* Linear Algebra [3M]

Topics :-

→ Determinants
→ Inverse of a matrix
→ Rank of matrix
→ Homogeneous & non-homogeneous linear eq^n.
→ Eigen values & Eigen vectors.
→ Cayley - Hamilton theorem.

Textbook :-
Matrices by A. R. Vasistha

* Determinants *

* Properties of Determinants :-

Point I : Value of the determinant will not be change if rows and columns are interchange.
i.e., $|A| = |A^T|$

Ex.

$$|A| = \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = 10 - 4 = 6$$

$$|A^T| = \begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} = 10 - 4 = 6$$

$\therefore |A| = |A^T|$

Point II : Value of the determinant is multiplied by -1 if two rows & two columns are interchange.

Ex.

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

$$|A| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2$$

Point III : Value of the determinant can be zero in the following cases :-

i) The elements in two rows or two columns are identical.

Ex.

same

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = 0$$

ii) The elements in two rows or two columns are proportional to each other.

Ex.

proportional

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 3 & 6 & 9 \end{vmatrix} = 0$$

iii) All elements in a row or column are zeros.

Ex.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 7 & 3 \end{vmatrix} = 0$$

iv) The elements in a determinant are of consicutive order (continuous order)

→ valid for ~~ont~~ 3×3 & high order matrices.

Ex.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$$

we can start 1st element from any no.

Point IV: The determinant of upper triangular, lower triangular, diagonal, scalar & Identity matrix is the product of its diagonal elements.

1) Upper triangular matrix

Ex.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{vmatrix} = 1 \times (-4) \times 7 = -28$$

2) Lower triangular matrix

Ex.

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 7 \end{vmatrix} = 1 \times 3 \times 7 = 21$$

3) Diagonal matrix

Ex.

$$|A| = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 3 \times 2 \times 5 = 30$$

A matrix is diagonal iff at least one diagonal element should be non zero & all other non-diagonal elements should be zero.

4) Scalar matrix

Ex.

$$|A| = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 3 \times 3 \times 3 = 27$$

→ Diagonal elements should be same.

→ Non - diagonal elements should be zero.

5) Identity matrix

Ex.

$$|I| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \times 1 \times 1 = 1$$

→ Determinant of Identity matrix is always 1.

Point V : If A is $N \times N$ matrix then

$$|KA| = K^n |A|$$

where n = order of matrix A

Ex.

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2$$

$$|3A| = 3^2 \cdot |A| = 9 \times (-2) = -18$$

Point VI : If each element of a row or column contains sum of 2 elements then the determinant can be expressed as sum of two determinants of same order

Ex.

$$|A| = \begin{vmatrix} 1 & 1^2 & 1^3+1 \\ 2 & 2^2 & 2^3+4 \\ 3 & 3^2 & 3^3+5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1^2 & 1^3 \\ 2 & 2^2 & 2^3 \\ 3 & 3^2 & 3^3 \end{vmatrix} + \begin{vmatrix} 1 & 1^2 & 1 \\ 2 & 2^2 & 4 \\ 3 & 3^2 & 5 \end{vmatrix}$$

* Note 1: Consider the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Method I:

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Method II:

$$\begin{array}{ccccc} \ominus & & \ominus & & \ominus \\ a_{13} & a_{11} & a_{12} & a_{13} & a_{11} \\ & a_{21} & a_{22} & a_{23} & \\ a_{33} & a_{31} & a_{32} & a_{33} & a_{31} \\ \oplus & & \oplus & & \oplus \end{array}$$

$$|A| = a_{13}a_{21}a_{32} + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} - a_{33}a_{21}a_{12} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11}$$

→ If determinant contains more no. of zeros use method I.

Ex. 1> Find determinant of

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$\begin{array}{ccccc} 2 & 1 & 2 & 2 & 1 \\ & 2 & 1 & 2 & \\ 1 & 2 & 2 & 1 & 2 \end{array}$$

$= 8 + 1 + 8 - 4 - 4 - 4$
$= 5$

Ex: 2> $A = \begin{pmatrix} 5 & 0 & 2 \\ \bar{0} & 3^{+} & \bar{0} \\ 2 & 0 & 1 \end{pmatrix}$

$= 3\,(5 - 4) = 3$

Ex. 3> The following represents eq^n of straight line

$$\begin{vmatrix} x & 2 & 4 \\ y & 8 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

The line passes through

a> (0,0) ✓b> (3,4) c> (4,3) d> (4,4)

$x(8) - 2(y) + 4(y-8)$
$8x - 2y + 4y - 32 = 0$
$8x + 2y = 32$
$4x + y = 16$
$\because 4(3) + 4 = 16$
$x = 3$ & $y = 4$

Note 2:— Consider the matrix

$$A = \begin{array}{c} \begin{matrix} + & - & + & - \end{matrix} \\ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}_{4\times4} \end{array}$$

Method I :— (Complicated)

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

$$+ a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

Method II :— (Preferable)

~~Con~~

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}_{4\times4}$$

Procedure:

Step 1 : Among the given element of a matrix, select any non zero element.

step 2 : Make all elements above & below or left & right of the selected element as zero using row & column operations

Therefore, the matrix is

$$A = \begin{pmatrix} a_{11}^{+} & a_{12} & a_{13} & a_{14} \\ 0^{-} & b_{22} & b_{23} & b_{24} \\ 0^{+} & b_{32} & b_{33} & b_{34} \\ 0^{-} & b_{42} & b_{43} & b_{44} \end{pmatrix}$$

→ 1^{st} column has more no of zeros so ...
the determinant along 1^{st} column

$$|A| = a_{11} \begin{vmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{vmatrix}$$

Ex. 1) Find the determinant of

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 9 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

$R_2 \rightarrow R_2 - R_1$
$R_3 \rightarrow R_3 - 2R_1$

$$A = \begin{pmatrix} 1^{+} & -1 & 0 & 1 \\ 0^{-} & 3 & 3 & 3 \\ 0^{+} & 5 & 4 & 7 \\ 0^{-} & 1 & 0 & 2 \end{pmatrix}$$

$$|A| = 1 \begin{vmatrix} 3 & 3 & 3 \\ 5 & 4 & 7 \\ 1^{+} & 0^{-} & 2^{+} \end{vmatrix}$$

$$1[21 - 12] + 2[12 - 15]$$
$$9 + 2(-3)$$
$$9 - 6 = 3$$

Problem 1 :- If A has m rows & m+5 columns & B has n rows and 11-n columns. The orders of A and B if AB and BA are defined ?

$\Rightarrow$

$(A)_{m \times (m+5)} \cdot (B)_{n \times (11-n)}$

$(B)_{n \times (11-n)} \cdot (A)_{m \times (m+5)}$

$\therefore \quad m+5 = n \qquad\qquad m - n = -5$

$\quad\;\; 11 - n = m \qquad\qquad m + n = 11$

$\qquad\qquad\qquad\qquad\quad 2m = 6$

$\qquad\qquad\qquad\qquad\quad m = 3 \qquad \therefore n = 8$

Therefore, orders are $A_{(3,8)}$ & $B_{(8,3)}$ resp.

Problem 2 :- If $A = (a_{ij})_{m\times n}$ such that $a_{ij} = i+j, \forall i,j$ then sum of all element of A is ?

$\Rightarrow$

$$A = \begin{pmatrix} 1+1 & 1+2 & 1+3 & \cdots & 1+n \\ 2+1 & 2+2 & 2+3 & \cdots & 2+n \\ \vdots & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & \\ m+1 & m+2 & m+3 & \cdots & m+n \end{pmatrix}$$

according to point VI

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \\ 3 & 3 & & 3 \\ \vdots & \vdots & & \vdots \\ m & m & \cdots & m \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & & n \\ \vdots & & & & \vdots \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$$

$$\frac{m(m+1)}{2} \quad \frac{m(m+1)}{2} + \dots + \frac{m(m+1)}{2} \quad \Bigg| \quad \frac{n(n+1)}{2} + \frac{n(n+1)}{2} + \dots + \frac{n(n+1)}{2}$$

for n columns $= \frac{n \cdot m(m+1)}{2}$ | for m rows $= \frac{m \cdot n(n+1)}{2}$

Sol:

$$\frac{mn(m+1)}{2} + \frac{mn(n+1)}{2}$$

$$\frac{mn}{2}\left[m+1+n+1\right]$$

$$\frac{mn}{2}\left[m+n+2\right]$$

Problem 8 :- If $A = (a_{ij})_{3\times3}$, $B = (b_{ij})_{3\times3}$ such that

$b_{ij} = 2^{i+j} a_{ij} \quad \forall\ i, j$; $|A| = 2$; $|B| = ?$

a) 2^{10} b) 2^{11} c) 2^{12} d) 2^{13}

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 2$$

$$|B| = \begin{vmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{vmatrix}$$

$$= 2^2 \cdot 2^3 \cdot 2^4 \begin{vmatrix} 1 a_{11} & 2 a_{12} & 2^2 a_{13} \\ 1 a_{21} & 2 a_{22} & 2^2 a_{23} \\ 1 a_{31} & 2 a_{32} & 2^2 a_{33} \end{vmatrix}$$

$$2^9 \cdot 2 \cdot 2^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$\frac{n(n+1)}{2}$

$\frac{n(n+1)}{2}$

$= 2^{12} \cdot 2 = \underline{2^{13}}$

Problem 4:— If $A = (a_{ij})_{n\times n}$ such that

i) $a_{ij} = i^2 - j^2, \ \forall\ i, j$

ii) $a_{ij} = i - j, \ \forall\ i, j$

Find sum of all elements of A

$\Longrightarrow$

i)
$$A = \begin{pmatrix} 0 & -3 & -8 & \cdots & (1^2 - n^2) \\ 3 & 0 & -5 & \cdots & (2^2 - n^2) \\ 8 & 5 & 0 & \cdots & (3^2 - n^2) \\ \vdots \\ (n^2-1^2) & (n^2-2^2) & (n^2-3^2) & \cdots & 0 \end{pmatrix}$$

A is skew symmetric matrix.

$\rightarrow$ In skew symmetric matrix all diagonal elements must be zero & non-diagonal elements should be real no.

Benefit :$\rightarrow$ Sum of all elements of skew symmetric matrix is always zero.

Ex.

i) $A = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}_{2\times 2}$ $\therefore$ Sum of all elements of skew sym. matrix $= 0$

2) $A = \begin{pmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{pmatrix}_{3\times 3}$ $\therefore$ Sum $= 0$

$\rightarrow$ If ith row, jth column elem. of a matrix is in the form $a_{ij} = i^n - j^n$ $(n>0)$ then corresponding matrix is always skew symm.

