14/12/20
Monday, December 14, 2020
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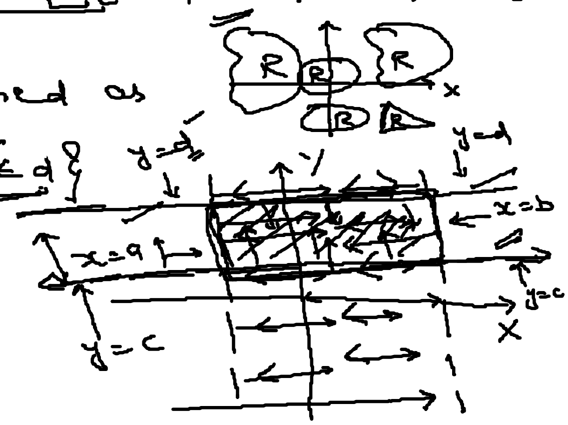
Unit-5 :

Multiple integral. Double integral: Let $f(x,y)$ be a function of x, y & we can integrate it over region R [A part of XY plane] as stated below:



① Let region R is rectangular defined as

$$R = \{ (x,y) : a \le x \le b, \quad c \le y \le d \}$$

$$I = \iint_{R} f(x,y)\,dx\,dy$$

$$\Rightarrow I = \int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dx\,dy$$

$c \le y \le d$
$y = c,\ y = d$

$a \le x \le b$
$x = a$ & $x = b$
$a \le x \le b$

$$I = \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx$$

Q-1 $\quad I = \int_{y=-1}^{2}\int_{x=1}^{2} x^2 y^3\,dx\,dy$

Sol. $\quad I = \int_{y=-1}^{2}\int_{x=1}^{2} x^2 y^3\,dx\,dy = \int_{y=-1}^{2} y^3 \left(\frac{x^3}{3}\right)_{x=1}^{2} dy$

$$= \int_{y=-1}^{2} y^3 \left[\frac{8-1}{3}\right] dy = \frac{7}{3}\left.\frac{y^4}{4}\right]_{y=-1}^{2} = \frac{7}{3}\times\left[\frac{16-1}{4}\right]$$

$$= \frac{7}{\cancel{3}}\times\frac{\cancel{15}\,5}{4} = \frac{35}{4}$$

OR $\quad I = \int_{x=1}^{2}\int_{y=-1}^{2} x^2 y^3\,dy\,dx = \int_{x=1}^{2} x^2 \left(\frac{y^4}{4}\right)_{y=-1}^{2} dx$

$$= \frac{16-1}{4}\int_{x=1}^{2} x^2\,dx = \frac{15}{4}\times\left(\frac{x^3}{3}\right)_{x=1}^{2} = \frac{\cancel{15}\,5}{4}\times\frac{7}{\cancel{3}}$$

$$= \frac{35}{4}$$

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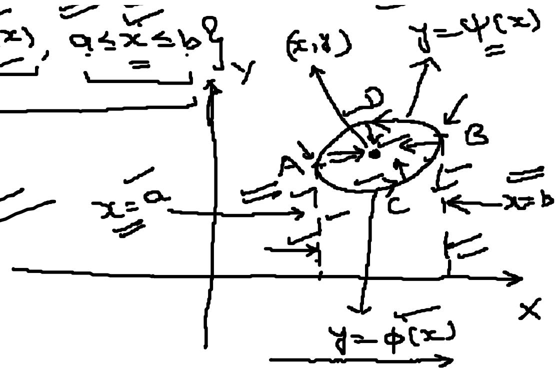
②: Let R be a non-rectangular region:

(2a) If region R is defined as

$R = \{(x,y) : \phi(x) \le y \le \psi(x),\ a \le x \le b\}$

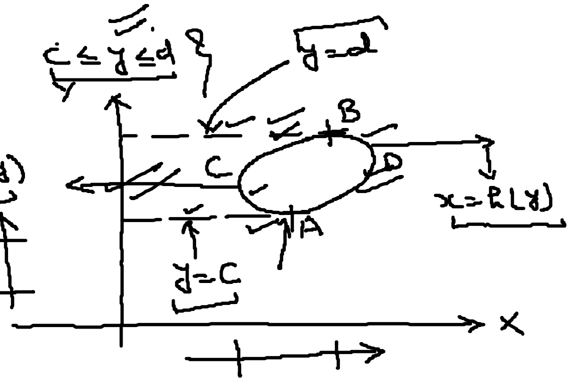
$$\therefore I = \int_{x=a}^{b} \int_{y=\phi(x)}^{\psi(x)} f(x,y)\,dy\,dx$$



(2b) If R is defined as

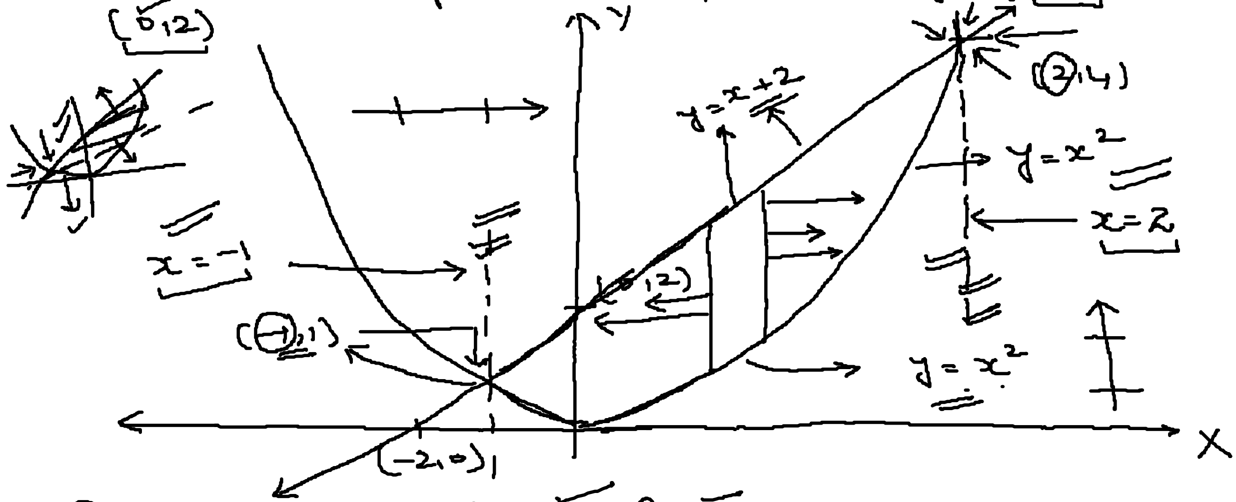
$R = \{(x,y) : g(y) \le x \le h(y),\ c \le y \le d\}$

