Decomposition and smoothing

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Outline

- Time series decomposition
- Simple exponential smoothing
- Holt's winters method

Time series components

- Trend: long term increase or decrease in the data.
- Seasonal: a seasonal pattern exists when a series is influenced by seasonal factors.
- Cyclic: A cyclic pattern exists when data exhibit rises and falls that are not of fixed period.

Time series components

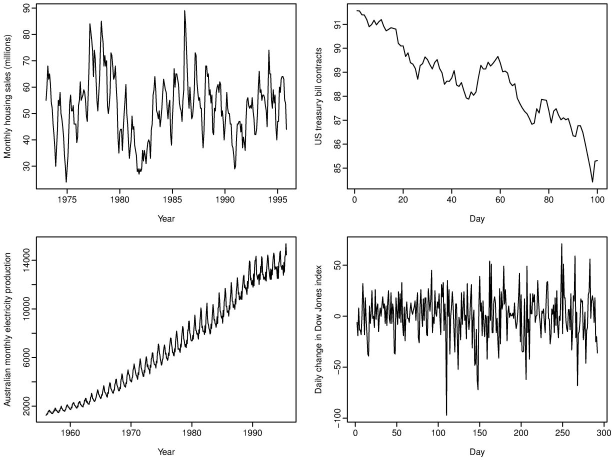


Figure: Examples with different combinations of the components.

TS decomposition

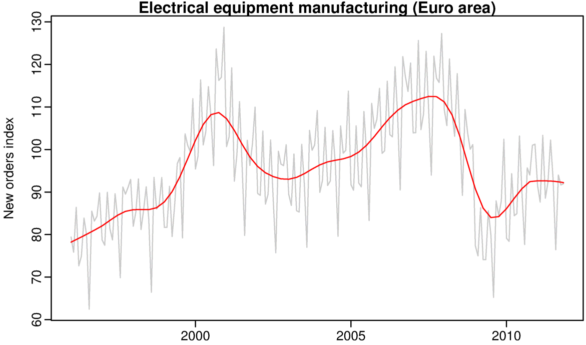
- Classical decomposition

$$X_t = S_t + T_t + E_t$$

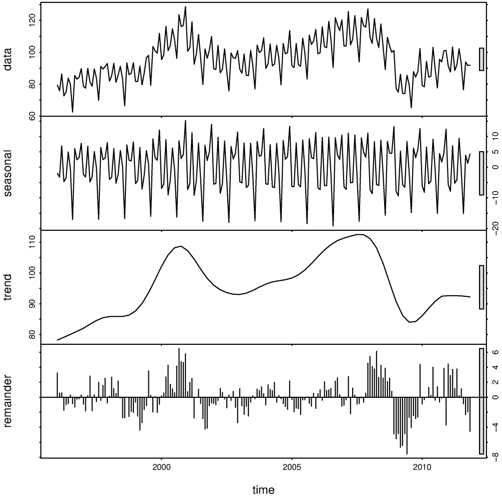
- Trend (T_t) - Long term movement in the mean.
- Seasonal variation (S_t) - Cyclical fluctuations due to calendar.
- Errors (E_t) - random and all other unexplained variations.

Electrical equipment manufacturing example.

```
fit <- stl(elecequip, s.window=5)
plot(elecequip, col="gray",
 main="Electrical equipment manufacturing",
 ylab="New orders index", xlab="")
lines(fit$time.series[,2],col="red",ylab="Trend")
```

