



CSE408

Asymptotic notations

Lecture #4

Asymptotic Notations



- The efficiency analysis framework concentrates on the order of growth of an algorithm's basic operation count as the principal indicator of the algorithm's
- To compare and rank such orders of growth, computer scientists use three notations: (*big oh*), (*big omega*), and (*big theta*) efficiency

O-notation

DEFINITION A function $t(n)$ is said to be in $O(g(n))$, denoted $t(n) \in O(g(n))$, if $t(n)$ is bounded above by some constant multiple of $g(n)$ for all large n , i.e., if there exist some positive constant c and some nonnegative integer n_0 such that

$$t(n) \leq cg(n) \quad \text{for all } n \geq n_0.$$

Example



As an example, let us formally prove one of the assertions made in the introduction: $100n + 5 \in O(n^2)$. Indeed,

$$100n + 5 \leq 100n + n \text{ (for all } n \geq 5) = 101n \leq 101n^2.$$

Thus, as values of the constants c and n_0 required by the definition, we can take 101 and 5, respectively.

Note that the definition gives us a lot of freedom in choosing specific values for constants c and n_0 . For example, we could also reason that

$$100n + 5 \leq 100n + 5n \text{ (for all } n \geq 1) = 105n$$

to complete the proof with $c = 105$ and $n_0 = 1$.

Ω -notation

DEFINITION A function $t(n)$ is said to be in $\Omega(g(n))$, denoted $t(n) \in \Omega(g(n))$, if $t(n)$ is bounded below by some positive constant multiple of $g(n)$ for all large n , i.e., if there exist some positive constant c and some nonnegative integer n_0 such that

$$t(n) \geq cg(n) \quad \text{for all } n \geq n_0.$$

Example



Here is an example of the formal proof that $n^3 \in \Omega(n^2)$:

$$n^3 \geq n^2 \quad \text{for all } n \geq 0,$$

i.e., we can select $c = 1$ and $n_0 = 0$.

Θ -notation

DEFINITION A function $t(n)$ is said to be in $\Theta(g(n))$, denoted $t(n) \in \Theta(g(n))$, if $t(n)$ is bounded both above and below by some positive constant multiples of $g(n)$ for all large n , i.e., if there exist some positive constants c_1 and c_2 and some nonnegative integer n_0 such that

$$c_2g(n) \leq t(n) \leq c_1g(n) \quad \text{for all } n \geq n_0.$$

Example



For example, let us prove that $\frac{1}{2}n(n-1) \in \Theta(n^2)$. First, we prove the right inequality (the upper bound):

$$\frac{1}{2}n(n-1) = \frac{1}{2}n^2 - \frac{1}{2}n \leq \frac{1}{2}n^2 \quad \text{for all } n \geq 0.$$

Second, we prove the left inequality (the lower bound):

$$\frac{1}{2}n(n-1) = \frac{1}{2}n^2 - \frac{1}{2}n \geq \frac{1}{2}n^2 - \frac{1}{2}n \frac{1}{2}n \quad (\text{for all } n \geq 2) = \frac{1}{4}n^2.$$

Hence, we can select $c_2 = \frac{1}{4}$, $c_1 = \frac{1}{2}$, and $n_0 = 2$.

Asymptotic order of growth



A way of comparing functions that ignores constant factors and small input sizes

- $O(g(n))$: class of functions $f(n)$ that grow no faster than $g(n)$
- $\Theta(g(n))$: class of functions $f(n)$ that grow at same rate as $g(n)$
- $\Omega(g(n))$: class of functions $f(n)$ that grow at least as fast as $g(n)$

