CS6190: Probabilistic Modeling Homework 1
Exponential Families, Conjugate Priors

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Written Part

1. **Expectation of sufficient statistics:** Consider a random variable X from a continuous exponential family with natural parameter $\eta = (\eta_1, \ldots, \eta_n)$. Recall this means the pdf is of the form:

$$p(x) = h(x) \exp\left(\eta \cdot T(x) - A(\eta)\right)$$

(a) Show that $E\left[T(x)|\eta\right] = \nabla A(\eta) = \left(\frac{\partial A}{\partial \eta_1}, \ldots, \frac{\partial A}{\partial \eta_d}\right)$.

Hint: Start with the identity $\int p(x)dx = 1$, and take the derivative with respect to η.

$$\begin{aligned}
\int p(x)dx &= 1 \\
\Rightarrow \int h(x) \exp\left(\eta \cdot T(x) - A(\eta)\right) dx &= 1 \\
\nabla \left(\int h(x) \exp\left(\eta \cdot T(x) - A(\eta)\right) dx\right) &= 0 \\
\int h(x) \exp\left(\eta \cdot T(x) - A(\eta)\right) . \left(T(x) - \nabla A(\eta)\right) dx &= 0 \\
\int T(x) h(x) \exp\left(\eta \cdot T(x) - A(\eta)\right) dx &= \nabla A(\eta) \int h(x) \exp\left(\eta \cdot T(x) - A(\eta)\right) dx \\
\int T(x) p(x) dx &= \nabla A(\eta) \int p(x) dx \\
E\left[T(x)\right] &= \nabla A(\eta)
\end{aligned}$$

(b) Verify this formula works for the Gaussian distribution with unknown mean, μ, and known variance, σ^2.

Hint: Start by thinking about what the natural parameter η and the function $A(\eta)$ are, then verify that the expectation of the Gaussian is the same as $\nabla A(\eta)$.

The pdf for a Gaussian distribution with unknown mean, μ, and known variance, σ^2 is given by:

$$
\begin{aligned}
p(x) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \\
&= \frac{1}{\sqrt{2\pi}} \exp\left[\ln\left(\frac{1}{\sigma}\right)\right] \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \\
&= \frac{1}{\sqrt{2\pi}} \exp\left[-\ln(\sigma) - \frac{(x-\mu)^2}{2\sigma^2}\right] \\
&= \frac{1}{\sqrt{2\pi}} \exp\left[-\ln(\sigma) - \frac{x^2}{2\sigma^2} + \frac{2x\mu}{2\sigma^2} - \frac{\mu^2}{2\sigma^2}\right] \\
&= \frac{1}{\sqrt{2\pi}} \exp\left[\frac{x\mu}{\sigma^2} - \frac{x^2}{2\sigma^2} - \left(\ln(\sigma) + \frac{\mu^2}{2\sigma^2}\right)\right] \\
&= \frac{1}{\sqrt{2\pi}} \exp\left[\begin{bmatrix} \frac{\mu}{\sigma^2} \\ \frac{-1}{2\sigma^2} \end{bmatrix} \cdot \begin{bmatrix} x \\ x^2 \end{bmatrix} - \left(\ln(\sigma) + \frac{\mu^2}{2\sigma^2}\right)\right]
\end{aligned}
$$

Based on the form of the exponential family pdf, we can see that:

$$
\begin{aligned}
A(\eta) &= \ln(\sigma) + \frac{\mu^2}{2\sigma^2}, \\
\eta &= \begin{bmatrix} \frac{\mu}{\sigma^2} \\ \frac{-1}{2\sigma^2} \end{bmatrix}, \text{ and} \\
T(x) &= \begin{bmatrix} x \\ x^2 \end{bmatrix}
\end{aligned}
$$