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Date
Page

* Note 3 :-
A matrix A is said to be symmetric if
$A^T = A$ or $a_{ij} = a_{ji}$

* Note 4 :-
A matrix A is said to be skew symmetric
if $A^T = -A$ or $a_{ij} = -a_{ji}$

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Problem 1 :- The value of $\begin{vmatrix} 1+b & b & 1 \\ b & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix} = ?$

$C_1 \to C_1 + C_2$

$$\begin{vmatrix} 1+2b & b & 1 \\ 1+2b & 1+b & 1 \\ 1+2b & 2b & 1 \end{vmatrix} = 0$$

proportional

Problem 2 :- Find the determinant of $\begin{vmatrix} 1/a & a & bc \\ 1/b & b & ca \\ 1/c & c & ab \end{vmatrix}$

$$= \begin{vmatrix} bc/abc & a & bc \\ ca/abc & b & ca \\ ab/abc & c & ab \end{vmatrix}$$

$$\frac{1}{abc} \begin{vmatrix} bc & a & bc \\ ca & b & ca \\ ab & c & ab \end{vmatrix} = 0$$

same

→ If there are no numbers inside determinant try to make 1 perticular row or column same

<u>Problem 8</u> :- find the value of $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3 + C_4$

$$= \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix}$$

$$= x+3a \begin{vmatrix} 1 & a & a & a \\ 1 & x & a & a \\ 1 & a & x & a \\ 1 & a & a & x \end{vmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$; $R_4 \to R_4 - R_1$

$$= x+3a \begin{vmatrix} 1 & a & a & a \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix}$$

according to pt IV

$$= (x+3a)\{1 \times (x-a) \times (x-a) \times (x-a)\}$$

$$= (x+3a)(x-a)^3$$

* <u>Short cut method</u> :-

<u>Procedure</u> :-

i) If the diagonal elements are one category of same elements & non diagonal elements are other category of same elements

2) select the 1st row

3) add all elements of 1st row

4) take the product of $(1^{st} - 2^{nd})(1^{st} - 3^{rd})$
$(1^{st} - 4^{th}) \ldots$

5) (3) × (4)

i.e.,

$$|A| = \begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$$

Problem: The value of $A = \begin{pmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{pmatrix}$

$$|A| = \begin{pmatrix} x+10 & 2 & 3 & 4 \\ x+10 & 2+x & 3 & 4 \\ x+10 & 2 & 3+x & 4 \\ x+10 & 2 & 3 & 4+x \end{pmatrix} = x+10 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{pmatrix}$$

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 - R_1 \quad R_4 \to R_4 - R_1$

$$= (x+10) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

$= (x+10)\ 1 \times x \times x \times x$

$= \underline{\underline{x^3 (x+10)}}$

Problem 2:

$$A = \begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3 + C_4$

$$= \begin{vmatrix} 1+a+b+c+d & b & c & d \\ 1+a+b+c+d & 1+b & c & d \\ 1+a+b+c+d & b & 1+c & d \\ 1+a+b+c+d & b & c & 1+d \end{vmatrix}$$

$$= (1+a+b+c+d) \begin{vmatrix} 1 & b & c & d \\ 1 & 1+b & c & d \\ 1 & b & 1+c & d \\ 1 & b & c & 1+d \end{vmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$; $R_4 \to R_4 - R_1$

$$= (1+a+b+c+d) \begin{vmatrix} 1 & b & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= 1+a+b+c+d \left[1 \left[1 \right] \right]$$

$$= 1+a+b+c+d$$

Problem 3: Find the value of $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$C_2 \to C_2 - C_1 \qquad C_3 \to C_3 - C_1$

$$
\begin{array}{cccc}
 & + & - & + \\
A = & \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}
\end{array}
$$

$= 1\left[(b-a)(c^2-a^2) - (c-a)(b^2-a^2)\right]$

$= (b-a)(c+a)(c-a) - (c-a)(b+a)(b-a)$

$= (b-a)(c-a)\left[c + \cancel{a} - b - \cancel{a}\right]$

$= (b-a)(c-a)(c-b)$

$= \{-(a-b)\}(c-a)\{-(b-c)\}$

$= \underline{(a-b)(b-c)}(c-a)$

* Short cut method [for $(2\times2)(3\times3)\ldots(n\times n)$]

If matrix is in the form $\begin{vmatrix} 1 & 1 & 1 & 1 & \ldots \\ a & b & c & d & \ldots \\ a^2 & b^2 & c^2 & d^2 & \ldots \\ a^3 & b^3 & c^3 & d^3 & \ldots \\ \vdots \end{vmatrix}$

then select the 2^{nd} row

& take $(2^{nd} - 1^{st})(3^{rd} - 2^{nd})\ldots(\text{last} - 1^{st})$

i.e., $(a-b)(b-c)(c-d)$

Date 14/7/2012
Page

* Inverse of a matrix *

Let $A = (a_{ij})$ be $n \times n$ matrix

i) **Minor:** Minor of an element a_{ij} is denoted by M_{ij} and is defined as

$$M_{ij} = (n-1)^{th} \text{ order determinant.}$$

ii) **Cofactor:** Cofactor of an element a_{ij} is denoted by A_{ij} and is defined as

$$A_{ij} = (-1)^{i+j} \cdot M_{ij}$$

Ex. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 7 & -2 \end{pmatrix}$$

Consider the 2nd row 2nd element

$$a_{22} = 3$$

→ Minor of a_{22} is

$$M_{22} = \begin{pmatrix} 1 & 2 \\ 4 & -2 \end{pmatrix} = -2-8 = -10$$

→ Cofactor of a_{22} is

$$A_{22} = (-1)^{2+2} \times (-10) = -10$$

* Invertible matrix:

A matrix A is said to be invertible if we can find some other matrix B such that $AB = BA = I$ then B is called inverse of matrix A.

$AB = I$

$A^{-1}AB = A^{-1}I$

$IB = A^{-1}I$

$$\boxed{B = A^{-1}}$$

* Singular matrix :—
A matrix is said to be singular if $|A| = 0$

* Non Singular matrix :—
A matrix is said to be non singular if $|A| \neq 0$
Inverse of matrix exists if $|A| \neq 0$.

Note 1:
$$A^{-1} = \frac{adj.A}{|A|}$$

$\Rightarrow adj.A = |A| \cdot A^{-1}$

$\Rightarrow A \cdot adj.A = A \cdot |A| \cdot A^{-1}$

$$\Rightarrow \boxed{A \cdot adj.A = |A| \cdot I = adj.A \cdot A}$$

Note 2:
$$A^{-1} = \frac{adj.A}{|A|}$$

$\Rightarrow (adj.A)^{-1} \cdot A^{-1} = (adj.A)^{-1} \cdot \frac{adj.A}{|A|}$

$\Rightarrow (adj.A)^{-1} \cdot A^{-1} = \frac{I}{|A|}$ $\quad \because A \cdot A^{-1} = I$

$\Rightarrow (adj.A)^{-1} \cdot A^{-1} \cdot A = \frac{I.A}{|A|}$

$$\Rightarrow \boxed{(adj.A)^{-1} = \frac{A}{|A|}}$$

classmate
Date ______
Page ______

Note 3: Let A be an $n \times n$ matrix.
We know that

$$adj.\,A = |A|.\,A^{-1}$$

$$\Rightarrow |adj.\,A| = |\,|A|.\,A^{-1}|$$

$$\begin{array}{cc} \downarrow & \downarrow \\ k & A \end{array}$$

$$\boxed{|kA| = k^n |A|}$$

$$\Rightarrow |A|^n.\,|A^{-1}|$$

$$\Rightarrow |A|^n.\,|A|^{-1}$$

$$\therefore \boxed{|adj.\,A| = |A|^{n-1}}$$

Replacing A by $adj.\,A$ in the above relation

$$|adj.\,adj.\,A| = |adj.\,A|^{n-1}$$
$$= \{|A|^{n-1}\}^{n-1}$$

$$\therefore \boxed{|adj.\,adj.\,A| = |A|^{(n-1)^2}}$$

Simillarly,

$$|adj.\,adj.\,adj.\,A| = |A|^{(n-1)^3}$$ & so on..

Note 4: We know that

$$A.\,adj.\,A = |A|.\,I$$

replacing A by $adj.\,A$

$$\Rightarrow adj.\,A(adj.\,\cancel{A}\,adj.\,A) = |adj.\,A|.\,I$$

$$\Rightarrow adj.\,A\,(adj.\,adj.\,A) = |A|^{n-1}.\,I$$

pre multiply both side by A

$$\Rightarrow A.\,adj.\,A(adj.\,adj.\,A) = A.\,|A|^{n-1}\,I$$

$$\boxed{A.\,I = A}$$

$$\Rightarrow |A|.\,I\,(adj.\,adj.\,A) = |A|^{n-1}.\,A$$

$$\Rightarrow |A| \cdot (\text{adj. adj. } A) = |A|^{n-1} \cdot A$$

$$\Rightarrow \text{adj. adj } A = \frac{|A|^{n-1} \cdot A}{|A|}$$

$$\Rightarrow \text{adj. adj. } A = \boxed{|A|^{n-2} \cdot A}$$

Note 5 :- A matrix A is said to be orthogonal if

$$\boxed{A \cdot A^T = A^T \cdot A = I}$$

$$A \cdot A^T = I$$

$$A^{-1} \cdot A \cdot A^T = A^{-1} I$$

$$I \cdot A^T = A^{-1} I$$

$$\boxed{A^T = A^{-1}}$$

Note 6 :- If A is an orthogonal matrix then A^{-1} & A^T are also orthogonal matrices.

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Problem 1 : For the matrix $M = \begin{pmatrix} 3/5 & 4/5 \\ x & 3/5 \end{pmatrix}$ such that $M^T = M^{-1}$. The value of x is ?

$$\frac{3}{5}(x) + \frac{4}{5}\left(\frac{3}{5}\right) = 0$$

$$\boxed{x = -4/5}$$

Note : If A is an orthogonal matrix then its rows & columns are pair wise orthogonal.
But converse of the stmt. may or may not be true.
i.e., If rows & columns of matrix are pair wise orthogonal, then the matrix may or may not be orthogonal.

Problem 2 : If $A = (a_{ij})_{5\times5}$ such that

i) $a_{ij} = i - j$, $\forall\, i, j$

ii) $a_{ij} = i^2 - j^2$, $\forall\, i, j$

find A^{-1} in each case.

$\Longrightarrow$ i) $a_{ij} = i - j$

$a_{ji} = j - i$

$= -(i-j)$

$= -a_{ij}$

$\therefore \boxed{a_{ij} = -a_{ji}}$ $\therefore$ $A \Rightarrow$ skew symmetric matrix

$\therefore \Rightarrow A^T = -A$

$\Rightarrow |A^T| = |-A|$ as $|KA| = K^n |A|$

$\Rightarrow = (-1)^5 |A|$

$\therefore |A^T| = -1\,|A|$ $|A| = |A^T| \rightarrow$ property No 1

$|A| = -|A|$

$|A| + |A| = 0$

$2|A| = 0$

$2 \neq 0$ $\therefore |A| = 0$

$\therefore A^{-1}$ does not exist.

Note: ① Inverse of any odd order skew symm matrix does not exist

Reason : Since every odd order skew symm matrix is singular i.e, $|A| = 0$

② Inverse of even order skew symm. matrix exists

Reason : Since every even order skew symm. matrix is non singular i.e, $|A| \neq 0$

ii) $A = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}_{2\times 2}$

$A^T = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$

$\boxed{A^T = -A}$

$\therefore |A| = \begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} = 0 + 9 = (3)^2$

$\Rightarrow$ The determinant of even order skew symm. matrix is a perfect square.

Pro

$\Rightarrow$ If i^{th} row, j^{th} column element of a matrix is in the form

$a_{ij} = i^n - j^n \quad (n > 0)$, the corresponding matrix is always skew symmetric.

