Protection of a Graph

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Abstract

For vertex v of a simple n–vertex graph $G = (V, E)$ let $f(v)$ be the number of guards stationed at v. A guard at v can deal with a problem at any vertex in its closed neighbourhood.

We consider four strategies, *i.e.* properties of such functions under which the entire graph may be deemed protected. The four properties are domination, Roman domination, weak Roman domination which have been studied previously and a new concept called secure domination.

The four parameters which give the minimum number of guards required to protect the graph under the various strategies, are studied. Exact values or bounds are obtained for specific classes of graphs. We also give a characterisation of those secure dominating sets which are minimal.

1 Introduction

Suppose that one or more guards are stationed at some of the vertices of a simple n–vertex graph $G = (V, E)$ and that a guard at vertex v can deal with a problem at any vertex in its closed neighbourhood.

Let $f : V \to \{0, 1, 2, \ldots\}$ where $f(v)$ is the number of guards at v and for $i = 0, 1, 2, \ldots$, $V_i = \{v \in V \mid f(v) = i\}$. Imprecisely, we will identify f with the partition of V induced by f and write $f = (V_0, V_1, V_2, \ldots)$. The **weight** of f, $w(f) = \sum_{v \in V} f(v) = \sum_{i \geq 1} i\,|V_i|$ is the total number of guards

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used by f. We say that f is a **safe** function (more intuitively G is safe under f) if there is at least one guard available to handle a problem at any vertex. Formally $f = (V_0, V_1, V_2, \ldots)$ is safe if each $v \in V_0$ is adjacent to at least one vertex of $V - V_0$ (equivalently if $V - V_0$ is a dominating set of G).

We now define four subclasses of safe functions. The functions in each subclass protect the entire graph according to a certain strategy. In each case the parameter of interest will be the minimum weight of (*i.e.*, the minimum number of guards used by) a function in the subclass.

(i) Domination

A **dominating function** of G is a safe function $f = (V_0, V_1)$, *i.e.* a safe function with at most one guard per vertex. Clearly f is a dominating function if and only if V_1 is a dominating set of G and the minimum number of guards is $\gamma(G)$, the domination number of G. This method of protection has, of course, been studied extensively (see [2]).

(ii) Roman domination

A **Roman dominating function** (RDF) of G is a safe function $f = (V_0, V_1, V_2)$ such that each $v \in V_0$ is adjacent to at least one vertex of V_2. The minimum number of guards used in any RDF is called the **Roman domination number** of G and is denoted by $\gamma_R(G)$. This method of protection has historical motivation [4] and is studied in [1].

(iii) Weak Roman domination

Suppose that guards are stationed according to the function f. When a guard at $v \in V - V_0$ moves along an edge to deal with a problem at $u \in V_0$, the new numbers of guards at vertices are given by the function g_{uv} (abbreviated to g wherever possible):

$$g(w) = \begin{cases} f(w) - 1, & \text{if } w = v \\ 1, & \text{if } w = u \\ f(w), & \text{otherwise.} \end{cases}$$

A **weak Roman dominating function** (WRDF) of G is a safe function $f = (V_0, V_1, V_2)$ such that, for each $u \in V_0$, there exists v such that

$$v \in N(u) \cap (V - V_0) \quad \text{and} \quad g_{uv} \text{ is safe.} \tag{1}$$

Informally, for every vertex u without a guard, there is a guard at an adjacent vertex v who can move to u and the new assignment of guards is still safe.

For $u \in V_0$ and $v \in V - V_0$ satisfying (1), we say that $\boldsymbol{u}$ **is defended by** $\boldsymbol{v}$ (or $\boldsymbol{v}$ **defends** $\boldsymbol{u}$). Observe that $u \in V_0$ is defended by any $v \in N(u) \cap V_2$.

The minimum number of guards in any WRDF is called the **weak Roman domination number** of G and is denoted by $\gamma_r(G)$. This concept of protection was introduced and studied in [3] because (trivially) for any G, $\gamma_r(G) \leq \gamma_R(G)$ and the inequality is strict for many graphs.

