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Unit - VI : Fourier Series

Fourier Series : Let $f(x)$ be periodic function with period $\boxed{2l}$ and is defined in interval $[\alpha, \alpha+2l]$ then fourier series expansion of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$\alpha + 2l - \alpha = 2l$

where a_0, a_n & b_n are called as fourier coefficients

$$a_0 = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\,dx \;;\; \boxed{a_n} = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$\& \; b_n = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

These are called as Euler's formulae.

Periodic function : A function $f(x)$ is said to be periodic if $f(x) = f(x+T)$, and period of $f(x)$ is T

e.g. $f(x) = \sin x$ as $\sin x = \sin(2\pi + x)$ i.e. $f(x) = f(2\pi + x)$

$\therefore$ $\sin x$ is periodic with period 2π | $\sin(2\pi + x) = \sin x$

$\cos x$ is also periodic with period 2π.

Q.> Find fourier series expansion of $f(x) = (2x - x^2)$ in $0 \le x \le 2$

Sol.> We know fourier series of $f(x)$ in $[\alpha, \alpha+2l]$ is | $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Here $f(x) = 2x - x^2$, and interval is $[0, 2] \Rightarrow l = 1$

Now $a_0 = \frac{1}{1}\int_0^2 (2x - x^2)\,dx = 1\left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - 8/3 = 4/3$

Now $a_n = \frac{1}{1}\int_0^2 \underbrace{(2x - x^2)}_{u}\underbrace{\cos\left(\frac{n\pi x}{1}\right)}_{v}dx$ ILATE

$$= \Bigg[\underbrace{(2x - x^2)}_{u}\left(\frac{\sin(n\pi x)}{n\pi}\right) - \underbrace{(2 - 2x)}_{u}\left(\frac{-\cos(n\pi x)}{(n\pi)^2}\right) + \underbrace{(-2)}_{u}\left(-\frac{\sin(n\pi x)}{(n\pi)^3}\right) - 0\Bigg]_0^2$$

$$= \left(0 - (2-4)\frac{\cos(n\pi 2)}{(n\pi)^2} + 0\right)$$

$$- \left(0 - 2\left(-\frac{\cos(0)}{(n\pi)^2}\right) + 0\right)$$

$\sin 2n\pi = 0$

$$= \frac{2\cos(2n\pi)}{(n\pi)^2} - \frac{1}{(n\pi)^2} = \frac{2\cos(2n\pi)-1}{(n\pi)^2} = \frac{1}{(n\pi)^2}$$

$\cos(n\pi) = (-1)^n$
$\cos(2n\pi) = (-1)^{2n} = +1$

and $$b_n = \frac{1}{1}\int_0^2 \underbrace{(2x-x^2)}_{u} \underbrace{\sin\left(\frac{n\pi x}{1}\right)}_{v} dx$$

$$= \Big[\underbrace{(2x-x^2)}_{u} \underbrace{\left(\frac{-\cos(n\pi x)}{n\pi}\right)}_{v} - \underbrace{(2-2x)}_{u} \left(-\frac{\sin(n\pi x)}{(n\pi)^2}\right) + \underbrace{(-2)}\left(\frac{\cos(n\pi x)}{(n\pi)^3}\right) - 0 \Big]_{x=0}^{2}$$

$$= \left[\left(0 - 0 + (-2)\frac{\cos(2n\pi)}{(n\pi)^3}\right) - \left(0 - 0 + (-2)\frac{(1)}{(n\pi)^3}\right)\right]$$

$$b_n = \frac{-2}{(n\pi)^3} + \frac{2}{(n\pi)^3} = 0$$

Now, Fourier series of $f(x) = 2x - x^2$ in $[0,2]$ is

$$f(x) = (2x-x^2) = \frac{4/3}{2} + \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2}\cos\left(\frac{n\pi x}{1}\right) + \sum_{n=1}^{\infty}(0)$$

$$\Rightarrow \underline{(2x-x^2)} = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{\cos(n\pi x)}{(n\pi)^2}$$

$$= \frac{2}{3} + \left[\frac{\cos(\pi x)}{\pi^2} + \frac{\cos(2\pi x)}{(2\pi)^2} + \frac{\cos(3\pi x)}{(3\pi)^2} + \cdots \infty \right]$$