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Problem 3:— If X and Y are two non zero matrices of the same order such that $XY = (0)_{n\times n}$, then

A) $|X| \neq 0$, $|Y| = 0$

B) $|X| = 0$, $|Y| \neq 0$

c) $|X| \neq 0$; $|Y| \neq 0$

D) $|X| = 0$; $|Y| = 0$

$\Rightarrow$ $XY = 0$

take determinants on b.s

$|XY| = 0$

$|X||Y| = 0$

then $|X| = 0$ or $|Y| = 0$ or both $|X| = 0$ & $|Y| = 0$

$\therefore$ © is omitted

Let $|X| = 0$, $|Y| \neq 0$ $\therefore Y^{-1} \Rightarrow$ exist

$$XY = 0 \Rightarrow XYY^{-1} = 0 \cdot Y^{-1}$$
$$\Rightarrow XI = 0$$
$$\Rightarrow X = 0 \quad \text{(contradiction to hypothesis}$$
$$\therefore \boxed{|Y| = 0}$$

If we choose $|Y| = 0$, $|X| \neq 0$ then we get

$$\boxed{|X| = 0}$$

$$\therefore \boxed{|X| = |Y| = 0}$$

Problem 4: If A, B, C, D, E, F, G are non-singular matrices of the same order such that $CEDBGAF = I$ then B^{-1} is —

$$\Rightarrow \quad CEDBGAF = I$$
$$C^{-1}CEDBGAF = C^{-1}I$$
$$IEDBGAF = C^{-1}I$$
$$EDBGAF = C^{-1}$$
$$E^{-1}EDBGAF = E^{-1}C^{-1}$$
$$IDBGAF = E^{-1}C^{-1}$$
$$D^{-1}DBGAF = D^{-1}E^{-1}C^{-1}$$
$$BGAF = D^{-1}E^{-1}C^{-1}$$
$$BGAFF^{-1} = D^{-1}E^{-1}C^{-1}F^{-1}$$
$$BGAI = D^{-1}E^{-1}C^{-1}F^{-1}$$
$$BGAA^{-1} = D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}$$
$$BG = D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}$$
$$BGG^{-1} = D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}G^{-1}$$
$$\boxed{B = D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}G^{-1}}$$

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

$$\therefore B^{-1} = (D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}G^{-1})^{-1}$$
$$\boxed{B^{-1} = GAFCED}$$

* Short cut method :—

$$\underline{CED}\,B\,\underline{GAF} = GAFCED$$

B^{-1}

classmate

Date ______
Page ______

$$\underline{CEDBGAF} = FCEDBG$$

$$\uparrow A^{-1}$$

Ily $\quad \underline{CEDBGAF} = CEDBGA$

$$\uparrow F^{-1}$$

Prob No. 6. Let k be a +ve real no. & let

$$A = \begin{pmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{pmatrix}_{3\times3} \qquad B = \begin{pmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{pmatrix}_{3\times3}$$

find i) det.(adj. B) ii) det.(adj. A)

iii) If det(adj. A) = 10^6, the value of k = ?

$\Longrightarrow$

i) $|adj\, B| = |B|^{n-1} = 0$

$\because$ determinant of odd order skew symm. matrix is zero.

ii) $|adj.\, A| = |A|^{n-1} = |A|^{3-1} = |A|^2$ ——— eq^n (1)

$$|A| = \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$$

$R_2 \to R_2 + R_3$

$$= \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 0 & 1+2k & -2k-1 \\ -2\sqrt{k} & 2k & -1 \end{vmatrix} \qquad -(1+2k)$$

$C_2 \to C_2 + C_3$

$$= \begin{vmatrix} 2k-1 & 4\sqrt{k} & 2\sqrt{k} \\ 0 \ominus & 0 \oplus & \ominus -(1+2k) \\ -2\sqrt{k} & 2k-1 & -1 \end{vmatrix}$$

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2M

classmate
Date
Page

$$\Rightarrow -\left\{-(1+2k)\left((2k-1)^2+8k\right)\right\}$$

$$\cancel{(1+2k)\left((2k+1)(2k-1)\right)}$$

$$\Rightarrow (1+2k)\left(4k^2-4k+1+8k\right)$$

$$\Rightarrow (1+2k)\left(4k^2+4k+1\right)$$

$$\Rightarrow (2k+1)(2k+1)^2 = (2k+1)^3$$

put in eq^n (1)

$$\Rightarrow \left|(2k+1)^3\right|^2 = (2k+1)^6$$

iii) $|\text{adj}.A| = 10^6$

$(2k+1)^6 = 10^6$

$2k+1 = 10 \qquad 2k = 9 \qquad \therefore \boxed{k = 9/2}$

Problem No. 5: If A, B, C, D are non singular matrices of the same order such that $ABCD = I$ then B^{-1} is ?

$ABCD = I$

B^{-1}

$\therefore \boxed{B^{-1} = CDA}$

Prob. No. 7:—

Find the inverse of foll. matrices —

EE-05 2M 1) $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{pmatrix}$

EE-95 2M H.W. 3) $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

EE-98 2M H.W. 2) $A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

$\Rightarrow$

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{\text{adj. } A}{|A|}$$

$$\Rightarrow |A| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{vmatrix}$$

$$= 1\left[2+3\right] - 1\left[6-2\right] = 5 - 4 = \underline{1}$$

$\Rightarrow$ adj. A =

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{pmatrix} \begin{array}{l} \longrightarrow \text{III} \\ \longrightarrow \text{I} \\ \longrightarrow \text{II} \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 1 \\ -1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 2 \\ 1 & 3 & 0 & 1 \end{pmatrix} \quad \therefore \text{adj } A = \begin{pmatrix} 5 & -3 & 1 \\ -6 & 4 & -1 \\ 4 & -3 & 1 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 5 & -3 & 1 \\ -6 & 4 & -1 \\ 4 & -3 & 1 \end{pmatrix}$$

Procedure : (3×3)

Select the middle row middle element & move in anticlockwise direction to complete 1 cycle.

The corresponding element will be return in the 1st column seperately.

classmate
Date
Page

2) Select the 3^{rd} row middle element & move in anti-clockwise direction to complete 1 cycle
The corresponding element will be return in the 2^{nd} column seperately.

3) Repeat the same process with 1^{st} row also.

4) Copy 1^{st} column as a last column & find det. of smaller matrices.

Procedure : (2 x 2)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \boxed{A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

EC-2008
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Prob. No. 8 :— Given an orthogonal matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$ then $(AA^T)^{-1}$ is — ?

Orthogonal $\Longrightarrow$ $(AA^T) = I$

$(AA^T)^{-1} = I^{-1}$

but Inverse / adj. of Identity matrix is Identity matrix only.

$\therefore (AA^T)^{-1} = I$

Prob. No. 9 Find the inverse of $A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow$ Here A is an orthogonal matrix therefore.

$A^{-1} = A^T$

* Rank of Matrix *

* **Sub matrix :** A matrix obtained by deleting some rows or columns or both is called as sub matrix.

Ex. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\underbrace{B = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \quad B_2 = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} \quad B_3 = \begin{pmatrix} 4 & 5 & 6 \\ 1 & -1 & 0 \end{pmatrix}}_{\text{sub matrices of A}}$$

* **Minor :** The determinant of square sub matrix is called its minor.

Ex.

$$\left.\begin{aligned} |B_1| &= \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5-8 = -3 \\ |B_2| &= \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 12-15 = -3 \end{aligned}\right\} \text{Minors of A}$$

* **Rank :** If the determinant of highest possible square matrix is not equal to zero then the order of the determinant is called Rank of matrix.

Ex. Find the rank of

$$A = \begin{pmatrix} 1 & 3 & -2 & 1 \\ 2 & -1 & 4 & 2 \\ 1 & -11 & 14 & 5 \end{pmatrix}_{3\times 4}$$

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix}_{3\times 3} = 1[-14+44] - 3[28-4] - 2[-22+1]$$
$$= 30 - 72 + 42$$
$$= 0$$

$$\begin{vmatrix} 3 & -2 & 1 \\ -1 & 4 & 2 \\ -11 & 14 & 5 \end{vmatrix}_{3\times 3} = 3[20-28] + 2[-5+22] + 1[-14+44]$$
$$= -24 + 34 + 30 = 40 \neq 0$$

$\therefore$ Rank of A $= \rho(A) = 3$ i.e., order of matrix

* Row Echelon form:-

A matrix A is said to be in Row Echelon form iff

i) zero rows should occupy the last rows, if any.

ii) the no. of zero's before a non zero element of each row is less than no. of such zeros before a non zero element of the next row.

Ex

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 7 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{6\times 6}$$

Condition 1 satisfy

— 2 —

classmate
Date
Page

Note: The rank of Row Echelon form matrix is equal to no. of non zero rows.

$\boxed{\rho(A) = 4}$ ← Linearly Independent Row/Vector

→ These non zero rows are called Linearly Independent rows/vector.

→ To reduce any matrix into row echelon form we should use only row operations.

→ Every upper triangular matrix is in R.E. form but every R.E. form will not be an upper triangular matrix.

Ex. 1

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 7 \end{pmatrix}$$

↓

Upper Δ^{lar} matrix
∴ A is in R.E. form

Ex. 2

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

↓

Not Upper Δ^{lar} matrix
but A is in R.E. Form

Upper Δ^{lar} matrix $\Rightarrow$ R.E. form   R.E. form $\not\Rightarrow$ Upper Δ^{lar} matrix

* Column Echelon form :–

A matrix A is said to be in column echelon form iff

i) zero ~~rows~~ columns should occupy the last columns, if any.

ii) The no. of zeros above a non zero element of each column is less than the no. of zeros above a non zero element of the next column

Ex. Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}$$

Note :-

Rank of matrix in Column Echelon form is equal to no of non zero columns.

$$\therefore \boxed{\rho(A) = 4}$$

→ To reduce any matrix into column echelon form, we should use only Column Operations

→ Every lower Δ^{lar} matrix will be in column echelon form.

but every C.E form will not be a lower Δ^{lar} matrix.

Ex. 1

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 9 & 4 & 5 \end{pmatrix}$$

Lower Δ^{lar} matrix

Lower Δ^{lar} M $\Rightarrow$ C.E.form

Ex. 2

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & [illegible] & [illegible] \\ 2 & [illegible] & [illegible] \end{pmatrix}$$

C.E. Form $\not\Rightarrow$ Lower Δ^{lar} matrix

Example: Consider the matrix $A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & -1 & 6 \\ 1 & -11 & 14 \end{pmatrix}$

$\Rightarrow$ Row Echelon form

$R_2 \rightarrow R_2 - 2R_1$ $\quad$ $R_3 \rightarrow R_3 - R_1$

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{pmatrix} \qquad R_3 \longrightarrow R_3 - 2R_1$$

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 7 & 8 \\ 0 & 0 & 0 \end{pmatrix} \qquad \therefore \boxed{\rho(A) = 2}$$

$\Rightarrow$ Column Echelon form

$C_2 \rightarrow C_2 - 3C_1$ $\quad$ $C_3 \rightarrow C_3 + 2C_1$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -7 & 8 \\ 1 & -14 & 16 \end{pmatrix} \qquad C_3 \longrightarrow 7C_3 + 8C_2$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -7 & 0 \\ 1 & -14 & 0 \end{pmatrix} \qquad \therefore \boxed{\rho(A) = 2}$$

Note: Rank of matrix = no of non zero rows &
no of non zero columns.

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Prob. No. 7

2) $A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ $A^{-1} = \frac{adj.A}{|A|}$

$|A| = \begin{vmatrix} 5 & 0 & 2 \\ 0^{\ominus} & 3^{\oplus} & 0^{\ominus} \\ 2 & 0 & 1 \end{vmatrix} = 3[5-4] = 3$

$adj.|A| = \begin{pmatrix} 3 & 0 & 0 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 5 & 0 \\ 3 & 0 & 0 & 3 \end{pmatrix}$

$adj.A = \begin{pmatrix} 3 & 0 & -6 \\ 0 & 1 & 0 \\ -6 & 0 & 15 \end{pmatrix}$ $\therefore A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & -6 \\ 0 & & 0 \\ -6 & 0 & 15 \end{pmatrix}$

3) $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $A^{-1} = \frac{Adj.A}{|A|}$

$|A| = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0^{\oplus} & 0^{\ominus} & 1^{\oplus} \end{vmatrix} = 1[1+1] \quad 2$

$adj.A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

$\therefore A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

15/7/12

* Information regarding Rank of matrix :

1) $\rho(O_{n\times n}) = 0$
2) $\rho(I_{n\times n}) = n$
3) $\rho\{\text{adj. } I_{n\times n}\} = n$
4) $\rho(A) = \rho(A^T)$
5) $\rho(A+B) \leq \rho(A) + \rho(B)$
6) $\rho(A-B) \geq \rho(A) - \rho(B)$
7) $\rho(AB) \geq \rho(A) + \rho(B) - n$, if A & B are $n\times n$ matrice
8) $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$
9) If A is an $m\times n$ matrix, then $\rho(A) \leq \min(m, n)$
10) If $\rho(A_{n\times n}) = n$, then $\rho(\text{adj. } A) = n$
11) If $\rho(A_{n\times n}) = n-1$, then $\rho(\text{adj. } A) = 1$
12) If $\rho(A_{n\times n}) = n-2$, then $\rho(\text{adj. } A) = 0$

2m

Problem 1: If $A = (a_{ij})_{m\times n}$ such that $a_{ij} = i\cdot j$, $\forall i, j$ then $\rho(A) = ?$

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 4 & 6 & \cdots & 2n \\ 3 & 6 & 9 & \cdots & 3n \\ \vdots & & & & \vdots \\ m & 2m & 3m & \cdots & mn \end{pmatrix}_{m\times n}$$

to find $\rho(A)$ convert the matrix into Row Echelon form

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_m \to R_m - mR_1$

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

$\therefore \rho(A) =$ No. of non zero rows $= 1$

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Problem 2:- If $x = (x_1, x_2 \dots x_n)^T$ is n-tuple non zero vector then

i) $\rho(xx^T)$

ii) $\rho(x^Tx)$

→ i) $\rho(xx^T)$

$x = (x_1, x_2 \dots x_n)^T$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n \times 1} \qquad x^T = (x_1, x_2 \dots x_n)_{1 \times n}$$

$\rho(xx^T) \leq \min\{\rho(x), \rho(x^T)\}$

$\rho(xx^T) \leq \min\{1, 1\}$

$\Rightarrow \rho(xx^T) \leq 1$ < 1 ✓, 0 ✗ ($x \rightarrow$ Non zero vector)

$\therefore \boxed{\rho(xx^T) = 1}$

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Problem 3:- The rank of 5×6 matrix Q is 4 then which of the foll. stmt is true.