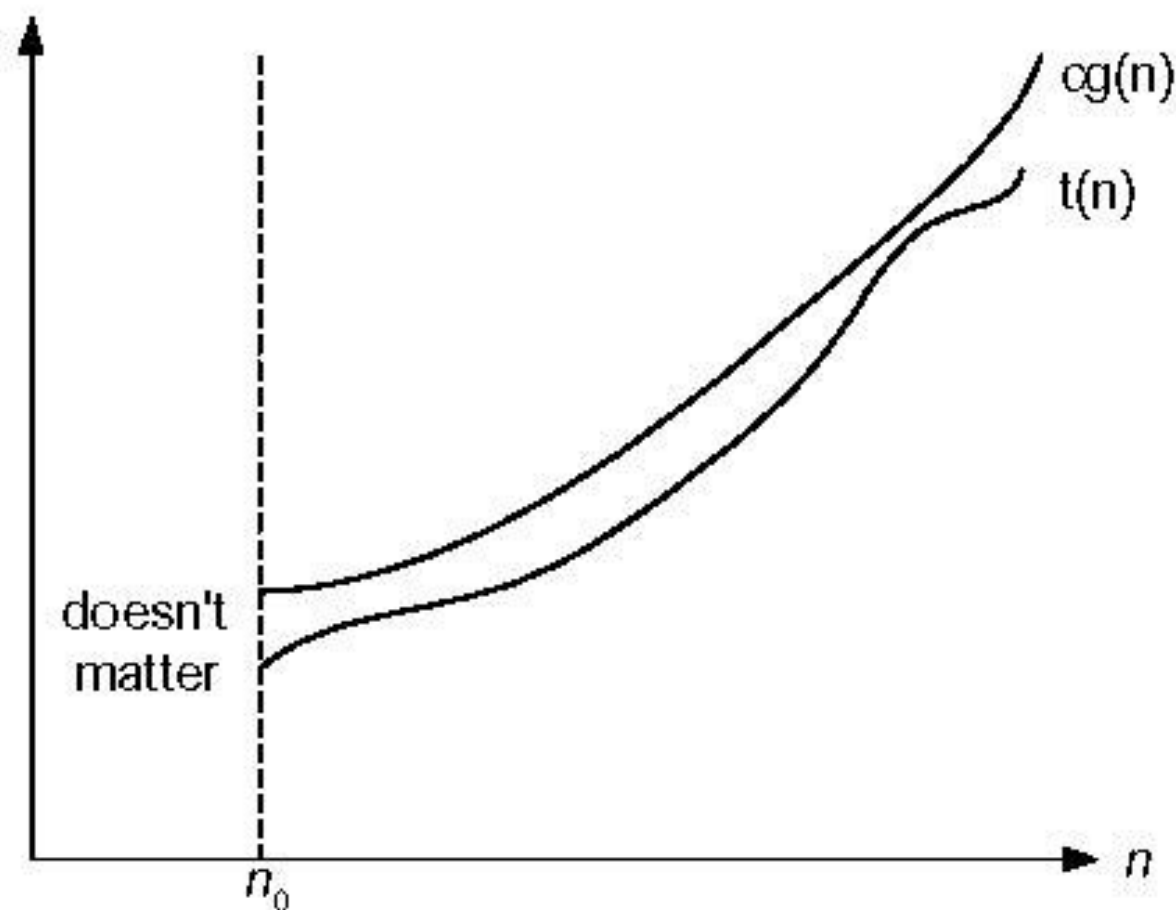


Figure 2.1 Big-oh notation: $t(n) \in O(g(n))$

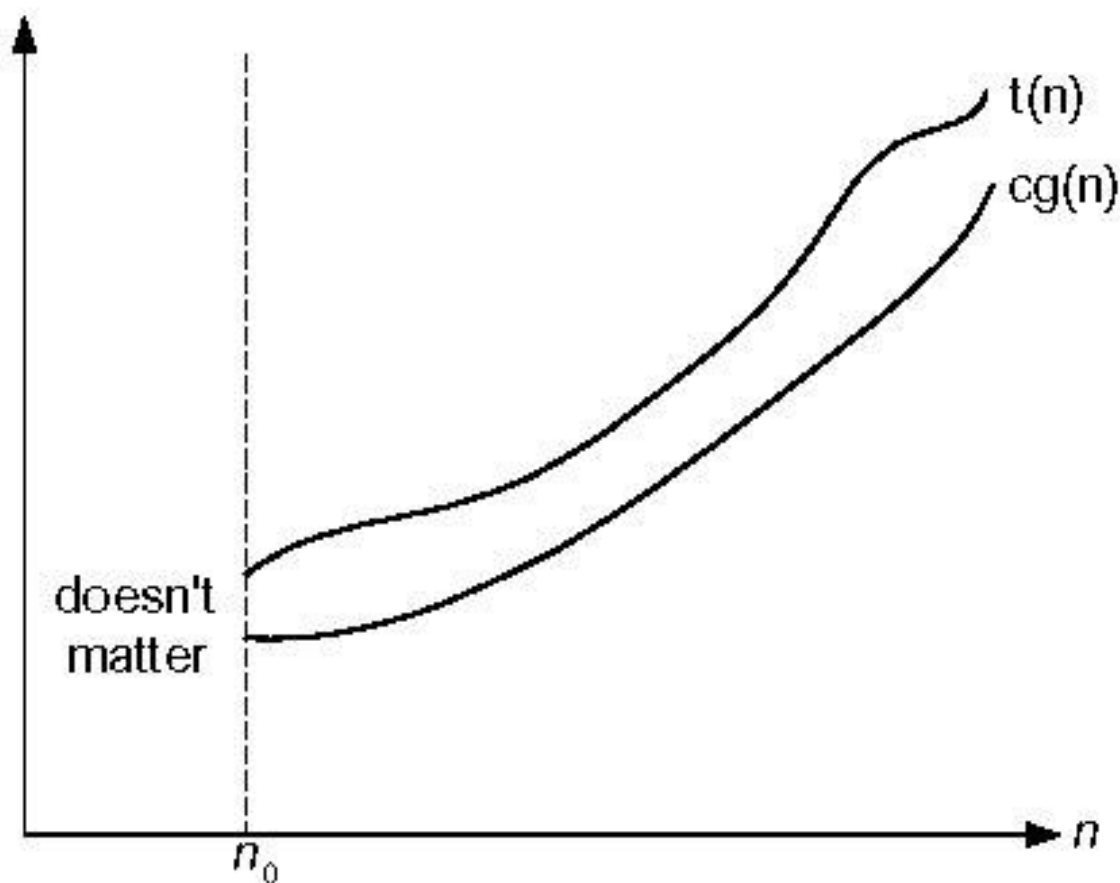


Fig. 2.2 Big-omega notation: $t(n) \in \Omega(g(n))$

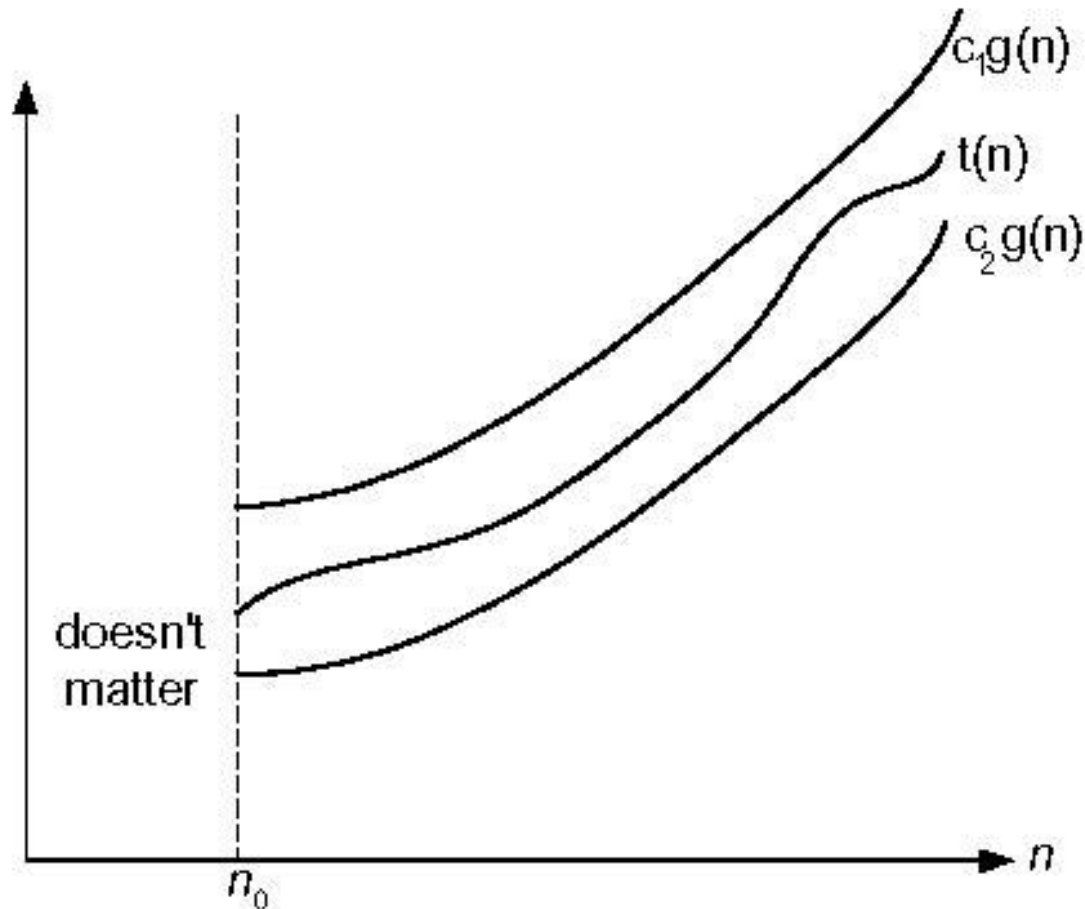


Figure 2.3 Big-theta notation: $t(n) \in \Theta(g(n))$

Some properties of asymptotic order of growth



- $f(n) \in O(f(n))$
- $f(n) \in O(g(n))$ iff $g(n) \in \Omega(f(n))$
- If $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$, then $f(n) \in O(h(n))$

