CS6190: Probabilistic Modeling Homework 1
Exponential Families, Conjugate Priors

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07 February, 2018

Written Part

1. **Expectation of sufficient statistics:** Consider a random variable X from a continuous exponential family with natural parameter $\eta = (\eta_1, \ldots, \eta_n)$. Recall this means the pdf is of the form:

$$p(x) = h(x) \exp\left(\eta \cdot T(x) - A(\eta)\right)$$

(a) Show that $E\left[T(x)|\eta\right] = \nabla A(\eta) = \left(\frac{\partial A}{\partial \eta_1}, \ldots, \frac{\partial A}{\partial \eta_d}\right)$.

Hint: Start with the identity $\int p(x)dx = 1$, and take the derivative with respect to η.

$$\begin{aligned}
\int p(x)dx &= 1 \\
\Rightarrow \int h(x) \exp\left(\eta \cdot T(x) - A(\eta)\right) dx &= 1 \\
\nabla \left(\int h(x) \exp\left(\eta \cdot T(x) - A(\eta)\right) dx \right) &= 0 \\
\int h(x) \exp\left(\eta \cdot T(x) - A(\eta)\right) . \left(T(x) - \nabla A(\eta)\right) dx &= 0 \\
\int T(x) h(x) \exp\left(\eta \cdot T(x) - A(\eta)\right) dx &= \nabla A(\eta) \int h(x) \exp\left(\eta \cdot T(x) - A(\eta)\right) dx \\
\int T(x) p(x) dx &= \nabla A(\eta) \int p(x)dx \\
E\left[T(x)\right] &= \nabla A(\eta)
\end{aligned}$$

(b) Verify this formula works for the Gaussian distribution with unknown mean, μ, and known variance, σ^2.

Hint: Start by thinking about what the natural parameter η and the function $A(\eta)$ are, then verify that the expectation of the Gaussian is the same as $\nabla A(\eta)$.

The pdf for a Gaussian distribution with unknown mean, μ, and known variance, σ^2 is given by:

$$
\begin{aligned}
p(x) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \\
&= \frac{1}{\sqrt{2\pi}} \exp\left[\ln\left(\frac{1}{\sigma}\right)\right] \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \\
&= \frac{1}{\sqrt{2\pi}} \exp\left[-\ln(\sigma) - \frac{(x-\mu)^2}{2\sigma^2}\right] \\
&= \frac{1}{\sqrt{2\pi}} \exp\left[-\ln(\sigma) - \frac{x^2}{2\sigma^2} + \frac{2x\mu}{2\sigma^2} - \frac{\mu^2}{2\sigma^2}\right] \\
&= \frac{1}{\sqrt{2\pi}} \exp\left[\frac{x\mu}{\sigma^2} - \frac{x^2}{2\sigma^2} - \left(\ln(\sigma) + \frac{\mu^2}{2\sigma^2}\right)\right] \\
&= \frac{1}{\sqrt{2\pi}} \exp\left[\begin{bmatrix}\frac{\mu}{\sigma^2} \\ \frac{-1}{2\sigma^2}\end{bmatrix} \cdot \begin{bmatrix} x \\ x^2 \end{bmatrix} - \left(\ln(\sigma) + \frac{\mu^2}{2\sigma^2}\right)\right]
\end{aligned}
$$

Based on the form of the exponential family pdf, we can see that:

$$
\begin{aligned}
A(\eta) &= \ln(\sigma) + \frac{\mu^2}{2\sigma^2}, \\
\eta &= \begin{bmatrix}\frac{\mu}{\sigma^2} \\ \frac{-1}{2\sigma^2}\end{bmatrix}, \text{ and} \\
T(x) &= \begin{bmatrix} x \\ x^2 \end{bmatrix}
\end{aligned}
$$

Based on the equation for $E[T(x)]$ above:

