

CS 7910 Computational Complexity

Assignment 7

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1. **(10 points)** We have studied a 2-approximation algorithm for the vertex cover problem in class. Give an example of a graph such that the size of the vertex cover computed by our algorithm is always equal to $2 * OPT$, where OPT is the size of the vertex cover in an optimal solution.

This actually proves that the approximation ratio 2 is “tight” for our algorithm.

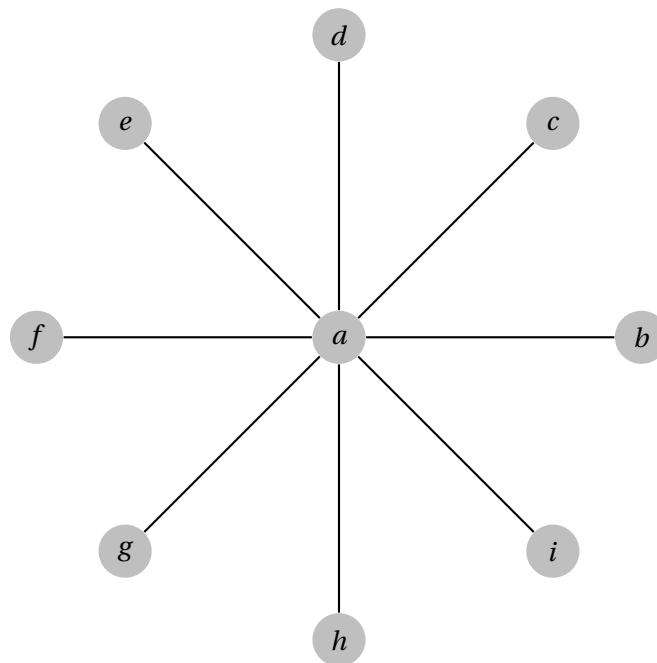


Figure 1: An example of a star shaped graph

Figure 1 shows an example of a star shaped graph where node a is connected to every other node by an edge. In such a graph, the set of vertices with just the node a will be

the vertex cover since the node will cover all the edges. If we use the 2-approximation algorithm, the vertex cover will include the two vertices in a randomly selected edge. So in such a graph, the 2-approximation algorithm will always return a vertex set of size 2, which will be twice the optimum solution of size 1.

2. **(10 points)** Let G be an undirected graph of n vertices and m edges. Let V denote the vertex set of G . Recall that we have proved the following observation about the relationship between a vertex cover and an independent set.

Observation 1 A subset $S \subseteq V$ is a vertex cover of G if and only if the subset $S' = V - S$ is an independent set of G .

In class we have designed a 2-approximation algorithm for computing a vertex cover. Suppose we use the following approximation algorithm to compute an independent set of G .

First, we use the algorithm studied in class to compute a vertex cover S of G . Then, we return $S' = V - S$ as the solution for our independent set problem. By the above observation, S' is an independent set of G .

Since the approximation ratio of the algorithm for computing the vertex cover is 2, is the approximation ratio of the above algorithm for computing an independent set also 2? Please justify your answer.

Let VC_2 and IS_2 be the Vertex Cover and Independent Set found by the 2-approximation algorithm as described above. Let the corresponding optimal solutions be denoted by VC_{OPT} and IS_{OPT} .

We know the following are true

$$\begin{aligned} |VC_2| + |IS_2| &= |V| \\ |VC_{OPT}| + |IS_{OPT}| &= |V| \\ |VC_2| &\leq 2|VC_{OPT}| \end{aligned}$$

It follows from the above that

$$\begin{aligned} |V| - |IS_2| &\leq 2(|V| - |IS_{OPT}|) \\ 2|IS_{OPT}| &\leq 2|V| - |V| + |IS_2| \end{aligned}$$

$$|IS_{OPT}| \leq \frac{1}{2}(|V| + |IS_2|) \tag{1}$$

For the independent set obtained by the above algorithm to have an approximation ratio of 2, the following should be true

$$|IS_{OPT}| \leq 2|IS_2| \tag{2}$$

Equation 1 can be changed as follows without affecting the inequality.

$$|IS_{OPT}| \leq 2|IS_2| + \frac{1}{2}|V| \tag{3}$$

Since equation 3 cannot be reduced to equation 2, I would say that the independent set does not have an approximation ration of 2.