Discrete Applied Mathematics 309 (2022) 32–45

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The secure domination number of Cartesian products of small graphs with paths and cycles

Michael Haythorpe*, Alex Newcombe

Flinders University, 1284 South Road, Tonsley, SA 5042, Australia

ARTICLE INFO

Article history:
Received 16 March 2021
Received in revised form 1 November 2021
Accepted 7 November 2021
Available online 29 November 2021

Keywords:
Secure domination
Graph
Paths
Cycles
Cartesian product

ABSTRACT

The secure domination numbers of the Cartesian products of two small graphs with paths or cycles is determined, as well as for Möbius ladder graphs. Prior to this work, in all cases where the secure domination number has been determined, the proof has either been trivial, or has been derived from lower bounds established by considering different forms of domination. However, the latter mode of proof is not applicable for most graphs, including those considered here. Hence, this work represents the first attempt to determine secure domination numbers via the properties of secure domination itself, and it is expected that these methods may be used to determine further results in the future.

1. Introduction

Consider an undirected graph G containing vertex set V and edge set E. Then $S \subseteq V$ is said to be a *dominating set* if, for every vertex $v \in V$, we have either $v \in S$, or there exists $w \in V$ such that $\{v, w\} \in E$ and $w \in S$. In the latter case, we say that w *covers* v. The size of the smallest dominating set in G is called the *domination number of* G, and is denoted by $\gamma(G)$. The *domination problem* is to determine the domination number of a given graph. The decision problem variant of this problem, asking whether a dominating set exists with cardinality no more than a given constant, is known to be NP-complete [10].

The domination problem has obvious real-world applications. A common example is to imagine a set of locations which must remain under observation. We could place a guard at each location; however, it may be possible for a guard at one location to see another location. We can represent this situation by a graph G where the vertices correspond to the locations, and an edge $\{v, w\}$ exists if a guard at location v can observe location w as well. Then, for any dominating set S for G, placing guards at the locations in S ensures every location is under observation. Of course, it is likely to be desirable to do this with as few guards as possible.

In 2005, Cockayne et al. [8] considered a new variant of the domination problem. They defined a *secure dominating set* $S \subseteq V$ to be one in which, for every vertex $v \in V$, we have either $v \in S$, or there exists $w \in V$ such that w covers v, and $(S \setminus \{w\}) \cup \{v\}$ is a dominating set. In the latter case, we say that w *guards* v. The interpretation is perhaps best illustrated by continuing the example above. Suppose that there is a disturbance at one of the locations. If a guard is present at the location, they deal with the disturbance. If not, then one of the guards from a neighbouring location must move to this location in order to deal with the disturbance. It is obviously desirable if the full set of locations is still under observation when this occurs. Then, the cardinality of the smallest secure dominating set in G is called the *secure domination number*

* Corresponding author.
E-mail addresses: michael.haythorpe@flinders.edu.au (M. Haythorpe), alex.newcombe@flinders.edu.au (A. Newcombe).

https://doi.org/10.1016/j.dam.2021.11.008
0166-218X/

of G, denoted by $\gamma_s(G)$, and the *secure domination problem* is to determine the secure domination number of a given graph. The secure domination problem is also known to be NP-complete [18].

Algorithms exist to solve the secure domination problem, including linear-time algorithms for trees [5] and block graphs [16], as well as algorithms for general graphs [2–4]. However, there are very few infinite families of graphs for which the secure domination number is known. When introducing the secure domination problem, Cockayne et al. [8] determined the secure domination number for complete multipartite graphs (including complete graphs), paths, and cycles. In 2019, Valveny and Rodriguez-Velázquez [17] expanded on this by determining the secure domination number for each of the following: the corona product of any graph with a discrete graph, the Cartesian product of two equal-sized stars, and the Cartesian product of any complete graph (other than K_2) with either a path, cycle, or star.

In all of these results, the proofs were either trivial (such as for complete graphs), or took advantage of general lower bounds. Most commonly, the lower bounds came from first considering the *weak Roman domination number*, which is itself a lower bound for the secure domination number. This is worth mentioning because none of the existing results were determined by considering the specific properties unique to secure dominating sets. Of course, this approach can only be fruitful for graph families for which the lower bound obtained from another type of domination happen to coincide with the secure domination number. In general, this is not the case.

The types of graphs considered in this work have been studied in the context of other types of domination. Notably, the domination number of the Cartesian product of two paths (a grid graph) has been extensively studied in [1,6,9] and has now been effectively solved by Gonçlaves et al. [11]. The domination number of the Cartesian product of other combinations of paths and cycles has been investigated in [9,12,13,15]. The Roman domination number of similar graphs has been investigated in [7,9,14,20]. Although there are many results known about these graphs for other types of domination, none of these lead to the corresponding result for secure domination.

We now add to this literature by considering Cartesian products of P_2 and P_3 with arbitrarily large paths or cycles. We adopt the convention that P_n and C_n respectively refer to paths and cycles containing n vertices. We also use the notation M_{2n} to refer to a Möbius ladder graph with $2n$ vertices; these are only defined for an even number of vertices. Upper bounds for $\gamma_s(P_2\square P_n)$ and $\gamma_s(P_3\square P_n)$ were given by Cockayne et al. [8], while the conjectured value for $\gamma_s(P_2\square C_n)$ was given by Valveny and Rodriguez-Velázquez [17] and separately by Winter [19]. Nobody has previously considered $P_3\square C_n$. We will determine the secure dominating number for each of these graph families, and as a corollary to the $P_2\square C_n$ case, we are also able to determine the secure domination number for M_{2n}. It is our expectation that these techniques will prove useful for determining the secure domination numbers of other infinite graph families as well.

2. Notation and useful results

We begin by establishing some valuable results and techniques which will be useful during the proofs in the proceeding sections. Although there have been some important bounds established in previous work, these have mainly been on the secure domination number itself. For the sake of the upcoming proofs, it is valuable to establish results for a given secure dominating set. The first item in the following lemma was pointed out by [17] and has been utilised several times before.

Lemma 1. *Suppose that S is a secure dominating set for a graph G containing vertex set V and edge set E. Then if any combination of the following modifications is made to G, S is still secure dominating for the resulting graph.*

1. *Adding a new edge between existing vertices in V.*
2. *Deleting an edge vw for $v \in S$ and $w \in S$.*
3. *Deleting a vertex v (and its incident edges) for $v \notin S$.*

Proof. In order for S to be secure dominating in a graph, the following must be satisfied. For every $v \notin S$, there must be a $w \in S$ such that $vw \in E$, and $(S \setminus \{w\}) \cup \{v\}$ is dominating. In this situation, we will say that w *guards* v.

Consider any vertex $v \notin S$. In G, there exists a w which guards v. This remains true even if another edge is added, and hence S is still secure dominating.

Next, consider two vertices $v \in S$ and $w \in S$. Neither vertex requires another to guard them, and hence the deletion of vw does not prevent S from being secure dominating.

Finally, consider a vertex $v \notin S$. If it is deleted, the requirement to find a vertex which guards v is removed. Its incident edges are also deleted, however since $v \notin S$, it could never have been the case that v guarded another vertex. Hence, S is still secure dominating. □

Recall that the *open neighbourhood* of a vertex v is the set of vertices that are adjacent to v and is denoted $N(v)$.

Lemma 2. *Suppose that S is a secure dominating set for a graph G. If an edge vw, with $v \notin S$ and $w \notin S$, satisfies at least one of the following*

1. $|N(v) \cap S| > 1$ *and* $|N(w) \cap S| > 1$;
2. $|N(v) \cap N(w) \cap S| \neq 1$,

then, edge vw may be deleted and S is still secure dominating for the resulting graph.

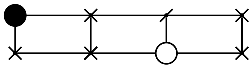

Fig. 1. The situation described in the Example below.

Proof. Consider any vertex $a \in S$ that guards v. Because vw satisfies either 1. or 2. there must exist another vertex $a' \in N(w) \cap S$ where $a' \neq a$. Thus $S \setminus \{a\} \cup \{v\}$ is dominating in the graph with edge vw deleted. The same argument can be applied to any vertex that guards w and thus S is secure dominating in the graph with edge vw deleted. □

The following corollary emerges immediately from the third item in Lemma 1, along with the fact that $\gamma_s(G_1 \cup G_2) = \gamma_s(G_1) + \gamma_s(G_2)$.

Corollary 3. *Suppose that S is a secure dominating set for a graph G containing vertex set V, and there exists $U \subset V$ such that if an edge ab exists for $a \in U$ and $b \notin U$, then $a \notin S$. Then if G_2 is constructed by deleting U and its adjacent edges, $|S| \geq \gamma_s(G_2) + |U \cap S|$.*

Since the results we will establish in this paper are for graphs which emerge from Cartesian products of P_2 or P_3, with another graph, the resulting graphs may be thought of as containing many copies of P_2 or P_3. For $P_2 \square P_n$ and $P_3 \square P_n$, it will sometimes be important to refer to specific copies. In such cases, we will use the notation P_2^i and P_3^i to refer to the ith copy, for $i \in \{1, 2, \ldots, n\}$. In other cases, we simply want to consider a set of m consecutive copies of arbitrary starting position. In such cases, we will use the notation $G^1, \ldots G^m$, and still refer to G^i as the ith copy within the set. In both cases, we will also use the notation v_j^i to refer to the jth vertex in the ith copy, where $j = 1$ refers to the top vertex (in the context of Cartesian products with paths) or the outer vertex (in the context of Cartesian products with cycles) in an embedding in the plane.

