30/12/20
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8:52 AM

Unit-VI Fourier Series

$\alpha + 2l - \alpha$

Let $f(x)$ be a periodic function with period $2l$ and is defined over the interval $[\alpha, \alpha+2l]$ then fourier series expansion of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where a_0, a_n, b_n are fourier Coefficients

$$a_0 = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\,dx, \quad a_n = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

These are called as Euler's formulae.

Periodic function: A function $f(x)$ is s.t.b. periodic if $f(x) = f(x+T)$ $\forall x \in D_f$, then period of function is T.

e.g. $f(x) = \sin x$

$\because \sin(x) = \sin(x + 2\pi)$ $\forall x \in$ Domain of $\sin x$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\sin(2\pi + x) = \sin x$

Q→ $f(x) = 2x - x^2$ in $[0, 3]$. Find fourier series expansion

Sol→ Here $f(x) = 2x - x^2$ and $2l = 3 \Rightarrow l = 3/2$

We know fourier series of $f(x)$ in $[\alpha, \alpha+2l]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{3}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{3}\right) \quad \text{(1)}$$

Now $a_0 = \frac{1}{(3/2)}\int_0^3 (2x - x^2)\,dx = \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3$

$= \frac{2}{3}\left[\left(9 - \frac{27}{3}\right) - 0\right] = 0$

ILATE

Now $a_n = \frac{1}{3/2}\int_0^3 \underbrace{(2x - x^2)}_{u} \underbrace{\cos\left(\frac{2n\pi x}{3}\right)}_{v} dx$

$$= \frac{2}{3}\left[(2x - x^2)\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)} - (2 - 2x)\left[\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right] + (-2)\left[-\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right] - 0 \right]_0^3$$

$\sin(2n\pi)$

$$+ \underbrace{(-2)}_{u} \left[- \frac{\sin\left(\frac{2n\pi x}{3}\right)}{\underbrace{\left(\frac{2n\pi}{3}\right)^3}_{v}} \right] - 0 \Bigg]_0^3$$

$$= \frac{2}{3} \left[\left(0 + (-4) \frac{\cos(2n\pi)}{\left(\frac{2n\pi}{3}\right)^2} + 0 \right) - \left(0 + 2 \frac{1}{\left(\frac{2n\pi}{3}\right)^2} 0 \right) \right]$$

$\sin(2n\pi) = 0$, $n \in I$

$\cos(n\pi) = (-1)^n$

$\cos(2n\pi) = (-1)^{2n} = 1$

$$= \frac{2}{3} \left[\frac{-4 \times 9}{4n^2\pi^2} - \frac{2 \times 9}{4n^2\pi^2} \right]$$

$$= \frac{2}{3} \frac{-54 \cdot 9}{4n^2\pi^2} = -\frac{9}{n^2\pi^2}$$

$$\text{and } b_n = \frac{1}{3/2} \int_0^3 \underbrace{(2x - x^2)}_{u} \underbrace{\sin\left(\frac{2n\pi x}{3}\right)}_{v} dx$$

$$= \frac{2}{3} \left[(2x - x^2) \left[- \frac{\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)} \right] - (2 - 2x) \left[- \frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2} \right] + (-2) \left[\frac{\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3} \right] \right]_0^3$$

$$= \frac{2}{3} \left[\left((-3) \left(- \frac{\cos(2n\pi)}{\frac{2n\pi}{3}} \right) + 0 - 2 \frac{\cos(2n\pi)}{\left(\frac{2n\pi}{3}\right)^3} \right) - \left(0 + 0 - 2 \times \frac{1}{\left(\frac{2n\pi}{3}\right)^3} \right) \right]$$

$$= \frac{2}{3} \left[\frac{3 \times 3}{2n\pi} \times 1 - \frac{2 \times 9}{4n^2\pi^2} \times 1 + \frac{2 \times 27}{8n^3\pi^3} \right]$$

Hence F.S is

$$(2x - x^2) = 0 + \sum_{n=1}^{\infty} \left(\frac{-9}{n^2\pi^2} \right) \cos\left(\frac{2n\pi x}{3}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{3}\right)$$

