26/11/20
Thursday, November 26, 2020
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Unit-4

Function of two variables: A function of two variables $f(x,y): D(\subseteq \mathbb{R}^2) \to \mathbb{R}$ is a rule which assign a unique value z to each point $(x,y) \in D$ & we write it as $z = f(x,y)$.

$\mathbb{R}\times\mathbb{R} = \{(x,y)\}$

$f: A(\subseteq \mathbb{R}) \to B(\subseteq \mathbb{R})$ is a rule, which assign a unique value y to each $x \in A$ and we write it as $y = f(x)$, $x = ?$

$f(x,y,z): D(\subseteq \mathbb{R}^3) \to \mathbb{R}$ is rule a rule which assign a unique value w to each point $(x,y,z) \in D(\subseteq \mathbb{R}^3)$ such that $w = f(x,y,z)$

$\mathbb{R}^2 = \mathbb{R}\times\mathbb{R} = \{(x,y): x\in\mathbb{R}, y\in\mathbb{R}\}$

$A\times B = \{(a,b): a\in A, b\in B\}$

$f(x) = e^x, \tan x, \log x, \sin x$

Area $= xy$

Volume $= lbh$

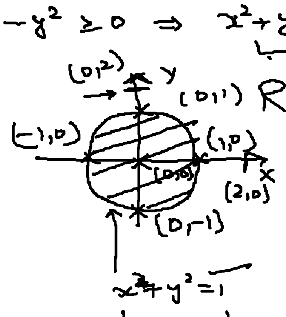
$(1,2)$

Area $= 1\times 2 = 2 \in \mathbb{R}$

Q. $f(x,y) = \sqrt{1-x^2-y^2}$, find D_f

$D_f \subseteq \mathbb{R}^2$

Sol. $f(x,y)$ will be defined if $1-x^2-y^2 \geq 0 \Rightarrow x^2+y^2 \leq 1$



$f(x) = \frac{1}{x}$, $x = 0$

$x^2+y^2=1$

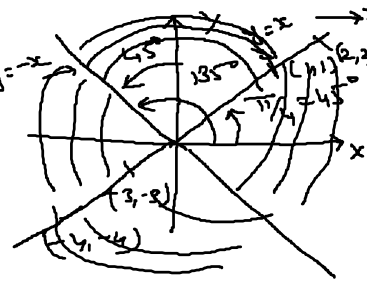
$f(2,0) = \sqrt{1-4-0} = \sqrt{-3}$

$f(0,2) = \sqrt{1-0-4} = \sqrt{-3}$

Here domain of function $f(x,y)$ or D_f is the set of all points which are inside & on the boundary of circle $x^2+y^2=1$

Q. $f(x,y) = \frac{1}{x^2-y^2}$, find D_f.

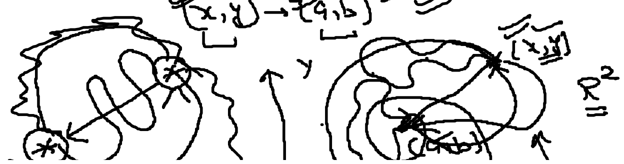
Sol. Take $x^2-y^2=0 \Rightarrow (x-y)(x+y)=0 \Rightarrow x-y=0 \Rightarrow x=y$ | $x+y=0 \Rightarrow x=-y$



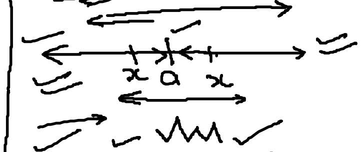
So, domain of given function $f(x,y)$ is set of all points of $\mathbb{R}^2$ or xy-plane except the points which lies on the lines $y=x$ & $y=-x$

H.W. **Q.** $f(x,y) = \log[1+x+y]$, find domain of $f(x,y)$.

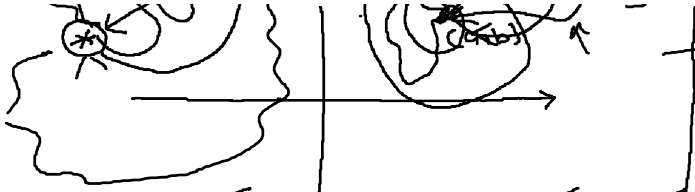
Limits: $\lim\limits_{(x,y)\to(a,b)} f(x,y)$



$\lim\limits_{x\to a} f(x)$ — Exists if



$\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x)$



$$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = K$$

$$\boxed{\lim_{x\to a} f(x) = K}$$

\# $\lim f(x,y)$ will exist, if $(x,y)\to(a,b)$ $(x,y)\to(a,b)$ along any path & the limiting value is same & finite [always]

\# We say, $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist, if there is a path along which limit does not exist, or limit is not same as the the value along other paths

\# Limit exist if it is independent of the choice of path. and if limit is path dependent, it does not exist

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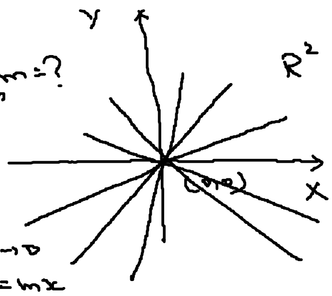
Q. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$



Sol → Consider a family of paths $y = mx$ (m = ?)

$$= \lim_{x\to 0} \frac{x}{\sqrt{x^2+m^2x^2}}$$

$$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}\sqrt{1+m^2}} = \lim_{x\to 0} \frac{1}{\sqrt{1+m^2}}$$

$x\to 0$, $y = mx$, $mx \to 0$, $y \to 0$

$$= \frac{1}{\sqrt{1+m^2}}$$

Here limit depends upon m, so it is path dependent & hence limit does not exist.

\# $y = x$, $(m=1)$, $\frac{1}{\sqrt{1+1}} = 1/\sqrt{2}$

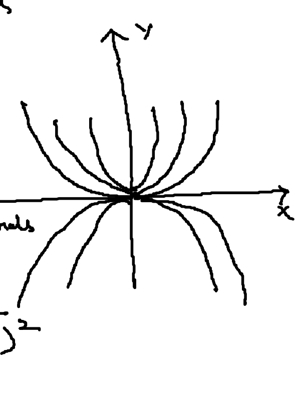
\# $y = 2x$, $(m=2)$, $\frac{1}{\sqrt{1+4}} = 1/\sqrt{5}$

Q → $\lim_{(x,y)\to(0,0)} \frac{x^4 y^2}{(x^4+y^2)^2}$

Sol → Consider a family of paths $y = mx^2$

+ve ↑ upwards

−ve ↓ Downwards



$$= \lim_{x\to 0} \frac{x^4\, m^2x^4}{[x^4+m^2x^4]^2}$$

$$= \lim_{x\to 0} \frac{m^2 \cancel{x^8}}{\cancel{x^8}[1+m^2]^2} = \lim_{x\to 0} \frac{m^2}{(1+m^2)^2}$$

$$= \frac{m^2}{(1+m^2)^2}$$

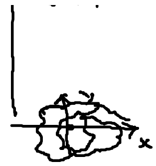
Hence, limit depends upon, so it is path dependent, hence limit does not exist

