03/12/20
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8:57 AM

Q. find first order partial derivatives for following functions

(1) $f(x,y,z) = (xy)^{\sin z}$ at $(3,5,\pi/2)$ (2) $f(x,y,z) = e^{x/y} + e^{z/y}$ at $(1,1,1)$

Sol. $f(x,y,z) = x^{\sin z}\, y^{\sin z}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x^{\sin z}\cdot y^{\sin z}\right] = y^{\sin z}\frac{\partial}{\partial x}\left(x^{\sin z}\right)$$

$$= y^{\sin z}\,\sin z\, x^{\sin z - 1}$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(3,5,\pi/2)} = 5^1 \times 1 \times 3^0 = 5$$

$f(x,y,z)$
$\downarrow$
$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

Const

f(variable)

$\frac{d}{dx}x^n = nx^{n-1}$

Now

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x^{\sin z}\cdot y^{\sin z}\right] = x^{\sin z}\frac{\partial}{\partial y}\left(y^{\sin z}\right)$$

$$= x^{\sin z}\,\sin z\cdot y^{\sin z - 1}$$

$$\Rightarrow \left(\frac{\partial f}{\partial y}\right)_{(3,5,\pi/2)} = 3^1 \times 1 \times 5^{1-1} = 3$$

Variable

(Constant)
$\downarrow$
$\frac{d}{dx}a^x = a^x \log a$

Now $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[(xy)^{\sin z}\right] =$

$$= (xy)^{\sin z}\log(xy)\,\frac{\partial}{\partial z}(\sin z)$$

$$= (xy)^{\sin z}\log(xy)\cos z$$

$$\Rightarrow \left(\frac{\partial f}{\partial z}\right)_{(3,5,\pi/2)} = \underline{0}$$

(2) $f(x,y,z) = e^{x/y} + e^{z/y}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[e^{x/y} + e^{z/y}\right] = \frac{\partial}{\partial x}e^{x/y} + \frac{\partial}{\partial x}e^{z/y}$$

$$= e^{x/y}\cdot\frac{1}{y}\cdot 1 + 0$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(1,1,1)} = e$$

$\frac{\partial}{\partial y}\left(e^{x/y}\right) = e^{x/y}\frac{\partial}{\partial y}\left(\frac{x}{y}\right)$
$= e^{x/y}\cdot x\cdot\frac{\partial}{\partial y}\left(\frac{1}{y}\right)$
$= e^{x/y}\cdot x\cdot\left(-\frac{1}{y^2}\right)$

Now $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[e^{x/y} + e^{z/y}\right] = \frac{\partial}{\partial y}\left(e^{x/y}\right) + \frac{\partial}{\partial y}\left(e^{z/y}\right)$

$$= e^{x/y}\cdot x\left(-\frac{1}{y^2}\right) + e^{z/y}\, z\left(-\frac{1}{y^2}\right)$$

$3^{2/3} \rightarrow$

$$\Rightarrow \left(\frac{\partial f}{\partial y}\right)_{(1,1,1)} = -e - e = -2e$$

Now $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[e^{x/y} + e^{z/y}\right] = \frac{\partial}{\partial z}\left(e^{x/y}\right) + \frac{\partial}{\partial z}\left(e^{z/y}\right)$

$$= 0 + e^{z/y}\cdot\frac{1}{y}\cdot 1 = \frac{e^{z/y}}{y}$$

$$\Rightarrow \left(\frac{\partial f}{\partial z}\right)_{(1,1,1)} = e$$

$\rightarrow e^{x/y} \leftrightarrow \not\leftrightarrow e^{x-y}$
$\left(e^x\right)^{1/y}$
$e^x \cdot e^{-y}$
$= \frac{e^x}{e^y}$
$\neq e^{x/y}$

Composite Function & chain rule:

Let $\boxed{z = f(u,v)}$, $\boxed{u = g(x,y),\ v = h(x,y)}$

$z \rightarrow (u,v) \rightarrow (x,y)$

$\rightarrow z$ is Composite Function of (x,y)

$y = f(x)$

$y = x^2$ or e^x

$\boxed{z \to (u,v)}$ + $\boxed{\begin{matrix} u \to (x,y) \\ v \to (x,y) \end{matrix}}$ $\qquad$ $\overline{\text{or}} \log x$

$z \to (u,v) \to (x,y)$ $\qquad$ $z \to (u,v) \to (x,y)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \quad \to \text{chain rule}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\,\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\,\frac{\partial v}{\partial y}$$

\# $z = f(u,v,w)$, $u = g(x,y)$, $v = h(x,y)$, $w = k(x,y)$

$z \to (u,v,w) \to (x,y)$

C.F.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x} \quad \to \text{chain Rule}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial y}$$

\# $z = f(u)$, $u = g(x,y)$ $\qquad$ $\frac{dz}{dx}$ ⓨ

$z \to u \to (x,y)$

C.F.

