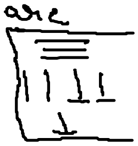
MTH-165

Count (objects)

Unit 1: Matrix: A matrix is an arrangement of (mn) numbers in square brackets, having m rows [Row is line which is parallel to earth] and each row consist n elements [or n columns] [Columns are lines which are perpendicular to earth], we call it as matrix of order $m \times n$.



S.i 1, 2, 3, 4, 5, 6

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3\times 2}$

No. of elements in matrix of order $m \times n = mn$

No. of Rows → No. of Columns

$A = [a_{ij}]_{m\times n}$

Capital letter

Row, Column

general element of matrix

gives an element at the position of ith Row & jth Column

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Special Type of Matrices:

① **Row Matrix:** Matrix having one row. $A = [a_{ij}]_{1\times n}$

eg. $A = [1\ 2\ 3]_{1\times 3}$

$B = [-2\ 2]_{1\times 2}$

② **Column Matrix:** Matrix having one Column. $A = [a_{ij}]_{m\times 1}$

eg. $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2\times 1}$, $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$

③ **Square matrix:** A matrix is square if No. of rows = No. of columns

$A = [a_{ij}]_{n\times n}$ eg $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$

④ **Rectangular Matrix:**

A matrix is rectangular if No. of rows ≠ No of Columns eg $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$

\# $A = [a_{ij}]_{m\times n}$ $\begin{cases} \text{if } m = n \Rightarrow \text{Square Matrix} \\ \text{if } m \neq n \Rightarrow \text{Rectangular Matrix} \end{cases}$

\# **Diagonal Matrix:** A square matrix is s.t.b. diagonal if $a_{ij} = 0$ when $i \neq j$

Primary, Secondary

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}_{3\times 3}$$

07/10/20: # **Scalar Matrix:** A diagonal matrix is s.t.b scalar if all diagonal entries are equal.

$$A = [a_{ij}]_{n\times n}, \begin{cases} a_{ij} = 0, & i \neq j \\ a_{ij} = k, & i = j \end{cases}$$

$[2\ 0$

$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Identity matrix : A scalar matrix is s.t.b. identity if all diagonal entries are one.

OR Unit Matrix

$A = [a_{ij}]_{n\times n}$, $\begin{cases} a_{ij} = 0, & i \neq j \\ a_{ij} = 1, & i = j \end{cases}$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

I_n denotes identity matrix of order $n\times n$

Null Matrix : A matrix is s.t.b. null matrix if all enteries are zero

or # zero matrix

$A = [a_{ij}]_{m\times n}$, $a_{ij} = 0, \forall i,j$ (for all)

$O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Square → Diagonal → Scalar → Identity

$O_{m\times n}$ represent null matrix of order $m\times n$

Symmetric Matrix : $A = [a_{ij}]_{n\times n}$ is s.t.b symmetric if $a_{ij} = a_{ji}$

$a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$

$A = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 6 & -7 \\ 5 & -7 & 8 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 & 5 \\ -2 & 6 & -7 \\ 5 & -7 & 8 \end{bmatrix}$

Skew-symmetric matrix : $A = [a_{ij}]_{n\times n}$ is s.t.b skew-symmetric matrix if $a_{ij} = -a_{ji}$

$a_{ij} = -a_{ji}$

For diagonal elements, $i = j$

$a_{ii} = -a_{ii}$

$\Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0 \Rightarrow \boxed{a_{11} = 0, a_{22} = 0, a_{33} = 0, a_{44} = 0}$

$\Rightarrow$ Diagonal enteries in skew-symmetric matrix are always zero.

$a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$

$A = \begin{bmatrix} 0 & -7 & 8 \\ 7 & 0 & 1/2 \\ -8 & -1/2 & 0 \end{bmatrix}_{3\times 3}$

Upper triangular : $A = [a_{ij}]_{n\times n}$ is s.t.b U.T. if $a_{ij} = 0$ for $i > j$

$A = \begin{bmatrix} 8 & 5 & -1/2 \\ 0 & -7 & 1 \\ 0 & 0 & -1 \end{bmatrix}_{3\times 3}$

Lower triangular: $A=[a_{ij}]_{m\times n}$ is s.t.b L.T. if $a_{ij}=0$ for $i<j$

$$A=\begin{bmatrix}5&0&0\\2&1&0\\7&3&5\end{bmatrix} \quad a_{12}$$

operations on Matrices:

① **Addition**: $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{m\times n}$

$C=A+B=[C_{ij}]_{m\times n}$ where $C_{ij}=a_{ij}+b_{ij}$

(ii) **Subtraction**: $D=A-B=[d_{ij}]_{m\times n}$, $d_{ij}=a_{ij}-b_{ij}$

(iii) **Scalar multiplication**: $A=[a_{ij}]_{m\times n}$, K is any scalar, then $KA=[Ka_{ij}]_{m\times n}$

$$A=\begin{bmatrix}1&2\\-2&4\end{bmatrix}_{2\times 2} \Rightarrow 2A=\begin{bmatrix}2&4\\-4&8\end{bmatrix}_{2\times 2}$$

(iv) **Multiplication**: $A=[a_{ij}]_{m\times p}$ & $B=[b_{jk}]_{p\times n}$

$C=AB=[c_{ij}]_{m\times n}$, $c_{ij}=\sum_{j=1}^{p} a_{ij}b_{jk}$ AB, AB, BA

$A=\begin{bmatrix}1&2&1\\0&1&2\end{bmatrix}_{2\times 3}$, $B=\begin{bmatrix}1\\0\\1\end{bmatrix}_{3\times 1}$

$C=AB=[C_{ij}]_{2\times 1}=\begin{bmatrix}4\\2\end{bmatrix}_{2\times 1}$

BA is not possible

$AB\neq BA$ Non-Commutative

(v) **Transpose**: $A=[a_{ij}]_{m\times n}$, then $A^T=A'=[a'_{ji}]_{n\times m}$

A^T or A' will be transpose of A if $a_{ij}=a'_{ji}$

$$A=\begin{bmatrix}1&2&3\\2&4&6\end{bmatrix}_{2\times 3} \quad A^T=\begin{bmatrix}1&2\\2&4\\3&6\end{bmatrix}_{3\times 2}=A' \quad a_{11}, a'_{11}$$

$(A')'=(A^T)^T=A$

$(A\pm B)'=A'\pm B'$ # $(KA)'=KA'$

$(AB)'=B'A'$ → Reversal Law

$A=\frac{1}{2}(A+A')+\frac{1}{2}(A-A')$

$$\underset{\text{square}}{A}=\underbrace{\tfrac{1}{2}(A+A')}_{\text{symmetric}}+\underbrace{\tfrac{1}{2}(A-A')}_{\text{skew-symmetric}}$$

$\boxed{C=\frac{1}{2}(A+A')}$

Symmetric: A square matrix $C'=\left(\frac{1}{2}(A+A')\right)$

is symmetric if $A' = A$

Skew-symmetric : A square matrix is skew-symmetric if
$A' = -A$

$C' = \left(\frac{1}{2}(A+A')\right)'$
$= \frac{1}{2}\left(A' + (A')'\right)$
$= \frac{1}{2}\left[A' + A\right]$
$= C$

Determinant : It is a value (numeric) associated to a square matrix.