$$\therefore I = \int_{y=c}^{d} \int_{x=g(y)}^{h(y)} f(x,y)\,dx\,dy$$



Q. Evaluate $I = \iint x^2\,dx\,dy$, over the region R bounded by $y = x^2$, $y = x+2$

Sol. $y = x^2$ is an upwards parabola with vertex at (0,0) and $y = x+2$ is a straight line which passes through (-2,0) & (0,2)



Point of intersection of $y = x^2$ & $y = x+2$ are

$\Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2, -1$

$y = 4, 1$

$\therefore$ Points are (-1,1) & (2,4)

Limits of region R are $-1 \le x \le 2$

$x^2 \le y \le x+2$

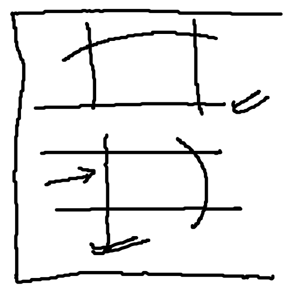
Now $I = \int^{2} \int^{x+2} x^2\,dy\,dx = \int^{2} x^2\,(y)^{x+2}\,dx$

Now $I = \int_{x=-1}^{2}\int_{y=x^2}^{x+2} x^2\,dy\,dx = \int_{x=-1}^{2} x^2\,(y)_{x^2}^{x+2}\,dx$

$$= \int_{x=-1}^{2} x^2\left[x+2-x^2\right]dx = \left[\frac{x^4}{4}+\frac{2x^3}{3}-\frac{x^5}{5}\right]_{x=-1}^{2}$$

$$= \left[\left(\frac{16}{4}+\frac{16}{3}-\frac{32}{5}\right)-\left(\frac{1}{4}-\frac{2}{3}+\frac{1}{5}\right)\right]$$

Area: Area of a region $R = \iint_{R} dx\,dy$



Q→ Find the area of region bounded by curves $x=y^2$, $x+y-2=0$

Sol→ Here $x=y^2$ is right handed parabola with vertex at $(0,0)$ & $x+y=2$ or $\frac{x}{2}+\frac{y}{2}=1$ is a straight line passing through $(2,0)$ & $(0,2)$

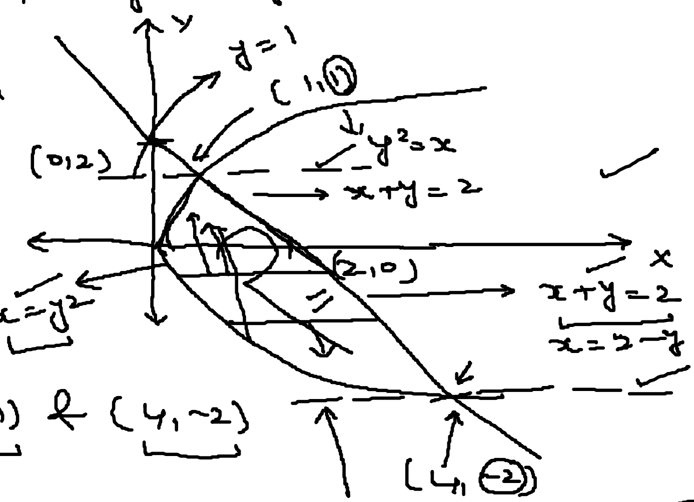
Points of intersection of $x=y^2$ & $x+y=2$ are $y^2+y-2=0$

$\Rightarrow (y+2)(y-1)=0$

$\Rightarrow y=-2, 1$

$\Rightarrow x=4, 1$

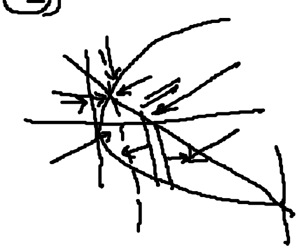
$\Rightarrow$ Points are $(1,1)$ & $(4,-2)$



Limits of region R are

$-2 \le y \le 1$

$y^2 \le x \le 2-y$



$\therefore$ Area of region $R = \int_{y=-2}^{1}\int_{x=y^2}^{2-y} dx\,dy$

$$= \int_{y=-2}^{1} (x)_{x=y^2}^{2-y}\,dy = \int_{y=-2}^{1}\left[2-y-y^2\right]dy$$

$$= \left(2y-\frac{y^2}{2}-\frac{y^3}{3}\right)_{y=-2}^{1}$$

$$= \left[\left(2-\frac{1}{2}-\frac{1}{3}\right)-\left(-4-\frac{4}{2}+\frac{8}{3}\right)\right]$$

H.W ①

Q. $\iint_R (x^2+y^2)\,dx\,dy$, R : $\underbrace{0 \le y \le \sqrt{1-x^2},\ 0 \le x \le 1}$

Sol. $I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$

17/12/20 Q. Change the order of integration & hence evaluate

$I = \int_{y=0}^{1} \int_{x=y}^{y^{1/3}} e^{x^2}\,dx\,dy$

Given $dx\,dy$ → Goal → $dy\,dx$

Sol. Here limits are $y \le x \le y^{1/3}$ & $0 \le y \le 1$

Here $x = y^{1/3} \Rightarrow y = x^3$, it passes through origin.