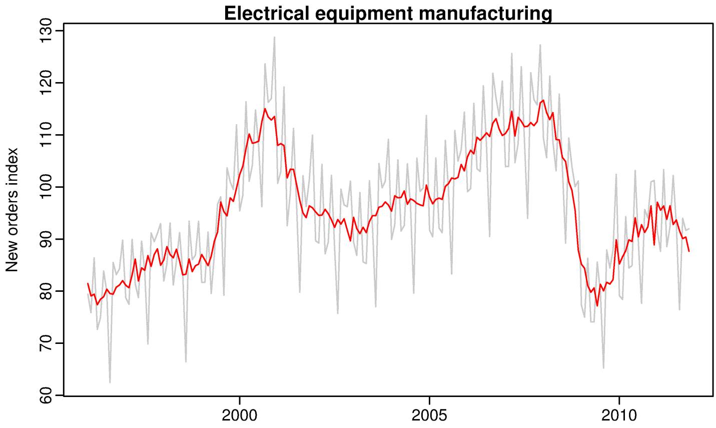


Additive decomposition using STL

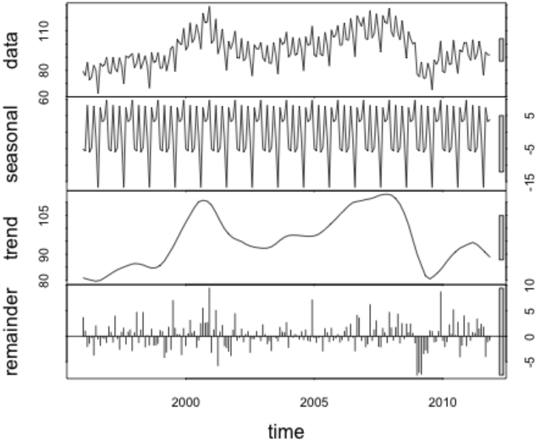


Seasonal adjusted data

We can obtain seasonal adjusted data by $y_t - S_t$.

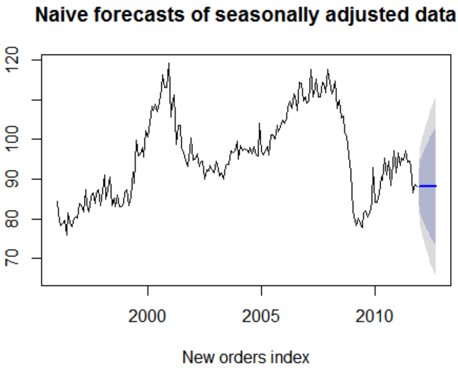


Application of STL while seasonal component does not change over time. Check STL options.



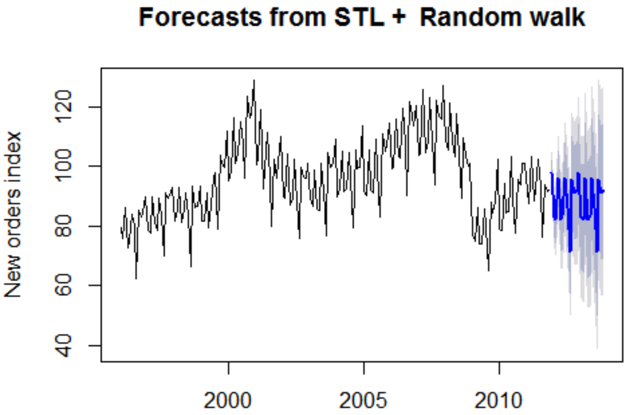
Forecast with decomposition

- Assuming an additive decomposition, the decomposed time series can be written as $y_t = \hat{S}_t + \hat{A}_t$, where $\hat{A}_t = \hat{T}_t + \hat{E}_t$ is the seasonally adjusted component.
- Forecast the seasonal component and seasonally adjusted component separately.



Forecast with decomposition

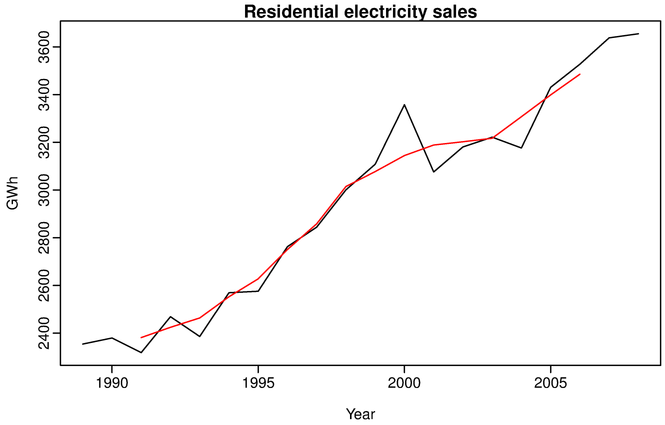
Adding back the seasonal component.