~~a)~~ Q will have 4 L.I rows & 4 L.I columns

b) Q will have 4 L.I rows & 5 L.I columns

c) QQ^T is invertible

d) Q^TQ is invertible

$\rho(Q_{5 \times 6}) = 4$

$\therefore$ 4 non zero rows or columns

option (a) ✓

option (b) ✗

If det of matrix $\neq 0$ then order of ...

option (c) $(Q)_{5\times6}$. $(Q)^T_{6\times5}$

$(QQ^T)_{5\times5} \rightarrow$ Invertible

$(QQ^T)^{-1}$ exists

$|(QQ^T)_{5\times5}| \neq 0 \Rightarrow \rho(QQ^T) = 5$ (contradictn to stmt)

option (d) $(Q^T)_{6\times5}$ $(Q)_{5\times6}$

$(Q^TQ)_{6\times6} \rightarrow$ Invertible

$(Q^TQ)^{-1}$ exists

$|(Q^TQ)_{6\times6}| \neq 0 \Rightarrow \rho(Q^TQ) = 6$ (contradiction to stmt)

* Linearly Dependent & Independent vectors *

$\Rightarrow$ Two vectors x_1 & x_2 are L.D. if one vector is expressed as multiple of other vector.

x_1 & $x_2 \Rightarrow$ same directional vectors

Example

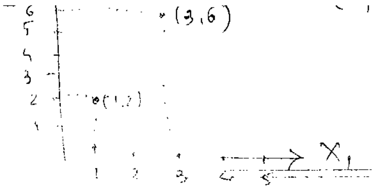
$x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $x_2 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

$x_2 = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\boxed{x_2 = 3x_1}$ or $\boxed{x_1 = \frac{1}{3}x_2}$

$\therefore x_1, x_2 \Rightarrow$ L.D.

collinear.



$\Rightarrow$ Two vectors in R^2 are L.D. if and only if they are collinear.

$\Rightarrow$ Three vectors in R^3 are L.D iff they are coplanar.

$\Rightarrow$ Two vectors x_1 x_2 are L.I. iff it is not possible to express one vector as a multiple of other vector.

Example:

$x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad x_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$x_1 \neq k x_2$ OR $x_2 \neq k x_1$

$\therefore$ $x_1, x_2 \Rightarrow$ L.I.

* Linearly Dependent vectors : (for 2 or more than 2 vectors)

A set of r n-vectors $x_1, x_2, \dots x_r$ are said to be linearly dependent if there exist r scalars $k_1, k_2, \dots, k_r$ such that

$k_1 x_1 + k_2 x_2 + \dots + k_r x_r = 0$

where $k_1, k_2, \dots, k_r$ not all zeros.

(atleast one k value is a non zero no.)

* Linearly Independent vectors :

A set of r n-vectors $x_1, x_2, \dots x_r$ are said to be linearly independent if there exist r scalar $k_1, k_2, \dots k_r$ such that

$k_1 x_1 + k_2 x_2 + \dots + k_r x_r = 0$

all zeros.

* Criteria for L.I & L.D

1) If $\rho(A)$ = no. of given vectors OR $|A| \neq 0$, the given vectors are said to be L.I.

Ex.

Consider the vectors

$x_1 = (1 \ 2 \ 2)$, $x_2 = (2 \ 1 \ 2)$, $x_3 = (2 \ 2 \ 1)$

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 1(1-4) - 2(2-4) + 2(4-2)$$

$$= -3 + 4 + 4$$

$$= 5$$

$\therefore |A| = 5 \neq 0 \implies$ L.I.

2) If $\rho(A)$ < no. of given vectors OR $|A| = 0$, the given vectors are said to be L.D

$n > n$

Ex.

Consider the vectors

$x_1 = (1 \ 3 \ -2)$, $x_2 = (2 \ -1 \ 4)$, $x_3 = (1 \ -11 \ 14)$

$$|A| = \begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix} = 1(-14+44) - 3(28-4) - 2(-22+1)$$

$$= 30 - 72 + 42$$

$|A| = 0 \implies$ L.D.

3) If the given vectors are L.D. then any one of the vector can be expressed as linear combination of other vectors

diff. of

4) If the given vectors are L.I then it is not possible to write any one of the vector as linear combination of other vectors.

at least one elem. must be nonzero

5) Every nxn zero vector is L.I. vector.

Ex.

Consider the vector

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$kx = 0$$

$$k \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Non zero vector

$$\boxed{\therefore k = 0}$$

$\therefore x \rightarrow$ L.I vector

columns/rows of Identity matrix

6) The set of unit vectors are always L.I.

Ex. Consider the set of vectors

$x_1 = (1, 0, 0)$, $x_2 = (0, 1, 0)$, $x_3 = (0, 0, 1)$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$|A| = 1 \neq 0 \implies$ L.I vectors

7) The set of vectors having atleast one zero vector are L.D.

Ex. Consider the set of vectors

$x_1 = (1, 2, 3)$, $x_2 = (0\ 0\ 0)$, $x_3 = (1\ {-1}\ 4)$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & -1 & 4 \end{vmatrix} = 0 \quad \therefore x_1 = x_2 = x_3 \Rightarrow \text{L.D.}$$

$|A| = 0 \implies$ L.D.

No. of vectors = r
No. of elements in each vector = n

8) A set of r vectors with $r < n$ components (elements) are always L.I., provided the vectors should not be in the same direction.
Ex. Consider the vectors
$(1 \ -1 \ 2 \ 1), (1 \ 2 \ 3 \ 4), (2 \ 3 \ 4 \ 9)$

$r = 3$
$n = 4$

here, $r < n \Longrightarrow$ L.I

9) A set of r vectors with $r > n$ components then given vectors are L.D.
Ex. Consider the vectors

$x_1 = (1 \ 1 \ 2)$
$x_2 = (1 \ -1 \ 0)$ $\quad r = 4$
$x_3 = (1 \ -1 \ 5)$ $\quad n = 3$
$x_4 = (9 \ -5 \ 4)$ $\quad$ here, $r > n \Longrightarrow$ L.D.
$\therefore x_1, x_2, x_3, x_4$ are L.D.

10) A set of r vectors with $r = n$ components may be L.I or L.D

* Dimension & Basis of the vectors

* Dimension :- It is defined as no. of L.I. vectors.
Dimension = No. of L.I. vectors = no. of non zero rows in Row Echelon form = no. of non zero columns in Column Echelon Form.

* Basis :- It is defined as the set of L.I. vectors.
Basis = set of L.I vectors = set of non zero rows in R.E. form = set of non zero columns in C.E. form.

Problem 1: Test whether the following vectors are L.D or L.I. Also find their dimension & basis.

$(1 \;\; 1 \;\; -1 \;\; 0)$, $(4 \;\; 6 \;\; -3 \;\; 1)$, $(-6 \;\; 2 \;\; 2 \;\; 2)$, $(9 \;\; 9 \;\; -6 \;\; 3)$

$r = 4$ $n = 4$ here $r = n$ ∴ may be LD or L.I

$$A = \begin{vmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ -6 & 2 & 2 & 2 \\ 9 & 9 & -6 & 3 \end{vmatrix}$$

$R_2 \to R_2 - 4R_1$ $R_3 \to R_3 + 6R_1$ $R_4 \to R_4 - 9R_1$

$$A = \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 3 & 3 \end{vmatrix}$$

$-3+4$

$2+6$ $2+6$

$-6+9$

zero

... row only not previous row

$R_2 \longleftrightarrow R_3$

$$A = \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{vmatrix}$$

$R_4 \to R_4 - 3R_3$

$$A = \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

3 L.I. vectors

whenever a matrix is reduced to E.F form then Rank of matrix = Dimension of matrix.

Set of L.I vectors are called basis

$\rho(A) = 3 <$ No. of given vectors (4)

$\therefore x_1\ x_2\ x_3\ x_4 \Longrightarrow$ L.D

$$\begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 1 & 1 \end{vmatrix} \neq 0$$

Dimension = 3

Basis = { (1 1 −1 0), (0 8 −4 2), (0 0 1 1) }

Gate (2m)

Problem: If $q_1, q_2, \ldots, q_m$ are n-dimensional vectors with $m < n$. The vectors are L.D. The matrix Q is $q_1, q_2, \ldots q_m$ as columns. The rank of Q = ?

a) m b) n c) between m & n d) $< m$

$$q_1 = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$Q = \begin{array}{cccc} q_1 & q_2 & q_3 & \cdots \; q_m \end{array}_{n \times m} \Rightarrow (Q)_{n\times m}$$

$\rho(Q_{n\times m}) \leq \min(n, m)$

but $m < n$ — given

$\therefore \rho(Q_{n\times m}) \leq m$ — m ✗, $< m$ ✓ from criteria (2)

$\therefore \boxed{\rho(Q) < m}$

* Nullity of a matrix :-

* Nullity :- It is denoted by $N(A)$.
& is defined as the difference betn order of matrix and rank of matrix.
i.e.,

$$\boxed{N(A) = n(A) - \rho(A)}$$

$N(A) \rightarrow$ Nullity, $n(A) \rightarrow$ order, $\rho(A) \rightarrow$ Rank

⟹ Nullity of a non-singular matrix is always zero.

Let A be an $n \times n$ non singular matrix.
Then,

$$|A_{n\times n}| \neq 0$$

$$\therefore \rho(A) = n$$

$$N(A) = n(A) - \rho(A)$$

$$= n - n$$

$$\therefore N(A) = 0$$

Problem: The nullity of $A = \begin{pmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{pmatrix} = 1$. The value of $k = ?$

$$N(A) = n(A) - \rho(A)$$

$$1 = 3 - \rho(A)$$

$$\rho(A) = 2$$

for 3×3 matrix
iff $\rho(A) = 3$ then $|A| \neq 0$

$$|A| = 0 \Rightarrow$$

$$\begin{vmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = 0$$

$$k = -1$$

$ax + by + cz = 0 \implies$ Homogeneous

$ax + by + cz = 2 \implies$ Non - Homogeneous

classmate

Date

Page

Problem : The rank of matrix A is 5 & nullity of matrix is 3 then order of matrix = ?

$$N(A) = n(A) - \rho(A)$$

$$3 = n(A) - 5$$

$$n(A) = 8$$

* Non-homogeneous system of Linear Equation *

Consider the following non homogeneous system of m linear eq^{ns} in n unknowns.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \text{I}$$

Procedure

1) Write the given system of eq^{ns} in the form

$$AX = B$$

i.e.,

$k = 8$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{\text{coefficient matrix}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\text{sol}^n\text{ matrix}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{\text{column matrix of constants}}$$

2) Write the elements of matrix B in the last column of matrix A. The resulting matrix is called Augmented matrix & is denoted by $(A|B)$

$$(A|B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

3) Reduce the augmented matrix $(A|B)$ into Row Echelon form & hence find rank of A & rank of $(A|B)$

4) If $\rho(A) < \rho(A|B)$ OR $\rho(A|B) \neq \rho(A)$, the given system of equations ~~are said to~~ have no sol^n. (inconsistent).

