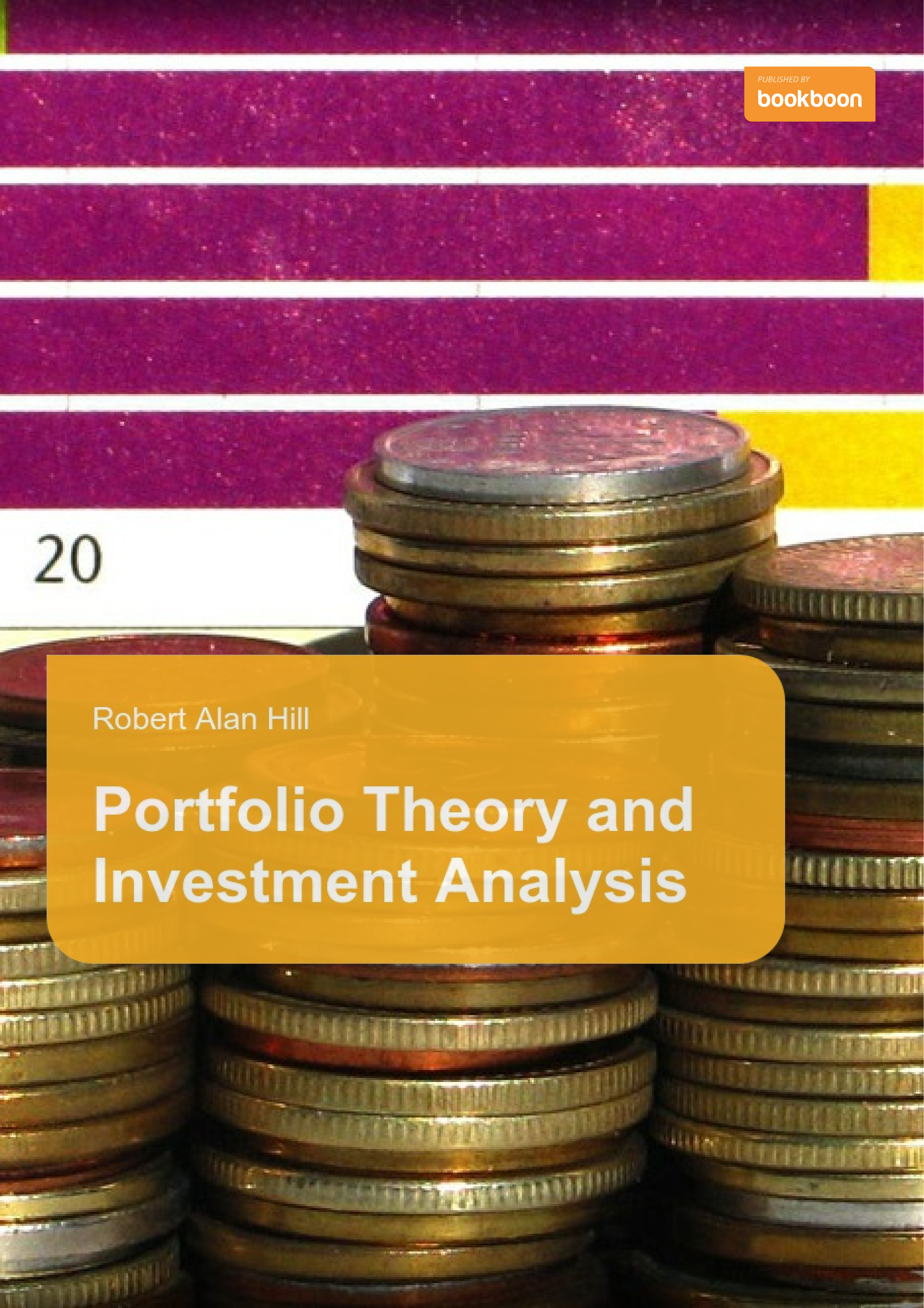


PUBLISHED BY bookboon

Robert Alan Hill

Portfolio Theory and Investment Analysis

Robert Alan Hill

Portfolio Theory and Investment Analysis

Portfolio Theory and Investment Analysis
2nd edition
© 2014 Robert Alan Hill & bookboon.com
ISBN 978-87-403-0606-4

Contents


ANYTIME, ANYWHERE
LEARNING ABOUT SAP SOFTWARE HAS NEVER BEEN EASIER.
SAP Learning Hub – the choice of when, where, and what to learn
SAP Learning Hub
SAP


NO-LIMITS LEARNING
LEVERAGE SOCIAL LEARNING, COLLABORATION, QUALITY CONTENT, AND HANDS-ON PRACTICE.
SAP Learning Hub
SAP

1 An Overview

Introduction

Once a company issues shares (common stock) and receives the proceeds, it has no *direct* involvement with their subsequent transactions on the capital market, or the price at which they are traded. These are matters for negotiation between existing shareholders and prospective investors, based on their own financial agenda.

As a basis for negotiation, however, the company plays a pivotal *agency* role through its implementation of investment-financing strategies designed to maximise profits and shareholder wealth. What management do to satisfy these objectives and how the market reacts are ultimately determined by the law of supply and demand. If corporate returns exceed market expectations, share price should rise (and vice versa).

But in a world where ownership is divorced from control, characterised by economic and geo-political events that are also beyond management's control, this invites a question.

> How do companies determine an optimum portfolio of investment strategies that satisfy a multiplicity of shareholders with different wealth aspirations, who may also hold their own diverse portfolio of investments?

1.1 The Development of Finance

As long ago as 1930, Irving Fisher's *Separation Theorem* provided corporate management with a lifeline based on what is now termed Agency Theory.

He acknowledged implicitly that whenever ownership is divorced from control, direct communication between management (*agents*) and shareholders (*principals*) let alone other stakeholders, concerning the likely profitability and risk of every corporate investment and financing decision is obviously impractical. If management were to implement optimum strategies that satisfy each shareholder, the company would also require prior knowledge of every investor's stock of wealth, dividend preferences and risk-return responses to their strategies.

According to Fisher, what management therefore, require is a model of *aggregate* shareholder behaviour. A theoretical abstraction of the real world based on simplifying assumptions, which provides them with a methodology to communicate a diversity of corporate wealth maximising decisions.

To set the scene, he therefore assumed (not unreasonably) that all investor behaviour (including that of management) is *rational* and *risk averse*. They prefer high returns to low returns but less risk to more risk. However, risk aversion does not imply that rational investors will not take a chance, or prevent companies from retaining earnings to gamble on their behalf. To accept a higher risk they simply require a commensurately higher return, which Fisher then benchmarked.

> Management's minimum rate of return on incremental projects financed by retained earnings should equal the return that existing shareholders, or prospective investors, can earn on investments of equivalent risk elsewhere.

He also acknowledged that a company's acceptance of projects internally financed by retentions, rather than the capital market, also denies shareholders the opportunity to benefit from current dividend payments. Without these, individuals may be forced to sell part (or all) of their shareholding, or alternatively borrow at the market rate of interest to finance their own preferences for consumption (income) or investment elsewhere.

To circumvent these problems Fisher assumed that if capital markets are *perfect* with no barriers to trade and a free flow of information (more of which later) a firm's *investment* decisions can not only be *independent* of its shareholders' *financial* decisions but can also satisfy their wealth maximisation criteria.

In Fisher's perfect world:

- Wealth maximising firms should determine optimum *investment* decisions by *financing* projects based on their *opportunity* cost of capital.
- The opportunity *cost* equals the *return* that existing shareholders, or prospective investors, can earn on investments of equivalent risk elsewhere.
- Corporate projects that earn rates of return less than the opportunity cost of capital should be rejected by management. Those that yield equal or superior returns should be accepted.
- Corporate earnings should therefore be distributed to shareholders as dividends, or retained to fund new capital investment, depending on the relationship between project profitability and capital cost.
- In response to rational managerial dividend-retention policies, the final consumption-investment decisions of rational shareholders are then determined independently according to their personal preferences.
- In perfect markets, individual shareholders can always borrow (lend) money at the market rate of interest, or buy (sell) their holdings in order to transfer cash from one period to another, or one firm to another, to satisfy their income needs or to optimise their stock of wealth.

Activity 1

Based on Fisher's Separation Theorem, share price should rise, fall, or remain stable depending on the inter-relationship between a company's project returns and the shareholders desired rate of return. Why is this?

For detailed background to this question and the characteristics of perfect markets you might care to download "Strategic Financial Management" (both the text and exercises) from bookboon.com and look through their first chapters.

1.2 Efficient Capital Markets

According to Fisher, in perfect capital markets where ownership is divorced from control, the separation of corporate dividend-retention decisions and shareholder consumption-investment decisions is not problematical. If management select projects using the shareholders' desired rate of return as a cut-off rate for investment, then at worst corporate wealth should stay the same. And once this information is communicated to the outside world, share price should not fall.

Of course, the Separation Theorem is an abstraction of the real world; a model with questionable assumptions. Investors do not always behave rationally (some speculate) and capital markets are not perfect. Barriers to trade do exist, information is not always freely available and not everybody can borrow or lend at the same rate. But instead of asking whether these assumptions are divorced from reality, the relevant question is whether the model provides a sturdy framework upon which to build.

Certainly, theorists and analysts believed that it did, if Fisher's impact on the subsequent development of finance theory and its applications are considered. So much so, that despite the recent global financial meltdown (or more importantly, because the events which caused it became public knowledge) it is still a basic tenet of finance taught by Business Schools and promoted by other vested interests world-wide (including governments, financial institutions, corporate spin doctors, the press, media and financial web-sites) that:

Capital markets may not be *perfect* but are still reasonably *efficient* with regard to how analysts *process* information concerning corporate activity and how this changes market values once it is conveyed to investors.

An efficient market is one where:

- Information is universally available to all investors at a low cost.
- Current security prices (debt as well as equity) reflect all relevant information.
- Security prices only change when new information becomes available.

Based on the pioneering research of Eugene Fama (1965) which he formalised as the "efficient market hypothesis" (EMH) it is also widely agreed that *information processing efficiency* can take *three forms* based on *two types* of analyses.

The weak form states that *current* prices are determined solely by a *technical* analysis of *past* prices. Technical analysts (or *chartists*) study historical price movements looking for cyclical patterns or trends likely to repeat themselves. Their research ensures that significant movements in current prices relative to their history become widely and quickly known to investors as a basis for immediate trading decisions. Current prices will then move accordingly.

The semi-strong form postulates that current prices not only reflect price history, but all *public* information. And this is where *fundamental analysis* comes into play. Unlike chartists, *fundamentalists* study a company and its business based on historical records, plus its current and future performance (profitability, dividends, investment potential, managerial expertise and so on) relative to its competitive position, the state of the economy and global factors.

In weak-form markets, fundamentalists, who make investment decisions on the expectations of individual firms, should therefore be able to "out-guess" chartists and profit to the extent that such information is not assimilated into past prices (a phenomenon particularly applicable to companies whose financial securities are infrequently traded). However, if the semi-strong form is true, fundamentalists can no longer gain from their research.

The strong form declares that current prices fully reflect *all information*, which not only includes all publically available information but also *insider* knowledge. As a consequence, unless they are lucky, even the most privileged investors cannot profit in the long term from trading financial securities before their price changes. In the presence of strong form efficiency the market price of any financial security should represent its intrinsic (true) value based on anticipated returns and their degree of risk.

> So, as the EMH strengthens, speculative profit opportunities weaken. Competition among large numbers of increasingly well-informed market participants drives security prices to a consensus value, which reflects the best possible forecast of a company's uncertain future prospects.

Which strength of the EMH best describes the capital market and whether investors can ever "beat the market" need not concern us here. The point is that whatever levels of efficiency the market exhibits (weak, semi-strong or strong):

- Current prices reflect all the relevant information used by that market (price history, public data and insider information, respectively).
- Current prices only change when new information becomes available.

It follows, therefore that prices must follow a "random walk" to the extent that new information is *independent* of the last piece of information, which they have already absorbed.

- And it this phenomenon that has the most important consequences for how management model their strategic investment-financing decisions to maximise shareholder wealth

Activity 2

Before we continue, you might find it useful to review the Chapter so far and briefly summarise the main points.

1.3 The Role of Mean-Variance Efficiency

We began the Chapter with an idealised picture of investors (including management) who are rational and risk-averse and formally analyse one course of action in relation to another. What concerns them is not only profitability but also the likelihood of it arising; a *risk-return* trade-off with which they feel comfortable and that may also be unique.

Thus, in a sophisticated mixed market economy where ownership is divorced from control, it follows that the objective of strategic financial management should be to implement optimum investment-financing decisions using risk-adjusted wealth maximising criteria, which satisfy a multiplicity of shareholders (who may already hold a diverse portfolio of investments) by placing them all in an equal, optimum financial position.

No easy task!

But remember, we have not only assumed that investors are rational but that capital markets are also reasonably efficient at processing information. And this greatly simplifies matters for management. Because today's price is *independent* of yesterday's price, efficient markets have *no memory* and individual security price movements are *random*. Moreover, investors who comprise the market are so large in number that no one individual has a comparative advantage. In the short run, "you win some, you lose some" but long term, investment is a *fair game* for all, what is termed a "martingale". As a consequence, management can now afford to take a *linear* view of investor behaviour (as new information replaces old information) and model its own plans accordingly.

What rational market participants require from companies is a diversified investment portfolio that delivers a maximum return at minimum risk.

What management need to satisfy this objective are investment-financing strategies that maximise corporate wealth, validated by simple *linear* models that statistically quantify the market's risk-return *trade-off.*

Like Fisher's Separation Theorem, the concept of linearity offers management a lifeline because in *efficient* capital markets, rational investors (including management) can now assess anticipated investment returns (r_i) by reference to their probability of occurrence, (p_i) using classical statistical theory.

If the returns from investments are assumed to be *random*, it follows that their *expected return* (R) is the expected monetary value (EMV) of a symmetrical, *normal* distribution (the familiar "bell shaped curve" sketched overleaf). Risk is defined as the *variance* (or dispersion) of individual returns: the greater the variability, the greater the risk.

Unlike the mean, the statistical measure of dispersion used by the market or management to assess risk is partly a matter of convenience. The *variance* (VAR) or its square root, the *standard deviation* ($\sigma = \sqrt{VAR}$) is used.

When considering the *proportion* of risk due to some factor, the variance ($VAR = \sigma^2$) is sufficient. However, because the standard deviation (σ) of a normal distribution is measured in the same units as (R) the expected value (whereas the variance (σ^2) only summates the squared deviations around the mean) it is more convenient as an *absolute* measure of risk.