Fourier series of Even & odd Functions: Let us find Fourier series of $f(x)$ in interval $[-l, l]$ $\boxed{(l-(-l)) = 2l}$

Fourier series will be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)}_{\text{odd}}\,dx$, $a_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)}_{\text{odd}} \underbrace{\cos\left(\frac{n\pi x}{l}\right)}_{\text{even}} dx$ → odd

$$b_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)}_{\text{even}} \underbrace{\sin\left(\frac{n\pi x}{l}\right)}_{\text{odd}} dx \to \text{odd}$$

Case 1: If $f(x)$ is even function i.e. $f(-x) = f(x)$, then

$$b_n = 0$$

Case 2: If $f(x)$ is an odd function i.e. $f(-x) = -f(x)$

then $a_0 = 0$ & $a_n = 0$

$\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$ $f(x)$ is odd

Find Fourier series of

Q→ $f(x) = x^3$ in $-\pi < x < \pi$

Sol→ Now $f(-x) = (-x)^3 = -x^3 = -f(x)$

$\Rightarrow f(-x) = -f(x) \Rightarrow f(x)$ is an odd function $\quad l = \pi$

Since we are looking for fourier series of $f(x) = x^3$ which is an odd function, in $[-\pi, \pi]$, so $a_0 = a_n = 0$

and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 \sin\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right) dx$

$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if $f(x)$ is even

$(-\pi, \pi)$ 2π

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 \sin(nx)\,dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} \underbrace{x^3}_{u} \underbrace{\sin(nx)}_{v}\,dx$$

$$= \frac{2}{\pi}\left[x^3 \left(\frac{-\cos(nx)}{n}\right) - (3x^2)\left(-\frac{\sin(nx)}{n^2}\right) + 6x\left(\frac{+\cos(nx)}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right) + 0 \right]_{x=0}^{\pi}$$

$$= \frac{2}{\pi}\left[\left(-\frac{(\pi)^3 \cos(n\pi)}{n} + \frac{6\pi \cos(n\pi)}{n^3}\right) - 0\right]$$

$\sin(n\pi) = 0$, $n \in I$

$\cos(n\pi) = (-1)^n$, $n \in I$

$$= \frac{2}{\pi}\left[\frac{-\pi^3 (-1)^n}{n} + \frac{6\pi (-1)^n}{n^3} \right]$$

$$b_n = 2\left[\frac{-\pi^2 (-1)^n}{n} + \frac{6(-1)^n}{n^3} \right]$$

Now F.S. is $f(x) = x^3 = \sum_{n=1}^{\infty} 2\left[\frac{\pi^2 (-1)^n}{n} + \frac{6(-1)^n}{n^3}\right] \sin\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right)$

$$\Rightarrow f(x) = x^3 = \sum_{n=1}^{\infty} 2\left[\frac{\pi^2}{n} + \frac{6}{n^3}\right](-1)^n \sin(nx)$$

Q→ Find F.S. of $f(x) = 9 - x^2$ in $[-3, 3]$

Sol→ $f(-x) = 9 - (-x)^2 = 9 - x^2 = f(x) \Rightarrow f(-x) = f(x)$

$\therefore f(x)$ is an even function.

Here F.S. of $f(x)$ in $[-3, 3]$ will be $\quad [l = 3]$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right)$$

Here $f(x)$ is an even function & we are expanding F.S. from

$[-3,3]$, so $b_n = 0$

Now $a_0 = \frac{1}{3}\int_{-3}^{3}(9-x^2)dx = \frac{2}{3}\int_0^3 (9-x^2)dx = \frac{2}{3}\left[9x - \frac{x^3}{3}\right]_0^3$

$= \frac{2}{3}\left[27 - 9\right] = \frac{2}{\cancel{3}} \times \cancel{18}^{6} = 12$

and $a_n = \frac{1}{3}\int_{-3}^{3}\underbrace{(9-x^2)\cos\left(\frac{n\pi x}{3}\right)}_{\text{Even}}dx = \frac{2}{3}\int_0^3 \underbrace{(9-x^2)}\underbrace{\cos\left(\frac{n\pi x}{3}\right)}dx$