and $x = y$, it is a straight line which passes through origin & it bisect first & 3rd quadrant

Points of intersection of $y = x^3$ & $y = x$ are

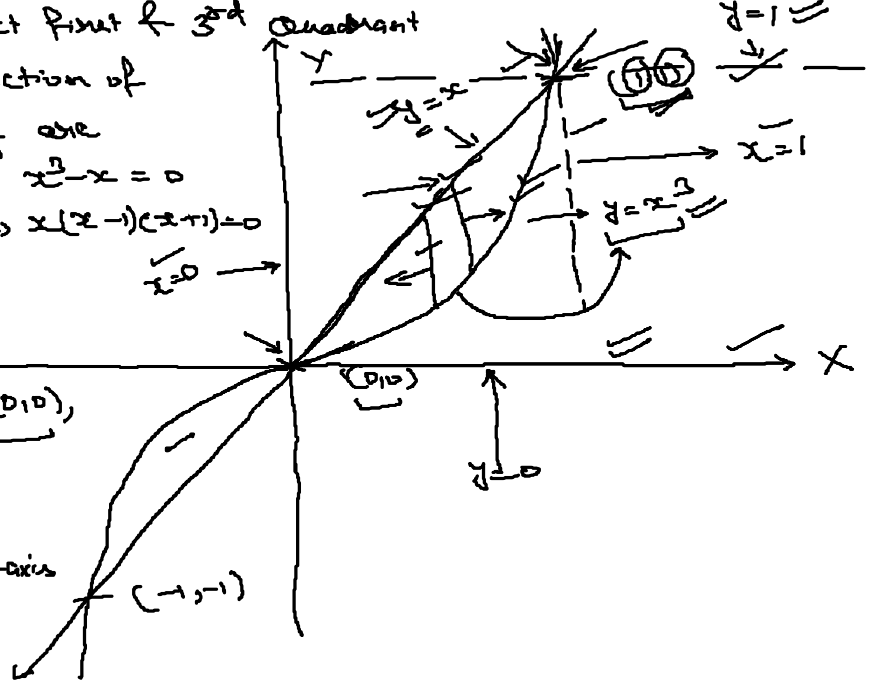
$\Rightarrow x^3 = x \Rightarrow x^3 - x = 0$

$\Rightarrow x(x^2-1) = 0 \Rightarrow x(x-1)(x+1) = 0$

$\Rightarrow x = 0, 1, -1$

$\Rightarrow y = 0, 1, -1$

∴ Points are $(0,0)$, $(1,1)$, $(-1,-1)$



Now $y=0$ is x-axis & $y=1$ is straight line parallel to x-axis at a distance of 1 unit & above x-axis

Limits of the region with changed order are

$0 \le x \le 1$, $x^3 \le y \le x$

∴ Now $I = \int_{x=0}^{1} \int_{y=x^3}^{x} e^{x^2}\,dy\,dx = \int_{x=0}^{1} e^{x^2} (y)_{y=x^3}^{x}\,dx$

$= \int_{x=0}^{1} e^{x^2}[x - x^3]\,dx = \int_{x=0}^{1} e^{x^2}[1-x^2]\,x\,dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

when $x = 0 \Rightarrow t = 0$

$x = 1 \Rightarrow t = 1$

$$= \int_{t=0}^{1} \underbrace{e^t}_{v} \underbrace{[1-t]}_{u} \frac{dt}{2} \qquad x=1 \Rightarrow t=1 \quad \text{ILATE}$$

$$= \frac{1}{2}\left[(1-t)\, e^t - \int (-1)\, e^t \, dt \right]_{t=0}^{1}$$

$$= \frac{1}{2}\left[(1-t)\, e^t + e^t \right]_{t=0}^{1}$$

$$= \frac{1}{2}\left[(0+e^1) - (e^0+e^0) \right] = \frac{1}{2}[e-2]$$

Q → Change the order of integration & hence evaluate

$$I = \int_{y=0}^{1} \int_{x=y}^{\sqrt{2-y^2}} \frac{y\, dy\, dx}{\sqrt{x^2+y^2}}$$

Given → $dx\,dy$

GOAL → $dy\,dx$

Sol → Limits are $y \le x \le \sqrt{2-y^2}$ & $0 \le y \le 1$

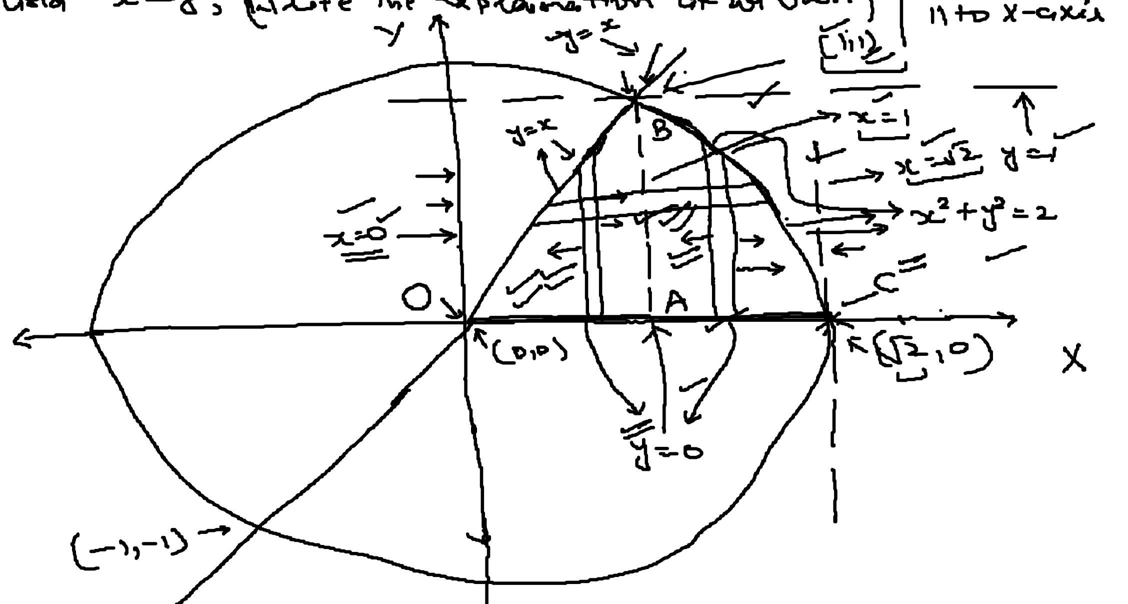
Here $x = \sqrt{2-y^2} \Rightarrow x^2 = 2 - y^2 \Rightarrow x^2+y^2 = 2 = (\sqrt{2})^2$