Moreover, the standard deviation (σ) possesses another attractive statistical property. Using confidence limits drawn from a Table of *z* statistics, it is possible to establish the *percentage probabilities* that a random variable lies within *one, two or three standard deviations above, below* or *around* its expected value, also illustrated below.

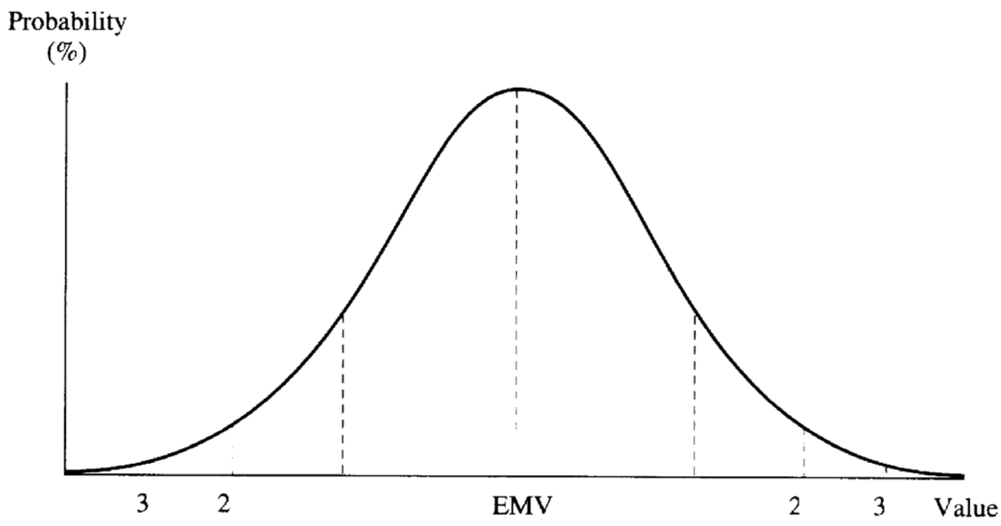


Figure 1.1: The Symmetrical Normal Distribution, Area under the Curve and Confidence Limits

Armed with this statistical information, investors and management can then accept or reject investments according to the degree of confidence they wish to attach to the likelihood (risk) of their desired returns. Using decision rules based upon their optimum criteria for *mean-variance efficiency,* this implies management and investors should pursue:

- Maximum expected return (R) for a given level of risk, (s).
- Minimum risk (s) for a given expected return (R).

Thus, our conclusion is that if modern capital market theory is based on the following three assumptions:

1) Rational investors,
2) Efficient markets,
3) Random walks.

The normative wealth maximisation objective of strategic financial management requires the optimum selection of a portfolio of investment projects, which maximises their expected return (R) commensurate with a degree of risk (σ) acceptable to existing shareholders and potential investors.

Activity 3

If you are not familiar with the application of classical statistical formulae to financial theory, read Chapter Four of "Strategic Financial Management" (both the text and exercises) downloadable from bookboon.com

Each chapter focuses upon the two essential characteristics of investment, namely expected return and risk. The calculation of their corresponding statistical parameters, the mean of a distribution and its standard deviation (the square root of the variance) applied to investor utility should then be familiar.

We can then apply simple mathematical notation: (r_i, p_i, R, VAR, σ and U) to develop a more complex series of ideas throughout the remainder of this text.

1.4 The Background to Modern Portfolio Theory

From our preceding discussion, rational investors in reasonably efficient markets can assess the likely profitability of *individual* corporate investments by a statistical weighting of their expected returns, based on a *normal* distribution (the familiar bell-shaped curve).

- Rational-risk averse investors expect either a *maximum* return for a *given* level of risk, or a *given* return for *minimum* risk.
- Risk is measured by the standard deviation of returns and the overall expected return is measured by its weighted probabilistic average.

Using mean-variance efficiency criteria, investors then have *three* options when managing a *portfolio* of investments depending on the performance of its individual components.

1) To trade (buy or sell),
2) To hold (do nothing),
3) To substitute (for example, shares for loan stock).

However, it is important to note that what any individual chooses to do with their portfolio constituents cannot be resolved by *statistical* analyses alone. Ultimately, their behaviour depends on how they interpret an investment's risk-return trade off, which is measured by their *utility curve*. This calibrates the individual's *current* perception of risk concerning uncertain *future* gains and losses. Theoretically, these curves are simple to calibrate, but less so in practice. Risk attitudes not only differ from one investor to another and may be unique but can also vary markedly over time. For the moment, suffice it to say that there is no *universally* correct decision to trade, hold, or substitute one constituent relative to another within a financial investment portfolio.

Review Activity

1. Having read the fourth chapters of the following series from bookboon.com recommended in Activity 3:

Strategic Financial Management (SFM),
Strategic Financial Management; Exercises (SFME).

- In *SFM*: pay particular attention to Section 4.5 onwards, which explains the relationship between *mean*-variance analyses, theconcept of *investor* utility and the application of *certainty equivalent* analysis to investment appraisal.
- In *SFME*: work through Exercise 4.1.

2. Next download the free companion text to this e-book:
Portfolio Theory and Financial Analyses; Exercises (PTFAE), 2010.

3. Finally, read Chapter One of *PTFAE*.

It will test your understanding so far. The exercises and solutions are presented logically as a guide to further study and are easy to follow. Throughout the remainder of the book, each chapter's exercises and equations also follow the same structure of this text. So throughout, you should be able to complement and reinforce your theoretical knowledge of modern portfolio theory (MPT) at *your own pace.*

Summary and Conclusions

Based on our Review Activity, there are two interrelated questions that we have not yet answered concerning any wealth maximising investor's risk-return trade off, irrespective of their behavioural attitude towards risk.

> What if investors don't want "to put all their eggs in one basket" and wish to diversify beyond a *single* asset portfolio?
>
> How do financial management, acting on their behalf, incorporate the *relative* risk-return trade-off between a *prospective* project and the firm's *existing* asset portfolio into a quantitative model that still maximises wealth?

To answer these questions, throughout the remainder of this text and its exercise book, we shall analyse the evolution of Modern Portfolio Theory (MPT).

Statistical calculations for the expected risk-return profile of a *two-asset* investment portfolio will be explained. Based upon the mean-variance efficiency criteria of Harry Markowitz (1952) we shall begin with:

- The risk-reducing effects of a diverse two-asset portfolio,
- The optimum two-asset portfolio that minimises risk, with individual returns that are perfectly (negatively) correlated.

We shall then extend our analysis to *multi-asset* portfolio optimisation, where John Tobin (1958) developed the *capital market line* (CML) to show how the introduction of risk-free investments define a "frontier" of efficient portfolios, which further reduces risk. We discover, however, that as the size of a portfolio's constituents increase, the mathematical calculation of the variance is soon dominated by covariance terms, which makes its computation unwieldy.

Fortunately, the problem is not insoluble as you will discover if you download "The Capital Asset Pricing Model" 2. edition, 2014, from bookboon.com when you complete this text. Ingenious, subsequent developments, such as the *specific* capital asset pricing model (CAPM) formulated by Sharpe (1963) Lintner (1965) and Mossin (1966), the option-pricing model of Black and Scholes (1973) and *general* arbitrage pricing theory (APT) developed by Ross (1976), all circumvent the statistical problems encountered by Markowitz.

By dividing *total* risk between *diversifiable* (unsystematic) risk and *undiversifiable* (systematic or market) risk, what is now termed Modern Portfolio Theory (MPT) explains how rational, risk averse investors and companies can price securities, or projects, as a basis for profitable portfolio trading and investment decisions. For example, a profitable trade is accomplished by buying (selling) an undervalued (overvalued) security relative to an appropriate stock market index of *systematic* risk (say the FT-SE All Share).This is measured by the *beta* factor of the individual security relative to the market portfolio. As we shall also discover it is possible for companies to define project betas for project appraisal that measure the systematic risk of specific projects.

So, there is much ground to cover. Meanwhile, you should find the diagram in the Appendix provides a useful road-map for your future studies.

Selected References

1. Jensen, M.C. and Meckling, W.H., "Theory of the Firm: Managerial Behaviour, Agency Costs and Ownership Structure", *Journal of Financial Economics*, 3, October 1976.
2. Fisher, I., *The Theory of Interest*, Macmillan (London), 1930.
3. Fama, E.F., "The Behaviour of Stock Market Prices", *Journal of Business*, Vol. 38, 1965.
4. Markowitz, H.M., "Portfolio Selection", *Journal of Finance*, Vol. 13, No. 1, 1952.
5. Tobin, J., "Liquidity Preferences as Behaviour Towards Risk", *Review of Economic Studies*, February 1958.
6. Sharpe, W., "A Simplified Model for Portfolio Analysis", *Management Science*, Vol. 9, No. 2, January 1963.
7. Lintner, J., "The valuation of risk assets and the selection of risk investments in stock portfolios and capital budgets", *Review of Economic Statistics*, Vol. 47, No. 1, December, 1965.
8. Mossin, J., "Equilibrium in a capital asset market", *Econometrica*, Vol. 34, 1966.
9. Hill, R.A., bookboon.com

    - *Strategic Financial Management (SFM)*, 2009.
    - *Strategic Financial Management: Exercises (SFME)*, 2009.
    - *Portfolio Theory and Financial Analyses (PTFA)*, 2010.
    - *Portfolio Theory and Financial Analyses: Exercises (PTFAE)*, 2010.
    - *The Capital Asset Pricing Model (CAPM)*, 2. edition, 2014.

2 Risk and Portfolio Analysis

Introduction

We have observed that *mean-variance efficiency* analyses, premised on investor rationality (maximum return) and risk aversion (minimum variability), are not always sufficient criteria for investment appraisal. Even if investments are considered in isolation, wealth maximising accept-reject decisions depend upon an individual's perception of the riskiness of its expected future returns, measured by their personal *utility curve*, which may be unique.

Your reading of the following material from the bookboon.com companion texts, recommended for Activity 3 and the Review Activity in the previous chapter, confirms this.

- *Strategic Financial Management* (*SFM*): Chapter Four, Section 4.5 onwards,
- *SFM*; *Exercises (SFME)*: Chapter Four, Exercise 4.1,
- *SFM*: *Portfolio Theory and Analyses; Exercises (PTAE)*: Chapter One.

Any conflict between mean-variance efficiency and the concept of investor utility can only be resolved through the application of *certainty equivalent* analysis to investment appraisal. The ultimate test of *statistical* mean-variance analysis depends upon *behavioural* risk attitudes.

So far, so good, but there is now another complex question to answer in relation to the search for future wealth maximising investment opportunities:

> Even if there is only one new investment on the horizon, including a choice that is either *mutually exclusive*, or if *capital is rationed*, (i.e. the acceptance of one precludes the acceptance of others).
>
> How do individuals, or companies and financial institutions that make decisions on their behalf, incorporate the *relative risk*-return trade-off between a *prospective* investment and an *existing* asset portfolio into a quantitative model that still maximises wealth?

2.1 Mean-Variance Analyses: Markowitz Efficiency

Way back in 1952 without the aid of computer technology, H.M.Markowitz explained why rational investors who seek an *efficient* portfolio (one which minimises risk without impairing return, or maximises return for a given level of risk) by introducing new (or off-loading existing) investments, cannot rely on mean-variance criteria alone.

> Even before *behavioural* attitudes are calibrated, Harry Markowitz identified a *third statistical* characteristic concerning the risk-return relationship between individual investments (or in management's case, capital projects) which justifies their inclusion within an *existing* asset portfolio to maximise wealth.

To understand Markowitz' train of thought; let us begin by illustrating his simple *two asset case*, namely the construction of an *optimum* portfolio that comprises two investments. Mathematically, we shall define their expected returns as $R_i(A)$ and $R_i(B)$ respectively, because their size depends upon which one of two future economic "states of the world" occur. These we shall define as S_1 and S_2 with an equal probability of occurrence. If S_1 prevails, $R_1(A) > R_1(B)$. Conversely, given S_2, then $R_2(A) < R_2(B)$. The numerical data is summarised as follows:

Return\State	S_1	S_2
$R_i(A)$	20%	10%
$R_i(B)$	10%	20%

Activity 1

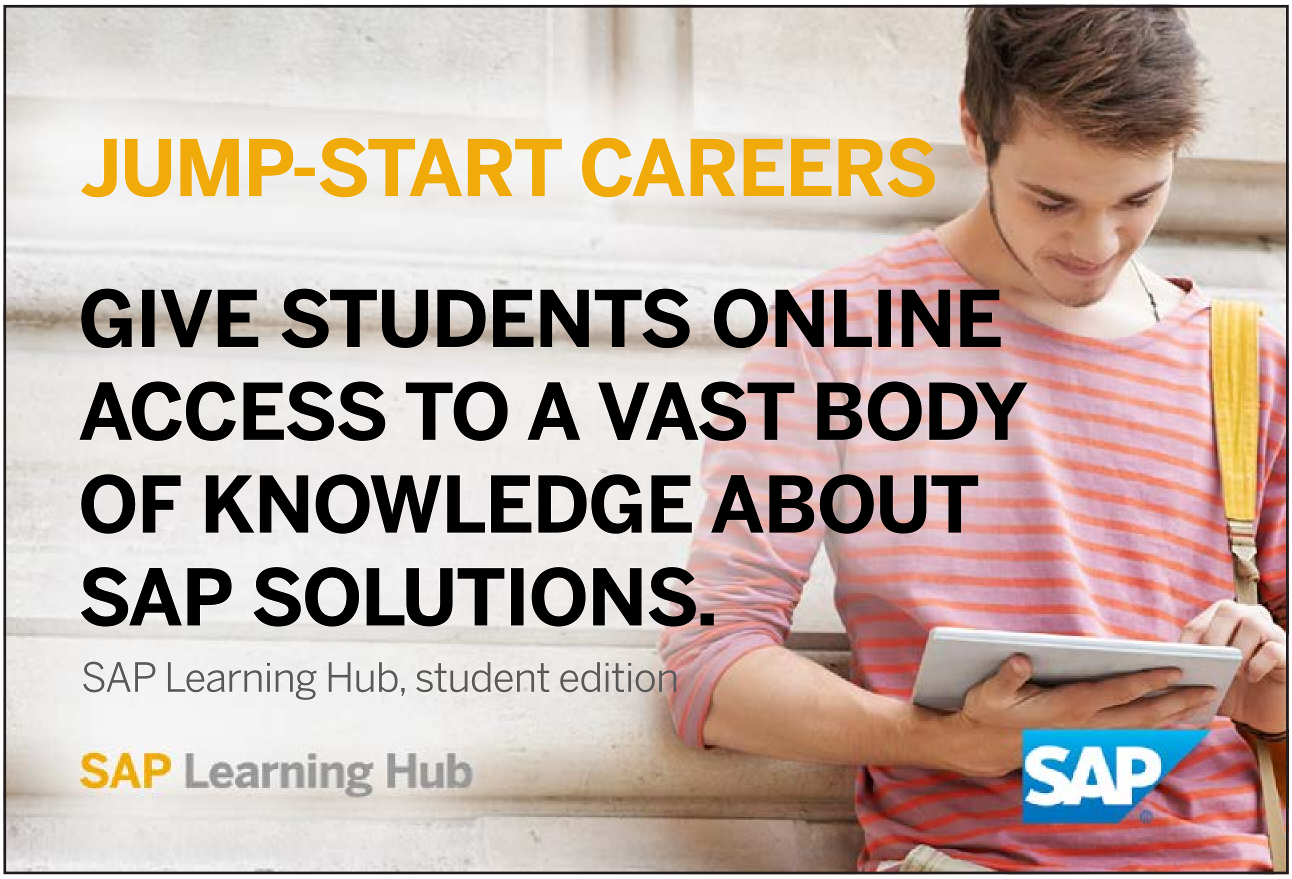
The overall expected return R(A) for investment A (its mean value) is obviously 15 per cent (the weighted average of its expected returns, where the weights are the probability of each state of the world occurring. Its risk (range of possible outcomes) is between 10 to 20 per cent. The same values also apply to B.