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x} \quad \to \text{Chain Rule}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y}$$

\# $z = f(u,v)$, $u = g(t)$, $v = h(t)$

$\boxed{z \to (u,v) \to t}$

C.F

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\cdot\frac{du}{dt} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dt} \quad \to \text{chain Rule}$$

Q → $w = z\log y + y\log z + xyz$, $x = \sin t$, $y = t^2+1$, $z = \cos^{-1} t$

find $\frac{dw}{dt}$ at $t = 0$

Sol → $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$w \to (x,y,z) \to t$

C.F.

$$\Rightarrow \frac{dw}{dt} = \left[0 + 0 + yz\cdot 1\right]\cos t + \left[\frac{z}{y} + \log z\cdot 1 + xz\right]\cdot[2t] + \left[\log y + \frac{y}{z} + xy\right]\left[-\frac{1}{\sqrt{1-t^2}}\right]$$

04/12/20 $$\Rightarrow \frac{dw}{dt} = (t^2+1)\cos^{-1}t\cdot\cos t + 2t\left[\frac{\cos^{-1}t}{t^2+1} + \log\cos^{-1}t + \sin t\cdot\cos^{-1}t\right] + \left[\log(t^2+1) + \frac{t^2+1}{\cos^{-1}t} + \sin t\,(t^2+1)\right]\left[-\frac{1}{\sqrt{1-t^2}}\right]$$

$\Rightarrow \left(\frac{dw}{dt}\right) = ?$ (Calculate at your own)

$t=0$

Q → If $z = f(ax+by)$, then $b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = 0$

Sol → $z = f(ax+by)$

Take $ax+by = t$

$z \to t \to (x,y)$ C.F.

$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$\Rightarrow z = f(t)$, $t = ax+by$

Now $\frac{\partial z}{\partial x} = \frac{dz}{dt}\cdot\frac{\partial t}{\partial x} = \frac{dz}{dt}(a)$ —(i)

Now $\frac{\partial z}{\partial y} = \frac{dz}{dt}\cdot\frac{\partial t}{\partial y} = \frac{dz}{dt}(b)$ —(ii)

Now $b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = ba\cancel{\frac{dz}{dt}} - ab\cancel{\frac{dz}{dt}} = 0$

Q → $u = f(x-y, y-z, z-x)$, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Sol → $u = f(x-y, y-z, z-x)$

Take $r = x-y$, $s = y-z$, $t = z-x$

$r = (x-y)z^0$

$u \to (r,s,t) \to x,y,z$ C.F

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$

$\Rightarrow u = f(r,s,t)$, $r = x-y$, $s = y-z$, $t = z-x$

Now $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot(1) + \frac{\partial u}{\partial s}\cdot(0) + \frac{\partial u}{\partial t}\cdot(-1) = \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial t}}$ —(i)

Again $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$

$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot(-1) + \frac{\partial u}{\partial s}\cdot(1) + \frac{\partial u}{\partial t}\cdot(0) = -\cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial s}}$ —(ii)

and $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$

$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot(0) + \frac{\partial u}{\partial s}\cdot(-1) + \frac{\partial u}{\partial t}(1) = -\cancel{\frac{\partial u}{\partial s}} + \cancel{\frac{\partial u}{\partial t}}$ (iii)

Now (i)+(ii)+(iii) $\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Q → $z = y + f(u)$, $u = \frac{x}{y}$ then $u\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$

Sol → $z = y + f\left(\frac{x}{y}\right)$

Now $\frac{\partial z}{\partial x} = 0 + f^{1}\left(\frac{x}{y}\right)\cdot\frac{\partial}{\partial x}\left(\frac{x}{y}\right)$

$= f^{1}(x/y)\frac{1}{y}$

Now $\frac{\partial z}{\partial y} = 1 + f^{1}(x/y)\frac{\partial}{\partial y}(x/y)$

$= 1 + f^{1}(x/y)\cdot x\left(-\frac{1}{y^2}\right)$

Now $u\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{x}{y}\cdot\frac{1}{y}\cancel{f^{1}[x/y]}$

$+ 1 - \frac{x}{y^2}\cancel{f^{1}(x/y)}$

$= 1$

$f(x/y)$

$x/y = u$

$f(u)$, $u = x/y$

$f \to u \to x,y$ C.F

$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

$\frac{d}{dx}[f(u)]$

$= \frac{d}{du}[f(u)]\cdot\frac{\partial u}{\partial x}$

$\frac{\partial f}{\partial x} = \frac{df}{du}\cdot\frac{\partial u}{\partial x}$

$= \frac{df}{d(x/y)}$

$= f^{1}(u)$

Q → $z = \log[u^2 + v]$, $u = e^{x+y^2}$, $v = x + y^2$, $2y\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$

Sol →

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$z \to (u, v) \to (x, y)$

C.F

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{1}{u^2+v}(2u)\cdot e^{x+y^2}\cdot(1) + \frac{1}{u^2+v}(1)\cdot(1) \quad \text{--- (1)}$$

