# A Quantized Analogue of the Markov-Krein Correspondence

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#### Outline

Introduction

Main Result

Rep Theory of U(N)

Technical Restatement

Semiclassical Limit

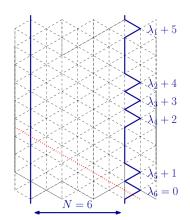
**Bijections** 

Methods

Conclusion

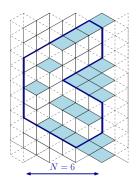
#### Random Lozenge Tilings

Fix a sequence of integers  $\lambda_1 \ge \cdots \ge \lambda_N$ , known as a *signature*. Consider domain enclosed by x=0, and x=N with "spikes" at vertical coordinates  $\lambda_i + N - i$ .



## Random Lozenge Tilings

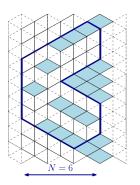
Interested in tilings of this domain with lozenges:  $\Box$   $\diamondsuit$   $\Box$ 



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Combinatorial constraints force there to be exactly M horizontal lozenges on x = M for  $1 \le M \le N$ , and they must interlace with the ones on x = M + 1.

Consider the uniform distribution on all tilings.



## Gelfand-Tsetlin patterns

Let the vertical positions of the horizontal lozenges on x = M be  $\lambda_i^{(M)} + M - i$ .

Tilings are in bijection with *Gelfand-Tsetlin Patterns*, i.e. sequences

$$\lambda = \lambda^{(N)} \succ \lambda^{(N-1)} \succ \cdots \succ \lambda^{(1)},$$

where we say  $\lambda \succ \mu$  if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_N$$
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.

We are interested in the distribution of  $\lambda^{(M)} := \pi_{N,M}\lambda$ .

Consider a sequence of signatures  $\lambda(N)$  that converge to a limit shape f:

$$\lim_{N\to\infty}\sum_{j=1}^N\left|\frac{\lambda_j(N)}{N}-f(j/N)\right|=0,\quad \sup_{j,N}\left|\frac{\lambda_j(N)}{N}\right|<\infty$$

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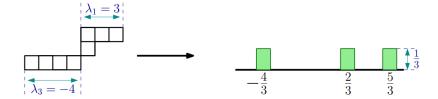
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We are interested in the case M=N-1. The limit shape is not interesting as it converges to f. We need to consider some *discrete derivative*, capturing difference between  $\lambda$  and  $\pi_{N,N-1}\lambda$ .

#### **Counting Measures**

Let  $\mathbb{GT}_N$  denote the set of signatures of length N. For  $\lambda \in \mathbb{GT}_N$ 

$$m[\lambda] := \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right)$$



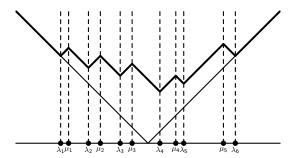
Convergence of  $\lambda(N)$  to limit shape implies weak convergence of  $m[\lambda(N)]$  to a limit measure  $\mathbf{m}$ .

## Signed Counting Measure and Continual Young Diagrams

For  $\lambda \in \mathbb{GT}_N$  and  $\mu \in \mathbb{GT}_{N-1}$ , define

$$d[\lambda,\mu] = \sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right) - \sum_{i=1}^{N-1} \delta\left(\frac{\lambda_i + N - 1 - i}{N}\right).$$

If  $\lambda \succ \mu$ , then we have associated continual Young diagram  $w[\lambda,\mu]$ , where  $d=\frac{1}{2}w''$ .



#### Main Theorem

Our main result (stated informally here) relates the signed measure corresponding to  $\lambda(N)$  and  $\pi_{N,N-1}\lambda(N)$  to the limit shape of  $\lambda(N)$ .

#### Theorem (G., Yao)

Fix a sequence of signatures  $\lambda(N)$  converging to some limit shape encoded by  $\mathbf{m}$ .

The random measure  $d[\lambda(N), \pi_{N,N-1}\lambda(N)]$  converges weakly to a deterministic measure  $\mathbf{d}$  related to  $\mathbf{m}$  through

$$\exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} \int_{\mathbb{R}} x^k \mathbf{d}(dx)\right) = \frac{1}{z} \left(-1 + \exp\left(z \sum_{k=0}^{\infty} z^k \int_{\mathbb{R}} x^k \mathbf{m}(dx)\right)\right).$$

Furthermore, this relation induces a bijection between measures with density in [0,1] to certain continual Young diagrams.