Mean-variance analysis therefore informs us that because R(A) = R(B) and σ (A) = σ (B), we should all be *indifferent* to either investment. Depending on your behavioural attitude towards risk, one is perceived to be as good (or bad) as the other. So, either it doesn't matter which one you accept, or alternatively you would reject both.

- Perhaps you can confirm this from your reading for earlier Activities?

However, the question Markowitz posed is whether there is an *alternative* strategy to the exclusive selection of either investment or their wholesale rejection? And because their respective returns do not move in *unison* (when one is good, the other is bad, depending on the state of the world) his answer was yes.

By not "putting all your eggs in one basket", there is a *third* option that in our example produces an *optimum* portfolio *i.e.* one with the *same* overall return as its constituents but with *zero* risk.



If we *diversify* investment and *combine* A and B in a *portfolio* (P) with half our funds in each, then the overall portfolio return R(P) = 0.5R(A) + 0.5R(B) still equals the 15 per cent mean return for A and B, whichever state of the world materialises. Statistically, however, our new portfolio not only has the same return, R(P) = R(A) = R(B) but the risk of its constituents, σ(A) = σ(B), is also eliminated entirely. Portfolio risk; σ(P) = 0. Perhaps you can confirm this?

Activity 2

As we shall discover, the previous example illustrates an ideal portfolio scenario, based upon your entire knowledge of investment appraisal under conditions of risk and uncertainty explained in the *SFM* texts referred to earlier. So, let us summarise their main points

- An uncertain investment is one with a *plurality* of cash flows whose probabilities are *non- quantifiable*.
- A *risky* investment is one with a *plurality* of cash flows to which we attach *subjective probabilities*.
- Expected returns are assumed to be characterised by a *normal* distribution (i.e. they are *random* variables).
- The *probability density function* of returns is defined by the *mean-variance* of their distribution.
- An *efficient choice* between individual investments *maximises* the discounted return of their anticipated cash flows and *minimises* the standard deviation of the return.

So, without recourse to further statistical analysis, (more of which later) but using your knowledge of investment appraisal:

Can you define the objective of portfolio theory and using our previous numerical example, briefly explain what Markowitz adds to our understanding of mean-variance analyses through the *efficient diversification* of investments?

For a given overall return, the objective of efficient portfolio diversification is to determine an overall standard deviation (level of risk) that is lower than any of its individual portfolio constituents.

According to Markowitz, three significant points arise from our simple illustration with one important conclusion that we shall develop throughout the text.

1) We can combine risky investments into a less risky, even risk-free, portfolio by "not putting all our eggs in one basket"; a policy that Markowitz termed *efficient* diversification, and subsequent theorists and analysts now term *Markowitz efficiency* (praise indeed).
2) A portfolio of investments may be preferred to all or some of its constituents, irrespective of investor risk attitudes. In our previous example, no rational investor would hold either investment exclusively, because diversification can maintain the *same* return for *less* risk.

3. Analysed in isolation, the risk-return profiles of individual investments are insufficient criteria by which to assess their true value. Returning to our example, A and B initially seem to be equally valued. Yet, an investor with a substantial holding in A would find that moving funds into B is an attractive proposition (and *vice versa*) because of the *inverse* relationship between the *timing* of their respective risk-return profiles, defined by likely states of the world. When one is good, the other is bad and *vice versa*.

According to Markowitz, risk may be *minimised*, if not eliminated entirely without compromising overall return through the diversification and selection of an *optimum* combination of investments, which defines an *efficient* asset portfolio.

2.2 The Combined Risk of Two Investments

So, in general terms, how do we derive (model) an optimum, efficient diversified portfolio of investments?

To begin with, let us develop the "two asset case" where a company have funds to invest in two profitable projects, A and B. One proportion x is invested in A and $(1-x)$ is invested in B.

We know from Activity 1 that the *expected return from a portfolio* R(P) is simply a weighted average of the expected returns from two projects, R(A) and R(B), where the weights are the proportional funds invested in each. Mathematically, this is given by:

(1) $$R(P) = x\,R(A) + (1 - x)\,R(B)$$

But, what about the likelihood (probability) of the portfolio return R(P) occurring?

Markowitz defines the *proportionate* risk of a two-asset investment as the *portfolio variance:*

(2) $$VAR(P) = x^2\,VAR(A) + (1-x)^2\,VAR(B) + 2x(1-x)\,COV(A, B)$$

Percentage risk is then measured by the *portfolio standard deviation* (i.e. the square root of the variance):

(3) $$\sigma(P) = \sqrt{VAR(P)} = \sqrt{[\,x^2\,VAR(A) + (1-x)^2\,VAR(B) + 2x(1-x)\,COV(A, B)]}$$

Unlike the risk of a *single* random variable, the variance (or standard deviation) of a *two-asset* portfolio exhibits *three* separable characteristics:

1) The risk of the constituent investments measured by their respective variances,
2) The squared proportion of available funds invested in each,
3) The relationship between the constituents measured by twice the *covariance*.

The *covariance* represents the variability of the combined returns of individual investments around their mean. So, if A and B represent two investments, the degree to which their returns ($r_i A$ and $r_i B$) vary together is defined as:

$$(4) \qquad COV(A,B) = \sum_{i=1}^{n} \{ [(r_i A - R(A)] [(r_i B - R(B)] p_i \}$$

For each observation i, we multiply three terms together: the deviation of $r_i(A)$ from its mean R(A), the deviation of $r_i(B)$ from its mean R(B) and the probability of occurrence p_i. We then add the results for each observation.

Returning to Equations (2) and (3), the covariance enters into our portfolio risk calculation *twice* and is *weighted* because the *proportional* returns on A vary with B and *vice versa*.

Depending on the state of the world, the logic of the covariance itself is equally simple.

- If the returns from two investments are *independent* there is no observable relationship between the variables and knowledge of one is of no use for predicting the other. The variance of the two investments combined will equal the sum of the individual variances, *i.e.* the covariance is *zero*.

- If returns are *dependent* a relationship exists between the two and the covariance can take on either a positive or negative value that affects portfolio risk.

1) When each paired deviation around the mean is negative, their product is positive and so too, is the covariance.
2) When each paired deviation is positive, the covariance is still positive.
3) When one of the paired deviations is negative their covariance is negative.

Thus, in a state of the world where individual returns are *independent* and whatever happens to one affects the other to opposite effect, we can reduce risk by diversification without impairing overall return.

Under condition (iii) the portfolio variance will obviously be less than the sum of its constituent variances. Less obvious, is that when returns are *dependent*, risk reduction is still possible.

To demonstrate the application of the statistical formulae for a two-asset portfolio let us consider an equal investment in two corporate capital projects (A and B) with an equal probability of producing the following paired cash returns.

P_i	A %	B %
0.5	8	14
0.5	12	6

We already know that the expected return on each investment is calculated as follows:

$R(A) = (0.5 \times 8) + (0.5 \times 12) = \underline{10\%}$

$R(B) = (0.5 \times 14) + (0.5 \times 6) = \underline{10\%}$

Using Equation (1), the *portfolio return* is then given by:

$R(P) = (0.5 \times 10) + (0.5 \times 10) = \underline{10\%}$

Since the portfolio return equals the expected returns of its constituents, the question management must now ask is whether the decision to place funds in both projects in equal proportions, rather than A or B exclusively, reduces risk?

To answer this question, let us first calculate the variance of A, then the variance of B and finally, the covariance of A and B. The data is summarised in Table 2.1 below.

With a negative covariance value of minus 8, combining the projects in equal proportions can obviously reduce risk. The question is by how much?

Probability	Deviations A	Deviations B	VAR(A)	VAR(B)	COV(A,B)
p_i	$(r_i - R)$	$(r_i - R)$	$(r_i - R)^2 p_i$	$(r_i - R)^2 p_i$	$[r_i A - R(A)][(r_i B - R(B)] p_i$
0.5	(2)	4	2	8	(4)
0.5	2	(4)	2	8	(4)
1.0	0	0	4	16	(8)

Table 2.1: The Variances of Two Investments and their Covariance

Using Equation (2), let us now calculate the portfolio variance:

$$VAR(P) = (0.5^2 \times 4) + (0.5^2 \times 16) + (2 \times 0.5)(0.5 \times -8) = 1$$

And finally, the *percentage* risk given by Equation (3), the portfolio standard deviation:

$$\sigma(P) = \sqrt{VAR(P)} = \sqrt{1} = \underline{1\%}$$

Activity 3

Unlike our original example, which underpinned Activities 1 and 2, the current statistics reveal that this portfolio is not *riskless* (*i.e.* the percentage risk represented by the standard deviation σ is not zero). But given that our investment criteria remain the same (either *minimise* σ, *given* R; or *maximise* R given σ) the next question to consider is how the portfolio's risk-return profile compares with those for the individual projects. In other words is diversification beneficial to the company?

If we compare the standard deviations for the portfolio, investment A and investment B with their respective expected returns, the following relationships emerge.

$$\sigma(P) < \sigma(A) < \sigma(B); \textit{ given } R(P) = R(A) = R(B)$$

These confirm that our decision to place funds in both projects in equal proportions, rather than either A or B exclusively, is the correct one. You can verify this by deriving the standard deviations for the portfolio and each project from the variances in the Table 2.1.

Investments	R %	σ %
P (0.5A + 0.5B)	10	1.00
A	10	2.00
B	10	4.00

2.3 The Correlation between Two Investments

Because the covariance is an *absolute* measure of the correspondence between the movements of two random variables, its interpretation is often difficult. Not all paired deviations need be negative for diversification to produce a degree of risk reduction. If we have small or large negative or positive values for individual pairs, the covariance may also assume small or large values either way. So, in our previous example, COV(A, B) = minus 8. But what does this mean exactly?

Fortunately, we need not answer this question? According to Markowitz, the statistic for the *linear correlation coefficient* can be substituted into the third covariance term of our equation for portfolio risk to simplify its interpretation. With regard to the mathematics, beginning with the variance for a two asset portfolio:

(2) $\text{VAR(P)} = x^2\ \text{VAR(A)} + (1\text{-}x)^2\ \text{VAR(B)} + 2x\ (1\text{-}x)\ \text{COV(A,B)}$

Let us define the correlation coefficient.

(5) $\text{COR(A,B)} = \dfrac{\text{COV(A,B)}}{\sigma\ \text{A}\ \sigma\ \text{B}}$

Now rearrange terms to redefine the covariance.

(6) $\text{COV(A,B)} = \text{COR(A,B)}\ \sigma\ \text{A}\ \sigma\ \text{B}$

Clearly, the portfolio variance can now be measured by the substitution of Equation (6) for

the covariance term in Equation (2).

(7) $\text{VAR(P)} = x^2\ \text{VAR(A)} + (1\text{-}x)^2\ \text{VAR(B)} + 2x\ (1\text{-}x)\ \text{COR(A,B)}\ \sigma\ \text{A}\ \sigma\ \text{B}$

The standard deviation of the portfolio then equals the square root of Equation (7):

(8) $\sigma\ \text{(P)} = \sqrt{\ }\ \text{VAR (P)} = \sqrt{\ }\ [\ x^2\ \text{VAR(A)} + (1\text{-}x)^2\ \text{VAR (B)} + 2x(1\text{-}x)\ \text{COR(A,B)}\ \sigma\ \text{A}\ \sigma\ \text{B}]$

Activity 4

So far, so good; we have proved mathematically that the correlation coefficient can replace the covariance in the equations for portfolio risk.

But, given your knowledge of statistics, can you now explain why Markowitz thought this was a significant contribution to portfolio analysis?

Like the standard deviation, the correlation coefficient is a *relative* measure of variability with a convenient property. Unlike the covariance, which is an *absolute* measure, it has only *limited values between +1 and -1*.This arises because the coefficient is calculated by taking the covariance of returns and dividing by the product (multiplication) of the individual standard deviations that comprise the portfolio. Which is why, for two investments (A and B) we have defined:

(5) $\text{COR(A,B)} = \dfrac{\text{COV(A,B)}}{\sigma\ \text{A}\ \sigma\ \text{B}}$

> The correlation coefficient therefore measures the extent to which two investments vary together as a *proportion* of their respective standard deviations. So, if two investments are *perfectly* and *linearly* related, they deviate by *constant proportionality*.

Of course, the interpretation of the correlation coefficient still conforms to the logic behind the covariance, but with the advantage of limited values.

- If returns are *independent*, i.e. no relationship exists between two variables; their correlation will be zero (although, as we shall discover later, risk can still be reduced by diversification).
- If returns are *dependent*:

1) A perfect, positive correlation of +1 means that whatever affects one variable will equally affect the other. Diversified risk-reduction is *not possible.*
2) A perfect negative correlation of -1 means that an *efficient* portfolio can be constructed, with *zero* variance exhibiting *minimum* risk. One investment will produce a return above its expected return; the other will produce an equivalent return below its expected value and *vice versa*.
3) Between +1 and –1, the correlation coefficient is determined by the proximity of direct and inverse relationships between individual returns So, in terms of risk reduction, even a low positive correlation can be beneficial to investors, depending on the allocation of total funds at their disposal.

Providing the correlation coefficient between returns is less than +1, all investors (including management) can profitably diversify their portfolio of investments. Without compromising the overall return, relative portfolio risk measured by the standard deviation will be less than the weighted average standard deviation of the portfolio's constituents.