It is a circle with center $(0,0)$ & radius $= \sqrt{2}$

and $x = y$, (write the explaination at ur own)

$y=0$ is x-axis

& $y=1$ is line ∥ to x-axis



Points of intersection of $y=x$ & $x^2+y^2=2$ are

$$\Rightarrow x^2+x^2 = 2 \Rightarrow 2x^2 = 2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$
$$\Rightarrow y = \pm 1$$

∴ Points are $(1,1)$ & $(-1,-1)$

Over the region OAB, limits are $0 \le x \le 1$, $0 \le y \le x$

and limits of region ABC are, $1 \le x \le \sqrt{2}$

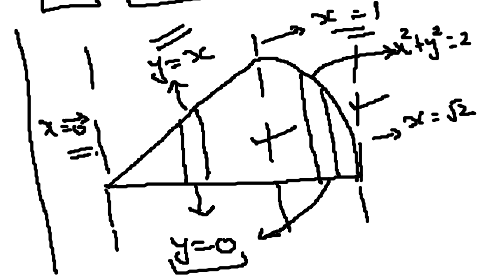
$x^2+y^2=2$

$y = \pm\sqrt{2-x^2}$

$y = +\sqrt{2-x^2}$

$$\therefore\ I = \int_{x=0}^{1}\int_{y=0}^{x} \frac{y}{\sqrt{x^2+y^2}}\,dy\,dx + \int_{x=1}^{\sqrt{2}}\int_{y=0}^{\sqrt{2-x^2}} \frac{y}{\sqrt{x^2+y^2}}\,dy\,dx$$

$0 \le y \le \sqrt{2-x^2}$



$$\Rightarrow\ I = I_1 + I_2 \quad \text{—(1)}$$

Now $$I_2 = \frac{1}{2}\int_{x=1}^{\sqrt{2}}\int_{y=0}^{\sqrt{2-x^2}} 2y\,(x^2+y^2)^{-1/2}\,dy\,dx$$

$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$= \cancel{\frac{1}{2}}\int_{x=1}^{\sqrt{2}} \left[\frac{(x^2+y^2)^{1/2}}{\cancel{1/2}}\right]_{y=0}^{\sqrt{2-x^2}} dx$$

18/12/20 Change of variable : Let $I = \iint_R f(x,y)\,dx\,dy$, let us change the variables $x = \phi(u,v)$, $y = \psi(u,v)$

$$\therefore\ I = \iint_{R'} f(\phi(u,v), \psi(u,v))\,|J|\,du\,dv$$

$J\left(\frac{x,y}{u,v}\right) = J =$ Jacobian of x, y w.r.t u, v $= \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

Special Case : Changing from (x,y) [Cartesian Co-ordinates] to (r,θ) [Polar Co-ordinates]

$$I = \iint_R f(x,y)\,dx\,dy$$

$x = r\cos\theta$, $y = r\sin\theta$

$$\therefore\ I = \iint_R f(r\cos\theta, r\sin\theta)\,|J|\,dr\,d\theta = \iint_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Now $$J = \frac{\partial(x,y)}{\partial(r,\theta)} = J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r[\cos^2\theta + \sin^2\theta] = r$$

$= r\,[\cos\theta + \sin\theta]$
$= r$

$$\Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = r$$

\# Area of region R $= \iint_R dx\,dy = \iint_R r\,dr\,d\theta$

Q.> Evaluate $\iint_R (x^2+y^2)\,dx\,dy$, $R: 0 \le y \le \sqrt{1-x^2}$, $0 \le x \le 1$ by changing to polar co-ordinates

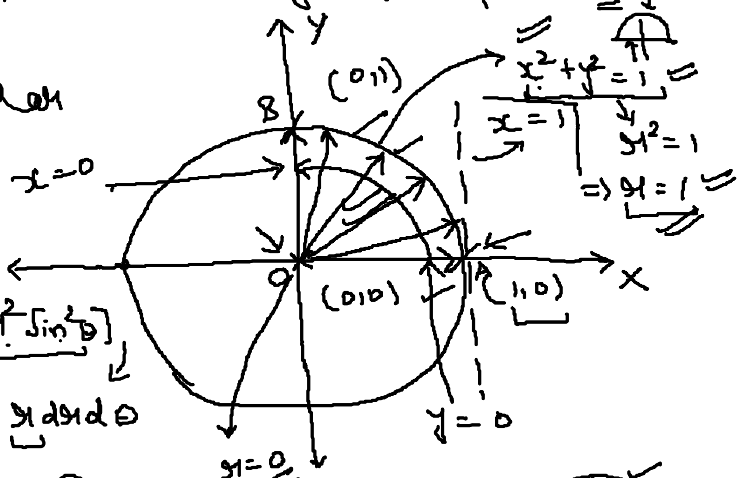
Sol. Here limits are $0 \le y \le \sqrt{1-x^2}$, $0 \le x \le 1$

Consider $y = \sqrt{1-x^2} \Rightarrow y^2 = 1-x^2 \Rightarrow x^2+y^2=1$, it is a circle with center $(0,0)$ & radius $= 1$

and $y=0$ is x-axis, $x=0$ is y-axis, $x=1$ is a line $\parallel$ to y-axis at a distance of 1 unit & on right side of y-axis.