$\lim_{x\to a} f(x) = f(a)$

Continuity: $f(x,y)$ is continuous at (a,b), if

Continuity: $f(x,y)$ is continuous at (a,b), if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$



Q. $f(x,y) = \begin{cases} \dfrac{(x-y)^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ Check the continuity of $f(x,y)$ at $(0,0)$

Sol. $f(x,y)$ will be continuous at $(0,0)$ if $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

Here $f(0,0) = 0$

Now $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \dfrac{(x-y)^2}{x^2+y^2}$

Put $y = mx$

$$= \lim_{x\to 0} \frac{(x-mx)^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{\cancel{x^2}[1-m]^2}{\cancel{x^2}[1+m^2]} = \frac{(1-m)^2}{1+m^2}$$

Limit depends upon m, so it is path dependent, hence limit does not exist. Hence, $f(x,y)$ is not continuous at $(0,0)$

Q. $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^3+y^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ Check the continuity at $(0,0)$

Sol. $f(x,y)$ will be continuous at $(0,0)$, if $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

Here, $f(0,0) = 0$

Now $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^3+y^3}$

Put $y = mx$

$$= \lim_{x\to 0} \frac{x^2 m^2 x^2}{x^3+m^3x^3} = \lim_{x\to 0} \frac{m^2 x^{\cancel{4}}}{\cancel{x^3}[1+m^3]}$$

$$= \lim_{x\to 0} \frac{m^2 x}{1+m^3}$$

If $m=-1$, limit does not exist. Hence, limit does not exist.

$y = -x$

$$\lim_{x\to 0} \frac{x^4}{x^3 - x^3} = \lim_{x\to 0} \frac{x^4}{0} = \infty$$

$m = -1$

$$\lim_{x\to 0} \frac{x}{1-1}$$

$$\lim_{x\to 0} \frac{x}{0} = \lim_{x\to 0} \infty = \infty$$

Q. $f(x,y) = \begin{cases} \dfrac{1}{1+e^{y/x}} + y^2, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ Check the continuity at $(0,0)$

Sol. $f(x,y)$ will be continuous at $(0,0)$, if $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

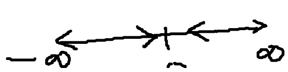
Here $f(0,0) = 0$

Now $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \left[\dfrac{1}{1+e^{y/x}} + y^2\right]$

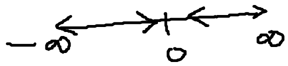
$$= \lim_{(x,y)\to(0,0)} \frac{1}{1+e^{y/x}} + \lim_{(x,y)\to(0,0)} y^2$$

$$= \lim_{x\to 0} \frac{1}{1+e^{y/x}} + \lim_{y\to 0} y^2$$

$$= \lim_{x\to 0} \frac{1}{1+e^{y/x}} + 0 \quad \text{--- (1)}$$



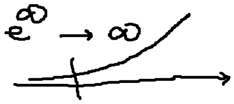
$$= \lim_{x\to 0} \frac{1}{1+e^{1/x}} + 0 \quad \text{—} (1)$$



L.H.L : $\lim_{x\to 0^-} \frac{1}{1+e^{1/x}}$, Put $x = 0 - h$, $h \to 0$, $h > 0$

$$= \lim_{h\to 0} \frac{1}{1+e^{-1/h}} = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

$e^{\infty} \to \infty$



$e^{-\infty} = \frac{1}{e^{\infty}} \Rightarrow \frac{1}{\infty} \to 0$

R.H.L : $\lim_{x\to 0^+} \frac{1}{1+e^{1/x}}$, Put $x = 0 + k$, $k \to 0$, $k > 0$

$$= \lim_{k\to 0} \frac{1}{1+e^{1/k}} = \frac{1}{1+e^{\infty}} = \frac{1}{\infty} \Rightarrow 0$$

$\therefore$ Limiting of $f(x,y)$ at $(0,0)$ does not exist. Hence $f(x,y)$ is not continuous at $(0,0)$.

Q → $\lim_{(x,y,z)\to(0,0,0)} \log\left[\frac{z}{xy}\right]$ surfaces

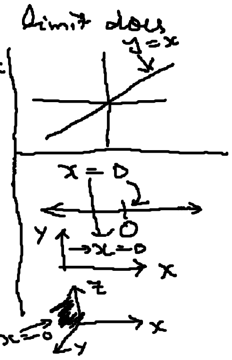
$x,y \to z$
$y,z \to x$
$x,z \to y$

Consider the paths $z = mx^2$, $y = x$

01/12/20 $= \lim_{x\to 0} \log\left[\frac{mx^2}{x \cdot x}\right] = \lim_{x\to 0} \log m$
$= \log m$

It depends upon m, so it is path dependent, hence limit does not exist

$z = x^2, y = mx$
$z = m_1 x^2$, $y = m_2 x$



Q → $\lim_{(x,y,z)\to(0,0,0)} \frac{xy^2z^2}{x^4+y^4+z^8}$

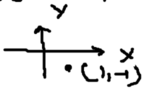
Sol → Take $z^2 = x$, $y = mx$

$$= \lim_{x\to 0} \frac{x\, m^2x^2\, x}{x^4 + m^4x^4 + x^4}$$

$$= \lim_{x\to 0} \frac{m^2 \cancel{x^4}}{\cancel{x^4}[1+m^4+1]} = \frac{m^2}{m^4+2}$$

Limit does not exist, as it depends upon.

Q → $f(x,y) = \begin{cases} \frac{x^2-2xy+y^2}{x-y}, & (x,y) \neq (1,-1) \\ 0, & (x,y) = (1,-1) \end{cases}$ Check the continuity at $(1,-1)$



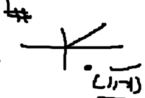
Sol → $f(x,y)$ will be continuous at $(1,-1)$, if $\lim_{(x,y)\to(1,-1)} f(x,y) = f(1,-1)$

Now $f(1,-1) = 0$

Again $\lim_{(x,y)\to(1,-1)} f(x,y) = \lim_{(x,y)\to(1,-1)} \frac{x^2-2xy+y^2}{x-y} = \frac{1-2(-1)+1}{1-(-1)} = \frac{4}{2} = 2$

$\therefore \lim_{(x,y)\to(1,-1)} f(x,y) \neq f(1,-1)$, hence $f(x,y)$ is not continuous at $(1,-1)$.

