Limit of a Function

Let $y = f(x)$ be a function of x and let 'a' be any real number.

We must first understand what a 'limit' is. A limit is the value, function approaches, as the variable within that function (usually 'x') gets nearer and nearer to a particular value. In other words, when x is very close to a certain number, what is $f(x)$ very close to?

Meaning of '$x \to a$'

Let x be a variable and 'a' be a constant. If x assumes values nearer and nearer to 'a', then we say that 'x tends to a' or 'x approaches a' and is written as '$x \to a$'. By $x \to a$, we mean that $x \neq a$ and x may approach 'a' from left or right, which is explained in the example given below.

Let us look at an example of a limit: What is the limit of the function $f(x) = x^3$ as x approaches 2? The expression 'the limit as x approaches to 2' is written as: $\lim\limits_{x \to 2}$ Let us check out some values of $\lim\limits_{x \to 2}$ as x increases and gets closer to 2, without even exactly getting there.

When $x = 1.9$, $f(x) = 6.859$
When $x = 1.99$, $f(x) = 7.88$
When $x = 1.999$, $f(x) = 7.988$
When $x = 1.9999$, $f(x) = 7.9988$

As x increases and approaches 2, $f(x)$ gets closer and closer to 8 and since x tends to 2 from left this is called 'left-hand limit' and is written as $\lim\limits_{x \to 2^-}$.

Now, let us see what happens when x is greater than 2.

When $x = 2.1$, $f(x) = 9.261$
When $x = 2.01$, $f(x) = 8.12$
When $x = 2.001$, $f(x) = 8.01$
When $x = 2.0001$, $f(x) = 8.001$

As x decreases and approaches 2, $f(x)$ still approaches 8. This is called 'right-hand limit' and is written as $\lim\limits_{x \to 2^+}$

$\overrightarrow{x}$ 2 2 $\overleftarrow{x}$

We get the same answer while finding both, left and right hand limits. Hence we write that $\lim\limits_{x \to 2} x^3 = 8$.

Meaning of the Symbol: $\lim\limits_{x \to a} f(x) = l$

Let $f(x)$ be a function of x where x takes values closer and closer to 'a' ($\neq a$), then $f(x)$ will assume values nearer and nearer to l. Hence we say, $f(x)$ tends to the limit 'l' as x tends to a.

The following are some of the simple algebraic rules of limits.

1. $\lim\limits_{x \to a} kf(x) = k \lim\limits_{x \to a} f(x)$
2. $\lim\limits_{x \to a}[f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$
3. $\lim\limits_{x \to a}[f(x) \cdot g(x)] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$
4. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{x \to a}{\lim\limits_{x \to a} g(x)} (\lim\limits_{x \to a} g(x) \neq 0)$

NOTES

1. If the left hand limit of a function is not equal to the right hand limit of the function, then the limit does not exist.
2. A limit equal to infinity is not the same as a limit that does not exist.

Continuous Functions

Let $f: A \to B$ be any given function and let $c \in A$. We say f is continuous at c, if given $\in > 0$, there exists $\delta > 0$ such that $|f(x) - f(c)| < \in$ whenever $|x - c| < \delta$

In words, this means that, if x is very close to c in domain, then $f(x)$ is very close to $f(c)$ in range.

Equivalently f is continuous at c. If $\lim_{x\to c} f(x) = f(c)$

We observe

1. $c \in A$, i.e., $f(c)$ must exist
2. $\lim_{x\to c} f(x)$ exists
3. $f(c)$ and $\lim_{x\to c} f(x)$ are equal.

If any of these three conditions fail, then f is discontinuous at $x = c$.

Algebra of Continuous Functions

If f, g be two continuous functions at c, then $f + g$, $f - g$, fg are also continuous at $x = c$.

To solve a problem of continuous functions at a point a, you can take the following approach.

1. Find the value $f(x)$ at $x = a$. If a is in the domain of f, $f(a)$ must exist. If a is not in the domain, then $f(a)$ does not exist. In such a case, f is not continuous at $x = a$.
2. Find $\lim_{x\to a} f(x)$. For this you have to first find $\lim_{x\to \infty} f(x) = l_1$(say) and $\lim_{x\to a^+} f(x) = l_2$(say). If $l_1 \neq l_2$ then $\lim_{x\to a} f(x)$ does not exist and so f is not continuous at $x = a$. If $l_1 = l_2$, then $\lim_{x\to a} f(x)$ exists.
3. If $\lim_{x\to a} f(x)$ exists and also $f(a)$ exists.

Then verify whether $\lim_{x\to a} f(x) = f(a)$.

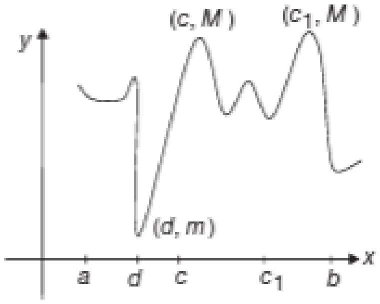
If $\lim_{x\to a} f(x) = f(a)$. Then f is continuous, otherwise it is not continuous at $x = a$.

Problems on continuous functions can be grouped into the following categories.

1. Using $\in$, δ notation.
2. Using existence of right and left hand limits.
3. To find the value of the unknown in $f(x)$ when f is given to be continuous at a point.
4. To find $f(a)$ when f is given to be continuous at $x = a$.

For functions that are continuous on (a, b) the following holds:

f is bounded and attains its bounds at least once on $[a, b]$, i.e., for some $c, d \in [a, b]$,
M = supremum of $f = f(c)$ and m = Infimum of $f = f(d)$



NOTE

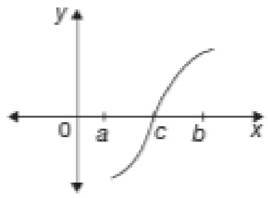
The converse may not be true as $f(x) = \begin{cases} 1; & 0 < x \le 1 \\ -1; & 1 < x \le 2 \end{cases}$ is bounded on [1, 2] but it is not continuous at $x = 1$.

Intermediate-value Theorem

If f is continuous on $[a, b]$ and $f(a) \neq f(b)$ then f takes every value between $f(a)$ and $f(b)$.