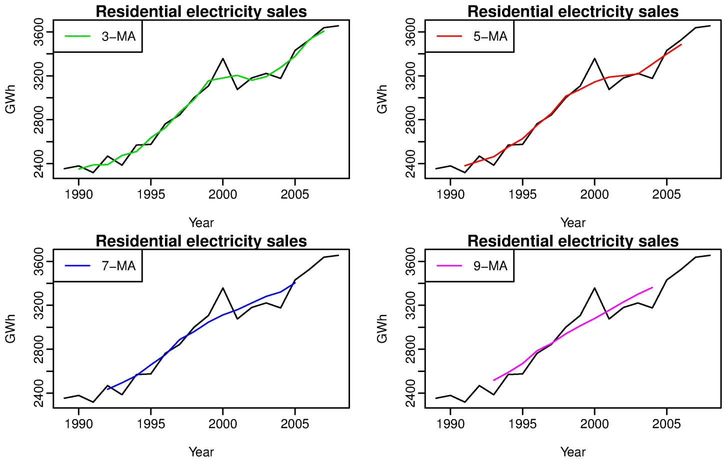
Moving average smoothing

A moving average of order m can be written as

$$\hat{T}_t = \frac{1}{m} \sum_{j=-k}^{k} y_{t+j}, \;\; m = 2k + 1.$$



Moving average at different orders



Moving average of moving average

- One reason for MA of MA is to make an ever-order MA symmetric.
- A 2×4 -MA can be written as

$$\frac{1}{2}[\frac{1}{4}(y_{t-2} + y_{t-1} + y_t + y_{t+1}) + \frac{1}{4}(y_{t-1} + y_t + y_{t+1} + y_{t+2})]$$

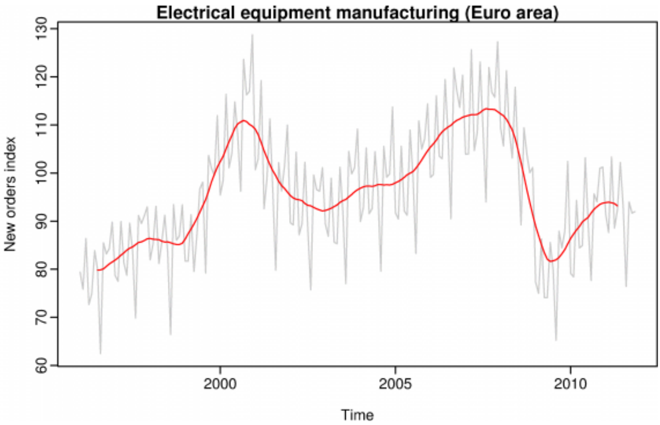
$$= \frac{1}{8}y_{t-2} + \frac{1}{4}y_{t-1} + \frac{1}{4}y_t + \frac{1}{4}y_{t+1} + \frac{1}{8}y_{t+2}.$$

- In general, an ever order MA should be followed by an even one, and vice versa.

Estimating trend cycle with seasonal data

- The 2×4-MA can be used to estimate quarterly data.
- $2 \times m$-MA v.s. $m + 1$MA.
- If the seasonal period is even and of order m, use $2 \times m$ MA to estimate the trend cycle.
- If it's odd , then use m-MA to estimate the trend cycle.
- Thus 2×12-MA for monthly trend, and 7-MA for daily trend.

Application of 2×12-MA.



- Simple exponential smoothing(SES) is suitable for forecasting data with no trend or seasonal patterns.