In the upcoming proofs, a common technique will be to consider how many vertices from each copy are contained in S. In general, we will say that if a copy contains i vertices in S, then it is a *type i copy*, and we will use m_i to denote the number of type i copies for $0 \leq i \leq 3$. Clearly, for $P_2 \square P_n$ and $P_2 \square C_n$, $m_3 = 0$ by definition. Then, it is clear that $m_0 + m_1 + m_2 + m_3 = n$, but also $m_1 + 2m_2 + 3m_3 = |S|$. Combining these, we obtain the below equation, which will be regularly used to establish bounds on m_i.

$$m_0 = n - |S| + m_2 + 2m_3. \tag{1}$$

We will use the term *block* of length l to refer to a set of l consecutive copies such that the first and last copies have type 0, and the other copies do not have type 0. Clearly, we can use the vertices of a block as the set U in Corollary 3. Furthermore, it will sometimes be useful to take advantage of the following result.

Lemma 4. *Suppose that S is a secure dominating set for a graph $G \square C_n$. Then if we have a block of length l containing the vertices U, then $|S| \geq \gamma_s(G \square C_{n-l}) + |U \cap S|$. Also, if we have a single type 0 copy, then $|S| \geq \gamma_s(G \square C_{n-1})$.*

Proof. We handle both cases simultaneously. As with Corollary 3, we can delete the edges which connect U to the rest of the graph, and S remains secure dominating for the new, disconnected graph. Then, from item 1 of Lemma 1 we can add edges to create a new graph G_2 which is the union of the subgraph induced by the vertices of U, and a smaller Cartesian product with l fewer copies of G. The result then follows immediately. □

Lemma 4 is a special case of a more general technique, where a subgraph satisfying the conditions of Corollary 3 can be essentially smoothed, rather than simply deleted. We will use a specific application of this technique in Lemma 28.

We will use the term *pattern* to refer to a set of consecutive copies which have specified types. For instance, if we say that S contains the pattern 101, it implies that there is a set of three consecutive copies where the first is of type 1, the second is of type 0, and the third is of type 1. In many cases, we will establish that certain patterns cannot exist in S, usually taking advantage of symmetry to avoid considering unnecessarily many cases. In order to illustrate this, we provide a short example here, which also serves to introduce the graphical notation that will be used.

Example. Suppose S is a secure dominating set for $P_2 \square P_n$. We will show that S cannot contain the pattern 1010. Suppose that it does. We consider $G^1, \ldots, G^4$, a set of four consecutive copies of P_2, obeying the pattern 1010. Consider G^1 first; due to symmetry, it does not matter which vertex is in S. Hence, we will assume that $v_1^1 \in S$. Then, G_2 is of type 0 and so by the domination condition, it is clear that $v_2^3 \in S$ (in order to cover v_2^2). Note that since G^3 is type 1, this also implies that $v_1^3 \notin S$. Then, if there is an attack at v_2^2, the guard at v_2^3 must move to assist, but this leaves v_1^3 unguarded, because G^4 is also type 0 copy. Hence, this situation cannot occur, and so S cannot contain the pattern 1010. This situation is displayed in Fig. 1. Note that we use a solid circle to denote vertices which are fixed in S by assumption, and hollow circles to denote vertices which we subsequently argue must appear in S. Likewise, we use crosses to denote those vertices which are not in S by assumption, and a strikeout to denote vertices which we subsequently argue must not appear in S.

Finally, we note that each of the upcoming proofs will use induction. Hence, it will be necessary to establish the secure domination number for a certain number (depending on the proof) of small cases. Rather than present these small cases here, we will rely on the exact formulation given in Burdett and Haythorpe [2] to handle these.

Fig. 2. The vertex patterns for $P_2 \square P_n$, which can be repeated as many times as needed, alternating between them. The result can then be modified accordingly, as described above.

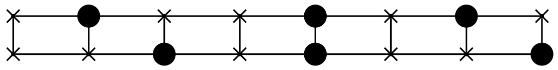

Fig. 3. The vertex pattern which can be repeated as many times as needed for $P_2 \square C_n$, $n = 0 \bmod 8$.

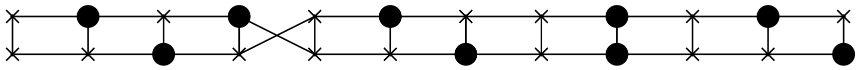

Fig. 4. The vertex pattern for M_{2n}, $n = 4 \bmod 8$, which may then be followed by as many copies of the vertex pattern from Fig. 3 as needed.

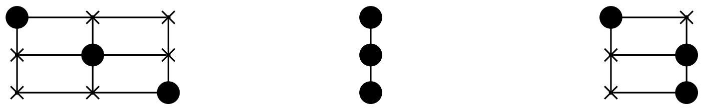

Fig. 5. The vertex patterns for $P_3 \square C_n$. The first item can be repeated as many times as needed. Then if $n = 1 \bmod 3$, the second item is used as the final copy. If $n = 2 \bmod 3$, the third item is used as the final two copies.

3. Upper bounds

In the proceeding sections, we will be establishing lower bounds. In order to obtain equalities, we will also require upper bounds, which we provide here.

Theorem 5. *We have the following upper bounds:*

$$\begin{aligned}
\gamma_s(P_2 \square P_n) &\leq \left\lceil \frac{3n+1}{4} \right\rceil \text{ for } n \geq 2,\\
\gamma_s(P_2 \square C_n) &\leq \frac{3n}{4} \text{ for } n = 0 \bmod 8, n \geq 8,\\
\gamma_s(M_{2n}) &\leq \frac{3n}{4} \text{ for } n = 4 \bmod 8, n \geq 12,\\
\gamma_s(P_3 \square P_n) &\leq n+2 \text{ for } n \geq 2,\\
\gamma_s(P_3 \square C_n) &\leq 3\left\lceil \frac{n}{3} \right\rceil \text{ for } n \geq 3.
\end{aligned}$$

Proof. The upper bound for $P_3 \square P_n$ was established in Cockayne et al. [8]. For the other cases, we will provide drawings which corresponding to a secure dominating set S which meets the upper bounds. Verifying that S is secure dominating is left as an exercise to the reader.

For $P_2 \square P_n$, the vertex patterns in the first and second items shown in Fig. 2 can be repeated, alternating between them. Suppose that we do so, obtaining a vertex pattern for a graph with $4\left\lceil \frac{n}{4} \right\rceil$ vertices. We now describe how to modify this to obtain a valid S for $P_2 \square P_n$. If $n = 0 \bmod 4$, simply add either of the vertices from the final copy to S. If $n = 1 \bmod 4$, delete the first, and the final two copies. For $n = 2 \bmod 4$, delete the final two copies. For $n = 3 \bmod 4$, delete the final copy.

For $P_2 \square C_n$, if $n = 0 \bmod 8$, then the vertex pattern shown in Fig. 3 can be repeated as many times as needed.

For M_{2n}, if $n = 4 \bmod 8$, then the vertex pattern shown in Fig. 4 can be followed by as many copies of the vertex pattern shown in Fig. 3 as needed.

For $P_3 \square C_n$, the first item in Fig. 5 shows the vertex pattern which can be repeated as many times as needed. Suppose that we do so, obtaining a vertex pattern for a graph with $3\left\lfloor \frac{n}{3} \right\rfloor$ vertices. We now describe how to modify this to obtain a valid S for $P_3 \square C_n$. If $n = 0 \bmod 3$, no modification is necessary. If $n = 1 \bmod 3$, the second item in Fig. 5 should be added as a final copy. If $n = 2 \bmod 3$, the third item in Fig. 5 should be added as the final two copies. □

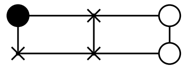

Fig. 6. The situation described in Lemma 6.

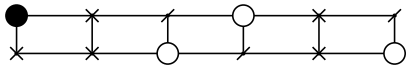

Fig. 7. The situation described in Lemma 7.

4. The secure domination number of $P_2\square P_n$

We now consider the graph $P_2\square P_n$ for $n \geq 2$. In Section 3, we established that $\gamma_s(P_2\square P_n) \leq \left\lceil \frac{3n+1}{4} \right\rceil$ for $n \geq 2$. We will now prove that equality holds.