> **Review Activity**
>
> Using the statistics generated by Activity 3, confirm that the substitution of the correlation coefficient for the covariance into our revised equations for the portfolio variance and standard deviation does not change their values, or our original investment decision?

Let us begin with a summary of the previous mean-variance data for the two-asset portfolio:

R(P) = 0.5 R(A) + 0.5 R(B)	VAR(P)	VAR(A)	VAR(B)	COV(A,B)
10%	1	4	16	(8)

The correlation coefficient is given by:

$$(5) \qquad COR(A,B) = \frac{COV(A,B)}{\sigma A \ \sigma B} = \frac{-8}{\sqrt{4}.\sqrt{16}} = \underline{-1}$$

Substituting this value into our revised equations for the portfolio variance and standard deviation respectively, we can now confirm our initial calculations for Activity 3.

$$(7) \qquad VAR(P) = x^2 \ VAR(A) + (1\text{-}x)^2 \ VAR(B) + 2x \ (1\text{-}x)COR(A,B) \ \sigma \ A \ \sigma \ B$$
$$= (0.5^2 \times 4) + (0.5^2 \times 16) + \{2 \times 0.5(1\text{-}0.5) \times -1(2 \times 4)\}$$
$$= 1$$

$$(8) \qquad \sqrt{VAR(P)} = \sqrt{1.0}$$
$$= 1.00 \ \%$$

Thus, the company's original *portfolio* decision to place an equal proportion of funds in both investments, rather than either A or B *exclusively*, still applies. This is also confirmed by a summary of the following inter-relationships between the risk-return profiles of the portfolio and its constituents, which are identical to our previous Activity.

$\sigma(P) = 1.00\% < \sigma(A) = 2.00\ \% < \sigma(B) = 4.00\%$; *given* $R(P) = R(A) = R(B) = 10\%$

Summary and Conclusions

It should be clear from our previous analyses that the risk of a *two-asset* portfolio is a function of its covariability of returns. Risk is at a *maximum* when the correlation coefficient between two investments is +1 and at a *minimum* when the correlation coefficient equals -1. For the vast majority of cases where the correlation coefficient is between the two, it also follows that there will be a *proportionate* reduction in risk, relative to return. Overall portfolio risk will be less than the weighted average risks of its constituents. So, investors can still profit by diversification because:

$\sigma(P) > \sigma(P) > \sigma(P)$, given $R(P)$, if $COR(A,B) = +1, 0,$ or -1, respectively.

Selected References

1. Hill, R.A., bookboon.com
   - *Strategic Financial Management (SFM)*, 2009.
   - *Strategic Financial Management: Exercises (SFME)*, 2009.
   - *Portfolio Theory and Financial Analyses (PTFA)*, 2010.
   - *Portfolio Theory and Financial Analyses: Exercises (PTFAE)*, 2010.
   - *The Capital Asset Pricing Model (CAPM)*, 2. edition, 2014.
2. Markowitz, H.M., "Portfolio Selection", *The Journal of Finance,* Vol. 13, No. 1, March 1952.

3 The Optimum Portfolio

Introduction

In an efficient capital market where the random returns from two investments are normally distributed (symmetrical) we have explained how rational (risk averse) investors and companies who seek an optimal portfolio can maximise their utility preferences by *efficient* diversification. Any combination of investments produces a trade-off between the two statistical parameters that define a normal distribution; their expected return and standard deviation (risk) associated with the *covariability* of individual returns. According to Markowitz (1952) this is best measured by the *correlation coefficient* such that:

> *Efficient* diversified portfolios are those which *maximise* return for a *given* level of risk, or *minimise* risk for a *given* level of return for different correlation coefficients.

The purpose of this chapter is to prove that when the correlation coefficient is at a minimum and portfolio risk is minimised we can derive an *optimum portfolio* of investments that maximises there overall expected return.

3.1 The Mathematics of Portfolio Risk

You recall from Chapter Two that substituting the *relative* linear correlation coefficient for the *absolute* covariance term into a two-asset portfolio's standard deviation simplifies the wealth maximisation analysis of the risk-return trade-off between the covariability of returns. Whenever the coefficient falls below one, there will be a *proportionate* reduction in portfolio risk, relative to return, by diversifying investment.

For example, given the familiar equations for the return, variance, correlation coefficient and standard deviation of a two-asset portfolio:

(1) $R(P) = x\ R(A) + (1-x)\ R(B)$

(2) $VAR(P) = x^2\ VAR(A) + (1-x)^2\ VAR(B) + 2x\ (1-x)\ COV(A,B)$

(5) $COR(A,B) = \dfrac{COV(A,B)}{\sigma A\ \sigma B}$

(8) $\sigma(P) = \sqrt{VAR(P)} = \sqrt{[\,x^2\ VAR(A) + (1-x)^2\ VAR(B) + 2x(1-x)\ COR(A,B)\ \sigma A\ \sigma B]}$

Harry Markowitz (*op. cit.*) proved mathematically that:

$\sigma(P) > \sigma(P) > \sigma(P)$, given R(P), if COR(A,B) = +1, 0, or -1, respectively.

However, he also illustrated that if the returns from two investments exhibit *perfect positive, zero*, or *perfect negative* correlation, then portfolio risk measured by the standard deviation using Equation (8) can be simplified further.

To understand why, let us return to the original term for the portfolio variance:

(2) $VAR(P) = x^2\ VAR(A) + (1-x)^2\ VAR(B) + 2x\ (1-x)\ COV(A,B)$

Because the correlation coefficient is given by:

(5) $COR(A,B) = \dfrac{COV(A,B)}{\sigma A\ \sigma B}$

We can rearrange its terms, just as we did in Chapter Two, to redefine the covariance:

(6) $COV(A,B) = COR(A,B)\ \sigma A\ \sigma B$

The portfolio variance can now be measured by the substitution of Equation (6) for the covariance term in Equation (2), so that.

(7) $\mathrm{VAR(P)} = x^2\,\mathrm{VAR(A)} + (1\text{-}x)^2\,\mathrm{VAR(B)} + 2x\,(1\text{-}x)\,\mathrm{COR(A,B)}\,\sigma\,\mathrm{A}\,\sigma\,\mathrm{B}$

The standard deviation of the portfolio then equals the square root of Equation (7):

(8) $\sigma\,(\mathrm{P}) = \surd\,\mathrm{VAR(P)} = \surd\,[\,x^2\,\mathrm{VAR(A)} + (1\text{-}x)^2\,\mathrm{VAR(B)} + 2x(1\text{-}x)\,\mathrm{COR(A,B)}\,\sigma\,\mathrm{A}\,\sigma\,\mathrm{B}]$

Armed with this information, we can now confirm that:

> If the returns from two investments exhibit perfect, positive correlation, portfolio risk is simply the *weighted average* of its constituent's risks and at a maximum.
>
> $\sigma(\mathrm{P}) = \mathrm{x}\,\sigma(\mathrm{A}) + (1\text{-x})\,\sigma(\mathrm{B})$

If the correlation coefficient for two investments is positive and COR(A,B) also equals plus one, then the correlation term can disappear from the portfolio risk equations without affecting their values. The portfolio variance can be rewritten as follows:

(9) $\mathrm{VAR(P)} = x^2\,\mathrm{VAR(A)} + (1 - x)^2\,\mathrm{VAR(B)} + 2x\,(1\text{-}x)\,\sigma(\mathrm{A})\,\sigma(\mathrm{B})$

Simplifying, this is equivalent to:

(10) $\mathrm{VAR(P)} = [x\,\sigma(\mathrm{A}) + (1\text{-}x)\,\sigma(\mathrm{B})]^2$

And because this is a *perfect square*, our probabilistic estimate for the risk of a two-asset portfolio measured by the standard deviation given by Equation (8) is equivalent to:

(11) $\sigma(\mathrm{P}) = \surd\,\mathrm{VAR(P)} = x\,\sigma(\mathrm{A}) + (1\text{-}x)\sigma(\mathrm{B}))$

To summarise:

> Whenever COR(A, B) = +1 (perfect positive) the portfolio variance VAR(P) and its square root, the standard deviation σ(P), simplify to the weighted average of the respective statistics, based on the probabilistic returns for the individual investments.

But this is not all. The substitution of Equation (6) into the expression for portfolio variance has two further convenient properties. Given:

(6) $COV(A,B) = COR(A,B)\ \sigma A\ \sigma B$

If the relationship between two investments is *independent* and exhibits *zero* correlation, the portfolio variance given by Equation (7) simplifies to:

(12) $VAR(P) = x^2\ VAR(A) + (1\text{-}x)^2\ VAR(B)$

And its corresponding standard deviation also simplifies:

(13) $\sigma(P) = \sqrt{[x^2\ VAR(A) + (1\text{-}x)^2\ VAR(B)]}$

Similarly, with *perfect inverse* correlation we can deconstruct our basic equations to simplify the algebra.

Activity 1

When the correlation coefficient for two investments is perfect positive and equals one, the correlation term disappears from equations for portfolio risk without affecting their values. The portfolio variance VAR(P)) and its square root, the standard deviation σ(P), simplify to the *weighted average* of the respective statistics.

Can you manipulate the previous equations to prove that if COR(A,B) equals *minus* one (perfect negative) they still correspond to a weighted average, like their perfect positive counterpart, but with one fundamental difference?

Let us begin again with the familiar equation for portfolio variance.

(7) $\text{VAR(P)} = x^2 \text{ VAR(A)} + (1\text{-}x)^2 \text{ VAR(B)} + 2x\ (1\text{-}x) \text{ COR(A,B) } \sigma \text{ A } \sigma \text{ B}$

If the correlation coefficient for two investments is negative and COR(A,B) also equals minus one, then the coefficient can disappear from the equation's third right hand term without affecting its value. It can be rewritten as follows with only a change of sign (positive to negative):

(14) $\text{VAR(P)} = x^2 \text{ VAR(A)} + (1\text{-}x)^2 \text{ VAR(B)} - 2x\ (1\text{-}x)\ \sigma\text{(A)}\ \sigma\text{(B)}$

Simplifying, this is equivalent to:

(15) $\text{VAR(P)} = [x\ \sigma\text{(A)} - (1\text{-}x)\ \sigma\text{(B)}]^2$

And because this is a *perfect square*, our probabilistic estimate for the risk of a two-asset portfolio measured by the standard deviation is equivalent to:

(16) $\sigma\text{(P)} = x\ \sigma\text{(A)} - (1\text{-}x)\ \sigma\text{(B)}$

The only difference between the formulae for the risk of a two-asset portfolio where the correlation coefficient is at either limit (+1 or –1) is simply a matter of sign (positive or negative) in the right hand term for σ (P).

3.2 Risk Minimisation and the Two-Asset Portfolio

When investment returns exhibit perfect positive correlation a portfolio's risk is at a maximum, defined by the weighted average of its constituents. As the correlation coefficient falls there is a proportionate reduction in portfolio risk relative to this weighted average. So, if we diversify investments; risk is minimised when the correlation coefficient is minus one.

To illustrate this general proposition, Figure 3.1 roughly sketches the various *two*-asset portfolios that are possible if corporate management combine two investments, A and B, in various proportions for different correlation coefficients.

Specifically, the *diagonal* line A (+1) B; the *curve* A (E) B and the *"dog-leg"* A (-1) B are the focus of all possible risk-return combinations when our correlation coefficients equal plus one, zero and minus one, respectively.

Thus, if project returns are perfectly, positively correlated we can construct a portfolio with any risk-return profile that lies along the *horizontal* line, A (+1) B, by varying the proportion of funds placed in each project. Investing 100 percent in A produces a minimum return but minimises risk. If management put all their funds in B, the reverse holds. Between the two extremes, having decided to place say two-thirds of funds in Project A, and the balance in Project B, we find that the portfolio lies one third along A (+1) B at point +1.

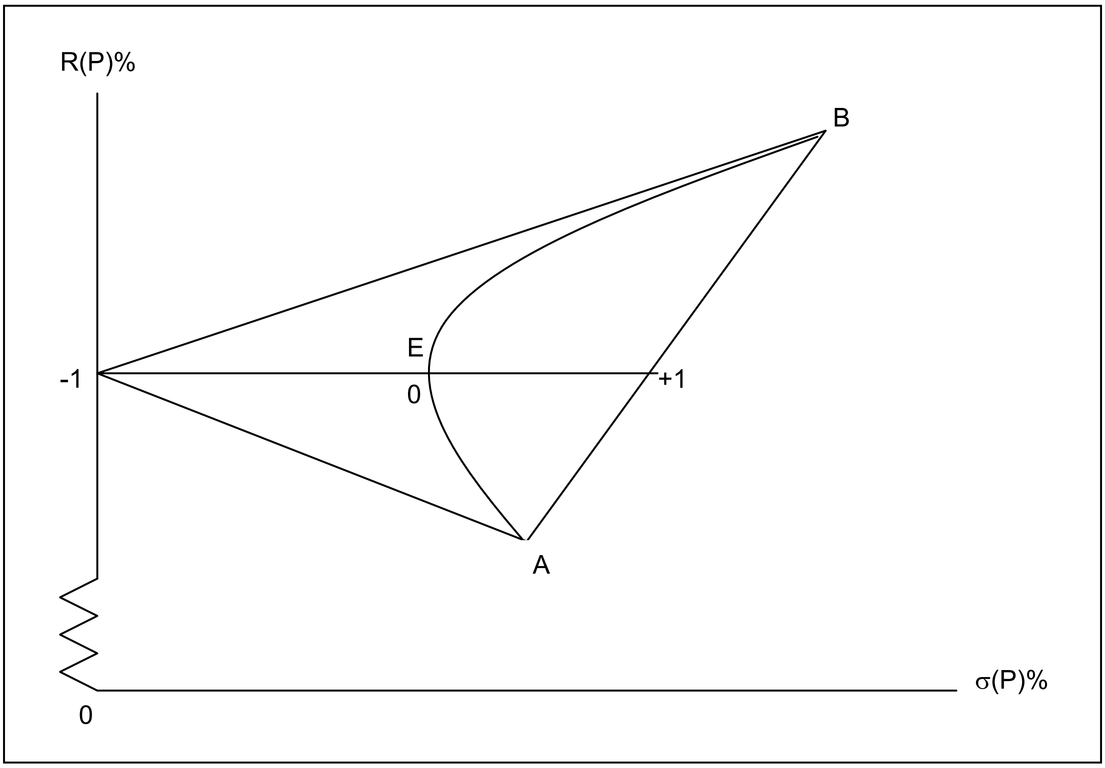


Figure 3.1: The Two Asset Risk-Return Profile and the Correlation Coefficient

Similarly, if the two returns exhibit perfect negative correlation, we could construct any portfolio that lies along the line A (-1) B. However, because the correlation coefficient equals minus one, the line is no longer straight but a *dog-leg* that also touches the vertical axis where s(P) equals zero. As a consequence, our choice now differs on two counts.