Fourier series expansion of even & odd functions:

Let $f(x)$ be a periodic function with period $2l$ & is defined over $[-l, l]$, then F.S. of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\,dx$, $\quad a_n = \frac{1}{l} \int_{-l}^{l} \underbrace{f(x)}_{\text{odd}} \cos\left(\frac{n\pi x}{l}\right) dx$

$$b_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)}_{\text{even}}\, \underbrace{\sin\left(\frac{n\pi x}{l}\right)}_{\text{odd}} dx$$

even × odd = odd

Case 1 : If $f(x)$ is an even function i.e. $f(-x) = f(x)$

$\therefore b_n = 0$

Case 2 : If $f(x)$ is an odd function i.e.
$f(-x) = -f(x)$

$\therefore \quad a_0 = 0 \ \& \ a_n = 0$

$\int_{-a}^{a} f(x)dx = 0$ if $f(x)$ is odd function

Q→ Find fourier series of $f(x) = x^3$ in $[-\pi, \pi]$

Sol→ $f(-x) = (-x)^3 = -x^3 = -f(x) \Rightarrow f(-x) = -f(x)$

$\therefore f(x)$ is an odd function

Hence F.S. of $f(x)$ in $[-\pi, \pi]$ is $\quad l = \pi \quad \because 2l = 2\pi$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\pi}\right)$$

Here $a_0 = 0$ & $a_n = 0 \quad \because f(x)$ is odd function

$$\text{and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^3}_{u} \underbrace{\sin\left(\frac{n\pi x}{\pi}\right)}_{v} dx$$

$$= \frac{1}{\pi}\left[x^3 \cdot \left[\frac{-\cos nx}{n}\right] - (3x^2)\left[-\frac{\sin nx}{n^2}\right] + 6x\left[\frac{\cos nx}{n^3}\right] - 6\left[\frac{\sin nx}{n^4}\right] + 0\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{-\pi^3 \cos(n\pi)}{n} + \frac{6\pi \cos n\pi}{n^3}\right) - \left(-\frac{(-\pi)^3 \cos(-n\pi)}{n} + \frac{6(-\pi)\cos(-n\pi)}{n^3}\right)\right]$$

$\sin(n\pi) = 0$, $n \in I$

$\sin(-\theta) = -\sin\theta$

$$= \frac{1}{\pi}\left[\frac{-\pi^3 \cos n\pi}{n} + \frac{6\pi \cos n\pi}{n^3} - \frac{\pi^3 \cos n\pi}{n} + \frac{6\pi \cos n\pi}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2\pi^3 \cos n\pi}{n} + \frac{12\pi \cos n\pi}{n^3}\right]$$

$$= \frac{-2\pi^2 (-1)^n}{n} + \frac{6(-1)^n}{n^3}$$

$\cos(-\theta) = \cos\theta$

$\cos(n\pi) = (-1)^n$

$$\text{Hence } \underbrace{f(x)}_{x^3} = \sum_{n=1}^{\infty}\left(-\frac{2\pi^2}{n} + \frac{6}{n^3}\right)(-1)^n \sin(nx)$$

in interval $[-\pi, \pi]$

$\underline{\underline{x}}$

31/12/20 Q. Find Fourier series of $f(x) = 9 - x^2$ in $-3 \le x \le 3$ $\quad \Big| \; 3-(-3) = 6$

Sol → $f(-x) = 9 - (-x)^2 = 9 - x^2 = f(x) \Rightarrow f(-x) = f(x)$

∴ $f(x)$ is an even function.