For small cases, we are able to use the exact formulation from [2], and indeed, we have done so to show that equality holds for $n \leq 16$. However, it could be the case that there is some value $k \geq 17$ such that equality holds for $n = 2, \ldots, k-1$, but $\gamma_s(P_2\square P_k) \leq \left\lceil \frac{3k-3}{4} \right\rceil = \left\lfloor \frac{3k}{4} \right\rfloor$. If so, then there exists a secure dominating set S for $P_2\square P_k$ such that $|S| = \left\lfloor \frac{3k}{4} \right\rfloor$. We will now show that this is impossible.

Recall that $P_2\square P_k$ contains k copies of P_2, which we call $P_2^1, P_2^2, \ldots, P_2^k$. We will refer to P_2^1 and P_2^k as "end" copies, P_2^2 and P_2^{k-1} as "second-end" copies, and all other copies as "internal copies". In this case, Eq. (1) becomes:

$$m_0 = \left\lceil \frac{k}{4} \right\rceil + m_2. \tag{2}$$

Note that, since $k \geq 17$, we have $m_0 \geq 5$. Hence, there is at least one internal copy of type 0.

Lemma 6. *Suppose that S is a secure dominating set for $P_2\square P_k$ such that $|S| = \left\lceil \frac{3k-3}{4} \right\rceil$. Furthermore, suppose that for all $n = 2, \ldots, k-1$, we have $\gamma_s(P_2\square P_n) = \left\lceil \frac{3n+1}{4} \right\rceil$. Then if P_2^2 is of type 0, it must be the case that P_2^1 is of type 1, and P_2^3 is of type 2. Similarly, if P_2^{k-1} is of type 0, it must be the case that P_2^k is of type 1, and P_2^{k-2} is of type 2.*

Proof. Suppose that P_2^2 is of type 0. In order to satisfy the domination condition, P_2^1 must either be of type 1 or type 2. Suppose it is of type 2, then we can delete P_2^1 and P_2^2, obtaining $P_2\square P_{k-2}$. Then, from Corollary 3 we have $|S| \geq \left\lceil \frac{3k-5}{4} \right\rceil + 2$, which is a contradiction. Hence, P_2^1 is of type 1.

Then, due to symmetry, we can select either vertex of P_2^1 to be in S. Assume $v_1^1 \in S$. Then, in order to satisfy the domination property, $v_2^3 \in S$. Also, if there is an attack at v_2^1, the guard at v_1^1 must move to assist, which implies that $v_1^3 \in S$ in order to avoid leaving v_1^2 unguarded. Hence P_2^3 must be of type 2. This situation is displayed in Fig. 6.

Due to symmetry, an analagous argument can be made for P_2^{k-1}, completing the proof. $\square$

Lemma 7. *Suppose that S is a secure dominating set for $P_2\square P_k$. Then S cannot contain the pattern 101101.*

Proof. Suppose that we have six consecutive copies of P_2, call them $G^1, \ldots, G^6$, obeying the pattern 101101. We denote v_1^i and v_2^i to be the top and bottom vertices of G^i, respectively. Due to symmetry, it does not matter which vertex of G^1 is in S. Assume $v_1^1 \in S$. By the domination condition, $v_2^3 \in S$. Then, if there is an attack at v_2^2, the guard at v_2^3 must move to assist, which implies that $v_1^4 \in S$ in order to avoid leaving v_1^3 unguarded. Finally, by the domination condition, $v_2^6 \in S$. This situation is displayed in Fig. 7. Now, suppose there is an attack at v_1^3. Either the guard at v_2^3, or the guard at v_1^4 must move to assist. In the former case, v_2^2 is left unguarded. In the latter case, v_1^5 is left unguarded. This implies that S is not secure dominating, which is a contradiction, and hence S cannot contain the pattern 101101. $\square$

Lemma 8. *Suppose that S is a secure dominating set for $P_2\square P_k$ such that $|S| = \left\lceil \frac{3k-3}{4} \right\rceil$. Furthermore, suppose that for all $n = 2, \ldots, k-1$, we have $\gamma_s(P_2\square P_n) = \left\lceil \frac{3n+1}{4} \right\rceil$. Then, $k = 0$ mod 4, and any internal copy P_2^m can only be of type 0 if m mod $4 \in \{2, 3\}$.*

Proof. Recall from (2) that there is at least one internal copy of type 0. Consider such a copy, say P_2^m for some $m \in \{3, \ldots, k-2\}$. If we delete this copy, we obtain the union of $P_2\square P_{m-1}$ and $P_2\square P_{k-m}$. Then, from Corollary 3 we have $|S| \geq \left\lceil \frac{3m-2}{4} \right\rceil + \left\lceil \frac{3k-3m+1}{4} \right\rceil$. By checking all combinations of k and m mod 4, it can be seen that this is a contradiction except for the following two cases: either $k = 0$ mod 4 and $m \in \{2, 3\}$ mod 4, or $k = 3$ mod 4 and $m = 2$ mod 4.

Suppose $k = 3$ mod 4. Recall from (2) that $m_0 = \left\lceil \frac{k}{4} \right\rceil + m_2$. However, internal copies P_2^m can only be of type 0 if $m = 2$ mod 4. Hence, at most $\left\lceil \frac{k}{4} \right\rceil - 2$ internal copies are of type 0. Hence, $2 + m_2$ of $P_2^1, P_2^2, P_2^{k-1}, P_2^k$ must be of type 0.

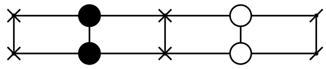

Fig. 8. The situation described in Lemma 9.

Suppose that P_2^1 is of type 0. Then, by the domination condition, P_2^2 must be of type 2, which implies that both P_2^{k-1} and P_2^k must be type 0, which violates the domination condition. Hence, P_2^1 is not of type 0. An analagous argument can be made to conclude that P_2^k is not of type 0. Hence, both P_2^2 and P_2^{k-1} must be of type 0, and $m_2 = 0$. However, according to Fig. 6 this is impossible, and hence $k \neq 3 \bmod 4$, completing the proof. $\square$

In the next proof, we utilise the concept of a "block", defined in Section 2. We similarly define an "end-block" to be a sequence of consecutive copies of P_2 such that the first is P_2^1 and the last is of type 0, or such that the first is of type 0 and the last is P_2^k. Also, the end-copy in an end-block (either P_2^1 or P_2^k) should not be type 0; if it is, then this is just a block, rather than an end-block. As such, an end-block contains only one copy of type 0, whereas a block contains two copies of type 0. As with a block, if the end-block contains l copies of P_2, we say it has length l.

Lemma 9. *Suppose that S is a secure dominating set for $P_2 \square P_k$ such that $|S| = \left\lceil \frac{3k-3}{4} \right\rceil$. Furthermore, suppose that for all $n = 2, \dots, k-1$, we have $\gamma_s(P_2 \square P_n) = \left\lceil \frac{3n+1}{4} \right\rceil$. Then, $m_2 = 0$.*

Proof. Suppose that $m_2 > 0$, then there is a type 2 copy of P_2. Then since $m_0 > 2$, this copy of P_2 must either be contained within a block or an end-block. Suppose first that it is contained within an end-block, of length l. Then the end-block must contain at least l vertices in S. We can delete this end-block, obtaining $P_2 \square P_{k-l}$. Then, from Corollary 3 we have $|S| \geq \left\lceil \frac{3k-3l+1}{4} \right\rceil + l$ which is a contradiction. Hence, the type 2 copy must be contained within a block.

Suppose that the block containing the type 2 copy has length l. By the definition of a block, $l \geq 3$. Then the block contains at least $l-1$ vertices in S. We can delete this block, obtaining a union of two graphs $P_2 \square P_a$ and $P_2 \square P_b$, where $a + b = k - l$. Then, from Corollary 3 we have $|S| \geq \left\lceil \frac{3a+1}{4} \right\rceil + \left\lceil \frac{3b+1}{4} \right\rceil + l - 1 \geq \left\lceil \frac{3a+3b+1}{4} \right\rceil + l - 1 = \left\lceil \frac{3k+l-3}{4} \right\rceil$. This is a contradiction unless $l = 3$. Hence, the block must have the pattern 020.

Now, suppose that all copies contained within this block are internal copies. Then the pattern 020 contradicts Lemma 8, as the two type 0 copies are distance two apart. Hence, the block must not contain all internal copies. Also, the block cannot start at P_2^2, or else P_2^4 would be of type 0 which contradicts Lemma 8. Similarly, since $k = 0 \bmod 4$ by Lemma 8, the block cannot start at P_2^{k-3}. Hence, there are only two possible options. Either the block starts at P_2^1, or P_2^{k-2}. Suppose it starts at P_2^1, and consider $P_2^2, P_2^3, P_2^4, P_2^5$. If there is an attack at v_1^1, the guard at v_1^2 must move to assist, which implies that $v_1^4 \in S$ in order to avoid leaving v_1^3 unguarded. An analagous argument can be made for an attack at v_2^1, implying that $v_2^4 \in S$. Hence, P_2^4 must be of type 2 as well. Then, since all type 2 copies must exist in a pattern 020, this implies that P_2^5 is of type 0, which contradicts Lemma 8. This situation is displayed in Fig. 8. By symmetry, because $k = 0 \bmod 4$, the same argument can be made for a block starting at P_2^{k-2}. Hence the pattern 020 cannot exist in S, and so $m_2 = 0$. $\square$

We are now ready to pose the main theorem of this section.