8/10/20 # $|[a_{11}]_{1\times1}| = a_{11}$ $\Rightarrow |[-1]_{1\times1}| = -1$

$\left|\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2}\right| = a_{11}a_{22} - a_{12}a_{21}$

Minor: Let a_{ij} be the general of a matrix A, then determinant of the matrix which we get from orginal matrix A, by deleting ith row & jth Coloumn and we denote it by M_{ij}

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$$

$$M_{23} = \text{Minor of } a_{23} = \left|\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}_{2\times2}\right| = a_{11}a_{32} - a_{12}a_{31}$$

Co-factor : Cofactor of element a_{ij} in matrix A is denoted A_{ij} & is defined as $A_{ij} = \underbrace{(-1)^{i+j}}_{\text{Sign}} M_{ij}$

Sign $\to +1$, $\to -1$

$(-1)^{i+j}$
$(-1)^{1}, (-1)^{2}$
$(-1)^{3}, (-1)^{4}$
-1 $+1$

$$A_{ij} = \begin{cases} M_{ij}, & i+j \text{ is even number} \\ -M_{ij}, & i+j \text{ is odd number} \end{cases}$$

$$\text{Co-factor of } a_{23} = A_{23} = (-1)^{2+3} M_{23} = -M_{23}$$

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = \det(A)$$
$$= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$$

$|A| = |A^T|$ # $|AB| = |A||B|$ # $|A \pm B| \neq |A| \pm |B|$

$|\alpha A| = \alpha^n |A|$, where α is any scalar, n is the order of matrix A

$|A_{3\times3}| = 3 \Rightarrow |2A| = 2^3 |A| = 8 \times 3 = 24$

Sub-Matrices: A matrix obtained by deleting rows or columns or both from any matrix A, is called submatrix of A.

e.g. $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 7 & 8 \end{bmatrix}_{2\times 3} \longrightarrow$ Rectangular

Few submatrices of A are

$$\begin{bmatrix} 3 & 4 \\ 7 & 8 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 5 & 7 & 8 \end{bmatrix}, \begin{bmatrix} 5 & 7 \end{bmatrix}$$

Minor: $\left|\begin{bmatrix} 3 & 4 \\ 7 & 8 \end{bmatrix}_{2\times 2}\right|$ = Minor of order 2 = $\left|\begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix}\right|$

$\left|\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\right|$

3 → Minor of order 2

Minors of order 1 → $|[2]_{1\times 1}|$, $|[3]_{1\times 1}|$, $|[8]_{1\times 1}|$

6 → Minors of order 1

# $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}, & a_{22} & a_{23} \\ a_{31}, & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \longrightarrow$ $|A|$ = Minor of order 3
Count = 1

A is a submatrix of itself

Minors of order 2 ⇒ $\left|\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}_{2\times 2}\right|$

$\left|\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}_{2\times 2}\right|$

→ Count = 9

Minor of order 1 : → $|[a_{11}]_{1\times 1}|$, $|[a_{12}]_{1\times 1}|$

→ Count = 9

19 Minor

Rank: Let A be any matrix, then rank of A is the order of largest order minor, which is non-vanishing (Non-zero)

Minor of order 3, Minor of order 2, Minor of order 1

Rank A = 1 or 2 or ~~3~~

Minor of order 2 whose value is non-zero

↓ $\neq 0$ 9 Non-zero   Value = 0

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[ ] 3x3 → Minor 3 = 0 → Rank ≠ 3

Minor 3 ≠ 0 → Rank = 3

→ Rank ≠ 3
Rank < 3

Minor → Highest
→ Non-Zero

2 order

Minor's

$\underline{\underline{1}} \to (9) \to \neq 0$ → Rank = 1 → Null-Matrix ⇒ (0 = Rank)

Minor of order 2 → Rank → $\underline{\underline{2, 1}}$

(9) $\neq 0$ → Rank = 2

All (9) = 0

Rank $\neq$ 2

Rank = 1

Rank of a null Matrix Zero.

Q→ find the rank $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 2 & 6 & 5 \end{bmatrix}_{3\times 3}$

Sol→ $|A|$ = Minor of order 3 $= 1[20-30] - 2[5-10] + 3[6-8]$

$= -10 - 2(-5) + 3(-2)$

$= -10 + 10 - 6$

$= -6 \neq 0$

$\therefore \text{Rank}(A) = \rho(A) = r(A) = \underline{3}$ ← Rank

Q→ Find rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}_{3\times 3}$

Sol→ Minor of order 3 $= |A| = 1[20-12] - 2[5-4] + 3[6-8]$

$= 8 - 2 - 6 = 8 - 8 = 0$

$\Rightarrow \text{Rank}(A) \neq 3$

Non-Zero

$\boxed{\begin{aligned} &\text{Rank}(A) < 3 \\ &\text{Rank}(A) = 2, 1 \end{aligned}}$

Minor of order 2 $= \left|\begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix}_{2\times 2}\right| = 20 - 12 = 8 \neq 0$

$\rho(A) = r(A) = \text{Rank}(A) = 2$

Nine minor of order 2 → (8)

(# Method of determinant)

If matrix A is of order $\boxed{m \times n}$ then

$r(A) \leq \min\{m, n\}$

$\boxed{\min(m,n)}$

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ → 2, 3

Rank = ?

⇒ (2)

$m \times n$ → (n)

= (m)

Elementary operations: (Row & Column)

Row elementary operations on matrices

(1) $R_i \leftrightarrow R_j$ [Interchanging ith & jth row]

(ii) $R_i \to kR_i$ [multiplication of ith row by k]
(scalar)

(iii) $R_i \to R_i + kR_j$ (k^{th} multiple of jth row element are added to ith row)
(scalar)

[These operations can be applied to columns as well
$C_i \leftrightarrow C_j$, $C_i \to kC_i$, $C_i \to C_i + kC_j$]

[# You can't apply both operations at a time for finding the rank]

Echelon form : A matrix is said to be in row echelon form if number of zeros in a row before the non-zero entry increases, when we go downwards into the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \to \text{Row Echelon form} \quad \text{Rank} = 3$$

$$\begin{bmatrix} 2 & 1 & \boxed{0} \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3} \quad \text{Rank} = 2$$

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$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\boxed{3\times3}} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 5 \end{bmatrix} \to \text{Columns Echelon form}$$

for row echelon only row operations are permitted
for column ——— column ————.