5) If $\rho(A|B) = \rho(A) =$ no. of unknowns, the given system of eq^n. have unique solution.

6) If $\rho(A|B) = \rho(A) <$ no. of unknowns, the given system of eq^n. have infinite no. of solutions.

7) If the given system of equations have a sol^n (unique or infinite sol^n).
The sol^n. can be found by reducing the matrix AB into Row Echelon form & by using back code substitution, the variables $x_1, x_2, \dots x_n$ can be found.

Note :- If the total no. of eq^{ns} < total no. of variables, the given system of eq^n. have infinite no. of sol^{ns}.

These infinite no. of sol^{ns} can be found by assigning $(n-r)$ variables as arbitrary constants.
These $(n-r)$ sol^{ns} are Linearly Independent sol^{ns}.

classmate

Date

Page

$x + y = 3$

$r = 1$, $n = 2$

$r < n$, $n - r = 2 - 1 = 1$

put $\boxed{y = c} \rightarrow$ L.I sol^n

$x + y = 3$

$x + c = 3$

$x = 3 - c$

$x = 3 - c$ & $y = c$

Note: Consider the system of eq^n.

$ax + by = e$

$cx + dy = f$

The above system of eq^{ns} have

1) No sol^n if $\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f}$

2) Unique sol^n if $\frac{a}{c} \neq \frac{b}{d}$

3) Infinite sol^n if $\frac{a}{c} = \frac{b}{d} = \frac{e}{f}$

ECE 2010 (1m)

Problem 1: The system of eq^{ns} $4x + 2y = 7$

$2x + y = 6$ have

$\frac{4}{2} = \frac{2}{1} \neq \frac{7}{6}$ $\left(\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f} \right)$

$\therefore$ The given system of eq^{ns} have no sol^n.

Problem 2: $x + 2y = 5$

$2x + 3y = 9$

$\frac{1}{2} \neq \frac{2}{3} \neq \frac{5}{9}$, $\frac{1}{2} \neq \frac{2}{1}$ $\left(\frac{a}{c} \neq \frac{b}{d} \right)$

$\therefore$ Unique sol^n

Classmate
Date
Page

roblem B: $x + y = 3$
$3x + 3y = 9$

$\frac{1}{3} = \frac{1}{3} = \frac{3}{9}$ $\therefore$ Infinite sol^n

Problem C: How many solutions does the following system of eq^{ns} have
i) $-x + 5y = -1$
$x - y = 2$
$x + 3y = 3$

a) Infinite b) Exactly 2 c) Unique sol^n d) No sol^n

$$(A|B) = \left(\begin{array}{cc|c} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{array}\right)$$

$R_2 \to R_2 + R_1$ $R_3 \to R_3 + R_1$

$$= \left(\begin{array}{ccc} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{array}\right)$$

$R_3 \to R_3 - 2R_2$

$$\left(\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

$\rho(A|B) = 2$ $\rho(A) = 2$
$\rho(A|B) = \rho(A) =$ No. of unknowns $= 2$
$\therefore$ Unique sol^n

(2m)

Problem 5: Consider following non homogeneous system of Linear eq^n in 3 variables x_1, x_2, x_3.

$$2x_1 - x_2 + 3x_3 = 1$$
$$3x_1 + 2x_2 + 5x_3 = 2$$
$$-x_1 + 4x_2 + x_3 = 3$$

The above system of eq^n have —

a) No sol^n b) Unique sol^n c) More than 1 but finite no of sol^{ns} d) Infinite no. of sol^{ns}

$$(A|B) = \begin{pmatrix} 2 & -1 & 3 & 1 \\ 3 & 2 & 5 & 2 \\ -1 & 4 & 1 & 3 \end{pmatrix}$$

0 sol^n

$R_2 \to 2R_2 - 3R_1$ $\quad R_3 \to 2R_3 + R_1$

$$(A|B) = \begin{pmatrix} 2 & -1 & 3 & 1 \\ 0 & 7 & 1 & 1 \\ 0 & 7 & 5 & 7 \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$(A|B) = \begin{pmatrix} 2 & -1 & 3 & 1 \\ 0 & 7 & 1 & 1 \\ 0 & 0 & 4 & 6 \end{pmatrix}$$

$\rho(A|B) = 3$ $\quad \rho(A) = 3$

No. of unknowns $= 3$

$\therefore$ Unique sol^n

2010 (2m)

Problem 6: For the set of eq^{ns} $x_1 + 2x_2 + x_3 + x_4 = 2$
$3x_1 + 6x_2 + 3x_3 + 3x_4 = 6$

which of the foll. stmt is true —

1) There exist only trivial sol^n
2) There are no sol^{ns}
3) Unique non trivial sol^n non zero sol^n
4) Infinite no of non ~~trivial~~ sol^n

classmate

Date

Page

$\Longrightarrow$ No of eq^{ns} $(r) = 2$

No of variables $(n) = 4$

here $r < n$

Infinite sol^n (Non trivial)

Problem 7: The value of x_3 obtained by solving the foll. system of eq^{ns}.

$$x_1 + 2x_2 - 2x_3 = 4$$
$$2x_1 + x_2 + x_3 = -2$$
$$-x_1 + x_2 - x_3 = 2$$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 2 & 1 & 1 & -2 \\ -1 & 1 & -1 & 2 \end{array}\right)$$

$R_2 \to R_2 - 2R_1$ $\qquad$ $R_3 \to R_3 + R_1$

$$(A|B) = \left(\begin{array}{cccc} 1 & 2 & -2 & 4 \\ 0 & -3 & 5 & -10 \\ 0 & 3 & -3 & 6 \end{array}\right)$$

$R_3 \to R_3 + R_2$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 0 & -3 & 5 & -10 \\ 0 & 0 & 2 & -4 \end{array}\right)$$

$$2x_3 = -4$$
$$\boxed{x_3 = -2}$$

(To find x_2

$$-3x_2 + 5x_3 = -10$$
$$-3x_2 - 10 = -10$$
$$-3x_2 = 0$$
$$\boxed{x_2 = 0}$$)

2011 (cm)

Problem 8 : find the values of λ, μ for which the following system of eq^{ns}

$$x + y + z = 6$$
$$x + 4y + 6z = 20$$
$$x + 4y + \lambda z = \mu$$ have

1) No sol^n
2) Unique sol^n
3) infinite sol^n

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{array}\right)$$

$R_2 \to R_2 - R_1$ $\quad$ $R_3 \to R_3 - R_1$

$$(A|B) = \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 3 & \lambda - 1 & \mu - 6 \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$(A|B) = \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 0 & \lambda - 6 & \mu - 20 \end{pmatrix}$$

$\cancel{\rho(A) = 3}$
$\cancel{\rho(A|B) = 3}$

1) **No sol^n** : If $\rho(A) < \rho(A|B)$

If $\lambda = 6$, $\mu \neq 20$

$\rho(A) = 2$ $\quad$ $\rho(A|B) = 3$

$\Rightarrow \rho(A) < \rho(A|B) \longrightarrow$ No sol^n

2) **Unique sol^n** : If $\rho(A) = \rho(A|B) =$ No of unknowns

For $\lambda \neq 6$ & $\mu \longrightarrow$ any valu $\mu = 20$ or $\mu \neq 20$

classmate

Date ______

Page ______

3) Infinite no. of sol^n :— If $\rho(A) = \rho(A|B) <$ No. of unknowns

No. of unknowns = 3

$\rho(A)$ & $\rho(A|B)$ should be less than 3 must be 2 bcoz rank can't be 0 & 1

It is possible if

$\lambda = 6$ & $\mu = 20$ $\Longrightarrow$ Infinite sol^n set

H.W

Problem 9 : Find the values of A & B for which the following system of eq^{ns}

$$x + 2y + 3z = 6$$
$$x + 3y + 5z = 9$$
$$2x + 5y + az = b$$

have

1) No sol^n 2) Unique sol^n 3) Infinite sol^{ns}

Gate 2M

Problem 10 : For what values of α & β, the following system of eq^{ns}

$$x + y + z = 5$$
$$x + 3y + 3z = 9$$
$$x + 2y + \alpha z = \beta$$

have infinite no of sol^n ?

a) $\alpha = 2$, $\beta = 7$ c) $\alpha = 3$, $\beta = 4$

b) $\alpha = 7$, $\beta = 2$ d) $\alpha = 4$, $\beta = 3$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array}\right)$$

$R_2 \longrightarrow R_2 - R_1$ $R_3 \longrightarrow R_3 - R_1$

$$= \left(\begin{array}{cccc} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & \alpha - 1 & \beta - 5 \end{array}\right)$$

$R_3 \longrightarrow 2R_3 - R_2$

Non-homogeneous System eqⁿ ⟹ draw augmented matrix

classmate

Date

Page

$$A = \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 2\alpha-4 & 2\beta-14 \end{pmatrix}$$

$\rho(A) = \rho(A|B) <$ No. of unknowns

$\rho(A) = \rho(A|B) < 3$

$2\alpha - 4 = 0$ $\qquad$ $2\beta - 14 = 0$

$2\alpha = 4$ $\qquad$ $2\beta = 14$

$\boxed{\alpha = 2}$ $\qquad$ $\boxed{\beta = 7}$

Problem 11 : If A is 3×4 matrix & the non homogeneous system of equations $AX = B$ is inconsistent (No sol The highest possible rank of A is

⟶

$(A|B)_{3\times5}$

$\rho\{(A|B)\}_{3\times5} \le \min(3,5)$

$\rho\{(A|B)\} \le 3$

$= 3$ Highest Possible Rank

But it is given that the given system of eqⁿˢ are inconsistent (No solⁿ)

For inconsistent → $\rho(A) < \rho(A|B)$

$\rho(A) < 3$ $\qquad$ $\rho(A) \to 2, 1, 0$

Highest possible rank of A $= \underline{\underline{2}}$

* Homogeneous system of Linear Eq^n *

Consider the following homogeneous system of Linear eq^{ns} in m equations & n unknowns

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = 0 \\ \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = 0 \end{array}\right\} \text{(I)}$$

Procedure to solve problems :-

1) Write the given system of eq^{ns} in the form $AX = 0$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

2) Reduce the matrix A into either row or column echelon form (but always row echelon form is preferable) OR find the determinant of matrix $|A|$.

3) If $\rho(A)$ = No. of unknowns (variables) OR $|A| \neq 0$ ($A \to$ Non singular matrix), the given system of eq^n have trivial sol^n ($x = y = z = 0$)

4) If $\rho(A) <$ No. of unknowns OR $|A| = 0$, the given system of eq^n have infinite no. of sol^n.

All these infinite no. of sol^n can be found by assigning $(n-r)$ variables as arbitrary constant. These $(n-r)$ sol^{ns} are called L.I. sol^{ns}.

$\boxed{AX = B}$

- $\boxed{B = 0}$
  - $\rho(A) = n$ or $|A| \neq 0$ → Trivial sol^n $(x = y = z = 0)$ → ~~L.D~~
  - $\rho(A) < n$ or $|A| = 0$ → Non-Trivial → Infinite no. of non zero sol^n → ~~L.I~~
- $\boxed{B \neq 0}$
  - $\rho(A) < \rho(A|B)$ → No sol^n
  - $\rho(A|B) = \rho(A) = n$ → Unique sol^n
  - $\rho(A|B) = \rho(A) < n$ → Infinite no. non zero sol^{ns}.