$y = mx$
$y^2 = mx$
#



$(y+1) = m(x-1)$

Q → $f(x,y) = \begin{cases} \frac{xy(x-y)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Check the Continuity at $(0,0)$

| $(y+1) = m(x-1)^2$
| $(x-1) = m(y+1)^2$

Sol. $f(0,0) = 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\underbrace{(x,y)\to(0,0)}} \frac{xy(x-y)}{x^2+y^2}$$

Take $y = mx$

$$= \lim_{x\to 0} \frac{x\,mx\,(x-mx)}{x^2+m^2x^2} = \lim_{x\to 0} \frac{x^3\, m(1-m)}{x^2[1+m^2]}$$

$$= \lim_{x\to 0} \underline{x}\,\underbrace{\left[\frac{m(1-m)}{1+m^2}\right]} = 0$$

| $m^2+1=0$
| $m^2=-1$
| $m=\pm i$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$, $f(x,y)$ is Continuous at $(0,0)$

Partial Derivatives: $z = f(\underbrace{x,y})$, we can differentiate it w.r.t both x & y, as mentioned below.

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

| $y = f(x)$
| $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
| x, y
| Area $= xy$
| Area $= xy$, $3x$
| $\{1 \to 1.01\}$, $\{2 \to 2.02\}$, $x+3$
| $\{4 \to 4.4\}$

Q→ $f(x,y) = x^4 - x^2y^2 + y^4$, find first order partial derivatives at $(-1,1)$

Sol.→ $f(x,y) = x^4 - x^2y^2 + y^4$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x^4 - x^2y^2 + y^4\right] = \frac{\partial}{\partial x}(x^4) - \frac{\partial}{\partial x}(x^2y^2) + \frac{\partial}{\partial x}(y^4)$$

$$= 4x^3 - y^2(2x) + 0 = 4x^3 - 2xy^2$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(-1,1)} = -4 - 2(-1)(1)^2 = -4+2 = -2$$

Again
$$\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left[x^4 - x^2y^2 + y^4\right] = \frac{\partial}{\partial y}(x^4) - \frac{\partial}{\partial y}(x^2y^2) + \frac{\partial}{\partial y}(y^4)$$
$$= 0 - x^2(2y) + 4y^3$$

$$\Rightarrow \left(\frac{\partial f}{\partial y}\right)_{(-1,1)} = -2+4 = 2$$

① Q. Find first order Partial derivatives

Q→ $f(x,y) = \log\left[\frac{x}{y}\right]$ at $(2,3)$ ② $f(x,y) = x^2 e^{y/x}$ at $(4,2)$

Sol. ② $f(x,y) = x^2 e^{y/x}$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x^2 e^{y/x}\right] = x^2\, e^{y/x}\left(\frac{-y}{x^2}\right) + e^{y/x}(2x)$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(4,2)} = 16\, e^{\frac{1}{2}}\left(-\frac{2}{16}\right) + e^{1/2}(8)$$

$$= 8\sqrt{e} - 2\sqrt{e} = 6\sqrt{e}$$

Again
$$\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left[x^2 e^{y/x}\right] = x^2 \frac{\partial}{\partial y}\left(e^{y/x}\right) = x^2\left[e^{y/x}\,\frac{1}{x}\cdot 1\right]$$

$$\Rightarrow \left(\frac{\partial f}{\partial y}\right)_{(4,2)} = 4e^{1/2} = 4\sqrt{e}$$

| $\frac{\partial}{\partial y}(y/x)$
| $= \frac{1}{x}\cdot 1$

Q→ $f(x,y,z) = (xy)^{\ln z}$ find first order partial derivatives at

Q→ $f(x,y,z) = (xy)^{\sin z}$ find first order partial derivatives at $(3,5,\pi/2)$

Sol.→ $f(x,y,z) = x^{\sin z}\, y^{\sin z}$

$$\frac{\partial f}{\partial x} = y^{\sin z} \frac{\partial}{\partial x} x^{\sin z} = y^{\sin z} \sin z\, x^{\sin z - 1}$$

$$\frac{\partial f}{\partial y} = x^{\sin z} \frac{\partial}{\partial y} y^{\sin z} = x^{\sin z} \cdot \sin z\, y^{\sin z - 1}$$

$$\frac{\partial f}{\partial z} = (xy)^{\sin z} \cdot \log(xy) \cdot \cos z$$

$\frac{d}{dx} a^x = a^x \log a$

02/12/20 Find first order partial derivatives of following functions

① $f(x,y,z) = e^{x/y} + e^{z/y}$ at $(1,1,1)$

② $f(x,y,z) = (xy)^{\sin z}$ at $(5,3,\pi/2)$

$f(x,y,z)$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

Sol → ① $f(x,y,z) = e^{x/y} + e^{z/y}$

Sol.→ $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[e^{x/y} + e^{z/y}\right] = \frac{\partial}{\partial x}\left[e^{x/y}\right] + \frac{\partial}{\partial x}\left[e^{z/y}\right]$$

$$= e^{x/y} \frac{\partial}{\partial x}\left(\frac{x}{y}\right) + 0 = e^{x/y} \cdot \frac{1}{y}(1)$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(1,1,1)} = e$$

||ly $$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[e^{x/y} + e^{z/y}\right] = \frac{\partial}{\partial y}\left(e^{x/y}\right) + \frac{\partial}{\partial y}\left(e^{z/y}\right)$$

$$= e^{x/y} \frac{\partial}{\partial y}\left(\frac{x}{y}\right) + e^{z/y} \frac{\partial}{\partial y}\left(z/y\right) = e^{x/y} \cdot x\left(-1/y^2\right) + e^{z/y} \cdot z\left(-1/y^2\right)$$

$$\Rightarrow \left(\frac{\partial f}{\partial y}\right)_{(1,1,1)} = -e - e = -2e$$

and $$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[e^{x/y} + e^{z/y}\right] = \frac{\partial}{\partial z}\left(e^{x/y}\right) + \frac{\partial}{\partial z}\left(e^{z/y}\right)$$

$$= 0 + e^{z/y} \frac{\partial}{\partial z}\left(z/y\right) = e^{z/y} \cdot \frac{1}{y} \cdot 1 = \frac{e^{z/y}}{y}$$

$$\Rightarrow \left(\frac{\partial f}{\partial z}\right)_{(1,1,1)} = e$$

Composite functions & chain Rule :

Let $z = f(u,v)$, $u = g(x,y)$, $v = h(x,y)$, then we say z is a composite function of x, y.

$$z \to (u,v) \to (x,y)$$

$$\boxed{z \to (u,v)} + \boxed{(u,v) \to (x,y)}$$

$$\boxed{z \to (u,v) \to (x,y)} \to z \to (u,v) \to (x,y)$$

$\left\{\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right\}$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$ → Chain Rule

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$z \to (u, v) \to (x, y)$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

\# $w = f(r, s, t)$, $r = g(x, y)$, $s = h(x, y)$, $t = k(x, y)$

$w \to (r, s, t) \to (x, y)$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x} \to \text{Chain Rule}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$$

\# $w = f(t)$, $t = g(x, y)$

$w \to t \to (x, y)$

$$\frac{\partial w}{\partial x} = \frac{dw}{dt}\cdot\frac{\partial t}{\partial x} \to \text{Chain Rule}$$

$$\frac{\partial w}{\partial y} = \frac{dw}{dt}\frac{\partial t}{\partial y}$$

\# $w = f(x, y)$, $x = g(t)$, $y = h(t)$

$w \to (x, y) \to t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \to (\text{Chain Rule})$$