$\left\{\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right\} \to ?$

Now
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{1}{(u^2+v)}(2u)\, e^{x+y^2}\cdot(0+2y) + \frac{1}{(u^2+v)}[0+1]\cdot[2y]$$

Now
$$2y\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 2y\left[\frac{2u\cdot e^{x+y^2}+1}{u^2+v}\right] - \frac{4uy\,e^{x+y^2}+2y}{u^2+v}$$

$$= \frac{\cancel{4uy\,e^{x+y^2}} + \cancel{2y} - \cancel{4uy\,e^{x+y^2}} - \cancel{2y}}{u^2+v} = 0$$

Implicit Function: $\{f(x, y) = c\} \to$ Implicit Function

$[f \to (x, y) \to (x)]$

C.F.

Implicit → $x^2 + y^2 = 4$

OR

Explicit → $y = \sqrt{4 - x^2}$ $x = \sqrt{4 - y^2}$

$f(x, y) = c$

→ $xy = 4$

$y = 4/x$ or $x = \frac{4}{y}$

$y = f(x)$

or $z = f(x, y)$

or $w = f(x, y, z)$

$[f \to (x, y) \to (x)]$

$[f \to (x, y) \to (y)]$

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

$$\Rightarrow 0 = \frac{\partial f}{\partial x}\cdot 1 + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$

$f_x \to \frac{\partial f}{\partial x}$, $f_y \to \frac{\partial f}{\partial y}$

Q → $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, find $\frac{dy}{dx}$

Sol →
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \Rightarrow f(x, y) = c$$

where $f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ & $c = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = -\frac{\left[\frac{2x}{a^2}\right]}{\left[\frac{2y}{b^2}\right]}$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{b^2 x}{a^2 y}\right)$$

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OR $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, find $\frac{dy}{dx}$

diff w.r.t x

$$\Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-\frac{\cancel{2}x}{a^2}}{\frac{\cancel{2}y}{b^2}} = \frac{-b^2x}{a^2y}$$

Q → If $x^y + y^x = \alpha$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

Sol → $x^y + y^x - \alpha = 0$, Here $f(x,y) = x^y + y^x - \alpha$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\left[\frac{y\,x^{y-1} + y^x \log y}{x^y \log x + x\,y^{x-1}}\right]$$

{OR Solve it by using the method of 10+2 }

Q → If $x^3 + 3xy - 2y^2 + 3xz + z^2 = 0$ —(1), find $\left(\frac{\partial z}{\partial x}\right)_y = 0$ & $\left(\frac{\partial z}{\partial y}\right)_x$

Sol → diff w.r.t. x, treating y as Constant

$$\Rightarrow 3x^2 + 3y\cdot 1 - 0 + 3\left[x\left(\frac{\partial z}{\partial x}\right)_y + z\cdot 1\right] + 2z\left(\frac{\partial z}{\partial x}\right)_y = 0$$

$$\Rightarrow (3x+2z)\left(\frac{\partial z}{\partial x}\right)_y = -3x^2 - 3y - 3z$$

$$\Rightarrow \left(\frac{\partial z}{\partial x}\right)_y = \frac{-[3x^2+3y+3z]}{3x+2z}$$

Again diff (1) w.r.t. y, by treating x as Constant

$$\Rightarrow 0 + 3x\cdot 1 - 4y + 3x\left(\frac{\partial z}{\partial y}\right)_x + 2z\left(\frac{\partial z}{\partial y}\right)_x = 0$$

$$\Rightarrow \left(\frac{\partial z}{\partial y}\right)_x = \frac{4y-3x}{3x+2z}$$

H.W. Q → If $\cot^{-1}\left(\frac{x}{y}\right) + y^3 + 1 = 0$, find $\frac{dy}{dx}$ & $\frac{dx}{dy}$

Higher order partial derivatives :

$f(x,y) \to \frac{\partial f}{\partial x}(x,y)$, $\frac{\partial f}{\partial y}(x,y)$ — 1st order partial derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = r$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} = s$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = s$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = t$$

$f_{xy} \neq f_{yx}$

Q → $f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, find second derivative of $f(x,y,z)$ at $(1,1,1)$

Sol → $\frac{\partial f}{\partial x} = \frac{1}{y}\cdot 1 + 0 + z\left(-\frac{1}{x^2}\right)$

(x,y,z) $f(x,y,z)$

$$\frac{\partial f}{\partial x} = \frac{1}{y}\cdot 1 + 0 + z\left(-\frac{1}{x^2}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 0 + z\left(\frac{2}{x^3}\right)$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -1/y^2 + 0$$

$$\frac{\partial^2 f}{\partial z\,\partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = 0 - \frac{1}{x^2}\cdot 1$$

(x,y,z) $f(x,y,z)$ $\frac{\partial f}{\partial x}$ $\frac{\partial f}{\partial y}$ $\frac{\partial f}{\partial z}$