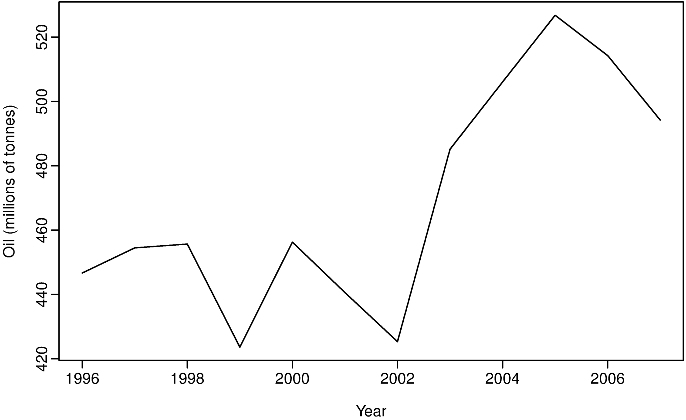


Figure: Oil production SAUDI from 1996-2007.

Models

- Previously we discussed the naive method:

$$\hat{y}_{T+h} = y_T \text{ for all } h.$$

  Thus the most recent obs. is the only important one for prediction purposes.

- The other method we mentioned, the average method:

$$\hat{y}_{T+h} = \frac{1}{T}\sum_{t=1}^{T} y_t.$$

  Thus all obs. are of equal importance for forecasting.

- Naturally, intuition tells us that a better forecasting method should lie in between these two extremes. More recent obs. should be more important and assigned more weights.

SES model

- Forecasts are calculated using weighted averages with exponential decaying weights.

$$\hat{y}_{T+1|T} = \alpha y_T + \alpha(1-\alpha) y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \cdots + (1-\alpha)^T y_{1|0}.$$

α is a smoothing parameter.

Obs.	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
y_T	0.2	0.4	0.6	0.8
y_{T-1}	0.16	0.24	0.24	0.16
y_{T-2}	0.128	0.144	0.096	0.032
y_{T-3}	0.1024	0.0864	0.0384	0.0064

- The weight goes down exponentially. When $\alpha = 1$, this reduces to the naive estimate.

Weighted average form

- By definition, the forecast at $t+1$ equals to a weighted average between the most recent obs. y_t and the most recent forecast:
$$\hat{y}_{t+1|t} = \alpha y_t + (1-\alpha)\hat{y}_{t|t-1}.$$

$$\begin{aligned}
\hat{y}_{2|1} &= \alpha y_1 + (1-\alpha)\ell_0 \\
\hat{y}_{3|2} &= \alpha y_2 + (1-\alpha)\hat{y}_{2|1} \\
\hat{y}_{4|3} &= \alpha y_3 + (1-\alpha)\hat{y}_{3|2} \\
&\vdots \\
\hat{y}_{T+1|T} &= \alpha y_T + (1-\alpha)\hat{y}_{T|T-1}
\end{aligned}$$

Weighted average form

By substitution we have

$$
\begin{aligned}
\hat{y}_{2|1} &= \alpha y_1 + (1-\alpha)\ell_0 \\
\hat{y}_{3|2} &= \alpha y_2 + (1-\alpha)\hat{y}_{2|1} \\
\hat{y}_{4|3} &= \alpha y_3 + (1-\alpha)\hat{y}_{3|2} \\
&\vdots \\
\hat{y}_{T+1|T} &= \alpha y_T + (1-\alpha)\hat{y}_{T|T-1}
\end{aligned}
$$

Weighted average form

Continuing substitution, we have

$$
\begin{aligned}
\hat{y}_{3|2} &= \alpha y_2 + (1-\alpha)\left[\alpha y_1 + (1-\alpha)\ell_0\right] \\
&= \alpha y_2 + \alpha(1-\alpha)y_1 + (1-\alpha)^2\ell_0 \\
\hat{y}_{4|3} &= \alpha y_3 + (1-\alpha)[\alpha y_2 + \alpha(1-\alpha)y_1 + (1-\alpha)^2\ell_0] \\
&= \alpha y_3 + \alpha(1-\alpha)y_2 + \alpha(1-\alpha)^2 y_1 + (1-\alpha)^3\ell_0 \\
&\ \vdots \\
\hat{y}_{T+1|T} &= \sum_{j=0}^{T-1} \alpha(1-\alpha)^j y_{T-j} + (1-\alpha)^T \ell_0.
\end{aligned}
$$

