16/11/20
Monday, November 16, 2020
8:56 AM

Unit-3

Rolle's Theorem : Let $f(x)$ be a function defined on $[a,b]$ and is satisfying following conditions

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f(x)$ is differentiable in (a,b)

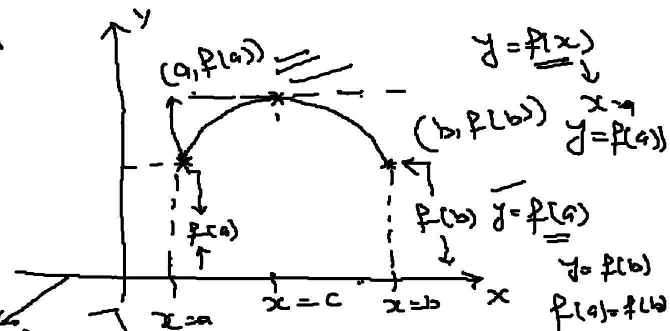
(iii) $f(a) = f(b)$

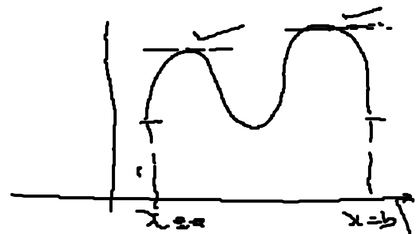
$[a,b]$, (a,b)

$[1,2]$, $(1,2)$

then there exist atleast one real point $c \in (a,b)$ such that

$f'(c) = 0 \rightarrow$ slope of x-axis





Slope of x-axis $= 0$

$\tan\theta$
$\tan\theta = 0$

$f'(x) \rightarrow$ Slope of tangent to the curve

$\tan\theta = $ slope

Q.) Verify Rolle's theorem $f(x) = \frac{8x^2}{3} - 2x$, $x \in [0, \frac{3}{4}]$

Sol. $f(x) = \frac{8x^2}{3} - 2x$, $x \in [0, 3/4]$

$f(x)$ is well-defined on $[0, 3/4]$ $\because f(x)$ is polynomial function.

(i) Since $f(x)$ is a polynomial function, so it is continuous for $x \in [0, 3/4]$

(ii) Now $f'(x) = \frac{16x}{3} - 2$ is well-defined on $(0, 3/4)$

$\Rightarrow f(x)$ is differentiable on $(0, 3/4)$

(iii) $f(0) = 0$ & $f(3/4) = \frac{8}{3}(3/4)^2 - 2(3/4)$

$= \frac{8}{3} \times \frac{9}{16} - \frac{3}{2} = 0$

$f(0) = 0 = f(3/4)$

Since all the conditions of Rolle's thm. are true, so Rolle's thm. is applicable.

Now, let $f'(c) = 0 \Rightarrow \frac{16c}{3} - 2 = 0$ $\quad (0, 3/4)$

$\Rightarrow c = \frac{6}{16} = \frac{3}{8} \in (0, 3/4)$

$\Rightarrow$ Hence, Rolle's thm. is verified

Q→ Discuss the applicability of Rolle's thm. $f(x) = \frac{x(x-2)}{x-1}$, $[0,2]$

Sol→ $f(x) = \frac{x[x-2]}{x-1}$, Here D_f = Domain of $f(x)$ is $R-\{1\}$
Hence, $f(x)$ is not defined at $x=1$ and $x=1 \in [0,2]$
$\therefore$ $f(x)$ is not defined for all $x \in [0,2]$, so Rolle's thm. is not applicable.