Q→ $z = f(ax+by)$, then show that $b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = 0$

Sol→ $z = f(ax+by)$

Take $ax+by = t$

$\Rightarrow z = f(t)$, $t = ax+by$

$z \to t \to (x, y)$

$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

Now $\frac{\partial z}{\partial x} = \frac{dz}{dt}\cdot\frac{\partial t}{\partial x} = (a\cdot 1 + 0)\frac{dz}{dt} = a\frac{dz}{dt}$ — (i)

Again $\frac{\partial z}{\partial y} = \frac{dz}{dt}\cdot\frac{\partial t}{\partial y} = \frac{dz}{dt}\cdot[0 + b\cdot 1] = b\frac{dz}{dt}$ — (ii)

Now $\overset{LHS}{=} b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = b\left[a\frac{dz}{dt}\right] - a\left[b\frac{dz}{dt}\right] = 0 = RHS$

Q→ $w = x^2+y^2$, $x = \frac{t^2-1}{t}$, $y = \frac{t}{t^2+1}$, find $\frac{dw}{dt}$ at $t=1$

Sol→ $\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

$w \to (x, y) \to t$

$\frac{dw}{dt} = ?$

$\Rightarrow \frac{dw}{dt} = (2x+0)\left[1 + \frac{1}{t^2}\right] + [0 + 2y]$

$\frac{(t^2+1)\cdot 1 - t\cdot 2t}{(t^2+1)^2}$

$\frac{t^2-1}{t} \to t - \frac{1}{t}$

$$\frac{dw}{dt} = 2\left[\frac{t^2-1}{t}\right]\left(\frac{t^2+1}{t^2}\right) + 2\left[\frac{t}{t^2+1}\right]\frac{1-t^2}{(t^2+1)^2}$$

$$\left(\frac{dw}{dt}\right)_{t=1} = 0 + 0 = 0$$

Q→ $w = z\log y + y\log z + xyz$, $x = \sin t$, $y = t^2+1$, $z = \cos^{-1} t$

find $\frac{dw}{dt}$ at $t=0$

$w \to (x,y,z) \to (t)$

Sol→ $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$

$$\Rightarrow \frac{dw}{dt} = yz\cdot\cos t + \left[\frac{z}{y} + \log z + xz\right](2t)$$

$$+\left[\log y\cdot 1 + \frac{y}{z} + xy\cdot 1\right]\frac{(-1)}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dw}{dt} = (t^2+1)\cos^{-1}t\cdot\cos t + 2t\left[\frac{\cos^{-1}t}{t^2+1} + \log\cos^{-1}t + \sin t\cos^{-1}t\right]$$

$$-\frac{1}{\sqrt{1-t^2}}\left[\log(1+t^2) + \frac{1+t^2}{\cos^{-1}t} + (1+t^2)\sin t\right]$$

$$\Rightarrow \left(\frac{dw}{dt}\right)_{t=0} = 1\times\frac{\pi}{2}\times 1 + 0 - \left[0 + \frac{1}{\pi/2} + 0\right]$$

$$= \frac{\pi}{2} - \frac{2}{\pi} = \frac{\pi^2-4}{2\pi}$$

03/12/20: Q If $u = f(x-y,\ y-z,\ z-x)$, then $\boxed{\frac{\partial u}{\partial x}} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Sol→ $u = f(x-y, y-z, z-x)$ $\quad r = (x-y)z^0$

Take $x-y = r$, $y-z = s$, $z-x = t$

⇒ $u = f(r,s,t)$, $r = x-y$, $s = y-z$, $t = z-x$

$u \to (r,s,t) \to (x,y,z)$

C.F.

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$

Now $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot 1 + \frac{\partial u}{\partial s}\cdot(0) + \frac{\partial u}{\partial t}\cdot(-1)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial t}} \quad \text{—(i)}$$

Now $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$

$$= \frac{\partial u}{\partial r}\cdot(-1) + \frac{\partial u}{\partial s}\cdot(1) + \frac{\partial u}{\partial t}(0)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial s}} \quad \text{—(ii)}$$

Now $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot(0) + \frac{\partial u}{\partial s}\cdot(-1) + \frac{\partial u}{\partial t}(1)$$

$$\Rightarrow \frac{\partial u}{\partial z} = -\cancel{\frac{\partial u}{\partial s}} + \cancel{\frac{\partial u}{\partial t}} \quad \text{—(iii)}$$

Now $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Q→ $z = \log[u^2+v]$, $u = e^{x+y^2}$, $v = x+y^2$, then $2y\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$

Sol→ $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$ $\quad z \to (u,v) \to (x,y)$

C.F.

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{1}{(u^2+v)}(2u+0)\, e^{x+y^2}[1+0] + \frac{1}{u^2+v}[0+1]\cdot 1$$

$\left\{\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right\}$

$$= \frac{2u\cdot e^{x+y^2}+1}{u^2+v} = \frac{2u^2+1}{u^2+v} \quad \Big|\; \because u = e^{x+y^2}$$

Now $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{2u}{u^2+v}\cdot e^{x+y^2}[0+2y] + \frac{1}{u^2+v}\cdot 2y$

$$\frac{\partial z}{\partial y} = \frac{4uy\, e^{x+y^2} + 2y}{u^2+v}$$

Now $2y\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{4u^2y + 2y}{u^2+v} - \frac{4u^2y+2y}{u^2+v} = 0$

OR

$z = \log[u^2+v]$, $u = e^{x+y^2}$, $v = x+y^2$, P.T. $2y\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$

Sol. $z = \log\left[\left(e^{x+y^2}\right)^2 + x+y^2\right]$

$\Rightarrow z = \log\left[e^{2(x+y^2)} + x+y^2\right]$

{Do this at your own}

$(x^3)^2 = x^3\cdot x^3 = x^{3+3} = x^{2\cdot 3}$

Q. $w = \sqrt{x^2+y^2+z^2}$, $x = u\cos v$, $y = u\sin v$, $z = uv$

P.T. $u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = \frac{u}{\sqrt{1+v^2}}$

$w \to (x,y,z) \to (u,v)$

C.F.

