UTAH STATE UNIVERSITY, COMPUTER SCIENCE DEPARTMENT

CS 7910 Computational Complexity Assignment 4

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1. (20 points) In this exercise, we consider the following *Efficient Recruiting Problem.*

   Suppose you are helping to organize a summer sports camp, and the following problem comes up. The camp is supposed to have at least one counselor who is skilled at each of the n sports covered by the camp (baseball, volleyball, basketball, and so on). They have received job applications from m potential counselors. For each of the n sports, there is some subset of the m applicants qualified in that sport. The question is: For a given number $k < m$, is it possible to hire at most k of the counselors such that for each of the n sports, at least one hired counselor is qualified in it. We call this the *Efficient Recruiting Problem.*

   Prove that the Efficient Recruiting problem is NP-Complete.

   This problem looks very similar to the Set Cover (SC) problem, which we showed in class to be NP-Complete. The Set Cover problem is that given a universal set U, and a set S of subsets of U, is it possible to find a set of k elements or less of S, such that the union of these elements is equal to the universal set U. We can designate an instance of the Set Cover problem to be (U, S, k).

   Let the set of n sports be designated by N, the set of m councillors by M and let k' be the number of councillors that need to be hired, such that $k' < m$ and we have at least one councillor for a sport. We can designate this instance of *Efficient Recruiting Problem* (ERP) to be (N, M, k').

   Given a certificate for the solution to an ERP problem instance, which will be a set of councillors S_m, we can verify in time $O(|S_m| * |N|)$, that is polynomial time that, the solution is correct. This shows that the ERP problem is in NP.

   In order to reduce SC to ERP, we can convert every element in the universal set U to an element in the set of sports N. We can convert each element in the set S of subsets of U to an element in the set of councillors M. Each element in the set of councillors will cover the sports corresponding to the elements of the universal set U covered by an element of set S. We can set $k' - 1 = k$. So we reduce the case where a union of at most k elements of S covers the universal set U, to the case where a union of at most k sports councillors from the set M covers all the sports in the set of sports N. This reduction can clearly be done in polynomial time as it is a on-to-one reduction.

In the case where k subsets S can cover the universal set U, correspondingly, a set of k councillors can cover all the sports in the set N. Also in the case where k councillors can cover all the sports in the set N, k subsets S can cover the universal set U.

The above two paragraphs show that ERP is in NP-Hard. Since we know that it is also in NP, it follows that ERP is NP-Complete.

2. (20 points) In this exercise, we consider the following *Hitting Set Problem*.

   Consider a set $A = \{a_1, a_2, ..., a_n\}$ of n elements, and a collection $B_1, B_2, ..., B_m$ of m subsets of A (i.e., $B_i \subseteq A$ for each $1 \le i \le m$).

   We say that a subset $H \subseteq A$ is a *hitting set* for the collection $\{B_1, B_2, ..., B_m\}$ if H contains at least one element from each B_i (i.e., $A \cap B_i$ is not empty).

   Given a number k, the *Hitting Set Problem* is to decide whether there is a hitting set H of size at most k.

   Prove that the *Hitting Set Problem* is NP-Complete.

   The Set Cover (SC) problem, a known NP-Complete problem, can be reduced to the Hitting Set (HS) problem. Let (U, S, k) be an instance of the SC problem, where U is the universal set of elements, S is the set of subsets of U and k is the maximum number of elements of S, that can be used to cover all the elements in U.

   Let (B, H, k') be an instance of the HS problem where B is the collection of m subsets of a universal set A, H is a subset of the universal set A and k' is the maximum number of elements of H, that can be used to cover all the elements in B.

   In order to reduce SC to HS, we can start with an instance of SC and convert each element in the set U to an element in the set B. Each element of set S can be converted to an element in set H. We can set $k = k'$. This conversion can easily be done in polynomial time as it is a one-to-one conversion.

   Consider the case when the SC instance above is true. As a result of the conversion above, we can see that in this case the set H in the HS instance (B, H, k) will cover every instance of B. In the case when when the HS instance is true, we can see that in the equivalent SC instance, the set S will cover the set U. This means that $Set\ Cover \le_P Hitting\ Set$. So HS will be in the NP-Hard set of problems.

   Given a certificate H of the HS problem, we can verify in $O(|H| * |B|)$, or in polynomial time, that the solution is correct. This means that the HS problem is in the set NP.

   Since HS is also known to be NP-Hard, it follows that HS is NP-Complete.