$[0,1) \cup (1,2]$

Q→ Discuss the applicability of Rolle's thm. for the function

$[3/2, 5]$ $[0, 1/2]$ $[2,4]$ $[2,6]$

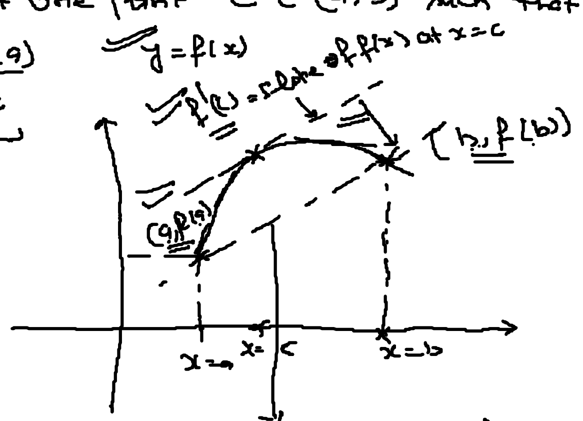
(i) $f(x) = |x-1|$ $x \in [0,2]$ (ii) $f(x) = \tan x$, $x \in [0, \pi/2]$

Sol'→ (i) Rolle's thm is not applicable as $f(x) = |x-1|$ is not differentiable at $x=1 \in (0,2)$ [Prove it]

(ii) $f(x) = \tan x$, $x \in [0, \pi/2]$
$f(x)$ is not defined as $x = \pi/2 \in [0, \pi/2]$, so Rolle's thm. is not applicable.

H.W # $f(x) = x$, $x \in [1,2]$, check the applicability of Rolle's thm
$f(1) = 1$, $f(2) = 2$

Lagrange's mean value theorem: Let $f(x)$ be defined in $[a,b]$ such that
(i) $f(x)$ is continuous in $[a,b]$
(ii) $f(x)$ is differentiable in (a,b)
then there exist atleast one point $c \in (a,b)$ such that
$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



Slope = $\frac{f(b) - f(a)}{b-a}$
of chord joining end points of the curve

18/11/20

Q→ Examine the applicability of LMVT for $f(x) = x^2 - 4x - 3$

in $[1,4]$.

<u>Sol.</u> $f(x) = x^2 - 4x - 3$ is well-defined over $[1,4]$ | $D_f = \mathbb{R}$

(i) $f(x)$ is a polynomial function, so it is continuous $[1,4]$

(ii) $f'(x) = 2x - 4$ exist on $(1,4)$. Hence $f(x)$ is differentiable on $(1,4)$

$\therefore$ LMVT is applicable.

Consider $f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow 2c - 4 = \frac{f(4)-f(1)}{4-1}$

$\Rightarrow 2c - 4 = \frac{(\cancel{16}-\cancel{16}-3)-(1-4-3)}{3} = \frac{3}{3} = 1$

$\Rightarrow c = 5/2 \in (1,4) \Rightarrow$ LMVT got verified.

H.W.

Q→ Find a point on parabola $y = (x+3)^2$ where tangent is parallel to the chord joining $(-3,0)$ & $(-4,1)$ $\quad 0 = f(-3)$, $1 = f(-4)$

Hint: $f(x) = (x+3)^2$ & interval is $[-3,-4]$

Q→ At what points on the curves, is the tangent parallel to x-axis ?

(i) $y = x^2$ in $[-2,2]$ (ii) $y = 16 - x^2$ in $[-1,1]$

(iii) $y = \cos x - 1$ on $[0, 2\pi]$

Q→ find the points of the curve $y = x^2 - 6x + 1$, at which tangent is parallel to the chord joining $(1,-4)$ & $(3,-8)$.

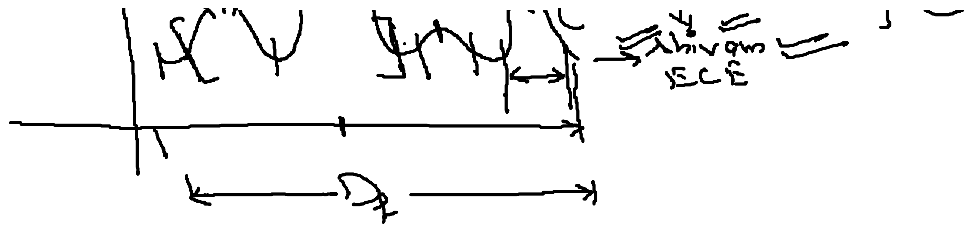
Maxima & Minima :

Absolute maxima & minima (Global): let $f(x)$ be a function with domain is D_f, then a point $x = c \in D_f$ is point of absolute maxima (or minima) if $f(c) \geq f(x)$ (or $f(c) \leq f(x)$) $\forall x \in D_f$.