Theorem 10. *Consider the graph $P_2 \square P_n$. Then for $n \geq 2$,*

$$\gamma_s(P_2 \square P_n) = \left\lceil \frac{3n+1}{4} \right\rceil.$$

Proof. We use the exact formulation from [2] to confirm the result for $n \leq 16$. Then, suppose there is some value $k \geq 17$ such that Theorem 10 holds for $n = 2, \dots, k-1$, but $\gamma_s(P_2 \square P_k) \leq \left\lceil \frac{3k-3}{4} \right\rceil$. Then, there is a secure dominating set S such that $|S| = \left\lceil \frac{3k-3}{4} \right\rceil$. By Lemma 8, it must be the case that $k = 0 \bmod 4$, and all internal copies P_2^m of type 0 must satisfy $m \in \{2, 3\} \bmod 4$. Furthermore, from Lemma 9, $m_2 = 0$, and so from Fig. 6, P_2^2 and P_2^{k-1} cannot be of type 0. It is also easy to check that neither P_2^1 nor P_2^k can be of type 0, or else P_2^2 or P_2^{k-1} need to be type 2 to satisfy the domination condition.

Given the above, we now consider all possible copies which might be type 0. They include P_2^3, P_2^{k-2}, and all internal pairs of the form P_2^m, P_2^{m+1} for $m = 2 \bmod 4$. There are $\frac{k}{4} - 2$ such internal pairs. Now, suppose that for one of these internal pairs, both are of type 0. Then it is clear from the domination condition that their neighbouring copies must be of type 2, which contradicts $m_2 = 0$. Hence, each internal pair contains at most one type 0 copy. However, since $k = 0 \bmod 4$ and $m_2 = 0$, (2) reduces to $m_0 = \frac{k}{4}$. Hence we must have exactly one type 0 copy in all internal pairs, and P_2^3 and P_2^{k-2} must both be type 0 as well.

Now, consider $P_2^2, \dots, P_2^7$. From the previous paragraph, we know that exactly one of P_2^6 and P_2^7 must be type 0. If P_2^6 is type 0, then P_2^7 must be type 1, and this results in the pattern 101101, contradicting Lemma 7. Hence, P_2^6 must be type 1 and P_2^7 type 0. The same argument can be made for $P_2^6, \dots, P_2^{11}$, concluding that P_2^{10} must be type 1 and P_2^{11} type 0. Continuing this argument, we arrive at P_2^{k-6} being type 0 and P_2^{k-5} being type 1, but P_2^{k-2} is also type 1. That means

$P_2^{k-6}, \ldots, P_2^{k-1}$ has the pattern 101101, contradicting Lemma 7. Hence, in all cases, a contradiction is reached, and so it cannot be the case that $|S| \leq \left\lceil \frac{3k-3}{4} \right\rceil$. Hence, the lower bound for $|S|$ coincides with the upper bound from Section 3, completing the proof. □

5. The secure domination number of $P_2 \square C_n$

We begin by noting that $P_2 \square C_n$ can be thought of as $P_2 \square P_n$ with the addition of two edges. Hence $\gamma_s(P_2 \square C_n) \leq \gamma_s(P_2 \square P_n)$. The question then is, is the inequality ever strict, and if so, under which conditions? In this section, we will prove that equality holds in all cases except when $n = 0$ mod8. Specifically, we will prove that

$$\gamma_s(P_2 \square C_n) = \begin{cases} \frac{3n}{4} & \text{if } n = 0 \text{ mod } 8, \\ \left\lceil \frac{3n+1}{4} \right\rceil & \text{otherwise.} \end{cases}$$

Suppose that there is some value k such that $\gamma_s(P_2 \square C_k) < \gamma_s(P_2 \square P_k)$. That is, there exists some secure dominating set S for $P_2 \square C_k$, such that $|S| = \left\lceil \frac{3k-3}{4} \right\rceil$. Note that this implies that S is not a secure dominating set for $P_2 \square P_k$. Recall that $P_2 \square C_k$ contains k copies of P_2, and each copy contains 0, 1 or 2 vertices in S. Note that $P_2 \square C_k$ can be drawn as two concentric cycles connected by inner edges; we will arbitrarily designate one of these cycles to be the "outer" cycle and the other to be the "inner" cycle. To distinguish between these, we will refrain from using the term "type 1" here, and instead use "type 1i" and "type 1o" to indicate a copy of P_2 has only the inner vertex in S, or only the outer vertex in S, respectively. Hence, we will say that each copy of P_2 will be one of four types: type 0, type 1i, type 1o or type 2.

Lemma 11. *Suppose S is a secure dominating set for $P_2 \square C_k$ such that $|S| = \left\lceil \frac{3k-3}{4} \right\rceil$. Then, no two neighbouring copies of P_2 may be of the same type.*

Proof. Suppose two copies are of the same type. Then, the two edges between them could be deleted, and by Lemmas 1 and 2, S would still be secure dominating in the resulting graph. However, the resulting graph is $P_2 \square P_k$, which leads to a contradiction. □

Lemma 12. *Suppose S is a secure dominating set for $P_2 \square C_k$ such that $|S| = \left\lceil \frac{3k-3}{4} \right\rceil$. Further, suppose there is a block (or a set of consecutive blocks) of length l which contains c vertices in S. Then, $c < \frac{3l}{4}$, and if $c = \frac{3l-1}{4}$ then $k = 0$ mod 4.*

Proof. We can delete the block (or set of consecutive blocks) obtaining $P_2 \square P_{k-l}$. Then, from Corollary 3 we have $|S| \geq \left\lceil \frac{3k-3l+1}{4} \right\rceil + c = \left\lceil \frac{3k+4c-3l+1}{4} \right\rceil$. It is clear that this contradicts the assumption if $4c - 3l + 1 \geq 1$, and hence $c < \frac{3l}{4}$. Also, if $c = \frac{3l-1}{4}$ we have $\left\lceil \frac{3k}{4} \right\rceil \leq \left\lceil \frac{3k-3}{4} \right\rceil$ which implies that $k = 0$ mod 4. □

Lemma 13. *Suppose S is a secure dominating set for $P_2 \square C_k$ such that $|S| = \left\lceil \frac{3k-3}{4} \right\rceil$. If any copy of P_2 is of type 2, then its neighbours must both be type 0, and it must be the case that $k = 0$ mod 4.*

Proof. Consider such a type 2 copy of P_2. It must exist inside a block with length $l \geq 3$. Then it must contain at least $l-1$ vertices in S. Applying Lemma 12, we obtain a contradiction unless $l = 3$. Hence, any block containing the type 2 copy must have the pattern 020. Then, in this case $l = 3$ and $c = 2$, and so $c = \frac{3l-1}{4}$. Hence, from Lemma 12, $k = 0$ mod 4. □

Lemma 14. *Suppose S is a secure dominating set for $P_2 \square C_k$ such that $|S| = \left\lceil \frac{3k-3}{4} \right\rceil$. Then S cannot contain the pattern 202, or the pattern 010.*

Proof. Suppose that S contains the pattern 202. Then, from Lemma 13, both type 2 copies must have an additional type 0 neighbour, and so S contains the pattern 02020. However, if we apply Lemma 12 for $l = 5$ and $c = 4$, we obtain a contradiction. Hence, S cannot contain the pattern 202.

Next, suppose that S contains the pattern 010. We will denote these three copies by G^1, G^2, G^3, and also consider the next three neighbouring copies G^4, G^5, G^6. We will denote the outer and inner vertices of G^i by v_1^i and v_2^i, respectively. By symmetry, it does not matter whether $v_1^2 \in S$ or $v_2^2 \in S$; we will assume the former. Then, by the domination property, $v_2^4 \in S$. Also, if there is an attack at v_2^2, the guard at v_1^2 must move to assist, which implies that $v_1^4 \in S$ in order to avoid leaving v_1^3 unguarded. Hence, G^4 is of type 2, and by Lemma 13 we know that G^5 is of type 0.

Then, if there is an attack at v_2^3, the guard at v_2^4 must move to assist, which implies that $v_2^6 \in S$ in order to avoid leaving v_2^5 unguarded. Likewise, if there is an attack at v_1^3, the guard at v_1^2 cannot move to assist without leaving v_2^2 unguarded. Hence, the guard at v_1^4 must move to assist, which implies that $v_1^6 \in S$ in order to avoid leaving v_1^5 unguarded. Therefore, G^6 is of type 2, and S contains the pattern 202, which as established above is impossible. Hence, S cannot contain the pattern 010. This situation is displayed in Fig. 9. □

Lemma 15. *Suppose S is a secure dominating set for $P_2 \square C_k$ such that $|S| = \left\lceil \frac{3k-3}{4} \right\rceil$. Then $k = 0$ mod 4.*

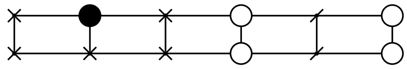

Fig. 9. The situation described in the second part of Lemma 14.

