CS6190: Probabilistic Modeling Homework 0

Gopal Menon

13 January, 2018

1. Let X and Y be continuous real-valued random variables. Prove the following:

   (a) $E\left[E\left[X|Y\right]\right] = E\left[X\right]$

$$
\begin{aligned}
E\left[E\left[X|Y\right]\right] &= \int_{-\infty}^{\infty} E\left[X|Y=y\right] p_Y(y)dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\ p_{X|Y=y}(x)dx\ p_Y(y)dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\ p_{X|Y=y}(x)p_Y(y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\ p_{XY}(x,y)dxdy \\
&= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} p_{XY}(x,y)dy\ dx \\
&= \int_{-\infty}^{\infty} x\ p_X(x)dx \\
&= E\left[X\right]
\end{aligned}
$$

   Here $p_X(x)$, $p_Y(y)$ and $p_{X|Y=y}(x)$ are the probability density functions for X, Y, and the conditional probability of X given the value of Y. $p_{XY}(x,y)$ is the joint probability distribution function of X and Y. I needed to look up the proof [1] as I was not aware that the first step shown above could be done and due to this, I was getting an extra y in the final result. The result I got before I looked up the proff was $yE\left[X\right]$.

   (b) $Var(X) = E\left[Var(X|Y)\right] + Var(E\left[X|Y\right])$

$$
\begin{aligned}
E\left[Var(X|Y)\right] + Var(E\left[X|Y\right]) &= E\left[E\left[X^2|Y\right] - E\left[X|Y\right]E\left[X|Y\right]\right] \\
&\quad + E\left[E\left[X|Y\right]E\left[X|Y\right]\right] - E\left[E\left[X|Y\right]\right]E\left[E\left[X|Y\right]\right] \\
&= a
\end{aligned}
$$

2.

References

[1] *Conditional Expectation*, www.statlect.com/fundamentals-of-probability/conditional-expectation.