # 7 Basic concepts in Aerodynamics 2/20/24

Substantial Derivative



Taylor series $\rho_2 = \rho_1 + \frac{\partial \rho}{\partial x}(x_2 - x_1) + \frac{\partial \rho}{\partial y}(y_2 - y_1) + \frac{\partial \rho}{\partial z}(z_2 - z_1) + \frac{\partial \rho}{\partial t}(t_2 - t_1)$

(Simplest, 1st order)

Relating ρ_1 & ρ_2

$$\underbrace{\frac{\rho_2 - \rho_1}{t_2 - t_1}}_{\text{Average time ROC density}} = \frac{\partial \rho}{\partial x}\frac{(x_2 - x_1)}{(t_2 - t_1)} + \frac{\partial \rho}{\partial y}\frac{(y_2 - y_1)}{(t_2 - t_1)} + \frac{\partial \rho}{\partial z}\frac{(z_2 - z_1)}{(t_2 - t_1)} + \frac{\partial \rho}{\partial t}$$

$\lim_{t_2 \to t_1} \frac{x_2 - x_1}{t_2 - t_1} = u$ etc.

Note: $\frac{\partial \rho}{\partial t}$ is one term of $\frac{D\rho}{Dt}$ (through a fixed point)

$\lim_{t_2 \to t_1} \frac{\rho_2 - \rho_1}{t_2 - t_1} = \frac{D\rho}{Dt}$

$\frac{D\rho}{Dt}$ In. ROC in density fluid through space

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial x}u + \frac{\partial \rho}{\partial y}v + \frac{\partial \rho}{\partial z}w + \frac{\partial \rho}{\partial t}$$

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \qquad V = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\frac{D\rho}{Dt} = \frac{\partial}{\partial t} + (V \cdot \nabla)$$ * form of substantial derivative = Time ROC following a fluid element

$$\frac{DT}{Dt} = \underbrace{\frac{\partial T}{\partial t}}_{\text{local derivative (time ROC fixed pt)}} + \underbrace{(V\nabla)T}_{\text{* convective derivative (space)}}$$

can all be written in the form

* Steady state, while local derivative = 0, still change through space

Continuity Equation: Substantial Derivative (cons. of mass)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

$$\underbrace{\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho}_{} + \rho \nabla \cdot \vec{V} = 0$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0$$

momentum Equation : Substantial Derivative (x comp.)

$$\frac{\partial(\rho u)}{\partial t} + \nabla(\rho u \vec{V}) = -\frac{\partial p}{\partial x} + \rho f_x + (F_{vis\,x})$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial t} \qquad \nabla(u\rho\vec{V}) = u\vec{\nabla}\cdot(\rho V) + (\rho\vec{V})\cdot\nabla u$$

$$u\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial t} + u\vec{\nabla}\cdot(\rho\vec{V}) + (\rho\vec{V})\cdot\nabla u = -\frac{\partial p}{\partial x} + \rho f_x + F_{vis\,x}$$

$$\rho\frac{du}{dt} + \cancel{u}\underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\vec{V}\right]}_{\text{cont.}} = -\frac{\partial p}{\partial x} + \rho f_x + F_{xvis}$$

$$\rho\underbrace{\left(\frac{\partial u}{\partial t} + \vec{V}\cdot\nabla u\right)}_{} = -\frac{\partial p}{\partial x} + \rho f_x + F_{xvis}$$

energy equation $\frac{Du}{Dt}$

$$\rho\frac{D(e+\frac{V^2}{2})}{Dt} = \rho\dot{q} - \nabla\cdot(p\vec{V}) + \rho(f\cdot\vec{V}) + Q_{vis} + W_{vis}$$

Pathlines & streamlines

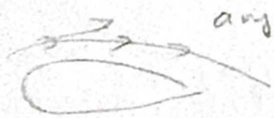
(long exposure)

pathlines : time exposure photograph of fluid element

unsteady flow : elements through same point not same (from fluctuation of flow field)

steady state : pathline same for diff elements

streamline : single frame of motion of flow



any point, the direction = direction of local velocity tangent vector

unsteady flow : streamline pattern diff at diff time

steady state pathline = streamline
(mag + direction of $\vec{V}$ = fixed + steady)

Streamline + pathline mathematical equations

$d\vec{s}, \vec{V}$, parallel

$d\vec{s} \times \vec{V} = 0$ $(|d\vec{s}||\vec{V}|\sin 0 = 0)$

$d\vec{s} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx & dy & dz \\ u & v & w \end{vmatrix}$

$d\vec{s} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ $\quad \vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$

