CS 7910 Computational Complexity Assignment 10

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1. **(20 points)** In this exercise, we design an approximation algorithm for the dominating set problem. We have proved in class that the dominating set problem is NP-Complete. Here we consider its optimization problem.

   Given an undirected graph G of n vertices, a subset S of vertices of G is a dominating set if each vertex v of G is either in S or connects to a vertex of S by an edge. The problem is to find a dominating set of G of minimum size.

   Design a polynomial-time approximation algorithm for the problem with approximation ratio $O(logn)$. In other words, if OPT is the size of the optimal dominating set and C is the size of the dominating set found by your algorithm, then it should hold that $C \leq O(logn) \cdot OPT$, which is equivalent to $C = O(OPT \cdot logn)$ by the definition of the big-O notation.

   Consider the following approximate algorithm for finding a dominating set, that takes a graph G having a set of V vertices and E edges as an input parameter

Algorithm 1 Greedy Dominating Set Approximation Algorithm

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1:  procedure GREEDY DOMINATING SET APPROXIMATION ALGORITHM(G)
2:      Create remaining vertices set R = V
3:      Create an empty set of vertices S = φ
4:      while R ≠ φ do
5:          Select vertex v from R such that set S_i consisting of vertex v
6:          and all vertices connected to v by an edge maximizes S_i ∩ R
7:          R = R − S_i
8:          S = S ∪ v
9:      end while
10:     return S
11: end procedure
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   For every element $x \in S_i \cap R$ in algorithm 1, we can associate a cost $C_x = \frac{1}{|S_i \cap R|}$. If we add up the cost for every element in set S_i we will get a total of 1. The size of the dominating

set found by this algorithm will be the number of vertices in set S. Since the dominating set will cover all vertices either by including them in S or by having an edge to it from a vertex in set S, the number of vertices C in the dominating set will be given by

$$C = \sum_{x \in V} C_x$$

We need to show that this algorithm will give us a dominating set of size

$$C \leq O(logn) \cdot OPT$$

where OPT is the number of vertices in the optimal dominating set.

We will start by proving the following lemma for any subset of vertices S_k

$$\sum_{x \in S_k} C_x \leq H(d_k) \tag{1}$$

where H is the harmonic function defined as

$$H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} = \theta(logn) \tag{2}$$

and

$$d_k = |S_k|$$

Let

$$S_k = \{x_1, x_2, ..., x_{d_k}\}$$

be the set S_k with its elements ordered by the time they are first removed from the set of remaining vertices R. Consider an element x_j of S_k. Suppose x_j is first removed in an iteration when we remove S_i from R. $x_j \in S_i \cap R$ at the beginning of the iteration. And $x_j, x_{j+1}, ..., x_{d_k}$ have not been removed from R yet. Since the set S_i may or may not be the same as S_k,

$$|S_i \cap R| \geq |S_k \cap R| \geq d_k - j + 1$$

We can see that

$$C_{x_j} = \frac{1}{|S_i \cap R|} \leq \frac{1}{d_k - j + 1}$$

for any $1 \leq j \leq d_k$.
So

$$C_{x_1} \leq \frac{1}{d_k}, C_{x_2} \leq \frac{1}{d_k - 1}, ..., C_{x_{d_{k-1}}} \leq \frac{1}{2}, C_{x_{d_k}} \leq 1 \tag{3}$$

Adding everything in 3, we get

$$\sum_{x \in S_k} C_x \leq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{d_{k-1}} + \frac{1}{d_k} = H(d_k) \tag{4}$$

So we can see from 4 that 1 is proved. Let A^* be the set of vertex sets formed by the union of all sets with each set consisting of a vertex in the optimal solution along with

all the vertices connected to it by edges in the graph G. For the size C of the dominating set found by the algorithm

$$C = \sum_{x \in V} C_x \leq \sum_{S_k \in A^*} \sum_{x \in S_k} C_x \tag{5}$$

Using 1, 5 reduces to

$$\begin{aligned} C &\leq \sum_{S_k \in A^*} H(d_k) \leq \sum_{S_k \in A^*} H(d_{max}) \\ C &\leq H(d_{max}) \cdot \left|A^*\right| \\ C &\leq O(log(d_{max})) \cdot OPT \end{aligned}$$

Since d_{max} (the size of the largest set of a vertex and all vertices connected to it by edges) is limited by the number of vertices n in graph G,

$$C \leq O(log(n)) \cdot OPT$$

$$C = O(OPT \cdot log(n)) \tag{6}$$

Equation 6 proves the requirement.