- It is possible to construct a *risk-free* portfolio.
- No rational, risk averse investor would be interested in those portfolios which offer a *lower* expected return for the *same* risk.

As you can observe from Figure 3.1, the investment proportions lying along the line -1 to B offer higher returns for a given level of risk relative to those lying between -1 and A. Using the terminology of Markowitz based on mean-variance criteria; the first portfolio set is *efficient* and acceptable whilst the second is *inefficient* and irrelevant. The line -1 to B, therefore, defines the *efficiency frontier* for a two-asset portfolio.

Where the two lines meet on the vertical axis (point -1 on our diagram) the portfolio standard deviation is zero. As the *horizontal* line (-1, 0, +1) indicates, this *riskless* portfolio also conforms to our decision to place two-thirds of funds in Project A and one third in Project B.

Finally, in most cases where the correlation coefficient lies somewhere between its extreme value, every possible two-asset combination always lies along a *curve*. Figure 3.1 illustrates the risk-return trade-off assuming that the portfolio correlation coefficient is *zero*. Once again, because the data set is not perfect positive (less than +1) it turns back on itself. So, only a proportion of portfolios are efficient; namely those lying along the E-B frontier. The remainder, E-A, is of no interest whatsoever. You should also note that whilst risk is not eliminated entirely, it could still be *minimised* by constructing the appropriate portfolio, namely point E on our curve.

3.3 The Minimum Variance of a Two-Asset Portfolio

Investors trade financial securities to earn a return in the form of dividends and capital gains. Companies invest in projects to generate net cash inflows on behalf of their shareholders. Returns might be higher or lower than anticipated and their variability is the cause of investment risk. Investors and companies can reduce risk by diversifying their portfolio of investments. The preceding analysis explains why risk minimisation represents an *objective* standard against which investors and management compare their variance of returns as they move from one portfolio to another.

To prove this proposition, you will have observed from Figure 3.1 that the decision to place two-thirds of our funds in Project A and one-third in Project B falls between E and A when COR(A,B) = 0. This is defined by point 0 along the horizontal line (-1, 0, +1).

Because portfolio risk is minimised at point E, with a higher return above and to the left in our diagram, the decision is clearly *suboptimal*. At one extreme, speculative investors or companies would place all their money in Project B at point B hoping to maximise their return (completely oblivious to risk). At the other, the most risk-averse among them would seek out the proportionate investment in A and B which corresponds to E. Between the two, a higher expected return could also be achieved for any degree of risk given by the curve E-A. Thus, all investors would move up to the efficiency frontier E-B and depending upon their risk attitudes choose an appropriate combination of investments above and to the right of E.

However, without a graph, let alone data to fall back on, this raises another fundamental question.

> How do investors and companies mathematically model an *optimum* portfolio with minimum variance from first principles?

According to Markowitz (*op. cit*) the mathematical derivation of a two-asset portfolio with *minimum* risk is quite straightforward.

Where a proportion of funds x is invested in Project A and (1-x) in Project B, the portfolio variance can be defined by the familiar equation:

$$(7) \qquad \text{VAR(P)} = x^2\ \text{VAR(A)} + (1-x)^2\ \text{VAR(B)} + 2x\ (1-x)\ \text{COR(A,B)}\ \sigma\ \text{A}\ \sigma\ \text{B}$$

The value of x, for which Equation (7) is at a *minimum*, is given by *differentiating* VAR(P) with respect to x and setting DVAR(P) / $\Delta\ x$ = 0, such that:

$$(17) \qquad x = \frac{\text{VAR(B)} - \text{COR(A,B)}\ \sigma(\text{A})\ \sigma(\text{B})}{\text{VAR(A)} + \text{VAR(B)} - 2\ \text{COR(A,B)}\ \sigma(\text{A})\ \sigma(\text{B})}$$

Since all the variables in the equation for minimum variance are now known, the risk-return trade-off can be solved. Moreover, if the correlation coefficient equals *minus one*, risky investments can be combined into a *riskless* portfolio by solving the following equation when the standard deviation is *zero*.

(18) $$\sigma(P) = \sqrt{[x^2 \, VAR(A) + (1-x)^2 \, VAR(B) + 2x(1-x) \, COR(A,B) \, \sigma A \, \sigma B]} = \underline{0}$$

Because this is a *quadratic* in one unknown (x) it also follows that to *eliminate* portfolio risk when COR(A,B) = -1, the proportion of funds (x) invested in Project A should be:

(19) $$x = 1 - \frac{\sigma(A)}{\sigma(A) + \sigma(B)}$$

Activity 2

Algebraically, mathematically and statistically, we have covered a lot of ground since Chapter Two. So, the previous section, like those before it, is illustrated by the numerical application of data to theory in the *bookboon* companion text.

Portfolio Theory and Financial Analyses; Exercises (PTFAE):
Chapter Three, 2010.

You might find it useful at this point in our analysis to cross-reference the appropriate Exercises (3.1 and 3.2) before we continue?

3.4 The Multi-Asset Portfolio

In efficient capital markets where the returns from *two* investments are normally distributed (symmetrical) we have explained how rational (risk averse) investors and companies who require an optimal portfolio can maximise their utility preferences by diversification. Any combination of investments produce a trade-off between the statistical parameters that define a normal distribution; the expected return and standard deviation (risk) associated with the covariance of individual returns.

Efficient diversified portfolios are those which *maximise* return for a *given* level of risk, or *minimise* risk for a given level of return for different correlation coefficients.

However, most investors, or companies and financial managers (whether they control capital projects or financial services (such as insurance premiums, pension funds or investment trusts) may be responsible for numerous investments. It is important, therefore, that we extend our analysis to portfolios with more than two constituents.

Theoretically, this is not a problem. According to Markowitz (*op cit.*) if individual returns, standard deviations and the covariance for each pair of returns are known, the portfolio return R(P), portfolio variance VAR(P) and a probabilistic estimate of portfolio risk measured by the standard deviation s(P), can be calculated.

For a *multi-asset* portfolio where the number of assets equals n and x_i represents the proportion of funds invested in each, such that:

$$\sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0$$

We can define the portfolio return and variance as follows

$$(20) \quad R(P) = \sum_{i=1}^{n} x_i R_i$$

$$(21) \quad VAR(P) = \sum_{i=1}^{n} x_i^2 \, VAR_i + \sum_{i \neq j}^{n} \sum x_i x_j \, COVij$$

The covariance term, COVij determines the degree to which variations in the return to one investment, i, can serve to offset the variability of another, j. The standard deviation is then derived in the usual manner.

(22) $\sigma(P) = \sqrt{VAR(P)}$

Assuming we now wish to *minimise* portfolio risk for any given portfolio return; our financial objective is equally straightforward:

(23) MIN: $\sigma(P)$, Given $R(P) = K$ (constant)

This mathematical function combines Equation (22) which is to be *minimised*, with a constraint obtained by setting Equation (20) for the portfolio return equal to a *constant* (K):

Figure 3.2 illustrates all the different risk-return combinations that are available from a hypothetical multi investment scenario.

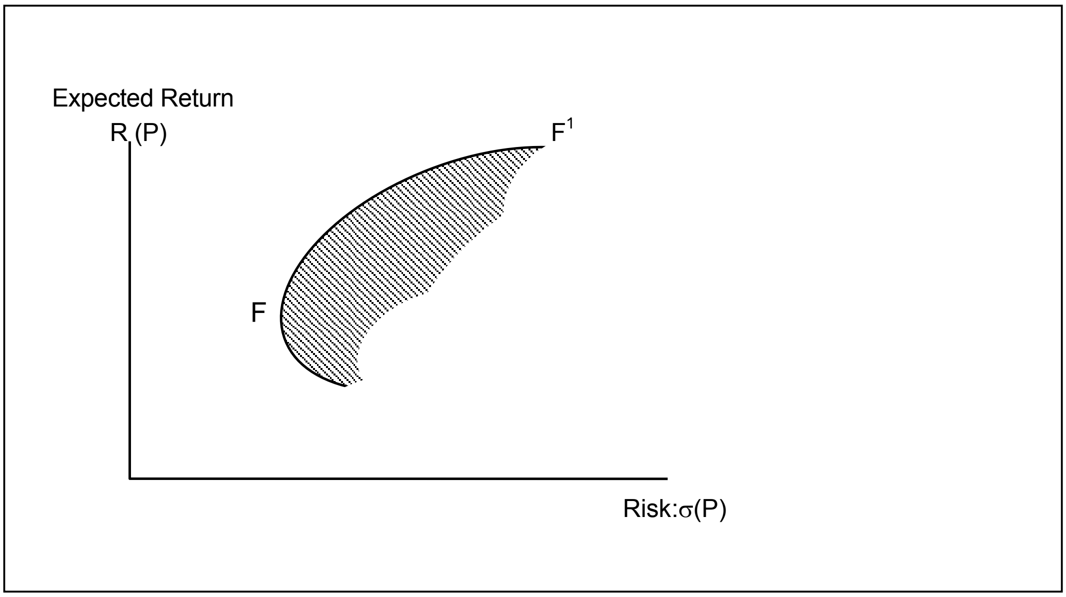


Figure 3.2: The Portfolio Efficiency Frontier: The Multi-Asset Case

The first point to note is that when an investment comprises a large number of assets instead of two, the possible portfolios now lie within on area, rather than along a line or curve. The area is constructed by plotting (infinitely) many lines or curves similar to those in Figure 3.1.

However, like a two-asset portfolio, rational, risk-averse investors or companies are not interested in all these possibilities, but only those that lie along the upper boundary between F and F^1. The portfolios that lie along this frontier are efficient because each produces the highest expected return for its given level of risk. To the right and below, alternative portfolios yield inferior results. To the left, no possibilities exist. Thus, an optimum portfolio for any investor can still be determined at an appropriate point on the efficiency frontier providing the individual's attitude toward risk is known.

So how is this calibrated?

3.5: The Optimum Portfolio

We have already observed that the *calculation* of statistical means and standard deviations is separate from their behavioural *interpretation*, which can create anomalies. For example, a particular problem we encountered within the context of investment appraisal was the "risk-return paradox" where one project offers a *lower return for less risk*, whilst the other offers a *higher return for greater risk*. Here, investor rationality (maximum return) and risk aversion (minimum variability) may be *insufficient* behavioural criteria for project selection. Similarly, with portfolio analysis:

> If two different portfolios lie on the efficiency frontier, it is impossible to choose between them without information on investor risk attitudes.

One solution is for the investor or company to consider a value for the portfolio's expected return R(P), say $R(p_i)$ depicted schematically in Figure 3.3.

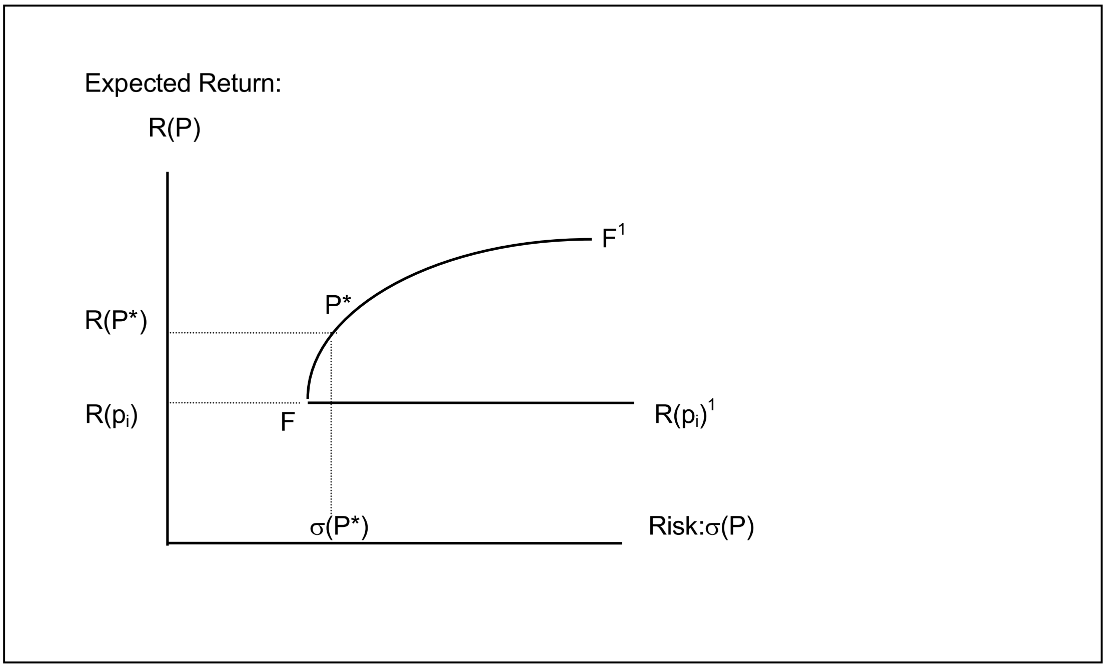


Figure 3.3: The Multi-Asset Efficiency Frontier and Investor Choice

All $R(p_i)$, $\sigma(p_i)$ combinations for different portfolio mixes are then represented by points along the horizontal line $R(p_i)$ - $R(p_i)^1$ for which $R(P) = R(p_i)$. The leftmost point on this line, F then yields the portfolio investment mix that satisfies Equation (23) for our objective function:

MIN: σ(P), Given R(P) = K (constant)

By repeating the exercise for all other possible values of R(P) and obtaining every efficient value of $R(p_i)$ we can then trace the entire opportunity locus, F-F[1].The investor or company then subjectively select the investment combination yielding a maximum return, subject to the constraint imposed by the degree of risk they are willing to accept, say P* corresponding to R(P*) and s(P*) in the diagram.

Review Activity

As an optimisation procedure, the preceding model is theoretically sound. However, without today's computer technology and programming expertise, its practical application was a lengthy, repetitive process based on trial and error, when first developed in the 1950s. What investors and companies needed was a portfolio selection technique that actually incorporated their risk preferences into their analyses. Fortunately, there was a lifeline.

As we explained in the Summary and Conclusions of Chapter Two's Exercise text, (PTFAE) rational risk-averse investors, or companies, with a *two-asset* portfolio will always be willing to accept higher risk for a larger return, but *only up to a point*. Their precise cut-off rate is defined by an *indifference curve* that calibrates their risk attitude, based on the concept of *expected utility*.

We can apply this analysis to a *multi-asset* portfolio of investments. However, before we develop the mathematics, perhaps you might care to look back at Chapter Two (*PTFAE*)and the simple two-asset scenario before we continue.

