

# CS6190: Probabilistic Modeling Homework 0

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1. Let  $X$  and  $Y$  be continuous real-valued random variables. Prove the following:

(a)  $E[E[X|Y]] = E[X]$

$$\begin{aligned} E[E[X|Y]] &= \int_{-\infty}^{\infty} E[X|Y=y] p_Y(y) dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x p_{X|Y=y}(x) dx p_Y(y) dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x p_{X|Y=y}(x) p_Y(y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x p_{XY}(x, y) dx dy \\ &= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} p_{XY}(x, y) dy dx \\ &= \int_{-\infty}^{\infty} x p_X(x) dx \\ &= E[X] \end{aligned}$$

Here  $p_X(x)$ ,  $p_Y(y)$  and  $p_{X|Y=y}(x)$  are the probability density functions for  $X$ ,  $Y$ , and the conditional probability of  $X$  given the value of  $Y$ .  $p_{XY}(x, y)$  is the joint probability distribution function of  $X$  and  $Y$ . I needed to look up the proof [1] as I was not aware that the first step shown above could be done and due to this, I was getting an extra  $y$  in the final result. The result I got before I looked up the proof was  $yE[X]$ .

(b)  $Var(X) = E[Var(X|Y)] + Var(E[X|Y])$

$$\begin{aligned} E[Var(X|Y)] + Var(E[X|Y]) &= E[E[X^2|Y] - E[X|Y]E[X|Y]] \\ &\quad + E[E[X|Y]E[X|Y]] - E[E[X|Y]]E[E[X|Y]] \\ &= E[E[X^2|Y]] - E[E[X|Y]E[X|Y]] \\ &\quad + E[E[X|Y]E[X|Y]] - E[X]E[X] \\ &= E[X^2] - E[X]E[X] \\ &= Var(X) \end{aligned}$$

Before I worked out the solution above, I was using the definition  $Var(X) = E[X^2] - \mu_X^2$  and that did not help. After I realized that  $\mu_X = E[X]$ , the solution fell in place.

2. Consider random variables  $X$  and  $Y$  with joint pdf

$$p(x, y) = \begin{cases} x + y, & \text{for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the marginal pdf  $p(x)$ ?

$$\begin{aligned}
 p(x) &= \int_{-\infty}^{\infty} p(x, y) dy \\
 &= \begin{cases} \int_0^1 (x + y) dy, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
 &= \begin{cases} \left( xy + \frac{1}{2}y^2 \right) \Big|_0^1, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
 &= \begin{cases} x + \frac{1}{2}, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}
 \end{aligned}$$

(b) What is the conditional pdf  $p(y|x)$ ?

$$\begin{aligned}
 p(y|x) &= \frac{p(x, y)}{p(x)} \\
 &= \begin{cases} \frac{x + y}{x + \frac{1}{2}}, & \text{for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}
 \end{aligned}$$

(c) What is the conditional expectation  $E[Y|X]$ ?

$$\begin{aligned}
 E[Y|X] &= \int_{-\infty}^{\infty} p_{Y|X}(y) dy \\
 &= \begin{cases} \int_0^1 \frac{x + y}{x + \frac{1}{2}} dy, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
 &= \begin{cases} \frac{xy + \frac{1}{2}y^2}{x + \frac{1}{2}} \Big|_0^1, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
 &= \begin{cases} \frac{x + \frac{1}{2}}{x + \frac{1}{2}}, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
 &= \begin{cases} 1, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}
 \end{aligned}$$

(d) What is the covariance  $Cov(X, Y)$ ?

$$Cov(X, Y) = E[XY] - E[X] E[Y]$$

$$\begin{aligned}
 E[XY] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyp(x, y) dx dy \\
 &= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} yp(x, y) dy dx
 \end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} yp(x, y)dy &= \begin{cases} \int_0^1 y(x+y)dy, \text{ for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} \int_0^1 (xy + y^2)dy, \text{ for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} \left. \frac{1}{2}xy^2 + \frac{1}{3}y^3 \right|_0^1, \text{ for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{2}x + \frac{1}{3}, \text{ for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases} \\
\int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} yp(x, y)dydx &= \begin{cases} \int_0^1 x \left( \frac{1}{2}x + \frac{1}{3} \right) dx, \text{ for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} \int_0^1 \left( \frac{1}{2}x^2 + \frac{1}{3}x \right) dx, \text{ for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} \left. \left( \frac{1}{2} \frac{1}{3}x^3 + \frac{1}{3} \frac{1}{2}x^2 \right) \right|_0^1, \text{ for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} \left( \frac{1}{6} + \frac{1}{6} \right), \text{ for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{3}, \text{ for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases} \\
&= E[XY] \\
E[X] &= \int_{-\infty}^{\infty} xp(x)dx \\
&= \begin{cases} \int_{-\infty}^{\infty} x \left( x + \frac{1}{2} \right) dx, \text{ for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} \int_{-\infty}^{\infty} \left( x^2 + \frac{1}{2}x \right) dx, \text{ for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} \left. \left( \frac{1}{3}x^3 + \frac{1}{2} \frac{1}{2}x^2 \right) \right|_0^1, \text{ for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{3} + \frac{1}{4}, \text{ for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} \frac{7}{12}, \text{ for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases}
\end{aligned}$$

$$\begin{aligned}
p(y) &= \int_{-\infty}^{\infty} p(x, y) dx \\
&= \begin{cases} \int_0^1 (x + y) dx, & \text{for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \left( \frac{1}{2}x^2 + yx \right) \Big|_0^1, & \text{for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} y + \frac{1}{2}, & \text{for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

$$\begin{aligned}
E[Y] &= \int_{-\infty}^{\infty} yp(y) dy \\
&= \begin{cases} \int_{-\infty}^{\infty} y \left( y + \frac{1}{2} \right) dy, & \text{for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \int_{-\infty}^{\infty} \left( y^2 + \frac{1}{2}y \right) dy, & \text{for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \left( \frac{1}{3}y^3 + \frac{1}{2} \frac{1}{2}y^2 \right) \Big|_0^1, & \text{for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{3} + \frac{1}{4}, & \text{for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{7}{12}, & \text{for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

$$\begin{aligned}
Cov(X, Y) &= E[XY] - E[X] E[Y] \\
&= \begin{cases} \frac{1}{3} - \frac{7}{12} \frac{7}{12}, & \text{for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{48}{144} - \frac{49}{144}, & \text{for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} -\frac{1}{144}, & \text{for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

## References

- [1] *Conditional Expectation*, [www.statlect.com/fundamentals-of-probability/conditional-expectation](http://www.statlect.com/fundamentals-of-probability/conditional-expectation).