$\frac{\partial w}{\partial u} = ?$

$\Rightarrow \frac{\partial w}{\partial v} = ?$

Sol. By chain rule

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\Rightarrow \frac{\partial w}{\partial u} = \frac{x}{w}(\cos v)\cdot 1 + \frac{y}{w}\cdot\sin v\cdot 1 + \frac{z}{w}\cdot v\cdot 1$$

$$\Rightarrow \frac{\partial w}{\partial u} = \frac{x\cos v + y\sin v + zv}{w} \quad (i)$$

$w = \sqrt{x^2+y^2+z^2}$

$\Rightarrow w^2 = x^2+y^2+z^2$

Partially diff w.r.t. x

$\Rightarrow 2w\frac{\partial w}{\partial x} = 2x$

$\Rightarrow \frac{\partial w}{\partial x} = \frac{x}{w}$

Now $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$

$$\Rightarrow \frac{\partial w}{\partial v} = \frac{x}{w}\cdot u(-\sin v) + \frac{y}{w}\cdot u\cos v + \frac{z}{w}\cdot u\cdot 1$$

$$\Rightarrow \frac{\partial w}{\partial v} = \frac{-ux\sin v + uy\cos v + uz}{w} \quad (ii)$$

Illy $\frac{\partial w}{\partial y} = \frac{y}{w}$

& $\frac{\partial w}{\partial z} = \frac{z}{w}$

Now $u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v}$

$$= \frac{ux\cos v + uy\sin v + \cancel{uzv} + uvx\sin v - uvy\cos v - \cancel{uvz}}{w}$$

$\boxed{f(x,y) = f(y,x)}$

$f(x,y) = x^2+y^2$

$\Rightarrow f(y,x) = y^2+x^2 = f(x,y)$

$$= \frac{u(u\cos v)\cos v + u(u\sin v)\sin v + uv(u\cos v)\sin v - uv(u\sin v)\cos v}{w}$$

$w = \sqrt{x^2+y^2+z^2}$

$$= \frac{u^2[\cos^2 v + \sin^2 v]}{w} = \frac{u^2}{w} = \frac{u^{\cancel{2}}}{\cancel{u}\sqrt{1+v^2}} = \frac{u}{\sqrt{1+v^2}}$$

$w = \sqrt{x^2+y^2+z^2}$
$= \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2v^2}$
$= \sqrt{u^2[\cos^2 v + \sin^2 v] + u^2v^2}$
$= \sqrt{u^2 + u^2v^2}$
$= u\sqrt{1+v^2}$

OR $w = \sqrt{x^2+y^2+z^2}$ ← $x = u\cos v$, $y = u\sin v$, $z = uv$

$w \to (x,y,z) \to (u,v)$

$\Rightarrow \underbrace{w = u\sqrt{1+v^2}}$ $\quad \frac{\partial w}{\partial u} \to ?$

Solve it at your own $\quad \frac{\partial w}{\partial v} \to ?$

Q → 05/12/20 $z = y + f(u)$, $u = \frac{x}{y}$, show that $u\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$

Sol → $z = y + f\left(\frac{x}{y}\right)$

$\frac{\partial z}{\partial x} = 0 + f'\left(\frac{x}{y}\right)\cdot\left(\frac{1}{y}\cdot 1\right)$ & $\frac{\partial z}{\partial y} = 1 + f'\left(\frac{x}{y}\right)x\left(-\frac{1}{y^2}\right)$

Now $u\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{x}{y}\cdot\frac{1}{y}\cancel{f'\left(\frac{x}{y}\right)} + 1 - \frac{x}{y^2}\cancel{f'(x/y)} = 1$

Implicit function: $f(x,y) = c$ → Implicit function

$f(x,y) = C$ → Implicit ✓

$y = f(x)$ or $x = g(y)$ → Explicit ✓

$xy = 3$ → Implicit | $y = 3/x$ or $x = 3/y$ → Explicit

$f(u)$ $u = x/y$

$f \to u \to (x,y)$

C.F.

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

$\frac{\partial f}{\partial x} = \frac{df}{du}\cdot\frac{\partial u}{\partial x}$

$= \boxed{\frac{df}{d(x/y)}}\frac{\partial}{\partial x}(x/y)$

$= \frac{d f(x/y)}{d(x/y)}\left(\frac{1}{y}\right)\cdot 1$

$xy = 3$

$\boxed{y = 3/x}$

$f(x,y) = C$ — ①

[eg $x^2 + 2xy - 3y^2 = 0$] → find $\frac{dy}{dx} = ?$

$\frac{dy}{dx} = ?$

$f \to (x,y) \to (x)$

C.F. → $\frac{df}{dx} = ?$

$\frac{df}{dx} = 0 \Rightarrow \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0$

$\Rightarrow \frac{\partial f}{\partial x}\cdot 1 + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$

Q → $x^2 + 2xy - 3y^2 = 0$, find $\frac{dy}{dx}$

Sol → diff w.r.t. x

$\Rightarrow 2x + 2\left[x\frac{dy}{dx} + y\cdot 1\right] - 6y\frac{dy}{dx} = 0$ $\quad$ Method of 10+2

$\Rightarrow (2x - 6y)\frac{dy}{dx} = -2x - 2y$

$\therefore \frac{dy}{dx} = \frac{-x+y}{}$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{3y-x}$$

$f(x,y)=c$

$\frac{dy}{dx} = -\frac{f_x}{f_y}$

OR $\quad x^2+2xy-3y^2=0$

$f(x,y) = c$

Here $f(x,y)= x^2+2xy-3y^2$

Now $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\left[\frac{2x+2y.1-0}{0+2x.1-6y}\right] = \frac{x+y}{3y-x}$

Q→ $x^y+y^x=\alpha$, find $\frac{dy}{dx}$

Solve this problem by using the idea of 10+2

Sol.→ $x^y+y^x-\alpha=0$

$f(x,y)$ $\quad c$

Here $f(x,y) = x^y+y^x-\alpha \Rightarrow \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\left[\frac{yx^{y-1}+y^x\log y-0}{x^y\log x+xy^{x-1}-0}\right]$

$v^c \qquad c^v \to a^x$

$c^v \to a^y \to a^y\log a \qquad v^c \to y^n \to ny^{n-1}$

$x^y+y^x-\alpha=0$

$$\Rightarrow \frac{dy}{dx} = -\frac{[yx^{y-1}+y^x\log y]}{[x^y\log x+xy^{x-1}]}$$

Q→ $x^3+3xy-2y^2+3xz+z^2=0$, find $\left(\frac{\partial z}{\partial x}\right)_y$ & $\left(\frac{\partial z}{\partial y}\right)_x$

$\to 0$

Sol→ $x^3+3xy-2y^2+3xz+z^2=0 \quad$ —(i)

diff w.r.t x, by treating y as a constant

$$3x^2+3y.1-0+3\left[x\left(\frac{\partial z}{\partial x}\right)_y+z.1\right]+2z\left(\frac{\partial z}{\partial x}\right)_y=0$$

$$\Rightarrow (3x+2z)\left(\frac{\partial z}{\partial x}\right)_y = -3x^2-3y-3z$$

$$\Rightarrow \left(\frac{\partial z}{\partial x}\right)_y = \frac{-3x^2-3y-3z}{3x+2z}$$

Diff (i) w.r.t y, treating x as constant

$$0+3x.1-4y+3x\left(\frac{\partial z}{\partial y}\right)_x+2z\left(\frac{\partial z}{\partial y}\right)_x=0$$

$$\Rightarrow (3x+2z)\left(\frac{\partial z}{\partial y}\right)_x = 4y-3x$$

$$\Rightarrow \left(\frac{\partial z}{\partial y}\right)_x = \frac{4y-3x}{3x+2z}$$

Higher order derivatives.