# Review of Representation Theory of U(N)

The branching rule allows us to understand the distribution of  $\pi_{N,M}\lambda$  in terms of  $\lambda$ . We need the representation theory of  $\mathrm{U}(N)$  in order to use this.

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The branching rule allows us to understand the distribution of  $\pi_{N,M}\lambda$  in terms of  $\lambda$ . We need the representation theory of  $\mathrm{U}(N)$  in order to use this.

Let U(N) be the group of all unitary matrices. Let T be a representation of U(N), i.e. a homomorphism

$$T: \mathrm{U}(N) \to \mathit{GL}(V).$$

T is *irreducible* if V has no nontrivial subspace W such that  $T(U)W \subset W$  for all  $U \in \mathrm{U}(N)$ .

## Classification of Irreducible Representations/Characters

#### Theorem (Cartan, Weyl, 1920s)

Irreducible representations of  $\mathrm{U}(N)$  are paramaterized by N-tuples of integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N).$$

$$\lambda_1$$
 (signatures) 
$$\lambda_2$$
 
$$\lambda = (5, 3, 0, -3)$$

## Probability distribution arising from representation

Let  $T_{\lambda}: \mathrm{U}(N) \to \mathit{GL}(V^{\lambda})$  be the irreducible representation corresponding to a signature  $\lambda \in \mathbb{GT}_N$ .

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Given an arbitrary representation  $T:\mathrm{U}(N)\to GL(V)$ , say it splits into irreps as

$$V = \bigoplus_{\lambda \in \mathbb{GT}_N} m_{\lambda} V^{\lambda}.$$

Define a probability measure  $\rho[V]$  on  $\mathbb{GT}_N$  by

$$\rho[V](\lambda) = \frac{m_{\lambda} \dim V^{\lambda}}{\dim V}.$$

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#### Theorem (Branching Rule)

The distribution of  $\mu = \pi_{N,M}\lambda$  is given by  $\rho[V^{\lambda}|_{U(M)}]$ .

# Characters of representations of U(N)

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The character of V is a function  $\chi_V:\mathrm{U}(N)\to\mathbb{C}$  given by

$$\chi_V(U) = \operatorname{Tr} T(U).$$

Unitary matrices can be diagonalized by unitary matrices, so  $\chi_{\lambda}$  is only a function of the eigenvalues of U, so can be thought of as a function  $(S^1)^N \to \mathbb{C}$ .

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Turns out that

$$\chi_{V^{\lambda}}(u_1,\ldots,u_N) = s_{\lambda}(u_1,\ldots,u_N) = \frac{\det\left[u_i^{\lambda_j+N-j}\right]_{i,j=1}^N}{\prod_{1\leq i< j\leq N}(u_i-u_j)},$$

i.e. the *Schur function*  $s_{\lambda}$ , which is a symmetric Laurent polynomial.



#### LLN-appropriate sequences

Call a sequence  $\lambda(N)$  regular if it approaches a limit shape. Want top row to also be random, instead of deterministic. So instead look at sequence of representations V(N) of  $\mathrm{U}(N)$ , so sequence  $\rho[V(N)]$  of probability measures on  $\mathbb{GT}_N$ .

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Condition on V(N) is that its character is approximately multiplicative as  $N \to \infty$ .

#### LLN-appropriate sequences

#### Definition

The sequence V(N) is said to be LLN-appropriate if if there is a holomorphic function H(x) in the neighborhood of unity such that

$$\chi_{V(N)}(x) = \exp\left(\sum_{i=1}^{N} NH(x_i)\right) T_N(x),$$

where for any fixed k,

$$\lim_{N\to\infty}\frac{1}{N}\log T_N(x_1,\ldots,x_k,1^{N-k})\to 0$$

uniformly in some neighborhood of  $(1^k)$ .

## Schur function asymptotics

It was shown by Bufetov and Gorin that if  $\lambda(N)$  is a regular sequence of signatures, then  $V^{\lambda(N)}$  is an LLN-appropriate sequence.