Component form

- Time series components consist of level, trend, and seasonal component.
- For simple exponential smoothing, only level l_t is needed.

$$\begin{aligned} \text{Forecast equation} \qquad & \hat{y}_{t+1|t} = l_t \\ \text{Smoothing equation} \qquad & l_t = \alpha y_t + (1-\alpha) l_{t-1}, \end{aligned}$$

where l_t is the level of the time series at time t.

Error correction form

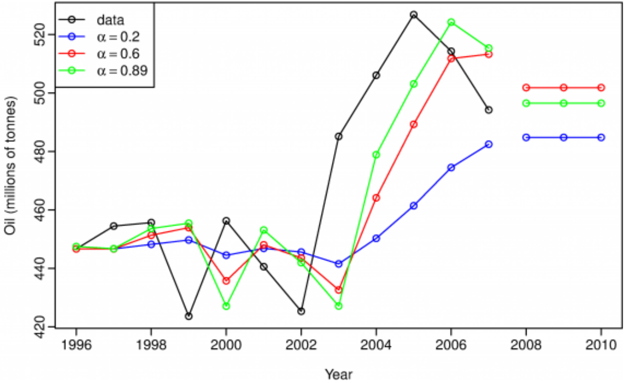
- We can rearrange the component form to get the following, where e_t is the one step within-sample forecast error at time t.

$$\begin{aligned} l_t &= l_{t-1} + \alpha(y_t - l_{t-1}) \\ &= l_{t-1} + \alpha e_t \end{aligned}$$

  where $e_t = y_t - l_{t-1} = y_t - \hat{y}_{t|t-1}$ for $t = 1, \cdots, T$.
- If the error at time t is negative, then the level at $t-1$ is over-estimated. The new level is then the previous level adjusted downwards.
- α controls the smoothness of the level forecast, smaller implies smoother.

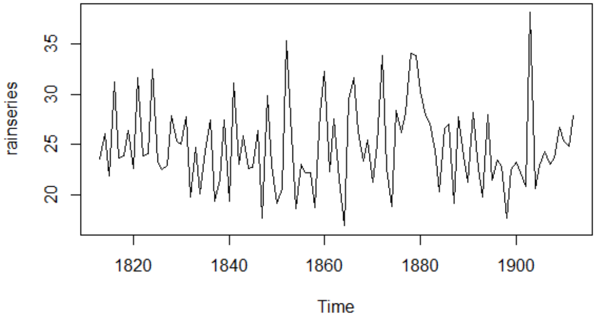
We can obtain the optimal α via minimizing the SSE

$$\mathrm{SSE} = \sum_{t=1}^{T} (y_t - \hat{y}_{t|t-1})^2 = \sum_{t=1}^{T} e_t^2.$$



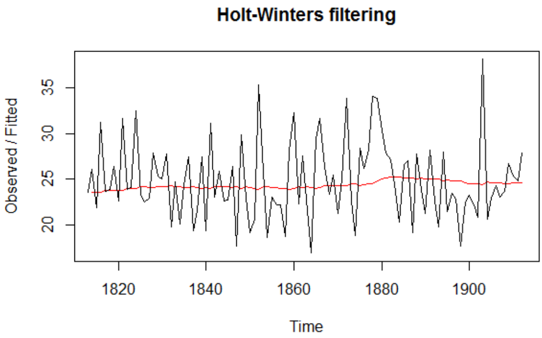
Simple exponential smoothing example

- If the time series contains no trend and seasonality, then we can use simple linear smoothing to make short term forecast.
- Degree of smoothing is controlled by alpha.
- The following figure shows the annual rainfall by inches for London



Simple exponential smoothing example

- From this plot, the mean stays at constant level, and doesn't have much seasonality.
- Use function HoltWinters(). For simple exponential smoothing, we set the parameters beta=FALSE and gamma=FALSE.
- The following figure shows the fitted line over original data.