Relative Maxima & Minima (local): Let $f(x)$ be a function with domain D_f & $x = c \in D_f$ is a point of relative maxima (or minima) if $f(c) \geq f(x)$ (or $f(c) \leq f(x)$) $\forall x \in (c-\delta, c+\delta)$, where $\delta > 0$ (However small)

$c-\delta \leftarrow c \rightarrow c+\delta \qquad 2.7 \leftarrow 3 \rightarrow 3.3 \quad \delta = 0.3$

MTH165



Test for finding relative maxima & minima of $f(x)$:

(A) First derivative test : for $y = f(x)$, (i) find $f'(x)$

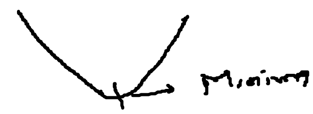
(ii) Put $f'(x) = 0$, solve the equation & get the stationary or critical point as $x = a, b, c$ - -

(iii) For $x = a$, study the sign of $f'(x)$ on left side & right side of a,

* if sign of $f'(x)$ changes from $-ve$ to $+ve$ then $x = a$ is the point of relative minima
* If sign of $f'(x)$ changes from $+ve$ to $-ve$ then $x = a$ is the point of maxima.

$f'(x) > 0$ → Increasing $f'(x) < 0$ → Decreasing

$f'(x)$ $x = a$
$-ve \rightarrow +ve$



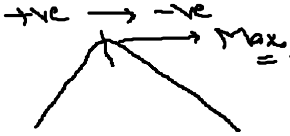
(B) 2nd derivative test :

If $y = f(x)$ is the function, then

(i) Find $f'(x)$ (ii) Take $f'(x) = 0$ and find stationary points as $x = a, b, - -$

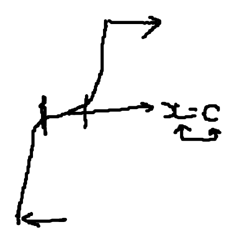
(iii) Find $f''(x)$ and for $x = a$, find $f''(a)$ and if $f''(a) > 0$, the $x = a$ is the point of minima and if $f''(a) < 0$ the $x = a$ is the point of maxima

$+ve \rightarrow -ve$



** If second derivative test fails $[f''(x) = 0]$, then go for first derivative test.

Point of inflexion : If $f'(x)$ does not change sign at a point $x = a$, then it is called as point of inflexion.



19/11/20

Q. Find maximum & minimum values of $f(x) = 9x^2 - 6x + 1$

Sol. $f(x) = 9x^2 - 6x + 1 \Rightarrow f'(x) = 18x - 6$

Take $f'(x) = 0 \Rightarrow 18x - 6 = 0 \Rightarrow x = 1/3$ is the stationary point.

Now $f'(x) = 18[x - 1/3]$ | $x < 1/3 \Rightarrow x - 1/3 < 0$

If $x < \frac{1}{3} \Rightarrow f'(x) = -ve$ | $x > 1/3 \Rightarrow x - 1/3 > 0$

If $x > 1/3 \Rightarrow f'(x) = +ve$

At $x = 1/3$, $f'(x)$ changes sign from $-ve$ to $+ve$, so $x = 1/3$ is the point minima

$\therefore$ Min. value $= f(1/3) = 9(1/9) - 6(1/3) + 1$
$= 1 - 2 + 1 = 0$

OR $f''(x) = 18$, Now $f''(1/3) = 18 > 0$, $\Rightarrow x = 1/3$ is the point of minima.