In particular, there is a Holomorphic function H (dependent on the limit shape), such that for any fixed k,

$$\lim_{N\to\infty}\frac{s_{\lambda(N)}(x_1,\ldots,x_k,1^{N-k})}{s_{\lambda}(1^N)}=H(x_1)+\cdots+H(x_k),$$

uniformly in a neighborhood of  $(1^k)$ .

## Main Examples of LLN-appropriate sequences

• Suppose  $\lambda^{(1)}(N), \dots, \lambda^{(r)}(N)$  are regular sequences. The following sequence of representations is LLN appropriate:

$$V(N) = V^{\lambda^{(1)}(N)} \otimes \cdots \otimes V^{\lambda^{(r)}(N)}.$$

• Suppose  $\lambda(N)$  is a regular sequence, and fix  $0 < \alpha < 1$ . The following sequence of representations is LLN appropriate:

$$V(N) = V^{\lambda(\lfloor N/\alpha \rfloor)}|_{U(N)}.$$

# Main Theorem (fully general version)

## Theorem (G., Yao)

Fix an LLN-appropriate sequence of representations V(N). Then, the random measure  $m[\lambda]$  converges weakly to a deterministic measure  $\mathbf{m}$  and the random measure  $\mathbf{d}[\lambda, \pi_{N,N-1}\lambda]$  converges weakly to a deterministic measure  $\mathbf{d}$ , where  $\lambda \sim \rho[V(N)]$ . The measures are related through

$$\exp\left(\sum_{k=1}^{\infty}\frac{z^k}{k}\int_{\mathbb{R}}x^k\mathbf{d}(dx)\right) = \frac{1}{z}\left(-1 + \exp\left(z\sum_{k=0}^{\infty}z^k\int_{\mathbb{R}}x^k\mathbf{m}(dx)\right)\right).$$

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The convergence of  $m[\lambda]$  was proved by Bufetov and Gorin.

#### Aside: Quantized Free Convolution

If  $\lambda^{(1)}(N)$  and  $\lambda^{(2)}(N)$  are regular sequences with limiting counting measure  $\mathbf{m}_1$  and  $\mathbf{m}_2$ , then the above implies that

$$m[V^{\lambda^{(1)}(N)} \otimes V^{\lambda^{(2)}(N)}] \rightarrow \mathbf{m}$$

for some measure m.

Bufetov and Gorin explicitly described the map

$$\textbf{m}_1,\textbf{m}_2\to\textbf{m}:=\textbf{m}_1\otimes\textbf{m}_2,$$

and coined it quantized free convolution.

#### Limit regimes for signatures

Consider a sequence of signatures  $\lambda(N)$ . In our case (the quantized case), the rows grow linearly in N.



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The semiclassical limit is when the rows grow superlinearly in N (studied by Biane, Collins-Sniady).



## Character asymptotics in the superlinear regime

Fix N for now, and fix real numbers  $a_1 > \cdots > a_N$ . Consider thick signatures given by  $\lambda_i = \lfloor a_i/\varepsilon \rfloor$ , where  $\varepsilon$  is some small parameter. We probe the normalized character of  $V^{\lambda}$  near the identity, i.e. at  $U = I + i\varepsilon H$  for some Hermitian matrix H.

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This corresponds to analyzing

$$\frac{s_{\lambda}(x_1,\ldots,x_N)}{s_{\lambda}(1^N)} = \frac{\det\left[x_i^{\lambda_j+N-j}\right]_{i,j=1}^N}{\prod_{i< j}(x_i-x_j)\prod_{i< j}\frac{\lambda_i-i-\lambda_j+j}{j-i}}$$

in the regime  $x_i = \exp(i\varepsilon b_i)$  for some real  $b_1 > \cdots > b_N$ .

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in the regime  $x_i = \exp(i\varepsilon b_i)$  for some real  $b_1 > \cdots > b_N$ . In the limit  $\varepsilon \to 0$ , this degenerates to

$$\frac{s_{\lambda}(x_1,\ldots,x_N)}{s_{\lambda}(1^N)} \to \frac{\det\left[e^{ia_ib_j}\right]_{i,j=1}^N}{\prod_{i< j}(ib_i-ib_j)\prod_{i< j}(a_i-a_j)}\prod_{i< i}(j-i).$$

# Harish-Chandra Integral

Let  $\mathcal{X}(a_1,\ldots,a_N)$  be the set of all Hermitian  $N\times N$  matrices, and consider the uniform measure on this set (conjugation of  $\operatorname{diag}(a_1,\ldots,a_N)$  by Haar distributed unitary matrices).