F.S. of $f(x) = 9 - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right)$

Here $f(x)$ is an even function & we are expanding Fourier series in $[-3, 3] \Rightarrow b_n = 0$ $\quad \Big| \; 2l = 6 \Rightarrow l = 3$

Now $a_0 = \frac{1}{3}\int_{-3}^{3}(9 - x^2)\,dx = \frac{2}{3}\int_{0}^{3}(9-x^2)\,dx$ $\quad \Big| \; \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if $f(x)$ is even

$= \frac{2}{3}\left[9x - \frac{x^3}{3}\right]_0^3 = \frac{2}{3}[27 - 9]$

$= \frac{2}{\cancel{3}} \times \cancel{18}\, 6 = \underline{\underline{12}}$

and $a_n = \frac{1}{3}\int_{-3}^{3} \underbrace{\underbrace{(9 - x^2)}_{\text{Even}} \underbrace{\cos\left(\frac{n\pi x}{3}\right)}_{\text{Even}}}_{\text{Even}} dx = \frac{2}{3}\int_{0}^{3} \underbrace{(9-x^2)}_{u} \underbrace{\cos\left(\frac{n\pi x}{3}\right)}_{v} dx$

$= \frac{2}{3}\left[(9-x^2)\left(\frac{\sin\left(\frac{n\pi x}{3}\right)}{\left(\frac{n\pi}{3}\right)}\right) - (-2x)\left(\frac{-\cos\left(\frac{n\pi x}{3}\right)}{\left(\frac{n\pi}{3}\right)^2}\right) + (-2)\left(\frac{-\sin\left(\frac{n\pi x}{3}\right)}{\left(\frac{n\pi}{3}\right)^3}\right)\right]_{x=0}^{3}$

$= \frac{2}{3}\left[\left(-6\,\frac{\cos(n\pi)}{\left(\frac{n\pi}{3}\right)^2}\right) - 0\right]$

$= \frac{2}{\cancel{3}}\left[-\frac{\cancel{6}^{\,2} \times 9}{n^2\pi^2}(-1)^n\right] = \frac{-36\,(-1)^n}{n^2\pi^2}$ $\quad \Big| \; \sin(n\pi) = 0,\ n \in I$

Now Fourier series of $f(x)$ will be

$$f(x) = 9 - x^2 = 6 + \sum_{n=1}^{\infty} \frac{-36\,(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{3}\right)$$

Q. Find F.S. of $f(x) = x - x^2$ in $[-\pi, \pi]$

Sol. Here $f(x) = x - x^2$, $2l = 2\pi \Rightarrow l = \pi$

∴ F.S. of $f(x)$ will be

$f(x) = x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\cancel{\pi}x}{\cancel{\pi}}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\cancel{\pi}x}{\cancel{\pi}}\right)$

$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$ — (1)

$\Big| \; f(-x) = -x - (-x)^2 = -x - x^2 = -[x + x^2]$
$\rightarrow f(-x) \neq f(x)$
$\rightarrow f(-x) \neq -f(x)$

Now $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\,dx - \int_{-\pi}^{\pi} x^2\,dx\right]$

$\Big| \; g(x) = x$, $h(x) = x^2$
$g(-x) = -x = -g(x)$
$h(-x) = (-x)^2$

$= \frac{1}{\pi}\left[0 - 2\int_{0}^{\pi} x^2\,dx\right]$

$$= -\frac{2}{\pi} \times \frac{\pi^3}{3} = -\frac{\pi^2}{3}$$

$= x^2$
$= f(x)$

Now $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \cos\left(\frac{n\pi x}{\pi}\right) dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x \cos(nx)}_{\underbrace{\text{Odd} \times \text{Even}}_{\text{Odd}}} dx - \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2 \cos(nx)}_{\underbrace{\text{Even} \quad \text{Even}}_{\text{Even}}} dx$$

$$= 0 - \frac{2}{\pi}\int_0^{\pi} x^2 \cos(nx)\, dx$$

$$= -\frac{2}{\pi}\left[x^2 \frac{\sin(nx)}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\left[\frac{2\pi \cos(n\pi)}{n^2} - 0\right] = \frac{-4(-1)^n}{n^2}$$

$\sin n\pi = 0$

Algebraic

ILATE $\longrightarrow$

Now $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \sin\left(\frac{n\pi x}{\pi}\right) dx$