$= \frac{2}{3}\left[(9-x^2)\frac{\sin\left(\frac{n\pi x}{3}\right)}{\left(\frac{n\pi}{3}\right)} - (-2x)\left[-\frac{\cos\left(\frac{n\pi x}{3}\right)}{\left(\frac{n\pi}{3}\right)^2}\right] + (-2)\left[-\frac{\sin\left(\frac{n\pi x}{3}\right)}{\left(\frac{n\pi}{3}\right)^3}\right]\right]_{x=0}^{3}$

Even * odd = odd
(+) * (−) = −
Even * Even = Even
(+) * (+) = +
odd * odd = Even
(−) * (−) = +

$\sin n\pi = 0$, $n \in I$

$\cos(n\pi) = (-1)^n$, $n \in I$

$= \frac{2}{3}\left[\left(0 - 2\times 3\,\frac{\cos(n\pi)}{\left(\frac{n\pi}{3}\right)^2} - 0\right) - 0\right]$

$= \frac{2}{\cancel{3}}\left[-\frac{\cancel{6}^{2}(-1)^n \times 9}{n^2\pi^2}\right] = -\frac{36(-1)^n}{n^2\pi^2}$

$\therefore$ F.S. of $f(x) = 9-x^2$ in $[-3,3]$ is

$\therefore 9-x^2 = 6 + \sum_{n=1}^{\infty} -\frac{36(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{3}\right)$

Q→ Find F.S. $f(x) = (x-x^2)$ in $[-\pi,\pi]$

Sol → Here $l = \pi$

$\therefore$ F.S. of $f(x)$ will be

$f(x) = x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right)$

$f(-x) = -x - (-x)^2$
$= -x - x^2$
$= -[x + x^2]$
$f(-x) \neq f(x)$
$f(-x) \neq -f(x)$

$\cos\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right) = \cos nx$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\left(\cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3}\right) - \left(\cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3}\right)\right]$

$= \frac{1}{\pi}\left[-\frac{2\pi^3}{3}\right] = -\frac{2\pi^2}{3}$

Now $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos(nx)\,dx$

$= \frac{1}{\pi}\left[\underbrace{(x-x^2)}\underbrace{\left(\frac{\sin nx}{n}\right)} - \underbrace{(1-2x)\left(-\frac{\cos nx}{n^2}\right)} + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{x=-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\left(\frac{(1-2\pi)\cos(n\pi)}{}\right) - \left(\frac{(1+2\pi)\cos(-n\pi)}{n^2}\right)\right]$ $\quad$ $\sin n\pi = 0$

$$= \frac{1}{\pi}\left[\frac{\cos(n\pi) - 2\pi\cos n\pi - \cancel{\cos n\pi} - 2\pi\cos n\pi}{n^2}\right]$$

$n \in I$

$\cos(-\theta) = \cos\theta$

$$= \frac{1}{\cancel{\pi}}\left[\frac{-4\cancel{\pi}\cos(n\pi)}{n^2}\right] = \frac{-4(-1)^n}{n^2}$$

Now $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx \, dx$

$$\Rightarrow b_n = \frac{1}{\pi}\left[(x - x^2)\left(-\frac{\cos nx}{n}\right) - (1-2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[\left(-(\pi - \pi^2)\frac{\cos n\pi}{n} - 2\frac{\cos(n\pi)}{n^3}\right) - \left(-(-\pi - \pi^2)\frac{\cos(-n\pi)}{n} - 2\frac{\cos(-n\pi)}{n^3}\right)\right]$$

$\sin(n\pi) = 0$, $n \in I$

$\sin(-n\pi) = -\sin(n\pi) = 0$

$$= \frac{1}{\pi}\left[\frac{(-\pi + \pi^2)}{n} - \cancel{\frac{2}{n^3}} + \frac{(-\pi - \pi^2)}{n} + \cancel{\frac{2}{n^3}}\right]\cos(n\pi)$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n} + \cancel{\frac{\pi^2}{n}} - \frac{\pi}{n} - \cancel{\frac{\pi^2}{n}}\right]\cos(n\pi)$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{-2\cancel{\pi}}{n}\right](-1)^n = -\frac{2}{n}(-1)^n$$

$\therefore$ F.S. of $f(x) = x - x^2 = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{(-4)(-1)^n}{n^2}\cos(nx) + \sum_{n=1}^{\infty}\frac{(-2)(-1)^n}{n}\sin(nx)$