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The Fourier transform of this measure is

$$\int_{A \in \mathcal{X}(a_1, \dots, a_N)} \exp(\operatorname{Tr}(AB)) dA = \frac{\det \left[e^{ia_ib_j}\right]_{i,j=1}^N}{\prod_{i < j} (ib_i - ib_j) \prod_{i < j} (a_i - a_j)} \prod_{i < j} (j - i),$$

where  $b_1 > \cdots > b_N$  are the eigenvalues of the Hermitian matrix B. This is known as the *Harish-Chandra integral*.

This suggests a heuristic limit to random matrices.

## Limit to Random Matrix Theory

- Signatures  $\leftrightarrow$  Eigenvalues
- Representations of U(N) → Unitarily invariant N × N random matrices
- Irreps  $V^{\lambda} \leftrightarrow \mathsf{Fixed}$  eigenvalue uniform measure  $\mathcal{X}(a_1,\ldots,a_N)$
- Projection operator → Principal submatrix

To deal with  $N o \infty$  limits, we instead work with

$$m_{\text{RMT}}[\lambda] = \frac{1}{N} \sum_{i=1}^{N} \delta\left(\varepsilon(N) \frac{\lambda_i}{N}\right)$$

where  $\varepsilon(N) = o(1)$ . This is purely heuristic, see Collins and Sniady for a proper treatment.

#### Heurstic limit of main result

Heuristically, we expect a random matrix analogue of our result with

$$\mathbf{m}_{\mathrm{RMT}}(dx) = \varepsilon(N)^k \mathbf{m}(dx), \quad \mathbf{d}_{\mathrm{RMT}}(dx) = \varepsilon(N)^k \mathbf{d}(dx),$$

where  $\mathbf{m}_{\mathrm{RMT}}$  is the limiting counting measure of the eigenvalues of a unitarily invariant random matrix, and  $\mathbf{d}_{\mathrm{RMT}}$  is the limiting signed measure of the eigenvalues and the eigenvalues of a principal  $(N-1)\times(N-1)$  matrix.

#### Herustic limit of main result

The connection formula then looks like

$$\exp\left(\sum_{k=1}^{\infty} \frac{z^{k}}{k} \int_{\mathbb{R}} x^{k} \mathbf{d}_{\mathrm{RMT}}(dx)\right)$$

$$= \frac{1}{\varepsilon(N)z} \left(-1 + \exp\left(\varepsilon(N)z \sum_{k=0}^{\infty} z^{k} \int_{\mathbb{R}} x^{k} \mathbf{m}_{\mathrm{RMT}}(dx)\right)\right),$$

which in the limit  $N \to \infty$  becomes

$$\exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} \int_{\mathbb{R}} x^k \mathbf{d}_{\mathrm{RMT}}(dx)\right) = \sum_{k=0}^{\infty} z^k \int_{\mathbb{R}} x^k \mathbf{m}_{\mathrm{RMT}}(dx).$$

These are only heuristics!

# Folk Theorem in RMT - Analogue of main result

#### **Theorem**

Let  $(M_N)$  be a sequence of unitarily invariant random matrices such that the spectral measure  $\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$  converges weakly in probability to a (deterministic) measure  $\mathbf{m}_{\mathrm{RMT}}$ .

Then,  $\sum_{i=1}^{N} \delta_{\lambda_i} - \sum_{i=1}^{N-1} \delta_{\mu_i}$  converges weakly in probability to a signed (deterministic) measure  $\mathbf{d}_{\mathrm{RMT}}$  which is related to  $\mathbf{m}$  through

$$\exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} \int_{\mathbb{R}} x^k \mathbf{d}_{\mathrm{RMT}}(dx)\right) = \sum_{k=0}^{\infty} z^k \int_{\mathbb{R}} x^k \mathbf{m}_{\mathrm{RMT}}(dx).$$

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Could be proved replicating our proof replacing Schur functions with Bessel functions, following Sun and Gorin. Was shown recently by Fujie and Hasebe combinatorially using moment method.