(iv) Secure domination

In Section 4 it will be seen that the possibility of two guards at vertices in WRDFs does not always reduce the total number of guards required for protection. Hence we introduce the following concept.

A **secure dominating function** is a safe function $f = (V_0, V_1)$ such that, for each $u \in V_0$, there exists v such that (1) is satisfied. We have used functions f in all of the definitions to show that each of the strategies is a particular instance of a more general protection model. However (as was the case with the first strategy) it is more convenient to define the new concept by the properties of V_1. Observe that for $f = (V_0, V_1)$, statement (1) is equivalent to $v \in N(u) \cap V_1$ and $(V_1 - \{v\}) \cup \{u\}$ is dominating. Hence we make the following definition.

The set V_1 is a **secure dominating set** (**SDS**) if, for each $u \in V - V_1$, there exists v such that

$$v \in N(u) \cap V_1 \quad \text{and} \quad (V_1 - \{v\}) \cup \{u\} \text{ is dominating.} \tag{2}$$

Thus $f = (V_0, V_1)$ is a secure dominating function if and only if V_1 is an SDS. We say that u is $\mathbf{V_1}$**–defended** by v (or v $\mathbf{V_1}$**–defends** u) if (2) is satisfied. In almost all places in the paper where this definition is used, the set V_1 is clear from the context and we will omit the prefix, *i.e.* we will abbreviate "V_1–defended" to "defended." Such abbreviations cannot be made in the proof of Theorem 4. Confusion with "defended" as defined in connection with WRDFs will also be avoided by the context.

The minimum cardinality of an SDS (= the minimum weight of a secure dominating function) is called the **secure domination number** of G and denoted by $\gamma_s(G)$.

Wherever possible we will use abbreviations γ for $\gamma(G)$, γ_r for $\gamma_r(G)$, Δ for $\Delta(G)$, *etc.* The four protection parameters γ, γ_R, γ_r and γ_s are related as follows:

Proposition 1 *The following inequalities hold for any G:*

$$\gamma \leq \gamma_r \begin{array}{l} \leq \gamma_R \leq 2\gamma \\ \leq \gamma_s \end{array}$$

Proof. The inequality involving γ_s is obvious and the remaining ones were observed in [1] and [3]. ■

Some properties of SDSs are obtained in Section 2. These are followed by lower bounds for protection parameters which involve order and maximum degree (Section 3). Lastly we consider some well–known classes of graphs in Section 4.

2 Secure dominating sets

In this section we establish some properties of SDSs. Further definitions are required. For $v \in X \subseteq V$, $w \in V - X$ is an $\boldsymbol{X}$**–external private neighbour** of v (abbreviated X–epn of v) if $N(w) \cap X = \{v\}$. Let $P(v, X)$ be the set of all X-epns of v.

Proposition 2 *Let X be a dominating set. Vertex $v \in X$ defends $u \in V - X$ if and only if $G[P(v, X) \cup \{u, v\}]$ is complete.*

Proof. Suppose v defends u. Then v is adjacent to each vertex of $P(v, X) \cup \{u\}$. Let $w \in P(v, X) - \{u\}$. Since $(X - \{v\}) \cup \{u\}$ is a dominating set and $X - \{v\}$ does not dominate w, we deduce that $uw \in E$. Finally, let $\{a, b\} \subseteq P(v, X) - \{u\}$. Then a is (uniquely) defended by v and $(X - \{v\}) \cup \{a\}$ is dominating. But $X - \{v\}$ does not dominate b and so $ab \in E$.

The converse is immediate since completeness implies that $(X - \{v\}) \cup \{u\}$ is a dominating set, *i.e.* u is defended by v as required. ■

Corollary 3 *X is an SDS if and only if for each $u \in V - X$, there exists $v \in X$ such that $G[P(v, X) \cup \{u, v\}]$ is complete.*

Proof. X is an SDS if and only if each $u \in V - X$ is defended and the result follows from Proposition 2. ■

The next result characterises those SDSs which are minimal. We need two further definitions. If X is dominating, let $S = \{v \in X \mid X - \{v\}$ is dominating$\}$ and for $u \in V - X$, $A(u, X) = \{v \in X \mid v\ X\text{–defends } u\}$.