$(w\,dy - v\,dz)\hat{i} + (u\,dz - w\,dx)\hat{j} + (v\,dx - u\,dy)\hat{k} = 0$

$w\,dy - v\,dz = 0$ $\quad u\,dz - w\,dx = 0$ $\quad v\,dx - u\,dy = 0$

Streamline equations

2D: $v\,dx = u\,dy \rightarrow \frac{dy}{dx} = \frac{v}{u}$ through $(0,5)$

velocity along a one direction (u, v, w) can be function of any var (x, y, z, t)

e.g. $\vec{V} = \quad u = \frac{y}{x^2+y^2} \quad v = \frac{-x}{x^2+y^2}$

$\frac{dy}{dx} = \left(\frac{-x}{\cancel{x^2+y^2}}\right)\left(\frac{\cancel{x^2+y^2}}{y}\right)$

$= \int y\,dy = \int -x\,dx$

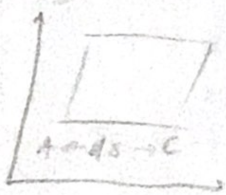
$= \frac{y^2}{2} = -\frac{x^2}{2} + C$

$y = \sqrt{-x^2 + C}$ $\quad C = 25$

$5^2 = C$

$y = \sqrt{-x^2 + 25}$

Angular velocity & vorticity



At "t" $A = V$

$C = V + \frac{\partial v}{\partial x} dx$

At "$t + \Delta t$" $A = V\Delta t$

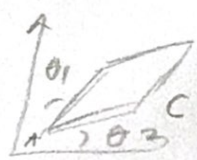
$C = (V + \frac{\partial v}{\partial x} dx)\Delta t$

Relative displacement between A & C

$(V + \frac{\partial v}{\partial x} dx)\Delta t - V\Delta t$

$= \frac{\partial v}{\partial x} dx\,\Delta t$

$\tan\Delta\theta_2 = \frac{\frac{\partial v}{\partial x} dx\Delta t}{dx} = \frac{\partial v}{\partial x}\Delta t$



adjacent 2 lines angular velocity = av. of change. $(\theta_1 \,\&\, \theta_2)$

$\Delta\theta_2 = \frac{\partial v}{\partial x}\Delta t$

$\Delta\theta_1 = -\frac{\partial u}{\partial y}\Delta t$

$\Rightarrow \frac{\partial\theta_1}{\partial t} = -\frac{\partial u}{\partial y}$

$\frac{\partial\theta_2}{\partial t} = \frac{\partial v}{\partial x}$

Angular velocity

2D: $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ avg

Curl vel. = ξ in vel. field

3D: $\begin{cases} \vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k} \\ \vec{\omega} = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}\right] \end{cases}$

Vorticity $\vec{\xi} = 2\vec{\omega}$

(a vel. field) $\vec{\xi} = \left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}\right]$

$\vec{\xi} = \nabla \times \vec{V}$ (curl of velocity) vector

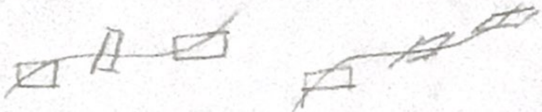
flow structure imp.

If $\nabla \times \vec{V} \neq 0$ every pt = rotational flow

$\nabla \times \vec{V} = 0$ at every pt. in flow = irrotational

Lecture #8 Basic Concepts in Aerodynamics 2/27/24

midterm 3/12 (tues)
5 true/false 10pts
4 mal problems 40 PTS



Rotational flow along streamline

Change of ∡s
(Sum ≠ 0 $\frac{d\theta}{dt} \neq 0$:
$\nabla \times \vec{V} \neq 0$, vorticity)

→ if sum of ∡ = 0
(even w/ deformation)
then irrotational flow

2D flow (z) - irrotational

$\xi = \xi \hat{k} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = 0$ $\quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ condition of irrotationality of 2D flow

e.g. $u = \frac{y}{(x^2+y^2)}$ $\quad v = \frac{-x}{(x^2+y^2)}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{x^2+y^2} & -\frac{x}{x^2+y^2} & 0 \end{vmatrix}$$

irrotational except at (but at origin) (0,0)

$$0\,\hat{i} + 0\,\hat{j} + \left(\frac{-x}{x^2+y^2}\frac{\partial}{\partial x} - \frac{y}{x^2+y^2}\frac{\partial}{\partial y}\right)\hat{k}$$

$$\left(\frac{-(x^2+y^2)+2x^2}{(x^2+y^2)^2} - \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}\right) = 0$$

symetric flow no gen. press. diff = no lift (stationary)