$f(x,y) \to \left(\frac{\partial f}{\partial x}\right) \to (x,y)$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = r$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\partial x} = f_{yx} = s$

$f(x,y) \to \left(\frac{\partial f}{\partial y}\right)$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\partial y} = f_{xy} = s$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = t$

(x,y) 1st order derivatives

In general $f_{xy} \neq f_{yx}$

Q → $f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, find second order derivatives at $(1,1,1)$

Sol. $\frac{\partial f}{\partial x} = \frac{1}{y}\cdot 1 + 0 + z\left(-\frac{1}{x^2}\right)$

$f(x,y,z) \to \frac{\partial f}{\partial x} \to f_{xx}, f_{xy}, f_{xz}$; f_y; f_z

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 0 + z\left(\frac{2}{x^3}\right) \Rightarrow (f_{xx})_{(1,1,1)} = 2$

$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\frac{1}{y^2} + 0 \Rightarrow (f_{yx})_{(1,1,1)} = -1$

$f_{zx} = \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = 0 - \frac{1}{x^2}\cdot 1 = -\frac{1}{x^2} \Rightarrow (f_{zx})_{(1,1,1)} = -1$

08/12/20 Homogeneous Function: A function $f(x,y)$ is said to be homogeneous with degree n, if it can be expressed as $x^n \phi\left(\frac{y}{x}\right)$ or $y^n g\left(\frac{x}{y}\right)$

e.g. $f(x,y) = x^2 + y^2 = x^2\left[1 + \frac{y^2}{x^2}\right]$, $\frac{y}{x} = t$

$= x^2(1+t^2) = x^2\phi(t) = x^2\phi\left(\frac{y}{x}\right)$ → Homogeneous degree = 2

OR $f(x,y) = y^2\left[1 + \frac{x^2}{y^2}\right]$, $\frac{x}{y} = u$

$= y^2(1+u^2) = y^2 h(u) = y^2 h\left(\frac{x}{y}\right)$

\# $f(x,y,z)$ → $x^n \phi\left(\frac{y}{x}, \frac{z}{x}\right)$ — $y^n h\left(\frac{x}{y}, \frac{z}{y}\right)$ — $z^n g\left(\frac{x}{z}, \frac{y}{z}\right)$ → Homogeneous with degree n

\# $f(x,y,z,w) \to x^n \phi\left(\frac{y}{x}, \frac{z}{x}, \frac{w}{x}\right)$

(\#*) $f(x,y)$ is said to be homogeneous with degree n, if $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

\# $f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x,y,z)$

\# Euler's theorem: Let $f(x,y)$ be a homogeneous function with degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

\# H → 3: $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f$

Q → If $u(x,y) = \log\left[\frac{\sqrt{x^2+y^2}}{x}\right]$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Sol. $u(\lambda x, \lambda y) = \log\left(\frac{\sqrt{\lambda^2 x^2 + \lambda^2 y^2}}{\lambda x}\right) = \log\left(\frac{\cancel{\lambda}\sqrt{x^2+y^2}}{\cancel{\lambda} x}\right)$

$= \log\left(\frac{\sqrt{x^2+y^2}}{x}\right) = u(x,y)$

\# $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

$\Rightarrow u(\lambda x, \lambda y) = \lambda^0 u(x,y)$

∴ $u(x,y)$ is homogeneous with degree $= n = 0$

By Euler's thm. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 0(u) = 0$

By Euler's thm. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 0(u) = 0$

Q→ $u(x,y) = \sqrt{y^2-x^2}\,\sin^{-1}\left[\frac{x}{y}\right]$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$

Sol.→ $u(\lambda x, \lambda y) = \sqrt{\lambda^2 y^2 - \lambda^2 x^2}\,\sin^{-1}\left[\frac{\lambda x}{\lambda y}\right] = \lambda\sqrt{y^2-x^2}\,\sin^{-1}\left(\frac{x}{y}\right)$

$\Rightarrow u(\lambda x, \lambda y) = \lambda^{1} u(x,y)$

∴ $u(x,y)$ is homogeneous with degree $= n = 1$

∴ By Euler's thm. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 1.u = u$

Cor. [Euler's theorem] → If $f(x,y)$ is a homogeneous function with degree n, $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$

Q→ $u(x,y) = \frac{y^3 - x^3}{y^2+x^2}$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$ &

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

Sol.→ $u(\lambda x, \lambda y) = \frac{\lambda^3 y^3 - \lambda^3 x^3}{\lambda^2 y^2 + \lambda^2 x^2} = \frac{\lambda^3}{\lambda^2}\left[\frac{y^3 - x^3}{y^2+x^2}\right]$

$\Rightarrow u(\lambda x, \lambda y) = \lambda^{1} u(x,y)$

$\Rightarrow u(x,y)$ is homogeneous with degree $= n = 1$

∴ By Euler's thm. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = u$

and $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 1(1-1)u = 0$

Q→ $u(x,y) = \boxed{\sin^{-1}\boxed{\left[\frac{x^2+y^2}{x+y}\right]}}$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$

Sol.→ $w = \sin u = \frac{x^2+y^2}{x+y}$

$w(\lambda x, \lambda y) = \frac{\lambda^2 x^2 + \lambda^2 y^2}{\lambda x + \lambda y} = \frac{\lambda^2}{\lambda}\left[\frac{x^2+y^2}{x+y}\right]$

$\Rightarrow w(\lambda x, \lambda y) = \lambda^{1} w(x,y)$

$\Rightarrow w$ is homogeneous function with degree $= n = 1$

$u(\lambda x, \lambda y) = \sin^{-1}\left[\frac{\lambda^2 x^2 + \lambda^2 y^2}{\lambda x + \lambda y}\right]$

$= \sin^{-1}\left[\frac{\lambda^2}{\lambda}\left(\frac{x^2+y^2}{x+y}\right)\right]$

$= \sin^{-1}\left[\lambda\left[\frac{x^2+y^2}{x+y}\right]\right]$

$u(\lambda x, \lambda y) = \lambda^{n} u(x,y)$

∴ By Euler's thm. $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = nw$

$\Rightarrow x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = w$

$\Rightarrow x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \sin u$

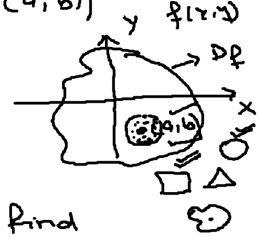
$\Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u} = \tan u$

Maxima & Minima :

Relative Maxima & Minima : Let $f(x,y)$ be a function of (x,y) then $(a,b) \in D_f$ is s.t.b point of relative maxima [or relative

minima), if $f(a,b) \geq f(x,y)$, $\forall (x,y)$ in some nbd. of (a,b)
$\left(f(a,b) \leq f(x,y) \;\; \forall (x,y) \in \text{some nbd. of } (a,b) \right)$