Problem 1: For what values of λ the system of eq^{ns}

$$x + y + z = 0$$
$$(\lambda+1)\, y + (\lambda+1) z = 0$$
$$(\lambda^2 - 1)\, z = 0$$

have 2 L.I sol^n ?

$\Rightarrow$

No. of L.I sol^n $= n - r = 2$

$= 3 - r = 2$

$\therefore \boxed{r = 1}$ → Rank of matrix

classmate
Date
Page

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ \cancel{\lambda+1} \; 0 & \lambda+1 & \lambda+1 \\ 3 & 0 & \lambda^2-1 \end{vmatrix} = 0 \Rightarrow \text{Upper } \Delta^{lar}$$

$\Rightarrow (\lambda+1)(\lambda^2-1) = 0$

$(\lambda+1)(\lambda+1)(\lambda-1) = 0$

$\lambda = -1, 1$ - Not correct
put in matrix in order to get
$\rho(A) = 1$

$\therefore \boxed{\lambda = -1}$

Problem 2 :- The rank of 3x3 matrix A is 1. The homogeneous system of eq^ns AX = 0 has
a) trivial sol^n b) 1 L.I. sol^n
c) 2 L.I. sol^ns d) 3 L.I. sol^ns

$(A)_{3\times3} \Rightarrow 3$ eq^ns & 3 variables

$n = 3 \quad r = 1 \Rightarrow r < n$
$\Downarrow$
Infinite sol^n

$\therefore$ L.I sol^n $= n - r$
$= 3 - 1$
$= 2$

2011 (pm)

Problem 3 :- The system of eq^ns: $2x_1 + x_2 + x_3 = 0$
$x_2 - x_3 = 0$
$x_1 + x_2 = 0$ have
a) No non trivial sol^n c) 5 Non trivial sol^ns
b) Unique non trivial sol^n d) Infinite non trivial sol^ns

$$\begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = 2[1] - 1[0+1] + 1[0-1]$$
$$= 2 - 1 - 1 = 0$$

Infinite sol^n

Problem 4: The system of eq^n

$$x + 3y - 2z = 0$$
$$2x - y + 4z = 0$$
$$x - 11y + 14z = 0$$

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix} = \cancel{+[-14+44] - 2[28-4] - 1[-22+1]}$$
$$= \cancel{30 - 48 + 21}$$

$$1[-14+44] - 3[28-4] + 2[-22+1]$$
$$= 0 \quad = 30 - 72 + 42 = 0$$

Problem 5: Find the values of k for which the following system of eq^{ns} have infinite no. of non trivial sol^{ns}.

$$\downarrow$$
$$|A| = 0$$

$$(3k-8)x + 3y + 3z = 0$$
$$3x + (3k-8)y + 3z = 0$$
$$3x + 3y + (3k-8)z = 0$$

$\Longrightarrow$ If $|A| = 0$, the given system of eq^{ns} have infinite no. of non zero sol^{ns}.

$$\begin{vmatrix} 3k-8 & 3\cancel{y} & 3\cancel{z} \\ 3 & 3k-8 & 3\cancel{z} \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

$$\Rightarrow (3k - 8 + 3\cancel{y} + 3\cancel{z})\ (3k - 8 - 3\cancel{y})\ (3k - 8 - 3\cancel{z}) = 0$$
$$(3k - 2)\ (3k - 11)\ (3k - 11) = 0$$

$$3k = 2 \qquad 3k = 11$$

$$\therefore k = \frac{2}{3}, \frac{11}{3}$$

classmate

Date
Page

H.W

Problem 6 : Find the real value of λ for which the foll system of eq^{ns} have non trivial sol^{ns}

infinite no. of non trivial sol^{n}

$$x + 2y + 3z = \lambda x$$
$$3x + y + 2z = \lambda y$$
$$2x + 3y + z = \lambda z$$

$\boxed{\lambda = 6}$

imp

* Eigen values & Eigen Vectors *

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\underbrace{A - \lambda I}_{\text{characteristic matrix}} = \begin{pmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix}$$

ch. det, ch. polynomial $\rightarrow$ $|A - \lambda I| = \begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 4$

$= \lambda^2 - 8\lambda + 16 - 4$

ch. eq^{n} $\leftarrow$ $|A - \lambda I| = 0 \implies \lambda^2 - 8\lambda + 12 = 0$

$\therefore \lambda = 6, 2$

$\downarrow$

$\therefore$ ch. roots or Eigen values

Null vector can not be Eigen vector

* **Eigen values:** let A be an $n \times n$ matrix. λ is a scalar (some constant). The matrix $A - \lambda I$ is called as characteristic matrix.

$|A - \lambda I|$ is called characteristic determinant or characteristic polyoomial.

The roots of this ch. det. are called char roots OR Eigen values OR Latent roots OR Proper values.

The set of eigen values of matrix A is called as spectrum of A.

* **Eigen Vector:** If λ is an eigen value of a matrix A then there exist a non zero vector X such that $AX = \lambda X$, then the non zero vector X is called as Eigen vector.

Note:

$$AX = \lambda X$$

$$AX = \lambda X I$$

$$AX - \lambda X I = 0$$

$$A\underline{X} - \lambda I \underline{X} = 0$$

$$\boxed{(A - \lambda I)\, X = 0}$$

Trivial sol^n $\Rightarrow$ $|A - \lambda I| \neq 0$

$\underbrace{\text{Infinite sol}^n}_{\text{Non trivial sol}^n}$ $\Rightarrow$ $|A - \lambda I| = 0$

$\Downarrow$

Eigen Vectors

Note: Consider the matrix

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

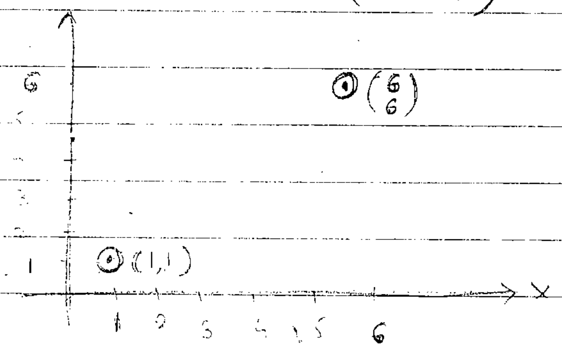
$$\underset{\downarrow \atop A}{\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}} \underset{\downarrow \atop X}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} 4+2 \\ 2+4 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} = \underset{\uparrow \atop \lambda}{6} \underset{\uparrow \atop X}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

(Non zero vector)

$X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is Eigen vector corresponding to eigen value $\lambda = 6$ for matrix $A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$

→ Any pt / vector exist along the same vector will also be an eigen vector corresponding to the same eigen value.

Ex. (2,2) (3,3) ... (7,7) (8,8) ...



Eg $A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$

$$\underset{\downarrow \atop A}{\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}} \underset{\downarrow \atop X}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} = \begin{pmatrix} 4+4 \\ 2+8 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \end{pmatrix} = 2 \underset{\downarrow \atop X}{\begin{pmatrix} 4 \\ 5 \end{pmatrix}}$$

Not same

$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is not eigen vector corresponding to eigen value 2

Problem 1: Find the eigen values & eigen vector of

$$A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

Symmetric matrix

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) - 16$$

$$= -9 - \cancel{3\lambda} + \cancel{3\lambda} + \lambda^2 - 16 = 0$$

$$\lambda^2 - 25 = 0$$

$$\lambda^2 = 25$$

$$\lambda = \pm 5$$

$\therefore$ Eigen values $= 5, -5$

case i) $(A - \lambda I) X = 0$

$$\begin{pmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigen vectors

put $\lambda = 5$

$$\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + 4x_2 = 0 \quad \text{---} (1)$$

$$4x_1 - 8x_2 = 0 \quad \text{---} (2)$$

eqn (1) $\Rightarrow$ divide b.s. by (-2)

$$x_1 - 2x_2 = 0$$

$$\frac{x_1}{x_2} = \frac{2}{1}$$

$x_1 = 2x_2$

$\therefore X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

classmate

Date ______

Page ______

Case ii) when $\lambda = -5$

$(A - \lambda I) X = 0$

$$\begin{pmatrix} 4-\lambda & 4 \\ 4 & -3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

put $\lambda = -5$

$$\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$8x_1 + 4x_2 = 0$ ———— (1)

$4x_1 + 2x_2 = 0$ ———— (2)

2 vectors are not dep.. only 1 vector is dep

$2x_1 + x_2 = 0$

$2x_1 = -x_2$

$\dfrac{x_1}{x_2} = \dfrac{-1}{2}$

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$X_1^T X_2 = 0$ or $X_1 X_2^T = 0 \Rightarrow X_1$ & X_2 are orthogonal

$$X_1^T \cdot X_2 = \begin{pmatrix} 2 & 1 \end{pmatrix}_{1\times 2} \begin{pmatrix} -1 \\ 2 \end{pmatrix}_{2\times 1}$$

$$(X_1^T \cdot X_2)_{1\times 1} = -2+2 = 0$$

$\therefore$ X_1 & $X_2 \Rightarrow$ orthogonal vectors

E.V. of Upper Δ, Lower Δ, diag... only
∴ diagonal elem. only

classmate
Date
Page

Note 1: The Eigen vectors corresponding to diff. eigen value of a real symmetric matrix are always orthogonal.

Note 2: If the Eigen vectors corresponding to diff. eigen values of any square matrix are always Linearly Independent.

$$x_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad x_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\therefore x_1 \neq k x_2 \quad \text{OR} \quad x_2 \neq k x_1$$

$$\therefore \left.\begin{matrix} x_1 \\ x_2 \end{matrix}\right\rangle \text{L.I.} \qquad (\text{acc}^n \text{ to criteria (4)})$$

In the above ex. the eigen values of matrix are 5, -5 the corresponding eigen vectors are $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

* Consider the matrix $A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$

$\lambda = 2, 2 \longrightarrow$ Repeated twice

when $\lambda = 2$ Max. no. of L.I. vectors

$(A - \lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

put $\lambda = 2$

$$\begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$0x_1 + 3x_2 = 0$$

$$3x_2 = 0 \qquad \therefore x_2 = 0$$

$x_1 \neq 0$

put $x_1 = c$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix} \text{ where } c \neq 0$$

Acc" to criteria (5)

every non zero vector is L.I vector.

Note:- If some of the eigen value of a matrix are repeated then the eigen vectors corresponding to repeated eigen values may be L.I or L.D.

If an eigen value λ is repeated n times the eigen vectors corresponding to repeated eigen values are always L.I. which are given by

$$p = n - r \quad ; \quad 1 \leq p \leq m \quad \{\text{max. m times}\}$$

$n \rightarrow$ no. of unknowns or variables

$r \rightarrow \rho(A - \lambda I)$

$\boxed{A_{2\times 2}}$

λ_1 & λ_2

- $\lambda_1 \neq \lambda_2$
  - x_1, x_2 → L.I
- $\lambda_1 = \lambda_2$
  - x_1 → L.I
  - x_1 & x_2 → max. 2 times → L.I

9.

$$\boxed{A_{3\times 3}}$$

$\downarrow$

$\lambda_1, \lambda_2, \lambda_3$

$\lambda_1 \neq \lambda_2 \neq \lambda_3$	$\lambda_1 \neq \lambda_2 \neq \lambda_3$	$\lambda_1 = \lambda_2 = \lambda_3$
$x_1 \quad x_2 \quad x_3$	$x_1 \quad x_2 \quad x_3$	$x_1 \quad\quad x_1, x_2 \quad\quad x_1, x_2, x_3$
L.I	L.I $\quad$ L.I or L.D	L.I $\quad$ L.I $\quad$ L.I
	L.I	

$$\begin{pmatrix} c_1+c_2 \\ c_2 \\ c_1 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$x_1 \quad x_2$

L.I

2011 wale Instr. (1m)

Note:-

Eigen vector of A

$$A\,X = \lambda\,X$$

Eigen value of A

$$A^{-1} \cdot A \cdot X = A^{-1} \cdot \lambda \cdot X$$

$$I \cdot X = A^{-1} \cdot \lambda \cdot X$$

$$X = \lambda \cdot A^{-1} X$$

$$\frac{X}{\lambda} = A^{-1} X$$

$$A^{-1} X = \frac{1}{\lambda} X$$

Eigen vector of A^{-1}

Eigen value of A^{-1}

If λ is eigen value of A & X is eigen vector of A but $\frac{1}{\lambda}$ is eigen value of A^{-1} and

classmate

Date

Page

X is eigen vector of A^{-1}. Therefore, A & A^{-1} have same eigen vectors.

