A Quantized Analogue of the Markov-Krein Correspondence

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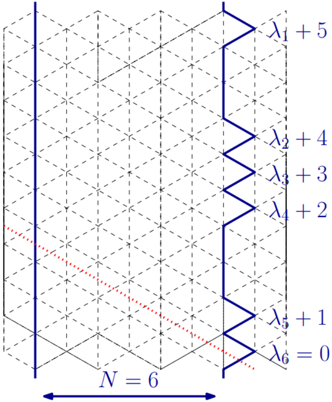
MIT Integrable Probability Seminar

May 11th, 2023

Outline

Random Lozenge Tilings

Fix a sequence of integers $\lambda_1 \geq \cdots \geq \lambda_N$, known as a *signature*. Consider domain enclosed by $x = 0$, and $x = N$ with "spikes" at vertical coordinates $\lambda_i + N - i$.



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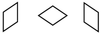
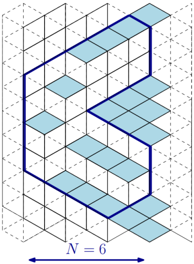
Interested in tilings of this domain with lozenges: 



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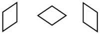
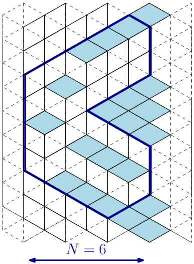
Interested in tilings of this domain with lozenges: 



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Consider the uniform distribution on all tilings. Studied extensively by Petrov.

Gelfand-Tsetlin patterns

Let the vertical positions of the horizontal lozenges on $x = M$ be $\lambda_i^{(M)} + M - i$.

Tilings are in bijection with *Gelfand-Tsetlin Patterns*, i.e. sequences

$$\lambda = \lambda^{(N)} \succ \lambda^{(N-1)} \succ \cdots \succ \lambda^{(1)},$$

where we say $\lambda \succ \mu$ if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_N.$$

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We are interested in the distribution of $\lambda^{(M)} := \pi_{N,M}\lambda$.

Behavior as $N \to \infty$

Consider a sequence of signatures $\lambda(N)$ that converge to a limit shape f:

$$\lim_{N\to\infty} \sum_{j=1}^{N} \left| \frac{\lambda_j(N)}{N} - f(j/N) \right| = 0, \quad \sup_{j,N} \left| \frac{\lambda_j(N)}{N} \right| < \infty$$

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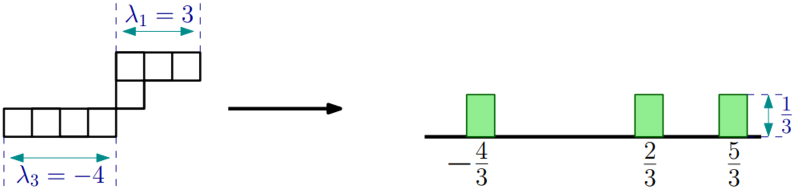
We are interested in the case $M = N - 1$. The limit shape is not interesting as it converges to f. We need to consider some *discrete derivative*, capturing difference between λ and $\pi_{N,N-1}\lambda$.

Counting Measures

Let $\mathbb{GT}_N$ denote the set of signatures of length N. For $\lambda \in \mathbb{GT}_N$

$$m[\lambda] := \frac{1}{N}\sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right)$$



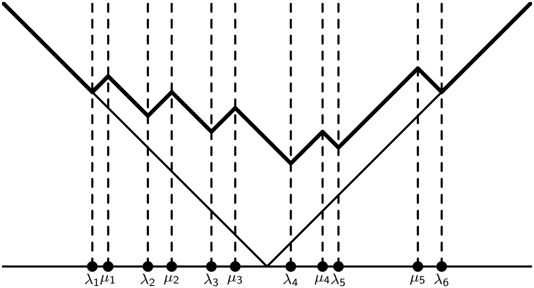
Convergence of $\lambda(N)$ to limit shape implies weak convergence of $m[\lambda(N)]$ to a limit measure $\mathbf{m}$. Call such a sequence of signatures *regular*.

Signed Counting Measure and Continual Young Diagrams

For $\lambda \in \mathbb{GT}_N$ and $\mu \in \mathbb{GT}_{N-1}$, define

$$d[\lambda, \mu] = \sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right) - \sum_{i=1}^{N-1} \delta\left(\frac{\lambda_i + N - 1 - i}{N}\right).$$

If $\lambda \succ \mu$, then we have associated continual Young diagram $w[\lambda, \mu]$, where $d = \frac{1}{2} w''$.