Theorem 4 *An SDS X is minimal if and only if, for each $s \in S$ with $N(s) \cap S \neq \emptyset$, there exists $u_s \in V - X$ such that, for each $v \in A(u_s, X) - \{s\}$, either*

(i) there exists $w \in V - X$ such that $N(w) \cap X = \{v, s\}$ and $u_s \notin N(w)$, or

(ii) $N(s) \cap X = \{v\}$ *and* $u_s \in N(v) - N(s)$.

Proof. Let X be a minimal SDS and consider any $s \in S$ with $N(s) \cap S \neq \emptyset$. By definition of S and minimality, $X - \{s\}$ is dominating but not an SDS.

$$\left.\begin{array}{c}\text{Hence there exists } u_s \in V - (X - \{s\}) \text{ such that for} \\ \text{each } y \in (X - \{s\}),\ u_s \text{ is not } (X - \{s\})\text{–defended by } y.\end{array}\right\} \qquad (3)$$

We show that $u_s \in V - X$. Suppose, to the contrary, that $u_s = s$ and let $z \in N(s) \cap S$. By (3) s is not $(X - \{s\})$–defended by z and so, by Proposition 2, z has an $(X - \{s\})$–epn w such that $s \notin N[w]$. But w is an X–epn of z, which contradicts the fact that $z \in S$. We conclude that $u_s \in V - X$.

Now consider any $v \in A(u_s, X) - \{s\}$. Then u_s is X–defended by v but, by (3), u_s is not $(X - \{s\})$–defended by v. Therefore, by Proposition 2, there exists $w \in V - (X - \{s\})$ such that

$$w \notin N(u_s), \quad w \in P(v, X - \{s\}), \quad w \notin P(v, X). \qquad (4)$$

It now follows from (4) that (i) holds if $w \neq s$ and (ii) holds if $w = s$.

To prove the converse, let X be an SDS satisfying the condition and $s \in X$. We must show that $X - \{s\}$ is not an SDS. There are three cases:

Case 1: $s \in X - S$.
Then $X - \{s\}$ is not dominating and hence is not an SDS.

Case 2: $s \in S$ with $N(s) \cap S = \emptyset$.
For each $v \in X \cap N(s)$, $v \in X - S$ and so v has an X–epn. Hence s is not $(X - \{s\})$–defended by v and therefore $X - \{s\}$ is not an SDS.

Case 3: $s \in S$ with $N(s) \cap S \neq \emptyset$.
In this case there exists $u_s \in V - X$ such that for each $v \in A(u_s, X) - \{s\}$ either (i) or (ii) holds. We show, with the following three subcases, that no $v \in X - \{s\}$ $(X - \{s\})$–defends u_s.

Subcase (a): $v \in A(u_s, X) - \{s\}$ and (i) holds.
Then w is an $(X - \{s\})$–epn of v and w is not adjacent to u_s. By Proposition 2, v does not $(X - \{s\})$–defend u_s.

Subcase (b): $v \in A(u_s, X) - \{s\}$ and (ii) holds.
Then s is an $(X - \{s\})$–epn of v and $s \notin N(u_s)$. Again, by Proposition 2, v does not $(X - \{s\})$–defend u_s.

Subcase (c): $v \in [N(u_s) \cap (X - \{s\})] - A(u_s, X)$.
Since $v \notin A(u_s, X)$, by Proposition 2, v has an X–epn w such that $w \notin N[u_s]$. But w is also an $(X - \{s\})$–epn of v and hence, by Proposition 2, v does not $(X - \{s\})$–defend u_s.

These cases show that, for all $s \in X$, $X - \{s\}$ is not an SDS, *i.e.* X is a minimal SDS, as required. ■

The following result is a direct consequence of Theorem 4.