Note similarity with $a \leq b$

- If $f_1(n) \in O(g_1(n))$ and $f_2(n) \in O(g_2(n))$, then
$$f_1(n) + f_2(n) \in O(\max\{g_1(n), g_2(n)\})$$

Establishing order of growth using limits



$$\lim_{n \rightarrow \infty} T(n)/g(n) = \begin{cases} 0 & T(n) \\ c > 0 & T(n) \\ \infty & T(n) \end{cases} \quad g(n)$$

Examples:

$$10n \quad \text{vs.} \quad n^2$$

$$n(n+1)/2 \quad \text{vs.} \quad n^2$$

L'Hôpital's rule and Stirling's formula



L'Hôpital's rule: If $\lim_{n \rightarrow \infty} f(n) = \lim_{n \rightarrow \infty} g(n) = \infty$ and the derivatives f' , g' exist, then

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)}$$

Example: $\log n$ vs. n

Stirling's formula: $n! \approx (2\pi n)^{1/2} (n/e)^n$

Example: 2^n vs. $n!$

EXAMPLE 1 Compare the orders of growth of $\frac{1}{2}n(n - 1)$ and n^2 . (This is one of the examples we used at the beginning of this section to illustrate the definitions.)

Example



$$\lim_{n \rightarrow \infty} \frac{\frac{1}{2}n(n-1)}{n^2} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{n^2 - n}{n^2} = \frac{1}{2} \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = \frac{1}{2}.$$

Since the limit is equal to a positive constant, the functions have the same order of growth or, symbolically, $\frac{1}{2}n(n-1) \in \Theta(n^2)$. ■

Example



EXAMPLE 2 Compare the orders of growth of $\log_2 n$ and \sqrt{n} . (Unlike Example 1, the answer here is not immediately obvious.)

Example



$$\lim_{n \rightarrow \infty} \frac{\log_2 n}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{(\log_2 n)'}{(\sqrt{n})'} = \lim_{n \rightarrow \infty} \frac{(\log_2 e) \frac{1}{n}}{\frac{1}{2\sqrt{n}}} = 2 \log_2 e \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0.$$

Since the limit is equal to zero, $\log_2 n$ has a smaller order of growth than \sqrt{n} . (Since $\lim_{n \rightarrow \infty} \frac{\log_2 n}{\sqrt{n}} = 0$, we can use the so-called **little-oh notation**: $\log_2 n \in o(\sqrt{n})$. Unlike the big-Oh, the little-oh notation is rarely used in analysis of algorithms.)



o-notation

The asymptotic upper bound provided by O -notation may or may not be asymptotically tight. The bound $2n^2 = O(n^2)$ is asymptotically tight, but the bound $2n = O(n^2)$ is not. We use o -notation to denote an upper bound that is not asymptotically tight. We formally define $o(g(n))$ (“little-oh of g of n ”) as the set

$o(g(n)) = \{f(n) : \text{for any positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \leq f(n) < cg(n) \text{ for all } n \geq n_0\} .$

For example, $2n = o(n^2)$, but $2n^2 \neq o(n^2)$.

The definitions of O -notation and o -notation are similar. The main difference is that in $f(n) = O(g(n))$, the bound $0 \leq f(n) \leq cg(n)$ holds for *some* constant $c > 0$, but in $f(n) = o(g(n))$, the bound $0 \leq f(n) < cg(n)$ holds for *all* constants $c > 0$. Intuitively, in the o -notation, the function $f(n)$ becomes insignificant relative to $g(n)$ as n approaches infinity; that is,

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0 . \tag{3.1}$$

EXAMPLE 3 Compare the orders of growth of $n!$ and 2^n .

Orders of growth of some important functions



- All logarithmic functions $\log_a n$ belong to the same class $\Theta(\log n)$ no matter what the logarithm's base $a > 1$ is
- All polynomials of the same degree k belong to the same class: $a_k n^k + a_{k-1} n^{k-1} + \dots + a_0 \in \Theta(n^k)$
- Exponential functions a^n have different orders of growth for different a 's
- $\text{order } \log n < \text{order } n^\alpha \ (\alpha > 0) < \text{order } a^n < \text{order } n! < \text{order } n^n$

Basic asymptotic efficiency classes



1	constant
$\log n$	logarithmic
n	linear
$n \log n$	n -log- n
n^2	quadratic
n^3	cubic
2^n	exponential
$n!$	factorial



Thank You !!!