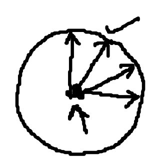
Now, shifting to polar co-ordinates (r,θ)

$\Rightarrow x = r\cos\theta,\ y = r\sin\theta$



$$\Rightarrow I = \iint_R [r^2\cos^2\theta + r^2\sin^2\theta]\, r\,dr\,d\theta$$

$$I = \iint_R r^3\,dr\,d\theta \quad -(1)$$



Now limits of region R in (r,θ)

$$\begin{bmatrix} 0 \le \theta \le \pi/2 \\ 0 \le r \le 1 \end{bmatrix}$$

$$\therefore I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^3\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\left(\frac{r^4}{4}\right)_{r=0}^{1} d\theta = \frac{1}{4}\int_{\theta=0}^{\pi/2} d\theta$$

$$= \frac{1}{4} \times \frac{\pi}{2}$$

$4 \cdot 2 = \underline{8}$

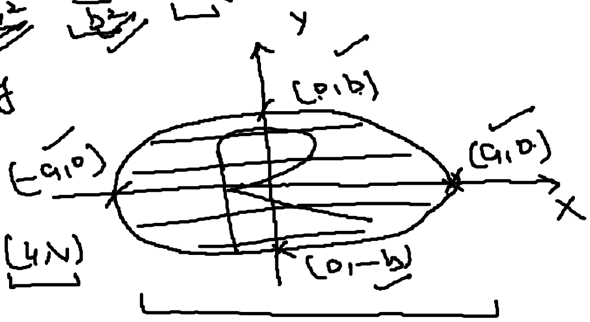
Cartesian System → Rectangular

Polar System → Circular system

Q→ $\iint_R \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dx\,dy$, R is the region with boundary $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sol → Here $I = \iint_R \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dx\,dy$

Here region R is bounded by ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



changing to new variable (u,v)

$\frac{x}{a} = u$ & $\frac{y}{b} = v$

$\Rightarrow x = au$ & $y = bv$

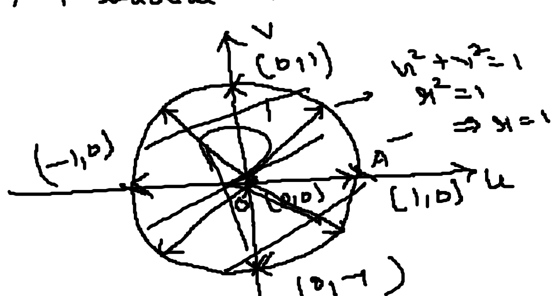
Now $I = \iint_{R'} \sqrt{1-u^2-v^2}\, |J|\, du\,dv$

where $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a\cdot 1 & 0 \\ 0 & b\cdot 1 \end{vmatrix} = ab$

$\Rightarrow I = \iint_{R'} \sqrt{1-u^2-v^2}\; ab\,du\,dv$

where, R' is the region bounded by $u^2+v^2=1$, it is a circle with centre is (0,0) & radius = 1

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \to u^2+v^2=1$



$r \geq 0$
$\therefore r \neq -1$

Changing to polar co-ordinates (r,θ)

$\therefore u = r\cos\theta$, $v = r\sin\theta$

$u^2+v^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$\Rightarrow I = ab \iint_{R'} \sqrt{1-r^2}\; r\,dr\,d\theta$

Limits of R' are $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$

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$\Rightarrow I = \frac{ab}{2}\int_0^{2\pi}\int_0^1 (1-r^2)^{1/2}(-2r)\,dr\,d\theta$ ← $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{}$

$$\Rightarrow I = ab \iint_{R'} \sqrt{1-r^2}\; r\,dr\,d\theta$$

$+r^2\sin^2\theta$

$= r^2$

Limits of R' are $0 \le \theta \le 2\pi$, $0 \le r \le 1$

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$$\Rightarrow I = -\frac{ab}{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} (1-r^2)^{1/2}(-2r)\,dr\,d\theta$$

$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$

$$= -\frac{ab}{2}\int_{\theta=0}^{2\pi} \left[\frac{[1-r^2]^{3/2}}{3/2}\right]_{r=0}^{1} d\theta$$

$$= -\frac{ab}{2} \times \frac{2}{3}\left[0 - 1\right] 2\pi = +\frac{2\pi ab}{3}$$

\# **Triple Integral**: Let $f(x,y,z)$ is to be integrated over some solid V, we do it as explained below:

Case 1: If $V: \{(x,y,z): a \le x \le b,\ c \le y \le d,\ e \le z \le f$

where a,b,c,d,e,f are all real constants $\}$

Cuboid

$\therefore I = \iiint_V f(x,y,z)\,dx\,dy\,dz$, then it can be evaluated in any order.

Case 2: If $V = \{(x,y,z): f_1(x,y) \le z \le f_2(x,y),\ g_1(x) \le y \le g_2(x),$

$a \le x \le b\ \}$

$$\Rightarrow I = \int_{x=a}^{b}\int_{y=g_1(x)}^{g_2(x)}\int_{z=f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Const. $x \quad y \quad z$

$\phi_1(x,z) \le y \le \phi_2(x,z)$

$\psi_1(x) \le z \le \psi_2(x)$

$a \le x \le b$

Order → first → y, 2nd → z, 3rd → x

Const. $x \quad y \quad z \quad x$

Q→ Evaluate $I = \int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a} (x^2+y^2+z^2)\,dx\,dy\,dz$

