CS6190: Probabilistic Modeling Homework 0

Gopal Menon

20 January, 2018

1. Let X and Y be continuous real-valued random variables. Prove the following:

   (a) $E[E[X|Y]] = E[X]$

$$\begin{aligned}
E[E[X|Y]] &= \int_{-\infty}^{\infty} E[X|Y=y]\, p_Y(y)dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\; p_{X|Y=y}(x)dx\; p_Y(y)dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\; p_{X|Y=y}(x)p_Y(y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\; p_{XY}(x,y)dxdy \\
&= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} p_{XY}(x,y)dy\; dx \\
&= \int_{-\infty}^{\infty} x\; p_X(x)dx \\
&= E[X]
\end{aligned}$$

   Here $p_X(x)$, $p_Y(y)$ and $p_{X|Y=y}(x)$ are the probability density functions for X, Y, and the conditional probability of X given the value of Y. $p_{XY}(x,y)$ is the joint probability distribution function of X and Y. I needed to look up the proof [1] as I was not aware that the first step shown above could be done and due to this, I was getting an extra y in the final result. The result I got before I looked up the proof was $yE[X]$.

   (b) $Var(X) = E[Var(X|Y)] + Var(E[X|Y])$

$$\begin{aligned}
E[Var(X|Y)] + Var(E[X|Y]) &= E\left[E\left[X^2|Y\right] - E[X|Y]\,E[X|Y]\right] \\
&\quad + E[E[X|Y]\,E[X|Y]] - E[E[X|Y]]\,E[E[X|Y]] \\
&= E\left[E\left[X^2|Y\right]\right] - E[E[X|Y]\,E[X|Y]] \\
&\quad + E[E[X|Y]\,E[X|Y]] - E[X]\,E[X] \\
&= E\left[X^2\right] - E[X]\,E[X] \\
&= Var(X)
\end{aligned}$$

   Before I worked out the solution above, I was using the definition $Var(X) = E\left[X^2\right] - \mu_X^2$ and that did not help. After I realized that $\mu_X = E[X]$, the solution fell in place.

2. Consider random variables X and Y with joint pdf

$$p(x,y) = \begin{cases} x+y, & \text{for } 0 \le x \le 1, \text{and } 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the marginal pdf $p(x)$?

$$\begin{aligned}
p(x) &= \int_{-\infty}^{\infty} p(x,y)dy \\
&= \begin{cases} \int_0^1 (x+y)dy, \text{for } 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} \left(xy + \frac{1}{2}y^2\right)\Big|_0^1, \text{for } 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} x + \frac{1}{2}, \text{for } 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases}
\end{aligned}$$

(b) What is the conditional pdf $p(y|x)$?

$$\begin{aligned}
p(y|x) &= \frac{p(x,y)}{p(x)} \\
&= \begin{cases} \frac{x+y}{x+\frac{1}{2}}, \text{for } 0 \le x \le 1, \text{and } 0 \le y \le 1 \\ 0, \text{otherwise} \end{cases}
\end{aligned}$$

(c) What is the conditional expectation $E[Y|X]$?

$$\begin{aligned}
E[Y|X] &= \int_{-\infty}^{\infty} p_{Y|X}(y)dy \\
&= \begin{cases} \int_0^1 \frac{x+y}{x+\frac{1}{2}}dy, \text{for } 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{xy+\frac{1}{2}y^2}{x+\frac{1}{2}}\Big|_0^1, \text{for } 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{x+\frac{1}{2}}{x+\frac{1}{2}}, \text{for } 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} 1, \text{for } 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases}
\end{aligned}$$

(d) What is the covariance $Cov(X,Y)$?