A & A^m have same eigen vectors ($m > 0$) corresponding to eigen values

$AX = \lambda X$

$AAX = A\lambda X$

$A^2 X = \lambda \lambda X$

$A^2 X = \lambda^2 X$

17/07/2012

* Properties of Eigen Values & Eigen vectors :-

1) Sum of eigen values is equal to $\underset{\text{sum of digonal elem}}{\text{trace of}}$ matrix.

i.e., if $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigen values of a matrix A then

(i) Trace $A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

2) (ii) $|A| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \ldots \lambda_n$

Ex. Consider the matrix

$\lambda_1 + \lambda_2 = 7$

$\lambda_1 \cdot \lambda_2 = 6$

$$A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$$

$$\begin{matrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{matrix}$$

$(5-\lambda)(2-\lambda) - 4$

$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = \lambda^2 - 7\lambda + 6$

$\lambda = 1, 6$

i) $\longrightarrow$ $1 + 6 = 5 + 2$ $\qquad \therefore$ Trace of $A = 7$

ii) $\longrightarrow$ $1 \times 6 = 10 - 4 = 6$ $\qquad |A| = 6$

3) The eigen values of upper triangular or lower triangular or diagonal or scalar or identity matrix is its diagonal elements.

classmate

Date

Page

1) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{pmatrix}$; $\lambda = 1, -4, 7$

$\downarrow$

Upper Δ^{lar}

2) $A = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 5 & 0 \\ 0 & 8 & 9 \end{pmatrix}$; $\lambda = 3, 5, 9$

$\downarrow$

Lower Δ^{lar}

3) $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$; $\lambda = 3, 4, 5$

$\downarrow$

Diagonal matrix

4) $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$; $\lambda = 3, 3, 3$

$\downarrow$

scalar matrix

5) $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $\lambda = 0, 0, 0$

$\downarrow$

scalar / Null matrix

6) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $\lambda = 1, 1, 1$

$\downarrow$

Identity matrix

4) The eigen values of A & A^T are same

5) The eigen values of A & $P^{-1}AP$ are same where P is a non singular matrix.

6) The eigen values of real symmetric matrix are real.

7) The eigen values of skew symmetric matrix are purely imaginary OR zeros.

8) The eigen values of orthogonal matrix are of unit modulus i.e., ± 1

9) If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also one of its eigen value

10) If λ is an eigen value of matrix A then

i) $k\lambda$ is an eigen value of kA.

ii) $\frac{1}{\lambda}$ is also an eigen value of A^{-1}.

iii) λ^2 — " — A^2.

iv) λ^m — " — A^m.

v) $\frac{|A|}{\lambda}$ is an eigen value of adj. A

vi) $\lambda \pm k$ — " — $A \pm kI$

vii) $(\lambda \pm k)^2$ — " — $(A \pm kI)^2$

viii) $\frac{1}{\lambda \pm k}$ — " — $(A \pm kI)^{-1}$

Note: If A is a singular matrix i.e., $|A| = 0$ then one of its eigen value should be zero

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = |A|$$
$$\downarrow$$
$$0 \cdot \lambda_2 \cdot \lambda_3 = |A|$$
$$|A| = 0 \qquad \therefore |A| \longrightarrow \text{singular}$$

2m

Problem 1: $A = \begin{pmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{pmatrix}$ are .

$j = \sqrt{-1}$

A) $3,\ 3+5j,\ 6-j$

B) $-6+5j,\ 3-j,\ 3+j$

C) $3-j,\ 3+j,\ 5+j$

d) $3,\ -1+3j,\ -1-3j$

by Prop. 1 :

$$\text{Trace } A = \lambda_1 + \lambda_2 + \lambda_3 \qquad [\text{cross check}]$$
$$-1-1+3 = 3 - 1 + 3j - 1 - 3j$$
$$1 = 1$$

If (1) or more opt^n satisfy more ... (1, 2) ...

Problem 2: The eigen values & eigen vector of a 2×2 matrix are given by

Eigen value — Eigen vector

$\lambda_1 = 8$ — $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 4$ — $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

the matrix is ——

a) $\begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$ b) $\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix}$

To find eigen value from eigen vector

Formula, $(A - \lambda I) X = 0$

1998 (2M)

Problem 3: The vector $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is an eigen vector of the matrix

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

The eigen value corresponding to the eigen vector is —

$\Rightarrow$

$$(A - \lambda I) X = 0$$

$$\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

λ value consider any 1 row & multiply

$$-1 - 4 + \lambda = 0$$

$$\boxed{\lambda = 5}$$

Problem 4: For the matrix $\begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}$ the eigen value corresponding to the eigen vector $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ is

$$(A - \lambda I) X = 0$$

$$\begin{pmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2 + 4 - \lambda = 0$$

$$\lambda = 6$$

Problem 5: The min & max eigen values of a matrix $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ are -2 & 6 resp. what would be the 3^{rd} eigen value.

$$\text{Trace } A = \lambda_1 + \lambda_2 + \lambda_3$$

$$\Rightarrow 1+5+1 = -2+6+\lambda_3$$

$$\boxed{\lambda_3 = 3}$$

Problem 6: The matrix $\begin{pmatrix} 1 & 2 & 4 \\ 3 & 0 & 6 \\ 1 & 1 & p \end{pmatrix}$ has an eigen value $= 3$. Sum of other two eigen values is —

$$1+0+p = 3 + \lambda_2 + \lambda_3$$

$$p+1-3 = \lambda_2 + \lambda_3$$

$$\boxed{\lambda_2 + \lambda_3 = p-2}$$

2011

Problem 7: Consider the following matrix

$$A = \begin{pmatrix} 2 & 3 \\ x & y \end{pmatrix}$$ The eigen values of A are 4 & 8. The values of x, y are

prop 1 $\longrightarrow$ $2+y = 4+8$

$$y = 12-2 = 10$$

prop. 2 $\longrightarrow$ $\cancel{2 \times 10 = 20}$ $\quad 4 \times 8 = 32$

$$\begin{vmatrix} 2 & 3 \\ x & 10 \end{vmatrix} = 32 \qquad x = -4$$

$$20 - 3x = 32$$

$$20 - 32 = 3x \qquad \therefore \cancel{x} = \frac{-10}{3}$$

$$3x = -12 \quad \boxed{x = -4}$$

$2y - 3x = 4 \times 8 \Rightarrow 2 \times 10 - 3x = 32$

$-3x = 32 - 20$

$-3x = 12$

$\boxed{x = -4}$

Pro 8 :- The eigen values of 3X3 matrix are given by 1, -3, 9. find

i) Trace $(A^2 + A^{-1} - adj.A)$

ii) det $(A^2 + A^{-1} - adj.A)$

$\Rightarrow \quad |A| = 1 \times -3 \times 9 = -27$

$A^2 + A^{-1} - adj.A$

$\rightarrow \quad \lambda^2 + \frac{1}{\lambda} - \frac{|A|}{\lambda}$ by prop. (10)

1

-3

9

$\lambda^2 + \frac{1}{\lambda} - \frac{|A|}{\lambda}$

$(1)^2 + \frac{1}{1} - \frac{(-27)}{1} = \underline{\underline{29}}$

$(-3)^2 + \frac{1}{(-3)} - \frac{(\cancel{-}27)}{(\cancel{-}3)}$

$9 - \frac{1}{3} - 9 = \underline{\underline{-1/3}}$

$9^2 + \frac{1}{9} - \frac{(-\cancel{27})^3}{\cancel{9}}$

$81 + \frac{1}{9} + 3$

$\frac{729 + 1 + 27}{9} = \underline{\underline{\frac{757}{9}}}$

i) Trace $(A^2 + A^{-1} - adj.A) = 29 - \frac{1}{3} + \frac{757}{9}$

ii) $\det(A^2 + A^{-1} - \text{adj}\, A) = 29 \times \left(\frac{-1}{3}\right) \times \frac{757}{9}$

Prob. 9 :- Given $A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{pmatrix}$, the eigen values of $3A^3 + 5A^2 - 6A + 2I$ are

Upper Δ^{lar}

$\therefore \lambda = 1, 3, -2$

$$3A^3 + 5A^2 - 6A + 2I \Longrightarrow 3\lambda^3 + 5\lambda^2 - 6\lambda + 2$$

put $\lambda = 1$

$$3(1)^3 + 5(1)^2 - 6(1) + 2$$
$$3 + 5 - 6 + 2 = 4$$

put $\lambda = 3$

$$3 \times 27 + 5 \times 9 - 18 + 2$$
$$81 + 45 - 16 = 126 - 16 = 110$$

put $\lambda = -2$

$$-(3 \times 8) + 20 + 12 + 2$$
$$-24 + 24 = 0$$
$$3 \times (-8) + 5 \times 4 + 12 + 2$$
$$-24 + 20 + 12 + 2 = 10$$

2009 (em)

Prob. No 10 :- The eigen values of 2×2 matrix A are given by -2 & -3 resp. the eigen values of $(x + I)^{-1} \cdot (X + 5I)$

This is not possible

$-\frac{57}{9}$ $\Longrightarrow$ $(X+I)^{-1}(X+5I) = (X+I)^{-1}\{(X+I) + 4I\}$

$$= (X+I)^{-1} \cdot (X+I) + 4I \cdot (X+I)^{-1}$$
$$= I + 4(X+I)^{-1}$$

$A + kI$

$$4(X+I)^{-1} + I \Longrightarrow \frac{4}{1+\lambda} + 1$$

$$\Rightarrow \frac{4}{1+\lambda} + 1$$

put $\lambda = -2$ $\qquad \frac{4}{1-2} + 1 = -3$

put $\lambda = -3$ $\qquad \frac{4}{1-3} + 1 = -1$

Prob 11 :- The eigen vector of a 3×3 matrix are $\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}$ are orthogonal. What will be the 3rd orthogonal eigen vector.