Let  $\mathcal{M}[a,b]$  denote the set of probability measures supported on the interval [a,b]. For any  $\mu \in \mathcal{M}[a,b]$ , define

$$\mu_k = \int_{-\infty}^{\infty} t^k d\mu(t).$$

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A continual Young diagram is defined to be a function  $w: \mathbb{R} \to \mathbb{R}$  that satisfies

- $|w(x_1) w(x_2)| \le |x_1 x_2|$  for all  $x_1, x_2 \in \mathbb{R}$ .
- There exists  $x_0 \in \mathbb{R}$  such that  $w(x) = |x x_0|$  for sufficiently large  $x_0$ .

For an interval [a, b], let  $\mathcal{D}[a, b]$  denote the set of continual Young diagrams satisfying  $w(x) = |x - x_0|$  for all  $x \notin [a, b]$ .

For any  $w \in \mathcal{D}[a, b]$ , define

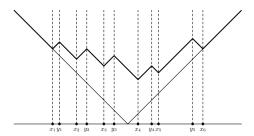
$$p_k(w) = \frac{1}{2} \int_a^b t^k w''(t) dt.$$

# Rectangular Young Diagrams

For interlacing sequences  $(x_i)$  and  $(y_i)$ , i.e.

$$x_1 \geq y_1 \geq x_2 \geq \cdots \geq y_{N-1} \geq x_N$$

let  $w^{(x_i),(y_i)}$  denote the *rectangular Young diagram*, as shown below.

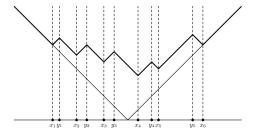


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Convergence of  $d[\lambda, \mu] \to \mathbf{d}$  implies convergence of  $w^{(\lambda_i),(\mu_i)} \to w$  (Bufetov).



# Theorem (Krein, Nudelman (1977) and Kerov (1993))

There is a bijective correspondence between  $\mathcal{M}[a,b]$  and  $\mathcal{D}[a,b]$  such that  $\mu \leftrightarrow w$  if and only if

$$\sum_{k=0}^{\infty} \mu_k z^k = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(w)}{k} z^k\right).$$

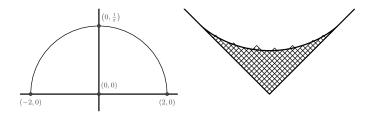
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Let  $w_{\rm RMT}$  be the continual Young diagram corresponding to  $\mathbf{d}_{\rm RMT}$ . Then  $\mathbf{m}_{\rm RMT}$  and  $w_{\rm RMT}$  are paired under this bijection.

Using the GUE, we see that semicircle law and VKLS curve are paired under this bijection.



The VKLS curve is the limiting diagram of the Plancherel measure, which assigns probability mass

$$\mathbb{P}(\lambda) = \frac{(f^{\lambda})^2}{\mathsf{N}!}$$

to a partition  $\lambda$  of size N.



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Theorem (G., Yao)

There is a bijective correspondence between  $\mathcal{M}[a,b]$  and  $\mathcal{D}[a,b]$  such that  $\mu \leftrightarrow w$  if and only if

$$\frac{1}{z}\left(-1+\exp\left(z\sum_{k=0}^{\infty}\psi_kz^k\right)\right)=\exp\left(\sum_{k=1}^{\infty}\frac{p_k(w)}{k}z^k\right).$$

### Theorem (G., Yao)

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Let  $\mathbf{m}$  be the measure from the main theorem, and let w be the diagram corresponding to  $\mathbf{d}$  from the main theorem (i.e.  $\mathbf{d} = \frac{1}{2}w''$ ). Then,  $\mathbf{m}$  and w are paired under this bijection.

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Let  $w_{\varepsilon} \in \widetilde{\mathcal{D}}[\mathsf{a}, \mathsf{b}]$  be the QMK pair of  $\psi_{\varepsilon}$ , and define  $d\hat{w}_{\varepsilon}(t) = w_{\varepsilon}(t/\varepsilon)$ .

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We show that  $\hat{w}_{\varepsilon} \to w$ , where  $w \in \mathcal{D}[a, b]$  is the ordinary MK pair of  $\mu$ .

# Example of the Qauntized Markov-Krein Bijection

Let  $\psi \in \widetilde{\mathcal{M}}$  be the density corresponding to the *one-sided* Plancherel character with parameter  $\gamma$ , which is one of the extreme characters of  $U(\infty)$ .