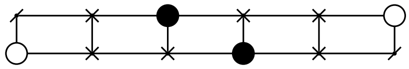

Fig. 10. The situation described in Lemma 15.

Proof. Suppose $k \neq 0$ mod 4. We know from Lemma 13 that no copies of P_2 are of type 2. Hence, all blocks have the pattern 01...10 for some number of 1s. We know from Lemma 14 that S cannot contain the pattern 010. We will now show that S cannot contain the pattern 0110 either in this case.

Consider six consecutive copies of P_2, denoted by $G^1, \dots, G^6$. Suppose that G^2 and G^5 are of type 0, one of G^3 and G^4 is of type 1i, and the other of type 1o. Due to symmetry, it does not matter which, so we will assume that $v_1^3 \in S$ and $v_2^4 \in S$. This situation is displayed in Fig. 10. Then, by the domination property $v_2^1 \in S$ and $v_1^6 \in S$. Also, since there are no copies of type 2, we have $v_1^1 \notin S$ and $v_2^6 \notin S$. Then, suppose there is an attack at v_2^3. Either the guard at v_1^3, or the guard at v_2^4 must move to assist. In the former case, v_1^2 is left unguarded. In the latter case, v_2^5 is left unguarded. Hence, this is impossible, and S cannot contain the pattern 0110. Therefore, all blocks contain at least three blocks of type 1i or type 1o.

Noting that $m_2 = 0$, we have $m_1 = \left\lceil \frac{3k-3}{4} \right\rceil$. Also, from the above paragraph, we have $m_1 \geq 3m_0$. Combining these together, we get $\frac{3k}{4} \leq \left\lceil \frac{3k-3}{4} \right\rceil$ which violates the initial assumption that $k \neq 0$ mod 4. $\square$

We have now established that if $\gamma_s(P_2 \Box C_k) < \gamma_s(P_2 \Box P_k)$, it must be the case that $k = 0$ mod 4. Note that this, in turn, implies that $|S| = \frac{3k}{4}$. We will now go a step further and prove that, in fact, it must be the case that $k = 0$ mod 8. Consider any block in S. From the results established so far, it is clear that the internal copies in the block are either alternating copies of type 1i and type 1o, or there is a single internal copy of type 2. We will refer to the former as a *type 1 block* of a certain length, and the latter as a *type 2 block*.

Lemma 16. *Suppose S is a secure dominating set for $P_2 \Box C_k$ such that $|S| = \frac{3k}{4}$. Suppose S contains a type 1 block of length 4. Then there must be a type 2 block on at least one side of this block.*

Proof. Suppose the opposite is true. Then S contains the pattern 101101. However, the proof of Lemma 7 can be repeated in this situation too, leading to the conclusion that this is impossible. $\square$

Now, denote by b_i the number of type 1 blocks of length i. Then the following result emerges immediately from Lemma 16, where equality occurs if every type 2 block has type 1 blocks of length 4 on both sides.

Corollary 17. *Suppose S is a secure dominating set for $P_2 \Box C_k$ such that $|S| = \frac{3k}{4}$. Then $b_4 \leq 2m_2$.*

Lemma 18. *Suppose S is a secure dominating set for $P_2 \Box C_k$ such that $|S| = \frac{3k}{4}$. Then all type 1 blocks must have length either 4 or 5, and $b_4 = 2m_2$.*

Proof. We know from Lemma 14 that the type 1 blocks cannot have length 3. By considering the number of vertices each block contributes to S, we have $2m_2 + \sum_{i=4}^{k}(i-2)b_i = \frac{3k}{4}$. Also, by considering the number of copies of P_2 each block contains, we have $2m_2 + \sum_{i=4}^{k}(i-1)b_i = k$. Combining these, we can obtain $b_4 = 2m_2 + \sum_{i=6}^{k}(i-5)b_i$. Then, the result emerges from Corollary 17 and the nonnegativity of b_i. $\square$

Lemmas 16 and 18 imply that S is exclusively made up of two kinds of patterns; specifically, the patterns 01102011 and 0111. Note that each pattern ends with either a type 1i or type 1o copy. We will say the pattern has "parity i" in the former case, and "parity o" in the latter case. This pattern is then followed by another pattern which begins with a type 0 copy.

Lemma 19. *Suppose S is a secure dominating set for $P_2 \Box C_k$ such that $|S| = \frac{3k}{4}$. Then b_5 must be an even number.*

Proof. Suppose S contains the pattern 11020110 with parity i. Note that the pattern must be preceded by another pattern which ends with 10, and followed by another pattern which begins with a 1. We now consider eleven consecutive copies of P_2, call them $G^1, \dots, G^{11}$ and denote the outer and inner vertices of G^i by v_1^i and v_2^i respectively. They have types obeying the pattern 10110201101, and G^3 has type 1i. This situation is displayed in Fig. 11. By the domination condition, G^1 is type 1o. If there is an attack at v_2^2, the guard at v_2^3 must move to assist, which implies that $v_1^4 \in S$ in order to avoid

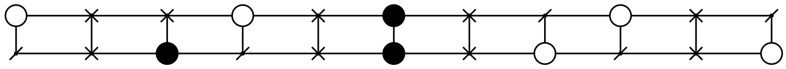

Fig. 11. The situation described in the second part of Lemma 19.

leaving v_1^3 unguarded. Then, if there is an attack at v_2^5, the guard at v_2^6 must move to assist, which implies that $v_2^8 \in S$ in order to avoid leaving v_2^7 unguarded. Then, if there is an attack at v_2^7, the guard at v_2^6 cannot move to assist without leaving v_2^5 unguarded. Therefore, the guard at v_2^8 must move to assist, which implies that $v_1^9 \in S$ in order to avoid leaving v_1^8 unguarded. Finally, by the domination condition, G^{11} is type 1i. As a result, we see that if the pattern 11020110 has parity i, the next pattern also has parity i. The analagous result can be obtained if the pattern 11020110 has parity o.

Now, suppose S contains the pattern 1110 with parity i. Then clearly the three type 1 copies in the pattern are of type 1i, 1o, and 1i respectively. Then by the domination condition, the next pattern must start with a type 1o copy. Hence, if the pattern 1110 has parity i, the next pattern has parity o. The analogous result can be obtained if the pattern 1110 has parity o.

Since the parity is not changed by 11020110, and is changed by 1110, there must be an even number of 1110 patterns. □

We are now ready to prove the main theorem of this section.

Theorem 20. *Consider $P_2 \square C_n$ for $n \geq 3$. Then,*

$$\gamma_s(P_2 \square C_n) = \begin{cases} \frac{3n}{4} & n = 0 \bmod 8, \\ \left\lceil \frac{3n+1}{4} \right\rceil & \text{otherwise.} \end{cases}$$

Proof. From Lemma 19, we know that if $n \neq 0$ mod 8 then $\gamma_s(P_2 \square C_n) = \gamma_s(P_2 \square P_n)$. Also, from Section 3 we know that if $n = 0$ mod 8, we have $\gamma_s(P_2 \square C_n) \leq \frac{3n}{4}$. Then all that remains is to show that $\gamma_s(P_2 \square C_n) \geq \frac{3n}{4}$ for $n = 0$ mod 8.

Suppose there is a secure dominating set S for $P_2 \square C_n$, $n = 0$ mod 8, such that $|S| \leq \frac{3n}{4} - 1$. Then, certainly there is at least one type 0 block. It can be deleted to obtain $P_2 \square P_{n-1}$. Then, from Corollary 3 we have $|S| \geq \left\lceil \frac{3n-2}{4} \right\rceil$. Since $n = 0$ mod 4, we know that $\left\lceil \frac{3n-2}{4} \right\rceil = \frac{3n}{4}$, which contradicts the initial assumption. □

We conclude this section by remarking that the secure domination number for $P_2 \square C_n$ has been conjectured twice previously, in different contexts, both of which agree with Theorem 32. Valveny and Rodriguez-Velázquez [17] considered $P_2 \square C_n$ directly, while Winter [19] gave a conjecture for the secure domination number of the generalised Petersen graph $P(n, 1)$ which is isomorphic to $P_2 \square C_n$. Interestingly, the latter gave the conjectured formula as $\left\lceil \frac{n+7}{8} \right\rceil + \left\lceil \frac{n+4}{8} \right\rceil + \left\lceil \frac{n+2}{4} \right\rceil + \left\lceil \frac{n+1}{4} \right\rceil$. By checking all choices of n mod 8, it can be seen that this coincides with Theorem 20.