Equivalently, if f is continuous on $[a, b]$ and $f(a) < k < f(b)$ or $f(b) < k < f(a)$, then there exists $c \in (a, b)$ such that $f(c) = k$.

Equivalently, If $f(a)$ and $f(b)$ are of opposite signs then there exists $c \in (a, b)$ such that $f(c) = 0$.



$f(a) < 0$ and $f(b) > 0$, clearly $f(c) = 0$.

NOTES

1. If $f(x)$ is continuous in $[a, b]$ then f takes all values between m and M at least once as x moves from a to b, where M = Supremum of f on $[a, b]$ and m = infimum of f on $[a, b]$.
2. If $f(x)$ is continuous in $[a, b]$, then $|f|$ is also continuous on $[a, b]$, where $|f|(x) = |f(x)|\, x \in [a, b]$.
3. Converse may not be true

For instance, $f(x) = \begin{cases} 1; & 0 < x \le 3 \\ -1; & 3 < x \le 5 \end{cases}$

is not continuous at $x = 3$, but $|f|(x) = 1 x \in [0, 5]$, being a constant function is continuous [0, 5].

Inverse-function Theorem

If f is a continuous one-to-one function on $[a, b]$ then f^{-1} is also continuous on $[a, b]$.

Uniform Continuity A function f defined on an interval I is said to be uniformly continuous on I if given $\in > 0$ there exists a $\delta > 0$ such that if x, y are in I and $|x - y| < \delta$ then $|f(x) - f(y)| < \in$.

NOTE

Continuity on $[a, b]$ implies uniform continuity whereas continuity on (a, b) does not mean uniform continuity.

Types of Discontinuity If f is a function defined on an interval I, it is said to have

(TD1) **a removable discontinuity** at $p \in I$, if $\lim_{x\to p} f(x)$ exists, but is not equal to $f(p)$.

(TD2) **a discontinuity of first kind from the left** at p if $\lim_{x\to p^-} f(x)$ exists but is not equal to $f(p)$.

(TD3) **a discontinuity of first kind from the right** at p if $\lim_{x\to p^+} f(x)$ exists but is not equal to $f(p)$.

(TD4) **a discontinuity of first kind** at p if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to p^+} f(x)$ exists but they are unequal.

(TD5) **a discontinuity of second kind from the left** at p if $\lim_{x\to p-} f(x)$ does not exist.

(TD6) **a discontinuity of second kind from the right** at p if $\lim_{x\to p^+} f(x)$ does not exist.

(TD7) **a discontinuity of second kind** at p if neither $\lim_{x\to p^-} f(x)$ nor $\lim_{x\to p^+} f(x)$ exist.

Examples for each type are presented in the following table:

Type	Example	Point of Discontinuity
TD1	$f(x) = \frac{x^2-1}{x-1}, x \neq$ $f(1) = 3$	$x = i$
TD2	$f(x) = x + 3$ for $0 < x < 1$ $f(x) = 5$ for $x \geq 1$	$x = 1$
TD3	$f(x) = x + 3$, for $x > 2$ $f(x) = 8$ for $x \leq 2$	$x = 2$
TD4	$f(x) = \begin{cases} x+3; & x > 2 \\ 7; & x = 2 \\ x-3; & x < 2 \end{cases}$	$x = 2$
TD5	$f(x) = \tan x$ for $x < \pi/2$ $f(x) = 1$, for $x \geq \pi/2$	$x = \frac{\pi}{2}$
TD6	$f(x) = 1$, for $x \leq \pi/2$ $f(x) = \tan x$ for $x > \pi/2$	$x = \frac{\pi}{2}$
TD7	$f(x) = 1/x$ at $x \neq 0$ $f(0)$ $= 3$ at $x = 0$	$x = 0$

NOTES

1. Every differentiable function is continuous, but the converse is not true.
The example of a function which is continuous but not differentiable at a point $f(x) = |x - 3|$ for $x \in R$ is continuous at $x = 3$, but it is not differentiable at $x = 3$.

2. The function may have a derivative at a point, but the derivative may not be continuous.
For example the function

$$f(x) = \begin{cases} x^3 \sin\frac{1}{x}; & x \neq 0 \\ 0; & x = 0 \end{cases}$$ has the derivative function

as

$$f'(x) = \begin{cases} 3x^2 \sin\frac{1}{x} - x\cos\frac{1}{x}; & x \neq 0 \\ 0; & x = 0 \end{cases}$$

However $\lim_{x\to 0} f'(x)$ doesn't exist.

SOLVED EXAMPLES

Example 1

Discuss the continuity of the function at $x = 1$ where $f(x)$ is defined by

$$f(x) = \frac{3x-2}{x} \text{ for } 0 < x \leq 1$$

$$= \frac{\sin(x-1)}{(x-1)} \text{ for } x > 1$$

Solution

Consider the left and right handed limits

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} \frac{3x-2}{x} = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} \frac{\sin(x-1)}{x-1}$$

$$= \lim_{(x-1)\to 0} \frac{\sin(x-1)}{(x-1)} = 1 \text{ and } f(1)$$

$$= \frac{3(1)-2}{1} = 1$$

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$\therefore f$ is continuous at $x = 1$.

Example 2

If $f(x) = \dfrac{(2^x-1)^2}{(\sin 2x)\log(1+x)}$ for $x \neq 0$ and $f(x) = \log 2$ for $x = 0$, discuss the continuity at $x = 0$.

Solution

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{(2^x-1)}{(\sin 2x)\log(1+x)}$$

$$= \lim_{x\to 0}\frac{\left(\frac{2^x-1}{x}\right)^2}{\frac{\sin 2x}{2x}(2)\frac{\log(1+x)}{x}}$$

$$= \lim_{x\to 0}\frac{\left(\frac{2^x-1}{x}\right)^2}{2\left(\frac{\sin 2x}{2x}\right)\log(1+x)^{\frac{1}{x}}}$$

$$= \frac{1}{2}\frac{\lim_{x\to 0}\left(\frac{2^x-1}{x}\right)^2}{\left(\lim_{x\to 0}\frac{\sin 2x}{2x}\right)\left(\log\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right)}$$

$$= \frac{1}{2}(\log 2)^2.$$

But given $f(x) = 2\log 2$ at $x = 0$

$\therefore \lim_{x\to 0} f(x) \neq f(0)$

$\therefore f(x)$ is not continuous at $x = 0$.