Rank : If a matrix is in row (or column) echelon form, then number of non-zero rows (or columns) is the rank of the matrix.

Q→ $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}_{2\times3}$ Find rank(A)

Sol. $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

$R_1 \to R_1 - R_2$

$\sim \begin{bmatrix} -1 & -2 & -3 \\ 3 & 5 & 7 \end{bmatrix}$

$R_2 \to R_2 + 3R_1$

$\sim \begin{bmatrix} -1 & -2 & -3 \\ 0 & -1 & -2 \end{bmatrix}$

$\therefore$ Rank(A) = 2

Equivalent matrices : matrices A & B are equivalent if they are of same order and one of them can be obtained from the other through elementary operations.
[Equivalent matrices have same rank]
we write it as $A \sim B$

Q → $A = \begin{bmatrix} -1 & 3 & -4 \\ -1 & -3 & 4 \\ 2 & 6 & -8 \end{bmatrix}_{3\times 3}$ find $\rho(A)$ or $r(A)$

Sol → $R_2 \to R_2 + R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank$[A] = 1$

Q → $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \\ 5 & 4 & -5 \end{bmatrix}_{4\times 3}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - 5R_1$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 5 \\ 0 & -3 & 5 \\ 0 & -6 & 10 \end{bmatrix}$$

$R_3 \to R_3 - R_2$, $R_4 \to R_4 - 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Rank}[A] = 2$$

Q → $\begin{bmatrix} 2 & 1 & 5 & -1 \\ -1 & 2 & 5 & 3 \\ 3 & 2 & 9 & -1 \end{bmatrix}$ find Rank [H.W.]

Solution to system of non-homogeneous linear equations.

Consider m linear equations in n variables (unknowns)

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

This system can be written as

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

$$AX = B$$

Coefficient Matrix

Matrix of unknown

Augmented Matrix: $[A|B] = [A:B]$

$$= \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]_{m\times(n+1)}$$

Case 1 : If Rank of $[A]$ = Rank$[A:B]$ = n [No. of variables] then given system of equations will be consistent & have a unique solution.

Case 2 : If $\text{Rank}(A) = \text{Rank}(A:B) \neq n$ [No. of variables] then given system is consistent and it will have infinite many solutions.

Case 3 : If $\text{Rank}(A) \neq \text{Rank}[A:B]$, then given system of equations is inconsistent and it has no solution.

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Solve
$$\left.\begin{aligned} x-y+z&=2\\ 2x+3y-z&=5\\ x+y-z&=0 \end{aligned}\right\} \text{ — (i)}$$

Sol. Given system can be written in matrix form as

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad AX = B$$

Now, Augmented matrix $= [A:B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & 3 & -1 & 5 \\ 1 & 1 & -1 & 0 \end{array}\right]_{3\times 4}$

$R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 5 & -3 & 1 \\ 0 & 2 & -2 & -2 \end{array}\right]$$

$R_3 \to \frac{1}{2} R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 5 & -3 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 5 & -3 & 1 \end{array}\right]$$

$R_3 \to R_3 - 5R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 6 \end{array}\right] \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad AX = B$$

$\therefore \text{Rank}[A:B] = 3$

$\text{Rank}(A) = 3$

$\therefore \text{Rank}[A:B] = \text{Rank}(A) = 3$ (No. of variables)

$\Rightarrow$ Given system is consistent and it will have unique solution.

$x - y + z = 2$ — (ii)

$y - z = -1$ — (iii)

$2z = 6$ — (iv)

$\boxed{z = 3}$, $\boxed{y = z - 1 = 3 - 1 = 2}$, $\boxed{\begin{aligned} x &= y - z + 2 \\ &= 2 - 3 + 2 \\ &= 1 \end{aligned}}$

Q → Solve $4x - 2y + 6z = 8$

$$\left.\begin{array}{l} x+y-3z=-1 \\ 15x-3y+9z=21 \end{array}\right\} \quad -(i)$$

Sol. $\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$

$AX = B$

Now $[A:B] = \left[\begin{array}{ccc|c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right]_{3\times 4}$

$R_1 \leftrightarrow R_2$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{array}\right]$

$R_2 \to R_2 - 4R_1, \quad R_3 \to R_3 - 15R_1$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{array}\right]$

$R_3 \to R_3 - 3R_2$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\text{Rank}[A:B] = 2$
$\text{Ran}(A) = 2$

$\Rightarrow \text{Rank}[A:B] = \text{Rank}(A) = 2$
$\neq 3$ (No. of variables)

$\Rightarrow$ Given system is consistent & will have infinite many solution.

$x+y-3z=-1 \quad -(ii)$

$-6y+18z=12 \quad -(iii)$

n = No of variables $= 3$

r = Rank $= 2$

$n-r = 3-2 = 1$
No. of independent variables

(iii) $\Rightarrow \quad 6y = 18z - 12$

$\Rightarrow y = 3z-2$

Let $\boxed{z = k}$, k is any real

$\Rightarrow \boxed{y = 3k-2}$

(ii) $\Rightarrow x = -y+3z-1 = -3k+2+3k-1$

$\Rightarrow \boxed{x=1}$

$\boxed{x=1,\ y=3k-2,\ z=k,}$ where k is any real number

$k=0$
$(x=1, y=-2, z=0)$
$k=1$
$x=1,\ y=1,\ z=1$

Q →
$$\left.\begin{array}{l} 2x-3y+7z=5 \\ 3x+y-3z=13 \\ 2x+19y-47z=32 \end{array}\right\} \quad -(i)$$

Sol. $\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$ =

$X = A^{-1}B$
$|A|$

$\therefore [A:B] = \left[\begin{array}{ccc|c} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array}\right]$

$R_1 \to R_1 - R_2$

$$\sim \left[\begin{array}{ccc|c} -1 & -4 & 10 & -8 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array}\right]$$

$R_2 \to R_2 + 3R_1, \; R_3 \to R_3 + 2R_1$

$$\sim \left[\begin{array}{ccc|c} -1 & -4 & 10 & -8 \\ 0 & -11 & 27 & -11 \\ 0 & 11 & -27 & 16 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\sim \left[\begin{array}{ccc|c} -1 & -4 & 10 & -8 \\ 0 & -11 & 27 & -11 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

$\therefore$ Rank $[A:B] = 3$

Rank $(A) = 2$

$\therefore$ Rank $[A:B] \neq$ Rank (A)

$\Rightarrow$ System is inconsistent & it has no solution.

