CS6190: Probabilistic Modeling Homework 0

Gopal Menon

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1. Let X and Y be continuous real-valued random variables. Prove the following:

   (a) $E\left[E\left[X|Y\right]\right] = E\left[X\right]$

   $$
   \begin{aligned}
   E\left[E\left[X|Y\right]\right] &= \int_{-\infty}^{\infty} E\left[X|Y=y\right] p_Y(y) dy \\
   &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, p_{X|Y=y}(x) dx\, p_Y(y) dy \\
   &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, p_{X|Y=y}(x) p_Y(y) dx dy \\
   &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, p_{XY}(x,y) dx dy \\
   &= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} p_{XY}(x,y) dy\ dx \\
   &= \int_{-\infty}^{\infty} x\, p_X(x) dx \\
   &= E\left[X\right]
   \end{aligned}
   $$

   Here $p_X(x)$, $p_Y(y)$ and $p_{X|Y=y}(x)$ are the probability density functions for X, Y, and the conditional probability of X given the value of Y. $p_{XY}(x,y)$ is the joint probability distribution function of X and Y. I needed to look up the proof [1] as I was not aware that the first step shown above could be done and due to this, I was getting an extra y in the final result. The result I got before I looked up the proof was $yE\left[X\right]$.

   (b) $Var(X) = E\left[Var(X|Y)\right] + Var(E\left[X|Y\right])$

   $$
   \begin{aligned}
   E\left[Var(X|Y)\right] + Var(E\left[X|Y\right]) &= E\left[E\left[X^2|Y\right] - E\left[X|Y\right] E\left[X|Y\right]\right] \\
   &\quad + E\left[E\left[X|Y\right] E\left[X|Y\right]\right] - E\left[E\left[X|Y\right]\right] E\left[E\left[X|Y\right]\right] \\
   &= E\left[E\left[X^2|Y\right]\right] - E\left[E\left[X|Y\right] E\left[X|Y\right]\right] \\
   &\quad + E\left[E\left[X|Y\right] E\left[X|Y\right]\right] - E\left[X\right] E\left[X\right] \\
   &= E\left[X^2\right] - E\left[X\right] E\left[X\right] \\
   &= Var(X)
   \end{aligned}
   $$

   Before I worked out the solution above, I was using the definition $Var(X) = E\left[X^2\right] - \mu_X^2$ and that did not help. After I realized that $\mu_X = E\left[X\right]$, the solution fell in place.

2. Consider random variables X and Y with joint pdf

   $$
   p(x,y) = \begin{cases} x + y, & \text{for } 0 \le x \le 1, \text{and } 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}
   $$

(a) What is the marginal pdf $p(x)$?

$$p(x) = \int_{-\infty}^{\infty} p(x, y) dy$$

$$= \begin{cases} \int_{0}^{1} (x + y) dy, \text{ for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases}$$

$$= \begin{cases} \left(\frac{1}{2} x^2 + xy \right) \Big|_{0}^{1}, \text{ for } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{2} + y, \text{ for } 0 \leq y \leq 1 \\ 0, \text{ otherwise} \end{cases}$$

(b) What is the conditional pdf $p(y|x)$?

(c) What is the conditional expectation $E[Y|X]$?

(d) What is the covariance $Cov(X, Y)$?

References

[1] *Conditional Expectation*, www.statlect.com/fundamentals-of-probability/conditional-expectation.