09/12/20 Method to find relative maxima & minima:

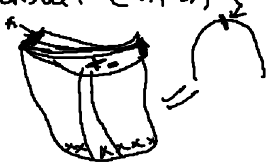
For a function $f(x,y)$

(i) $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ (ii) $\frac{\partial f}{\partial x}=0$ & $\frac{\partial f}{\partial y}=0$, solve these equations simultaneously, to find stationary points (a_1,b_1), (a_2,b_2) - -

(iii) $r=\frac{\partial^2 f}{\partial x^2}$, $s=\frac{\partial^2 f}{\partial x \partial y}$, $t=\frac{\partial^2 f}{\partial y^2}$

(iv) At (a_1,b_1), find the value of $rt-s^2$ & r

If $rt-s^2>0$, $r>0$, then (a_1,b_1) is relative minima
If $rt-s^2>0$, $r<0$, then (a_1,b_1) is relative maxima
If $rt-s^2<0$, then (a_1,b_1) is a saddle point.
If $rt-s^2=0$, then test fails, and not able to conclude anything about (a_1,b_1) & further investigations are needed.



Horse saddle

Q.→ $f(x,y)= xy+\frac{9}{x}+\frac{3}{y}$, find relative maxima & minima

Sol.→ $\frac{\partial f}{\partial x}= y.1-\frac{9}{x^2}+0=0$ — (i) , $\frac{\partial f}{\partial y}= x.1+0-\frac{3}{y^2}=0$ — (ii)

from (i) $y=\frac{9}{x^2}$, put in (ii)

$\therefore$ (ii) $\Rightarrow x-\frac{3}{(9/x^2)^2}=0 \Rightarrow x-\frac{3}{81/x^4}=0 \Rightarrow x-\frac{x^4}{27}=0$

$\Rightarrow 27x-x^4=0 \Rightarrow x[3^3-x^3]=0$

$\Rightarrow x[3-x][3^2+x^2+3x]=0$

$\Rightarrow x=0,3$

If $x=0$, $y\to\infty \Rightarrow$ Reject this

and if $x=3$, $y=\frac{9}{9}=1 \Rightarrow$ Stationary point is $(3,1)$

$x^2+3x+9=0$
$\Rightarrow x=\frac{-3\pm\sqrt{9-4\times1\times9}}{2\times1}$
$=\frac{-3\pm\sqrt{9-36}}{2}$
$=\frac{-3\pm\sqrt{-27}}{2}$
Complex roots, so reject these

Now $r=\frac{\partial^2 f}{\partial x^2}=(-9)(-2x^{-3}) = 18/x^3$

$s=\frac{\partial^2 f}{\partial x\partial y}=1$, $t=\frac{\partial^2 f}{\partial y^2}=(-3)(-2y^{-3})=\frac{6}{y^3}$

At $(3,1)$, $r=\frac{18}{3^3}=\frac{18}{27}=2/3$, $s=1$, $t=\frac{6}{1^3}=6$

Now $rt-s^2= \frac{2}{3}\times 6-(1)^2 = 4-1= 3>0$, $r=2/3>0$

$\therefore$ $(3,1)$ is the point of relative minima

Min. Value $= f(3,1)= 3\times1+\frac{9}{3}+\frac{3}{1}$ | $xy+9/x+3/y$
$= 3+3+3=9$

Q.→ $f(x,y)= x^2+xy+y^2+\frac{1}{x}+\frac{1}{y}$, find relative max & min.

Sol.→ $\frac{\partial f}{\partial x} = 2x+y.1+0-1/x^2+0=0 \Rightarrow 2x^3+x^2y-1=0$

x^2
$2x^3$

$\frac{\partial f}{\partial y} = 0 + x\cdot 1 + 2y + 0 - \frac{1}{y^2} = 0 \Rightarrow xy^2 + 2y^3 - 1 = 0$ (ii)

(i) − (ii) $\Rightarrow (2x^3 - 2y^3) + (x^2y - xy^2) = 0$

$\Rightarrow 2[x-y][x^2+y^2+xy] + xy[x-y] = 0$

$\Rightarrow (x-y)[2x^2+2y^2+2xy+xy] = 0$

$\Rightarrow (x-y)[2x^2+2y^2+3xy] = 0$

If $x-y=0 \Rightarrow x=y$, put in (i)

$\therefore 2x^3 + x^3 - 1 = 0 \Rightarrow 3x^3 = 1$

$\Rightarrow x^3 = 1/3 \Rightarrow x = (1/3)^{1/3} = \frac{1}{3^{1/3}}$

and $y = \frac{1}{3^{1/3}} \Rightarrow$ Stationary Point is $\left(\frac{1}{3^{1/3}}, \frac{1}{3^{1/3}}\right)$

or $2x^2+2y^2+3xy=0$

$\Rightarrow 2x^2 + 3yx + 2y^2 = 0$

$\Rightarrow x = \frac{-3y \pm \sqrt{9y^2 - 4\times 2\times 2y^2}}{2\times 2}$

$= \frac{-3y \pm \sqrt{-7y^2}}{4}$

Complex roots, so reject these

Now $r = \frac{\partial^2 f}{\partial x^2} = 2 + \frac{2}{x^3}$

$s = \frac{\partial^2 f}{\partial x \partial y} = 1$

$t = \frac{\partial^2 f}{\partial y^2} = 2 + \frac{2}{y^3}$

Now, at $\left(\frac{1}{3^{1/3}}, \frac{1}{3^{1/3}}\right)$, $r = 2 + \frac{2}{1/3} = 8$, $s = 1$, $t = 2 + \frac{2}{1/3} = 8$

Now $rt - s^2 = 8\times 8 - (1)^2 = 63 > 0$, $r = 8 > 0$

$\therefore \left(\frac{1}{3^{1/3}}, \frac{1}{3^{1/3}}\right)$ is of relative minima | $x^2+xy+y^2+\frac{1}{x}+\frac{1}{y}$

$\therefore$ Min. value $= f\left(\frac{1}{3^{1/3}}, \frac{1}{3^{1/3}}\right) = \frac{1}{3^{2/3}} + \frac{1}{3^{2/3}} + \frac{1}{3^{2/3}} + \frac{1}{\left(\frac{1}{3^{1/3}}\right)} + \frac{1}{\left(\frac{1}{3^{1/3}}\right)}$

$= \frac{3}{3^{2/3}} + 3^{1/3} + 3^{1/3}$

$= 3^{1/3} + 3^{1/3} + 3^{1/3} = 3\cdot 3^{1/3} = \underline{3^{4/3}}$ | $\frac{3^1}{3^{2/3}} = 3^{1-2/3}$

Lagrange's method of undetermined multiplier:

Let $f(x,y)$ is to be maximized or minimized under some condition $\phi(x,y) = c \longrightarrow \phi(x,y) - c = 0$