$\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y\,\partial x}$, $\frac{\partial^2 f}{\partial z\,\partial x}$

$$\therefore (f_{xx})_{(1,1,1)} = 2,\quad (f_{yx})_{(1,1,1)} = -1,\quad (f_{zx})_{(1,1,1)} = -1$$

\# Complete at your own

\# **Homogeneous function:** A function $f(x,y)$ is said to be homogeneous of degree n, if it can be expressed as $x^n\phi(y/x)$ or $y^n g(x/y)$

e.g. \# $f(x,y) = x^2+y^2 = x^2\left[1+y^2/x^2\right] = x^2\left[1+t^2\right]$, $(y/x = t)$

$= x^2 g(t) = x^2 g(y/x)$. Hence $f(x,y)$ is homogeneous with degree 2.

$f(x,y) = x^2+y^2 = y^2\left[1+\frac{x^2}{y^2}\right] = y^2\left[1+\left(\frac{x}{y}\right)^2\right]$, $\quad \frac{x}{y} = u$

$= y^2\left[1+u^2\right] = y^2 h(u) = y^2 h\left(\frac{x}{y}\right)$

Homogeneous with degree 2

\# $f(x,y,z)$ will be homogeneous with degree n, if it can be written as one of the following forms

$$x^n\phi\left(\frac{y}{x}, \frac{z}{x}\right) \text{ or } y^n g\left(\frac{x}{y}, \frac{z}{y}\right) \text{ or } z^n h\left(\frac{x}{z}, \frac{y}{z}\right)$$

\# $f(x,y)$ will be Homogeneous with degree n, if

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

e.g. $f(x,y) = x^2+y^2 \Rightarrow f(\lambda x, \lambda y) = (\lambda x)^2 + (\lambda y)^2$

$= \lambda^2\left[x^2+y^2\right]$

$= \lambda^2 f(x,y)$

\# If $f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x,y,z)$, then $f(x,y,z)$ will be homogeneous with degree n.

\# $f(x,y,z,t) \longrightarrow x^n\phi\left(\frac{y}{x}, \frac{z}{x}, \frac{t}{x}\right)$

$\rightarrow y^n g\left[\frac{x}{y}, \frac{z}{y}, \frac{t}{y}\right]$

OR

$\rightarrow z^n h\left[\frac{x}{z}, \frac{y}{z}, \frac{t}{z}\right]$

OR

$\rightarrow t^n k\left[\frac{x}{t}, \frac{y}{t}, \frac{z}{t}\right]$

$\rightarrow f(\lambda x, \lambda y, \lambda z, \lambda t) = \lambda^n f(x, y, z, t)$

Q→ Check if $u(x,y) = \frac{y^3 - x^3}{y^2 + x^2}$ is Homogeneous or not

Sol. → $u(x,y) = \frac{x^3\left[\frac{y^3}{x^3} - 1\right]}{x^2\left[y^2/x^2 + 1\right]} = x\left(\frac{t^3 - 1}{t^2 + 1}\right)$ where $t = \frac{y}{x}$ | $u(x,y) = x^n \phi\left(\frac{y}{x}\right)$, $y/x \to t$, $\to y^n h\left(\frac{x}{y}\right)$

$= x\, g(t) = x^1 g\left[y/x\right]$

Hence $u(x,y)$ is homogeneous with degree 1.

OR $u(x,y) = \frac{y^{\cancel{3}}}{\cancel{y^2}}\left[\frac{1 - x^3/y^3}{1 + x^2/y^2}\right] = y\left[\frac{1 - u^3}{1 + u^2}\right]$ $\frac{x}{y} = u$

$= y\,\phi(u) = y\,\phi\left(x/y\right) \Rightarrow$ Homogeneous of degree 1

OR $u(\lambda x, \lambda y) = \frac{\lambda^3 y^3 - \lambda^3 x^3}{\lambda^2 y^2 + \lambda^2 x^2} = \frac{\lambda^{\cancel{3}}}{\cancel{\lambda^2}}\left[\frac{y^3 - x^3}{y^2 + x^2}\right]$

$= \lambda\, u(x,y)$

$\Rightarrow u(\lambda x, \lambda y) = \lambda^1 u(x,y)$

Homogeneous of degree $= 1$

| $u(\lambda x, \lambda y) = \lambda^n u(x,y)$

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Euler's theorem: Let $f(x,y)$ be a Homogeneous function of degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$ | $f(x,y) \to H, 5$

Q→ If $u(x,y) = \log\left[\frac{\sqrt{x^2+y^2}}{x}\right]$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ | $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 5f$

Sol. → $u(\lambda x, \lambda y) = \log\left[\frac{\sqrt{(\lambda x)^2 + (\lambda y)^2}}{\lambda x}\right]$

$= \log\left[\frac{\cancel{\lambda}\sqrt{x^2+y^2}}{\cancel{\lambda} x}\right] = \log\left[\frac{\sqrt{x^2+y^2}}{x}\right]$ | $u(\lambda x, \lambda y) = \lambda^n u(x,y)$

$\Rightarrow u(\lambda x, \lambda y) = \lambda^0 u(x,y) \Rightarrow u(x,y)$ is a homogeneous function with degree zero

∴ By Euler's Thm.