Example 3

Find the value of k if

$$f(x) = \frac{2x^3-5x^2+4x+11}{x+1},\ \text{for } x \neq -1$$

And $f(-1) = k$ is continuous at $x = -1$.

Solution

Given $f(x)$ is continuous at $x = -1$

$$\Rightarrow \lim_{x\to -1} f(x) = f(-1) = k.$$

$$\Rightarrow \lim_{x\to -1} f(x) \lim_{x\to -1}\left[\frac{2x^3-5x^2+4x+11}{x+1}\right]$$

$$= \lim_{x\to -1}\frac{(x+1)(2x^2-7x+11)}{x+1}$$

$$= 2(-1)^2 - 7(-1) + 11$$

$$= 2 + 7 + 11 = 20$$

$$\therefore k = 20$$

Example 4

If $f(x) = \frac{x-4}{|x-4|}+a$, for $x < 4$, $= a+b$ for $x = 4$, $= \frac{x-4}{|x-4|}+b$, for $x > 4$

And $f(x)$ is continuous at $x = 4$, then find the values of a and b.

Solution

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}\frac{x-4}{|x-4|}+a$$

$$= \lim_{x\to 4^-}\frac{(x-4)}{-(x-4)}+a = -1+a$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}\frac{x-4}{|x-4|}+b$$

$$= \lim_{x\to 4^+}\frac{x-4}{(x-4)}+b = 1+b$$

Since given $f(x)$ is continuous at $x = 4$

$$\lim_{x\to 4^-} f(x) = f(4) = \lim_{x\to 4^+} f(x)$$

$\Rightarrow -1 + a = a + b = 1 + b \Rightarrow a = 1, b = -1$

Example 5

Examine the continuity of the given function at origin where,

$$f(x) = \begin{cases} \dfrac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Solution

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} = 0$$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0^+}\frac{x}{e^{-1/x}+1} = 0$$

Then,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x) = 0$$

Thus the function is continuous at the origin.

Derivatives

In this section we will look at the simplistic form of the definition of a derivative, the derivatives of certain standard functions and application of derivatives.

For a function $f(x)$, the ratio $\frac{[f(a+h)-f(a)]}{h}$ is the rate of change of $f(x)$ in the interval $[a, (a+h)]$.

The limit of this ratio as h tends to zero is called the derivative of $f(x)$. This is represented as $f'(x)$, i.e.,

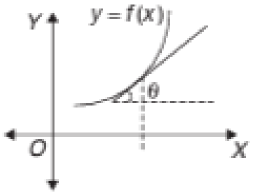
$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = f'(x)$$

The derivative $f'(x)$ is also represented as $\frac{d\{f(x)\}}{dx}$ or $\frac{d}{dx}\{f(x)\}$

Hence, if $y = f(x)$, i.e., y is a function of x, then $\frac{dy}{dx}$ is the derivative of y with respect to x.

NOTES

1. $\frac{dy}{dx}$ is the rate of change of y with respect to x.
2. If the function y can be represented as a general curve, and a tangent is drawn at any point where the tangent makes an angle θ with the horizontal (as shown in the figure), then $\frac{dy}{dx} = \tan\theta$, In other words, derivative of a function at a given point is the slope of the curve at that point, i.e., tans of the angle, the tangent drawn to the curve at that point, makes with the horizontal.



Standard Results

If $f(x)$ and $g(x)$ are two functions of x and k is a constant, then

1. $\frac{d}{dx}(c) = 0$ (c is a constant)
2. $\frac{d}{dx} k \cdot f(x) = k \frac{d}{dx} f(x)$ (k is a constant)
3. $\frac{d}{dx}(f(x) \pm g(x))$
   $= \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

Product Rule

4. $\frac{d}{dx}\{f(x) \cdot g(x)\} = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Quotient Rule

5. $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$

Chain Rule

6. If $y = f(u)$ and $u = g(x)$ be two functions, then $\frac{dy}{dx} = \left(\frac{dy}{du}\right) \times \left(\frac{du}{dx}\right)$

Derivatives of Some Important Functions

1. (a) $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$

   (b) $\frac{d}{dx}\left[\frac{1}{x^n}\right] = \frac{-n}{x^{n+1}}$

   (c) $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}; \ x \neq 0$
2. $\frac{d}{dx}[ax^n + b] = an \cdot x^{n-1}$
3. $\frac{d}{dx}[ax + b]^n = n\, a\, (ax + b)^{n-1}$
4. $\frac{d}{dx}[e^{ax}] = a \cdot e^{ax}$
5. $\frac{d}{dx}[\log x] = \frac{1}{x}; x > 0$
6. $\frac{d}{dx}[a^x] = a^x \log a; a > 0$
7. (a) $\frac{d}{dx}[\sin x] = \cos x$

   (b) $\frac{d}{dx}[\cos x] = -\sin x$

   (c) $\frac{d}{dx}[\tan x] = \sec^2 x$

   (d) $\frac{d}{dx}[\cot x] = -\mathrm{cosec}^2 x$

   (e) $\frac{d}{dx}[\sec x] = \sec x \cdot \tan x$

   (f) $\frac{d}{dx}[\mathrm{cosec}\, x] = -\mathrm{cosec}\, x \cdot \cot x$

Inverse Rule

If $y = f(x)$ and its inverse $x = f^{-1}(y)$ is also defined, then $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

Second Derivative

If $y = f(x)$, then the derivative of derivative of y is called as second derivative of y and is represented by $\frac{d^2y}{dx^2}$.

$\frac{d^2y}{dx^2} = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ where $\frac{dy}{dx}$ is the first derivative of y.