$$
\begin{aligned}
E[T(x)] &= \nabla A(\eta) \\
\Rightarrow E\left[\begin{bmatrix} x \\ x^2 \end{bmatrix}\right] &= \begin{bmatrix}\frac{\partial A}{\partial \eta_1} \\ \frac{\partial A}{\partial \eta_2}\end{bmatrix}, \text{ where } \eta_1 = \frac{\mu}{\sigma^2}, \text{ and } \eta_2 = \frac{-1}{2\sigma^2} \\
\begin{bmatrix} E[x] \\ E[x^2] \end{bmatrix} &= \begin{bmatrix} \frac{\partial\left(\ln(\sigma) + \frac{\mu^2}{2\sigma^2}\right)}{\partial \eta_1} \\ \frac{\partial\left(\ln(\sigma) + \frac{\mu^2}{2\sigma^2}\right)}{\partial \eta_2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial\left(\ln(\sigma) + \frac{\sigma^2}{2}\frac{\mu^2}{\sigma^4}\right)}{\partial \eta_1} \\ \frac{\partial\left(\ln\left(\sqrt{\frac{-1}{2\eta_2}}\right) - \mu^2\eta_2\right)}{\partial \eta_2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial\left(\ln(\sigma) + \frac{\sigma^2}{2}\eta_1^2\right)}{\partial \eta_1} \\ \frac{\partial\left(\ln\left(\frac{i}{\sqrt{2}\sqrt{\eta_2}}\right) - \mu^2\eta_2\right)}{\partial \eta_2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sigma^2}{2}.2\eta_1 \\ \frac{\sqrt{2}\sqrt{\eta_2}}{i}.\frac{i}{\sqrt{2}}.\frac{-1}{2}\eta_2^{\frac{-3}{2}} - \mu^2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sigma^2}{2}.2\frac{\mu}{\sigma^2} \\ \frac{-1}{2\eta_2} - \mu^2 \end{bmatrix} \\
&= \begin{bmatrix} \mu \\ \sigma^2 - \mu^2 \end{bmatrix} \\
\Rightarrow E[x] &= \mu, \text{ and} \\
E[x^2] &= \sigma^2 - \mu^2, \text{ which is not the case since } \sigma^2 = E[x^2] - \mu^2
\end{aligned}
$$

2. **Noninformative priors for the Poisson distribution:** Let $X \sim Pois(\lambda)$. Recall that the pmf of the Poisson is

$$P(X = k; \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

(a) Rewrite the above pmf in exponential family form. What is the natural parameter? What is the sufficient statistic?

$$\begin{aligned} P(X = k; \lambda) &= \frac{\lambda^k e^{-\lambda}}{k!} \\ &= \frac{1}{k!} \exp\left(\ln \lambda^k e^{-\lambda}\right) \\ &= \frac{1}{k!} \exp\left(k \ln \lambda - \lambda\right) \\ \eta &= \ln \lambda, \text{ natural parameter} \\ h(k) &= k, \text{ sufficient statistic} \end{aligned}$$

(b) Give at least two different options for noninformative priors for $p(\lambda)$.

i. **Uniform Prior:** In this case the prior is

$$p(\lambda) = c, \text{ for } 0 \leq x \leq \infty, \text{, where c is a constant}$$

ii. **Jeffrey's Prior:** In this case the prior is

$$
\begin{aligned}
\mathcal{I}(\lambda) &= E\left[\left(\frac{\partial}{\partial\lambda}\ln p(x|\lambda)\right)^2\right] \\
&= -E\left[\frac{\partial^2}{\partial\lambda^2}\ln p(x|\lambda)\right] \\
&= -E\left[\frac{\partial^2}{\partial\lambda^2}\ln\frac{\lambda^x e^{-\lambda}}{x!}\right] \\
&= -E\left[\frac{\partial^2}{\partial\lambda^2}\left(x\ln\lambda - \lambda - \ln x!\right)\right] \\
&= E\left[\frac{\partial}{\partial\lambda}\left(\frac{x}{\lambda} - 1\right)\right] \\
&= -E\left[\frac{-x}{\lambda^2}\right] \\
&= E\left[\frac{x}{\lambda^2}\right] \\
&= \frac{1}{\lambda^2}E\left[x\right] \\
&= \frac{1}{\lambda^2}\sum_{i=1}^{\infty} xp(x) \\
&= \frac{1}{\lambda^2}\sum_{i=1}^{\infty} x\frac{\lambda^x e^{-\lambda}}{x!} \\
&= \frac{e^{-\lambda}}{\lambda}\sum_{i=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!} \\
&= \frac{e^{-\lambda}}{\lambda}\left(\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \ldots\right) \\
&= \frac{e^{-\lambda}}{\lambda}e^{\lambda} \\
&= \frac{1}{\lambda} \\
p(\lambda) &= \sqrt{\mathcal{I}(\lambda)} \\
&= \sqrt{\frac{1}{\lambda}}
\end{aligned}
$$

(c) What are the resulting posteriors for your two options? Are they proper (i.e., can they be normalized)?