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$\begin{aligned}
E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyp(x,y)dxdy \\
&= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} yp(x,y)dydx
\end{aligned}$$

$$\int_{-\infty}^{\infty} yp(x,y)dy = \begin{cases} \displaystyle\int_0^1 y(x+y)dy, \text{for } 0 \leq x \leq 1, \text{and } 0 \leq y \leq 1 \\ 0, \text{otherwise} \end{cases}$$

$$= \begin{cases} \displaystyle\int_0^1 (xy+y^2)dy, \text{for } 0 \leq x \leq 1, \text{and } 0 \leq y \leq 1 \\ 0, \text{otherwise} \end{cases}$$

$$= \begin{cases} \displaystyle\frac{1}{2}xy^2 + \frac{1}{3}y^3\Big|_0^1, \text{for } 0 \leq x \leq 1, \text{and } 0 \leq y \leq 1 \\ 0, \text{otherwise} \end{cases}$$

$$= \begin{cases} \displaystyle\frac{1}{2}x + \frac{1}{3}, \text{for } 0 \leq x \leq 1, \text{and } 0 \leq y \leq 1 \\ 0, \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} yp(x,y)dydx = \begin{cases} \displaystyle\int_0^1 x\left(\frac{1}{2}x + \frac{1}{3}\right)dx, \text{for } 0 \leq x \leq 1, \text{and } 0 \leq y \leq 1 \\ 0, \text{otherwise} \end{cases}$$

$$= \begin{cases} \displaystyle\int_0^1 \left(\frac{1}{2}x^2 + \frac{1}{3}x\right)dx, \text{for } 0 \leq x \leq 1, \text{and } 0 \leq y \leq 1 \\ 0, \text{otherwise} \end{cases}$$

$$= \begin{cases} \displaystyle\left(\frac{1}{2}\frac{1}{3}x^3 + \frac{1}{3}\frac{1}{2}x^2\right)\Big|_0^1, \text{for } 0 \leq x \leq 1, \text{and } 0 \leq y \leq 1 \\ 0, \text{otherwise} \end{cases}$$

$$= \begin{cases} \displaystyle\left(\frac{1}{6} + \frac{1}{6}\right), \text{for } 0 \leq x \leq 1, \text{and } 0 \leq y \leq 1 \\ 0, \text{otherwise} \end{cases}$$

$$= \begin{cases} \displaystyle\frac{1}{3}, \text{for } 0 \leq x \leq 1, \text{and } 0 \leq y \leq 1 \\ 0, \text{otherwise} \end{cases}$$

$$= E\left[XY\right]$$

$$E\left[X\right] = \int_{-\infty}^{\infty} xp(x)dx$$

$$= \begin{cases} \displaystyle\int_{-\infty}^{\infty} x\left(x + \frac{1}{2}\right)dx, \text{for } 0 \leq x \leq 1, \text{and } 0 \leq y \leq 1 \\ 0, \text{otherwise} \end{cases}$$

$$= \begin{cases} \displaystyle\int_{-\infty}^{\infty} \left(x^2 + \frac{1}{2}x\right)dx, \text{for } 0 \leq x \leq 1, \text{and } 0 \leq y \leq 1 \\ 0, \text{otherwise} \end{cases}$$

$$= \begin{cases} \displaystyle\left(\frac{1}{3}x^3 + \frac{1}{2}\frac{1}{2}x^2\right)\Big|_0^1, \text{for } 0 \leq x \leq 1, \text{and } 0 \leq y \leq 1 \\ 0, \text{otherwise} \end{cases}$$

$$= \begin{cases} \displaystyle\frac{1}{3} + \frac{1}{4}, \text{for } 0 \leq x \leq 1, \text{and } 0 \leq y \leq 1 \\ 0, \text{otherwise} \end{cases}$$

$$= \begin{cases} \displaystyle\frac{7}{12}, \text{for } 0 \leq x \leq 1, \text{and } 0 \leq y \leq 1 \\ 0, \text{otherwise} \end{cases}$$