Half-Range Series : Let $f(x)$ be a periodic function with period as $2l$ & is defined over $[0, l]$, then $[l]$

① Half-Range Cosine series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$ & $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

② Half-Range sine of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

where $b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$

02/01/21

Q. Find Half-Range sine & Cosine series for $f(x) = 1+x$ in $[0,1]$

Sol. $f(x) = 1+x$ defined in the interval $[0,1] \Rightarrow \ell = 1$

$\therefore$ Half-Range Cosine series of $f(x)$ will be

$$f(x) = 1+x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{1}\right)$$

where $a_0 = \frac{2}{1}\int_0^1 (1+x)dx = 2\left[x+\frac{x^2}{2}\right]_0^1 = 2\left[1+\frac{1}{2}\right] = 3$

and $a_n = \frac{2}{1}\int_0^1 \underbrace{(1+x)}_{u}\underbrace{\cos\left(\frac{n\pi x}{1}\right)}_{v}dx$

$$= 2\left[\underbrace{(1+x)}_{u}\underbrace{\left(\frac{\sin(n\pi x)}{n\pi}\right)}_{} - \underbrace{1}_{u'}\left[\frac{-\cos(n\pi x)}{(n\pi)^2}\right]\right]_{x=0}^{1}$$

$$= 2\left[\left(0+\frac{\cos(n\pi)}{(n\pi)^2}\right) - \left(0+\frac{1}{(n\pi)^2}\right)\right]$$

$$a_n = 2\left[\frac{(-1)^n - 1}{(n\pi)^2}\right]$$

$\sin(n\pi) = 0$ if $n \in I$

$\cos(n\pi) = (-1)^n$

$\therefore$ Half-Range Cosine series of $f(x)$ is

$$f(x) = (1+x) = \frac{3}{2} + \sum_{n=1}^{\infty} 2\left[\frac{(-1)^n - 1}{(n\pi)^2}\right]\cos\left(\frac{n\pi x}{1}\right)$$

again, Half-Range sine series of $f(x) = 1+x$ will be

$$\therefore f(x) = (1+x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{1}\right)$$

where $b_n = \frac{2}{1}\int_0^1 \underbrace{(1+x)}_{u}\underbrace{\sin\left(\frac{n\pi x}{1}\right)}_{v}dx$

$$= 2\left[(1+x)\left[\frac{-\cos(n\pi x)}{n\pi}\right] - 1\cdot\left[-\frac{\sin(n\pi x)}{(n\pi)^2}\right]\right]_0^1$$

$$= 2\left[\left(\frac{-2\cos(n\pi)}{n\pi} - 0\right) - \left(\frac{-1}{n\pi}+0\right)\right]$$

$\boxed{\sin n\pi = 0}$

$$= 2\left[\frac{-2(-1)^n + 1}{n\pi}\right]$$

$\therefore$ Half-Range sine series of $f(x) = 1+x$ is

$$\therefore f(x) = 1+x = \sum_{n=1}^{\infty} 2\left[\frac{1-2(-1)^n}{n\pi}\right]\sin\left(\frac{n\pi x}{1}\right)$$

Q.) Find Fourier series of $f(x) = 1+x$ in $[0,1]$

Sol.) $f(x)=1+x$, Here $2l=1 \Rightarrow l=1/2$

∴ Fourier series of $f(x)=1+x$ in $[0,1]$ is

$$f(x)=1+x = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{1/2}\right)+\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{1/2}\right)$$

$$= \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos(2n\pi x)+\sum_{n=1}^{\infty} b_n \sin(2n\pi x)$$

where $a_0 = \frac{1}{(1/2)}\int_0^1 (1+x)\,dx$

$$a_n = \frac{1}{(1/2)}\int_0^1 (1+x)\cos\left(\frac{n\pi x}{1/2}\right)dx = 2\int_0^1 (1+x)\cos(2n\pi x)\,dx$$

$$b_n = \frac{1}{(1/2)}\int_0^1 (1+x)\sin\left(\frac{n\pi x}{1/2}\right)dx = 2\int_0^1 (1+x)\sin(2n\pi x)\,dx$$

Q.) $f(x) = \begin{cases} -x, & -2 \le x < 0 \\ x^2, & 0 \le x \le 2 \end{cases}$ Find Fourier series of $f(x)$