$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu = 0(u) = \underline{\underline{0}}$ | Degree $=0$

Q $\rightarrow$ $u(x,y) = \sqrt{y^2-x^2}\ \sin^{-1}\left[\frac{x}{y}\right]$, then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = u$

Sol $\rightarrow$ $u(\lambda x,\lambda y) = \sqrt{\lambda^2y^2-\lambda^2x^2}\ \sin^{-1}\left[\frac{\cancel{\lambda}x}{\cancel{\lambda}y}\right] = \lambda\sqrt{y^2-x^2}\ \sin^{-1}\left[\frac{x}{y}\right]$

$\Rightarrow u(\lambda x,\lambda y) = \lambda^1 u(x,y)$ | $u(\lambda x,\lambda y) = \lambda^n u(x,y)$

$\therefore$ $u(x,y)$ is homogeneous function with degree $= n = 1$

$\therefore$ By Euler's thm. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu = u$ | $\because n=1$

Cor. $\rightarrow$ [Euler's thm.] $\rightarrow$ If $u(x,y)$ is a homogeneous function with degree n, then $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

Q $\rightarrow$ $u(x,y) = \frac{y^3-x^3}{y^2+x^2}$, then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = u$ & $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = 0$

Sol $\rightarrow$ $u(\lambda x,\lambda y) = \frac{\lambda^3y^3-\lambda^3x^3}{\lambda^2y^2+\lambda^2x^2} = \frac{\lambda^3}{\lambda^2}\left[\frac{y^3-x^3}{y^2+x^2}\right]$

$\Rightarrow u(\lambda x,\lambda y) = \lambda^1 u(x,y) \Rightarrow u$ is a homogeneous with degree $= n = 1$

$\therefore$ By Euler's thm. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu = 1.u = \underline{\underline{u}}$ | $\because n=1$

& $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 1.(1-1)u = \underline{\underline{0}}$

Q $\rightarrow$ $u(x,y) = \sin^{-1}\left[\frac{x^2+y^2}{x+y}\right]$, then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \tan u$

Sol. $\rightarrow$ $w = \sin u = \frac{x^2+y^2}{x+y}$

Now $w(\lambda x,\lambda y) = \frac{\lambda^2x^2+\lambda^2y^2}{\lambda x+\lambda y}$

$= \frac{\lambda^{\cancel{2}}}{\cancel{\lambda}}\left[\frac{x^2+y^2}{x+y}\right] = \lambda w$

$\Rightarrow w(\lambda x,\lambda y) = \lambda^1 w$

$u(\lambda x,\lambda y) = \sin^{-1}\left[\frac{\lambda^2x^2+\lambda^2y^2}{\lambda x+\lambda y}\right]$

$= \sin^{-1}\left[\frac{\lambda^{\cancel{2}}}{\cancel{\lambda}}\left(\frac{x^2+y^2}{x+y}\right)\right]$

$= \sin^{-1}\left[\lambda\left(\frac{x^2+y^2}{x+y}\right)\right]$

$u(\lambda x,\lambda y) = \lambda^n u(x,y)$

$\Rightarrow w$ is a homogeneous function with degree $= n = 1$

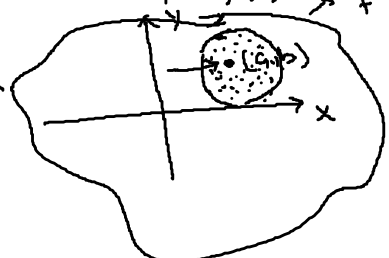
$\therefore$ By Euler's thm. $x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y} = nw = w$ | $\because n=1$

$\Rightarrow x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u) = \sin u$

$\Rightarrow x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y} = \sin u$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u} = \tan u$$

Maxima & Minima: (#) **Relative Maxima & Minima:** Let $f(x,y)$ be a function of x,y, then $(a,b) \in D_f$ is said to be point of relative maxima (relative minima) if $f(a,b) \geq f(x,y)$ $\forall$ $(x,y) \in$ some nbd of (a,b) ($f(a,b) \leq f(x,y)$ $\forall$ $(x,y) \in$ some nbd. of (a,b))



Method to find relative maxima & minima

Let $f(x,y)$ be the function

(i) find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

(ii) Put $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$, solve these equations to get the stationary points as $(a_1, b_1), (a_2, b_2) \cdots$

(iii) find $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

(iv) At (a_1, b_1), find the value of $rt - s^2$ & r both.

