CS6190: Probabilistic Modeling Homework 0

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1. Let X and Y be continuous real-valued random variables. Prove the following:

   (a) $E[E[X|Y]] = E[X]$

$$\begin{aligned}
E[E[X|Y]] &= \int_{-\infty}^{\infty} E[X|Y=y]\, p_Y(y)dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\ p_{X|Y=y}(x)dx\ p_Y(y)dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\ p_{X|Y=y}(x)p_Y(y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\ p_{XY}(x,y)dxdy \\
&= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} p_{XY}(x,y)dy\ dx \\
&= \int_{-\infty}^{\infty} x\ p_X(x)dx \\
&= E[X]
\end{aligned}$$

   Here $p_X(x)$, $p_Y(y)$ and $p_{X|Y=y}(x)$ are the probability density functions for X, Y, and the conditional probability of X given the value of Y. $p_{XY}(x,y)$ is the joint probability distribution function of X and Y. I needed to look up the proof [1] as I was not aware that the first step shown above could be done and due to this, I was getting an extra y in the final result. The result I got before I looked up the proof was $yE[X]$.

   (b) $Var(X) = E[Var(X|Y)] + Var(E[X|Y])$

$$\begin{aligned}
E[Var(X|Y)] + Var(E[X|Y]) &= E\left[E\left[X^2|Y\right] - E[X|Y]\,E[X|Y]\right] \\
&+ E[E[X|Y]\,E[X|Y]] - E[E[X|Y]]\,E[E[X|Y]] \\
&= E\left[E\left[X^2|Y\right]\right] - E[E[X|Y]\,E[X|Y]] \\
&+ E[E[X|Y]\,E[X|Y]] - E[X]\,E[X] \\
&= E\left[X^2\right] - E[X]\,E[X] \\
&= Var(X)
\end{aligned}$$

   Before I worked out the solution above, I was using the definition $Var(X) = E\left[X^2\right] - \mu_X^2$ and that did not help. After I realized that $\mu_X = E[X]$, the solution fell in place.

2.

References

[1] *Conditional Expectation*, www.statlect.com/fundamentals-of-probability/conditional-expectation.