Sol.→ $$I = \int_{z=0}^{c}\int_{y=0}^{b}\left[\frac{x^3}{3} + y^2x + z^2x\right]_{x=0}^{a} dy\,dz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b}\left[\frac{a^3}{3} + ay^2 + az^2\right] dy\,dz$$

$$= \int_{z=0}^{c}\left[\frac{a^3}{3}y + \frac{ay^3}{3} + az^2y\right]_{y=0}^{b} dz$$

$$= \int^{c} \left[\frac{a^3 b}{3} + \frac{ab^3}{3} + abz^2\right] dz$$

$$= \int_{z=0} \left[\frac{a^3 b}{3} + \frac{ab^3}{3} + \right]$$

$$= \left(\frac{a^3 b}{3} + \frac{ab^3}{3}\right) z + ab \frac{z^3}{3} \Big]_{z=0}^{c} = \frac{(a^3 b + ab^3)c}{3} + \frac{abc^3}{3}$$

$$= \frac{a^3bc + ab^3c + abc^3}{3} = \frac{abc[a^2+b^2+c^2]}{3}$$

Q→ $I = \int_{y=1}^{e} \int_{x=1}^{\log y} \int_{z=1}^{e^x} \log z \cdot 1 \, dz \, dx \, dy$ ILATE $\int \left(\frac{1}{z} \cdot z\right)$

$\int \log x \, dx$

$\int \log x \cdot 1 \, dx$

$\log 1 = 0$

$$= \int_{y=1}^{e} \int_{x=1}^{\log y} \left[z \log z - z \right]_{z=1}^{e^x} dx\, dy$$

$$= \int_{y=1}^{e} \int_{x=1}^{\log y} \left[e^x \cdot \log e^x - e^x - 1(-1) \right] dx\, dy$$

$$= \int_{y=1} \int_{x=1}^{\log y} \left[x e^x - e^x + 1 \right] dx\, dy$$

$$= \int_{y=1}^{e} \left[x e^x - 1 \cdot e^x - e^x + x \right]_{x=1}^{\log y} dy$$

$$= \int_{y=1}^{e} \left[\left(\log y \cdot e^{\log y} - 2 e^{\log y} + \log y \right) - (e - 2e + 1) \right] dy$$

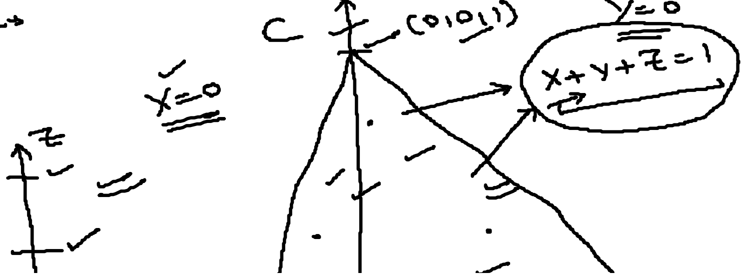
$$= \int_{y=1}^{e} \left[y \log y - 2y + \log y + e - 1 \right] dy$$

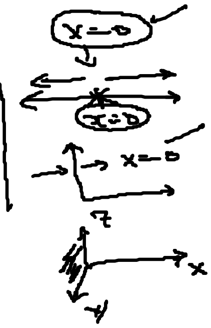
Complete at your own

Q→ Evaluate $\iiint_T \frac{dx\, dy\, dz}{(x+y+z+1)^3}$ with boundary of T

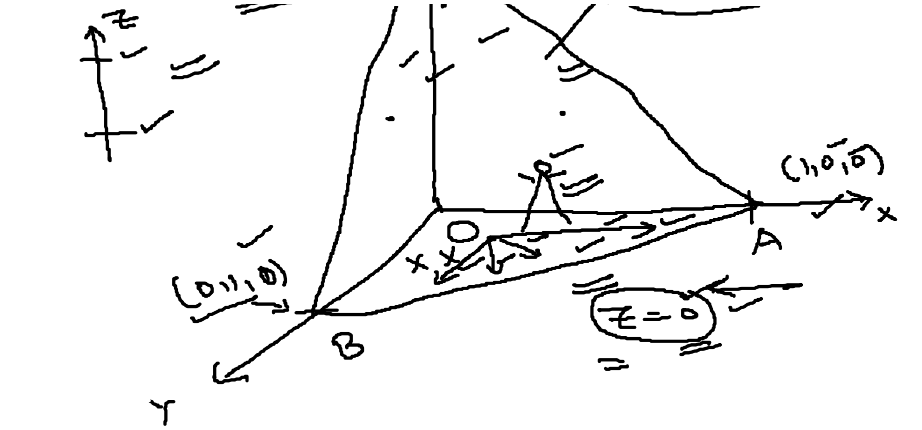
is $x=0,\ y=0,\ z=0,\ x+y+z=1$

Sol.→





$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$



$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$x+y+z=1$

Here $x=0$ is YZ-plane, $y=0$ is XZ-plane, $z=0$ is XY-plane and $x+y+z=1$ is a plane making intercept of 1-unit on all axes.

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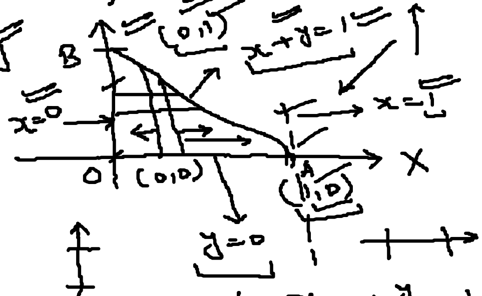
$$\begin{bmatrix} 0 \le z \le 1-x-y \\ 0 \le y \le 1-x \\ 0 \le x \le 1 \end{bmatrix}$$

$x+y+z=1$
$\Rightarrow z = 1-x-y$



Now $$I = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} (x+y+z+1)^{-3}\, dz\, dy\, dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} \left[\frac{(z+x+y+1)^{-2}}{(-2)}\right]_{z=0}^{1-x-y} dy\, dx$$

$\frac{x}{1}+\frac{y}{1}=1$

$\frac{x}{a}+\frac{y}{b}=1$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} \left(-\tfrac{1}{2}\right)\left[(2)^{-2} - (x+y+1)^{-2}\right] dy\, dx$$

