UTAH STATE UNIVERSITY, COMPUTER SCIENCE DEPARTMENT

CS 7910 Computational Complexity
Assignment 3

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1. In class we have studied a polynomial-time reduction from 3SAT to Clique and a polynomial-time reduction from 3SAT to Independent Set. Give a polynomial-time reduction from Clique to Independent Set.

   The clique problem is that given a graph G and an integer k, we need to find out if it is possible to create a subgraph G_k with k vertices where every vertex is connected to every other vertex through an edge. Such a graph that we construct is called a Clique.

   In order to construct an Independent Set (IS) from a Clique, we can construct a graph G' with equivalent vertices as in G. However the graph G' we construct will not have any edges wherever G will have edges. And G' will have edges between nodes wherever G does not have edges. Corresponding to subgraph G_k of graph G, we can find the subgraph ${G_k}'$ of graph G'.

   If the graph G has n vertices, the graph G' we construct will have the same number of vertices. So we can create the graph in $O(n)$ time. Since we can construct the graph G' from G in polynomial time, the reduction satisfies the first condition that it should be done in polynomial time.

   If G_k is a Clique, the graph ${G_k}'$ will not have any edges between its vertices and so it will be an IS. If the graph ${G_k}'$ is an IS with no edges between its vertices, the equivalent graph G_k will have an edge wherever one is missing in ${G_k}'$. And so G_k will be a Clique.

   So we have shown that $Clique \leq_P Independent\ \ Set$

2. Given two undirected graphs G_1 and G_2, let V_1 and V_2 be the vertex sets of the two graphs, respectively. An isomorphism of G_1 and G_2 is a bijection between the vertex sets V_1 and V_2, $f : V_1 \to V_2$, such that any two vertices u and v of G_1 are adjacent in G_1 if and only if the two vertices $f(u)$ and $f(v)$ of G_2 are adjacent in G_2.

Given two graphs G_1 and G_2, the graph isomorphism problem is to decide whether G_1 and G_2 are isomorphic.
It is easy to show that the graph isomorphism problem is in NP because given a bijection f as a certificate, we can easily check whether the certificate is valid in polynomial time.
However, it has been an open problem whether the graph isomorphism problem is in P or in NPC. Recently Babai has made significant progress by providing a so-called "quasipolynomial- time" algorithm for this problem (you may find it in the following manuscript). The manuscript has been submitted to the 48th Annual Symposium on the Theory of Computing (STOC 2016) for review and the result will be known soon.
Graph Isomorphism in Quasipolynomial Time, L. Babai, arXiv:1512.03547, 2016.
In this exercise, we are considering a "slightly different" problem, called subgraph isomorphism. Recall that a graph G' is a subgraph of another graph G if every vertex of G' is in G and every edge of G' is also in G.

Given two graphs G_1 and G_2, the subgraph isomorphism problem is to decide whether G_2 contains a subgraph G' that is isomorphic to G_1.

Prove that the subgraph isomorphism problem is NP-Complete.

Note: I will send you a hint about this problem on Monday. You may try to work on it before that.

Consider a graph G and an integer k. The Clique problem is to find G_k, a subgraph of G, such that every node in G_k, has an edge connecting it to every other node in G_k.

Convert an instance of the Clique problem to a specific instance of the subgraph isomorphism problem with graphs G and G_k, where G_k is the complete graph; i.e. $(G, k) \rightarrow (G, G_k)$. This can easily be done in polynomial time.

Any subgraph of G_k will be a complete graph. Since every node of G_k will be connected by an edge to every other node, the same will be the case with its subgraph. In this special case of subgraph isomorphism (G, G_k), whenever a node connects two edges in a subgraph of G_k, the corresponding edges in G will also be connected. The reason is that G_k is a subgraph of G. Also, whenever two nodes in G are connected by an edge, if these nodes exist in the subgraph of G_k, they will also be connected. So we have shown that if there exists a k-Clique in G, then there exists a subgraph in the special case of (G, G_k), where the subgraph of G_k is isomorphic to G (the subgraph isomorphism case exists).

On the other hand if G has a complete graph G_k as a subgraph, then G_k has to be a Clique.

Given a certificate for the special instance of the subgraph isomorphism problem, (G, G_k), or for that matter any instance of the subgraph isomorphism problem, it is can be easily verified in $O(n^2)$ time where n is the number of subgraph vertices, or in polynomial time, that the instance is correct. Given this and the fact that a Clique instance can be reduced to a subgraph isomorphism instance in polynomial time, and knowing that finding a Clique is an NP-Complete problem, we can conclude that finding an instance of the subgraph isomorphism problem is NP-Complete.