If $\boxed{rt - s^2 > 0}$ & $r > 0$, then (a_1, b_1) is of relative minima
If $\boxed{rt - s^2 > 0}$ & $r < 0$, then (a_1, b_1) is of relative maxima
If $rt - s^2 < 0$, then (a_1, b_1) is a saddle point
If $rt - s^2 = 0$, then test fails to conclude anything about the point (a_1, b_1) and we need further investigation.

10/12/20

Q. Find relative maxima & minima for $f(x,y) = xy + \frac{9}{x} + \frac{3}{y}$

Sol. $f(x,y) = xy + \frac{9}{x} + \frac{3}{y}$

$\rightarrow \frac{\partial f}{\partial x} = y.1 + 9\left(-\frac{1}{x^2}\right)$ & $\frac{\partial f}{\partial y} = x.1 + 0 + 3\left(-\frac{1}{y^2}\right)$

Take $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$\Rightarrow y - \frac{9}{x^2} = 0$ — (i) & $x - \frac{3}{y^2} = 0$ — (ii)

From (i) $y = \frac{9}{x^2}$, Put in (ii)

$\therefore$ (ii) $\Rightarrow x - \frac{3}{\left(\frac{9}{x^2}\right)^2} = 0 \Rightarrow x - \frac{\cancel{3}x^4}{\cancel{81}_{27}} = 0 \Rightarrow 27x - x^4 = 0$

$\Rightarrow x[3^3 - x^3] = 0 \Rightarrow x(3-x)[3^2 + x^2 + 3x] = 0$

$\Rightarrow x = 0, 3$

$\therefore$ If $x = 0 \Rightarrow y \rightarrow \infty$, reject the point

and if $x = 3 \Rightarrow y = \frac{\cancel{9}}{\cancel{3^2}} = 1 \Rightarrow$ Stationary point is $(3,1)$

$x^2 + 3x + 9 = 0$

$x = \frac{-3 \pm \sqrt{9 - 4 \times 1 \times 9}}{2 \times 1}$

$= \frac{-3 \pm \sqrt{-27}}{2}$

$\Rightarrow x = \frac{-3 \pm 3\sqrt{3}\,i}{2}$

These are complex roots, so reject these.

$-9x^{-2}$

$(-2)x^{-3}$

Now $r = \frac{\partial^2 f}{\partial x^2} = (-9)\left(\frac{-2}{x^3}\right) = \frac{18}{x^3}$

, $s = \frac{\partial^2 f}{\partial x \partial y} = 1$, $t = \frac{\partial^2 f}{\partial y^2} = (-3)\left(\frac{-2}{y^3}\right) = \frac{6}{y^3}$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 1, \quad t = \frac{\partial^2 f}{\partial y^2} = (-3)\left(\frac{-2}{y^3}\right) = \frac{6}{y^3}$$ there.

Now, at $(3,1)$, $r = \frac{18}{3^3} = \frac{18}{27} = \frac{2}{3}$, $s = 1$, $t = \frac{6}{1^3} = 6$

Now, $rt - s^2 = \left(\frac{2}{3}\right)(6) - (1)^2 = 3 > 0$, $r = 2/3 > 0$

$\therefore$ $(3,1)$ is the point of relative minima

$\therefore$ Min. Value $= f(3,1) = 3 \times 1 + \frac{9}{3} + \frac{3}{1}$
$= 3 + 3 + 3 = 9$

$xy + \frac{9}{x} + \frac{3}{y}$

Q → $f(x,y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$, find relative maxima & minima

Sol → $\frac{\partial f}{\partial x} = 2x + y.1 + 0 - \frac{1}{x^2} = 0 \Rightarrow 2x^3 + x^2 y - 1 = 0$ — (i)

& $\frac{\partial f}{\partial y} = x.1 + 2y - \frac{1}{y^2} = 0 \Rightarrow xy^2 + 2y^3 - 1 = 0$ (ii)

y^{-2}

(i)-(ii) $\Rightarrow (2x^3 - 2y^3) + (x^2 y - xy^2) = 0$

(i) $(-2x^3)$

(i) $(2y^3)$

$\Rightarrow 2[x-y][x^2+y^2+xy] + xy[x-y] = 0$

$\Rightarrow (x-y)[2x^2 + 2y^2 + 2xy + xy] = 0$

$\Rightarrow (x-y)[2x^2 + 2y^2 + 3xy] = 0$

If $x - y = 0 \Rightarrow x = y$

Put in (i) $\Rightarrow 2x^3 + x^2 \cdot x - 1 = 0$

$\Rightarrow 3x^3 = 1$

$\Rightarrow x^3 = 1/3 \Rightarrow x = (1/3)^{1/3} = \frac{1}{3^{1/3}}$

$\therefore y = x = \frac{1}{3^{1/3}}$

$\therefore$ Stationary point is $\left(\frac{1}{3^{1/3}}, \frac{1}{3^{1/3}}\right)$

and if $2x^2 + 2y^2 + 3xy = 0$

$\Rightarrow 2x^2 + (3y)x + (2y^2) = 0$

$\Rightarrow x = \frac{-3y \pm \sqrt{(-3y)^2 - 4 \times 2 \times 2y^2}}{2 \times 2}$

$= \frac{-3y \pm \sqrt{9y^2 - 16y^2}}{4}$

$\Rightarrow x = \frac{-3y \pm \sqrt{7}\, y\, i}{4}$

Reject these.