5.1. The secure domination number of M_{2n}

Much like $P_2 \square C_n$, the Möbius ladder graph M_{2n} can be thought of as $P_2 \square P_n$ with the addition of two edges. In this case, the two edges form a "twist", however the location of this twist is not fixed. Indeed, we can choose to think of the twist as occurring between any two neighbouring copies of P_2. As such, whenever we look at proofs which are contained within a subgraph of M_{2n}, we can choose to think of the twist as occurring elsewhere. It can be quickly checked that all of the proofs for $P_2 \square C_n$ prior to Lemma 19 are also applicable to M_{2n}. In particular, it is hence established that if there is a secure dominating set S for M_{2k} such that $|S| < \gamma_s(P_2 \square P_k)$, then S must be made up exclusively of two kinds of patterns; specifically, the patterns 01102011 and 0111. Then, since we can think of the twist occurring between any neighbouring copies of P_2, we imagine it occurring between two patterns. We can then follow the same argument as in Lemma 19, noting the twist, to obtain the following corollary.

Corollary 21. *Suppose S is a secure dominating set for M_{2k} such that $|S| = \frac{3k}{4}$. Then b_5 must be an odd number.*

Finally, we can use analagous arguments as in Theorem 20, substituting Corollary 21 for Lemma 19 and using the appropriate upper bound from Section 3 to obtain the following result. The only case not covered by the analagous arguments is $n = 4$, however we have confirmed that $\gamma_s(M_8) = 3$ using the exact formulation from [2].

Theorem 22. *Consider M_{2n} for $n \geq 3$. Then,*

$$\gamma_s(M_{2n}) = \begin{cases} \frac{3n}{4} & n = 4 \bmod 8, \\ \left\lceil \frac{3n+1}{4} \right\rceil & \text{otherwise.} \end{cases}$$

Fig. 12. The two cases considered in Lemma 23.

6. The secure domination number of $P_3 \square P_n$

We now consider $P_3 \square P_n$. From Cockayne et al. [8] we know that $\gamma_s(P_3 \square P_n) \leq n+2$ for all $n \geq 2$. In this section, we will prove the following for $n \geq 2$,

$$\gamma_s(P_3 \square P_n) = \begin{cases} n+1 & n \leq 8 \text{ or } n = 10, \\ n+2 & n = 9 \text{ or } n \geq 11. \end{cases}$$

We use the exact formulation from [2] to confirm the result for $n \leq 21$. Then, suppose there is some value $k \geq 22$ such that $\gamma_s(P_3 \square P_n) = n+2$ for $n = 11, \dots, k-1$ but $\gamma_s(P_3 \square P_k) \leq k+1$. Then, there is a secure dominating set S for $P_3 \square P_k$ such that $|S| = k+1$.

Recall that $P_3 \square P_k$ contains k copies of P_3, which we call $P_3^1, \dots, P_3^k$. We will refer to P_3^1 and P_3^k as "end copies", P_3^2 and P_3^{k-1} as "second-end copies", and all other copies as "internal copies".

Lemma 23. *If S is a secure dominating set for $P_3 \square P_k$ such that $|S| = k+1$, then $m_0 > 0$.*

Proof. From Eq. (1) we have $m_0 = -1 + m_2 + 2m_3$. Now, suppose that $m_0 = 0$. Then it is clear that $m_3 = 0$, $m_2 = 1$ and $m_1 = k-1$. Now suppose that none of $P_3^1, P_3^2, P_3^3, P_3^4$ are type 2. Clearly P_3^1 cannot be type 0, or else P_3^2 would need to be type 3 to satisfy the domination condition. Hence, P_3^1 is type 1. Up to symmetry, there are two cases to consider. Either one of the upper or lower vertices is in S, or the middle vertex is in S.

For the former case, we will assume $v_1^1 \in S$. Then, by the domination condition, $v_3^2 \in S$. Now, if there is an attack at v_3^1, the guard at v_3^2 must move to assist, which implies that $v_2^3 \in S$ in order to avoid leaving v_2^2 unguarded. Then, if there is an attack at v_2^1, the guard at v_1^1 must move to assist, leaving v_1^2 unguarded. Hence, this case is impossible.

For the second case, we have $v_2^1 \in S$. Then, suppose $v_1^2 \notin S$. If there is an attack at v_1^1, the guard at v_2^1 must move to assist, which implies that $v_3^2 \in S$ in order to avoid leaving v_3^1 unguarded. It is clear that either the upper or lower vertex of P_2^3 must be in S, and due to symmetry either choice is equivalent. We will assume $v_3^2 \in S$. Then, by the domination condition, $v_1^3 \in S$. Then, if there is an attack at v_1^2, the guard at v_1^3 must move to assist, which implies that $v_2^4 \in S$ in order to avoid leaving v_2^3 unguarded. Finally, if there is an attack at v_3^1, either the guard at v_2^1, or the guard at v_3^2, must move to assist. In the former case, v_1^1 is left unguarded. In the latter case, v_3^3 is left unguarded. Hence, this case is also impossible, and so it cannot be the case that none of $P_3^1, P_3^2, P_3^3, P_3^4$ are type 2. These two cases are displayed in Fig. 12.

An analagous argument can be made for $P_{k-3}^3, P_{k-2}^3, P_{k-1}^3, P_k^3$, and so $m_2 > 2$, which is a contradiction, completing the proof. $\square$

We are now ready to prove the main result of this section.

Theorem 24. *Consider $P_3 \square P_n$ for $n \geq 2$. Then,*

$$\gamma_s(P_3 \square P_n) = \begin{cases} n+1 & n \leq 8 \text{ or } n = 10, \\ n+2 & n = 9 \text{ or } n \geq 11. \end{cases}$$

Proof. We use the exact solver to verify Theorem 24 for $n \leq 21$. From Section 3 we know that $\gamma_s(P_3 \square P_n) \leq n+2$. Then, it suffices to show that for any $P_3 \square P_k$, $k \geq 22$, it is impossible to have a secure dominating set S such that $|S| = k+1$. Suppose that such an S exists. Then for some value of k, $P_3 \square P_k$ constitutes the minimal such example. From Lemma 23 we know that $m_0 > 0$.

Suppose that there is an internal copy of type 0, say P_3^m for $m = 3, \dots, k-2$. Then we can delete this copy, resulting in the union of $P_3 \square P_{m-1}$ and $P_3 \square P_{k-m}$. Since $k \geq 22$, at least one of $m-1$ and $k-m$ is either 9, or at least 11. Then, from Corollary 3 we have $|S| \geq k+2$, which contradicts our initial assumption. Therefore, there are no internal copies of type 0.

Suppose that P_3^2 is of type 0. In order to satisfy the domination condition, P_3^1 must have at least one vertex in S. Suppose P_3^1 is type 1, then $v_2^1 \in S$. However, if there is an attack at v_1^1, then the guard at v_2^1 must move to assist, leaving v_3^1 unguarded. Hence, P_3^1 must contain at least two vertices in S. Then, we can delete P_3^1, P_3^2, to obtain $P_3 \square P_{k-2}$. Then, from Corollary 3 we have $|S| \geq k+2$, which contradicts our initial assumption. An analagous argument can be made for P_3^{k-1}, and therefore there are no second-end copies of type 0.

At this stage, we know that either one or both of P_3^1 and P_3^k must be type 0. Suppose they both are. Then, in order to satisfy the domination condition, P_3^2 and P_3^{k-1} must both be type 3, which violates Eq. (1). Hence, exactly one of them must be type 0. Due to symmetry, the choice is equivalent, so we assume P_3^1 is type 0. Then, P_3^2 is type 3, and all remaining copies must be type 1. However, this implies that $P_3^{k-3}, P_3^{k-2}, P_3^{k-1}, P_3^k$ are all type 1, and using the same argument as used in the proof of Lemma 23, this is impossible. Hence, the initial assumption must be incorrect, completing the proof. □

7. The secure domination number of $P_3 \square C_n$

We now consider $P_3 \square C_n$. We will prove that:

$$\gamma_s(P_3 \square C_n) = \begin{cases} n+1 & \text{if } n = 4, 7, \\ 3\left\lceil \frac{n}{3} \right\rceil & \text{otherwise.} \end{cases}$$

We use the exact formulation from [2] to confirm the above for $n \leq 26$. Then, suppose there is a value $k \geq 27$ such that the above is true for $n = 3, \ldots, k-1$, but $\gamma_s(P_3 \square C_k) = 3\left\lceil \frac{n}{3} \right\rceil - 1$. We will first focus on the cases when $k = 1 \bmod 3$, and then handle the other cases afterwards. That is, there is a secure dominating set S for $P_3 \square C_k$ such that $|S| = k+1$.

Theorem 25. *Suppose that S is a secure dominating set for $P_3 \square C_n$, and that S contains the pattern 111111. Whichever vertex from the first copy is in S, the same vertex from the fourth copy is also in S.*

Proof. Suppose it is not true. In Fig. 13 we display, up to symmetry, all possible situations in which the first and fourth copies have different vertices in S, and no neighbouring copies share the same vertex in S. In cases 1, 2, 7 and 9, the domination property is not satisfied. For the remaining cases, we will demonstrate that an attack at a certain vertex forces a guard to move and leave another vertex unguarded. In each case, we refer to the six copies as $G^1, \ldots, G^6$ respectively, and the choice of vertices to be contained in S is fixed for $G^1, \ldots, G^4$. We will refer to the top, central, and bottom vertices of G^i as v_1^i, v_2^i and v_3^i respectively.