$\sin(-n\pi)$
$= -\sin(n\pi)$
$= 0$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{-\cos(nx)}{n}\right) - (1-2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos(nx)}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(-(\pi - \pi^2)\frac{\cos(n\pi)}{n} - 2\frac{\cos(n\pi)}{n^3}\right) - \left(\left(-\pi - (-\pi)^2\right)\left(-\frac{\cos(-n\pi)}{n}\right) - 2\frac{\cos(-n\pi)}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-\pi + \pi^2)(-1)^n}{n} - \cancel{\frac{2(-1)^n}{n^3}} + \frac{(-\pi - \pi^2)(-1)^n}{n} + \cancel{\frac{2(-1)^n}{n^3}}\right]$$

$\cos(-n\pi) = \cos(n\pi)$
$= (-1)^n$

$$= \frac{1}{\pi}\ \frac{(-\pi + \cancel{\pi^2} - \pi - \cancel{\pi^2})(-1)^n}{n}$$

$$= \frac{1}{\cancel{\pi}}(-2\cancel{\pi})\frac{(-1)^n}{n} = \frac{-2(-1)^n}{n}$$

$\therefore$ F.S. is $f(x) = x - x^2 = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n^2}\cos(nx) + \sum_{n=1}^{\infty}\frac{-2(-1)^n}{n}\sin(nx)$

Q → Find F.S. of $f(x) = \begin{cases} -x, & -2 \le x < 0 \\ x^2, & 0 < x \le 2 \end{cases}$

$\# \int e^{x} \cos nx\, dx$
$\int e^{ax} \cos bx\, dx = ?$

Sol. → Here $2l = 4 \Rightarrow l = 2$

$\therefore$ Fourier series of $f(x)$ will be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

Now $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx\right]$

$= \frac{1}{2}\left[\int_{-2}^{0}(-x)\,dx + \int_{0}^{2} x^2\,dx\right]$

$= \frac{1}{2}\left[-\left(\frac{x^2}{2}\right)_{-2}^{0} + \left(\frac{x^3}{3}\right)_{0}^{2}\right] = \frac{1}{2}\left[-\left(0-\frac{4}{2}\right) + \left(\frac{8}{3}-0\right)\right]$

$= \frac{1}{2}\left[2 + \frac{8}{3}\right] = \frac{1}{2} \times \frac{14}{3} = 7/3$

and $a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$

$= \frac{1}{2}\left[\int_{-2}^{0} f(x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx\right]$

$= \frac{1}{2}\left[\int_{-2}^{0} -x\cos\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{2} x^2\cos\left(\frac{n\pi x}{2}\right)dx\right]$

$= \frac{1}{2}\left[-\left(x\left(\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right) - 1\cdot\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right)\right)_{x=-2}^{0}\right.$

$\left.+\left[x^2\left(\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right) - 2x\cdot\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right) + 2\left(-\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^3}\right)\right]_{0}^{2}\right]$

$= \frac{1}{2}\left[-\left(\frac{1}{(n\pi/2)^2} - \left(\frac{(-2)\sin(-n\pi)}{n\pi/2} + \frac{\cos(-n\pi)}{(n\pi/2)^2}\right)\right)\right.$ $\sin(-n\pi)=0$

$\left. + \left(2\times 2\,\frac{\cos(n\pi)}{(n\pi/2)^2}\right)\right]$

$= \frac{1}{2}\left[\frac{\cos(n\pi)}{(n\pi/2)^2} - \frac{1}{(n\pi/2)^2} + \frac{4\cos(n\pi)}{(n\pi/2)^2}\right]$

$= \frac{1}{2}\left[5(-1)^n - 1\right]\frac{4^2}{n^2\pi^2} = \frac{2}{n^2\pi^2}\left[5(-1)^n - 1\right]$

*(Complete at your own) *(find b_n)

04/01/21

Half-Range Series : Let $f(x)$ be a periodic function with period 2ℓ & is defined over $[0,\ell]$, then $[\alpha, \alpha+2\ell]$ $\ell = \ell - 0$ $= \alpha + 2\ell - \alpha$ $[0,\ell]$

① Half-Range Cosine series of $f(x)$ is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{\ell}\right)$

where $a_0 = \frac{2}{\ell}\int_0^{\ell} \ldots$ & $a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$ & $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