8. (a) $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

   (b) $\frac{d}{dx}\mathrm{cosec}^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}$

   (c) $\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$

   (d) $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$

(e) $\frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2}$

(f) $\frac{d}{dx}\cot^{-1}x=\frac{-1}{1+x^2}$

9. (a) $\frac{d}{dx}\sinh x=\cosh x$

(b) $\frac{d}{dx}\cosh x=\sinh x$

(c) $\frac{d}{dx}\tanh x=\text{sech}^2 x$

(d) $\frac{d}{dx}\coth x=-\text{cosech}^2 x$

(e) $\frac{d}{dx}\text{sech}\, x=-\text{sech}\, x\tanh x$

(f) $\frac{d}{dx}\text{cosech} x=-\text{cosech} x\coth x$

10. (a) $\frac{d}{dx}\sinh^{-1}x=\frac{1}{\sqrt{1+x^2}}$

(b) $\frac{d}{dx}\cosh^{-1}x=\frac{1}{\sqrt{x^2-1}}$

(c) $\frac{d}{dx}\tanh^{-1}x=\frac{1}{1-x^2}$

(d) $\frac{d}{dx}\coth^{-1}x=\frac{-1}{x^2-1}$

(e) $\frac{d}{dx}\text{sech}^{-1}x=\frac{-1}{x\sqrt{1-x^2}}$

(f) $\frac{d}{dx}\text{cosech}^{-1}x=\frac{-1}{x\sqrt{x^2+1}}$

Successive Differentiation

If f is differentiable function of x and the derivative f' is also a differentiable function of x, then f'' is called the second derivative of f. Similarly 3rd, 4th ... nth derivative of f may be defined and are denoted by f''', f'''', ... f^n or $y_3, y_4 \cdots y_n$.

11. The n^{th} derivatives of some special functions:

(a) $\frac{d^n}{dx^n}x^n=n\,!$

(b) $\frac{d^n}{dx^n}x^m=\frac{m!}{(m-n)}x^{m-n}$ s(m being a positive integer more than n)

(c) $\frac{d^n}{dx^n}e^{ax}=a^n e^{ax}$

(d) $\frac{d^n}{dx^n}\left(\frac{1}{x-a}\right)=\frac{(-1)^n n!}{(x+a)^{n+1}};x\neq -a$

(e) $\frac{d^n}{dx^n}\log(x+a)=\frac{(-1)^{n-1}(n-1)!}{(x+a)}$; $(x+a)>0$

(f) $\frac{d^n}{dx^n}\sin(ax+b)=a^n\sin\left(\frac{n\pi}{2}+ax+b\right)$

(g) $\frac{d^n}{dx^n}\cos(ax+b)=a^n\cos\left(\frac{n\pi}{2}+ax+b\right)$

(h) $\frac{d^n}{dx^n}(e^{ax}\sin bx)$

$=(a^2+b^2)^{n/2}e^{ax}\sin\left(bx+n\tan^{-1}\frac{b}{a}\right)$

(i) $\frac{d^n}{dx^n}(e^{ax}\cos bx)$

$=(a^2+b^2)^{n/2}e^{ax}\cos\left(bx+n\tan^{-1}\frac{b}{a}\right)$

(j) $\frac{d^n}{dx^n}\left(\frac{1}{x^2+a^2}\right)=\frac{(-1)^n n}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta$

where $\theta=\tan^{-1}\left(\frac{x}{a}\right)$

(k) $\frac{d^n}{dx^n}(\tan^{-1}x)=(-1)^{n-1}(n-1)!\,\sin^n\theta\cdot\sin n\theta$

where $\theta=\cot^{-1}x$.

Application of Derivatives

Errors in Measurement

Problems relating to errors in measurement can be solved using the concept of derivatives. For example, if we know the error in measurement of the radius of a sphere, we can find out the consequent error in the measurement of the volume of the sphere. Without going into further details of theory, we can say dx = error in measurement of x and dy = consequent error in measurement of y, Where $y = f(x)$. Hence, we can rewrite $\frac{dy}{dx}=f'(x)$ as $dy=f'(x)\cdot dx$.

Thus, if we know the function $y = f(x)$ and dx, error in measurement of x, we can find out dy, the error in measurement of y.

NOTES

1. An error is taken to be positive when the measured value is greater than the actual value and negative when it is less.
2. Percentage error in y is given by $\left(\frac{dy}{y}\right)\times 100$.

Rate of Change

While defining the derivative, we have seen that derivative is the 'rate of change'. This can be applied to motion of bodies to determine their velocity and acceleration.

Velocity If we have s, the distance covered by a body expressed as a function of t, i.e., $s = f(t)$, then rate of change of s is called velocity (v). $v=\frac{ds}{dt}=f'(t)$.

Acceleration Rate of change of velocity is defined as acceleration. Since $v = f'(t)$ itself is a function of t, we can write $v = f'(t)$.

$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$, i.e., acceleration is the second derivative of the function $s = f(t)$.

Maxima and Minima

A function takes a maximum value or a minimum value when the slope of the tangent of the curve at that point is zero, i.e., when the first derivative of the function is zero. If $y = f(x)$, then y is maximum or minimum at the point $x = x_1$ if $\left(\frac{dy}{dx}\right)_{x=x_1} = 0$.

Thus we can find the value of x_1 by equating $\frac{dy}{dx} = 0$.

As mentioned above that y can have a maximum or a minimum value at $x = x_1$. Whether y is a maximum value or minimum is governed by the sign of the second derivative. The function y has a minimum value if the second derivative is positive. In other words, y is maximum at $x = x_1$ if $\frac{d^2y}{dx^2} < 0$ at $x = x_1 \cdot y$ is minimum at $x = x_1$ if $\frac{d^2y}{dx^2} > 0$ at $x = x_1$. $\left(\frac{dy}{dx}\right)_{x=x_1} = 0$. in both the cases discussed above.

The above discussion can be summerized as follows:

1. If $f'(c) = 0$ and $f''(c)$ is negative, then $f(x)$ is maximum for $x = c$
2. If $f'(c) = 0$ and $f''(c)$ is positive, then $f(x)$ is minimum for $x = c$
3. If $f'(c) = f''(c) = \cdots = f^{r-1}(c) = 0$ and $f^r(c) \neq 0$, then
   (a) If r is even, then $f(x)$ is maximum or minimum for $x = c$ according as $f^r(c)$ is negative or positive.
   (b) If r is odd, then there is neither maximum nor a minimum for $f(x)$ at $x = c$.