Q. $\left.\begin{array}{l} x - 2y + 3t = 2 \\ 2x + y + z + t = -4 \\ 4x - 3y + z + 7t = 8 \end{array}\right\}$

$$[A:B] = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 2 \\ 2 & 1 & 1 & 1 & -4 \\ 4 & -3 & 1 & 7 & 8 \end{array}\right]$$

$R_2 \to R_2 - 2R_1, \; R_4 \to R_4 - 4R_1$

$$\sim \left[\begin{array}{ccccc} 1 & -2 & 0 & 3 & 2 \\ 0 & 5 & 1 & -5 & -8 \\ 0 & 5 & 1 & -5 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 2 \\ 0 & 5 & 1 & -5 & -8 \\ 0 & 0 & 0 & 0 & 8 \end{array}\right] \Rightarrow \text{No solution}$$

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Q. Solve $\left.\begin{array}{l} x - 2y + 3t = 2 \\ 2x + y + z + t = 4 \\ 4x - 3y + z + 7t = 8 \end{array}\right\}$ —(i)

Sol.
$$\underset{A}{\left[\begin{array}{cccc} 1 & -2 & 0 & 3 \\ 2 & 1 & 1 & 1 \\ 4 & -3 & 1 & 7 \end{array}\right]} \underset{X}{\left[\begin{array}{c} x \\ y \\ z \\ t \end{array}\right]} = \underset{B}{\left[\begin{array}{c} 2 \\ 4 \\ 8 \end{array}\right]}$$

$$[A:B] = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 2 \\ 2 & 1 & 1 & 1 & 4 \\ 4 & -3 & 1 & 7 & 8 \end{array}\right]$$

$R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - 4R_1$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 2 \\ 0 & 5 & 1 & -5 & 0 \\ 0 & 5 & 1 & -5 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 2 \\ 0 & 5 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ Rank $[A:B] = 2 = $ Rank $(A) \neq 4$ (No. of variables) n

Given system is consistent & it has infinite solutions

$x - 2y + 3t = 2$ — (ii)

$5y + z - 5t = 0$ — (iii)

$n - r = 4 - 2 = 2$ No. of independent variables

x, y, z, t

y, z, t

Let $\boxed{y = k_1}$, $\boxed{t = k_2}$, k_1, k_2 are real numbers.

$\therefore \boxed{z = 5t - 5y = 5k_2 - 5k_1}$

and (ii) $\Rightarrow$ $x = 2y - 3t + 2$

$\boxed{x = 2k_1 - 3k_2 + 2}$

$\Rightarrow$ Solution is $x = 2k_1 - 3k_2 + 2$, $y = k_1$, $z = 5k_2 - 5k_1$, $t = k_2$

k_1, k_2 are real numbers

$\Rightarrow y = 0, t = 1, x = -1, z = 5$

$\Rightarrow y = 1, t = 0,$ ——

Q $\to$

$n = 4$

$r = 1$

Eq.'s (will get)

No. of independent $a = 1$, $b = 2$, $c = 3$, $d = 4$

Q $\to$ Investigate the value of λ & μ, so that given eq.'s

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$

(i) No solution (ii) Unique solution (iii) Infinite solutions

Sol $\to$ $[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$ $\to B$

A

$R_2 \to R_2 - R_1$, $R_3 = R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$\boxed{\lambda - 3 \neq 0}$

$R_3 \to R_3 - R_2$

$\lambda - 3 \neq 0$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] = 0$$

$\mu = 10$ $\mu = 10$

$= 0 \Rightarrow$

$\neq 0 \Rightarrow \mu \neq 10$

$\neq 0$

(iii) Infinite solution : for infinite solution !

Rank $[A:B]$ = Rank $[A] \neq 3$ [No. of variables

for Rank(A) to be 2, $\lambda - 3 = 0 \Rightarrow \boxed{\lambda = 3}$ $\qquad 2 \neq 3$

for Rank $[A:B]$ to be 2, $\mu - 10 \Rightarrow \boxed{\mu = 10}$

Hence for $\lambda=3$, $\mu=10$, we will get infinite solution

(i) No solution: for no solution $\text{Rank}[A] \neq \text{Rank}[A:B]$

If $\lambda=3 \Rightarrow \text{Rank}(A)=2$

and if $\mu \neq 10 \Rightarrow (\mu-10)\neq 0 \Rightarrow \text{Rank}[A:B]=3$

$\therefore$ For $\lambda=3$, $\mu\neq 10$, we will get no solution

(ii) Unique solution: for unique solution

$\text{Rank}[A] = \text{Rank}[A:B] = 3$ [No. of variables]

Rank [A] will be 3, if $\lambda-3\neq 0 \Rightarrow \lambda\neq 3$

Rank [A:B] will always 3, no matter what will be by μ, so μ can take any value

$\therefore$ For unique solution $\lambda\neq 3$, μ can take any value

Solution to system of homogeneous linear equations:

Consider a system of m equations in n variables

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

Matrix form of this system is

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$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m\times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{m\times 1}$$

$$AX=O$$

(x,y)

NH $\rightarrow x+y+3z=5x$

H $\rightarrow x+y+3z=0$

H $\rightarrow 2x+y+7z=0$

NH $\rightarrow 2x+y+7z=10$

$10x^0$

yz

$0x^5, 0x^7$

Case 1: If $\text{Rank}(A)=n$ (No. of variables), then system has unique solution and this will be trivial solution, that is (zero)

$x_1=x_2=\cdots=x_n=0$

Case 2: If $\text{Rank}(A)\neq n$ (No. of variables) then system has infinite many solutions

$[A:B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} & 0 \end{array}\right]$

$[A:O]$

$\Rightarrow \text{Rank}[A:O] = \text{Rank}(A)$

Q→ Solve

$$\left.\begin{aligned} 3x+y+2z&=0 \\ x-2y+3z&=0 \\ x+5y-4z&=0 \end{aligned}\right\} \text{(i)}$$

Sol. Matrix form of (i) is

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 5 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX=O$$

$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg\} \left\{ \begin{matrix} 2 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} \right]$

No solution

$\text{Rank}[A:B] \neq \text{Rank}(A)$

Now $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 5 & -4 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 1 & -2 & 3 \\ 3 & 1 & 2 \\ \cdots \end{bmatrix}$

Now $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 5 & -4 \end{bmatrix}$ Rank(A:B) ≠ Rank(A)

$R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 1 & -2 & 3 \\ 3 & 1 & 2 \\ 1 & 5 & -4 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 7 & -7 \\ 0 & 7 & -7 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ Rank (A) = 2 ≠ 3 (No. of variables)