(i) Consider an auxiliary function, $F(x,y) = f(x,y) + \lambda(\phi(x,y) - c)$

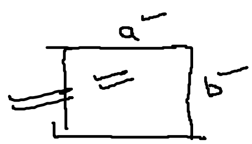
(ii) Find $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$

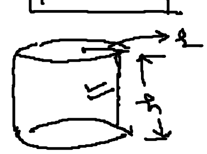
(iii) Solve $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, together with $\phi(x,y) = c$, to get the stationary points $(a_1, b_1), (a_2, b_2)$ - -

10/12/20 Q→ Divide a number into three parts such that product of first, square of second & cube of third is maximum

Sol. Let first part is x

and second part y





& third part z

$\therefore x+y+z = c$ —(i)

Now Product $= f(x,y,z) = xy^2z^3$

Auxiliary function $= F(x,y,z)$

$= xy^2z^3 + \lambda[x+y+z-c]$

V → Max, S → Fixed

15 → 4, 5, 6

0, 1, 14 ; 1, 2, 12 ; 2, 3, 10

Product → $2\times3^2\times10^3$; $1\times2^2\times12^3$; $0\times1^2\times14^3$

Now $\frac{\partial F}{\partial x} = y^2z^3\cdot1 + \lambda[1] = 0$ —(ii)

$\frac{\partial F}{\partial y} = 2xz^3y + \lambda[1] = 0$ —(iii)

$\frac{\partial F}{\partial z} = 3xy^2z^2 + \lambda[1] = 0$ —(iv)

from (ii), $\lambda = -y^2z^3$, from (iii) $\lambda = -2xyz^3$, from (iv) $\lambda = -3xy^2z^2$

Now $-y^2z^3 = -2xyz^3 = -3xy^2z^2$

Now $+y^{\cancel{2}}\cancel{z^3} = +2x\cancel{y}\cancel{z^3} \Rightarrow \boxed{y = 2x}$

and $+2\cancel{x}\cancel{y}z^{\cancel{3}} = +3\cancel{x}y^{\cancel{2}}\cancel{z^2} \Rightarrow \boxed{2z = 3y}$

Now, $x+y+z = c \Rightarrow \frac{y}{2} + y + \frac{3}{2}y = c \Rightarrow \frac{y+2y+3y}{2} = c$

$\Rightarrow 3y = c \Rightarrow \boxed{y = c/3}$, $\boxed{x = c/6}$, $z = \frac{3}{2}y$

$= \frac{\cancel{3}}{2}\times\frac{c}{\cancel{3}}$

$\Rightarrow \boxed{z = c/2}$

$\therefore$ $x = \frac{c}{6}$, $y = \frac{c}{3}$, $z = \frac{c}{2}$, Product $= f(x,y,z) = xy^2z^2$ will be maximum.

48 → c

xy^2z^3 → Max

(8, 16, 24)

2, 2, 44

1, 3, 44

2, 5, 43

2, 6, 40

Q → A rectangular box without top is to have a given volume. How should the box be made, so as to use least material.

Sol. → Let length of box = x

width —— = y

height —— = z

Volume = $xyz = c$ —(i)

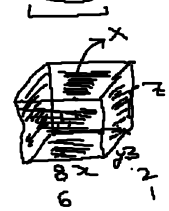
Surface area $= f(x,y,z) = [2yz + 2xz + xy]$

By Lagrange's method

$\therefore$ Auxiliary function =

$= F(x,y,z) = (xy + 2yz + 2zx) + \lambda[xyz - c]$

Volume → $24c$ (unit)^3



$\frac{\partial F}{\partial x} = (y\cdot1 + 0 + 2z\cdot1) + \lambda[yz\cdot1] = 0$ —(ii)

$\frac{\partial F}{\partial y} = (x + 2z + 0) + \lambda[xz\cdot1] = 0$ —(iii)

$\frac{\partial F}{\partial z} = (0 + 2y + 2x) + \lambda[xy\cdot1] = 0$ —(iv)

from (ii), $\lambda = \frac{-(y+2z)}{yz}$, from (iii) $\lambda = \frac{-(x+2z)}{xz}$

and (iv) $\lambda = \frac{-(2y+2x)}{xy}$

$\Rightarrow +\frac{(y+2z)}{yz} = +\frac{(x+2z)}{xz} = +\left(\frac{2y+2x}{xy}\right)$

From the equality of first two terms

$$\frac{y+2z}{yz} = \frac{x+2z}{xz}$$

$$\Rightarrow \cancel{xy} + 2zx = \cancel{xy} + 2zy \Rightarrow \cancel{2}\cancel{z}x = \cancel{2}\cancel{z}y$$

$$\Rightarrow \boxed{x = y}$$

||ly $\frac{x+2z}{\cancel{x}z} = \frac{2y+2x}{\cancel{x}y} \Rightarrow xy + 2zy = 2yz + 2zx$

$$\Rightarrow \cancel{x}y = 2z\cancel{x} \Rightarrow \boxed{y = 2z}$$

Now Consider, $xyz = C \Rightarrow y \cdot y \cdot (y/2) = C$

$$\Rightarrow y^3 = 2C \Rightarrow \boxed{y = (2C)^{1/3}}$$

$$\boxed{x = y = (2C)^{1/3}} \quad \& \quad \boxed{z = y/2 = \frac{(2C)^{1/3}}{2}}$$

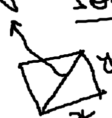
12/12/20 Q. Find the rectangle with constant perimeter whose diagonal is minimum

volume $= 24 = C$

$l = (48)^{1/3}$, $b = (48)^{1/3}$

$h = \frac{(48)^{1/3}}{2}$

$\sqrt{x^2+y^2}$



Sol → Let length of rectangle $= x$

width $= y$

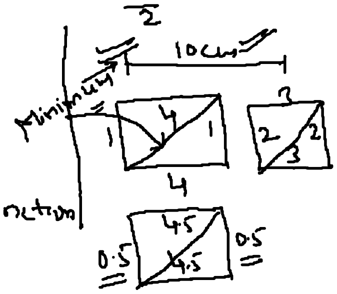
∴ $2x + 2y = 2C \Rightarrow x + y = C$ —(i)

Now $f(x,y) =$ Diagonal $= \sqrt{x^2+y^2}$

∴ By Lagrange method, so auxiliary function

$F(x,y) = \sqrt{x^2+y^2} + \lambda[x+y-C]$

[Do at your own]



OR : $f(x,y) = \sqrt{x^2+y^2} = \sqrt{x^2+(C-x)^2}$

∴ $g(x) = \sqrt{x^2+(C-x)^2}$

Do at your own

Q → Find the extreme value of $x^2+2xy+z^2$ under the constraints $2x+y=0$ & $x+y+z=1$

Sol → By Lagrange method, Consider the auxiliary function

$F(x,y,z) = x^2 + 2xy + z^2 + \lambda_1(2x+y) + \lambda_2(x+y+z-1)$

{Solve at your own}