Mean Value Theorems

Rolle's Theorem Let f be a function defined on $[a, b]$ such that

1. f is continuous on $[a, b]$;
2. f is differentiable on (a, b) and
3. $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$

Lagrange's Mean Value Theorem Let f be a function defined on $[a, b]$ such that

1. f is continuous on $[a, b]$,
2. f is differentiable on (a, b) then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Another Form If f is defined on $[a, a + h]$ such that

1. f is continuous on $[a, a + h]$.
2. f is differentiable on $(a, a + h)$ then there exists atleast one $\theta \in (0, 1)$ such that $f(a + h) = f(a) + hf'(a + \theta h)$.

Meaning of the sign of the derivative

SIGN OF $f'(x)$ on [a, b]	Meaning
$f'(x) \geq 0$	f is non-decreasing
$f'(x) > 0$	f is increasing
$f'(x) < 0$	f is non-increasing
$f'(x) < 0$	f is decreasing
$f'(x) = 0$	f is constant

Example: The function f, defined on R by $f(x) = x^3 - 15x^2 + 75x - 125$ is non-decreasing in every interval as $f'(x) = 3(x^2 - 10x + 15) = 3(x - 5)^2 \geq 0$

Thus f is non-decreasing on R.

Cauchy's Mean Value Theorem Let f and g be two functions defined on $[a, b]$ such that

1. f and g are continuous on $[a, b]$
2. f and g are differentiable on (a, b)
3. $g'(x) \neq 0$ for any $x \in (a, b)$ then there exists at least one real number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Taylor's Theorem

Let f be *a* real-valued function defined on $[a, a + h]$ such that

1. f^{n-1} is continuous on $[a, a + h]$
2. f^{n-1} is derivable on $(a, a + h)$, then there exists a number $\theta \in (0, 1)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots$$

$$+ \frac{h^{n-1}}{(n-1)} f^{n-1}(a) + R_n.$$

Where

$$R_n = \frac{h^n f^n(a + \theta h)}{n!}$$

(Lagranges' form of remainder)

$$R_n = \frac{h^n (1-\theta)^{n-1} f^n(a+\theta h)}{(n-1)!}$$

(Cauchy's form of remainder)

Maclaurin's Theorem Let $f \cdot [0, x] \to R$ such that

1. f^{n-1} is continuous on $[0, x]$,
2. f^{n-1} is derivable on $(0, x)$

Then there exists a real number $\theta \in (0, 1)$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{n!} f''(0) + \cdots$$

$$+ \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n.$$

Where

$$R_n = \frac{x^n}{n!} f^n(\theta x)$$

(Lagranges form of remainder)

$$R_n = \frac{x^n (1-\theta)^{n-1} f^n(\theta x)}{(n-1)!}$$

(Cauchy's form of remainder)

Maclaurin's Series Let $f(x)$ be a function which posses derivatives of all orders in the interval $[0, x]$, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)} f^{(n-1)}(0)$$

$$+ \frac{x^n}{n!} f^n(0) + \cdots \text{ is known as}$$

Maclaurin's infinite series.

Series expansions of some standard functions

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$
2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$
3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$
4. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$
5. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$
6. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \cdots + \frac{(-1)^{n-1} x^n}{n}$
7. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
8. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
9. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$
10. $(1-x)^{-\frac{1}{2}} = 1 + \frac{x}{2} + \frac{1 \cdot 3}{2 \cdot 3} x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot x^3 + \cdots$
11. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^{n-1}}{(2n-1)} x^{2n-1} + \cdots$
12. $\sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \cdots$

Example 6

For the function $f(x) = x(x^2 - 1)$ test for the applicability of Rolle's theorem in the interval $[-1, 1]$ and hence find c such that $-1 < c < 1$.

Solution

Given $f(x) = x(x^2 - 1)$

1. f is continuous in $[-1, 1]$
2. f is differentiable in $(-1, 1)$
3. $f(-1) = f(1) = 0$

$\therefore$ $f(x)$ satisfies the hypothesis of Rolle's theorems
$\therefore$ We can find a number c such that $f'(c) = 0$, i.e., $f'(x) = 3x^2 - 1$

$$f'(c) = 0 \;\Rightarrow\; 3c^2 - 1 = 0 \;\Rightarrow\; c = \pm\sqrt{\frac{1}{3}}$$

$$\Rightarrow\; c = \sqrt{\frac{1}{3}}$$

Example 7

If $f(x) = 2x^2 + 3x + 4$, then find the value of θ in the mean value theorem.

Solution

$f(a) = 2a^2 + 3a + 4$
$f(a+h) = 2(a^2 + 2ah + h^2) + 3a + 3h + 4$
$f(a+h) - f(a) = 4ah + 2h^2 + 3h = 2(2ah + h^2) + 3h$

$$\frac{f(a+h) - f(a)}{h} = 2(2a+h) + 3$$

$$= 4\left(a + \frac{h}{2}\right) + 3 \quad (1)$$

Now $f'(x) = 4x + 3$, $f^1(a + \theta h)$
$= 4a + 4h\theta + 3$ (2)

Comparing Eqs. (1) and (2) we have $4\left(a + \frac{h}{2}\right) + 3$

$$= 4a + 4h\theta + 3 \;\Rightarrow\; a + h\theta = a + \frac{h}{2}$$

$$\Rightarrow\; \theta = \frac{1}{2}$$

Partial Differentiation

Let u be a function of two variables x and y. Let us assume the functional relation as $u = f(x, y)$. Here x alone or y alone or both x and y simultaneously may be varied and in each case a change in the value of u will result. Generally the change in the value of u will be different in each of these three cases. Since x and y are independent, x may be supposed to vary when y remains constant or the reverse.

The derivative of u wrt x when x varies and y remains constant is called the partial derivative of u wrt x and is denoted by $\frac{\partial u}{\partial x}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right), \ \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right).$$

Total Differential Co-efficient

If u be a continuous function of x and y and if x and y receive small increments Δx and Δy, u will receive, in turn, a small increment Δu. This Δu is called total increment of u.

$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y)$

In the differential form, this can be written as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

du is called the total differential of u. If $u = f(x, y, z)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \cdot\frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

Implicit Function

If the relation between x and y be given in the form $f(x, y) = c$ where c is a constant, then the total differential co-efficient wrt x is zero.

Homogeneous Functions

Let us consider the function $f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_ny^n$. In this expression the sum of the indices of the variable x and y in each term is n. Such an expression is called a homogeneous function of degree n.