i. **Uniform Prior:** The posterior is

$$
\begin{aligned}
p(\lambda|x) &\propto p(x|\lambda)p(\lambda) \\
&= \prod_{i=1}^{n}\frac{\lambda^{x_i}e^{-\lambda}}{x_i!}.c, \text{based on the assumtion of i.i.d. samples} \\
&\propto \prod_{i=1}^{n}\frac{\lambda^{x_i}e^{-\lambda}}{x_i!}
\end{aligned}
$$

Even though the prior cannot be normalized, since it will not add up to 1, the posterior shown above, where the constant c has been dropped, can be normalized. The reason is that the posterior is the product of distributions that can themselves be normalized.

ii. **Jeffrey's Prior:** The posterior is

$$\begin{aligned} p(\lambda|x) &\propto p(x|\lambda)p(\lambda) \\ &= \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \cdot \sqrt{\frac{1}{\lambda}}, \text{based on the assumtion of i.i.d. samples} \\ &= \prod_{i=1}^{n} \frac{\lambda^{x_i - \frac{1}{2}} e^{-\lambda}}{x_i!} \end{aligned}$$

This can be normalized since it is the product of the likelihood, which is a product of Poisson distributions, and so can be normalized, and the prior, which can also be normalized.

3. **Non-conjugate priors:** Let X_i be from a Gaussian with known variance σ^2 and mean μ with uniform prior, i.e.,

$$\begin{aligned} \mu &\sim Unif(a, b) \\ X_i &\sim N(\mu, \sigma^2) \end{aligned}$$

What is the posterior pdf, $p(\mu|x_1, \ldots, x_n; \sigma^2, a, b)$?

Hint: There will be an integral that you won't be able to analytically solve (just leave it in integral form).

$$\begin{aligned} p(\mu|x_1, \ldots, x_n; \sigma^2, a, b) &= p(x_1, \ldots, x_n|\mu; \sigma^2)p(\mu; a, b) \\ &= \int_{-\infty}^{\infty} p(x|\mu; \sigma^2)p(\mu; a, b)dx \\ &= \int_{a}^{b} p(x|\mu; \sigma^2)\frac{1}{b-a}dx \\ &= \frac{1}{b-a}\frac{1}{\sqrt{2\pi}\sigma}\int_{a}^{b} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx \end{aligned}$$

R Coding Part

```r
# Density function for Inverse Gamma: conjugate prior for Gaussian sigma^2
density_inverse_gamma <- function(x_coordinates, alpha, beta) {

  return (beta^alpha / gamma(alpha) * x_coordinates^(-alpha - 1) * exp(-beta / x_coordinates))

}

# Density function for Normal-Inverse-Gamma with parameters mu0, lambda, alpha, beta
density_normal_inverse_gamma <-
  function(distribution_of_mean, distribution_of_variance, mu0 = 0, lambda = 1, alpha = 8, beta = 16)
{
  return (dnorm(distribution_of_mean, mean = mu0, sd = sqrt(distribution_of_variance / lambda)) *
            density_inverse_gamma(distribution_of_variance, alpha, beta))
}

# Read data for hippocampus volume and patient details
hippocampus_volume <- read.csv(file="oasis_hippocampus.csv", header=TRUE, sep=",")
patient_details <- read.csv(file="oasis_cross-sectional.csv", header=TRUE, sep=",")
```

```r
# Join both frames on identifer
patient_hippocampus_volume <- merge(hippocampus_volume, patient_details, by="ID")

# Select right hippocampal volume for control group
control_group_rt_hippo_vol <-
  subset(patient_hippocampus_volume, CDR==0.0, select=RightHippoVol)

# Select right hippocampal volume for mild dementia group
mild_dementia_group_rt_hippo_vol <-
  subset(patient_hippocampus_volume, CDR!=0.0 & CDR!=2.0 & !is.null(CDR),
         select=RightHippoVol)

# Select right hippocampal volume for dementia group
dementia_group_rt_hippo_vol <-
  subset(patient_hippocampus_volume, CDR==2.0, select=RightHippoVol)

library(fitdistrplus)
```

```
## Loading required package: MASS

## Loading required package: survival
```

```r
library(MASS)
library(survival)

# Model each group as a normal variable with its own mean and variance
control_group_rt_hippo_vol_normal_model <-
  fitdistr(control_group_rt_hippo_vol[[1]],"normal")
mild_dementia_group_rt_hippo_vol_normal_model <-
  fitdistr(mild_dementia_group_rt_hippo_vol[[1]],"normal")
dementia_group_rt_hippo_vol_normal_model <-
  fitdistr(dementia_group_rt_hippo_vol[[1]],"normal")
```