In case 3, if there is an attack at v_2^2, the guard at v_3^2 is forced to move to assist, leaving v_3^3 unguarded.

In case 4, if there is an attack at v_2^3, the guard at v_2^2 cannot move to assist without leaving v_3^2 unguarded. Hence, the guard at v_1^3 must move to assist, which implies that $v_1^5 \in S$ in order to avoid leaving v_1^4 unguarded. Then, if there is an attack at v_2^4, the guard at v_3^4 is forced to move to assist, leaving v_3^3 unguarded.

In case 5, in order to satisfy the domination condition, there must be a guard at v_1^5. Then, if there is an attack at v_1^4, the guard at v_1^5 must move to assist, which implies that $v_2^6 \in S$ in order to avoid leaving v_2^5 unguarded. Then, if there is an attack at v_2^4, either the guard at v_2^3 or the guard at v_3^4 must move to assist. In the former case, v_1^3 is left unguarded. In the latter case, v_3^5 is left unguarded.

In case 6, if there is an attack at v_2^3, the guard at v_1^3 must move to assist, which implies that $v_1^5 \in S$ in order to avoid leaving v_1^4 unguarded. Then, if there is an attack at v_3^3, either the guard at v_3^2 or the guard at v_3^4 must move to assist. In the former case, v_2^2 is left unguarded. In the latter case, v_2^4 is left unguarded.

In case 8, if there is an attack at v_2^3, the guard at v_3^3 must move to assist, leaving v_3^2 unguarded. □

The next corollary follows immediately from Theorem 25 and the fact that neighbouring copies cannot have the same vertices in S. To see this, suppose that neighbouring copies P^i, P^{i+1} have the same vertices in S. Then by Lemmas 1 and 2, we can delete all edges between P^i and P^{i+1} to obtain the graph $P_3 \square P_k$ for which S remains secure dominating. Thus $|S| \geq \gamma_s(P_3 \square P_k) \geq n+1$, which is a contradiction.

Corollary 26. *Suppose S is a secure dominating set for $P_3 \square C_k$, $k = 1 \bmod 3$, such that $|S| = k+1$. If S contains the pattern 1111, then the first three copies of P_3 in that pattern each contain a different vertex in S.*

Corollary 26 itself leads to another corollary.

Corollary 27. *Suppose S is a secure dominating set for $P_3 \square C_k$. If every copy of P_3 in S is of type 1, then $k = 0 \bmod 3$.*

As discussed in Section 2, the proof of the following lemma is a more general version of that for Lemma 4. Instead of removing a set of copies, we instead identify a subgraph which contains some, but not all, vertices from various consecutive copies, and which only connects to the rest of the graph via edges whose endpoints are not in S. Then we are able to "smooth" out this subgraph in a natural way. We will show here the full process of deleting the relevant edges, setting aside the disconnected subgraph, and adding in the new edges to complete the smoothing.

Lemma 28. *Suppose S is a secure dominating set for $P_3 \square C_k$, $k = 1 \bmod 3$, such that $|S| = k+1$, but that $\gamma_s(P_3 \square C_{k-3}) = k-1$. Then S does not contain the pattern 111111111.*

Proof. Suppose that S does contain the pattern 111111111. From Corollary 26, the first three copies in that pattern contain a different vertex in S, and then from Theorem 25 this is then repeated for the next three copies, and then the three copies after that. It is clear then that this turns into a diagonal pattern which begins at one of the first three copies, and then continues throughout the pattern. The diagonal pattern can either go from the top vertex down to the bottom vertex, or vice versa; due to symmetry, these are equivalent, so we will assume the former.

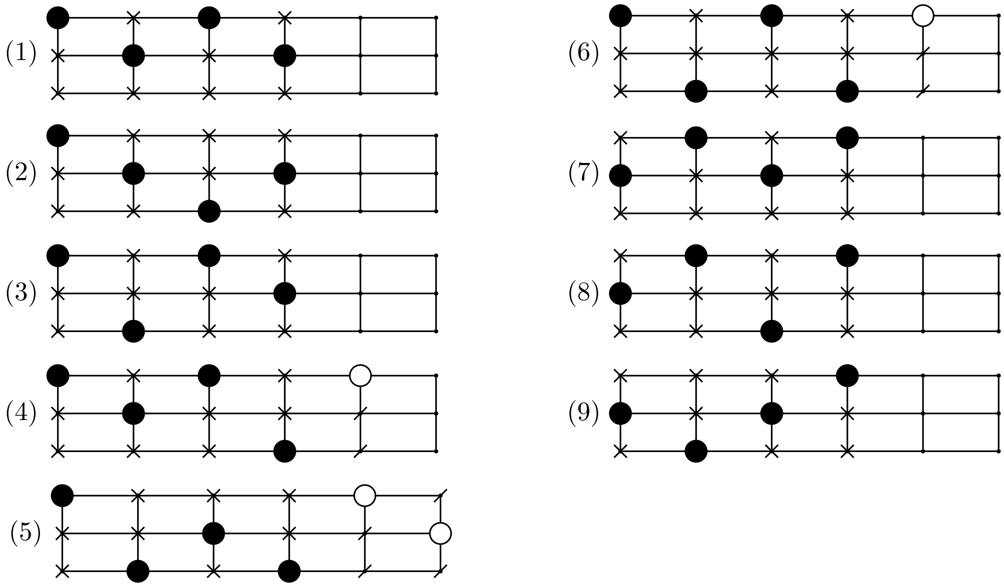

Fig. 13. 9 cases.

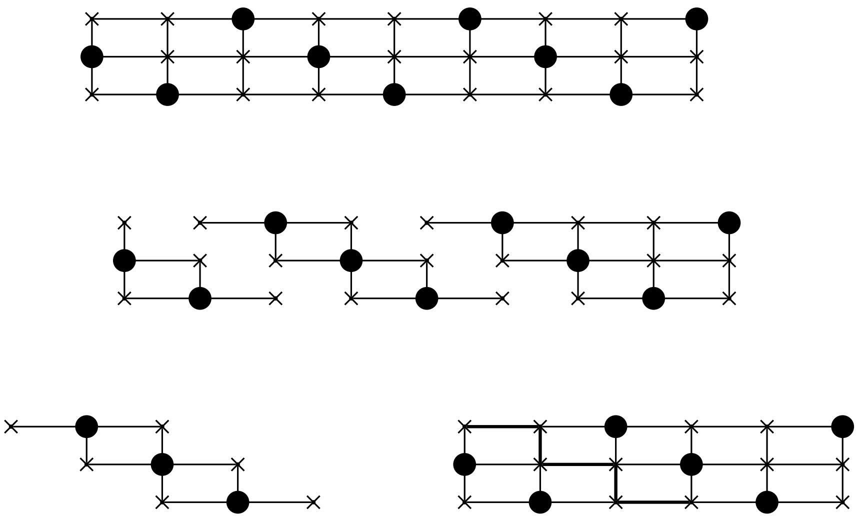

Fig. 14. The situation described in the second part of Lemma 28. The first part shows an example of the pattern 111111111. The second part shows the pattern after the ten edges are deleted. The third part shows the final situation, with the disconnected subgraph moved aside, and five new edges added to reconnected the pattern.

Now, suppose we look at one instance of this diagonal pattern starting at either the third, fourth or fifth copy. An example for the diagonal pattern starting at the third copy is displayed in part 1 of Fig. 14. Then it is possible to identify a set of five edges to the left of the pattern which are all between vertices not in S, and likewise five edges to the right of the pattern, that can be deleted, and by Corollary 3, S is still secure dominating in the resulting, disconnected graph. Finally, we can add in five edges to rejoin the pattern together, which again does not prevent S from being secure dominating in the resulting graph. What results is the union of $P_3\Box C_{k-3}$, plus another subgraph which contains exactly three vertices in S. Hence, $|S| \geq k-1+3$ which contradicts the initial assumption. Note that the process of deleting edges spanned seven copies of P_3, starting two copies to the left of the diagonal pattern, and ending four copies to the right. Then, since the diagonal pattern will start at either the third, fourth or fifth copy, this process can be performed if there are at least nine consecutive type 1 copies, and so S cannot contain the pattern 111111111. The process of deleting and adding edges is illustrated in parts 2 and 3 of Fig. 14. $\square$

Fig. 15. The two cases considered in Lemma 30.

Lemma 29. *Suppose S is a secure dominating set for $P_3\Box C_k$, $k = 1$ mod 3, such that $|S| = k+1$, but that $\gamma_s(P_3\Box C_{k-3}) = k-1$. Then $m_3 = 0$.*

Proof. Recall that the assumption is known to be false for $k \le 26$. Hence, $k \ge 27$. Then, suppose $m_3 > 0$. Consider any block containing a type 3 copy. Suppose the block has length l, then it contains at least l vertices in S. Then, from Lemma 4 we obtain $|S| \ge \gamma_s(P_3\Box P_{k-l}) + l$. By the results of Section 6, this is a contradiction unless $k - l \in \{4, 7\}$ and the block contains no type 2 copies and exactly one type 3 copy. However, from Lemma 28, such a block could only be at most length 19. Since $k \ge 27$, this is a contradiction.