Q→ $f(x) = (x-2)^4 (x+1)^3$ find Max. & Min

Sol→ $f'(x) = (x-2)^4 \, 3(x+1)^2 + 4(x-2)^3 (x+1)^3$

$= (x-2)^3 (x+1)^2 [3(x-2) + 4(x+1)]$

$f'(x) = (x-2)^3 (x+1)^2 (7x-2)$

Take $f'(x) = 0 \Rightarrow x = 2, -1, 2/7$ are stationary points

For $x = 2$, if $x < 2$, $f'(x) = (-ve)(+ve)(+ve)$
$= -ve$

$x < 2 \Rightarrow (x-2) < 0$, $x+1 < 3$, $7x < 14$, $7x - 2 < 12$

if $x > 2$, $f'(x) = (+ve)(+ve)(+ve) = +ve$

$\therefore f'(x)$ change sign from $-ve$ to $+ve$

$\therefore x = 2$ is point of minima

$\therefore$ Min value $= f(2) = 0$

if $x > 2 \Rightarrow x - 2 > 0$, $x+1 > 3$, $7x > 14$, $7x - 2 > 12$

For $x = -1$, if $x < -1$, $f'(x) = (-ve)(+ve)(-ve)$
$= +ve$

$f'(x) = (x-2)^3 (x+1)^2 (7x-2)$

$x < -1 \Rightarrow x + 1 < 0$, $x - 2 < -3$, $7x < -7$, $7x - 2 < -9$

if $x > -1$, $f'(x) = (-ve)(+ve)(-ve)$
$= +ve$

$x > -1 \Rightarrow x + 1 > 0$, $x - 2 > -3$, $7x > -7$, $7x - 2 > -9$

$\therefore f'(x)$ hasn't changed sign

$\therefore x = -1$, is the point of inflexion

For $x = 2/7$, if $x < 2/7$, $f'(x) = (-ve)(+ve)(-ve)$
$= +ve$

$x < 2/7$, $7x - 2 < 0$, $x - 2 < -12/7$

if $x > 2/7$, $f'(x) = (-ve)(+ve)(+ve) = -ve$

$x > 2/7$, $7x - 2 > 0$, $x - 2 > -\frac{12}{7}$

$\therefore$ at $x = 2/7$, $f'(x)$ changes sign from $+ve$ to $-ve \Rightarrow x = 2/7$ is the point of maxima

$\therefore$ Max. value $= f(2/7) = (2/7 - 2)^4 (2/7 + 1)^3$

Q→ Find two +ve numbers whose sum is 15 & the sum of squares is minimum squares is minimum

Sol → Let the first number is x
——— 2nd ——— $15-x$

Now $f(x) = x^2 + (15-x)^2$

$f'(x) = 2x + 2(15-x)(-1)$

$= 4x - 30$

Take $f'(x) = 0 \Rightarrow 4x - 30 = 0$

$x = 15/2$ is the stationary Point

$f''(x) = 4$, $f''(15/2) = 4 > 0$

$\therefore$ $x = 15/2$ is the Point of minima

$\therefore$ Two number $15/2$, $15/2$

15 , 0	$\frac{15}{2}, \frac{15}{2}$
14 , 1	$\frac{7}{9}, 15-7/9$
13 , 2	$3/7, 15-3/7$
12 , 3	$8/9, 15-8/9$
→ 11 , 4	$1/3, 15-1/3$

$(15)^2+0^2, (14)^2+1^2, (13)^2+2^2$

$\rightarrow (\frac{15}{2})^2 + (15/2)^2, (\frac{7}{9})^2 + (15-7/9)^2$

Least

Q → Show that of all rectangles with a given perimeter, the square has the largest area

Sol. Let length of rectangle $= x$
Width ——— $= y$

$\therefore$ $2x + 2y = 2c$ (Perimeter is fixed)