The joint posterior $p(\mu_j, \sigma_j^2|y_{ij})$ computation is given by:

$$
\begin{aligned}
p(\mu_j, \sigma_j^2|y_{ij}) &\propto p(y|\mu_0, \sigma_0)p(\mu, \sigma), \text{ where } p(\mu, \sigma) \sim N-IG(\mu_0, n_0, \alpha, \beta) \\
&= \prod_{i=1}^{n}\left(\frac{1}{\sqrt{2\pi\sigma^2}} exp\left[-\frac{(y_i-\mu_n)^2}{2\sigma^2}\right]\right)\frac{\sqrt{n_0}}{\sigma\sqrt{2\pi}}\frac{\beta^\alpha}{\Gamma(\alpha)}\left(\frac{1}{\sigma^2}\right)^{\alpha+1} exp\left[-\frac{2\beta+n_0(\mu-\mu_0)^2}{2\sigma^2}\right] \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\frac{n_0^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}\frac{\beta^\alpha}{\Gamma(\alpha)}(\sigma^2)^{-\frac{n}{2}-\frac{1}{2}-\alpha-1} exp\left[\frac{\sum_{i=1}^{n}(y_i-\mu)^2-n_0(\mu-\mu_0)^2-2\beta}{2\sigma^2}\right] \\
p(\sigma_j^2|y_{ij}) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\frac{n_0^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}\frac{\beta^\alpha}{\Gamma(\alpha)}(\sigma^2)^{-\frac{n}{2}-\frac{1}{2}-\alpha-1}\int_{-\infty}^{\infty} exp\left[\frac{\sum_{i=1}^{n}(y_i-\mu)^2-n_0(\mu-\mu_0)^2-2\beta}{2\sigma^2}\right] d\mu
\end{aligned}
$$

Since the integral of a normal distribution with a non-zero mean will be the same as the integral of a normal distribution with a zero mean (the area under the curve stays the same), this can be written as:

$$
\begin{aligned}
p(\sigma_j^2|y_{ij}) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\frac{n_0^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}\frac{\beta^\alpha}{\Gamma(\alpha)}(\sigma^2)^{-\frac{n}{2}-\frac{1}{2}-\alpha-1}\int_{-\infty}^{\infty} exp\left[\frac{-(n_0+n)\mu^2-2\beta}{2\sigma^2}\right] d\mu \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\frac{n_0^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}\frac{\beta^\alpha}{\Gamma(\alpha)}(\sigma^2)^{-\frac{n}{2}-\alpha-\frac{3}{2}}\frac{\sqrt{2\pi}\sigma}{\sqrt{n_0+n}} exp\left[\frac{-2\beta}{2\sigma^2}\right] \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\frac{n_0^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}\frac{\beta^\alpha}{\Gamma(\alpha)}(\sigma^2)^{-\frac{n}{2}-\alpha-1}\frac{\sqrt{2\pi}}{\sqrt{n_0+n}} exp\left[\frac{-\beta}{\sigma^2}\right]
\end{aligned}
$$

The above expression for the marginal posterior is of the Inverse Gamma distribution form with parameters

$$-\alpha_n - 1 = -\frac{n}{2} - \alpha - 1$$
$$\alpha_n = \alpha + \frac{n}{2}$$

and β_n remaining unchanged from that of the N-IG distribution. The value of β_n, as per the class notes is:

$$\beta_n = \beta + \frac{1}{2}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{2}\frac{n_0 n}{n_0 + n}(\mu_0 - \bar{x})^2$$