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The density of  $\psi$  is given by

$$\psi(x) = \frac{1}{\pi} \arccos \frac{x + \gamma}{2\sqrt{\gamma(x+1)}}$$
 for  $x \in [\gamma - 2\sqrt{\gamma}, \gamma + 2\sqrt{\gamma}]$ .

The corresponding diagram is given by

$$w(x) = \frac{2}{\pi} \left( (x - \gamma) \arcsin \left( \frac{x - \gamma}{2\sqrt{\gamma}} \right) + \sqrt{4\gamma - (x - \gamma)^2} \right).$$

The diagram is a scaled and shifted version of the VKLS curve.

# Example of the Qauntized Markov-Krein Bijection

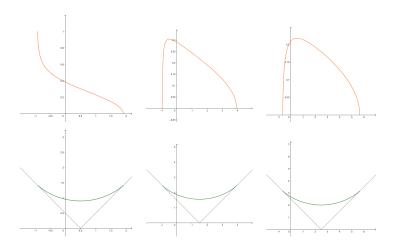


Figure: One-sided Plancherel character limiting measure and corresponding diagram for  $\gamma = 0.5, 1.5, 2.5$ 



# Schur Generating Functions

Mostly following *Quantized Free Convolution* by Bufetov and Gorin (2016).

Let  $\rho$  be a probability measure on  $\mathbb{GT}_N$ . It's *Schur Generating Function* (SGF) is

$$S_{\rho}(x_1,\ldots,x_N) = \sum_{\lambda \in \mathbb{GT}_N} \rho(\lambda) \frac{s_{\lambda}(x_1,\ldots,x_N)}{s_{\lambda}(1^N)}.$$

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For a representation  $V=\oplus_{\lambda}m_{\lambda}V^{\lambda}$  of  $\mathrm{U}(\mathit{N})$ , we see that

$$S_{\rho[V]}(x) = \sum_{\lambda \in \mathbb{GT}_N} \frac{m_\lambda \dim V^\lambda}{\dim V} \frac{s_\lambda(x)}{s_\lambda(1^N)} = \frac{1}{\dim V} \sum_{\lambda \in \mathbb{GT}_N} m_\lambda s_\lambda(x),$$

which is just the normalized character of V.

# Schur Generating Functions

This shows that

$$S_{\rho[V_N|_{\mathrm{U}(N-1)}]}(x_1,\ldots,x_{N-1})=S_{\rho[V_N]}(x_1,\ldots,x_{N-1},1).$$

In particular,

$$S_{\pi_{N,N-1}\delta_{\lambda}}(x_1,\ldots,x_{N-1})=\frac{s_{\lambda}(x_1,\ldots,x_{N-1},1)}{s_{\lambda}(1^N)}.$$

### **Differential Operators**

Define a differential operator on functions of  $(x_1, \ldots, x_N)$  by

$$\mathcal{D}_{N,k} := \frac{1}{\prod\limits_{1 \leq i < j \leq N} (x_i - x_j)} \left( \sum_{i=1}^N (x_i \partial_i)^k \right) \prod\limits_{1 \leq i < j \leq N} (x_i - x_j).$$

Key idea is that  $s_{\lambda}$  is an eigenfunction of this operator:

$$\mathcal{D}_{N,k}s_{\lambda}(x) = \left(\sum_{i=1}^{N} (\lambda_i + N - i)^k\right) s_{\lambda}(x).$$

## Extracting moments through operators

The above implies that

$$\mathcal{D}_{N,k}S_{\rho}(x) = \sum_{\lambda \in \mathbb{GT}_N} \frac{\rho(\lambda)}{s_{\lambda}(1^N)} N^{k+1} \left( \int x^k m[\lambda](dx) \right) s_{\lambda}(x),$$

SO

$$\mathbb{E}\left[\int_{\mathbb{R}} x^k m[\rho](dx)\right] = \left.\frac{1}{N^{k+1}} \mathcal{D}_{N,k} S_{\rho}(x)\right|_{x=1^N}.$$

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Using the above SGF projection relation, we also derive

$$\mathbb{E}\left[\int_{\mathbb{R}} x^k d[\rho](dx)\right] = \frac{1}{N^k} (\mathcal{D}_{N,k} - \mathcal{D}_{N-1,k}) S_{\rho}(x) \bigg|_{x=1^N}.$$