Euler's Theorem

If $f(x, y)$ is a homogeneous function of degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.

This is known as Euler's theorem on homogeneous function.

Maxima and Minima for Function of Two Variables

A function $f(x, y)$ is said to have a local maximum at a point (a, b), if $f(a + h, b + k) \le f(a, b)$ for all small values of h and k, i.e., $f(x, y)$ has a local maximum at (a, b), if $f(a, b)$ has a highest value in a neighbourhood of (a, b).

Similarly, $f(x, y)$ is said to have a local minimum at a point (a, b), if $f(x, y)$ has least value at (a, b) in a neighbourhood of (a, b).

Procedure to Obtain Maxima and Minima

Let $f(x, y)$ be a function of two variables for which we need to find maxima and minima.

1. Find $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$
2. Take $f_x = 0$ and $f_y = 0$ and solve them as simultaneous equations to get pairs of values for x and y, which are called stationary points.
3. Find $r = f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $s = f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$ and

   $t = f_{yy} = \frac{\partial^2 f}{\partial y^2}$ and find $rt - s^2$.
4. At a stationary point, say (a, b)
   (a) If $rt - s^2 > 0$, then (a, b) is called an extreme point of $f(x, y)$ at which $f(x, y)$ has either maximum or minimum which can be found as follows.

   Case 1: If $r < 0$, then $f(x, y)$ has a local maximum at $a, b)$

   Case 2: If $r > 0$, then $f(x, y)$ has a local minimum at (a, b).

   (b) If $rt - s^2 < 0$, then (a, b) is called as saddle point of $f(x, y)$ where $f(x, y)$ has neither maximum nor minimum at (a, b).

Example 8

Find the stationary points of the function $f(x, y) = x^2y + 3xy - 7$ and classify them into extreme and saddle points.

Solution

Given $f(x, y) = x^2y + 3xy - 7$

$\therefore f_x = \frac{\partial f}{\partial x} = 2xy + 3y$ and $f_y = \frac{\partial f}{\partial y} = x^2 + 3x$

Now $f_x = 0 \Rightarrow 2xy + 3y = 0$ and $f_y = 0$

$\Rightarrow x^2 + 3x = 0$

$\Rightarrow \quad y = 0$ and $x = \frac{-3}{2}$; $x(x+3)x = 0$ and $x = -3$

But for $x = \frac{3}{2}, f_y \ne 0$

$\therefore$ The stationary points of $f(x, y)$ are $(0, 0)$ and $(-3, 0)$

Now $r = f_{xx} = 2y$; $s = f_{xy} = 2x + 3$ and $t = f_{yy} = 0$

And $rt - s^2 = 2y \times 0 - (2x + 3)^2 = -(2x + 3)^2$

$\therefore rt - s^2 < 0$ at $(0, 0)$ as well as $(-3, 0)$

Hence the two stationary points $(0, 0)$ and $(-3, 0)$ are saddle points where $f(x, y)$ has neither maximum nor minimum.

Example 9

Find the maximum value of the function $f(x, y, z) = z - 2x^2 - 3y^2$ where $3xy - z + 7 = 0$.

Solution

Given $f(x, y, z) = z - 2x^2 - 3y^2$ (1)

Where $3xy - z + 7 = 0$ (2)

$\Rightarrow z = 3xy + 7$ (3)

Substituting the value of z in (1), we have $f = 3xy + 7 - 2x^2 - 3y^2$

$$\therefore f_x = \frac{\partial f}{\partial x} = 3y - 4x \text{ and } f_y = \frac{\partial f}{\partial y} = 3x - 6y$$

$f_x = 0 \Rightarrow 3y - 4x = 0$ and $f_y = 0 \Rightarrow 3x - 6y = 0$

$f_x = 0$ and $f_y = 0$ only when $x = 0$ and $y = 0$

$\therefore$ The stationary point is (0, 0)

Now $r = f_{xx} = \dfrac{\partial^2 f}{\partial x^2} = -4$; $s = f_{xy} = \dfrac{\partial^2 f}{\partial x \partial y} = 3$ and

$$t = f_{yy} = \frac{\partial^2 f}{\partial y^2} = -6$$

$\therefore rt - s^2 = (-4)(-6) - 3^2 = 24 - 9 = 15 > 0$ and $r = -4 < 0$

$\therefore f$ has a maximum value at (0, 0)

For $x = 0, y = 0$, from (3), $z = 3 \times 0 \times 0 + 7 \Rightarrow z = 7$

$\therefore$ The maximum value exists for $f(x, y, z)$ at (0, 0, 7) and that maximum value is $f(x, y, z)_{at\,(0,0,7)} = 7 - 2 \times 0^2 - 3 \times 0^2 = 7$.

Indefinite Integrals

If $f(x)$ and $g(x)$ are two functions of x such that $g'(x) = f(x)$, then the integral of $f(x)$ is $g(x)$. Further, $g(x)$ is called the antiderivative of $f(x)$.

The process of computing an integral of a function is called Integration and the function to be integrated is called integrand.

An integral of a function is not unique. If $g(x)$ is any one integral of $f(x)$, then $g(x) + c$ is also its integral, where C is any constant termed as constant of integration.

Some Standard Formulae

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c \; (n \neq -1)$

2. $\int (ax+b)^n dx = \dfrac{(ax+b)^{n+1}}{(n+1)a} + c \; (n \neq -1)$

3. $\int \dfrac{1}{x} dx = \log x + c$

4. $\int \dfrac{1}{ax+b} dx = \dfrac{\log(ax+b)}{a} + c$

5. $\int a^x dx = \dfrac{a^x}{\log a} + c$

6. $\int e^x dx = e^x + c$

7. $\int \sin x \, dx = -\cos x + c$

8. $\int \cos x \, dx = \sin x + c$

9. $\int \sec^2 x \, dx = \tan x + c$

10. $\int \text{cosec}^2 x \, dx = -\cot x + c$

11. $\int \sec x \; \tan x \, dx = \sec x + c$

12. $\int \text{cosec} x \; \cot x \, dx = \text{cosec} x + c$

13. $\int \tan x \, dx = \log(\sec x) + c$

14. $\int \cot x \, dx = \log(\sin x) + c$

15. $\int \sec x \, dx = \log(\sec x + \tan x) + c$
$$= \log \tan\left[\frac{\pi}{4} + \frac{x}{2}\right] + c$$