Main Theorem

Our main result (stated informally here) relates the signed measure corresponding to $\lambda(N)$ and $\pi_{N,N-1}\lambda(N)$ to the limit shape of $\lambda(N)$.

Theorem (G., Yao)

Fix a sequence of signatures $\lambda(N)$ converging to some limit shape encoded by $\mathbf{m}$*.*

The random measure $d[\lambda(N), \pi_{N,N-1}\lambda(N)]$ converges weakly to a deterministic measure $\mathbf{d}$ *related to* $\mathbf{m}$ *through*

$$\exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} \int_{\mathbb{R}} x^k \mathbf{d}(dx)\right) = \frac{1}{z}\left(-1 + \exp\left(z \sum_{k=0}^{\infty} z^k \int_{\mathbb{R}} x^k \mathbf{m}(dx)\right)\right).$$

Furthermore, this relation induces a bijection between measures with density in $[0,1]$ to certain continual Young diagrams.

Review of Representation Theory of $\mathrm{U}(N)$

The *branching rule* allows us to understand the distribution of $\pi_{N,M}\lambda$ in terms of λ. We need the representation theory of $\mathrm{U}(N)$ in order to use this.

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Let $\mathrm{U}(N)$ be the group of all unitary matrices. Let T be a representation of $\mathrm{U}(N)$, i.e. a homomorphism

$$T : \mathrm{U}(N) \to GL(V).$$

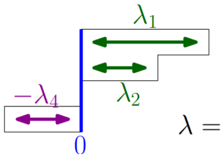
T is *irreducible* if V has no nontrivial subspace W such that $T(U)W \subset W$ for all $U \in \mathrm{U}(N)$.

Classification of Irreducible Representations/Characters

Theorem (Cartan, Weyl, 1920s)

Irreducible representations of $\mathrm{U}(N)$ *are paramaterized by* N*-tuples of integers*

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N).$$



(signatures)

$$\lambda = (5, 3, 0, -3)$$

Probability distribution arising from representation

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Given an arbitrary representation $T : \mathrm{U}(N) \to GL(V)$, say it splits into irreps as

$$V = \bigoplus_{\lambda \in \mathbb{GT}_N} m_\lambda V^\lambda.$$

Define a probability measure $\rho[V]$ on $\mathbb{GT}_N$ by

$$\rho[V](\lambda) = \frac{m_\lambda \dim V^\lambda}{\dim V}.$$

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Theorem (Branching Rule)

The distribution of $\mu = \pi_{N,M}\lambda$ *is given by* $\rho[V^\lambda|_{U(M)}]$.

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The character of V is a function $\chi_V : \mathrm{U}(N) \to \mathbb{C}$ given by

$$\chi_V(U) = \mathrm{Tr}\, T(U).$$

Unitary matrices can be diagonalized by unitary matrices, so χ_λ is only a function of the eigenvalues of U, so can be thought of as a function $(S^1)^N \to \mathbb{C}$.

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Turns out that

$$\chi_{V^\lambda}(u_1, \ldots, u_N) = s_\lambda(u_1, \ldots, u_N) = \frac{\det \left[u_i^{\lambda_j + N - j} \right]_{i,j=1}^N}{\prod_{1 \le i < j \le N} (u_i - u_j)},$$

i.e. the *Schur function* s_λ, which is a symmetric Laurent polynomial.

LLN-appropriate sequences

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Condition on $V(N)$ is that its character is approximately multiplicative as $N \to \infty$.

LLN-appropriate sequences

Definition

The sequence $V(N)$ is said to be LLN-appropriate if if there is a holomorphic function $H(x)$ in the neighborhood of unity such that

$$\chi_{V(N)}(x) = \exp\left(\sum_{i=1}^{N} NH(x_i)\right) T_N(x),$$

where for any fixed k,

$$\lim_{N\to\infty} \frac{1}{N} \log T_N(x_1, \ldots, x_k, 1^{N-k}) \to 0$$

uniformly in some neighborhood of (1^k).