#### Moment method

Given an LLN-appropriate sequence V(N) with associated function H(x), we have explicit formulas for the moments of the limiting measures  $\mathbf{m}$  and  $\mathbf{d}$ :

$$m_k := \int x^k \mathbf{m}(dx) = \frac{1}{k+1} \oint_1 \frac{1}{w} \left( wH'(w) + \frac{w}{w-1} \right)^{k+1} \frac{dw}{2\pi \mathbf{i}}$$

and

$$d_k := \int x^k \mathbf{d}(dx) = \oint_1 \frac{1}{w-1} \left( wH'(w) + \frac{w}{w-1} \right)^k \frac{dw}{2\pi \mathbf{i}}.$$

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In order to prove weak convergence to m, d, it suffices to show that

$$\mathbb{E}\left[\int_{\mathbb{R}} x^k m[\rho[V(N)]](dx)\right] \to m_k$$

and

$$\mathbb{E}\left[\left(\int_{\mathbb{R}} x^k m[\rho[V(N)]](dx)\right)^2\right] \to m_k^2,$$

and similarly for  $d_k$ .



# Moment Computations

Given the almost-multiplicative form of the SGFs (due to LLN-appropriateness), the problem reduces to a careful analysis of the leading terms of the operators

$$\mathcal{D}_{N,k}, \mathcal{D}_{N,k} - \mathcal{D}_{N-1,k}, \mathcal{D}_{N,k}^2, (\mathcal{D}_{N,k} - \mathcal{D}_{N-1,k})^2, [\mathcal{D}_{N,k}, \mathcal{D}_{N-1,k}]$$

acting on the function

$$\exp\left(N\sum_{i=1}^N H(x_i)\right)$$
.

We do this by explicitly expanding all the above operators combinatorially using the Leibniz rule, and analyzing the orders of all resulting terms.

# Completing the proof

One can show that the formulas provided for  $m_k$  and  $d_k$  satisfy the QMK relation for any holomorphic H:

$$\exp\left(\sum_{k\geq 1} d_k \frac{z^k}{k}\right) = \frac{1}{z} \left(\exp\left(z\sum_{k\geq 0} z^k m_k\right) - 1\right).$$

The bijection theorem is fairly straightforward to show using integral representation theorems for various measures, similar to the Cauchy-Stietjes transform. All the relevant representation theorems are found in Nudelman and Krein (1977).

#### **Future Directions**

• Is there a nice geometric description of  $\widetilde{D}[a,b]$ ? The relation  $\frac{1}{u}\exp\left(\sum_{k=1}^{\infty}\frac{p_k(w)}{k}u^{-k}\right)>-1$  for  $u\in\mathbb{C}\setminus[a,b]$  is unsatisfying given that  $\widetilde{M}[a,b]$  has a nice geometric description.

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- In the RMT result,  $m[\lambda] \to \mathbf{m}$  is a sufficient condition to show  $d[\lambda] \to \mathbf{d}$ . In the quantized case,  $m[\lambda] \to \mathbf{m}$  is suspected to be weaker than LLN-appropriate, so it would be interesting if the assumption of LLN-appropriateness could be relaxed to just  $m[\lambda] \to \mathbf{m}$ . I as of now don't know any explicit counterexamples to  $[m[\lambda] \to \mathbf{m}] \Longrightarrow \text{LLN-appropriate}$ .

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- The operator approach should be usable to analyze the fluctuations of the signed measure  $d[\rho]$ . In the RMT case, the fluctuations were identified with a derivative of the GFF, would be interesting if there are any similar descriptions of the fluctuations of  $d[\rho]$ .

# Summary

- Interested in measures on signatures induced by uniform distribution on GT-patterns.
- Proved a generating function relationship between the limit shape of the top row and the difference of the counting measures of the first and second rows.
- This relation extends to a bijection from measures with density in [0, 1] to certain continual Young diagrams.
- Alluded to similar RMT result on unitarily invariant random matrices using semiclassical limit.

This work can be found at arXiv:2011.10724.

### Acknowledgements

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All figures not generated by me were taken from *Representations* of classical Lie groups and quantized free convolution by Alexey Bufetov and Vadim Gorin (arXiv:1311.5780). Most of the methods were heavily inspired from this work as well.