Now $r = \frac{\partial^2 f}{\partial x^2} = 2 + \frac{2}{x^3}$

$s = \frac{\partial^2 f}{\partial x \partial y} = 1$, $t = \frac{\partial^2 f}{\partial y^2} = 2 + \frac{2}{y^3}$

$\therefore$ At $\left(\frac{1}{3^{1/3}}, \frac{1}{3^{1/3}}\right)$, $r = 2 + \frac{2}{1/3} = 8$, $s = 1$, $t = 2 + \frac{2}{1/3} = 8$

Now $rt - s^2 = 8 \times 8 - (1)^2 = 63 > 0$, $r = 8 > 0$

$\therefore$ $\left(\frac{1}{3^{1/3}}, \frac{1}{3^{1/3}}\right)$ is the point of relative minima

$x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$

$\Rightarrow$ Min. value $= f\left(\frac{1}{3^{1/3}}, \frac{1}{3^{1/3}}\right) = \frac{1}{3^{2/3}} + \frac{1}{3^{2/3}} + \frac{1}{3^{2/3}}$
$+ \frac{1}{1/3^{1/3}} + \frac{1}{1/3^{1/3}}$

$= \frac{3}{3^{2/3}} + 3^{1/3} + 3^{1/3}$

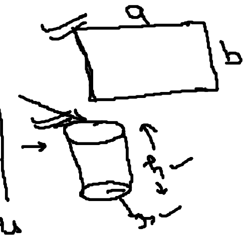
$= 3^{1/3} + 3^{1/3} + 3^{1/3} = 3 \cdot 3^{1/3} = 3^{4/3}$

Lagrange's method of undetermined multiplier:

Let a function $f(x,y)$ is to be maximized or minimized under a condition $\phi(x,y) = c$, so construct an auxiliary function

$\rightarrow F[x,y] = f[x,y] + \lambda [\phi(x,y) - c] \leftarrow$ $\hookrightarrow \phi(x,y) - c = 0$

$\Rightarrow$ Solve $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$ with $\phi[x,y] = c$, and get the stationary points (a_1, b_1), (a_2, b_2) - - -



11/1/9/20 Q→ Divide a number into three parts such that product of first with square of second & cube of third is maximum.

Sol → Let the number be c, such that, after dividing c into three parts, x be its first part, y be 2nd part & z be the third part

$\therefore \quad x + y + z = c \quad$ —(i)

Now Product $= f(x,y,z) = xy^2z^3$

24
1, 2, 21 / 2, 3, 19, 3, 4, 17
$3/2, 1/2, 22$
$(1)(2)^2(21)^3$
$2(3)^2(19)^3$
$3(4)^2(17)^3$

$\therefore$ By Lagrange method, auxiliary function is

$F[x,y,z] = xy^2z^3 + \lambda[x+y+z-c]$

$\Rightarrow \frac{\partial F}{\partial x} = y^2z^3 \cdot 1 + \lambda[1] = 0 \quad$ —(ii) $\Rightarrow \lambda = -y^2z^3$

$\frac{\partial F}{\partial y} = 2xyz^3 + \lambda[1] = 0 \quad$ —(iii) $\Rightarrow \lambda = -2xyz^3$

$\frac{\partial F}{\partial z} = 3xy^2z^2 + \lambda[1] = 0 \quad$ —(iv) $\Rightarrow \lambda = -3xy^2z^2$

$\Rightarrow \quad +y^2z^3 = +2xyz^3 = +3xy^2z^2$

Consider $y^{\cancel{2}}\cancel{z^3} = 2x\cancel{y}\cancel{z^3} \Rightarrow \boxed{y = 2x}$

Again, $2\cancel{x}\cancel{y}z^{\cancel{3}} = 3\cancel{x}y^{\cancel{2}}\cancel{z^2} \Rightarrow \boxed{2z = 3y}$

Now (i) $\Rightarrow \quad x + y + z = c \Rightarrow \frac{y}{2} + y + \frac{3}{2}y = c \Rightarrow 3y = c$

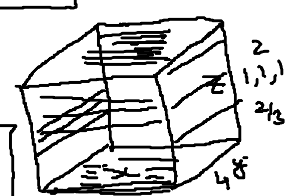
$\Rightarrow \boxed{y = c/3}$, $\boxed{x = y/2 = c/6}$ $\boxed{z = \frac{3}{2}y = \frac{c}{2}}$

$\boxed{xy^2z^3}$

$\boxed{4, 8, 12}$

Q→ A rectangular box without top, is to have a given volume. How should the box be made, so as to use least material