Sol. Here total interval is $[-2,2] \Rightarrow 2l=4 \Rightarrow l=2$

Now F.S. of $f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)+\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$

Now $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} f(x)\,dx+\int_0^2 f(x)\,dx\right]$

$$=\frac{1}{2}\left[\int_{-2}^{0}(-x)\,dx+\int_0^2 x^2\,dx\right] = \frac{1}{2}\left[-\left(\frac{x^2}{2}\right)_{-2}^{0}+\left(\frac{x^3}{3}\right)_0^2\right]$$

$$= \frac{1}{2}\left[-\left(0-\frac{4}{2}\right)+\left(\frac{8}{3}-0\right)\right] = \frac{1}{2}\left[+2+8/3\right] = 7/3$$

Now $a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\left[\int_{-2}^{0} -x\cos\left(\frac{n\pi x}{2}\right)dx+\int_0^2 x^2\cos\left(\frac{n\pi x}{2}\right)dx\right]$

$$=\frac{1}{2}\left[-\left\{x\left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right]-1\cdot\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right]_{-2}^{0}\right.$$

$$\left.+\left\{x^2\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}-2x\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{(n\pi/2)^2}\right)+2\cdot\left(-\frac{\sin\left(\frac{n\pi x}{2}\right)}{(n\pi/2)^3}\right)\right\}_0^2\right]$$

$$=\frac{1}{2}\left[-\left(0+\frac{1}{(n\pi/2)^2}\right)-\left(0+\frac{\cos(-n\pi)}{(n\pi/2)^2}\right)\right]$$

$$+\left[\left(0+4\frac{\cos(n\pi)}{(n\pi/2)^2}+0\right)\right.$$

$\sin(-n\pi) = -\sin(n\pi) = -0$

$\cos(-n\pi)=\cos(n\pi)$

$$- \quad 0 \Big] \Big]$$

$\cos(-n\pi) = \cos(n\pi) = (-1)^n$

$$= \frac{1}{2}\left[-\frac{4}{n^2\pi^2} + \frac{(-1)^n 4}{n^2\pi^2} + \frac{4(-1)^n \times 4}{n^2\pi^2} \right]$$

$$a_n = \frac{1}{2}\left[\frac{20(-1)^n - 4}{n^2\pi^2}\right] = \frac{10(-1)^n - 2}{n^2\pi^2}$$

Now $b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx$

$$= \frac{1}{2}\left[-\int_{-2}^{0} x \sin\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{2} x^2 \sin\left(\frac{n\pi x}{2}\right)dx \right]$$

* Complete at ur own

Q→ find both Half-Range sine & Cosine series for
$f(x) = e^{-x}$ in $0 \le x \le 2$

Sol→ Here $l = 2$

∴ Half Range Cosine series of $f(x)$ is

$$f(x) = e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

Here $a_0 = \frac{2}{2}\int_0^2 e^{-x}dx = 1\left[\frac{e^{-x}}{(-1)}\right]_0^2 = -\left[e^{-2} - 1\right]$
$= 1 - e^{-2} = 1 - \frac{1}{e^2}$
$= \frac{e^2 - 1}{e^2}$

$a = -1$, $b = \frac{n\pi}{2}$

and $a_n = \frac{2}{2}\int_0^2 e^{-x}\cos\left(\frac{n\pi x}{2}\right)dx$

$$= \frac{e^{-x}}{(-1)^2 + \left(\frac{n\pi}{2}\right)^2}\left[-\cos\left(\frac{n\pi x}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi x}{2}\right) \right]_0^2$$

$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$= \frac{1}{1 + \frac{n^2\pi^2}{4}}\left[e^{-2}\left(-\cos(n\pi) + 0\right) - \left(e^{-0}\left(-\cos 0 + 0\right)\right)\right]$$

$$a_n = \frac{4}{4 + n^2\pi^2}\left[\frac{-(-1)^n}{e^2} + 1\right]$$

∴ Half-Range Cosine series is

$$a_n = \frac{e^2 - 1}{\ } + \sum^{\infty} \left(\frac{4}{4 + n^2\pi^2}\right)\left[1 - \frac{(-1)^n}{e^2}\right]\cos\left(\frac{n\pi x}{2}\right)$$

Find Half Range Sine series at your own