$$
\begin{aligned}
p(y) &= \int_{-\infty}^{\infty} p(x,y)dx \\
&= \begin{cases} \int_0^1 (x+y)dx, \text{for } 0 \le x \le 1, \text{and } 0 \le y \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} \left(\frac{1}{2}x^2 + yx\right)\Big|_0^1, \text{for } 0 \le x \le 1, \text{and } 0 \le y \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} y + \frac{1}{2}, \text{for } 0 \le x \le 1, \text{and } 0 \le y \le 1 \\ 0, \text{otherwise} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
E[Y] &= \int_{-\infty}^{\infty} yp(y)dy \\
&= \begin{cases} \int_{-\infty}^{\infty} y\left(y + \frac{1}{2}\right)dy, \text{for } 0 \le x \le 1, \text{and } 0 \le y \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} \int_{-\infty}^{\infty} \left(y^2 + \frac{1}{2}y\right)dy, \text{for } 0 \le x \le 1, \text{and } 0 \le y \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} \left(\frac{1}{3}y^3 + \frac{1}{2}\frac{1}{2}y^2\right)\Big|_0^1, \text{for } 0 \le x \le 1, \text{and } 0 \le y \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{3} + \frac{1}{4}, \text{for } 0 \le x \le 1, \text{and } 0 \le y \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{7}{12}, \text{for } 0 \le x \le 1, \text{and } 0 \le y \le 1 \\ 0, \text{otherwise} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
Cov(X,Y) &= E[XY] - E[X]E[Y] \\
&= \begin{cases} \frac{1}{3} - \frac{7}{12}\frac{7}{12}, \text{for } 0 \le x \le 1, \text{and } 0 \le y \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{48}{144} - \frac{49}{144}, \text{for } 0 \le x \le 1, \text{and } 0 \le y \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} -\frac{1}{144}, \text{for } 0 \le x \le 1, \text{and } 0 \le y \le 1 \\ 0, \text{otherwise} \end{cases}
\end{aligned}
$$

3. Let $X \sim Exp(\lambda)$, i.e., the exponential distribution with pdf $p(x) = \lambda exp(-\lambda x)$. Let $Y = \sqrt{X}$.
   (a) What is the density function $p(y)$?

$$
\begin{aligned}
Y &= \sqrt{X} \\
f(x) &= \sqrt{x} \\
f^{-1}(y) &= y^2 \\
p(y) &= \left|\frac{d}{dy}(f^{-1}(y))\right| p(f^{-1}(y)) \\
&= \left|\frac{d}{dy}y^2\right| p(y^2) \\
&= \begin{cases} 2y\lambda exp(-\lambda y^2), \text{for } y \geq 0 \\ 0, \text{otherwise} \end{cases}
\end{aligned}
$$

(b) What is the cdf, $F(y) = P(Y \leq y)$? Verify that $F(0) = 0$ and $F(\infty) = 1$.

The cdf can be found by integrating the pdf.

$$
\begin{aligned}
F(y) = P(Y \leq y) &= \int_0^y 2y\lambda exp(-\lambda y^2)dy \\
\text{Substituting } u = -\lambda y^2, \text{ we get } dy &= -\frac{1}{2\lambda y}du \\
\int 2y\lambda exp(-\lambda y^2)dy &= \int 2y\lambda exp(u) \times -\frac{1}{2\lambda y}du \\
&= -\int exp(u)du \\
&= -exp(u) \\
&= -exp(-\lambda y^2) \\
F(y) &= -exp(-\lambda y^2)\Big|_0^y \\
&= -exp(-\lambda y^2) - (-exp(-\lambda 0^2)) \\
&= -exp(-\lambda y^2) + 1 \\
&= 1 - exp(-\lambda y^2) \\
F(0) &= 1 - exp(-\lambda 0^2) \\
&= 1 - 1 \\
&= 0 \\
F(\infty) &= 1 - exp(-\infty) \\
&= 1 - 0 \\
&= 1
\end{aligned}
$$

(c) What is the quantile function F^{-1}?

If q_p is the quantile value given by the quantile function $F^{-1}(q_p)$, based on the cdf above:

$$
\begin{aligned}
q_p &= 1 - exp(-\lambda (F^{-1}(q_p))^2) \\
exp(-\lambda (F^{-1}(q_p))^2) &= 1 - q_p \\
-\lambda (F^{-1}(q_p))^2 &= \ln(1 - q_p) \\
(F^{-1}(q_p))^2 &= \frac{-1}{\lambda}\ln(1 - q_p) \\
F^{-1}(q_p) &= \sqrt{\frac{-1}{\lambda}\ln(1 - q_p)}
\end{aligned}
$$

(d) Compute the mean, $E[Y]$, and variance, $Var(Y)$. **Hint:** Use integration by parts.

References

[1] *Conditional Expectation*, www.statlect.com/fundamentals-of-probability/conditional-expectation.