Corollary 5 *If an SDS X has the property that $G[S]$ has no edges, then X is a minimal SDS.* ■

3 Lower bounds involving maximum degree

Proposition 6 *For any G with maximum degree $\Delta \geq 1$, $\gamma_R \geq 2n/(\Delta+1)$.*

Proof. Let $f = (V_0, V_1, V_2)$ be an RDF of G with weight γ_R. Since each $v \in V_0$ is adjacent to a vertex of V_2, we observe that

$$|V_0| \leq \Delta |V_2|. \tag{5}$$

Therefore

$$\begin{aligned}(\Delta+1)\gamma_R &= (\Delta+1)|V_1| + (\Delta+1)2|V_2| \\ &\geq (\Delta+1)|V_1| + 2|V_2| + 2|V_0| \quad \text{(by (5))} \\ &\geq 2|V_1| + 2|V_2| + 2|V_0| \\ &= 2n. \quad ■\end{aligned}$$

The bound of Proposition 6 is attained for all stars $K_{1,p}$.

The next two results bound γ_s for K_3–free and K_4–free graphs in terms of order and maximum degree.

Theorem 7 *If G is an order n triangle–free graph, and has maximum degree Δ, then*

$$\gamma_s \geq n(2\Delta - 1)/(\Delta^2 + 2\Delta - 1).$$

For each Δ, the bound is attained for infinitely many n.

Proof. Let X be an SDS. If $u \in P(v, X)$, then u is uniquely defended by v. Since G is triangle–free, Proposition 2 implies that $|P(v, X)| < 2$. Let X_i $(i = 0, 1)$ be the set of vertices of X which have i X–epns and $|X_i| = x_i$. Further suppose that

$$C = \{u \in V - X \mid |N(u) \cap X| \geq 2\} \text{ and } |C| = c.$$

Then

$$V = X_0 \cup X_1 \cup \left(\bigcup_{v \in X} P(v, X) \right) \cup C \quad \text{(disjoint union)}.$$

Therefore

$$n = x_0 + x_1 + x_1 + c,$$

i.e.,

$$c = n - x_0 - 2x_1. \tag{6}$$

By counting $e(C, X)$, the number of edges from C to X, we obtain

$$2c \le e(C, X) \le (\Delta - 1)x_1 + \Delta x_0.$$

We use (6) to eliminate c from this inequality and get

$$(\Delta + 1)x_0 + (\Delta + 3)x_1 \ge 2n. \tag{7}$$

No $u \in C$ is defended by $v \in X_1$ (otherwise Proposition 2 implies the existence of a K_3). Therefore each $u \in C$ is adjacent to X_0. Hence $\Delta x_0 \ge c$ and, from (6),

$$(\Delta + 1)x_0 + 2x_1 \ge n. \tag{8}$$

The minimum value of $x_0 + x_1$ subject to the constraints (7), (8) and $x_0, x_1 \ge 0$ is

$$(2\Delta - 1)n/(\Delta^2 + 2\Delta - 1)$$

as required. It is achieved by

$$x_0 = \frac{n(\Delta - 1)}{\Delta^2 + 2\Delta - 1}, \quad x_1 = \frac{n\Delta}{\Delta^2 + 2\Delta - 1}.$$

Let $\Delta \ge 1$ be fixed and $n = k(\Delta^2 + 2\Delta - 1)$, where $k \in \mathbb{Z}^+$. Let the graph G_k have vertex set $X_0 \cup X_1 \cup B \cup C$ (a disjoint union), where

$$|X_1| = |B| = \Delta k, \quad |X_0| = (\Delta - 1)k \ \text{ and } \ |C| = \Delta(\Delta - 1)k.$$

Add the edges of a matching from X_1 to B and join each $u \in C$ to one vertex in each of X_0 and X_1, so that each vertex of $X_0 \cup X_1$ has degree Δ. Then $X_0 \cup X_1$ is an SDS of G_k and the bound is attained. ■

Theorem 8 *If G is an order n, K_4-free graph, and has maximum degree Δ, then*

$$\gamma_s \ge n(2\Delta - 3)/(\Delta^2 + 2\Delta - 5).$$

For each $\Delta \ge 2$ the bound is attained for infinitely many n.