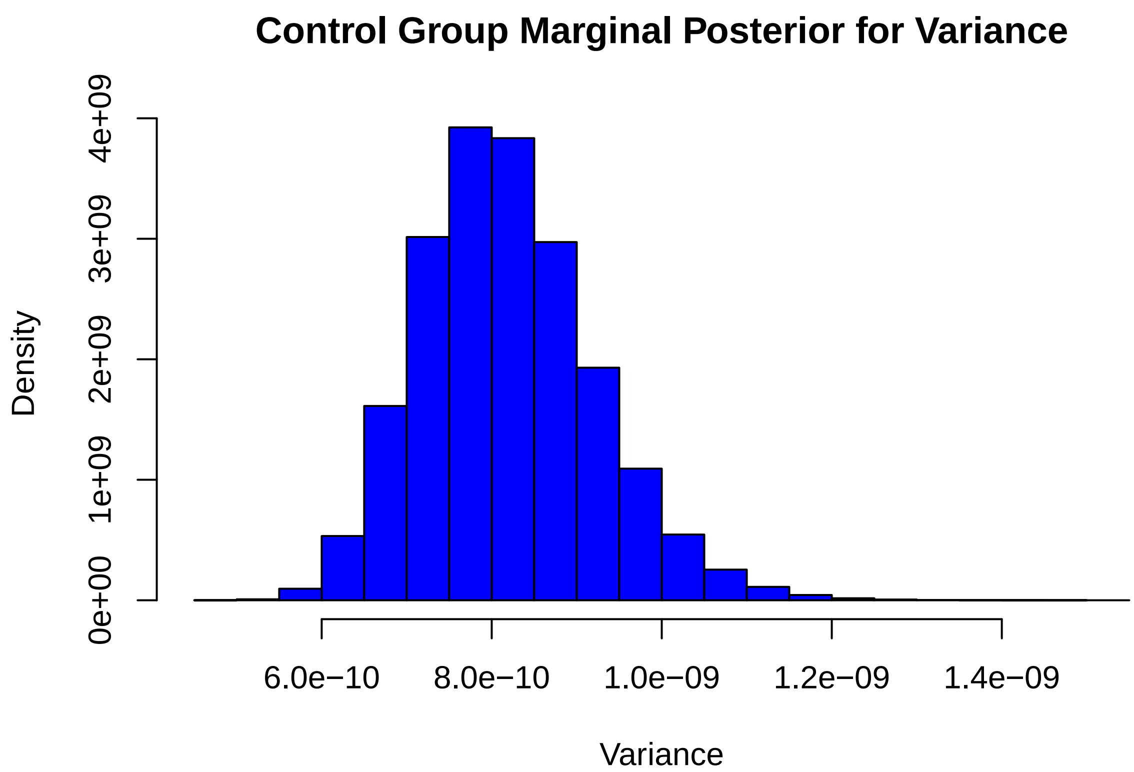
```r
# Find marginal posterior distribution of variance. Create a range for the variance that
# is 20% on both sides of the sample variance. Add up the posterior density for mean value
# that is three standard deviations on both sides.

# Parameters for the N-IG distribution
MU_NOUGHT = 0
N_NOUGHT = 10^(-6)
SHAPE_ALPHA_NOUGHT = 10^(-6)
SCALE_BETA_NOUGHT = 10^(-6)

# Compute the parameters for the marginal
control_grp_shape_alpha_n = SHAPE_ALPHA_NOUGHT + 0.5 * length(control_group_rt_hippo_vol[[1]])
control_grp_scale_beta_n = SCALE_BETA_NOUGHT +
  0.5 * sum((control_group_rt_hippo_vol[[1]] -
             control_group_rt_hippo_vol_normal_model$estimate[[1]])^2) +
          0.5 * (N_NOUGHT * length(control_group_rt_hippo_vol[[1]]) /
                   (N_NOUGHT + length(control_group_rt_hippo_vol[[1]]))) *
          (MU_NOUGHT - control_group_rt_hippo_vol_normal_model$estimate[[1]])^2

# Sample from the inverse gamma marginal posterior for variance
NUMBER_OF_SAMPLES = 10^6
samples_from_variance_marginal_posterior =
  1/rgamma(n=NUMBER_OF_SAMPLES, shape=control_grp_shape_alpha_n, scale=control_grp_scale_beta_n)

# Plot a histogram of the samples from the marginal posterior for variance
hist(samples_from_variance_marginal_posterior, freq=FALSE,
     main= "Control Group Marginal Posterior for Variance", xlab="Variance", ylab="Density", col="blue")
```




```r
library(MCMCpack)
```

```
## Loading required package: coda

## ##
## ## Markov Chain Monte Carlo Package (MCMCpack)

## ## Copyright (C) 2003-2018 Andrew D. Martin, Kevin M. Quinn, and Jong Hee Park

## ##
## ## Support provided by the U.S. National Science Foundation

## ## (Grants SES-0350646 and SES-0350613)
## ##
```

```r
control_grp_posterior_sigma =
  1/rgamma(n=1000, shape=control_grp_shape_alpha_n, scale=control_grp_scale_beta_n)
control_grp_posterior_density =
  dinvgamma(control_grp_posterior_sigma,
            shape=control_grp_shape_alpha_n, scale=control_grp_scale_beta_n)
```

I tried plotting the density function over the histogram (see code after histogram), but got all zero values. Also I did not plot the sample variance over the histogram as the value was very different from that shown on the histogram. The control group variance was 15,059,103 as opposed to the variance values on the histogram which are different in orders of magnitude.