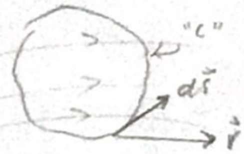
Circulation - (vorticity)

- fundemental to calc. lift
- rotation makes airflow not symetric.
- lift = produce circulation

"C" closed curve in flow field

$\vec{V}$ = velocity

$d\vec{s}$ = directed line segment.



negative since $\vec{V}$ = + but $d\vec{s}$ ↺

$\boxed{(\text{Circulation})\ \Gamma = -\oint \vec{V} \cdot d\vec{s}}$ $\quad \Gamma$ - line integral of $\vec{V}$ along closed curve

Parameters $\vec{V}$, $d\vec{s}$, diff. if any element change ⟶ generate diff lifts

circulation depend only on velocity field
choice of "C" curve

stokes' theorem

line int → surface int $\quad -\oint_S \vec{V} \cdot d\vec{s} = -\underbrace{\iint_S \nabla \times \vec{V}\, d\vec{s}}_{\text{vorticity}}$ $\quad \boxed{\Gamma = -\iint_S \xi\, d\vec{s}}$

many diff opt in surface as long as bound w/ "C"

circulation about curve "C" = open surface int. of vorticity bound by "C" differential of stream function

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \qquad u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}$$

$$d\psi = -v\,dx + u\,dy = 0$$

$$\boxed{\left(\frac{dy}{dx}\right)_{\psi\,constant} = \frac{v}{u}}$$

differential of velocity potential along equipotential = 0 (constant)

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \qquad \frac{\partial \phi}{\partial x} = u \quad \frac{\partial \phi}{\partial y} = v$$

$$d\phi = u\,dx + v\,dy = 0$$

$$\boxed{\left(\frac{dy}{dx}\right)_{\phi\,constan} = -\frac{u}{v}}$$

$$(dy/dx)_{\psi\,constant} = -\frac{1}{(dy/dx)_{\phi\,constant}}$$

relating velocity potential & stream function

($\perp$) - recepcol of ϕ_{const} =

Related w/ differential

Dfferences

[ϕ diff same direction of $\vec{V}$
[ψ diff. normal to $\vec{V}$ direction

[ϕ only irrotational
[ψ any rotational or irrotational

[ϕ = 3D flows
[ψ = 2D flows

Lecture # 10

Laplace Equation - 2nd order linear PDE . solutions = harmonic functions

$\nabla^2\phi = 0$ incompressible irrotational flow of continuity automatically satisfy Laplace equation

Stream function - 2D incompressible flow

$u = \frac{\partial\psi}{\partial y}$ $\quad v = -\frac{\partial\psi}{\partial x}$ $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial x}\right)$

$= \frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial y\partial x} = 0$ ← continuity equation satisfied automatically

also irrotational $\quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ (curl of velocity = 0 for irrotational flow)

Stream function also satisfy Laplace equation if irrotational, incompressible 2D

$$\frac{\partial}{\partial x}\left(-\frac{\partial\psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = 0$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \leftarrow \nabla^2\phi = 0 \text{ Laplace equation}$$

- Any irrotational + incompressible has ϕ & ψ (2D) which satisfy Laplace ∴ any Laplace soln rep. ϕ or ψ for (2D) irrotational, incompressible flow
- since Laplace = Linear ∴ any sum of particular soln of linear PDE also a solution $\quad \nabla^2\phi = 0$, soln $= \phi_1, \phi_2 \ldots \phi_n$ ∴ $\phi = \phi_1 + \phi_2 \cdots + \phi_n$

↳ complicated flow pattern (Irr + incomp) can be made by adding some elementary flows.

Laplace equation = governing equation → Must derive the right flow (soln) by defining <u>boundry conditions</u>

flow over airfoil

Boundry = surface of Body

freestream at ∞ distanee from body

- infinity boundary condition - far away from body all direction → from where freestream in uniform

V_∞ = aligned w/ x direction at ∞ $\begin{cases} u = \partial\phi/\partial x = \partial\psi/\partial y = V_\infty \\ v = \partial\phi/\partial y = -\partial\psi/\partial x = 0 \end{cases}$

(one directional)

- Surface boundry condition - flow can't pentrate surface ∴ $V_{surface}$ alway tangent to body surface (stagnet or tangent)

$\vec{V}\cdot\vec{n} = 0$

$(\nabla\phi)\cdot\vec{n} = 0$

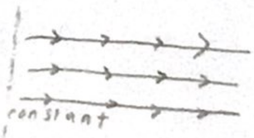
or

$\frac{\partial\phi}{\partial n} = 0$

body contour is streamline (ψ = constant) $\frac{dy}{dx} = \left(\frac{v}{u}\right)_{surface}$