Proof. Let X be an SDS. By hypothesis and Proposition 2, for each $v \in X$, $|P(v, X)| \le 2$. Define C and X_i, x_i for $i = 0, 1, 2$ as in the proof of Theorem 7. Then

$$V = X_0 \cup X_1 \cup X_2 \cup \left(\bigcup_{v \in X} P(v, X) \right) \cup C \quad \text{(disjoint union)}.$$

Therefore
$$n = x_0 + x_1 + x_2 + (x_1 + 2x_2) + c,$$
i.e.,
$$c = n - x_0 - 2x_1 - 3x_2. \tag{9}$$
The number $e(C, X)$ of edges between C and X satisfies
$$2c \leq e(C, X) \leq (\Delta - 2)x_2 + (\Delta - 1)x_1 + \Delta x_0.$$
Using (9) we obtain
$$(\Delta + 2)x_0 + (\Delta + 3)x_1 + (\Delta + 4)x_2 \geq 2n. \tag{10}$$
No $u \in C$ is defended by $v \in X_2$ (otherwise there is a K_4 by Proposition 2). Hence each $c \in C$ is adjacent to a vertex of $X_0 \cup X_1$. We deduce that
$$(\Delta - 1)x_1 + \Delta x_0 \geq c.$$
Using this and (9) we conclude that
$$(\Delta + 1)x_0 + (\Delta + 1)x_1 + 3x_2 \geq n. \tag{11}$$
It is therefore necessary to solve the linear program: minimize $(x_0 + x_1 + x_2)$ subject to (10), (11) and $x_0, x_1, x_2 \geq 0$.

The vertices of the feasible region are among the $\binom{5}{3}$ points of intersection of three of the five planes obtained with equality in the constraints. Thus the solution is elementary (if a little tedious) and we omit the details.

The minimum is $(2\Delta - 3)n/(\Delta^2 + 2\Delta - 5)$, as required, and is attained when
$$x_0 = 0, \quad x_1 = \frac{(\Delta - 2)n}{\Delta^2 + 2\Delta - 5}, \quad x_2 = \frac{(2\Delta - 3)n}{\Delta^2 + 2\Delta - 5}.$$
Fix $\Delta \geq 2$ and let $n = k(\Delta^2 + 2\Delta - 5)$, where $k \in \mathbb{Z}^+$. Let the graph G_k have vertex set
$$X_1 \cup X_2 \cup Y_1 \cup Y_2 \cup C \quad \text{(disjoint union)},$$
where
$$\begin{aligned}
|X_1| &= |Y_1| = (\Delta - 2)k, \\
|X_2| &= (\Delta - 1)k, \\
|Y_2| &= 2(\Delta - 1)k \text{ and} \\
|C| &= (\Delta - 1)(\Delta - 2)k.
\end{aligned}$$
Add edges so that:

(i) there is a matching from X_1 to Y_1;

(ii) $G_k[X_2 \cup Y_2] = (\Delta - 1)kK_3$, where each K_3 contains one vertex of X_2 and two of Y_2;

(iii) each $u \in C$ joins one vertex of each of X_1 and X_2 so that the degree of each vertex of $X_1 \cup X_2$ is Δ.

Then $X_1 \cup X_2$ is an SDS of G_k and the bound is attained. ■

4 Special graphs

In this section we give some precise values and bounds for protection parameters in frequently encountered classes of graphs. Known results are included for comparison. The first result requires no proof.

Proposition 9 *For the complete graph K_n, $\gamma = \gamma_r = \gamma_s = 1$ and $\gamma_R = 2$.* ■

Proposition 10 *For the complete bipartite graph $K_{p,q}$ where $p \leq q$:*

(a) $$\gamma = \begin{cases} 1, & p = 1 \\ 2, & p > 1. \end{cases}$$

(b) [2] $$\gamma_R = \begin{cases} 2, & p = 1 \\ 3, & p = 2 \\ 4, & p \geq 3. \end{cases}$$

(c) $$\gamma_r = \begin{cases} 2, & p = 1, 2, \quad q > 1 \\ 3, & p = 3 \\ 4, & p \geq 4. \end{cases}$$

(d) $$\gamma_s = \begin{cases} q, & p = 1 \\ 2, & p = 2 \\ 3, & p = 3 \\ 4, & p \geq 4. \end{cases}$$

Proof. Parts (a) and (b) are known results. Suppose that $P = \{u_1, \ldots, u_p\}$ and $Q = \{v_1, \ldots, v_q\}$ are the defining partite sets of $K_{p,q}$.