Schur function asymptotics

It was shown by Bufetov and Gorin that if $\lambda(N)$ is a regular sequence of signatures, then $V^{\lambda(N)}$ is an LLN-appropriate sequence.
In particular, there is a Holomorphic function H (dependent on the limit shape), such that for any fixed k,

$$\lim_{N\to\infty} \frac{s_{\lambda(N)}(x_1, \ldots, x_k, 1^{N-k})}{s_\lambda(1^N)} = H(x_1) + \cdots + H(x_k),$$

uniformly in a neighborhood of (1^k).

Main Examples of LLN-appropriate sequences

- Suppose $\lambda^{(1)}(N), \dots, \lambda^{(r)}(N)$ are regular sequences. The following sequence of representations is LLN appropriate:

$$V(N) = V^{\lambda^{(1)}(N)} \otimes \cdots \otimes V^{\lambda^{(r)}(N)}.$$

- Suppose $\lambda(N)$ is a regular sequence, and fix $0 < \alpha < 1$. The following sequence of representations is LLN appropriate:

$$V(N) = V^{\lambda(\lfloor N/\alpha \rfloor)}|_{U(N)}.$$

Main Theorem (fully general version)

Theorem (G., Yao)

Fix an LLN-appropriate sequence of representations $V(N)$. Then, the random measure $m[\lambda]$ converges weakly to a deterministic measure $\mathbf{m}$ *and the random measure $d[\lambda, \pi_{N,N-1}\lambda]$ converges weakly to a deterministic measure* $\mathbf{d}$*, where $\lambda \sim \rho[V(N)]$. The measures are related through*

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The convergence of $m[\lambda]$ was proved by Bufetov and Gorin.

Aside: Quantized Free Convolution

If $\lambda^{(1)}(N)$ and $\lambda^{(2)}(N)$ are regular sequences with limiting counting measure $\mathbf{m}_1$ and $\mathbf{m}_2$, then the above implies that

$$m[V^{\lambda^{(1)}(N)} \otimes V^{\lambda^{(2)}(N)}] \to \mathbf{m}$$

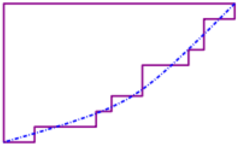
for some measure $\mathbf{m}$.
Bufetov and Gorin explicitly described the map

$$\mathbf{m}_1, \mathbf{m_2} \to \mathbf{m} := \mathbf{m}_1 \otimes \mathbf{m}_2,$$

and coined it quantized free convolution.

Limit regimes for signatures

Consider a sequence of signatures $\lambda(N)$. In our case (the quantized case), the rows grow linearly in N.



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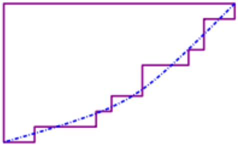

The semiclassical limit is when the rows grow superlinearly in N (studied by Biane, Collins-Sniady).

Basic idea of semiclassical limit

For fixed size unitary group $\mathrm{U}(N)$, we can make a direct connection to RMT by "thickening" the signatures.

In this case, representations of $\mathrm{U}(N)$ (which encode probability distributions on $\mathbb{GT}_N$) become $N \times N$ unitarily invariant random matrices (which encode probability distributions on vectors in $\mathbb{R}^N$).

Character asymptotics in the superlinear regime

Fix N for now, and fix real numbers $a_1 > \cdots > a_N$. Consider thick signatures given by $\lambda_i = \lfloor a_i/\varepsilon \rfloor$, where ε is some small parameter. We probe the normalized character of V^λ near the identity, i.e. at $U = I + i\varepsilon H$ for some Hermitian matrix H.

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This corresponds to analyzing

$$\frac{s_\lambda(x_1, \dots, x_N)}{s_\lambda(1^N)} = \frac{\det \left[x_i^{\lambda_j + N - j} \right]_{i,j=1}^N}{\prod_{i<j}(x_i - x_j) \prod_{i<j} \frac{\lambda_i - i - \lambda_j + j}{j - i}}$$

in the regime $x_i = \exp(i\varepsilon b_i)$ for some real $b_1 > \cdots > b_N$.

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in the regime $x_i = \exp(i\varepsilon b_i)$ for some real $b_1 > \cdots > b_N$. In the limit $\varepsilon \to 0$, this degenerates to

$$\frac{s_\lambda(x_1, \ldots, x_N)}{s_\lambda(1^N)} \to \frac{\det \left[e^{i a_i b_j} \right]_{i,j=1}^N}{\prod_{i<j}(ib_i - ib_j) \prod_{i<j}(a_i - a_j)} \prod_{i<j}(j - i).$$

Harish-Chandra Integral

Let $\mathcal{X}(a_1, \ldots, a_N)$ be the set of all Hermitian $N \times N$ matrices, and consider the uniform measure on this set (conjugation of $\mathrm{diag}(a_1, \ldots, a_N)$ by Haar distributed unitary matrices).