2. In this exercise, we consider a "dual" problem of the load balancing problem.

   Suppose there are m machines and n jobs such that each job i has a processing time t_i. Consider a job assignment that assigns each job to one of these machines. For each machine j, let T_j denote the total sum of the processing time of all jobs assigned to machine j, and we call T_j the workload of machine j. We call the value $min_{1 \leq j \leq m} T_j$ the *minimum workload* of all machines of the assignment.

   The *dual load balancing problem* is to compute a job assignment that *maximizes* the minimum workload of all machines.

   Remark. Recall that the load balancing problem is to find a job assignment that minimizes the maximum workload of all machines. Therefore, the two problem are "dual" to each other.

   a) **(5 points)** The dual load balancing problem defined above is an optimization problem. What is the decision version of this problem?

      The decision version of the problem is that given m machines and n jobs as described above, is it possible to find an assignment of jobs to machines such that the minimum workload is at least k.

   b) **(10 points)** Prove that the decision problem is NP-Complete.

      We know that the partition problem is in NP-Complete. As a recap, the partition problem is, given a set $A = \{a_1, a_2, ..., a_p\}$, to determine if it is possible to partition A into two subsets A_1 and A_2 ($A = A_1 \cup A_2$ and $A_1 \cap A_2 = \phi$) such that $\sum A_1 = \sum A_2$. If we can reduce the partition problem to our dual load balancing problem in polynomial time, then we can show that our problem is in NP-Hard. Let (A) be an instance of the partition problem, where A is a set of items as described above and let (m, n, k) be an instance of the dual load balancing problem, where m, n and k are as described in (a) above. Consider a special case of the dual load balancing problem where $m = 2$, $n = A$ and $k = \frac{1}{2} \sum A = \frac{1}{2} \sum t_i$.

i. We can see that it is possible to reduce the instance of the partition problem to our special instance of the dual load balancing problem in polynomial time.

ii. Consider the case where it is possible to partition the set A. In this case, we know that it will be possible to have a minimum workload of k. Consider the case where it is possible to have a minimum workload of k. The way the problem has been constructed, the workload of each machine will be k and this means that the instance of the partition problem will be true.

From (i) and (ii) above we can conclude that the dual load balancing problem is in NP-Hard since it can be reduced in polynomial time from a known NP-Complete problem.

Given an instance of a dual load balancing problem (m, n, k) and a certificate (an assignment list of jobs to each machine), we can verify in polynomial time that the certificate is correct. So the dual load balancing problem is in NP.

Since the dual load balancing problem is both in NP-Hard and NP, we can conclude that it is in NP-Complete.

c) **(15 points)** Let A be the sum of the processing time of all jobs, i.e., $A = \sum_{i=1}^{n} t_i$. We assume that $t_i \leq \frac{A}{2m}$ for each job i (intuitively, each t_i is not "too big"). Under this assumption, design a polynomial-time approximation algorithm for the problem with approximation ratio 2. In other words, if OPT is the minimum workload in an optimal solution and C is the minimum workload in your solution, then it holds that $C \geq \frac{1}{2} \cdot OPT$.

Consider the greedy algorithm discussed in class for allocation of jobs to machines. To recap, the algorithm makes one pass through the jobs in any order and each job is picked up and assigned to the machine that has the smallest load so far.

If it is possible to spread the jobs out equally among the machines, each machine will have a load equal to $\frac{A}{m}$. If it is not possible to spread out the jobs evenly among the machines, then it will mean that some machine(s) will have a load $> \frac{A}{m}$ and other machine(s) will have to have a load $< \frac{A}{m}$. This means that the minimum load assigned to a machine will have an upper limit of $\frac{A}{m}$. Or in other words

$$OPT \leq \frac{A}{m} \tag{7}$$

We know that the size of the largest job is limited by $\frac{A}{2m}$. The largest job can either be the only job assigned to a machine or it could be one of more than one job assigned to a machine. In both cases the minimum load assigned to a machine will be at least the largest job size. So

$$C \geq \frac{A}{2m}$$

$$C \geq \frac{1}{2} \cdot \frac{A}{m} \tag{8}$$

From 7 and 8, we can conclude that

$$C \geq \frac{1}{2} \cdot OPT \tag{9}$$

9 proves that the greedy algorithm has an approximation ratio of 2.