16. $\int \text{cosec} x \, dx = \log(\text{cosec} x + \cot x) + c$
$$= \log \tan \frac{x}{2} + c$$

17. $\int \dfrac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c \quad \text{or} \quad -\cos^{-1} x + c$

18. $\int \dfrac{1}{1+x^2} dx = \tan^{-1} x + c \quad \text{or} \quad -\cot^{-1} x + c$

19. $\int \dfrac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c \quad \text{or} \quad -\text{cosec}^{-1} x + c$

20. $\int \sinh x \, dx = \cosh x + c$

21. $\int \cosh x \, dx = \sinh x + c$

22. $\int \text{sech}^2 x \, dx = \tanh x + c$

23. $\int \text{cosech}^2 x \, dx = -\coth x + c$

24. $\int \text{sech}\, x \tanh x \, dx = -\text{sech}\, x + c$

25. $\int \text{sech}\, x \coth x \, dx = -\text{cosech} x + c$

26. $\int K f(x) dx = K \int f(x) dx + c$

27. $\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx + c$

28. $\int \dfrac{f'(x)}{f(x)} dx = \log[f(x)] + c$

29. $\int f(x)^n \cdot f'(x) dx = \dfrac{[f(x)]^{n+1}}{n+1} + c$

30. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+c$

31. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a}+c$ or

$\log|x+\sqrt{a^2+x^2}|+c$

32. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a}+c$ or

$\log|x+\sqrt{x^2+a^2}|+c$

33. $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

34. $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$

35. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$

36. $\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$

37. $\int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin h^{-1}\frac{x}{a}+c$

38. $\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\cos h^{-1}\frac{x}{a}+c$

39. $\int \log x\,dx = x(\log x-1) = x\log\left(\frac{x}{e}\right)+c$

40. $\int e^x[f(x)+f'(x)]\,dx = e^x f(x)+c$

Definite Integrals

The difference in the values of an integral of a function $f(x)$ for two assigned values say a, b of the independent variable x, is called the Definite Integral of $f(x)$ over the interval $[a, b]$ and is denoted by $\int_a^b f(x)dx$.

The number 'a' is called the lower limit and the number 'b' is the upper limit of integration.

Fundamental Theorem of Integral Calculus

If $f(x)$ is a function of x continuous in $[a, b]$, then $\int_a^b f(x)dx = g(b)-g(a)$ where $g(x)$ is a function such that $\frac{d}{dx}g(x) = f(x)$.

Properties of definite integrals

1. If $f(x)$ is a continuous function of x over $[a, b]$, and c belongs to $[a, b]$, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.
2. If $f(x)$ is continuous function of x over $[a, b]$, then $\int_a^b Kf(x)dx = K\int_a^b f(x)dx$.
3. If $f(x)$ is continuous function of x over $[a, b]$, then $\int_b^a f(x)dx = -\int_a^b f(x)dx$.
4. If $f(x)$ is continuous in some neighbourhood of a, then $\int_a^a f(x)dx = 0$.
5. If $f(x)$ and $g(x)$ are continuous in $[a, b]$, then $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
6. $\int_a^b f(x)dx = \int_a^b f(z)dz = \int_a^b f(t)dt$
7. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
8. $\int_{-a}^a f(x) = 0$, if $f(x)$ is odd
9. $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$ if $f(x)$ is even
10. $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$, if $f(2a-x) = f(x)$

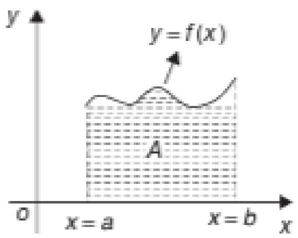
    $= 0$ if $f(2a-x) = -f(x)$
11. $\int_0^{na} f(x)dx = n\int_0^a f(x)dx$, if $f(a+x) = f(x)$

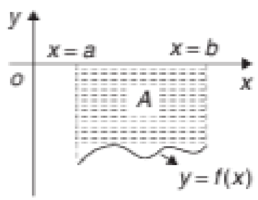
Applications of Integration

Area as a Definite Integral

1. The area enclosed by a curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis is given by:

$$A = \int_a^b |f(x)|dx = \begin{cases} \int_a^b f(x)dx, & \text{if } f(x)\ge 0,\ a\le x\le b \\ -\int_a^b f(x)dx, & \text{if } f(x)\le 0,\ a\le x\le b \end{cases}$$





2. Similarly, the area enclosed by the curve $x = g(y)$, the lines $y = c$ and $y = d$ and the y-axis is $A = \int_c^d |g(y)|dy$
3. When $f(x)\ge 0$ for $a\le x\le c$ and $f(x)\le 0$ for $c\le x\le b$, then the area enclosed by the curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis is $A = \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$

PREVIOUS YEARS' QUESTIONS

1. Evaluate $\int_0^{\infty} \frac{\sin t}{t} dt$ **[GATE, 2007]**

(A) π (B) $\frac{\pi}{2}$

(B) $\frac{\pi}{4}$ (D) $\frac{\pi}{8}$

2. A velocity vector is given as $\bar{v} = 5xy\bar{i} + 2y^2\bar{j} + 3yz^2\bar{k}$. The divergence of the this velocity vector at (1, 1, 1) is **[GATE, 2007]**

(A) 9 (B) 10

(C) 14 (D) 15

3. The value of $\int_0^3 \int_0^x (6 - x - y) dx\, dy$ is **[GATE, 2008]**

(A) 13.5 (B) 27.0

(C) 40.5 (D) 54.0

4. The inner (dot) product of two vectors $\bar{P}$ and $\bar{Q}$ is zero. The angle (degrees) between the two vectors is **[GATE, 2008]**