Case 1: $p = 1$
If $\gamma_r = 1$, the single guard must be at u_1 in order to dominate. However in that case no v_i is defended and so $\gamma_r \geq 2$. Deployment of two guards at u_1 shows that $\gamma_r \leq 2$, so that $\gamma_r = 2$. Suppose that X is an SDS of $K_{1,q}$. If $u_1 \notin X$, then since X dominates, $|X| \geq q$. If $u_1 \in X$, then no two vertices v_i, v_j are in $V - X$, for otherwise neither is defended. Again $|X| \geq q$. Since Q is an SDS, $\gamma_s = q$.

Case 2: $p = 2$
Since no single vertex dominates, $\gamma_r > 1$. The set P is an SDS and so (using Proposition 1) $\gamma_r = \gamma_s = 2$.

Case 3: $p = 3$
It is easy to check that $\gamma_r > 2$. The set P is an SDS and so $\gamma_r = \gamma_s = 3$.

Case 4: $p > 3$
Suppose that f is a WRDF with at most three guards. If all guards are in P (or all in Q), then f is not safe; a contradiction. Hence one of the sets (say P) has exactly one guard and some vertex of Q is not defended; also contradictory. Hence $\gamma_r \geq 4$. The set $\{u_1, u_2, v_1, v_2\}$ is an SDS and we deduce that $\gamma_r = \gamma_s = 4$. ■

Similar simple arguments establish the following results for complete multipartite graphs. We omit the proofs.

Proposition 11 *For the graph $K_{p_1,p_2,\ldots,p_t}$ where $p_1 \leq p_2 \leq \cdots \leq p_t$ and $t \geq 3$:*

(a) $\gamma = \begin{cases} 1, & p_1 = 1 \\ 2, & p_1 \geq 2. \end{cases}$ (b) $\gamma_R = \begin{cases} 2, & p_1 = 1 \\ 3, & p_1 = 2 \\ 4, & p_1 \geq 3. \end{cases}$

(c) $\gamma_r = \begin{cases} 2, & p_1 = 1,2 \\ 3, & p_1 \geq 3. \end{cases}$ (d) $\gamma_s = \begin{cases} 2, & p_1 = 1,\ p_2 \leq 2 \\ 2, & p_1 = 2 \\ 3, & \text{otherwise.} \end{cases}$ ■

We now show that paths and cycles are examples of graphs for which $\gamma_r = \gamma_s$.

Theorem 12 *For the graphs P_n and C_n:*

(a) $\gamma = \left\lceil \frac{n}{3} \right\rceil$. (b) [2] $\gamma_R = \left\lceil \frac{2n}{3} \right\rceil$.

(c) [3] $\gamma_r = \left\lceil \frac{3n}{7} \right\rceil$. (d) $\gamma_s = \left\lceil \frac{3n}{7} \right\rceil$.

Proof. Parts (a), (b) and (c) are known results. By (c) and Proposition 1, we have $\gamma_s(P_n)$ and $\gamma_s(C_n)$ are at least $\lceil 3n/7 \rceil$. Further, if H' is a spanning subgraph of any graph H, then $\gamma_s(H') \geq \gamma_s(H)$. We deduce that $\gamma_s(P_n) \geq \gamma_s(C_n)$. Therefore, in order to complete the proof, we need only exhibit an SDS X of P_n of size $\lceil 3n/7 \rceil$. Let $n = 7k + r$, where $k \geq 1$ and $0 \leq r \leq 6$. Further, let the vertex sequence of P_n be $(u_1, \ldots, u_n)$. Define

$$Y = \bigcup_{j=0}^{k-1} \{u_{7j+2}, u_{7j+4}, u_{7j+6}\}$$

and

$$Z = \begin{cases} \emptyset, & \text{if } r = 0 \\ \{u_{7k+1}\}, & \text{if } r = 1, 2 \\ \{u_{7k+1}, u_{7k+3}\}, & \text{if } r = 3, 4 \\ \{u_{7k+1}, u_{7k+3}, u_{7k+5}\}, & \text{if } r = 5, 6. \end{cases}$$

Then $X = Y \cup Z$ is an SDS of P_n of size $\lceil 3n/7 \rceil$. ■

The last few results establish upper bounds for protection parameters of products of paths, cycles and complete graphs.