Simple exponential smoothing example

- For each year, we can compute the error between the observed value and fitted value, thus the in sample sum of square errors(for all the years we have data with)

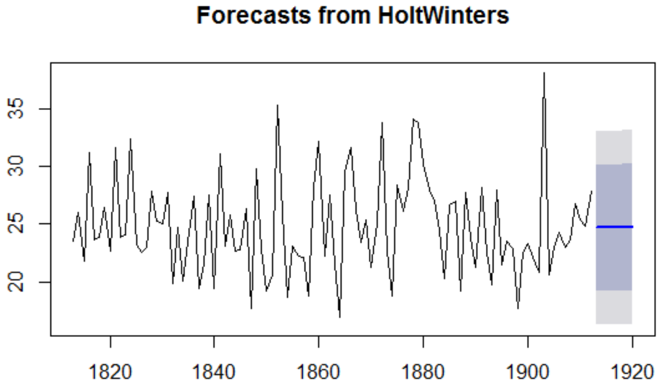
```
> rainseriesforecasts$SSE
[1] 1828.855
```

- For simple exponential smoothing, we use the first value (23.56) as the initial value. This can be specified in HoltWinters using "l.start" option.
- Once the model is fitted, we can use the forecast function to obtain future rainfall values.

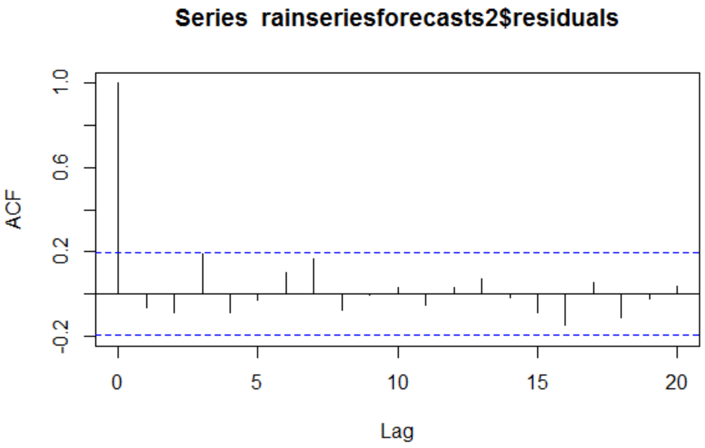
```
>  forecast(rainseriesforecasts, h=2)
     Point Forecast    Lo 80    Hi 80    Lo 95    Hi 95
1913       24.67819 19.17493 30.18145 16.26169 33.09470
1914       24.67819 19.17333 30.18305 16.25924 33.09715
```

Simple exponential smoothing example

- One can plot the predictions using plot.forecast



- Now we can check the residuals and find out whether we have taken care of all the autocorrelations.



Simple exponential smoothing example

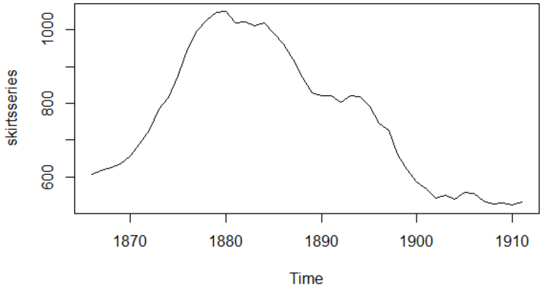
- To formally test if there are non-zero autocorrelations at lags 1-20, we can use the Box-Ljung test.

```
> Box.test(rainseriesforecasts2$residuals,
lag=20, type="Ljung-Box")

Box-Ljung test

data:  rainseriesforecasts2$residuals
X-squared = 17.4008, df = 20, p-value = 0.6268
```

- The P-value is 0.63, thus we do not reject the null hypothesis that the autocorrelations from lag 1-20 are 0.