In Chapter Two (*PTFAE*) we discovered that if an investor's or company's objective is to minimise the standard deviation of expected returns this can be determined by reference to a their utility *indifference curve*, which calibrates attitudes toward risk and return. Applied to portfolio analysis, the mathematical equation for any curve of indifference between portfolio risk and portfolio return for a rational investor can be written:

(24) $\quad VAR(P) = \alpha + \lambda\ R(P)$

Graphically, the value of l indicates the *steepness* of the curve and α indicates the *horizontal intercept*. Thus, the objective of the Markowitz portfolio model is to *minimise* α. If we rewrite Equation (24), for any indifference curve that relates to a portfolio containing n assets, this objective function is given by:

(25) $\quad MIN{:}\ \alpha = VAR\ (P) - \lambda\ R(P)$

For all possible values of $\lambda \geq 0$, where R(P) = K(constant), subject to the non-negativity constraints:

$\alpha_i \geq 0$, i = 1, 2, 3 …n

And the essential requirement that sources of funds equals uses and x_i be proportions expressed mathematically as:

$$\sum_{i=1}^{n} x_i = 1$$

Any portfolio that satisfies Equation (25) is *efficient* because no other asset combination will have a lower degree of risk for the requisite expected return.

An optimum portfolio for an individual investor is plotted in Figure 3.4. The efficiency frontier F - F[1] of risky portfolios still reveals that, to the right and below, alternative investments yield inferior results. To the left, no possibilities exist. However, we no longer determine an optimum portfolio for the investor by trial and error.

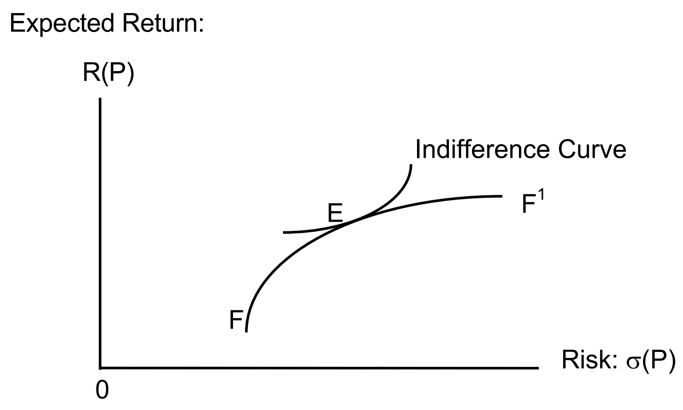


Figure 3.4: the Determination of an Optimum Portfolio: The Multi-Asset Case

The optimum portfolio *is at* the *point* where one of the curves for their equation of indifference (risk-return profile) is *tangential* to the frontier of efficient portfolios (point E on the curve F-F^1). This portfolio is optimal because it provides the best combination of risk and return to suit their preferences.

Summary and Conclusions

We have observed that the objective function of multi-asset portfolio analysis is represented by the following indifference equation.

$$(25)\ \text{MIN: } \alpha = \text{VAR(P)} - \lambda\, \text{R(P)}$$

This provides investors and companies with a standard, against which they can compare their preferred risk-return profile for any efficient portfolio.

However, its interpretation, like other portfolio equations throughout the Chapter assumes that the efficiency frontier has been correctly defined. Unfortunately, this in itself is no easy task.

Based upon the pioneering work of Markowitz (op. cit.) we explained how a rational and risk-averse investor, or company, in an efficient capital market (characterised by a normal distribution of returns) who require an optimal portfolio of investments can maximise utility, having regard to the relationship between the expected returns and their dispersion (risk) associated with the covariance of returns within a portfolio.

Any combination of investments produces a trade-off between the two statistical parameters; expected return and standard deviation (risk) associated with the covariability of individual returns. And according to Markowitz, this statistical analysis can be simplified.

> Efficient diversified portfolios are those which maximise return for a given level of risk, or minimise risk for a given level of return for different correlation coefficients.

The Markowitz portfolio selection model is theoretically sound. Unfortunately, even if we substitute the correlation coefficient into the covariance term of the portfolio variance, without the aid of computer software, the mathematical complexity of the variance-covariance matrix calculations associated with a multi-asset portfolio limits its applicability.

The *constraints* of Equation (25) are *linear* functions of the *n* variables x_i, whilst the *objective function* is an equation of the *second degree* in these variables. Consequently, methods of *quadratic* programming, rather than a simple *linear* programming calculation, must be employed by investors to *minimise* VAR(P) for various values of R (P) = K.

Once portfolio analysis extends beyond the two-asset case, the data requirements become increasingly formidable. If the covariance is used as a measure of the variability of returns, not only do we require estimates for the expected return and the variance for each asset in the portfolio but also estimates for the correlation matrix between the returns on all assets.

For example, if management invest equally in three projects, A, B and C, each deviation from the portfolio's expected return is given by:

$$[1/3\ r_iA - R(A)] + [1/3\ r_iB - R(B)] + [1/3\ r_iC - R(C)]$$

If the deviations are now squared to calculate the variance, the proportion 1/3 becomes $(1/3)^2$, so that:

VAR(P) = VAR[1/3 (A)+1/3 (B)+1/3 (C)]

= $(1/3)^2$ (the sum of three variance terms, plus the sum of six covariances).

For a twenty asset portfolio:

VAR(P) = $(1/20)^2$ (sum of twenty variance, plus the sum of 380 covariances).

As a *general rule*, if there are $\Sigma\ x_i = n$ projects, we find that:

(26) VAR (P) = $(1/n)^2$ (sum of n variance terms, plus the sum of n $(n\text{-}1)$ covariances.

In the covariance matrix (x_i ... x_n), x_i is paired in turn with each of the other projects (x_2 ... x_n) making (n-1) pairs in total. Similarly, (n -1) pairs can be formed involving x_2 with each other x_i and so forth, through to x_n making n (n-1) permutations in total.

Of course, *half of these pairs will be duplicates*. The set x_1, x_2 is identical with x_2, x_1. The n asset case therefore requires only 1/2 $(n^2 - n)$ distinct covariance figures altogether, which represents a substantial data saving in relation to Equation (26). Nevertheless, the decision-maker's task is still daunting, as the number of investments for inclusion in a portfolio increases.

Not surprising, therefore, that without today's computer technology, a search began throughout the late 1950s and early 1960s for simpler mathematical and statistical measures of Markowitz portfolio risk and optimum asset selection. As the *bookboon* Capital Asset Pricing Model companion Business text will reveal.

Selected References

1. Markowitz, H.M., "Portfolio Selection", *The Journal of Finance,* Vol. 13, No. 1, March 1952.
2. Hill, R.A., bookboon.com
    - *Strategic Financial Management (SFM)*, 2009.
    - *Strategic Financial Management: Exercises (SFME)*, 2009.
    - *Portfolio Theory and Financial Analyses (PTFA)*, 2010.
    - *Portfolio Theory and Financial Analyses: Exercises (PTFAE)*, 2010.
    - *The Capital Asset Pricing Model (CAPM)*, 2. edition, 2014.

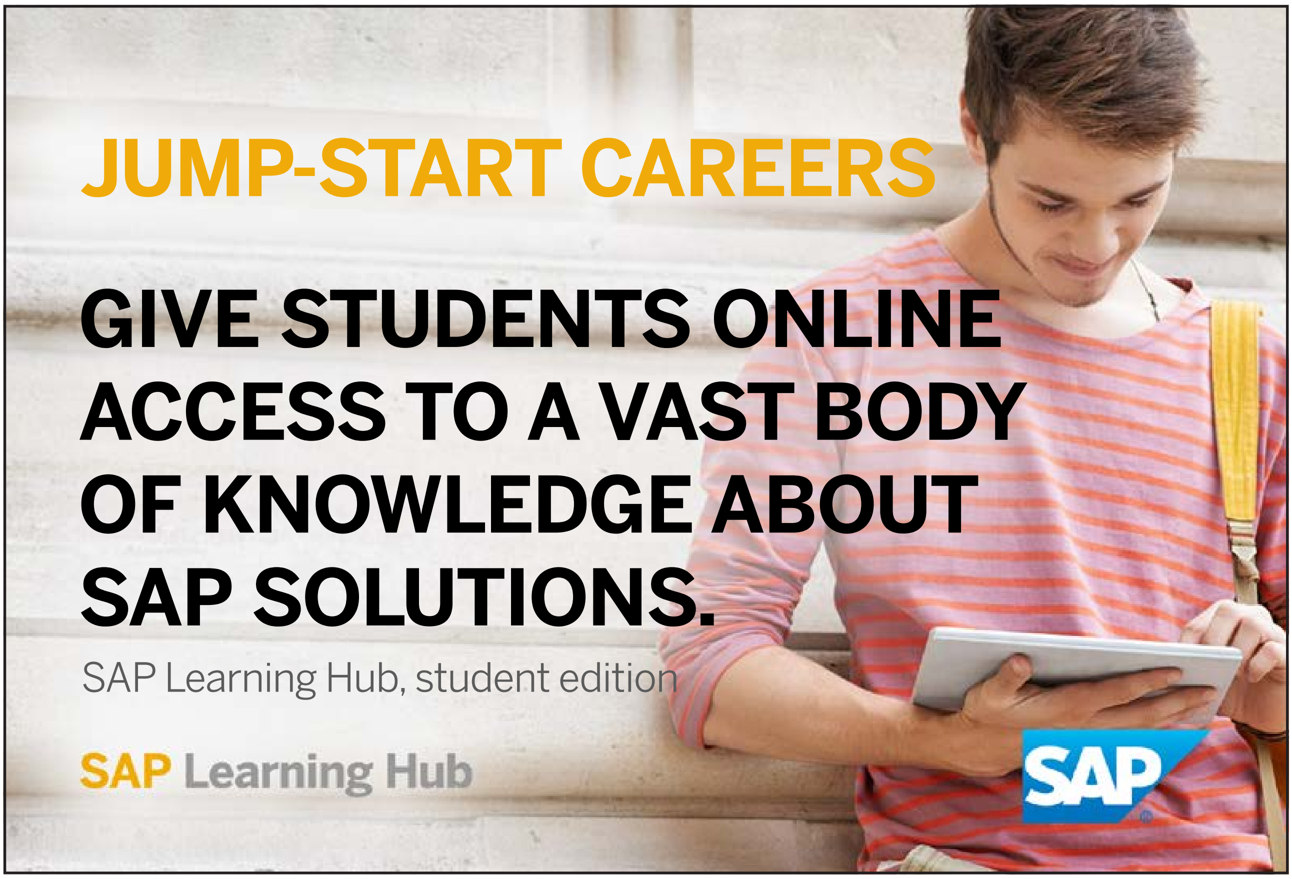
4 The Market Portfolio

Introduction

The objective of efficient portfolio diversification is to achieve an overall standard deviation lower than that of its component parts without compromising overall return.

> In an ideal world portfolio theory should enable:
>
> - Investors (private or institutional) who play the stock market to model the effects of adding new securities to their existing spread.
> - Companies to assess the extent to which the pattern of returns from new projects affects the risk of their existing operations.

For example, suppose there is a *perfect positive correlation* between two securities that comprise the market, or two products that comprise a firm's total investment. In other words, high and low returns always move sympathetically. It would pay the investor, or company, to place all their funds in whichever investment yields the highest return at the time. However, if there is *perfect inverse correlation*, where high returns on one investment are always associated with low returns on the other and *vice versa*, or there is *random* (*zero*) *correlation* between the returns, then it can be shown statistically that overall risk reduction can be achieved through diversification.



According to Markowitz (1952), if the correlation coefficient between any number of investments is less then one (perfect positive), the total risk of a portfolio measured by its standard deviation is lower than the weighted average of its constituent parts, with the greatest reduction reserved for a correlation coefficient of minus one (perfect inverse).

Thus, if the standard deviation of an individual investment is higher than that for a portfolio in which it is held, it would appear that some of the standard deviation must have been diversified away through correlation with other portfolio constituents, leaving a residual risk component associated with other factors.

Indeed, as we shall discover later, by reading the Capital Asset Pricing Model companion text, the reduction in *total* risk only relates to the *specific* risk associated with *micro-economic* factors, which are unique to individual sectors, companies, or projects. A proportion of *total* risk, termed *market* risk, based on *macro-economic* factors correlated with the market is inescapable.

The distinguishing features of specific and market risk had important consequences for the development of Markowitz efficiency and the emergence of Modern portfolio Theory (MPT) during the 1960s. For the moment, suffice it to say that whilst market risk is not diversifiable, theoretically, specific risk can be eliminated entirely if all rational investors diversify until they hold the *market portfolio*, which reflects the risk-return characteristics for every available financial security. In practice, this strategy is obviously unrealistic. But as we shall also discover later, studies have shown that with less than thirty diversified constituents it is feasible to reach a position where a portfolio's standard deviation is close to that for the market portfolio.

Of course, without today's computer technology and sophisticated software, there are still problems, as we observed in previous Chapters. The significance of covariance terms in the Markowitz variance calculation are so unwieldy for a well-diversified risky portfolio that for most investors, with a global capital market to choose from, it is untenable. Even if we substitute the correlation coefficient into the covariance of the portfolio variance, the mathematical complexity of the variance-covariance matrix calculations for a risky multi-asset portfolio still limits its applicability. So, is there an alternative?

4.1 The Market Portfolio and Tobin's Theorem

We have already explained that if an individual or company objective is to minimize the standard deviation of an investment's expected return, this could be determined by reference to indifference curves, which calibrate attitudes toward risk and return. In Chapter Three of the companion theory texts (*PTFA*) and (*PTFAE*) we graphed an equation of *indifference* between portfolio risk and portfolio return for any rational investor relative to their optimum portfolio.