$\int (x+a)^n\, dx = \frac{(x+a)^{n+1}}{n+1}$

$$= \int_{x=0}^{1} \left(-\tfrac{1}{2}\right)\left[\tfrac{1}{4}y - \frac{(y+x+1)^{-1}}{(-1)}\right]_{y=0}^{1-x} dx$$

$$= \int_{x=0}^{1} \left(-\tfrac{1}{2}\right)\left[\left(\tfrac{1}{4}(1-x) - \frac{(2)^{-1}}{(-1)}\right) - \left(0 - \frac{(x+1)^{-1}}{(-1)}\right)\right] dx$$

$$= \left(-\tfrac{1}{2}\right)\left[\tfrac{1}{4}\left(x - \tfrac{x^2}{2}\right) + \tfrac{1}{2}x - \log|x+1|\right]_{x=0}^{1}$$

$\int \frac{1}{x+1}\, dx = \log(x+1)$

$$= \left(-\tfrac{1}{2}\right)\left[\left(\tfrac{1}{4}\left(1-\tfrac{1}{2}\right) + \tfrac{1}{2} - \log 2\right) - 0\right]$$

$\log 1 = 0$

Q→ Evaluate $\iiint_T x^2 y\, dx\, dy\, dz$, $T: x^2+y^2 \le 1$, $0 \le z \le 1$

Sol→

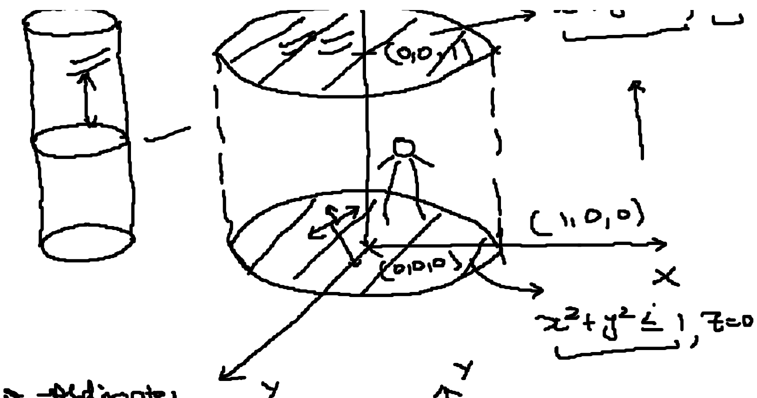
$x^2+y^2 \le 1$, $z=1$

Sol ->

Here $x^2+y^2 \le 1$ is a circular plate with centre at $(0,0)$ & radius $= 1$

Limits are

$0 \le z \le 1$

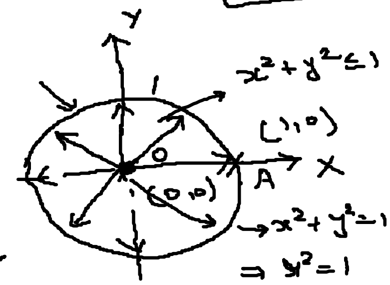


changing to polar co-ordinates (r,θ)

$\therefore \quad x = r\cos\theta, \; y = r\sin\theta$

$0 \le \theta \le 2\pi$

$0 \le r \le 1$



$$\therefore \; I = \int_{z=0}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\cos^2\theta \; r\sin\theta \; r\,dr\,d\theta\,dz$$

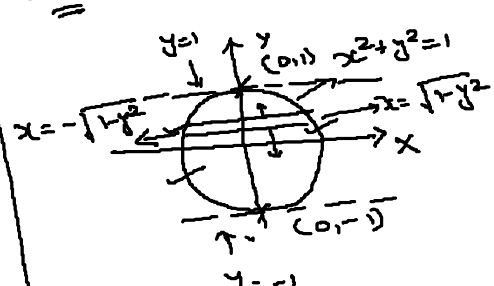
$$= -\frac{1}{5}\int_{z=0}^{1}\int_{\theta=0}^{2\pi} (\cos\theta)^2(-\sin\theta)\,d\theta\,dz$$

$$= -\frac{1}{5}\int_{z=0}^{1} \left[\frac{(\cos\theta)^3}{3}\right]_{\theta=0}^{2\pi} dz = 0$$

$\cos 2\pi = 1$
$\cos 0 = 1$

Volume: volume of any solid

$V = \iiint dx\,dy\,dz$



Q. Calculate the volume of cylinder which is in the previous question

Sol. volume $= \iiint_{V \to \text{cylinder}} dx\,dy\,dz$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{1} r\,dz\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\cdot 1\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\left(\frac{1}{2}\right)d\theta = \frac{1}{2}\times 2\pi = \pi$$

$-1 \le y \le 1$

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$ | $x^2+y^2=1$, $x = \pm\sqrt{1-y^2}$

vertical strip

$-1 \le x \le 1$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$\pi(1$

$\pi r^2 h = \pi(1)(1) = \pi$

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Change of variable: Let for solving $I = \iiint_V f(x,y,z)\,dx\,dy\,dz$

we are changing the variables from (x,y,z) to (u,v,w), using

$$\left[x = \phi(u,v,w),\ y = \psi(u,v,w),\ z = \xi(u,v,w)\right]$$

$$\therefore I = \iiint_{V'} f\left[\phi(u,v,w), \psi(u,v,w), \xi(u,v,w)\right] |J|\, du\,dv\,dw$$

where J = Jacobian of (x,y,z) w.r.t. (u,v,w) = $J\left(\frac{x,y,z}{u,v,w}\right)$

$$= \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

* **Special Case 1**: Changing from Cartesian Co-ordinates (x,y,z) to cylindrical co-ordinates (r,θ,z).

