CS6190: Probabilistic Modeling Homework 0

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1. Let X and Y be continuous real-valued random variables. Prove the following:

   (a) $E\left[E\left[X|Y\right]\right] = E\left[X\right]$

$$\begin{aligned}
E\left[E\left[X|Y\right]\right] &= \int_{-\infty}^{\infty} E\left[X|Y=y\right] p_Y(y)dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\ p_{X|Y=y}(x)dx\ p_Y(y)dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\ p_{X|Y=y}(x)p_Y(y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\ p_{XY}(x,y)dxdy \\
&= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} p_{XY}(x,y)dy\ dx \\
&= \int_{-\infty}^{\infty} x\ p_X(x)dx \\
&= E\left[X\right]
\end{aligned}$$

   Here $p_X(x)$, $p_Y(y)$ and $p_{X|Y=y}(x)$ are the probability density functions for X, Y, and the conditional probability of X given the value of Y. $p_{XY}(x,y)$ is the joint probability distribution function of X and Y. I needed to look up the proof [1] as I was not aware that the first step shown above could be done and due to this, I was getting an extra y in the final result. The result I got before I looked up the proof was $yE\left[X\right]$.

   (b) $Var(X) = E\left[Var(X|Y)\right] + Var(E\left[X|Y\right])$

$$\begin{aligned}
E\left[Var(X|Y)\right] + Var(E\left[X|Y\right]) &= E\left[E\left[X^2|Y\right] - E\left[X|Y\right]E\left[X|Y\right]\right] \\
&+ E\left[E\left[X|Y\right]E\left[X|Y\right]\right] - E\left[E\left[X|Y\right]\right]E\left[E\left[X|Y\right]\right] \\
&= E\left[E\left[X^2|Y\right]\right] - E\left[E\left[X|Y\right]E\left[X|Y\right]\right] \\
&+ E\left[E\left[X|Y\right]E\left[X|Y\right]\right] - E\left[X\right]E\left[X\right] \\
&= E\left[X^2\right] - E\left[X\right]E\left[X\right] \\
&= Var(X)
\end{aligned}$$

   Before I worked out the solution above, I was using the definition $Var(X) = E\left[X^2\right] - \mu_X^2$ and that did not help. After I realized that $\mu_X = E\left[X\right]$, the solution fell in place.

2. Consider random variables X and Y with joint pdf

$$p(x,y) = \begin{cases} x+y, & \text{for } 0 \le x \le 1, \text{and } 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the marginal pdf $p(x)$?

$$
\begin{aligned}
p(x) &= \int_{-\infty}^{\infty} p(x,y)dy \\
&= \begin{cases} \int_0^1 (x+y)dy, \text{for } 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} \left(xy + \frac{1}{2}y^2\right)\Big|_0^1, \text{for } 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} x + \frac{1}{2}, \text{for } 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases}
\end{aligned}
$$

(b) What is the conditional pdf $p(y|x)$?

$$
\begin{aligned}
p(y|x) &= \frac{p(x,y)}{p(x)} \\
&= \begin{cases} \frac{x+y}{x+\frac{1}{2}}, \text{for } 0 \le x \le 1, \text{and } 0 \le y \le 1 \\ 0, \text{otherwise} \end{cases}
\end{aligned}
$$

(c) What is the conditional expectation $E[Y|X]$?

$$
\begin{aligned}
E[Y|X] &= \int_{-\infty}^{\infty} p_{Y|X}(y)dy \\
&= \begin{cases} \int_0^1 \frac{x+y}{x+\frac{1}{2}}dy, \text{for } 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{xy+\frac{1}{2}y^2}{x+\frac{1}{2}}\Big|_0^1, \text{for } 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{x+\frac{1}{2}}{x+\frac{1}{2}}, \text{for } 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases} \\
&= \begin{cases} 1, \text{for } 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases}
\end{aligned}
$$

(d) What is the covariance $Cov(X,Y)$?

References

[1] *Conditional Expectation*, www.statlect.com/fundamentals-of-probability/conditional-expectation.