$\Rightarrow$ Hence given system has infinite solutions

$x - 2y + 3z = 0$ —(ii)

$\cancel{7}y - z = 0$ —(iii)

$n - r = 3 - 2 = 1$ ← No of variables

Let $\boxed{z = k}$, k is any real number

$\Rightarrow \boxed{y = z = k}$ & $x = 2y - 3z = 2k - 3k = -k$

$\Rightarrow \boxed{x = -k}$

$x = -k$, $y = k$, $z = k$ — k ∈ real

k = 0

x = y = z = 0

Q. $x + 2y + 3z = 0$

$3x + 4y + 4z = 0$

$7x + 10y + 12z = 0$ } —(1)

x = 0, y = 0, z = 0

k = 1 → x = −1, y = 1, z = 1

k = −1 → x = 1, y = z = −1

Sol → $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 7R_1$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ Rank [A] = 3 = No. of variable

$\Rightarrow$ System has unique solution & it will be trivial (zero) solution that is $x = y = z = 0$

Vectors:

$3\hat{i} + 2\hat{j} \longrightarrow (3\ 2), [3\ 2], \begin{bmatrix} 3 \\ 2 \end{bmatrix} \to$ 2D

$3\hat{i} + 2\hat{j} + 5\hat{k} \to (3\ 2\ 5), [3\ 2\ 5], \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \to$ 3D

N-dimensional vector is written as $(x_1, x_2, \ldots, x_n)$

Linearly dependent vectors. Let $V_1\ V_2\ \ldots\ V_n$ are ...

Linearly dependent vectors: Let $V_1, V_2, \ldots V_m$ are m, n-dimensional vectors, we say these are Linearly dependent if there exist, scalars $\alpha_1, \alpha_2, \ldots \alpha_m$ not all zero such that $\alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_m V_m = 0$

$V_1 \to (v_{11}, v_{12}, \ldots v_{1n})$

$V_2 \to (v_{21}, v_{22}, \ldots v_{2n})$

Linearly independent vectors: n-Dimensional vectors

We say $V_1, V_2 \ldots V_m$ are Linearly independent if $\alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_n V_n = 0$
$\Rightarrow [\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0]$

x, y

$0, 0$

$x + 0y = 0$

x, y

$2, 1$

$x + 2y = 0$

$x + 2y = 0$

$x = -2y$

$x + 2y^2 = 0$

$\Rightarrow x = -2y^2$

19/10/20 Q → Check if vectors $V_1 \leftrightarrow (1, 3, 4, 2)$, $V_2 \leftrightarrow (3, -5, 2, 2)$, $V_3 \leftrightarrow (2, -1, 3, 2)$ are L.I. or L.D. If, Linearly dependent find the relationship.

$V_1 + V_2 - 2V_3 = 0$

$0 = 0$

Sol. Let $\alpha_1, \alpha_2, \alpha_3$ be scalars, such that

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0$$

$$\Rightarrow \alpha_1 (1, 3, 4, 2) + \alpha_2 (3, -5, 2, 2) + \alpha_3 (2, -1, 3, 2) = 0$$

$$\Rightarrow (\alpha_1, 3\alpha_1, 4\alpha_1, 2\alpha_1) + (3\alpha_2, -5\alpha_2, 2\alpha_2, 2\alpha_2) + (2\alpha_3, -\alpha_3, 3\alpha_3, 2\alpha_3) = 0$$

$$\Rightarrow (\alpha_1 + 3\alpha_2 + 2\alpha_3,\ 3\alpha_1 - 5\alpha_2 - \alpha_3,\ 4\alpha_1 + 2\alpha_2 + 3\alpha_3,\ 2\alpha_1 + 2\alpha_2 + 2\alpha_3) = 0$$

$$\Rightarrow \left.\begin{aligned} \alpha_1 + 3\alpha_2 + 2\alpha_3 &= 0 \\ 3\alpha_1 - 5\alpha_2 - \alpha_3 &= 0 \\ 4\alpha_1 + 2\alpha_2 + 3\alpha_3 &= 0 \\ 2\alpha_1 + 2\alpha_1 + 2\alpha_3 &= 0 \end{aligned}\right\}$$

$$A = \text{Coefficient Matrix} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & -5 & -1 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -7 \\ 0 & -10 & -5 \\ 0 & -4 & -2 \end{bmatrix}$$

$R_2 \to -\frac{1}{7} R_2$, $R_3 \to -\frac{1}{5} R_3$, $R_4 \to -\frac{1}{2} R_4$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$, $R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \text{Rank}(A) = 2 \neq 3 \text{ [No. of unknowns]}$$

$\Rightarrow$ Given system has non-trivial solution. Hence, system has infinite many solution. Hence vectors are linearly dependent.

$n - r = 3 - 2 = 1$

$$\alpha_1 + 3\alpha_2 + 2\alpha_3 = 0$$

$$\{ 2\alpha_2 + \alpha_3 = 0 \}$$

Let $\boxed{\alpha_2 = k}$ where k is any non-zero real number

$\boxed{\alpha_3 = -2k}$ & $\alpha_1 = -3\alpha_2 - 2\alpha_3 = -3[k] - 2[-2k]$

$\Rightarrow \boxed{\alpha_1 = k.}$

Now, $\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0$

$\Rightarrow kV_1 + kV_2 - 2kV_3 = 0$

$\Rightarrow k[V_1 + V_2 - 2V_3] = 0 \quad | \; k \neq 0$

$\Rightarrow \boxed{V_1 + V_2 - 2V_3 = 0}$ is the linear relationship b/w V_1, V_2, V_3.

Q → Check if vectors $V_1 \Leftrightarrow (3, 2, 4)$, $V_2 \Leftrightarrow (1, 0, 2)$, $V_3 \Leftrightarrow (1, -1, -1)$ are L.I or L.D. If L.D find the relationship

Sol → Let $\alpha_1, \alpha_2, \alpha_3$ scalars, such that

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0 \quad | \; O = 0$$

$$\Rightarrow \alpha_1(3, 2, 4) + \alpha_2(1, 0, 2) + \alpha_3(1, -1, -1) = 0$$

$$\Rightarrow (3\alpha_1, 2\alpha_1, 4\alpha_1) + (\alpha_2, 0, 2\alpha_2) + (\alpha_3, -\alpha_3, -\alpha_3) = 0$$

$$\Rightarrow (3\alpha_1 + \alpha_2 + \alpha_3, \; 2\alpha_1 - \alpha_3, \; 4\alpha_1 + 2\alpha_2 - \alpha_3) = 0$$

$$\Rightarrow \left.\begin{aligned} 3\alpha_1 + \alpha_2 + \alpha_3 &= 0 \\ 2\alpha_1 - \alpha_3 &= 0 \\ 4\alpha_1 + 2\alpha_2 - \alpha_3 &= 0 \end{aligned}\right\}$$

Now, $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & -1 \\ 4 & 2 & -1 \end{bmatrix}$

$R_1 \to R_1 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & -1 \\ 4 & 2 & -1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -5 \\ 0 & -2 & -9 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & -2 & -5 \\ 0 & 0 & -4 \end{bmatrix}$$

$\therefore$ Rank(A) = 3 = Number of unknowns

$\Rightarrow$ Given system has trivial solution → [zero]

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0 \Rightarrow$ Hence vectors are linearly independent.

