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Rank of a Matrix

* The rank of a Matrix A is the order of the largest non-zero minor of A and it is denoted by $P(A)$ or $r(A)$.

* The rank of a non-zero matrix A is possible if:

a) There is atleast one minor of order 'r' which is non-zero.

b) Every other minor of Matrix A of higher order is zero.

Ques: Find the rank of the matrix A.

$$A = \begin{bmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{bmatrix}$$

sol^n: Echelon form:

To make the element zero of any row/column.

Applying $\Rightarrow R_2 \to 2R_2 + R_3$

$$\begin{bmatrix} 5 & 3 & 0 \\ 0 & 0 & 0 \\ -2 & 4 & 8 \end{bmatrix}$$

$\Rightarrow$ Non-zero rows = 2

$P(A) = 2$

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* Remark :

(I) If $|A|=0$ then the rank of a matrix is not equal to the order of a matrix.

(II) If $|A| \neq 0$ then rank = order of matrix

⇒ E.g of previous Ques:

$$|A| = \begin{bmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{bmatrix}$$

$$|A| = -56+56 = 0 \Rightarrow \rho(A) \neq 3$$

Now, check the Minor Determinent, If any of them is non-zero then it's rank is 2. If all is zero then rank is 1.

$$\therefore \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10-3 = 7 \neq 0$$

Rank = 2.

⇒ H.W Question :

(1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & 8 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix}$

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Ans. Method - 1.

(i) $|A| = \begin{array}{ccc} -1 & 8 & \\ \end{array}$
$$|A| = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & 8 & 3 \end{bmatrix}$$

$$|A| = 3(16+4) + 6(-2-8) + 3(4-4)$$

$$|A| = 60 - 60 + 0 = 0 \Rightarrow \delta(A) \neq 3$$

$$\text{Minors} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0 \quad \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} = 0$$

$$\cancel{\begin{bmatrix} 2 & 4 \\ -1 & 8 \end{bmatrix} \neq 0}. \quad \begin{bmatrix} 2 & 4 \\ -1 & 8 \end{bmatrix} \neq 0$$

$$\delta(A) = \cancel{3}.2$$

Method - 2.

(ii)
$$|A| = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & 8 & 3 \end{bmatrix}$$

Applying $R_1 \rightarrow 2R_1 - R_2$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 6 \\ -1 & 8 & 3 \end{bmatrix}$$

~~Applying~~ $\cancel{R_2 \rightarrow R}$.

$$\delta(A) = 2.$$

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(ii) $B = \begin{vmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{vmatrix}$

$C_1 \to 2C_1 - C_2$

$$\begin{vmatrix} 0 & 2 & -1 & 3 \\ 0 & 4 & 1 & -2 \\ 0 & 6 & 3 & -7 \end{vmatrix}$$

$\therefore \rho(B) = 3$

<u>Note</u>

In rectangular matrix, Apply operation then find the determinant.

$$\begin{array}{ccc} 1 & 3 & \\ \multicolumn{3}{c}{\begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ 6 & 3 & -7 \end{bmatrix}} \end{array}$$

$3[12-6] - 2[-6-6] - 7[2+4]$

$18 + 24 - 42 = 0.$

$\therefore$ Ans $\Rightarrow \rho(B) = 2.$

Ques: Find the rank of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & -1 & 1 \\ 4 & 5 & 5 & 3 \\ 5 & 4 & 1 & 3 \end{bmatrix}$$

$R_1 \to 2R_1 - R_2$

A→.
$$\begin{bmatrix} 0 & 3 & 7 & 1 \\ 2 & 1 & -1 & 1 \\ 4 & 5 & 5 & 3 \\ 5 & 4 & 1 & 3 \end{bmatrix}$$

$R_2 \to 2R_2 - R_3$

$$\begin{bmatrix} 0 & 3 & 7 & 1 \\ 0 & -3 & -7 & -1 \\ 4 & 5 & 5 & \\ 5 & & & \end{bmatrix}$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & -7 & -1 \\ \cancel{0}\,4 & 5 & 3 & 3 \\ 5 & 4 & 1 & 3 \end{bmatrix}$$

$(-1)\left[20 - 25\right] + 3\left[-15\right] + 3\left[12\right]$

$+ \cancel{4}5 - 45 + 36.$

$\cancel{5}$ $\quad (-1)\{-9\} - 45 + 36$

$(-1)\{-4\}$ $\quad = 0.$

$\rho(A) = 2.$

H.W on Question.

(1) $A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix}$

Ans. Apply; $C_4 \to C_4 - C_3$ & C_3

$$\begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 2 & 3 & -1 \\ 3 & 1 & 1 & 2 \end{bmatrix}$$

Apply, $C_1 \to C_1 + C_2$ & $C_3 \to \cancel{C_3 - 2C_2}\ 2C_2 - C_3$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 5 & 1 & 1 & -2 \end{bmatrix}$$

$C_4 \to C_4 - C_3$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 5 & 1 & 1 & -3 \end{bmatrix}$$

$C_1 \to 3C_1 + 5(C_4)$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & -3 \end{bmatrix}$$

$|A| \neq 0$ $\qquad \rho(A) = 3$

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* Lineary Independent [LI] and Linear dependent [LD]:

A finite set of element $\{v_1, v_2, --- v_n\}$ of the elements of V is said to be linearly dependent if there exist scalars $\alpha_1, \alpha_2, \alpha_3 --- \alpha_n$ not all zero, such that

$$\alpha_1 v_1 + \alpha_2 V_2 + ----- + \alpha_n v_n = 0 \quad \text{--- (1)}$$

If above is true only for $\alpha_1 = \alpha_2 = -- = \alpha_n = 0$ then set of vectors are linearly independent.