Holt's exponential smoothing example

- If the time series contains certain trend but no seasonality, then we can use Holt's exponential smoothing to make short term forecast.
- Smoothing is controlled by α and β. α is controlling the level of the smoothing, and β is controlling the slope of the smoothing.



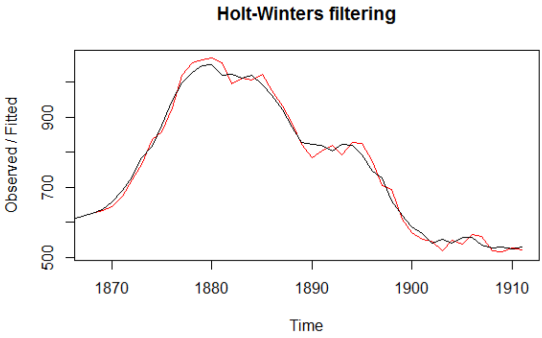
Holt's exponential smoothing example

- From this plot, the mean does not stay at constant level, which means trend exists. But there doesn't seem to be much seasonality.
- For Holt's exponential smoothing, we set the parameter gamma=FALSE.

```
> skirtsseriesforecasts
Holt-Winters exponential smoothing with trend and without season
 HoltWinters(x = skirtsseries, gamma = FALSE)
Smoothing parameters:
 alpha:  0.8383481
 beta :  1
 gamma:  FALSE
```

Holt's exponential smoothing example

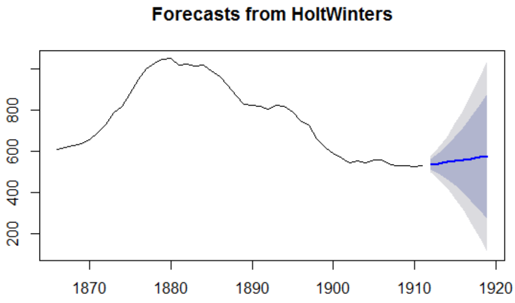
- The forecasts agree well with the real data.
- For Holt's exponential smoothing, we can specify the initial values for level and slope in HoltWinters using "l.start" and "b.start" option.



Holt's exponential smoothing example

- We can forecast several steps ahead as in simple exp. smoothing.

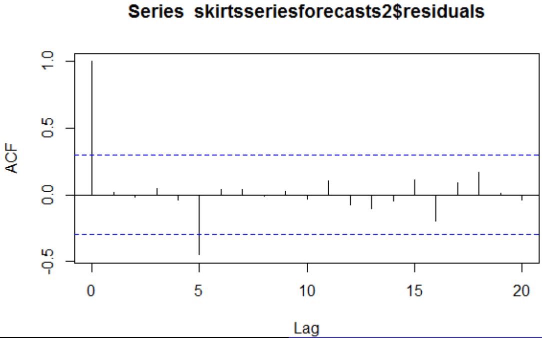
```
> forecast(skirtsseriesforecasts, h=2)
     Point Forecast    Lo 80    Hi 80    Lo 95    Hi 95
1912       534.9990 509.5521 560.4460 496.0813 573.9168
1913       540.6895 491.0105 590.3685 464.7120 616.6670
```



Holt's exponential smoothing example

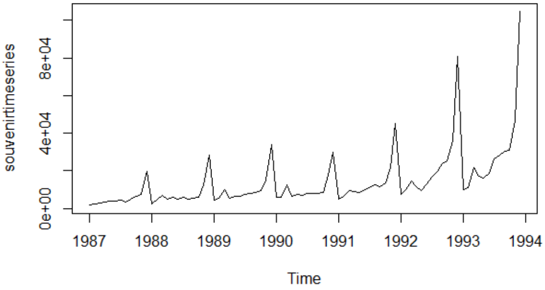
- Check the residuals.

```
> Box.test(skirtsseriesforecasts2$residuals,
lag=20, type="Ljung-Box")
Box-Ljung test
data:  skirtsseriesforecasts2$residuals
X-squared = 19.7312, df = 20, p-value = 0.4749
```