Based on the equation for $E[T(x)]$ above:

$$
\begin{aligned}
E[T(x)] &= \nabla A(\eta) \\
\Rightarrow E\left[\begin{bmatrix} x \\ x^2 \end{bmatrix}\right] &= \begin{bmatrix} \frac{\partial A}{\partial \eta_1} \\ \frac{\partial A}{\partial \eta_2} \end{bmatrix}, \text{ where } \eta_1 = \frac{\mu}{\sigma^2}, \text{ and } \eta_2 = \frac{-1}{2\sigma^2} \\
\begin{bmatrix} E[x] \\ E[x^2] \end{bmatrix} &= \begin{bmatrix} \frac{\partial\left(\ln(\sigma) + \frac{\mu^2}{2\sigma^2}\right)}{\partial \eta_1} \\ \frac{\partial\left(\ln(\sigma) + \frac{\mu^2}{2\sigma^2}\right)}{\partial \eta_2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial\left(\ln(\sigma) + \frac{\sigma^2}{2}\frac{\mu^2}{\sigma^4}\right)}{\partial \eta_1} \\ \frac{\partial\left(\ln\left(\sqrt{\frac{-1}{2\eta_2}}\right) - \mu^2\eta_2\right)}{\partial \eta_2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial\left(\ln(\sigma) + \frac{\sigma^2}{2}\eta_1^2\right)}{\partial \eta_1} \\ \frac{\partial\left(\ln\left(\frac{i}{\sqrt{2}\sqrt{\eta_2}}\right) - \mu^2\eta_2\right)}{\partial \eta_2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sigma^2}{2}.2\eta_1 \\ \frac{\sqrt{2}\sqrt{\eta_2}}{i}.\frac{i}{\sqrt{2}}.\frac{-1}{2}\eta_2^{\frac{-3}{2}} - \mu^2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sigma^2}{2}.2\frac{\mu}{\sigma^2} \\ \frac{-1}{2\eta_2} - \mu^2 \end{bmatrix} \\
&= \begin{bmatrix} \mu \\ \sigma^2 - \mu^2 \end{bmatrix} \\
\Rightarrow E[x] &= \mu, \text{ and} \\
E[x^2] &= \sigma^2 - \mu^2, \text{ which is not the case since } \sigma^2 = E[x^2] - \mu^2
\end{aligned}
$$

2. **Noninformative priors for the Poisson distribution:** Let $X \sim Pois(\lambda)$. Recall that the pmf of the Poisson is

$$P(X = k; \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

(a) Rewrite the above pmf in exponential family form. What is the natural parameter? What is the sufficient statistic?

$$\begin{aligned} P(X = k; \lambda) &= \frac{\lambda^k e^{-\lambda}}{k!} \\ &= \frac{1}{k!} \exp\left(\ln \lambda^k e^{-\lambda}\right) \\ &= \frac{1}{k!} \exp\left(k \ln \lambda - \lambda\right) \\ \eta &= \ln \lambda, \text{ natural parameter} \\ h(k) &= k, \text{ sufficient statistic} \end{aligned}$$

(b) Give at least two different options for noninformative priors for $p(\lambda)$.

i. **Uniform Prior:** In this case the prior is

$$p(\lambda) = c, \text{ for } 0 \leq x \leq \infty, \text{, where c is a constant}$$

ii. **Jeffrey's Prior:** In this case the prior is

$$
\begin{aligned}
\mathcal{I}(\lambda) &= E\left[\left(\frac{\partial}{\partial\lambda}\ln p(x|\lambda)\right)^2\right] \\
&= -E\left[\frac{\partial^2}{\partial\lambda^2}\ln p(x|\lambda)\right] \\
&= -E\left[\frac{\partial^2}{\partial\lambda^2}\ln\frac{\lambda^x e^{-\lambda}}{x!}\right] \\
&= -E\left[\frac{\partial^2}{\partial\lambda^2}\left(x\ln\lambda-\lambda-\ln x!\right)\right] \\
&= E\left[\frac{\partial}{\partial\lambda}\left(\frac{x}{\lambda}-1\right)\right] \\
&= -E\left[\frac{-x}{\lambda^2}\right] \\
&= E\left[\frac{x}{\lambda^2}\right] \\
&= \frac{1}{\lambda^2}E\left[x\right] \\
&= \frac{1}{\lambda^2}\sum_{i=1}^{\infty} xp(x) \\
&= \frac{1}{\lambda^2}\sum_{i=1}^{\infty} x\frac{\lambda^x e^{-\lambda}}{x!} \\
&= \frac{e^{-\lambda}}{\lambda}\sum_{i=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!} \\
&= \frac{e^{-\lambda}}{\lambda}\left(\frac{\lambda^0}{0!}+\frac{\lambda^1}{1!}+\frac{\lambda^2}{2!}+\dots\right) \\
&= \frac{e^{-\lambda}}{\lambda}e^{\lambda} \\
&= \frac{1}{\lambda} \\
p(\lambda) &= \sqrt{\mathcal{I}(\lambda)} \\
&= \sqrt{\frac{1}{\lambda}}
\end{aligned}
$$