$\Rightarrow y = c - x$

$\therefore$ $f(x) = \text{Area} = x(c-x) = cx - x^2$

$f'(x) = c - 2x$, Take $f'(x) = 0$

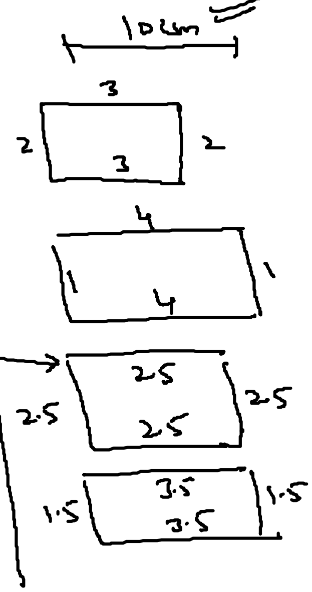
$\Rightarrow c - 2x = 0 \Rightarrow x = c/2$ is the stationary Point

$f''(x) = -2 \Rightarrow f''(c/2) = -2 < 0$

$\therefore$ $x = c/2$ is the Point of maxima

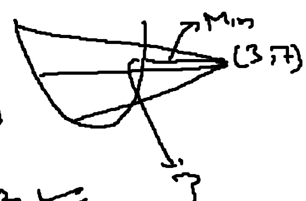
$\therefore$ $y = c - x = c - c/2 = c/2$

$\therefore$ Hence rectangle is square in nature having maximum area



Q → Find the Point on $y = x^2 + 7$, which is nearest to point $(3,7)$

Sol → Let the co-ordinates of point lying on curve $y = x^2+7$ be $(x, y) \Leftrightarrow (x, x^2+7)$



Now $d = \text{distance} = \sqrt{(x-3)^2 + (x^2+7-7)^2}$

$8, 2$

$\frac{8}{2} = 4$

$\Rightarrow D = d^2 = f(x) = x^2 + 9 - 6x + x^4$

$\Rightarrow D = f(x) = x^4 + x^2 - 6x + 9$

$4x^2+4x+6$

Now $f'(x) = 4x^3+2x-6$

Take $f'(x)=0 \Rightarrow 4x^3+2x-6=0$

Put $x=1$

$$\begin{array}{r|l} (x-1) & 4x^3+2x-6 \\ & 4x^3-4x^2 \\ \hline & 4x^2+2x-6 \\ & 4x^2-4x \\ \hline & 6x-6 \\ & 6x-6 \\ \hline \end{array}$$

$\Rightarrow (x-1)(4x^2+4x+6)=0$

$\Rightarrow x=1, \ \frac{-2\pm\sqrt{20}\,i}{4} \rightarrow$ (Rejected)

$2x^2+2x+3=0$

$x = \frac{-2\pm\sqrt{4-24}}{2\times 2}$

Here $x=1$, is the stationary point

$f''(x) = 12x^2+2$, $f''(1) = 14 > 0$

$\therefore$ $x=1$ is the point of minima

$\therefore$ Point of given curve which will be nearest to $(3,7)$ is $(1,8)$

and nearest distance $= \sqrt{4+1} = \sqrt{5}$

$y = x^2-6x+1$, Point $(1,-4)$ & $(3,-8)$ [$a=1$, $f(a)=-4$, $b=3$, $f(b)=-8$]

$f(x) = x^2-6x+1$, $[1,3]$

$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow 2c-6 = \frac{-8-(-4)}{3-1} = \frac{-4}{2} = -2$

$\Rightarrow 2c = 4 \Rightarrow \boxed{c=2} \rightarrow y=f(x) = x^2-6x+1$

$\Rightarrow$ Point is $(2, 2^2-6\times 2+1)$

$\Leftrightarrow (2,-7)$

20/11/20 Indeterminate forms:

($\frac{0}{0}$ form) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ — (1), in case if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

then expression (1) will reduce $\frac{0}{0}$ forms (Indeterminate form)