Theorem 13 $\gamma_r(P_m \times P_k) \le \gamma_s(P_m \times P_k) \le \left\lceil \frac{mk}{3} \right\rceil + 2.$

Proof. We construct an SDS of the grid graph $G = P_m \times P_k$ of size at most $\lceil mk/3 \rceil + 2$. Consider the standard plane embedding depicted in Figure 1(a). Place one guard at each vertex of the set V_1 of solid black vertices and let $V_0 = V - V_1$. Observe that $|V_1| = \lceil mk/3 \rceil$.

There are a variety of situations which depend on the residues of m and k modulo 3. For each pair of values of these residues, each $u \in V_0$ which is not in the closed neighbourhood of the bottom left or top right corners, is securely dominated by V_1, *i.e.* is dominated by V_1 and defended by some $v \in V_1$.

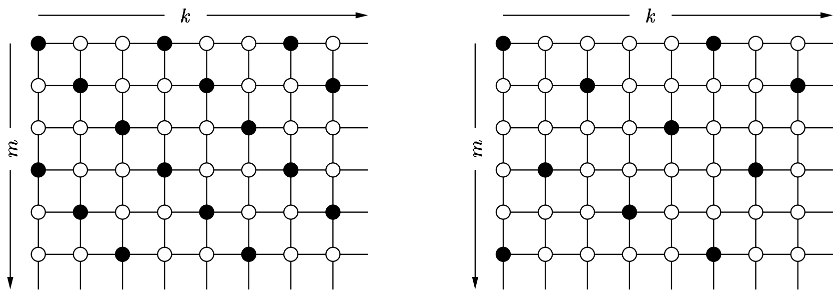
The three possible situations for the top right corner are shown in Figure 2. In each case add the single vertex depicted by the square, to V_1. Then, in each of the three cases, the corner is securely dominated by V_1. The situation in the bottom left corner is identical. Hence, in all cases we have exhibited an SDS of size at most $\lceil mk/3 \rceil + 2$, as required. ■

Corollary 14 $\gamma_r(C_m \times C_k) \le \gamma_s(C_m \times C_k) \le \left\lceil \frac{mk}{3} \right\rceil$. ■

The corners (*i.e.*, degree two vertices) which forced the addition of extra vertices to the black vertices of Figure 1(a) in the proof of Theorem 13, are not present in $C_m \times C_k$. Hence the solid black vertices form an SDS of size $\lceil mk/3 \rceil$, as required. The graph $C_m \times C_k$ has maximum degree 4 and is K_3–free. Hence Theorem 7 gives:

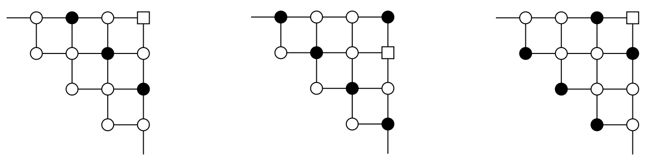
Corollary 15 $\gamma_s(C_m \times C_k) \ge \frac{7mk}{23}$. ■

Theorem 16 $\gamma_R(P_m \times P_k) \le 2\left(\left\lceil \frac{mk}{5} \right\rceil + \left\lceil \frac{m}{5} \right\rceil + \left\lceil \frac{k}{5} \right\rceil\right).$



(a) Secure domination (b) Roman domination

Figure 1: Partial construction of SDS and RDF for grids.



(a) $k \equiv 0$ (b) $k \equiv 1$ (c) $k \equiv 2$

Figure 2: Possible top right corners for SDS construction of grids.