(c) What are the resulting posteriors for your two options? Are they proper (i.e., can they be normalized)?

i. **Uniform Prior:** The posterior is

$$
\begin{aligned}
p(\lambda|x) &\propto p(x|\lambda)p(\lambda) \\
&= \prod_{i=1}^{n}\frac{\lambda^{x_i}e^{-\lambda}}{x_i!}.c, \text{based on the assumtion of i.i.d. samples} \\
&\propto \prod_{i=1}^{n}\frac{\lambda^{x_i}e^{-\lambda}}{x_i!}
\end{aligned}
$$

Even though the prior cannot be normalized, since it will not add up to 1, the posterior shown above, where the constant c has been dropped, can be normalized. The reason is that the posterior is the product of distributions that can themselves be normalized.

ii. **Jeffrey's Prior:** The posterior is

$$\begin{aligned}
p(\lambda|x) &\propto p(x|\lambda)p(\lambda) \\
&= \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \cdot \sqrt{\frac{1}{\lambda}}, \text{based on the assumtion of i.i.d. samples} \\
&= \prod_{i=1}^{n} \frac{\lambda^{x_i - \frac{1}{2}} e^{-\lambda}}{x_i!}
\end{aligned}$$

This can be normalized since it is the product of the likelihood, which is a product of Poisson distributions, and so can be normalized, and the prior, which can also be normalized.

3. **Non-conjugate priors:** Let X_i be from a Gaussian with known variance σ^2 and mean μ with uniform prior, i.e.,

$$\begin{aligned}
\mu &\sim Unif(a, b) \\
X_i &\sim N(\mu, \sigma^2)
\end{aligned}$$

What is the posterior pdf, $p(\mu|x_1, \ldots, x_n; \sigma^2, a, b)$?

Hint: There will be an integral that you won't be able to analytically solve (just leave it in integral form).

$$\begin{aligned}
p(\mu|x_1, \ldots, x_n; \sigma^2, a, b) &= p(x_1, \ldots, x_n|\mu; \sigma^2)p(\mu; a, b) \\
&= \int_{-\infty}^{\infty} p(x|\mu; \sigma^2)p(\mu; a, b)dx \\
&= \int_{a}^{b} p(x|\mu; \sigma^2)\frac{1}{b-a}dx \\
&= \frac{1}{b-a}\frac{1}{\sqrt{2\pi}\sigma}\int_{a}^{b} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx
\end{aligned}$$

R Coding Part

```r
# Density function for Inverse Gamma: conjugate prior for Gaussian sigma^2
density_inverse_gamma <- function(x_coordinates, alpha, beta) {

  return (beta^alpha / gamma(alpha) * x_coordinates^(-alpha - 1) * exp(-beta / x_coordinates))

}

# Density function for Normal-Inverse-Gamma with parameters mu0, lambda, alpha, beta
density_normal_inverse_gamma <-
  function(distribution_of_mean, distribution_of_variance, mu0 = 0, lambda = 1, alpha = 8, beta = 16)
{
  return (dnorm(distribution_of_mean, mean = mu0, sd = sqrt(distribution_of_variance / lambda)) *
            density_inverse_gamma(distribution_of_variance, alpha, beta))
}

# Read data for hippocampus volume and patient details
hippocampus_volume <- read.csv(file="oasis_hippocampus.csv", header=TRUE, sep=",")
patient_details <- read.csv(file="oasis_cross-sectional.csv", header=TRUE, sep=",")
```

```r
# Join both frames on identifer
patient_hippocampus_volume <- merge(hippocampus_volume, patient_details, by="ID")

# Select right hippocampal volume for control group
control_group_rt_hippo_vol <-
  subset(patient_hippocampus_volume, CDR==0.0, select=RightHippoVol)

# Select right hippocampal volume for mild dementia group
mild_dementia_group_rt_hippo_vol <-
  subset(patient_hippocampus_volume, CDR!=0.0 & CDR!=2.0 & !is.null(CDR),
         select=RightHippoVol)

# Select right hippocampal volume for dementia group
dementia_group_rt_hippo_vol <-
  subset(patient_hippocampus_volume, CDR==2.0, select=RightHippoVol)

library(fitdistrplus)
```

```
## Loading required package: MASS

## Loading required package: survival
```

```r
library(MASS)
library(survival)

# Model each group as a normal variable with its own mean and variance
control_group_rt_hippo_vol_normal_model <-
  fitdistr(control_group_rt_hippo_vol[[1]],"normal")
mild_dementia_group_rt_hippo_vol_normal_model <-
  fitdistr(mild_dementia_group_rt_hippo_vol[[1]],"normal")
dementia_group_rt_hippo_vol_normal_model <-
  fitdistr(dementia_group_rt_hippo_vol[[1]],"normal")
```