(ii) Half-Range sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

Q→ find both Half Range series for $f(x) = 1+x$, $0 \le x \le 1$

Sol→ Here $f(x) = 1+x$, $0 \le x \le 1$, $l = 1$

∴ Half-Range Cosine series of $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{1}\right)$

where $a_0 = \frac{2}{1}\int_0^1 (1+x)\,dx = 2\left[x + \frac{x^2}{2}\right]_0^1 = 2\left[1 + \frac{1}{2}\right]$
$= 2 \times 3/2 = 3$

and $a_n = \frac{2}{1}\int_0^1 (1+x)\cos\left(\frac{n\pi x}{1}\right)dx$

$$= 2\left[(1+x)\left[\frac{\sin(n\pi x)}{n\pi}\right] - 1.\left[\frac{-\cos(n\pi x)}{(n\pi)^2}\right]\right]_0^1$$

$$= 2\left[\left(0 + \frac{\cos(n\pi)}{(n\pi)^2}\right) - \left(0 + \frac{1}{(n\pi)^2}\right)\right] \qquad \begin{array}{|l} \sin n\pi = 0 \\ n \in I \end{array}$$

$$= 2\left[\frac{(-1)^n - 1}{(n\pi)^2}\right]$$

∴ Half-Range cosine series of $f(x) = 1+x = \frac{3}{2} + \sum_{n=1}^{\infty} 2\left[\frac{(-1)^n - 1}{(n\pi)^2}\right]\cos(n\pi x)$

and Half-Range sine series of $f(x) = (1+x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{1}\right)$

Now $b_n = \frac{2}{1}\int_0^1 (1+x)\sin\left(\frac{n\pi x}{1}\right)dx$

$$= 2\left[(1+x)\left(\frac{-\cos(n\pi x)}{n\pi}\right) - 1.\left(-\frac{\sin(n\pi x)}{(n\pi)^2}\right)\right]_0^1$$

$$= 2\left[\left(-2\frac{\cos(n\pi)}{n\pi} + 0\right) - \left(\frac{1.(-1)}{n\pi} + 0\right)\right]$$

$$= 2\left[\frac{1 - 2(-1)^n}{n\pi}\right]$$

∴ Half-Range sine series of $f(x) = 1+x = \sum_{n=1}^{\infty} 2\left[\frac{1 - 2(-1)^n}{n\pi}\right]\sin\left(\frac{n\pi x}{1}\right)$

M.Imp Q→ find Fourier series of $f(x) = 1+x$ in $[0,1]$

Sol: Here $f(x) = (1+x)$, $2l = 1 \Rightarrow l = 1/2$

\# Complete at your own

Complex form of Fourier Series :

Complex form of fourier series for the function $f(x)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{\frac{in\pi x}{l}}, \text{ where } i = \sqrt{-1}$$

where $C_n = \frac{1}{2l} \int_{-l}^{l} f(x)\, e^{-\frac{in\pi x}{l}}\, dx, \quad n = 0, \pm 1, \pm 2 \ldots$

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Q→ Find Complex fourier series of

$f(x) = e^{-x}, \quad -\pi < x < \pi$

Sol→ Complex fourier series of $f(x) = e^{-x}$ in $[-\pi, \pi]$ is

$$f(x) = e^{-x} = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{\pi}} = \sum_{n=-\infty}^{\infty} C_n e^{inx} \qquad \begin{aligned} 2l &= 2\pi \\ l &= \pi \end{aligned}$$

where $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-x} \cdot e^{-\frac{in\pi x}{\pi}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-x} \cdot e^{-inx} dx$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-(1+in)x} dx = \frac{1}{2\pi} \left[\frac{e^{-(1+in)x}}{-(1+in)} \right]_{-\pi}^{\pi}$$

$$= \frac{-1}{2\pi(1+in)} \left[e^{-(1+in)\pi} - e^{+(1+in)\pi} \right]$$

$$\therefore f(x) = e^{-x} = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi(1+in)} \left[e^{(1+in)\pi} - e^{-(1+in)\pi} \right] e^{inx}$$

Dirichlet's Condition : (Conditions for Fourier series)

1. $f(x)$ is periodic, single valued & finite
2. $f(x)$ has finite number discontinuities in one period
3. $f(x)$ must have finite number of maxima & minima