$\lim_{x\to a} \frac{f(x)}{g(x)}$ and it takes $\frac{0}{0}$ form, then, we can evaluate this limiting value as given below

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, if it again reduces to $\frac{0}{0}$ form, so we can repeat the procedure

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$ [L' Hospital Rule]

Q. Find $\lim_{x\to 0} \frac{a^x-b^x}{x} \longrightarrow$

10+2 method

$\boxed{\lim_{x\to 0} \frac{(a^x-1)}{x} - \lim_{x\to 0} \frac{(b^x-1)}{x}}$

Sol. $\lim_{x\to 0} \frac{a^x - b^x}{x}$ $\left[\frac{0}{0} \text{ form}\right]$

By L'Hospital Rule

$$\Rightarrow \lim_{x\to 0} \frac{a^x - b^x}{x} = \lim_{x\to 0} \frac{a^x \log a - b^x \log b}{1} = \log a - \log b$$
$$= \log(a/b)$$

Q. find $\lim_{x\to 1} \frac{x^x - x}{x-1-\log x}$

Sol. $\lim_{x\to 1} \frac{x^x - x}{x-1-\log x}$ $\left[\frac{0}{0} \text{ form}\right]$

By L'Hospital Rule

$$\lim_{x\to 1} \frac{x^x - x}{x-1-\log x} = \lim_{x\to 1} \frac{x^x[1+\log x]-1}{1-1/x}$$

$\frac{d}{dx}[x^x] = x^x[1+\log x]$

$\left[\frac{0}{0} \text{ form}\right]$

By L'Hospital Rule

$$= \lim_{x\to 1} \frac{x^x \cdot \frac{1}{x} + (1+\log x)\, x^x\, (1+\log x)}{1/x^2}$$

$$= \frac{1+1}{1} = 2$$

$\left(\frac{\infty}{\infty} \text{ form}\right)$ $\lim_{x\to a} \frac{f(x)}{g(x)} \to \left(\frac{\infty}{\infty} \text{ form}\right)$ even then we can apply L'Hospital Rule.

Q. $\lim_{x\to 0} \frac{\log x}{\cot x}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$ $\left[\begin{matrix}\log 0 \to -\infty \\ \cot 0 \to \infty\end{matrix}\right.$

log 10

L'Hospital Rule

$$\lim_{x\to 0} \frac{\log x}{\cot x} = \lim_{x\to 0} \frac{1/x}{-\operatorname{cosec}^2 x}$$

$$= -\lim_{x\to 0} \frac{\sin^2 x}{x} = -\lim_{x\to 0} \left(\frac{\sin x}{x}\right)(\sin x)$$

$$= -(1)(0) = 0$$

OR $\lim_{x\to 0} \frac{\sin^2 x}{x}$ $\left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x\to 0} \frac{2 \sin x \cos x}{1}$$

$$= 2 \times 0 \times 1 = 0$$

$[0 \cdot \infty \text{ form}]$

Q. $\lim_{x\to 0} \tan x \cdot \log x$ $[-0 \cdot \infty \text{ form}]$ $\to \frac{\infty}{(1/0)}$, $\frac{0}{(1/\infty)}$

$$= \lim_{x\to 0} \frac{\log x}{\cot x} \quad \left[\frac{\infty}{\infty} \text{ form}\right]$$

L'Hospital Rule

L' Hospital Rule

$$= \lim_{x\to 0} \frac{1/x}{-\operatorname{cosec}^2 x} = 0$$

$[\infty - \infty$ form$]$

Q. $\lim_{x\to 0}\left[\frac{1}{\sin x} - \frac{1}{x}\right]$ $[\infty-\infty$ form$]$

$$= \lim_{x\to 0} \frac{x-\sin x}{x\sin x} \quad \left[\frac{0}{0} \text{ form}\right]$$