$X = \begin{pmatrix}1\\0\\1\end{pmatrix}$ $\quad X_2 = \begin{pmatrix}1\\0\\-1\end{pmatrix}$. Let $X_3 = \begin{pmatrix}x\\y\\z\end{pmatrix}$

here, 3 vectors are orthogonal

$\therefore X^T \cdot X_3 = 0 \qquad X_2^T \cdot X_3 = 0$

$$\begin{pmatrix}1 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \begin{pmatrix}1 & 0 & -1\end{pmatrix} \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$x + z = 0$ —— (i) $\qquad x - z = 0$ —— (ii)

add (i) & (ii)

$2x = 0$

$\therefore x = 0$ $\quad$ put in (i)

$0 + z = 0$

$z = 0$

$y \neq 0$ $\quad \because$ zero vector can't be eigen vector

$\Rightarrow \therefore y = c$ $\qquad c$ — arbitrary constant & $c \neq 0$

∴ The 3rd eigen vector is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ c \\ 0 \end{pmatrix}$

$$x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

2008 (2m)

Prob. 12 :- The eigen vector of 2×2 matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ are given by $\begin{pmatrix} 1 \\ a \end{pmatrix}, \begin{pmatrix} 1 \\ b \end{pmatrix}$, what is $(a+b)$?

$$\lambda = 1, 2.$$

$AX = \lambda X$ OR $(A - \lambda I)X = 0$

put $\lambda = 1$

$$\underset{2\times2}{\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}} \underset{2\times1}{\begin{pmatrix} 1 \\ a \end{pmatrix}} = \cancel{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} 1 \begin{pmatrix} 1 \\ a \end{pmatrix}$$

~~1+2a 2a~~ $1 + 2a = 1$

& $2a = a$ $\boxed{a = 0}$

put $\lambda = 2$

$$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ b \end{pmatrix} = 2 \begin{pmatrix} 1 \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 2b \end{pmatrix}$$

$$1 + 2b = 2$$

$$2b = 1$$

$$\boxed{b = 1/2}$$

Method I :-

Case 1 : when $\lambda = 1$

$(A - \lambda I) X = 0$

$$\begin{pmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Prob.

$$\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$2x_2 = 0 \qquad \therefore x_2 = 0$

here $x_1 \neq 0$

$x_1 = c \qquad c \to$ non zero arbitrary const

$$X = \begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

generally c is replaced by 1

Case 2 : when $\lambda = 2$

$$\begin{pmatrix} 1-2 & 2 \\ 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-x_1 + 2x_2 = 0$

$2x_2 = x_1$

$\frac{2}{1} = \frac{x_1}{x_2}$

$$X_2 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

$x_1 = 2 \qquad x_2 = 1$

$$\frac{x_1}{x_2} = \frac{1}{(1/2)}$$

$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix} \Rightarrow a = 0$$

$$x_2 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ b \end{pmatrix} \quad b = 1/2$$

Prob. 13: Find eigen values & eigen vectors of foll.

a) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

$\Longrightarrow \lambda = 1, 2, 3$

const

for dis[illegible] values
→ Consider [illegible] rows?
→ select [illegible] middle elem
→ If [illegible] elem. then [illegible]

→ put [illegible]

→ put $\lambda = 1$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 \end{array}$$

$$\frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$x_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

other form of [illegible] vector

→ put $\lambda = 2$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 \end{array}$$

$$\frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{0}$$

$$x_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

→ put $\lambda = 3$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 1 & 0 & -2 & 1 \\ -1 & 2 & 0 & -1 \end{array}$$

$$\frac{x_1}{2} = \frac{x_2}{4} = \frac{x_3}{2}$$

$$x_3 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

b) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{pmatrix}$

$\lambda = 1, -4, 7$

put $\lambda = 1$

	x_1	x_2	x_3
2	3	0	2
-5	2	0	-5

$$\frac{x_1}{19} = \frac{x_2}{0} = \frac{x_3}{0} \qquad X_1 = \begin{pmatrix} 19 \\ 0 \\ 0 \end{pmatrix}$$

put $\lambda = -4$

	x_1	x_2	x_3
2	3	5	2
-8	2	0	-8

$$\frac{x_1}{28} = \frac{x_2}{-10} = \frac{x_3}{-40} \qquad X_2 = \begin{pmatrix} 28 \\ -10 \\ -40 \end{pmatrix}$$

put $\lambda = 7$

	x_1	x_2	x_3
2	3	-6	2
-11	2	0	-11

$$\frac{x_1}{37} = \frac{x_2}{12} = \frac{x_3}{66} \qquad X_3 = \begin{pmatrix} 37 \\ 12 \\ 66 \end{pmatrix}$$

Note :- To find the eigen vectors corresponding to nonrepeated eigen value of a matrix, we proceed as follows :

1) Select the 1^{st} two rows only
2) Start from the 1^{st} row middle no. & move in anticlockwise direction to complete 1 cycle. If any element exist in the main diagonal while we are moving in anticlockwise direction, then the eigen value should be subtracted from the corresponding diagonal elements.
These elements will be return seperately in row wise
3) Repeate the same procedure with 2^{nd} row also.
& find eigen vectors

* Cayley - Hamiltom theorem *

Statement :
Every square matrix satisfies its own characteristic eqn.
Ex. Consider the matrix $A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$

Its char. eqn. is $\lambda^2 - 8\lambda + 12 = 0$
by cayley - Hamiltom theorem, every square matrix satisfies its own char. eqn.

i.e., $A^2 - 8A + 12I = 0$

Note : (2X2)
Consider the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

$$\Longrightarrow \lambda^2 - \lambda[\text{trace of } A] + |A| = 0$$

by cayley-hamilton theorem, it is

$$A^2 - A[\text{trace of } A] + |A| \cdot I = 0$$

Note: (3×3)
Consider the 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$, its characteristic eq^n is —

General procedure:—

$$\lambda^3 - \lambda^2(\text{trace of } A) + \lambda\left\{ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \right\} - |A| = 0$$

Using cayley Hamilton theorem, we can find

1) Inverse of a matrix.
2) Powers of a matrix.

* Positive powers of matrix

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

char. eq^n $\longrightarrow$ $\lambda^2 - 8\lambda + 12 = 0$

Cayley Hamilton theorem $\longrightarrow$ $A^2 - 8A + 12I = 0$ ——— (1)

$$A^2 = 8A - 12I \quad \text{——— (2)}$$
$$A^3 = 8A^2 - 12A \longrightarrow (3)$$
$$A^4 = 8A^3 - 12A^2 \longrightarrow (4)$$
$$A^5 = 8A^4 - 12A^3 \longrightarrow (5)$$
$$\vdots$$

is —

* Negative powers of matrix

$$A^2 - 8A + 12I = 0$$
$$A^2 \cdot A^{-1} - 8 \cdot A \cdot A^{-1} + 12 \cdot I \cdot A^{-1} = 0$$
$$A - 8I + 12A^{-1} = 0$$
$$12A^{-1} = 8I - A$$
$$A^{-1} = \frac{1}{12}\left[-A + 8I\right]$$
$$A^{-1} \cdot A^{-1} = \frac{1}{12}\left[-A^{-1} \cdot A + 8IA^{-1}\right]$$
$$A^{-2} = \frac{1}{12}\left[-I + 8A^{-1}\right]$$

$$A^{-1} \cdot A^{-2} = \frac{1}{12}\left[-A^{-1} \cdot I + 8A^{-1} \cdot A^{-1}\right]$$

$$A^{-3} = \frac{1}{12}\left[-A^{-1} + 8A^{-2}\right]$$

Problem 1: — Find A^8 using Caylay hamiltom theorem for

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$|A| = -1 - 4 = -5$$

$$\text{trace of } A = 1 \cancel{*} (-1) = 0$$

$$A^2 - A\,(\text{trace of } A) + |A| \cdot I = 0$$

$$A^2 - A(0) + (-5) \cdot I = 0$$

$$\cancel{A^2 + A - 5I}$$

$$A^2 - 5I = 0$$

$$A^2 = 5I$$

$$(A^2)^4 = (5I)^4 = 5^4 \cdot I^4$$

$$A^8 = 625\, I$$

$$= 625 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^8 = \begin{pmatrix} 625 & 0 \\ 0 & 625 \end{pmatrix}$$

(4M)

Problem 2:

Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic eq^n.

Consider the matrix

$$A = \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix}$$

once we get char. eq^n after replacing λ by A 1st time we get char eq^n we should not multiply it by A^{-1}.

1) A satisfies the relation

ⓐ $A + 3I + 2A^{-1} = 0$ ⓑ $A^2 + 2A + 2I = 0$

ⓒ $(A+I)(A+2I) = 0$ ⓓ $\exp(A)$

$$|A| = 0 + 2 = 2$$

$$\text{trace of } A = -3 + 0 = -3$$

$$\Rightarrow A^2 - A(\text{trace of } A) + |A| \cdot I = 0$$

$$A^2 + 3A + 2I = 0$$

$$\cancel{A^2 + 2A}$$

$$A^2 + 2AI + A + 2I = 0$$

$$A(A+2I) + I(A+2I) = 0$$

$$(A+2I)(A+I) = 0$$

2) A^9 equals $511 > 510$

a) $511A + 510I$ c) $154A + 155I$

b) $309A + 104I$ d) $\exp(9A)$

$\Rightarrow$

$$A^2 + 3A + 2I = 0$$

$$A^2 = -3A - 2I \quad \text{——— (1)}$$

$$A^3 = -3A^2 - 2A \quad \text{——— (2)}$$

$(3 > 2)$

$A^{(\text{even})} \rightarrow$ all $-$ve elem

$A^{\text{odd}} \rightarrow$ all +ve elem

classmate

Date ______

Page ______

$$= -3(-3A - 2I) - 2A$$

$$\Rightarrow A^3 = 7A + 6I \longrightarrow (2) \quad (7>6)$$

$$\Rightarrow A^4 = 7A^2 + 6A$$

$$= 7[-3A - 2I] + 6A$$

$$= -21A - 14I + 6A$$

$$A^4 = -15A - 14I \longrightarrow (3)$$

$(15>14)$

$$\Rightarrow A^5 = -15A^2 - 14A$$

$$= -15[-3A - 2I] - 14A$$

$$= 45A + 30I - 14A$$

$$A^5 = 31A + 30I \longrightarrow (4)$$

$(31>30)$

Imp * $A^{(\text{Higher} \overset{+ve}{\wedge} \text{Number})} \Rightarrow A^{5000}, A^{10000}$

* Method :-

char eq^n $\Rightarrow \lambda^2 - (-3+0)\lambda + 2 = 0$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = -1, -2$$

$\Rightarrow$ If λ values are not repeated then

$$\lambda^n = a\lambda + b \longrightarrow (1)$$

for $\lambda = -1$ $\quad (-1)^n = -a + b \longrightarrow (2)$

for $\lambda = -2$ $\quad (-2)^n = -2a + b \longrightarrow (3)$

$$(-1)^n - (-2)^n = a$$

$$\therefore \boxed{a = (-1)^n - (-2)^n}$$

$$\therefore \boxed{b = (-1)^n + a}$$

Pro

$$b = (-1)^n + a$$
$$= (-1)^n + (-1)^n - (-2)^n$$
$$= 2(-1)^n - (-2)^n$$

put a & b in eq^n. (1)

$$\lambda^n = \left[(-1)^n - (-2)^n\right]\lambda + \left[2(-1)^n - (-2)^n\right]$$

by c-h theorem

$$\boxed{A^n = \left[(-1)^n - (-2)^n\right]A + \left[2(-1)^n - (-2)^n\right]I}$$

For ex put $n = 9$

$$A^n = \left[(-1)^9 - (-2)^9\right]A + \left[2(-1)^9 - (-2)^9\right]I$$

$$= (-1 + 512)A + (-2 + 512)I$$

$$A^9 = 511A + 510 \cdot I$$

Problem 3: The char. eq^n. of 3x3 matrix P is given by

$$\alpha(\lambda) = |\lambda I - P| = \lambda^3 + \lambda^2 + 2\lambda + 1$$

where I denoted the Identity matrix. The inverse of the matrix P will be —

a) $P^2 + P + 2I$ c) $-(P^2 + P + I)$

b) $P^2 + P + I$ d) $-(P^2 + P + 2I)$

$$\lambda^3 + \lambda^2 + 2\lambda + 1$$

$$\Longrightarrow P^3 + P^2 + 2P + 1 \cdot I = 0$$

$$\cancel{2P = -I - P^2 - P^3}$$

$$P^{-1} \cdot P^3 + P^{-1} \cdot P^2 + 2P^{-1}P + P^{-1} \cdot I = 0$$

$$P^2 + P + 2I + P^{-1} = 0$$

$$P^{-1} = -P^2 - P - 2I = -(P^2 + P + 2I)$$

imp
Problem (E. value & E. vectors)

How many L.I. eigen vectors of $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

$\lambda = 2, 2$

No. of L.I e.vectors $\Rightarrow 1 \le p \le \underset{\downarrow}{m}$ ← No. of times e. value repeated

$m = 2$

$p = n - r$

n → no of unknowns, r → $\rho(A - \lambda I)$

$$(A - \lambda I) = \begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} \qquad \lambda = 2$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\rho(A - \lambda I) = 1$

$p = n - r$

$= 2 - 1 = \boxed{1}$

—— x ——