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The Fourier transform of this measure is

$$\int_{A \in \mathcal{X}(a_1,\ldots,a_N)} \exp(\mathrm{Tr}(AB))dA = \frac{\det\left[e^{ia_ib_j}\right]_{i,j=1}^{N}}{\prod_{i<j}(ib_i - ib_j)\prod_{i<j}(a_i - a_j)} \prod_{i<j}(j-i),$$

where $b_1 > \cdots > b_N$ are the eigenvalues of the Hermitian matrix B. This is known as the *Harish-Chandra integral*.

This suggests a heuristic limit to random matrices.

Limit to Random Matrix Theory

- Signatures $\leftrightarrow$ Eigenvalues
- Representations of $\mathrm{U}(N)$ $\leftrightarrow$ Unitarily invariant $N \times N$ random matrices
- Irreps V^λ $\leftrightarrow$ Fixed eigenvalue uniform measure $\mathcal{X}(a_1, \ldots, a_N)$
- Projection operator $\leftrightarrow$ Principal submatrix

To deal with $N \to \infty$ limits, we instead work with

$$m_{\mathrm{RMT}}[\lambda] = \frac{1}{N} \sum_{i=1}^{N} \delta\left(\varepsilon(N) \frac{\lambda_i}{N}\right)$$

where $\varepsilon(N) = o(1)$. This is purely heuristic, see Collins and Sniady for a proper treatment.

Heuristic limit of main result

Heuristically, we expect a random matrix analogue of our result with

$$\mathbf{m}_{\mathrm{RMT}}(dx) = \varepsilon(N)^k \mathbf{m}(dx), \quad \mathbf{d}_{\mathrm{RMT}}(dx) = \varepsilon(N)^k \mathbf{d}(dx),$$

where $\mathbf{m}_{\mathrm{RMT}}$ is the limiting counting measure of the eigenvalues of a unitarily invariant random matrix, and $\mathbf{d}_{\mathrm{RMT}}$ is the limiting signed measure of the eigenvalues and the eigenvalues of a principal $(N-1) \times (N-1)$ matrix.

Heuristic limit of main result

The connection formula then looks like

$$\exp\left(\sum_{k=1}^{\infty}\frac{z^k}{k}\int_{\mathbb{R}}x^k\mathbf{d}_{\mathrm{RMT}}(dx)\right)$$
$$=\frac{1}{\varepsilon(N)z}\left(-1+\exp\left(\varepsilon(N)z\sum_{k=0}^{\infty}z^k\int_{\mathbb{R}}x^k\mathbf{m}_{\mathrm{RMT}}(dx)\right)\right),$$

which in the limit $N\to\infty$ becomes

$$\exp\left(\sum_{k=1}^{\infty}\frac{z^k}{k}\int_{\mathbb{R}}x^k\mathbf{d}_{\mathrm{RMT}}(dx)\right)=\sum_{k=0}^{\infty}z^k\int_{\mathbb{R}}x^k\mathbf{m}_{\mathrm{RMT}}(dx).$$

These are only heuristics!

Folk Theorem in RMT - Analogue of main result

Theorem

Let (M_N) be a sequence of unitarily invariant random matrices such that the spectral measure $\frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}$ converges weakly in probability to a (deterministic) measure $\mathbf{m}_{\mathrm{RMT}}$.

Then, $\sum_{i=1}^{N}\delta_{\lambda_i} - \sum_{i=1}^{N-1}\delta_{\mu_i}$ converges weakly in probability to a signed (deterministic) measure $\mathbf{d}_{\mathrm{RMT}}$ which is related to $\mathbf{m}$ through

$$\exp\left(\sum_{k=1}^{\infty}\frac{z^k}{k}\int_{\mathbb{R}} x^k \mathbf{d}_{\mathrm{RMT}}(dx)\right) = \sum_{k=0}^{\infty} z^k \int_{\mathbb{R}} x^k \mathbf{m}_{\mathrm{RMT}}(dx).$$

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Could be proved replicating our proof replacing Schur functions with Bessel functions, following Sun and Gorin. Was shown recently by Fujie and Hasebe combinatorially using moment method.