<u>Elementry Flows</u>

- uniform flow



ψ = constant

ϕ = constant

V_∞ $\begin{cases} \nabla\cdot\vec{v} = 0 & \text{continuity incompressible} \\ \nabla\times\vec{v} = 0 & \text{irrotatiotional} \end{cases}$

$u = V_\infty \quad v = 0$

$\frac{\partial\phi}{\partial x} = u = V_\infty \quad \frac{\partial\phi}{\partial y} = v = 0$

(intg)

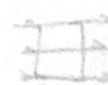
$\begin{cases} \phi = V_\infty x + f(y) \\ \phi = c + g(x) \end{cases} \longrightarrow \begin{matrix} g(x) = V_\infty x \\ f(y) = \text{constant} \end{matrix}$

$\boxed{\phi = V_\infty x + \text{constant}}$

$\frac{\partial\psi}{\partial y} = u = V_\infty \quad \frac{\partial\psi}{\partial x} = -v = 0$

$\psi = V_\infty y + \text{constant}$

circulation Γ for rect. $\ell \times h$ "curve"



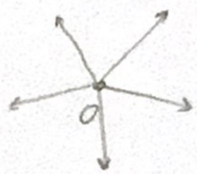
$\Gamma = -\oint_c \vec{v}\cdot d\vec{s} = -V_\infty L - 0\cdot h + V_\infty L + 0\cdot h = 0$

$\Gamma = -\oint_c \vec{v}\cdot d\vec{s} = -\vec{V}_\infty \underbrace{\oint_c d\vec{s}}_{\oint_c d\vec{s} \text{ (closed surface)}} = 0$

(Specific solution of Laplace Equation)

Source flow : 2D incompressible flow where all streamline are straight lines from central point "o"

$\vec{v}$ along each streamline vary inversely with distance from "o"



Polar cordinates : $V_r, V_\theta \rightarrow V_\theta = 0$

(1) source flow is physically possible impressible flow

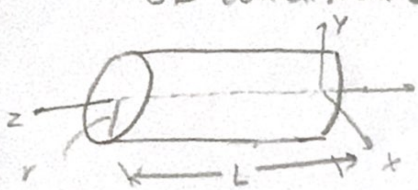
$\nabla \cdot \vec{v} = 0$ (continuity)

(2) Source flow irrotational at every point.

(Sourceflow $\vec{v}$ inversely proportional to radial distance)

$$\begin{cases} V_r = c/r \\ V_\theta = 0 \end{cases} \text{ velocity components}$$

3D conditions



consider $\dot{m}$ accross elementry cylinder surface

$ds = (r \cdot d\theta) \cdot L \leftarrow$ surface area

$\dot{m} = \rho V_r ds = \rho V_r (r d\theta) \cdot L$

Total $\dot{m}$: $\int_0^{2\pi} \rho V_r (r d\theta) \cdot L = \rho r L V_r \int_0^{2\pi} d\theta = 2\pi r L \rho V_r$

Total $\dot{V}$: $\frac{\dot{m}}{\rho} = 2\pi r L V_r$

volumetric flow rate/unit lengths $\Lambda = \frac{\dot{V}}{L} = 2\pi r V_r$

(source strength) · how strong source flow is

$$\begin{cases} V_r = \frac{\Lambda}{2\pi r} \\ V_r = \frac{c}{r} \end{cases} \longrightarrow c = \frac{\Lambda}{2\pi}$$

$$\begin{cases} \frac{\partial \phi}{\partial r} = V_r = \frac{\Lambda}{2\pi r} \\ \frac{1}{r}\frac{\partial \Phi}{\partial \theta} = V_\theta = 0 \end{cases} \longrightarrow \begin{matrix} \phi = \frac{\Lambda}{2\pi} \ln(r) + f(\theta) \\ \phi = \text{constant} + f(r) \end{matrix} \rightarrow \begin{matrix} f(r) = \frac{\Lambda}{2\pi} \ln r \\ f(\theta) = \text{constant} \end{matrix}$$

$$\boxed{\Phi = \frac{\Lambda}{2\pi} \ln r + \text{constant}}$$

$$\begin{cases} \frac{\partial \psi}{\partial r} = V_\theta = 0 \\ \frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_r = \frac{\Lambda}{2\pi r} \end{cases} \longrightarrow \begin{matrix} \psi = \text{constant} + f(\theta) \\ \psi = \frac{\Lambda}{2\pi}\theta + f(r) \end{matrix} \quad \Psi = \frac{\Lambda}{2\pi}\theta + \text{constant}$$