Q→ Find inverse of matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Sol. Consider $[A : I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - R_2,\ R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right]$$

$$\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}$$

$R_1 \to R_1 - R_3,\ R_2 \to R_2 + 2\ _3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -3 & 2 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right]$$

$R_3 \to (-1)R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

21/10/20 # Eigen values and eigen vectors of matrix

Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order $n \times n$, let λ be a scalar, let X be a column matrix of order $n \times 1$ of unknowns, Consider a system of homogeneous equations [having n unknowns & n equations] as

$$AX = \lambda X \Rightarrow AX - \lambda X = 0 \Rightarrow (A - \lambda I)X = 0 \quad \text{—(1)}$$

$A - \lambda I$

$A = [\ ]$

This system (given in (1)) has either a trivial solution or non-trivial solution
↳ (zero) ↳ (Infinite solutions)

$f(\lambda) = \lambda^2 - 2\lambda + 3$

$f(A) = A^2 - 2A + 3I$

Those values of λ which can give non-

$AX = O$

trivial solution of (i) are called as $A'X = 0$ eigen values of A. The non-trivial solution for any particular value of λ [that is the value of X] is called the eigen vector

Eigen vector can't be zero

$$\underset{n\times 1}{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(i) will non-trivial solution
$\Leftarrow$ Rank $(A - \lambda I) \neq n$ (Number of variables)
This condition will be met if
$\Rightarrow \longrightarrow |A - \lambda I| = 0$ (Necessary conditions)

$x_1 = ?$, $x_2 = ?$

$$\underline{X} = \begin{bmatrix} ? \\ ? \\ \vdots \end{bmatrix}_{n\times 1} \quad [\ ] \quad [\ ]$$

Coloumn vectors

Method to find eigen values & vectors of a matrix:

$$\left(\underset{n\times n}{A} - \lambda \underset{n\times n}{I}\right) \quad \boxed{n \times n}$$

(1) Construct $(A - \lambda I)$ matrix where λ is a scalar & I is identity matrix of order same as A

$$|A - \lambda I| \to n$$
$$0 = 0$$

(ii) Put $|A - \lambda I| = 0$, it is called characteristic eq. of matrix A, it will be a polynomial eq. in λ of degree n [if the order of A is $n\times n$), so solve this eq., find the roots (values of λ) and these roots or values of λ will be eigen values or latent roots or characteristic roots

Q. Find eigen values & vectors of $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

Diagonalizable

Sol. Consider $A - \lambda I = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{bmatrix}$$

Consider characteristic eq of matrix A, $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)\left[\cancel{2} - 3\lambda + \lambda^2 - \cancel{2}\right] - 1[-4 + 2\lambda] - 1[-2] = 0$$

$$\Rightarrow (3-\lambda)[\lambda^2 - 3\lambda] - 2\lambda + 4 + 2 = 0$$

$$\Rightarrow \lambda(3-\lambda)(\lambda - 3) + 2(-\lambda + 3) = 0$$

$$\boxed{\begin{aligned} &-\lambda[\lambda^2 + 9 - 6\lambda] - 2\lambda + 6 = 0 \\ &\Rightarrow -\lambda^3 - 9\lambda + 6\lambda^2 - 2\lambda + 6 = 0 \\ &\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \end{aligned}}$$

Characteristic eq. of A

$$(3-\lambda)[\lambda^2 - 3\lambda + 2] = 0$$

$$\Rightarrow (3-\lambda)(\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda = 1, 2, 3 \text{ are}$$

the eigenvalues

$A \to n \times n$
$X \to n \times 1$
$AX = 0$

for $\lambda = 1$, let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1}$ be the eigen vector of A

$\Rightarrow \quad (A - \lambda I)X = 0$

$\Rightarrow \quad \begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$R_2 \to R_2 + R_1$

$\Rightarrow \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$R_3 \to R_3 - R_2$

$\Rightarrow \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$n - r = 3 - 2 = 1$
$x = k$
$\Rightarrow y - z = 2k$
$y + z = 0$
$[1, 2, 3]$

$\Rightarrow \quad 2x + y - z = 0$
$\rightarrow \{ \; y + z = 0 \; \}$

let $z = k \Rightarrow y = -k$, $x = \dfrac{-y+z}{2} = \dfrac{+k+k}{2} = k$ $\quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

where k is any non-zero real number

$\Rightarrow \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ -k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\lambda = 1$
$k = 1 \to \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
$k = 2 \to \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$
$k = -3 \to \begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix}$
$\vec{V_1} = k\vec{V_2}$

for $\lambda = 1$, eigen vector is

$X = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\left[\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right]$

$\left[\frac{2}{\sqrt{12}}, -\frac{2}{\sqrt{12}}, \frac{2}{\sqrt{12}} \right] \to \left[\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right]$

For $\lambda = 2$, let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector of A

$\Rightarrow \quad (A - \lambda I)X = 0$

$\Rightarrow \quad \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$R_2 \to R_2 + 2R_1$

$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$R_3 \to R_3 - R_2$

$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$x + y - z = 0 \Rightarrow x - z = 0 \Rightarrow x = z$

$\boxed{y = 0}$

let $z = k$, $x = k$, where k is any non-zero real number

$\Rightarrow \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ for $\lambda = 2$, eigen vector is $X = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

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For $\lambda = 3$, let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector A

$\Rightarrow (A - \lambda I)X = 0$

$\Rightarrow \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$R_3 \to R_3 - R_1$

$\Rightarrow \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$R_1 \leftrightarrow R_2$

$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$n = 3, \; r = 2$

$n - r = 3 - 2 = 1$

$\Rightarrow -x - y + z = 0$

$\{ y - z = 0 \}$

let $z = k$, $y = k$ & $x = -y + z = -k + k = 0$

where k is non-zero real number

$\Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ k \\ k \end{bmatrix} = k\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow$ Eigen vector is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Q. $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix} \longrightarrow$ Not Diagonalizable