Markov-Krein Bijection

Let $\mathcal{M}[a,b]$ denote the set of probability measures supported on the interval $[a,b]$. For any $\mu \in \mathcal{M}[a,b]$, define

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A continual Young diagram is defined to be a function $w : \mathbb{R} \to \mathbb{R}$ that satisfies

- $|w(x_1) - w(x_2)| \leq |x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}$.
- There exists $x_0 \in \mathbb{R}$ such that $w(x) = |x - x_0|$ for sufficiently large x_0.

For an interval $[a,b]$, let $\mathcal{D}[a,b]$ denote the set of continual Young diagrams satisfying $w(x) = |x - x_0|$ for all $x \notin [a,b]$.

For any $w \in \mathcal{D}[a,b]$, define

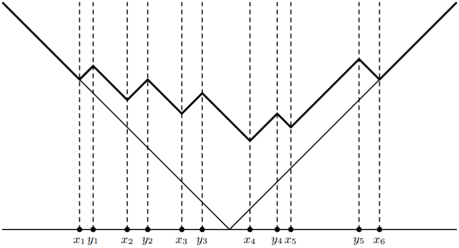
$$p_k(w) = \frac{1}{2}\int_a^b t^k w''(t)dt.$$

Rectangular Young Diagrams

For interlacing sequences (x_i) and (y_i), i.e.

$$x_1 \geq y_1 \geq x_2 \geq \cdots \geq y_{N-1} \geq x_N,$$

let $w^{(x_i),(y_i)}$ denote the *rectangular Young diagram*, as shown below.



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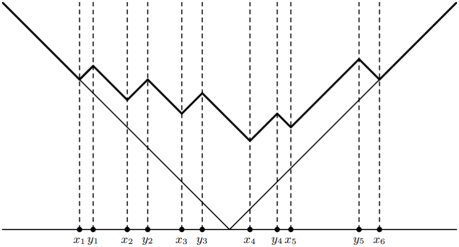

Convergence of $d[\lambda, \mu] \to \mathbf{d}$ implies convergence of $w^{(\lambda_i),(\mu_i)} \to w$ (Bufetov).

Markov-Krein Bijection

Theorem (Krein, Nudelman (1977) and Kerov (1993))

There is a bijective correspondence between $\mathcal{M}[a,b]$ and $\mathcal{D}[a,b]$ such that $\mu \leftrightarrow w$ if and only if

$$\sum_{k=0}^{\infty} \mu_k z^k = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(w)}{k} z^k\right).$$

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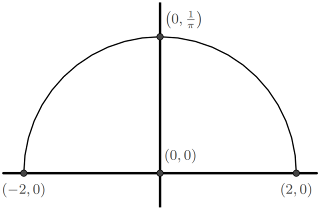
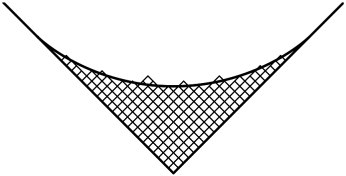
Let w_{RMT} be the continual Young diagram corresponding to $\mathbf{d}_{\mathrm{RMT}}$. Then $\mathbf{m}_{\mathrm{RMT}}$ and w_{RMT} are paired under this bijection.

Markov-Krein Bijection

Using the GUE, we see that semicircle law and VKLS curve are paired under this bijection.





The VKLS curve is the limiting diagram of the Plancherel measure, which assigns probability mass

$$\mathbb{P}(\lambda) = \frac{(f^{\lambda})^2}{N!}$$

to a partition λ of size N.

Qauntized Markov-Krein Bijection

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There is a bijective correspondence between $\widetilde{\mathcal{M}}[a, b]$ and $\widetilde{\mathcal{D}}[a, b]$ such that $\mu \leftrightarrow w$ if and only if

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Let $\mathbf{m}$ be the measure from the main theorem, and let w be the diagram corresponding to $\mathbf{d}$ from the main theorem (i.e. $\mathbf{d} = \frac{1}{2}w''$). Then, $\mathbf{m}$ and w are paired under this bijection.

Semiclassical Limit

The ordinary Markov-Krein bijection is recovered as a semiclassical limit.

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We show that $\hat{w}_\varepsilon \to w$, where $w \in \mathcal{D}[a, b]$ is the ordinary MK pair of μ.

