(8.1 pt2) Gram-Schmidt Algoritm

- Recall P ← Projection of $C(A)$ $\quad P = A(A^T A)^{-1} A^T$

- **Orthogonal matrix** (Q) $m \times n$ matrix

  if $Q^T Q = I_n$ ← Identity matrix

$$Q = [\underbrace{q_1, q_2, \cdots q_n}_{\text{column } \vec{v} \text{ of } Q}] \qquad Q^T = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix}$$

$$Q^T \cdot Q = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} \cdot [q_1, q_2 \cdots q_n] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If Q = orthogonal, then $C(Q)$ form orthonormal system (columns have length pairwise orthogonal)

$$\left.\begin{matrix} q_1^T q_1 = 1 & q_1^T q_2 = 0 & \cdots & q_1^T q_n = 0 \\ q_2^T q_1 = 0 & q_2^T q_2 = 1 & \cdots & q_2^T q_n = 0 \\ \vdots & & & \vdots \\ q_n^T q_1 = 0 & q_n^T q_2 = 0 & \cdots & q_n^T q_n = 1 \end{matrix}\right\}$$

square of length of q_i

$q_i^T q_i = 1$

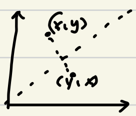
$q_i^T q_j = 0, \ i \neq j$

q_i, q_j = orthogonal

e.g.: $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \left.\begin{matrix} 0 \cdot 1 = 0 \\ 1 \cdot 0 = 0 \end{matrix}\right]$ orthogonal

$L_1 = 1 = L_2 = 1$ Orthogonal matrix

$$\begin{bmatrix} x \\ y \end{bmatrix} Q = \begin{bmatrix} y \\ x \end{bmatrix}$$



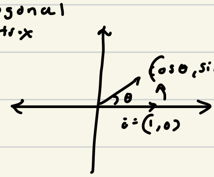
Q matrix gave mirror of x, y on line $y = x$

e.g.: $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\quad$ orthogonal matrix

$L = 1$ $\qquad L = 1$

$\cos^2\theta + \sin^2\theta = 1$ $\qquad \cos^2\theta + \sin^2\theta = 1$

- Length = 1
- dot product = 0



$$Q \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

i rotate θ CCW

Properties of orthogonal matrix

- If Q = orthogonal
  - $|Qv| = |v|$
  - $\measuredangle(Qv, Qw) = \measuredangle(v, w)$

mul. by Q does not change length or angle of vector

proof: $|Qv|^2 = (Qv)^T (Qv)$

(length not changed w/ Q) $= v^T \underbrace{(Q^T Q)}_{I_n} v = v^T v = |v|^2$

$$(Qv)^T \cdot (Qw) = v^T \underbrace{Q^T Q}_{I_n} w = v^T \cdot w$$

$P = A\underbrace{(A^TA)}_{}^{-1}A^T$

$\llcorner$ I_n to get then use Q^TQ then.... $P = QQ^T$

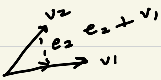
$C(A) = C(Q)$

<u>Takeaway</u>! if matrix = orthogonal $P = QQ^T$ which is simplified form of $P = A(A^TA)^{-1}A^T$

<u>Gram-Schmidt Algorithm</u>: $v_1, v_2, \dots, v_n \longrightarrow q_1, q_2, \dots, q_n$

> where • $q_1, q_2, \dots q_n$ = orthonormal • span$\{v_1 \dots v_n\}$ = span $\{q_1 \dots q_n\}$
(mag. = 1, all set vector mutually orthogonal)

> $v_1 \rightarrow e_1 = v_1$



$v_2 \rightarrow e_2$

$e_2 = v_2 - \text{Proj}_{v_1}(v_2)$
$\Downarrow$
$\dfrac{v_1 v_1^T}{v_1^T v_1} \cdot v_2$

$\longrightarrow e_2 = v_2 - \dfrac{v_1 v^T}{v_1^T v_1} \cdot v_2$ (orthogonal to v_1)

$q_1 = \dfrac{e_1}{|e_1|} \quad q_2 = \dfrac{e_2}{|e_2|}$

Orthonormal, make all vectors orthogonal to each other (e_1, e_2, e_3) then we get q by getting unit length ($q = e/|e|$)

$v_3 \rightarrow e_3$

$e_3 = v_3 - \text{Proj}_{v_1}(v_3) = v_3 - \dfrac{v_1 v^T}{v_1^T v_1} \cdot v_3$ (orthogonal to v_1)

$- \dfrac{v_2 v_2^T}{v_2^T v_2} \cdot v_3$ (also orthogonal to v_2)

$q_3 = \dfrac{e_3}{|e_3|}$

general form

$e_k = v_k - \dfrac{v_1 v_1^T}{v_1^T v} v_k - \dfrac{v_{k-1} v_{k-1}^T}{v_{k-1}^T v_{k-1}} \cdot v_k$

$q_k = \dfrac{e_k}{|e_k|}$

e.g. $v_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$e_1 = v_1 \qquad q_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \cdot \dfrac{1}{\sqrt{2}}$

$e_2 = v_2 - \dfrac{v_1 v^T}{v_1^T v_1} v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad q_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \dfrac{1}{\sqrt{3}}$

$e_3 = v_3 - \dfrac{v_1 v_1^T}{v_1^T v_1} v_3 - \dfrac{v_2 v_2^T}{v_2^T v_2} v_3 \longrightarrow v_3 - e_1 \dfrac{e_1^T v_3}{e_1^T e_1} - e_2 \dfrac{e_2^T v_3}{e_2^T e_2}$

$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \cancel{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}} \dfrac{0}{2} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \dfrac{1}{3} \quad e_3 = \begin{bmatrix} -4/3 \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \quad q_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \dfrac{1}{\sqrt{6}}$

8.1 #1 $\vec{u} = \begin{bmatrix} 0 \\ -1 \\ -2 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 0 & -1 & -2 & 0 \\ 1 & 3 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4}_{\text{Basis for vectors } W^\perp}$ $R(A)^\perp = N(A)$

8.1 #4 $\vec{v} = \begin{bmatrix} -5 \\ -5 \\ 2 \end{bmatrix}$ $l = t\vec{u}$ $\vec{u} = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}$

$Proj_{\vec{u}}(\vec{v}) = \frac{\vec{u}\,\vec{u}^T}{\vec{u}^T u} \cdot \vec{v} \xrightarrow[\text{(solve)}]{}$

8.1 #2 $L: \begin{bmatrix} -1 \\ 1 \\ -9 \end{bmatrix}$ $L^\perp = ??$

$\sim [1 \ -1 \ 9]$ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -9 \\ 0 \\ 1 \end{bmatrix} x_3$ (Basis)

8.1 #3 $W = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} y$ (Basis)

$v = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$ if $v \cdot w = 0$, then $v \perp w$

$-2+2=0$ ✓ $-2+2=0$ ✓ ($v \perp w$ since dot product basis $= 0$)

$v_1, \ldots, v_n = \mathbb{R}^m$

$A = [v_1 \ldots v_n]$ $Q = [q_1 \ldots q_n]$ ortnonormal

$A = Q \cdot U \leftarrow$ upper triangular

$Q^T A = \underbrace{Q^T Q}_{I_n} U$ (gram-schmidt Algorithm)

$U = Q^T A$ - upper triangular

$= \begin{bmatrix} q^T_1 \\ \vdots \\ q^T_n \end{bmatrix} [v_1 \ldots v_n] = \begin{bmatrix} q_1^T v_1 & \cdots & \cdots \\ q_2^T v_1 & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$ (upper triangle)