Sol. Characteristic eq. is $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)[4 + \lambda^2 - 4\lambda - 2] - 2[+1] + 2[2-\lambda] = 0$

$\Rightarrow (1-\lambda)[\lambda^2 - 4\lambda + 2] \quad -2 \quad +4 - 2\lambda = 0$

$\Rightarrow (1-\lambda)[\lambda^2 - 4\lambda + 2] - 2\lambda + 2 = 0$

$\Rightarrow (1-\lambda)[\lambda^2 - 4\lambda + 2 + 2] = 0 \Rightarrow (1-\lambda)(\lambda - 2)^2 = 0$

$\Rightarrow \lambda = 1, \underbrace{2, 2}$ are the eigen ~~vectors~~ values

For $\lambda = 1$, let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigenvector of A

$\Rightarrow (A - \lambda I)X = 0$

$\Rightarrow \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$R_1 \to R_1 - 2R_2$

$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$R_1 \leftrightarrow R_3$

$\Rightarrow \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow \quad -x+2y+z=0$

$\longrightarrow \{y+z=0\}$

$n=3, \ r=2$

$n-r=3-2=1$

Let $z=k, \ y=-k, \ x=2y+z=2(-k)+k=-k$

where k is non-zero real number

$$\Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ -k \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ for $\lambda=1$, eigen vector is $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

For $\lambda=2$, let $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector of A

$\Rightarrow (A-\lambda I)X=0$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \to R_3 - R_1$

$$\Rightarrow \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \to R_3 + 2R_2$

$$\Rightarrow \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow \quad -x+2y+2z=0$

$\boxed{z=0}$

$\Rightarrow \quad -x+2y=0 \Rightarrow x=2y$

Count of eigen vector for a repeated eigen values

$\boxed{n-r=3-2=1}$

Number of independent unknowns

Let $y=k$, $x=2k$, where k is non-zero real number

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda=[2,2]$

1 EV   2 EV

$\Rightarrow$ for $\lambda=2$, eigen vector is $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Q→ $A=\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \longrightarrow$ Diagonalizable

Sol→ Characteristic eq. is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow (-2-\lambda)[-\lambda+\lambda^2-12]-2[-2\lambda-6]-3[-4+1-\lambda]=0$

$\Rightarrow (-2-\lambda)(\lambda-4)(\lambda+3)+4[\lambda+3]+3[\lambda+3]=0$

$\Rightarrow (\lambda+3)[-2\lambda+8-\lambda^2+4\lambda+7]=0$

$\Rightarrow (\lambda+3)(\lambda^2-2\lambda-15)=0$

$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-5)=0$

$\Rightarrow \lambda=-3,-3,5$ are the eigen values

$\Rightarrow \lambda = -3, -3, 5$ are the eigen values

For $\lambda = 5$, Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector of A

$\Rightarrow (A - \lambda I) X = 0$

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_1 \leftrightarrow R_3$

$$\Rightarrow \begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 7R_1$

$$\Rightarrow \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow x + 2y + 5z = 0$

$y + 2z = 0$

$n = 3, r = 2$
$n - r = 1$

Let $z = k$, $y = -2k \Rightarrow x = -2y - 5z$
$= -2(-2k) - 5(k) = -k$

where k is any non-zero real number

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Eigen vector is $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

For $\lambda = -3$, let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigenvector of A

$\Rightarrow (A - \lambda I) X = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \to R_3 + R_1$, $R_2 \to R_2 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow x + 2y - 3z = 0$

Let $y = k_1$, $z = k_2 \Rightarrow x = -2y + 3z$
$= -2k_1 + 3k_2$

No. of eigenvectors

$n = 3$, $r = 1$

$n - r = 3 - 1 = 2$

No. of independent unknown

$k_1 = 0, k_2 = 1$

$k_1 = 1, k_2 = 0$

$k_1 = k_2 = 0$

where k_1, k_2 are real numbers and both can't be zero at same time

$$\Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2k_1 + 3k_2 \\ k_1 \end{bmatrix} = \begin{bmatrix} -2k_1 \\ k_1 \end{bmatrix} + \begin{matrix} 3 \\ 0 \end{matrix} \qquad \begin{bmatrix} \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2k_1 + 3k_2 \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -2k_1 \\ k_1 \\ 0 \end{bmatrix} + \begin{matrix} 3k_2 \\ 0 \\ k_2 \end{matrix} \qquad \begin{bmatrix} 2 \\ \\ \end{bmatrix}$$

$$= k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Eigen vectors for $\lambda = -3$ are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Q.> $A = \begin{bmatrix} 4 & 3 \\ 2 & 9 \end{bmatrix}$

Sol. characteristic eq is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 3 \\ 2 & 9-\lambda \end{vmatrix} = 0$$

$$\underbrace{A^2 - 13A + 30I}_{\gamma}$$
$$0 = 0$$

$$\Rightarrow 36 - 13\lambda + \lambda^2 - 6 = 0$$

$$\Rightarrow \lambda^2 - 13\lambda + 30 = 0 \rightarrow \text{characteristic eq.}$$

$$\Rightarrow (\lambda - 10)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 3, 10 \text{ are the eigen values}$$

For $\lambda = 3$, let $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be the eigen vector of A

$(A - \lambda I) X = 0$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$R_2 \to R_2 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad n - r = 2 - 1 = 1$$

$$x + 3y = 0$$

let $y = k \Rightarrow x = -3k$, where k is non-zero real number

$$\Rightarrow X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3k \\ k \end{bmatrix} = k \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Eigen vector is $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Properties of eigen values & vectors :

1. A & A^T (A') have same eigen values
2. For a diagonal matrix, eigen values are the diagonal entries.
3. For a triangular matrix, eigen values are the diagonal enteries.
4. For idempotent matrix ($A^2 = A$) eigen values are either 0 or 1.
5. Sum of eigen values = Sum of diagonal entries of the matrix

Trace = It is sum of eigen values or sum of diagonal elements.

23/10/20 6. Product of eigen values is equal to the determinant of

(7) If λ is the eigen value of the matrix A then λ^{-1} is the eigen value of A^{-1} matrix $A \to \frac{1}{\lambda}$

(8) If λ is the eigen value of matrix A, then $k\lambda$ (where k is any scalar) is eigen value of kA

(9) If λ is the eigen value of A, then λ^m is the eigen value of A^m

(10) If λ is the eigen value of A, then $\frac{|A|}{\lambda}$ is the eigen value of $adj(A)$.