Example of the Qauntized Markov-Krein Bijection

Let $\psi \in \widetilde{\mathcal{M}}$ be the density corresponding to the *one-sided Plancherel character* with parameter γ, which is one of the extreme characters of $U(\infty)$.

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The density of ψ is given by

$$\psi(x) = \frac{1}{\pi} \arccos \frac{x+\gamma}{2\sqrt{\gamma(x+1)}} \quad \text{for } x \in [\gamma - 2\sqrt{\gamma}, \gamma + 2\sqrt{\gamma}].$$

The corresponding diagram is given by

$$w(x) = \frac{2}{\pi} \left((x-\gamma) \arcsin \left(\frac{x-\gamma}{2\sqrt{\gamma}} \right) + \sqrt{4\gamma - (x-\gamma)^2} \right).$$

The diagram is a scaled and shifted version of the VKLS curve.

Example of the Qauntized Markov-Krein Bijection

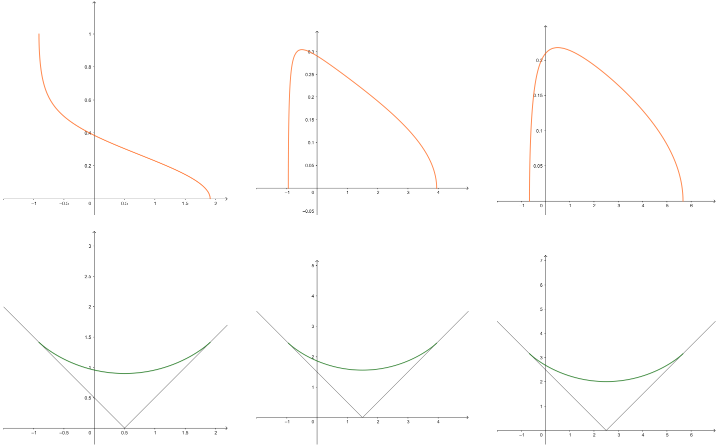

Figure: One-sided Plancherel character limiting measure and corresponding diagram for $\gamma = 0.5, 1.5, 2.5$

Schur Generating Functions

Mostly following *Quantized Free Convolution* by Bufetov and Gorin (2016).

Let ρ be a probability measure on $\mathbb{GT}_N$. It's *Schur Generating Function* (SGF) is

$$S_\rho(x_1, \dots, x_N) = \sum_{\lambda \in \mathbb{GT}_N} \rho(\lambda) \frac{s_\lambda(x_1, \dots, x_N)}{s_\lambda(1^N)}.$$

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For a representation $V = \oplus_\lambda m_\lambda V^\lambda$ of $\mathrm{U}(N)$, we see that

$$S_{\rho[V]}(x) = \sum_{\lambda \in \mathbb{GT}_N} \frac{m_\lambda \dim V^\lambda}{\dim V} \frac{s_\lambda(x)}{s_\lambda(1^N)} = \frac{1}{\dim V} \sum_{\lambda \in \mathbb{GT}_N} m_\lambda s_\lambda(x),$$

which is just the normalized character of V.

Schur Generating Functions

This shows that

$$S_{\rho[V_N|_{\mathrm{U}(N-1)}]}(x_1, \dots, x_{N-1}) = S_{\rho[V_N]}(x_1, \dots, x_{N-1}, 1).$$

In particular,

$$S_{\pi_{N,N-1}\delta_\lambda}(x_1, \dots, x_{N-1}) = \frac{s_\lambda(x_1, \dots, x_{N-1}, 1)}{s_\lambda(1^N)}.$$

Differential Operators

Define a differential operator on functions of $(x_1, \ldots, x_N)$ by

$$\mathcal{D}_{N,k} := \frac{1}{\prod_{1 \le i < j \le N} (x_i - x_j)} \left(\sum_{i=1}^{N} (x_i \partial_i)^k \right) \prod_{1 \le i < j \le N} (x_i - x_j).$$

Key idea is that s_λ is an eigenfunction of this operator:

$$\mathcal{D}_{N,k} s_\lambda(x) = \left(\sum_{i=1}^{N} (\lambda_i + N - i)^k \right) s_\lambda(x).$$

Extracting moments through operators

The above implies that

$$\mathcal{D}_{N,k} S_\rho(x) = \sum_{\lambda \in \mathbb{GT}_N} \frac{\rho(\lambda)}{s_\lambda(1^N)} N^{k+1} \left(\int x^k m[\lambda](dx) \right) s_\lambda(x),$$

so

$$\mathbb{E}\left[\int_{\mathbb{R}} x^k m[\rho](dx) \right] = \frac{1}{N^{k+1}} \mathcal{D}_{N,k} S_\rho(x) \Big|_{x=1^N}.$$