(A) 0 (B) 30

(C) 90 (D) 120

5. For a scalar function $f(x, y, z) = x^2 + 3y^2 + 2z^2$, the gradient at the point $P(1, 2, -1)$ is **[GATE, 2009]**

(A) $2\vec{i} + 6\vec{j} + 4\vec{k}$ (B) $2\vec{i} + 12\vec{j} - 4\vec{k}$

(C) $2\vec{i} + 12\vec{j} + 4\vec{k}$ (D) $\sqrt{56}$

6. The $\lim_{x \to 0} \frac{\sin\left[\frac{2}{3}\right]x}{x}$ is **[GATE, 2010]**

(A) $\frac{2}{3}$ (B) 1

(C) $\frac{3}{2}$ (D) ∞

7. Given function

$F(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$. The optimal value of $f(x, y)$ **[GATE, 2010]**

(A) is a minimum equal to $\frac{10}{3}$

(B) is a maximum equal to $\frac{10}{3}$

(C) is a minimum equal to $\frac{8}{3}$

(D) is a maximum equal to $\frac{8}{3}$

8. What is the value of the definite integral, $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} dx$? **[GATE, 2011]**

(A) 0 (B) $\frac{a}{2}$

(C) a (D) $2a$

9. If $\bar{a}$ and $\bar{b}$ are two arbitrary vectors with magnitudes a and b, respectively, $|\vec{a} \times \vec{b}|^2$ will be equal to **[GATE, 2011]**

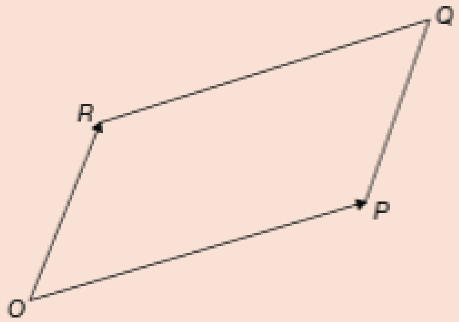
(A) $a^2b^2 - (\vec{a} \cdot \vec{b})^2$

(B) $ab - \vec{a} \cdot \vec{b}$

(C) $a^2b^2 + (\vec{a} \cdot \vec{b})^2$

(D) $ab + \vec{a} \cdot \vec{b}$

10. For the parallelogram $OPQR$ shown in the sketch, $\overrightarrow{OP} = a\bar{i} + b\bar{j}$ and $\overrightarrow{OR} = c\bar{i} + d\bar{j}$. The area of the parallelogram is



[GATE, 2012]

(A) $ad - bc$
(B) $ac + bd$
(C) $ad + bc$
(D) $ab - cd$

11. There is no value of x that can simultaneously satisfy both the given equations. Therefore, find the least square error solution to the two equations, i.e., find the value of x that minimizes the sum of squares of the errors in the two equations
$2x = 3$
$4x = 1$ [GATE, 2013]

12. The solution for $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta d\theta$ is [GATE, 2013]

(A) 0 (B) $\frac{1}{15}$
(C) 1 (D) $\frac{8}{3}$

13. $\lim_{x\to\infty}\left(\frac{x+\sin x}{x}\right)$ equals is [GATE, 2014]

(A) $-\infty$ (B) 0
(C) 1 (D) ∞

14. The expression $\lim_{x\to 0}\frac{x^{\alpha}-1}{\alpha}$ is equal to [GATE, 2014]

(A) lnx (B) 0
(C) xlnx (D) ∞

15. With reference to the conventional cartesian (x, y) coordinate system, the vertices of a triangles have the following coordinates: $(x_1, y_1) = (1, 0)$: $(x_2, y_2) = (2, 2)$: and (x_3, y_3), $= (4, 3)$. The area of the triangle is equal to [GATE, 2014]

(A) $\frac{3}{2}$ (B) $\frac{3}{4}$
(C) $\frac{4}{5}$ (D) $\frac{5}{2}$

16. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x}$ is equal to [GATE, 2015]

(A) e^{-2} (B) e
(C) 1 (D) e^2

17. While minimizing the function $f(x)$, necessary and sufficient conditions for a point, x_0 to be a minima are: [GATE, 2015]

(A) $f'(x_0) > 0$ and $f''(x_0) = 0$
(B) $f'(x_0) < 0$ and $f''(x_0) = 0$
(C) $f'(x_0) = 0$ and $f''(x_0) < 0$
(D) $f'(x_0) = 0$ and $f''(x_0) > 0$

18. The directional derivative of the field $u(x, y, z) = x^2 - 3yz$ in the direction for the vector $(\hat{i} + \hat{j} - 2\hat{k})$ at point $(2, -1, 4)$ is ______. [GATE, 2015]

19. The optimum value of the function $f(x) = x^2 - 4x + 2$ is [GATE, 2016]

(A) 2 (maximum) (B) 2 (minimum)
(C) −2 (maximum) (D) −2 (minimum)

20. The quadratic approximation of $f(x) = x^3 - 3x^2 - 5$ at the point $x = 0$ is [GATE, 2016]

(A) $3x^2 - 6x - 5$ (B) $-3x^2 - 5$
(C) $-3x^2 + 6x - 5$ (D) $3x^2 - 5$

21. What is the value of $\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{x^2+y^2}$? [GATE, 2016]

(A) 1 (B) −1
(C) 0 (D) Limit does not exist

22. The area between the parabola $x^2 = 8y$ and the straight line $y = 8$ is _____. [GATE, 2016]

23. The area of the region bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$ is [GATE, 2016]

(A) $\frac{59}{6}$ (B) $\frac{9}{2}$
(C) $\frac{10}{3}$ (D) $\frac{7}{6}$

24. The angle of intersection of the curves $x^2 = 4y$ and $y^2 = 4x$ at point (0, 0) is [GATE, 2016]

(A) 0° (B) 30°
(C) 45° (D) 90°

25. The value of $\int_0^{\infty}\frac{1}{1+x^2}dx + \int_0^{\infty}\frac{\sin x}{x}dx$ is [GATE, 2016]

(A) $\frac{\pi}{2}$ (B) π
(C) $\frac{3\pi}{2}$ (D) 1

Previous Years' Questions

1. B	**2.** D	**3.** A	**4.** C	**5.** B	**6.** A	**7.** A	**8.** B	**9.** A	**10.** A
11. 0.875	**12.** B	**13.** C	**14.** A	**15.** A	**16.** D	**17.** D	**18.** −5.72 to −5.70		**19.** D
20. B	**21.** D	**22.** 85.33	**23.** B	**24.** D	**25.** B				