$x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$\therefore\ I = \iiint_V f(x,y,z)\,dx\,dy\,dz$ changes to

$$I = \iiint_{V'} f(r\cos\theta, r\sin\theta, z)\ |J|\ dr\,d\theta\,dz$$

where $J = J\left(\frac{x,y,z}{r,\theta,z}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= 1\left[r\cos^2\theta + r\sin^2\theta\right] = r$ $\quad r \geq 0,\ |r| = r$

$$\Rightarrow I = \iiint_{V'} f(r\cos\theta, r\sin\theta, z)\ r\,dr\,d\theta\,dz$$

Special Case 2: Changing from Cartesian Co-ordinates (x,y,z) to spherical Co-ordinates (r,θ,ϕ)

$x = r\sin\phi\cos\theta,\quad y = r\sin\phi\sin\theta,\ z = r\cos\phi$

$\therefore\ I = \iiint_V f(x,y,z)\,dx\,dy\,dz$ changes to

$$\Rightarrow I = \iiint_{V'} f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\ |J|\ dr\,d\theta\,d\phi$$

where $J = J\left(\frac{x,y,z}{r,\phi,\theta}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \partial z & \partial z & \partial z \end{vmatrix}$

$$\frac{\partial r}{} \quad \frac{\partial \phi}{} \quad \frac{\partial \theta}{}$$

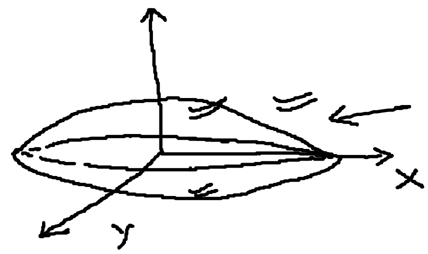
$$= r^2 \sin\phi$$

$$\Rightarrow I = \iiint_{V'} f(r\sin\phi\cos\theta,\ r\sin\phi\sin\theta,\ r\cos\phi)\, r^2\sin\phi\, dr\, d\theta\, d\phi$$

Q → $I = \iiint_{T} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\, dx\,dy\,dz$, boundary of T is $\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Sol → Here $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is an ellipsoid

Changing the variables from (x, y, z) (u, v, w).



$\Rightarrow \frac{x}{a} = u,\ \frac{y}{b} = v,\ \frac{z}{c} = w$

$\Rightarrow x = au,\ y = bv,\ z = cw$

$$\Rightarrow I = \iiint_{V'} \sqrt{1 - u^2 - v^2 - w^2}\, |J|\, du\,dv\,dw$$

where $J = J\left(\frac{x, y, z}{u, v, w}\right) = \frac{\partial(x, y, z)}{\partial(u, v, w)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$= abc$$

$$\Rightarrow I = \iiint_{V'} \sqrt{1 - u^2 - v^2 - w^2}\ abc\ du\,dv\,dw$$ $\because a, b, c$ are lengths

where V' is the solid bounded by $u^2 + v^2 + w^2 = 1$, it is a sphere with centre $(0, 0, 0)$ & radius = 1

Again, Changing from (u, v, w) to (r, θ, ϕ)

$\rightarrow u = r\sin\phi\cos\theta,\ v = r\sin\phi\sin\theta,\ w = r\cos\phi$

$$\Rightarrow I = abc \iiint_{V'} \sqrt{1 - r^2\sin^2\phi\cos^2\theta - r^2\sin^2\phi\sin^2\theta - r^2\cos^2\phi}\ |J|\, dr\,d\theta\,d\phi$$

where $J = J\left(\frac{u, v, w}{r, \theta, \phi}\right) = r^2 \sin\phi$

$$\Rightarrow I = abc \iiint_{V'} \sqrt{1 - r^2\sin^2\phi\,(\cos^2\theta + \sin^2\theta) - r^2\cos^2\phi}\ r^2\sin\phi\, dr\,d\theta\,d\phi$$

$$= abc \iiint_{V'} \sqrt{1 - r^2\sin^2\phi - r^2\cos^2\phi}\ r^2\sin\phi\, dr\, d\theta\, d\phi$$

$$= abc \iiint_{V'} \sqrt{1-r^2}\ r^2 \sin\phi\, dr\, d\theta\, d\phi$$

where limits of V' are $0 \le r \le 1$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$

$x^2+y^2+z^2 = a^2$

Limits are

$0 \le r \le a$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

$$\Rightarrow I = abc \int_{r=0}^{1}\int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} \sqrt{1-r^2}\ r^2 \sin\phi\, d\theta\, d\phi\, dr$$

$$= abc \int_{r=0}^{1}\int_{\phi=0}^{\pi} \sqrt{1-r^2}\ r^2 \sin\phi\ (2\pi)\, d\phi\, dr$$

$$= 2\pi abc \int_{r=0}^{1} \sqrt{1-r^2}\ r^2 \left(-\cos\phi\right)_{\phi=0}^{\pi} dr$$

$\cos\pi = -1$

$\cos 0 = 1$

$$= 2\pi abc \int_{r=0}^{1} \sqrt{1-r^2}\ r^2 \left[-\cos\pi - (-\cos 0)\right] dr$$

$$= 2\pi abc \int_{r=0}^{1} \sqrt{1-r^2}\ r^2\ (2)\, dr$$

$r = \sin t \Rightarrow dr = \cos t\, dt$

$r=0 \Rightarrow t=0$

$\&\ r=1 \Rightarrow t = \pi/2$

$$= 4\pi abc \int_{t=0}^{\pi/2} \cos t\ \sin^2 t \cdot \cos t\, dt$$

$$= \pi abc \int_{t=0}^{\pi/2} \left(2\sin t \cos t\right)^2 dt$$

$$= \pi abc \int_{t=0}^{\pi/2} \sin^2 2t\, dt = \pi abc \int_{t=0}^{\pi/2} \frac{1-\cos 4t}{2}\, dt$$

$$= \frac{\pi abc}{2}\left[t - \frac{\sin 4t}{4}\right]_{t=0}^{\pi/2}$$

$$= \frac{\pi abc}{2}\left[\frac{\pi}{2}\right] = \frac{\pi^2 abc}{4}$$