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Using the above SGF projection relation, we also derive

$$\mathbb{E}\left[\int_{\mathbb{R}} x^k d[\rho](dx) \right] = \frac{1}{N^k} (\mathcal{D}_{N,k} - \mathcal{D}_{N-1,k}) S_\rho(x) \Big|_{x=1^N}.$$

Moment method

Given an LLN-appropriate sequence $V(N)$ with associated function $H(x)$, we have explicit formulas for the moments of the limiting measures $\mathbf{m}$ and $\mathbf{d}$:

$$m_k := \int x^k \mathbf{m}(dx) = \frac{1}{k+1} \oint_1 \frac{1}{w} \left(wH'(w) + \frac{w}{w-1} \right)^{k+1} \frac{dw}{2\pi \mathbf{i}}$$

and

$$d_k := \int x^k \mathbf{d}(dx) = \oint_1 \frac{1}{w-1} \left(wH'(w) + \frac{w}{w-1} \right)^{k} \frac{dw}{2\pi \mathbf{i}}.$$

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In order to prove weak convergence to $\mathbf{m}, \mathbf{d}$, it suffices to show that

$$\mathbb{E}\left[\int_{\mathbb{R}} x^k m[\rho[V(N)]](dx) \right] \to m_k$$

and

$$\mathbb{E}\left[\left(\int_{\mathbb{R}} x^k m[\rho[V(N)]](dx) \right)^2 \right] \to m_k^2,$$

and similarly for d_k.

Moment Computations

Given the almost-multiplicative form of the SGFs (due to LLN-appropriateness), the problem reduces to a careful analysis of the leading terms of the operators

$$\mathcal{D}_{N,k}, \mathcal{D}_{N,k} - \mathcal{D}_{N-1,k}, \mathcal{D}_{N,k}^2, (\mathcal{D}_{N,k} - \mathcal{D}_{N-1,k})^2, [\mathcal{D}_{N,k}, \mathcal{D}_{N-1,k}]$$

acting on the function

$$\exp\left(N \sum_{i=1}^{N} H(x_i)\right).$$

We do this by explicitly expanding all the above operators combinatorially using the Leibniz rule, and analyzing the orders of all resulting terms.

Completing the proof

One can show that the formulas provided for m_k and d_k satisfy the QMK relation for any holomorphic H:

$$\exp\left(\sum_{k\geq 1} d_k \frac{z^k}{k}\right) = \frac{1}{z}\left(\exp\left(z\sum_{k\geq 0} z^k m_k\right) - 1\right).$$

The bijection theorem is fairly straightforward to show using integral representation theorems for various measures, similar to the Cauchy-Stietjes transform. All the relevant representation theorems are found in Nudelman and Krein (1977).

Future Directions

- Is there a nice geometric description of $\widetilde{D}[a,b]$? The relation $\frac{1}{u}\exp\left(\sum_{k=1}^{\infty}\frac{p_k(w)}{k}u^{-k}\right) > -1$ for $u \in \mathbb{R} \setminus [a,b]$ is unsatisfying given that $\widetilde{M}[a,b]$ has a nice geometric description.

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- The operator approach should be usable to analyze the fluctuations of the signed measure $d[\rho]$. In the RMT case, the fluctuations were identified with a derivative of the GFF, would be interesting if there are any similar descriptions of the fluctuations of $d[\rho]$.

Summary

- Interested in measures on signatures induced by uniform distribution on GT-patterns.
- Proved a generating function relationship between the limit shape of the top row and the difference of the counting measures of the first and second rows.
- This relation extends to a bijection from measures with density in $[0, 1]$ to certain continual Young diagrams.
- Alluded to similar RMT result on unitarily invariant random matrices using semiclassical limit.

This work can be found at arXiv:2011.10724.

Acknowledgements

Thank you to my mentor Dr. Andrew Ahn for guiding me through this project, and thank you to Prof. Alexei Borodin for useful suggestions regarding the presentation and general guidance!

All figures not generated by me were taken from *Representations of classical Lie groups and quantized free convolution* by Alexey Bufetov and Vadim Gorin (arXiv:1311.5780). Most of the methods were heavily inspired from this work as well.