UTAH STATE UNIVERSITY, COMPUTER SCIENCE DEPARTMENT

CS 7910 Computational Complexity
Assignment 3

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1. In class we have studied a polynomial-time reduction from 3SAT to Clique and a polynomial-time reduction from 3SAT to Independent Set. Give a polynomial-time reduction from Clique to Independent Set.

   The clique problem is that given a graph G and an integer k, we need to find out if it is possible to create a graph G_k with k vertices where every vertex is connected to every other vertex through an edge. Such a graph that we construct is called a Clique. In order to construct an Independent Set (IS) from a Clique, we can construct a graph $G_k{}'$ with equivalent vertices as in the Clique. However the graph we construct will not have any edges wherever the Clique will have edges. Since the clique will have edges between all vertices, the graph we construct will not have any edges at all. Since there will be no edges, the graph will become an IS.

   If the Clique G_k has k vertices, the graph $G_k{}'$ we construct will have the same number of vertices. So we can create the graph in $O(n)$ time. Since we can construct the graph $G_k{}'$ from the Clique G_k in polynomial times, the reduction satisfies the first condition that it should be done in polynomial time.

   If G_k is a Clique, the graph $G_k{}'$ will not have any edges between its vertices and so it will be an IS. If the graph $G_k{}'$ is an IS with no edges between its vertices, the equivalent graph G_k will have an edge wherever one is missing in $G_k{}'$. And so G_k will be a Clique.

   So we have shown that $Clique \leq_P Independent\ \ Set$

2. Given two undirected graphs G_1 and G_2, let V_1 and V_2 be the vertex sets of the two graphs, respectively. An isomorphism of G_1 and G_2 is a bijection between the vertex sets V_1 and V_2, $f : V_1 \to V_2$, such that any two vertices u and v of G_1 are adjacent in G_1 if and only if the two vertices $f(u)$ and $f(v)$ of G_2 are adjacent in G_2.
   Given two graphs G_1 and G_2, the graph isomorphism problem is to decide whether G_1 and G_2 are isomorphic.

It is easy to show that the graph isomorphism problem is in NP because given a bijection f as a certificate, we can easily check whether the certificate is valid in polynomial time.

However, it has been an open problem whether the graph isomorphism problem is in P or in NPC. Recently Babai has made significant progress by providing a so-called "quasipolynomial- time" algorithm for this problem (you may find it in the following manuscript). The manuscript has been submitted to the 48th Annual Symposium on the Theory of Computing (STOC 2016) for review and the result will be known soon.

Graph Isomorphism in Quasipolynomial Time, L. Babai, arXiv:1512.03547, 2016.

In this exercise, we are considering a "slightly different" problem, called subgraph isomorphism. Recall that a graph G' is a subgraph of another graph G if every vertex of G' is in G and every edge of G' is also in G.

Given two graphs G_1 and G_2, the subgraph isomorphism problem is to decide whether G_2 contains a subgraph G' that is isomorphic to G_1.

Prove that the subgraph isomorphism problem is NP-Complete.

Note: I will send you a hint about this problem on Monday. You may try to work on it before that.

Let G_1 be a graph and let G_{1_k} be a Clique that is a subgraph of G_1 with k vertices. We know that to find out if such a subgraph exists or not is an NP-Complete problem.

We can now reduce the Clique problem to the subgraph isomorphism problem. Construct a graph G_2 which has the same number of vertices as G_1 and the same edges between its corresponding vertices as G_1. The graphs G_1 and G_2 will be equivalent. Construct a G_2 subgraph G' that is equivalent to the clique G_{1_k}. Consider any two vertices in G'. Since G' is a Clique, the two vertices will be adjacent. The corresponding two vertices in Clique G_{1_k} which is a subgraph of G_1 will also be adjacent. Consider any two vertices in G_{1_k}. They will be adjacent and the corresponding vertices in G', which is a subgraph of G_2 will also be adjacent. So we have shown that G_2 contains a subgraph G' that is isomorphic to G_1.

The reduction from a Clique to a subgraph isomorphism problem can be done in time $O(|V|+|E|)$ where V and E are the vertex set and edge set of graph G_1. This is polynomial time.