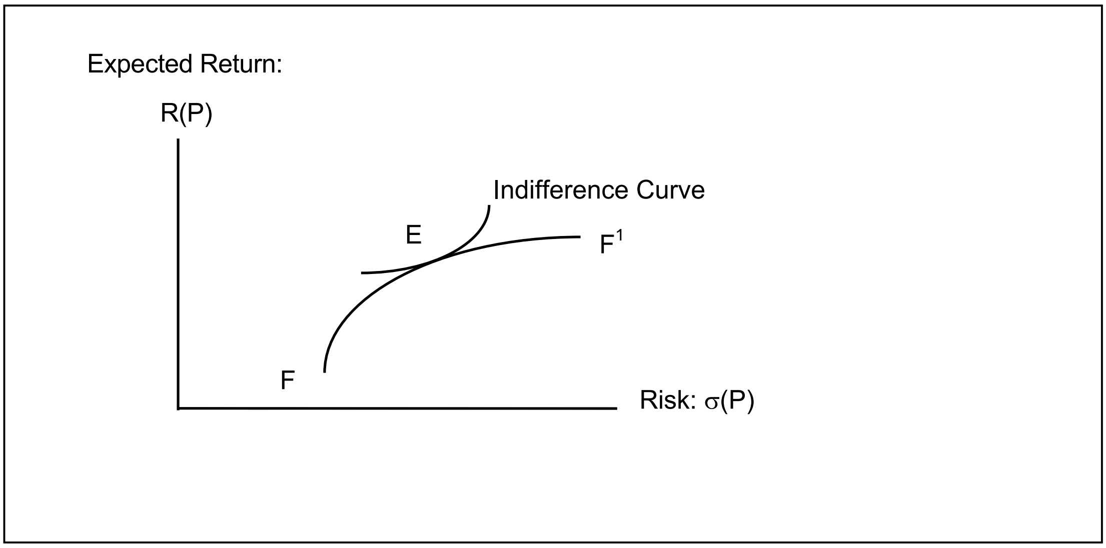


Figure 4.1: the Determination of an Optimum Portfolio: The Multi-Asset Case

Diagrammatically, you will recall that the *optimum* portfolio is determined at the *point* where one of the investor's indifference curves (risk-return profile) is *tangential* to the frontier of efficient portfolios. This portfolio (point E on the curve F-F^1 in Figure 4.1) is optimal because it provides the best combination of risk and return to suit their preferences.

However, apart from the computational difficulty of deriving optimum portfolios using variance-covariance matrix calculations (think 1950s theory without twenty-first century computer technology-software) this policy prescription only concerns *wholly* risky portfolios.

But what if *risk-free* investments (such as government stocks) are included in portfolios? Presumably, investors who are *totally* risk-averse would opt for a *riskless* selection of financial and government securities, including cash. Those who require an element of *liquidity* would construct a *mixed* portfolio that combines risk and risk-free investments to satisfy their needs.

Thus, what we require is a more sophisticated model than that initially offered by Markowitz, whereby the returns on new investments (risk-free or otherwise) can be compared with the risk of the market portfolio.

Fortunately, John Tobin (1958) developed such a model, built on Markowitz efficiency and the *perfect* capital market assumptions that underpin the Separation Theorem of Irving Fisher (1930) (with which you should be familiar).

Tobin demonstrates that in a perfect market where risky financial securities are traded with the option to lend or borrow at a risk-free rate, using risk-free assets, such as government securities.

> Investors and companies need not calculate a multiplicity of covariance terms. All they require is the covariance of a new investment's return with the overall return on the *efficient market portfolio*.

To understand what is now termed Tobin's *Separation Theorem*, suppose every stock market participant invests in all the market's risky securities, with their expenditure in each proportionate to the market's total capitalisation. Every investor's risky portfolio would now correspond to the market portfolio with a market return and market standard deviation, which we shall denote as M, r_m and σ_m, respectively.

Tobin maintains that in *perfect* capital markets that are *efficient*, such an investment strategy is completely rational. In *equilibrium*, security prices will reflect their "true" intrinsic value. In other words, they provide a return commensurate with a degree of risk that justifies their inclusion in the market portfolio. Obviously, if a security's return does not compensate for risk; rational investors will want to sell their holding. But with no takers, price must fall and the yield will rise until the risk-return trade-off once again merits the security's inclusion in the market portfolio. Conversely, excess returns will lead to buying pressure that raises price and depresses yield as the security moves back into equilibrium.

This phenomenon is portrayed in Figure 4.2, where M represents the 100 per cent risky market portfolio, which lies along the efficiency frontier of all risky investment opportunities given by the curve $F\text{-}F^1$.

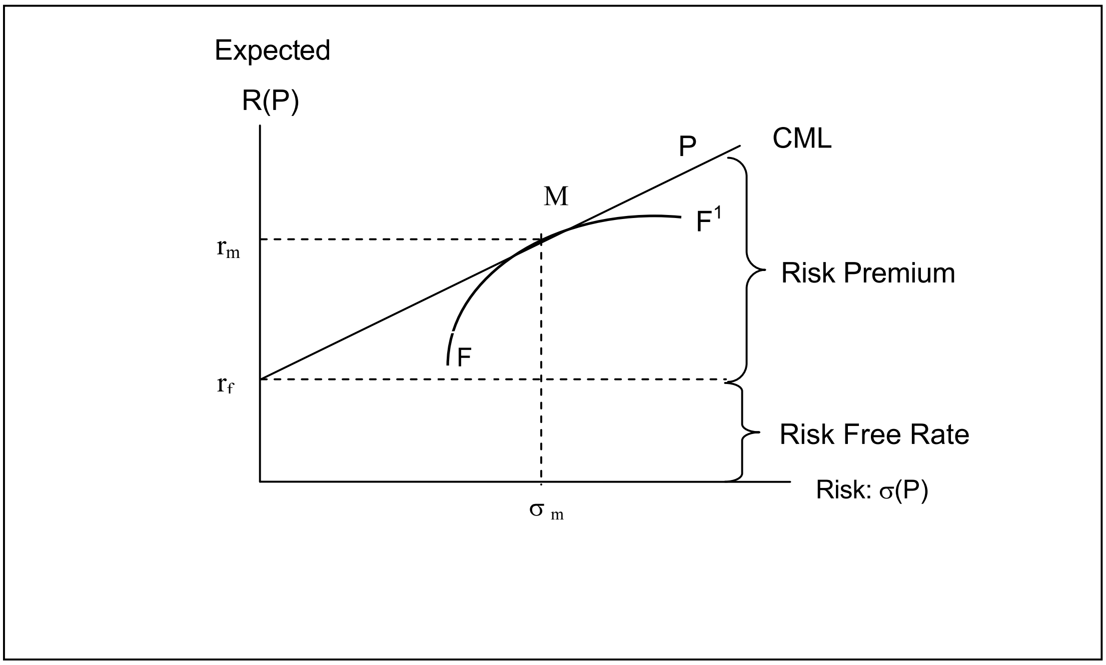


Figure 4.2: The Capital Market Line

Now, assume that all market participants can not only choose risky investments with the return r_m in the market portfolio M. They also have the option of investing in *risk-free* securities (such as short–term government stocks) at a risk-free rate, r_f. According to their aversion to risk and their desire for liquidity, we can now separate their preferences, (hence the term Separation Theorem). Investors may now opt for a *riskless* portfolio, or a *mixed* portfolio, which comprises any preferred combination of risk and risk-free securities.

Diagrammatically, investors can combine the market portfolio with risk-free investments to create a portfolio between r_f and M in Figure 4.2. If a line is drawn from the risk-free return r_f on the vertical axis of our diagram to the point of tangency with the efficiency frontier at point M, it is obvious that part of the original frontier (F-F^1) is now *inefficient*.

Below M, a higher return can be achieved for the same level of risk by combining the market portfolio with risk-free assets. Since r_f denotes a riskless portfolio, the line r_f -M represents increasing proportions of portfolio M combined with a reducing balance of investment at the risk-free rate.

Of course, as Fisher first explained way back in 1930, if capital markets are perfect (where borrowing and lending rates are equal) there is nothing to prevent individuals from borrowing at the risk-free rate to build up their investment portfolios. Tobin therefore adapted this concept to show that if investors could borrow at a risk-free rate and invest more in portfolio M using borrowed funds, they could construct a portfolio beyond M in Figure 4.2.

To show this, the line r_f to M has been extended to point P and beyond to CML. The effect eliminates the remainder of our original efficiency frontier. Any initial efficient portfolios lying along the curve M-F_1 are no longer desirable. With borrowing (leverage) there are always better portfolios with higher returns for the same risk. The line (r_f-M-CML) in Figure 4.2 is a new portfolio "efficiency frontier" for all investors, termed the *Capital Market Line* (CML).

Activity 1

To illustrate the purpose of the CML, let us assume that historically an investment company has *passively* held a market portfolio (M) of risky assets. This fund tracks the London FT-SE 100 (Footsie) on behalf of its clients.

However, with increasing global uncertainty the company now wishes to manage their portfolio *actively*, introducing risk-free investments into the mix and even borrowing funds if necessary.

Using Figure 4.2 for reference, briefly explain how the company's new strategy would redefine its optimum portfolio (or portfolios) if it is willing to borrow up to point P?

The portfolio lending-borrowing line (r_f-M-P) in Figure 4.2 is the new efficiency frontier (CML) for all the company's portfolio constituents. Portfolios lying along the CML between r_f and M are constructed by placing a proportion of their available funds in the market portfolio and the residual in risk-free assets. To establish a portfolio lying halfway up the line r_f-M, the company should divide funds equally between the two.

Portfolios lying along the CML beyond M (for example, P in the diagram) are constructed by placing all their funds in M, plus an amount borrowed at the risk-free rate (r_f). The amount borrowed would equal the ratio of the line r_f - M: M-P.

4.2 The CML and Quantitative Analyses

We have observed diagrammatically that if capital markets are efficient, all rational investors would ideally hold the market portfolio (M) irrespective of their risk attitudes. By finding the point of tangency between the efficiency frontier (F-F^1) and the capital market line (CML) then borrowing or lending at the risk-free rate (r_f) it is also possible for individual investors to achieve a desired balance between risk and return elsewhere on the CML.

Obviously, portfolios whose risk-return characteristics place it below the CML are inefficient and could be improved by altering their composition. It is also possible that an investor might "beat the market" (if only by luck rather than judgement) so that the portfolio's risk-return profile would lie above the CML, making it "super-efficient". However, if markets are efficient without access to insider information (as portfolio theorists assume) then this will be a temporary phenomenon.

Like the work of Fisher and Markowitz before him, Tobin's theorem is another landmark in the development of financial theory, which you ought to read at source. At the very least you need to be able to manipulate the following statistical equations, which are applied to an Exercise in Chapter Four of our companion text (*PTFAE*).

Portfolio Risk

So, let us begin by *redefining* our general portfolio risk formula based on Equation (7) for the standard deviation (which you first encountered in Chapter Two). Combining the market variance of returns (s^2_m) with the variance of risk-free investments (s^2_f):

(27) $$\sigma_p = \sqrt{[x^2\sigma^2_m + (1\text{-}x)^2\,\sigma^2_f + 2x\,(1\text{-}x)\,\sigma_m\,\sigma_f\,\mathrm{COR}\,(_{m,f})]}$$

The first point to note is that because the variability of risk-free returns is obviously zero, their variance (s^2_f) and standard deviation (s_f) equals zero. The second and third terms of Equation (27), which define the variance of the risk-free investment and the correlation coefficient, disappear completely. Thus, Equation (27) for the portfolio's standard deviation simplifies to:

(28) $$\sigma_p = \sqrt{(x^2\,\sigma^2_m)}$$

Rearranging the terms of Equation (28) with only one unknown and simplifying, we can also determine the proportion of funds (x) invested in the market portfolio. Given any investor's preferred portfolio and the market standard deviation of returns (s_p and s_m):

(28) $x = \sigma_p / \sigma_m$

Portfolio Return

In Chapter Two we defined the expected return for a two-asset portfolio R(P) as the weighted average of expected returns from two investments or projects, R(A) and R(B), where the weights are the proportional funds invested in each. Mathematically, this is given by:

(1) $R(P) = x\, R(A) + (1 - x)\, R(B)$

The equation can be adapted to calculate the expected return (r_p) for any portfolio that includes a combination of risky and risk-free investments, whose returns are (r_m) and (r_f) respectively.

(29) $r_p = x\, r_m + (1-x)\, r_f$

The Market Price of Risk or Risk Premium

Because the CML is a *simple linear regression* line, its slope (α_m) is a *constant*, measured by:

(30) $\alpha_m = (r_m - r_f) / \sigma_m$

The expected return for any portfolio on the CML (r_p) can also be expressed as:

(31) $r_p = r_f + [(r_m - r_f) / \sigma_m]\, \sigma_p$

Given r_f (the risk-free rate of return) which is the *intercept* illustrated in Figure 4.2 (where σ_p equals zero) r_m is still the market portfolio return and σ_m and σ_p define market risk and the risk of the particular portfolio, respectively.

The *constant* slope of the CML (α_m) defined by Equation (30) is called the *market price of risk*. It represents the *incremental* return ($r_m - r_f$) obtained by investing in the market portfolio (M) divided by market risk (σ_m). In effect it is the *risk premium* added to the risk-free rate (sketched in Figure 4.2) to establish the total return for any particular portfolio's risk-return trade off.

For example, with a risk premium α_m defined by Equation (30), the incremental return from a portfolio bearing risk (σ_p) in relation to market risk (σ_m) is given by:

$$\alpha_m (\sigma_p - \sigma_m)$$

This can be confirmed if we were to compare a particular portfolio return with that for the market portfolio. The difference between the two (r_p - r_m) equals the market price of risk (α_m) times the spread (σ_p - σ_m).

To *summarise*, the expected return for any efficient portfolio lying on the CML comprising the market portfolio, plus either borrowing or lending at the risk free rate can be expressed by simplifying Equations (30) and (31), so that:

$$(32)\ rp = rf + \alpha_m \,.\, \alpha p$$

In other words, the expected return of an efficient portfolio (r_p) equals the risk-free rate of return (r_f) plus a risk premium ($\alpha_m.\alpha_p$).This premium reflects the market's risk-return trade-off (α_m) combined with the portfolio's own risk (α_p).

4.3 Systematic and Unsystematic Risk

The objective of portfolio diversification is the selection of investment opportunities that reduce *total* portfolio risk without compromising *overall* return. The preceding analysis based on Markowitz efficiency and Tobin's Separation Theorem in perfect capital markets indicates that:

If the standard deviation (risk) of an individual investment is higher than that of the portfolio in which it is held, then part of the standard deviation must have been diversified away through correlation with other portfolio constituents.

A high level of diversification results in rational investors holding the market portfolio, which they will do in combination with lending or borrowing at the risk-free rate. This leaves only the element of risk that is correlated with the market as a whole. In other words portfolio risk equals market risk, which is undiversifiable

$$\sigma_p - \sigma_m$$

To clarify this point for future analysis, Figure 4.3 summarises the relationship between total risk and its component parts where.

Total risk is split between:

- *Systematic* or *market risk*, so called because it is endemic throughout the system (market) and is undiversifiable. It relates to general economic factors that affect all firms and financial securities, and explains why share prices tend to move in sympathy.
- *Unsystematic risk,* sometimes termed *specific*, *residual*, or *unique risk*, relates to specific (unique) economic factors, which impact upon individual industries, companies, securities and projects. It can be eliminated entirely through efficient diversification.