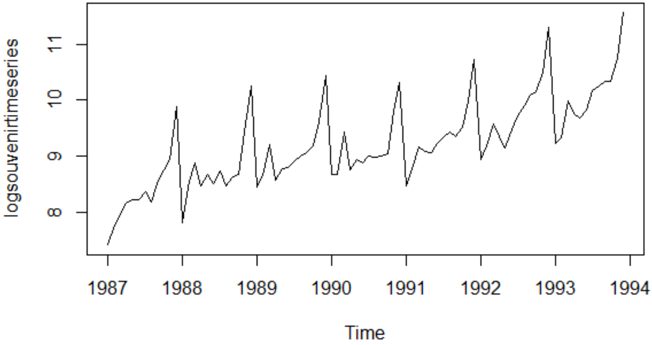
Holt-winters exponential smoothing example

- If the time series contains both trend and seasonality, then we can use Holt-winters exponential smoothing to make short term forecast.
- Smoothing is controlled by α, β and γ. α is controlling the level of the smoothing, and β is controlling the slope of the smoothing,and γ is controlling seasonality.



Holt-winters exponential smoothing example

- Clearly the variance is not stable, we do log transformation.



Holt-winters exponential smoothing example

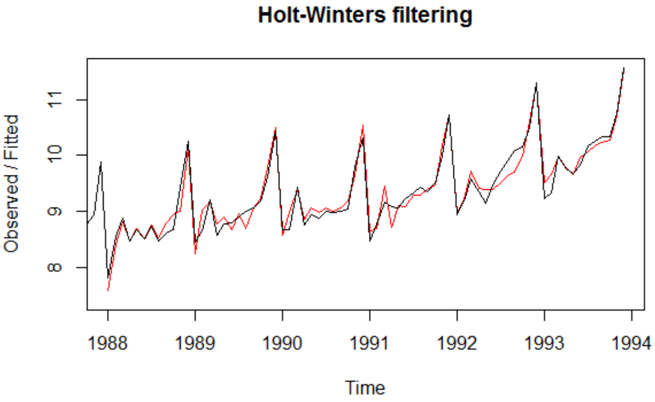
- From this plot, the mean does not stay at constant level, which means trend exists. There also exists clear seasonality.

```
souvenirtimeseriesforecasts=HoltWinters(logsouvenirtimeserie
> souvenirtimeseriesforecasts
Holt-Winters exponential smoothing with trend and additive s
 HoltWinters(x = logsouvenirtimeseries)
Smoothing parameters:
 alpha:  0.413418
 beta :  0
 gamma:  0.9561275
```

- Meaning of the parameters

Holt-winters exponential smoothing example

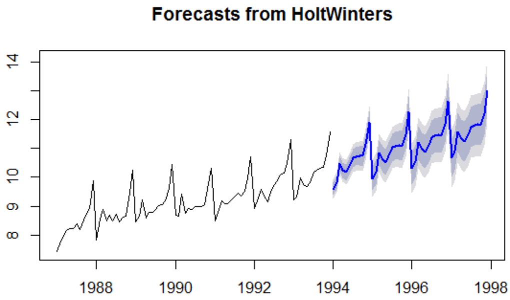
- The forecasts agree well with the real data, especially the seasonal peaks.



Holt-winters exponential smoothing example

- We can forecast several steps ahead as in simple exp. smoothing.

```
> forecast(souvenirtimeseriesforecasts, h=2)
         Point Forecast    Lo 80     Hi 80    Lo 95     Hi 95
Jan 1994       9.597062 9.381514  9.812611 9.267409  9.926715
Feb 1994       9.830781 9.597539 10.064024 9.474068 10.187495
```



- Check the residuals.

```
> Box.test(souvenirtimeseriesforecasts2$residuals,
 lag=20, type="Ljung-Box")

data:  souvenirtimeseriesforecasts2$residuals
X-squared = 17.5304, df = 20, p-value = 0.6183
```