$\left[24\right]$ [unit]^3



Sol → Let volume of the box $= V$.

let x be the length, y be the width & z be the height of rectangular box $\Rightarrow \quad xyz = V \quad$ —(i)

→ 6   3
→ 8   2
→ 12   3
→ 4   3/2
→ 24

Now, surface area $= f[x,y,z] = 2yz + 2xz + xy$

By Lagrange's Method, auxiliary function is

$F[x,y,z] = xy + 2yz + 2zx + \lambda[xyz - V]$

$\Rightarrow \frac{\partial F}{\partial x} = y + 0 + 2z + \lambda[yz] = 0 \quad$ —(ii) $\Rightarrow \lambda = -\frac{(y+2z)}{yz}$

$\frac{\partial F}{\partial y} = x + 2z + \lambda[xz] = 0 \quad$ —(iii) $\Rightarrow \lambda = -\frac{(x+2z)}{xz}$

$\frac{\partial F}{\partial z} = \ldots 2x + \lambda[xy] = 0 \quad$ —(iv) $\Rightarrow \lambda = -(2x+2y)$

$$\frac{\partial F}{\partial z} = 2y + 2x + \lambda[xy] = 0 \quad \text{—(iv)} \Rightarrow \lambda = -\frac{(2x+2y)}{xy}$$

$$\Rightarrow \; +\frac{(y+2z)}{yz} = +\frac{(x+2z)}{xz} = +\frac{(2x+2y)}{xy}$$

Consider $\frac{y+2z}{y\cancel{z}} = \frac{x+2z}{x\cancel{z}} \Rightarrow \cancel{xy} + 2zx = \cancel{xy} + 2zy$

$\Rightarrow \cancel{2z}x = \cancel{2z}y \Rightarrow \boxed{y = x}$

again $\frac{x+2z}{\cancel{x}z} = \frac{2x+2y}{\cancel{x}y} \Rightarrow xy + 2\cancel{zy} = 2xz + 2\cancel{yz}$

$\cancel{x}y = 2\cancel{x}z \Rightarrow \boxed{y = 2z}$

Now (i) $\Rightarrow xyz = V \Rightarrow y \cdot y \cdot \frac{y}{2} = V \Rightarrow y^3 = 2V \Rightarrow \boxed{y = (2V)^{1/3}}$

$\Rightarrow \boxed{x = y = (2V)^{1/3}} \Rightarrow \boxed{z = y/2 = \frac{(2V)^{1/3}}{2}}$

$V = 24$

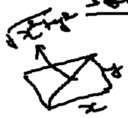
$x = (2V)^{1/3} = (48)^{1/3} = y$

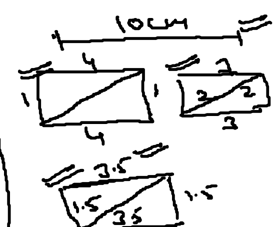
$z = \frac{(48)^{1/3}}{2}$

14/12/20 Q. Find the rectangle of constant perimeter, whose diagonal is minimum.

Sol. Let length of rectangle $= x$

width $= y$





$\therefore \; 2x + 2y = 2C \Rightarrow x + y = C$ —(i)

Now Diagonal $= f(x,y) = \sqrt{x^2+y^2}$

By Lagrange method, consider auxiliary function

$F(x,y) = \sqrt{x^2+y^2} + \lambda[x+y-C]$

Do it at your own.

OR $f(x,y) = \sqrt{x^2+y^2} = \sqrt{x^2+(C-x)^2}$

$\therefore \; g(x) = \sqrt{x^2+(C-x)^2}$

Complete it at your own

Q. Find the extreme value of $x^2 + 2xy + y^2$ under the constraints $2x + y = 0$ & $x + y + z = 1$

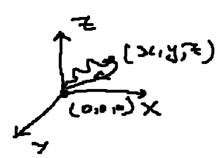
Sol. By Lagrange method, consider the auxiliary function

$F(x,y,z) = x^2 + 2xy + y^2 + \lambda_1(2x+y) + \lambda_2(x+y+z-1)$

Do this $\frac{\partial F}{\partial x} = 0, \; \frac{\partial F}{\partial y} = 0, \; \frac{\partial F}{\partial z} = 0$

Do it at your own

Q. $\lim_{(x,y,z)\to(0,0,0)} \log\left[\frac{z}{xy}\right]$



Sol. Take a family path $z = mx^2$ & $y = x$

$= \lim_{x\to 0} \log\left[\frac{m\cancel{x^2}}{\cancel{x \cdot x}}\right] = \lim_{x\to 0} \log m = \log m$

$\therefore$ As limit depend upon m, so it does not exist.

$\underline{\underline{\text{OR}}}$ $\quad z = x^2, \quad y = mx$

$\underline{\underline{\text{OR}}}$ $\quad z = m_1 x^2, \quad y = m_2 x$