The joint posterior $p(\mu_j, \sigma_j^2|y_{ij})$ computation is given by:

$$
\begin{aligned}
p(\mu_j, \sigma_j^2|y_{ij}) &\propto p(y|\mu_0, \sigma_0)p(\mu, \sigma), \text{ where } p(\mu, \sigma) \sim N - IG(\mu_0, n_0, \alpha, \beta) \\
&= \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi\sigma^2}} exp\left[-\frac{(y_i - \mu_n)^2}{2\sigma^2} \right] \right) \frac{\sqrt{n_0}}{\sigma\sqrt{2\pi}} \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2} \right)^{\alpha+1} exp\left[-\frac{2\beta + n_0(\mu - \mu_0)^2}{2\sigma^2} \right] \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{n_0^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-\frac{n}{2}-\frac{1}{2}-\alpha-1} exp\left[\frac{\sum_{i=1}^{n}(y_i - \mu)^2 - n_0(\mu - \mu_0)^2 - 2\beta}{2\sigma^2} \right] \\
p(\sigma_j^2|y_{ij}) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{n_0^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-\frac{n}{2}-\frac{1}{2}-\alpha-1} \int_{-\infty}^{\infty} exp\left[\frac{\sum_{i=1}^{n}(y_i - \mu)^2 - n_0(\mu - \mu_0)^2 - 2\beta}{2\sigma^2} \right] d\mu
\end{aligned}
$$

Since the integral of a normal distribution with a non-zero mean will be the same as the integral of a normal distribution with a zero mean (the area under the curve stays the same), this can be written as:

$$
\begin{aligned}
p(\sigma_j^2|y_{ij}) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{n_0^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-\frac{n}{2}-\frac{1}{2}-\alpha-1} \int_{-\infty}^{\infty} exp\left[\frac{-(n_0 + n)\mu^2 - 2\beta}{2\sigma^2} \right] d\mu \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{n_0^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-\frac{n}{2}-\alpha-\frac{3}{2}} \frac{\sqrt{2\pi}\sigma}{\sqrt{n_0 + n}} exp\left[\frac{-2\beta}{2\sigma^2} \right] \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{n_0^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-\frac{n}{2}-\alpha-1} \frac{\sqrt{2\pi}}{\sqrt{n_0 + n}} exp\left[\frac{-\beta}{\sigma^2} \right]
\end{aligned}
$$

The above expression for the marginal posterior is of the Inverse Gamma distribution form with parameters

$$-\alpha_n - 1 = -\frac{n}{2} - \alpha - 1$$
$$\alpha_n = \alpha + \frac{n}{2}$$

and β_n remaining unchanged from that of the N-IG distribution. The value of β_n, as per the class notes is:

$$\beta_n = \beta + \frac{1}{2}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{2}\frac{n_0 n}{n_0 + n}(\mu_0 - \bar{x})^2$$