Proof. We exhibit an RDF $f = (V_0, V_1, V_2)$ of the grid graph $P_m \times P_k$. Firstly, place two guards at each vertex of the set V_2 of black vertices in Figure 1(b). It is routine to show that $|V_2| \le \lceil mk/5 \rceil$. Each vertex of $V - V_2$ which is not on the boundary, is adjacent to a vertex of V_2. However, there are boundary vertices which are not in $N[V_2]$. Therefore, in order to complete the RDF, some boundary vertices will be assigned one guard (*i.e.*, will be placed in V_1).

Let X be the set of vertices on the bottom row, the vertex sequence of which is $(u_1, \ldots, u_n)$. Place $u_j \in X_1 = V_1 \cap X$ if $j = i + 2$ or $j = i - 3$, where $u_i \in V_2$. Then

$$\text{for all} \quad u \in X, \quad u \in N[V_2] \cup V_1. \tag{12}$$

Note that $|X_1| \le \lceil k/5 \rceil$. Similar placements of single guards (*i.e.*, vertices in V_1) at subsets X_2, X_3, X_4 (where $|X_2| \le \lceil k/5 \rceil$, $|X_3|, |X_4| \le \lceil m/5 \rceil$) of the top, left and right boundaries respectively, ensure that these three boundary sets also have property (12).

Hence, if $V_1 = X_1 \cup X_2 \cup X_3 \cup X_4$ and $V_0 = V - (V_1 \cup V_2)$, then $f = (V_0, V_1, V_2)$ is an RDF. Therefore

$$\gamma_R \le |V_1| + 2|V_2| \le 2\left(\left\lceil \frac{mk}{5} \right\rceil + \left\lceil \frac{m}{5} \right\rceil + \left\lceil \frac{k}{5} \right\rceil\right),$$

as required. ■

Corollary 17

$$\left\lceil \frac{2mk}{5} \right\rceil \le \gamma_R(C_m \times C_k) \le \gamma_R(P_m \times P_k) \le 2\left(\left\lceil \frac{mk}{5} \right\rceil + \left\lceil \frac{m}{5} \right\rceil + \left\lceil \frac{k}{5} \right\rceil\right).$$

Proof. The central inequality is true since $P_m \times P_k$ is a spanning subgraph of $C_m \times C_n$. The left most and right most relations follow from Proposition 6 and Theorem 16 respectively. ■

Theorem 18 *If* $2 \le m \le k$, *then*

$$\gamma_R(K_m \times K_k) = \begin{cases} 2m - 1 & \text{if } m = k \\ 2m & \text{if } m < k. \end{cases}$$

Proof. First observe that, since $\gamma_R \le 2\gamma = 2m$ (Proposition 1), the number of vertices with two guards is at most m. For $c \in \{1, \ldots, m\}$ let $\eta(c)$ be the minimum weight of a function in the class of RDFs $f = (V_0, V_1, V_2)$, where $|V_2| = c$. In this class $w(f) = 2c + |V_1|$, so that $\eta(c)$ is attained with minimum $|V_1|$, *i.e.* with maximum $|V_0|$. By the RDF property $V_0 \subseteq N(V_2) - V_2$, so that

$$|V_0| \le |N(V_2)| - |N(V_2) \cap V_2|.$$

Since $c \le m$, $|N(V_2)|$ is maximised and $|N(V_2) \cap V_2|$ has its minimum value, namely zero, when V_2 is independent, which implies that $|V_0| + |V_2| = |N(V_2)| + |V_2| = N[V_2]$.

To evaluate $N[V_2]$, observe that augmenting an $(i-1)$–independent set of $K_m \times K_k$ to an i–independent set increases the size of the closed neighbourhood by $(m + k - 1) - 2(i - 1) = m + k - 2i + 1$. Hence, to attain $\eta(c)$,

$$|V_0| + |V_2| = N[V_2] = \sum_{i=1}^{c} (m + k - 2i + 1) = c(m + k) - c^2$$

and hence $|V_1| = mk - c(m+k) + c^2$. Therefore

$$\begin{aligned}
\gamma_R &= \min_{1\le c\le m} \eta(c) \\
&= \min_{1\le c\le m} (2|V_2| + |V_1|) \\
&= \min_{1\le c\le m} (c^2 - c(m+k-2) + mk)
\end{aligned}$$

and the result follows by elementary calculus. ■

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