By L'Hospital Rule

$$= \lim_{x\to 0} \frac{1-\cos x}{x\cos x + \sin x} \quad \left[\frac{0}{0} \text{ form}\right]$$

By L'Hospital Rule

$$= \lim_{x\to 0} \frac{\sin x}{-x\sin x + \cos x \cdot 1 + \cos x}$$

$$= \frac{0}{2} = \underline{\underline{0}}$$

$[0^0, 1^\infty, \infty^0$ form$]$

Q. $\lim_{x\to \pi/2} (\sin x)^{\tan x}$ $[1^\infty$ form$]$

Sol. $y = \lim_{x\to\pi/2} (\sin x)^{\tan x}$

$$\Rightarrow \log y = \lim_{x\to\pi/2} \log[\sin x]^{\tan x}$$

$$\Rightarrow \log y = \lim_{x\to\pi/2} \tan x \log \sin x \quad [\infty\cdot 0 \text{ form}]$$

$$= \lim_{x\to\pi/2} \frac{\log \sin x}{\cot x} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$\Rightarrow \log y = \lim_{x\to\pi/2} \frac{\cot x}{-\operatorname{cosec}^2 x} = \lim_{x\to\pi/2} -\frac{\cos x}{\cancel{\sin x}} \sin^2 x$$

$$= \lim_{x\to 0} -\sin x\cos x = 0$$

$$\Rightarrow \log y = 0 \Rightarrow y = e^0 \Rightarrow \boxed{y=1}$$

Q. $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}$ $[1^\infty$ form$]$

Sol. $y = \lim \left[\frac{a^x+b^x+c^x}{\ }\right]^{1/x}$

$x \to 0 \quad (\quad 3 \quad)$

Sol. $y = \lim_{x \to 0} \left[\frac{a^x+b^x+c^x}{3}\right]^{1/x}$

$$\log y = \lim_{x\to 0} \frac{\log(a^x+b^x+c^x) - \log 3}{x} \quad \left[\frac{0}{0} \text{ form}\right]$$

By L'Hospital Rule

$$\log y = \lim_{x\to 0} \frac{\frac{1}{(a^x+b^x+c^x)}\left[a^x \log a + b^x \log b + c^x \log c\right]}{1}$$

$$= \frac{\log(abc)}{3} = \log(abc)^{1/3}$$

$$\Rightarrow \log y = \log(abc)^{1/3}$$

$$\boxed{y = (abc)^{1/3}} \Rightarrow \lim \left(\frac{a^x+b^x+c^x}{3}\right)^{1/x} = (abc)^{1/3}$$

Q. $\lim_{x\to 1} (1-x^2)^{1/\log(1-x)}$ $\quad [0^0 \text{ form}]$

Sol. Let $y = \lim_{x\to 1} (1-x^2)^{\frac{1}{\log(1-x)}}$

$$\Rightarrow \log y = \lim_{x\to 1} \frac{\log(1-x^2)}{\log(1-x)} \quad \left[\frac{\infty}{\infty} \text{ form}\right]$$

By L'Hospital Rule

$$= \lim_{x\to 1} \frac{\frac{1}{(1-x^2)}(-2x)}{\frac{1}{(1-x)}(-1)} = \lim_{x\to 1} \frac{2x\,\cancel{(1-x)}}{\cancel{(1-x)}(1+x)}$$

$$= \lim_{x\to 1} \frac{2x}{1+x} = \frac{2}{2} = 1$$

$$\Rightarrow \log y = 1 \Rightarrow \boxed{y = e^1 = e}$$

$$\Rightarrow \lim_{x\to 1} (1-x^2)^{\frac{1}{\log(1-x)}} = e$$

Q. find the values of following limits

(i) $\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x^3}$ (ii) $\lim_{x\to 0} \frac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2}$

(iii) $\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$ (iv) $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