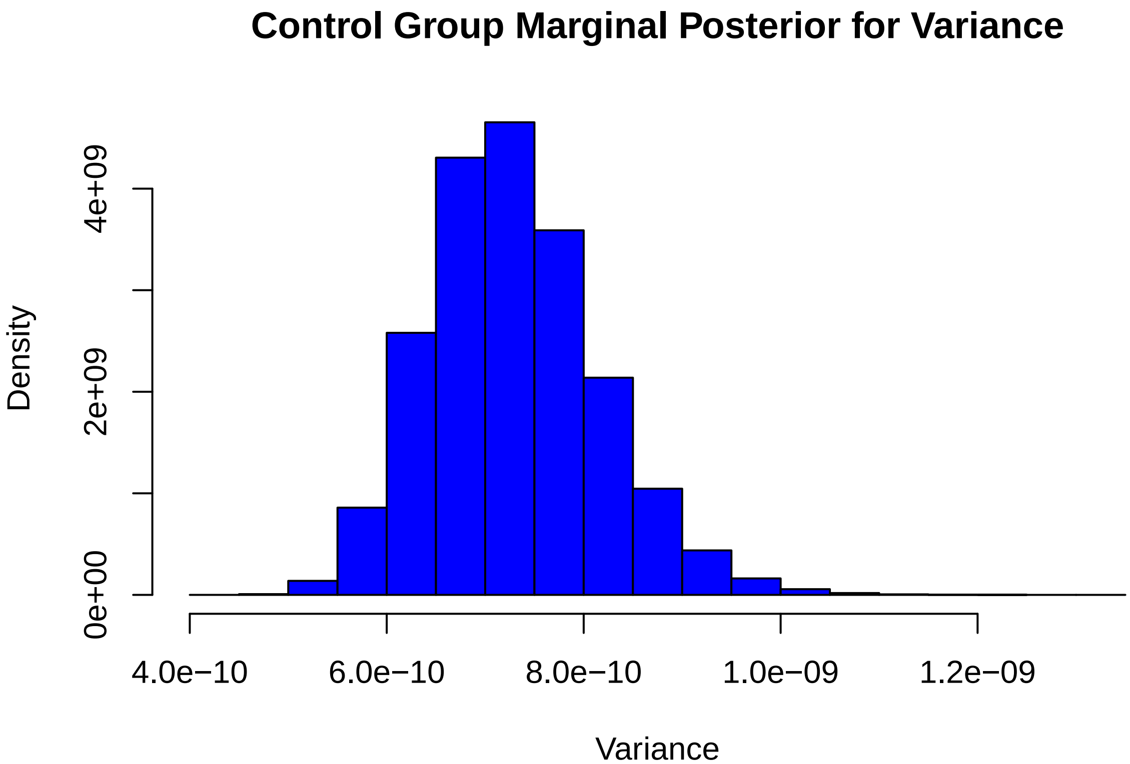
```r
# Find marginal posterior distribution of variance. Create a range for the variance that
# is 20% on both sides of the sample variance. Add up the posterior density for mean value
# that is three standard deviations on both sides.

# Parameters for the N-IG distribution
MU_NOUGHT = 0
N_NOUGHT = 10^(-6)
SHAPE_ALPHA_NOUGHT = 8
SCALE_BETA_NOUGHT = 16

# Compute the parameters for the marginal
control_grp_shape_alpha_n = SHAPE_ALPHA_NOUGHT + 0.5 * length(control_group_rt_hippo_vol[[1]])
control_grp_scale_beta_n = SCALE_BETA_NOUGHT +
  0.5 * sum((control_group_rt_hippo_vol[[1]] -
              control_group_rt_hippo_vol_normal_model$estimate[[1]])^2) +
          0.5 * (N_NOUGHT * length(control_group_rt_hippo_vol[[1]]) /
                   (N_NOUGHT + length(control_group_rt_hippo_vol[[1]]))) *
          (MU_NOUGHT - control_group_rt_hippo_vol_normal_model$estimate[[1]])^2

# Sample from the inverse gamma marginal posterior for variance
NUMBER_OF_SAMPLES = 10^6
samples_from_variance_marginal_posterior =
  1/rgamma(n=NUMBER_OF_SAMPLES, shape=control_grp_shape_alpha_n, scale=control_grp_scale_beta_n)

# Plot a histogram of the samples from the marginal posterior for variance
hist(samples_from_variance_marginal_posterior, freq=FALSE,
     main= "Control Group Marginal Posterior for Variance", xlab="Variance", ylab="Density", col="blue")
```




```
# Plot the density function
#lines(number_sequence, y_probability_density_trend, lty=1, lwd=3, col="red")
#legend("topright", legen=c("Random Number Density", "PDF of function Y"),
#       col=c("blue", "red"), lwd=3, lty=1)
```