In terms of our earlier analysis, systematic risk measures the extent to which an investment's return moves sympathetically (systematically) with all the financial securities that comprise the market portfolio (the *system*). It describes a particular portfolio's inherent sensitivity to global political and macro-economic volatility. The best recent example, of course, is the 2007 financial meltdown and subsequent economic recession. Because individual companies or investors have no control over such events, they require a rate of return commensurate with their relative systematic risk. The greater this risk, the higher the rate of return required by those with widely diversified portfolios that reflect movements in the market as a whole.

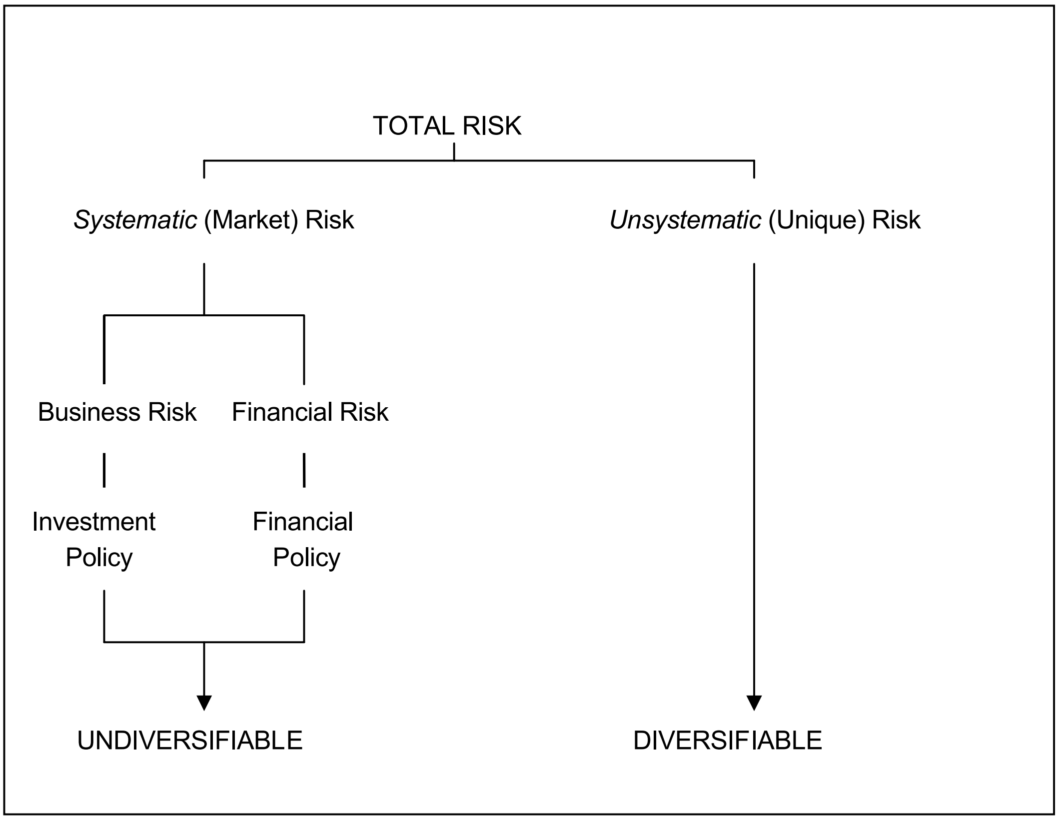


Figure 4.3: The Inter-relationship of Risk Concepts

In contrast, unsystematic risk relates to an individual security's price or even a project and is independent of market risk. Applied to individual companies, it is caused by micro-economic factors such as the level of profitability, product innovation and the quality of financial management. Because it is completely diversifiable (variations in returns cancel out over time) unsystematic risk carries no market premium. Thus, all the risk in a fully diversified portfolio is market or systematic risk.

You may have encountered systematic risk elsewhere in your studies under other names. For example, Figure 4.3 reveals that systematic risk comprises a company's *business risk* and possibly *financial risk.* Certainly if you have read the author's other *SFM* texts, you will recall that business risk reflects the unavoidable variability of project returns according to the nature of the investment (*investment policy*). This may be higher or lower than that for other projects, or the market as a whole. Systematic risk may also reflect a premium for financial risk, which arises from the proportion of debt to equity in a firm's capital structure (gearing) and the amount of dividends paid in relation to the level of retained earnings, (*financial policy*). Of course, there is considerable empirical support for the view that financial risk is irrelevant based on the seminal work of Modigliani-Miller (1958 and 1961) explained in *SFM*. Irrespective of whether financial policies matter, for the moment all we need say is that for all-equity firms with full dividend distribution policies, there is an academic consensus that business risk equals systematic (market) risk and is not diversifiable.

Review Activity

Given our analysis of Markowitz efficiency and the Separation of Tobin, briefly summarise the implications for optimum portfolio management?

Summary and Conclusions

Markowitz, explains how investors or companies can reduce risk but maintain their return by holding more than one investment providing their returns are not positively correlated. This implies that all rational investors will diversify their risky investments into a portfolio.

Tobin illustrates how the introduction of risk-free investments further reduces portfolio risk, using the CML to define a new frontier of efficient portfolios.

Consequently, all investors are capable of eliminating unsystematic risk by expanding their investment portfolios until they reflect the market portfolio.

Based on numerous studies, Figure 4.4 highlights the empirical fact that up to 95 percent of unsystematic risk can be diversified away by randomly increasing the number of investments in a portfolio to about thirty. With one investment, portfolio risk is represented by the sum of unsystematic and systematic risk, i.e. the investment's *total risk* as measured by its standard deviation. When the portfolio constituents reach double figures virtually all the risk associated with holding that portfolio becomes systematic or market risk. See Fisher and Lorie (1970) for one of the earliest and best reviews of the phenomenon.

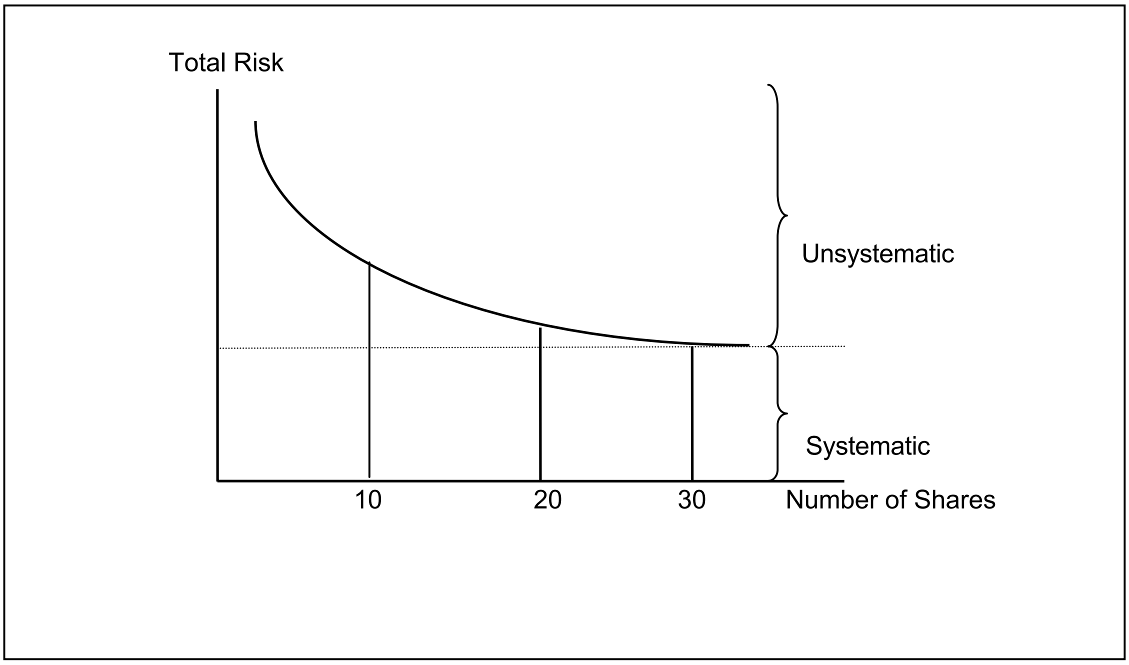


Figure 4.4: Portfolio Risk and Diversification

It should therefore come as no surprise that without to-days computer technology and software to solve their problems:

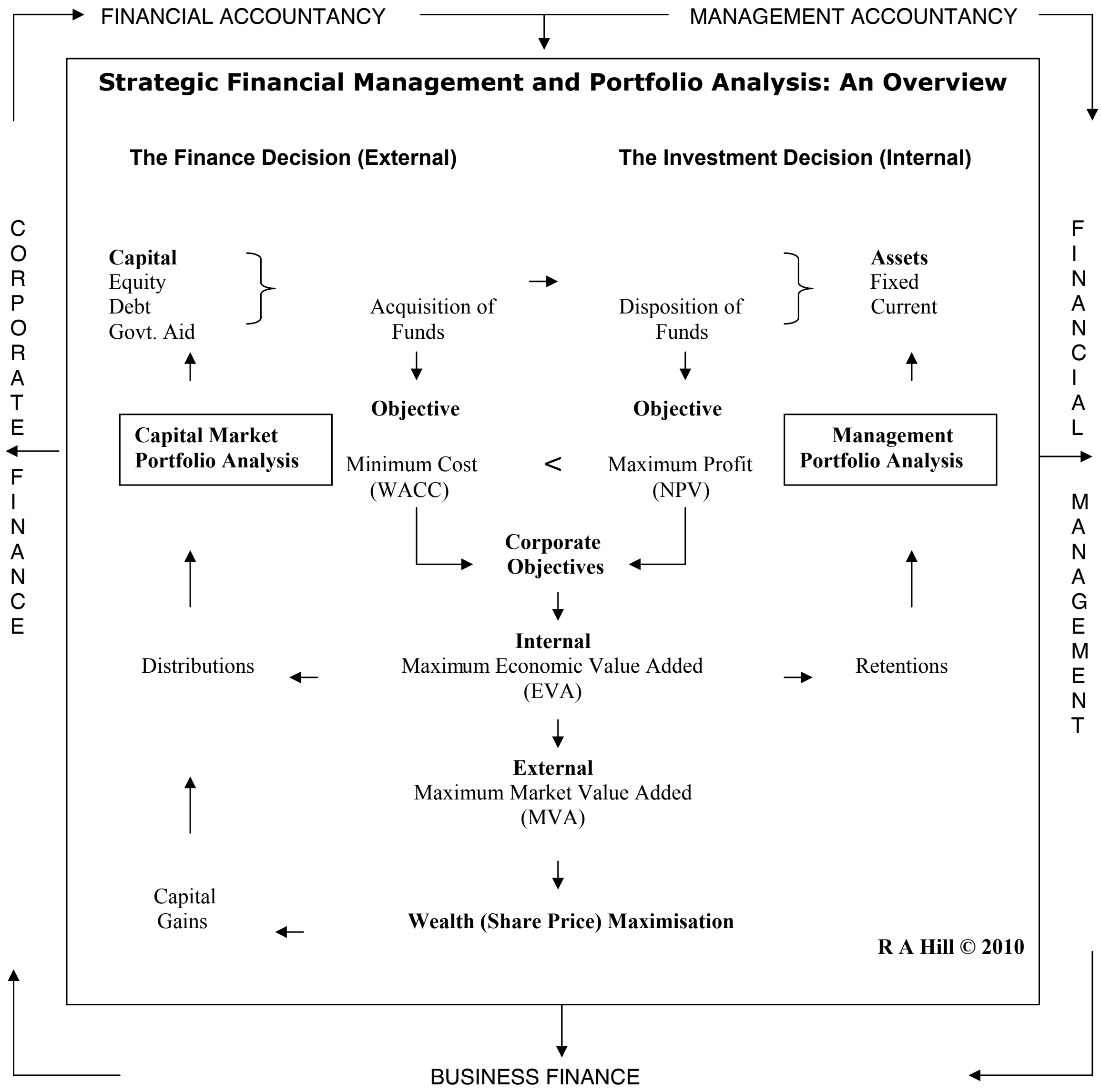
> Academic and financial analysts of the 1960s, requiring a much simpler model than that offered by Markowitz to enable them to diversify efficiently, were quick to appreciate the work of Tobin and the utility of the relationship between the systematic risk of either a project, a financial security, or a portfolio and their returns.

If you wish to explore these inter-relationships and the current status of Modern Portfolio Theory (MPT) based on a simplification of Markowitz efficiency and Tobin's Theorem, you should now download the *bookboon* text "The Capital Asset Pricing Model" 2. edtion, 2014, by the author as a basis for further study.

Selected References

1. Markowitz, H.M., "Portfolio Selection", *The Journal of Finance,* Vol. 13, No. 1, March 1952.
2. Tobin, J., "Liquidity Preferences as Behaviour Towards Risk", *Review of Economic Studies*, February 1958.
3. Fisher, I., *The Theory of Interest,* Macmillan (London), 1930.
4. Modigliani, F. and Miller, M.H., "The Cost of Capital, Corporation Finance and the Theory of Investment", *American Economic Review,* Vol. XLVIII, No. 4, September 1958.
5. Miller, M.H. and Modigliani, F., "Dividend Policy, Growth and the Valuation of Shares", *Journal of Business of the University of Chicago,* Vol. 34, No. 4, October 1961.
6. Fisher, L. and Lorie, J., "Some Studies of Variability of Returns on Investment in Common Stocks", *Journal of Business,* April 1970.
7. Hill, R.A., bookboon.com
   - *Strategic Financial Management (SFM)*, 2009.
   - *Strategic Financial Management: Exercises (SFME)*, 2009.
   - *Portfolio Theory and Financial Analyses (PTFA)*, 2010.
   - *Portfolio Theory and Financial Analyses: Exercises (PTFAE)*, 2010.
   - *The Capital Asset Pricing Model (CAPM)*, 2. edition, 2014.

5 Appendix


FINANCIAL ACCOUNTANCY
MANAGEMENT ACCOUNTANCY
Strategic Financial Management and Portfolio Analysis: An Overview
The Finance Decision (External)
The Investment Decision (Internal)
Capital
Equity
Debt
Govt. Aid
Acquisition of Funds
Disposition of Funds
Assets
Fixed
Current
Objective
Objective
Capital Market Portfolio Analysis
Minimum Cost (WACC)
<
Maximum Profit (NPV)
Management Portfolio Analysis
Corporate Objectives
Internal
Maximum Economic Value Added (EVA)
Distributions
Retentions
External
Maximum Market Value Added (MVA)
Capital Gains
Wealth (Share Price) Maximisation
R A Hill © 2010
CORPORATE FINANCE
FINANCIAL MANAGEMENT
BUSINESS FINANCE