(v) $\lim_{x\to 0} (a^x+b^x)^{1/x}$ (vi) $\lim_{x\to 0} \left[\frac{1}{x} - \frac{1}{e^x - 1}\right]$ $\quad x = 1, 2; \ ^2 \quad x = 0, a = 0$

(vii) $\lim_{x\to 2} \left[\frac{1}{x-2} - \frac{1}{\log(x-1)}\right]$ $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty$

25/11/20 Taylor's theorem with remainder :

$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \infty$

Let $f(x)$ be defined & have continuous derivative upto $(n+1)^{th}$ order in some interval I, containing a point a, then Taylor series expansion of $f(x)$ about $\boxed{x=a}$ is given as

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a)$$
$$+ \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}[a + \theta(x-a)]$$

where $0 < \theta < 1$

$x = a+h$
$x - a = h$

here $R_n = \frac{(x-a)^{n+1}}{n+1} f^{n+1}(a + \theta(x-a))$ is called as remainder or error term [Lagrange's form of remainder

Maclaurin theorem with remainder : If $a = 0$

then $$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^n(0)$$
$$+ \frac{x^{n+1}}{(n+1)!} f^{n+1}[\theta x]$$
where $0 < \theta < 1$ Remainder Term

If we take $x = a+h$ or $x - a = h$

then Taylor's theorem becomes
$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^n}{n!} f^n(a)$$
$$+ \frac{h^{n+1}}{(n+1)!} f^{n+1}(c), \quad a < c < a+h$$
$c = a + \theta h$
$0 < \theta < 1$

Q. If $f(x) = \log(1+x)$, $x > 0$, using {Maclaurin's theorem} show that $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3[1+\theta x]^3}$ $0 < \theta < 1$

Sol. We know Maclaurin's theorem about $x = 0$ is
$$f(x) = f(0) + \frac{x^1}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x)$$

Here $f(x) = \log(1+x) \Rightarrow f(0) = \log 1 = 0$

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = \frac{1}{1} = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \Rightarrow f''(0) = -\frac{1}{1} = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(\theta x) = \frac{2}{(1+\theta x)^3}$$

$\therefore$ By MacLaurin's theorem

$$\log(1+x) = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}\frac{2}{(1+\theta x)^3}$$

$$\Rightarrow \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}\cdot\frac{1}{(1+\theta x)^3}$$

H.W. Q. Show that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}e^{\theta x}$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots$ in the power of $(x-1)$

$(x-1)$

Q. Find series expansion of e^x about $x=1$, upto 4th degree term with remainder

Sol. Taylor theorem of $f(x)$ about $x=a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{iv}(a+(x-a)\theta)$$

$0<\theta<1$ — (i)

Now $f(x) = e^x$, $a = 1$

$$\therefore e^x = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \frac{(x-1)^4}{4!}f^{iv}[1+(x-1)\theta]$$

$0<\theta<1$ — (ii)

$f(x) = e^x \Rightarrow f(1) = e^1 = e$

$f'(x) = e^x \Rightarrow f'(1) = e^1 = e$

$f''(x) = e^x \Rightarrow f'(1) = e^1 = e$

$f'''(x) = e^x \Rightarrow f'(1) = e^1 = e$

$f^{iv}(x) = e^x \Rightarrow f^{iv}[1+(x-1)\theta] = e^{1+(x-1)\theta}, 0<\theta<1$

Put in (ii)

$$\Rightarrow e^x = e + (x-1)e + \frac{(x-1)^2}{2!}e + \frac{(x-1)^3}{3!}e + \frac{(x-1)^4}{4!}e^{1+(x-1)\theta}$$

$$e^x = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!}e^{(x-1)\theta}\right]$$

$0 \leq \theta < 1$

H.W. Q.1 Find series expansion of $\cos x$ about $x=0$, upto 4th degree term, using Maclaurin's theorem with remainder.

Q.1 Find series expansion of $\sin x$ about $x = \pi/2$ upto 5th degree term, using Taylor's theorem with remainder.