9.1: Change of Basis

$T: V \to W$ $\qquad$ $v^{\in V} \xrightarrow{T} w^{\in W}$

dim $= n$ $\qquad$ dim $= m$ $\qquad$ B - Basis in V $\{v_1, \dots v_n\}$ $\qquad$ $v = c_1 v_1 + \dots + c_n v_n$

$\mathbb{R}^n$ $\qquad$ $\mathbb{R}^m$ $\qquad$ B' - basis in W $\{w_1, \dots w_n)$

$Tv = b_1 w_1 + \dots + b_m w_n$

$v \xrightarrow{T} w$

$$[v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \xrightarrow{A} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = [Tv]_{B'}$$

$\mathbb{R}^n$ $\qquad$ $m \times n$ matrix $\qquad$ $\mathbb{R}^m$

$$\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = A \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

how to find A (matrix of T corresponding to B, B'?) $\rightarrow$ $A = [Tv_1, Tv_2 \dots Tv_n]$

Column vector in terms of Basis B'

$A = [T]_{B' \leftarrow B}$ $\qquad$ Change of Basis matrix

$Tv = w$

$$[T]_{B' \leftarrow B} [v]_B = [Tv]_{B'}$$

(v) (w)

e.g.: $T: P_1 \to P_2$ $\qquad$ B - standard Basis for P_1

$T p(x) = \int_0^x p(t)\,dt$ $\qquad$ B' - standard Basis for P_2

$[T]_{B' \leftarrow B}$? $\quad$ T_{P_1} $\quad$ $P = a_0 + a_1 x$

Recall: standard Basis: $P(x) = a_0 + a_1 x^1 \dots + a_n x^n$ $\quad$ $\{1, x, x^2, \dots x^n)$

$\dim P_1 = 2$ $\qquad$ $\dim P_2 = 3$

$B = \{1, x\}$ $\qquad$ $B' = \{1, x, x^2\}$

$T(1)(x) = \int_0^x dt = x \to 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2$

looking at coefficient from B in the B' coefficients = column vector

$$[T]_{B' \leftarrow B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$m \times n$ (3×2) $\qquad$ from element 1 of B $\qquad$ element 2 of B $\qquad$ (from $B' \leftarrow B$)

$T(x)(x) = \int_0^x t\,dt = \frac{x^2}{2} \to 0 \cdot 1 + 0 \cdot x + \frac{1}{2} x^2$

(just writing in standard Basis)

T_{P_1} $\quad P = a_0 + a_1 x \longleftrightarrow \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = [P]_B$

(m×n)

$(T_P)_{B'} = [T]_{B' \leftarrow B} \; P[B]$

$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ a_0 \\ a_1/2 \end{bmatrix}$

$T(p)(x) = (0 \cdot 1) + (a_0 \; x) + (a_1/2 \cdot x^2)$ $\quad = \int_0^x p(t)\,dt$

$= a_0 + a_1 t$

e.g.: $T_d : P_3 \rightarrow P_2 \quad [T_d]_{B' \leftarrow B}$

$P_2 \quad P_3$

$T_d(p) = P'$

$\dim B = 4 \qquad \dim B' = 3$

$B = [1, x, x^2, x^3] \qquad B' = [1, x, x^2]$

$T_d(1) = 0 - (0 \cdot 1) + (0 \cdot x) + 0 \cdot x^2)$

$T_d \; x = 1 \quad 1 \quad 0 \quad 0$

$T_d \; x^2 = 2x \quad 0 \quad 2 \quad 0$

$T_d \; x^3 = 3x^2 \quad 0 \quad 0 \quad 3$

$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = [T_d]_{B' \leftarrow B}$

$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = [P]_B$

$[T_P]_{B'} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$T(p)(x) = (a_1 \cdot 1) + (a_2 \cdot x) + (a_3 \cdot x^2)$

Properties

- $V_B \xrightarrow{T} W_{B'} \xrightarrow{S} V_{B''}$

$[S \cdot T]_{B'' \leftarrow B} = [S]_{B'' \leftarrow B'} \; [T]_{B' \leftarrow B}$

- $V_B \xrightarrow{T} W_{B'}$ - isomorphism (one to one, onto)

$W \xrightarrow{T^{-1}} V \quad [T^{-1}]_{B' \leftarrow B} = ([T]_{B' \leftarrow B})^{-1}$

e.g.: $V = W = P_2$

$B = \{1, x, x^2\}$

to check if basis, see if linearly independent

$B' = \{x, x^2+1, x^2-1\}$

$\alpha x + \beta(x^2+1) + \gamma(x^2-1) = 0$

$(\beta - \gamma)\cdot 1 + (\alpha)x + (\beta+\gamma)x^2 = 0$

all coefficient zero : linearly indep.

$$\left.\begin{aligned} \beta-\gamma &= 0 \\ \alpha &= 0 \\ \beta+\gamma &= 0 \end{aligned}\right\} \begin{aligned} \alpha &= 0 \\ \beta &= 0 \\ \gamma &= 0 \end{aligned}$$

$T = id$ - identity $[T(p) = p]$

$\dim B = 3 \qquad \dim B' = 3$

$T_{id}(1) = \quad 0 \quad \frac{1}{2} \quad -\frac{1}{2}$

$T_{id}(x) = \quad 1 \quad 0 \quad 0$

$T_{id}(x^2) = \quad 0 \quad \frac{1}{2} \quad \frac{1}{2}$

$$T(p)_{B' \leftarrow B} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$P_B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$1 = \alpha \cdot x + \beta(x^2+1) + \gamma(x^2-1)$

$\alpha = 0 \qquad \beta = 1 + \gamma$

$\beta + \gamma = 0 \qquad \beta = \frac{1}{2}$

$\beta - \gamma = 1 \qquad \gamma = -\frac{1}{2}$

$$[T_p]_{B'} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$x = \alpha \cdot x + \beta(x^2+1) + \gamma(x^2-1)$

$\alpha = 1$

$\beta + \gamma = 0$

$\beta - \gamma = 0$

$$\begin{bmatrix} a_1 \\ \frac{1}{2}a_0 + \frac{1}{2}a_2 \\ -\frac{a_0}{2} + \frac{a_2}{2} \end{bmatrix}$$

$x^2 = \alpha x + \beta(x^2+1) + \gamma(x^2-1)$

$\alpha = 0 \qquad \gamma = \frac{1}{2}$

$\beta + \gamma = 1$

$\beta - \gamma = 0 \qquad \beta = \frac{1}{2}$

$$T(p)(x) = (a_1 \cdot 1) + \left(\frac{a_0 + a_2}{2} \cdot x\right) + \left(\frac{a_2 - a_0}{2} \cdot x^2\right)$$

e.g. $V = M_{2\times 2}$

$W = M_{3\times 2}$

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\dim V = 4$

$\dim W = 6$

$T: M_{2\times 2} \to M_{3\times 2}$

$$B_{2\times 2} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$B'_{3\times 2} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$T_{B' \leftarrow B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \\ 0 & 2 \\ 3 & 3 \\ 0 & 0 \end{bmatrix}$$

(6×4)

$$T_{\substack{\text{matrix} \\ \text{mul.}}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{bmatrix} \to [1, 0, 2, 0, 3, 0]$$

$$T \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix} \to [0, 1, 0, 2, 0, 3]$$

etc...