The only remaining possibility is that there are no blocks, which implies that $m_0 \le 1$. If $m_0 = 0$ then from Eq. (1) we have $m_3 = 0$, violating the initial assumption. Hence, $m_0 = 1$, $m_1 = k-2$, $m_2 = 0$, and $m_3 = 1$. From Lemma 28, this implies that $k \le 18$, which is a contradiction. Hence, $m_3 = 0$. □

Lemma 30. *Suppose S is a secure dominating set for $P_3\Box C_k$, $k = 1$ mod 3, such that $|S| = k+1$, but that $\gamma_s(P_3\Box C_{k-3}) = k-1$. Then S does not contain the pattern 011.*

Proof. Suppose that S does contain the pattern 011. Consider four consecutive copies of P_3, denoted by $G^1, \dots, G^4$, and suppose that G^2 is type 0, while G^3 and G^4 both are type 1. Then due to symmetry, there are two cases to consider. Either the middle vertex of G^3 is in S, or one of the upper and lower vertices of G^3 is in S.

For the former case, we have $v_2^3 \in S$. In order to satisfy the domination condition, we must have $v_1^1 \in S$ and $v_3^1 \in S$. Then, from Lemma 29 we know that $m_3 = 0$, and so $v_2^1 \notin S$. Then, suppose there is an attack at v_2^2. The guard at v_2^3 must move to assist, which implies that both $v_1^4 \in S$ and $v_3^4 \in S$ in order to avoid leaving v_1^3 or v_3^3 unguarded.

For the latter case, due to symmetry, we can choose either $v_1^3 \in S$ or $v_3^3 \in S$; we will assume $v_1^3 \in S$. In order to satisfy the domination condition, we must have $v_2^1 \in S$, $v_3^1 \in S$, and $v_3^4 \in S$. Also, from Lemma 29 we know that $m_3 = 0$, and so $v_1^1 \notin S$. Then, suppose there is an attack at v_1^2. The guard at v_1^3 must move to assist, which leaves v_2^3 unguarded. Hence, in both cases, a contradiction is reached. These two cases are displayed in Fig. 15. □

Lemma 31. *Suppose S is a secure dominating set for $P_3\Box C_k$, $k = 1$ mod 3, such that $|S| = k+1$, but that $\gamma_s(P_3\Box C_{k-3}) = k-1$. If $k \ge 29$, then $m_0 \ge 2$.*

Proof. Suppose $m_0 = 0$. By Lemma 29, $m_3 = 0$, and then Eq. (1) implies that $m_1 = k-1$ and $m_2 = 1$. This implies there are $k-1$ type 1 copies in a row, and from Lemma 28 this means that $k \le 9$, contradicting the initial assumption.

Then, suppose $m_0 = 1$. Then since $m_3 = 0$, we have $m_1 = k-3$, $m_2 = 2$. Then, it can be seen from Lemmas 30 and 31 that $k \le 14$, which is also a contradiction. □

We are now ready to prove the main theorem of this section.

Theorem 32. *Consider $P_3\Box C_n$ for $n \ge 3$. Then,*

$$\gamma_s(P_3\Box C_n) = \begin{cases} n+1 & \text{if } n = 4, 7, \\ 3\left\lceil \frac{n}{3} \right\rceil & \text{otherwise.} \end{cases}$$

Proof. We use the exact formulation from [2] to confirm Theorem 32 for $n \le 26$. Now, suppose there is some value $k \ge 27$ such that Theorem 32 is true for $n = 3, \dots, k-1$, but not for $n = k$.

If $k = 1$ mod 3 then from Lemma 31, we have $m_0 \ge 2$, and hence there are blocks in S. Furthermore, from Lemma 29, we have $m_3 = 0$, and then Eq. (1) implies that $m_2 = m_0 + 1$. Hence, there is a block containing at least two copies of type 2. Suppose this block is of length l. Then we can trim out this block, removing at least l entries from S. Using Corollary 3, this is a contradiction unless $k - l \in \{4, 7\}$ and there are exactly two type 2 copies in the block. However, by Lemma 28, there can be at most eight type 1 copies in a row and also by Lemma 30, S cannot contain the pattern 011. Such a block could only have maximum length of 14 and since $k \ge 27$, this is impossible, and so $k \ne 1$ mod 3.

Suppose $k = 2$ mod 3. Then we have $\gamma_s(P_3\Box C_k) \le k$, and a secure dominating set S exists such that $|S| = k$. Suppose that there is any type 0 copy in S. Then from Lemma 4 we have $|S| \ge \gamma_s(P_3\Box C_{k-1})$. However, since $n = k$ is the first time that Theorem 32 is not true, this implies that $|S| \ge k+1$, which is a contradiction. Hence, there must not be any type 0 copies in S. Then Eq. (1) implies that every copy is of type 1. However, from Corollary 27, this implies that $k = 0$ mod 3 which is a contradiction. Hence, $k \ne 2$ mod 3.

The only remaining possibility is that $k = 0 \bmod 3$. Then we have $\gamma_s(P_3 \Box C_k) \leq k - 1$, and a secure dominating set S exists such that $|S| = k - 1$. This implies that at least one type 0 copy exists in S. From Lemma 4 we have $|S| \geq \gamma_s(P_3 \Box C_{k-1})$. However, since $n = k$ is the first time that Theorem 32 is not true, this implies that $|S| \geq k$, which is a contradiction, completing the proof. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] S. Alanko, S. Crevals, A. Isopoussu, P. Östergård, V. Pettersson, Computing the domination number of grid graphs, Elecron. J. Comb. 18 (2011) #P141.

[2] R. Burdett, M. Haythorpe, An improved binary programming formulation for the secure domination problem, Ann. Oper. Res. 295 (2020) 561–573.

[3] A.P. Burger, A.P. de Villiers, J.H. van Vuuren, A binary programming approach towards achieving effective graph protection, in: Proceedings of the 2013 ORSSA Annual Conference, ORSSA, 2013, pp. 19–30.

[4] A.P. Burger, A.P. de Villiers, J.H. van Vuuren, Two algorithms for secure graph domination, J. Comb. Math. Comb. Comput. 85 (2013) 321–339.

[5] A.P. Burger, A.P. de Villiers, J.H. van Vuuren, A linear algorithm for secure domination in trees, Discrete Appl. Math. 171 (2014) 15–27.

[6] T.Y. Chang, E.O. Hare, Domination numbers of complete grid graphs, I, Ars Combin. 38 (1) (1995) 994.

[7] E.J. Cockayne, P.A. Dreyer Jr., S.M. Hedetniemi, S.T. Hedetniemi, Roman domination in graphs, Discrete Math. 278 (2003) 11–22.

[8] E.J. Cockayne, P.J.P. Grobler, W.R. Grundlingh, J. Munganga, J.H. van Vuuren, Protection of a graph, Util. Math. 67 (2005) 19–32.

[9] P.A. Dreyer Jr., Applications and Variations of Domination in Graphs, (Ph.D. thesis), Rutgers University, New Jersey, 2000.

[10] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman, 1979.

[11] D. Gonçlaves, A. Pinlou, R. Michaël, S. Thomassé, The domination number of grids, SIAM J. Discrete Math. 25 (3) (2011) 1443–1453.

[12] S. Klavžar, N. Seifter, Dominating cartesian products of cycles, Discrete Appl. Math. 59 (1995) 129–136.

[13] M. Nandi, S. Parui, A. Adhikari, The domination numbers of cylindrical grid graphs, Appl. Math. Comput. 217 (2011) 4879–4889.

[14] P. Pavlič, J. Žerovnik, Roman domination number of the cartesian products of paths and cycles, Electron. J. Combin. 19 (3) (2012) #P19.

[15] P. Pavlič, J. Žerovnik, A note on the domination number of the cartesian products of paths and cycles, Kragujevac J. Math. 37 (2) (2013) 275–285.

[16] D. Pradhan, A. Jha, On computing a minimum secure dominating set in block graphs, J. Comb. Optim. 35 (2) (2018) 613–631.

[17] M. Valveny, J.A. Rodriguez-Velázquez, Protection of graphs with emphasis on Cartesian product graphs, FILOMAT 33 (1) (2019) 319–333.

[18] H. Wang, Y. Zhao, Y. Deng, The complexity of secure domination problem in graphs, Discuss. Math. Graph Theory 38 (2) (2018) 385–398.

[19] A. Winter, Domination, Total Domination and Secure Domination (Honours Thesis), University of South Australia, 2018.

[20] I.G. Yero, J.A. Rodriguez-Velázquez, Roman domination in cartesian product graphs and strong product graphs, Appl. Anal. Discrete Math. 7 (2013) 262–274.