$\boxed{AX = \lambda X}$ ← Eigen vector, Eigen value

$A = [a_{ij}]_{3\times3}$

$\lambda = 1$

$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\{ AX = \lambda X \}$ $\lambda > 1$, $\lambda < 1$

$AX = \lambda X$

$\lambda \to A$

$\frac{|A|}{\lambda} \to adj A$

Q. Find sum & product of eigen values of $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Sol. Eigen values of A are 3, 2, 5

$\Rightarrow$ Trace = 10 & Product = 30

Q. Find eigen value of $adj\,A$ & $\boxed{A^2 - 2A + I}$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

Sol. Eigen values of A are 2, 4, 3

Now $|A| = 24$

Now Eigen values of (adj A) are $\frac{|A|}{2}, \frac{|A|}{4}, \frac{|A|}{3}$

$\lambda = 2$ (λ=4) 12, 6, 8

Eigen value of A^2 are 4, 16, 9

Eigen values of 2A are 4, 8, 6

Eigen values of I are 1, 1, 1

Now, eigen values of $(A^2 - 2A + I)$ 1, 9, 4

Q. Let A be a 4x4 matrix with eigen values 1, -1, 2, -2. Find the eigen values of matrix $B = 2A + A^{-1} - I$ and also find $|B|$.

$-4 - \frac{1}{2} - 1$

Sol. Eigen values of A are 1, -1, 2, -2

———— 2A — 2, -2, 4, -4

———— A^{-1} — 1, -1, $\frac{1}{2}$, $-\frac{1}{2}$

———— I — 1, 1, 1, 1

Eigen values of $B = 2A + A^{-1} - I$ are 2, -4, $\frac{7}{2}$, $-\frac{11}{2}$

$\Rightarrow |B| = 2(-4)(\frac{7}{2})(-\frac{11}{2}) = 154$

Q. Let eigen values of A are 1, 2, -1. Find eigen values of $B = A - A^{-1} + A^2$ and find trace of B (H.W.)

Cayley Hamilton theorem : Every square matrix satisfies

Cayley-Hamilton theorem. : Every square matrix satisfies its own characteristic equation.

Q → If $A = \begin{bmatrix} 4 & 3 \\ 2 & 9 \end{bmatrix}$. Verify Cayley-Hamilton theorem and hence find A^{-1}.

Sol. → Characteristic eq. of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 3 \\ 2 & 9-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ Characteristic eq. will be $\lambda^2 - 13\lambda + 30 = 0$

Now $A^2 = \begin{bmatrix} 4 & 3 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} 22 & 39 \\ 26 & 87 \end{bmatrix}$ | $A^2 - 13A + 30I$

Now $A^2 - 13A + 30I = \begin{bmatrix} 22 & 39 \\ 26 & 87 \end{bmatrix} - 13\begin{bmatrix} 4 & 3 \\ 2 & 9 \end{bmatrix} + 30\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 22 & 39 \\ 26 & 87 \end{bmatrix} - \begin{bmatrix} 52 & 39 \\ 26 & 117 \end{bmatrix} + \begin{bmatrix} 30 & 0 \\ 0 & 30 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$\Rightarrow$ Cayley-Hamilton theorem is verified

Now $A^2 - 13A + 30I = 0$

Multiply with A^{-1}

$A - 13I + 30A^{-1} = 0$

$30A^{-1} = 13I - A = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 2 & 9 \end{bmatrix}$

$\Rightarrow A^{-1} = \frac{1}{30}\begin{bmatrix} 9 & -3 \\ -2 & 4 \end{bmatrix}$

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Q → If $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}_{3\times3}$ Verify that A satisfies Cayley-Hamilton theorem & hence find A^{-1}.

Sol. → Characteristic eq. of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)[15 - 8\lambda + \lambda^2 - 16] - 4[20 - 4\lambda] = 0$

$\Rightarrow \lambda^2 - 8\lambda - 1 - \lambda^3 + 8\lambda^2 + \lambda - 80 + 16\lambda = 0$

$\Rightarrow \lambda^3 - 9\lambda^2 - 9\lambda + 81 = 0$ is the Characteristic eq.

Now we shall check the value $(A^3 - 9A^2 - 9A + 81I)$

Complete at your own.

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① Linear dependence and independence of vectors using determinant :

If number of vectors & components in each vector are same, then construct a matrix (it will be a square matrix) with each vector as a row of the matrix. If determinant of matrix will be zero, then vectors will be linearly dependent, otherwise linearly independent.

Q→ check if vectors (3, 2, 4), (1, 0, 2), (1, −1, −1) are L.I or L.D.

Sol→ $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & -1 \end{bmatrix}_{3\times3}$

$|A| = 3[2] - 2[-1-2] + 4[-1] = 6+6-4 = 8 \neq 0$

⇒ Hence given vector are L.I

② Form a matrix by inserting each vector as a row of the matrix, reduce it to echelon form, then given vectors are L.I. if there is no row of all zero in the echelon form.

Q→ Check if the vectors (2, 3, 6, −3, 4), (4, 2, 12, −3, 6) (4, 10, 12, −9, 10) are L.I. or L.D.

Sol→ $A = \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 4 & 2 & 12 & -3 & 6 \\ 4 & 10 & 12 & -9 & 10 \end{bmatrix}$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 2R_1$

$\sim \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 0 & -4 & 0 & 3 & -2 \\ 0 & 4 & 0 & -3 & 2 \end{bmatrix}$

$R_3 \to R_3 + R_2$

$\sim \begin{bmatrix} 2 & 3 & 6 & -3 & 4 \\ 0 & -4 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

⇒ vectors are L.D.

Q→ Again check the L.D. or L.I of the (3, 2, 4), (1, 0, 2) (1, −1, −1)

Sol→ $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 4 \\ 1 & -1 & -1 \end{bmatrix}$

$R_2 \to R_2 - 3R_1,\ R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -2 \\ 0 & -1 & -3 \end{bmatrix}$

$R_2 \to \frac{1}{2} R_2$

$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & -3 \end{bmatrix}$

$R_3 \to R_3 + R_2$

$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -4 \end{bmatrix}$

vectors are L.I.

Diagonalization: A square matrix of order $n \times n$, is said to be diagonalizable if it has n linear independent eigen vectors

Gauss-Jordan Method for finding inverse of the matrix

Q→ $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ Find A^{-1} using G-J method

Sol.→ $$[A:I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right]$$

$R_1 \to R_1 - 2R_2$, $R_3 \to R_3 + 5R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right]$$

$R_3 \to \frac{1}{2} R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right]$$

$R_1 \to R_1 + R_3$, $R_2 \to R_2 - 2R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

Unit-2:

Derivative: Let $y = f(x)$ be a function of x, then derivative of y w.r.t x [represents rate of change of y w.r.t x] and is denoted as

$$\frac{dy}{dx} = \frac{d}{dx}f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$