H.W

ques: What is homogenious eq.?

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ques: Check the following vectors for LI or LD.

$$\{(2,2,1), (1,-1,1), (1,0,1)\}$$

soln let $x_1 = (2,2,1)$, $x_2 = (1,-1,1)$, $x_3 = (1,0,1)$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \quad \text{--- (1)}$$

$$\alpha_1 (2,2,1) + \alpha_2 (1,-1,1) + \alpha_3 (1,0,1) = 0$$

$$(2\alpha_1 + \alpha_2 + \alpha_3), (2\alpha_1 - \alpha_2 + \alpha_2), (\alpha_1 + \alpha_2 + \alpha_3) = 0$$

$$\left.\begin{aligned} 2\alpha_1 + \alpha_2 + \alpha_3 &= 0 \\ 2\alpha_1 - \alpha_2 + \alpha_3 &= 0 \\ \alpha_1 + \alpha_2 + \alpha_3 &= 0 \end{aligned}\right\} \quad \alpha_1 = \alpha_2 = \alpha_3 = 0$$

so it is L.I

* **Remarks:**

(a) ~~The~~ If $\rho(A)$ = order of A, then it will be called L.I.

$\therefore \quad |A| \neq 0$

(b) If $\rho(A) \neq$ order of A, then it will be called (D).

(c) In this we only have to apply row transformati

⇒ **Alternative method:**

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$|A| \Rightarrow 1(1) + 1(2) + 1(-2-2)$$

$$1 + 2 - 4 = -1$$

$$|A| \neq 0$$

$$\therefore \rho(A) = 3 \; ; \quad L.I.$$

H.W

Ques: A $\{(1,2,3,1), (2,1,-1,1), (4,5,5,3)\ (5,4,1,3)$

Ques: B. A = $\{$ ① $(1,2,3,4), (2,0,1,-2),$ $(3,2,4,2)$

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<u>method-1</u>

Ans (i) $\begin{bmatrix} -1 & 1 & 4 & 0 \\ -2 & -4 & -6 & -2 \\ -1 & 1 & 4 & 0 \\ 5 & 4 & 1 & 3 \end{bmatrix}$

$R_1 \to R_1 - R_2$ & $R_3 \to R_3 - R_4$

$\begin{bmatrix} -1 & 1 & 4 & 0 \\ 2 & 1 & -1 & 1 \\ -1 & 1 & 4 & 0 \\ 5 & 4 & 1 & 3 \end{bmatrix}$

$R_1 \to R_1 - R_3$

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 1 \\ -1 & 1 & 4 & 0 \\ 5 & 4 & 1 & 3 \end{bmatrix}$ $R_1 \to 3R_1 - R_3$ $\begin{bmatrix} 1 & -1 & -4 & 0 \\ -1 & 1 & 4 & 0 \\ 5 & 4 & 1 & 3 \end{bmatrix}$

$R_1 \to R_1 + R_2$

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 4 & 0 \\ 5 & 4 & 1 & 3 \end{bmatrix}$ → As rank $\rho(A) = 2$. which is not equal to order of matrix. So its L.D.

(ii) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & -2 \\ 3 & 2 & 4 & 2 \end{bmatrix}$ $R_1 \to R_1 - R_3$

↓

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & -2 \\ 3 & 2 & 4 & 2 \end{bmatrix}$ ← $R_1 \to R_1 + R_2$ $\begin{bmatrix} -2 & 0 & -1 & 2 \\ 2 & 0 & 1 & -2 \\ 2 & 2 & 4 & 2 \end{bmatrix}$

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* **Dimension:** The No. of non-zero rows.

* **Basis:** The set of vectors consisting of non-zero rows.

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* **Solution of linear system of equation:**

Consider the system of n-equation in 'n' unknowns such that

$$a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n = b_2$$
$$\text{— — — — — — —}$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

In matrix form, we can write as — (1)

$\boxed{AX = b}$ where $A = \begin{bmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ a_{12} & & & & \vdots \\ \vdots & & & & \vdots \\ a_{n1} & - & - & - & a_{nn} \end{bmatrix}$

$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$ & $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$

- If $b \neq 0 \rightarrow$ non-Homogenious
- If $b = 0 \Rightarrow$ Homogeneous
- Consistent $\rightarrow$ If it has atleast one solution
- Inconsistent = " " " " non-solution

⇒ Argumented method:

The argumented matrix is defined as

$$\boxed{C = [A : B] \quad \text{or} \quad C = [A / B]}$$

→ Coff. of variable

→ constants

- In this matrix there are three possibilities

i) The system has a unique solution.
   if $S(A) = S(C) = n$ → no of known

(ii) The system has no solution
   if $S(A) \neq S(C)$

(iii) The system has infinite number of solution.
   if $S(A) = S(C) < n$

Remark

We donot normally elementry coloumn transformation in solving the linear system of eq. bcoz when we interchange the two coloumn then the order of the unknown in the given system is also changed.

↳ Multi level.

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Ques: Using Gauss elimination method, find the values of x, y, z.

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$

Solⁿ let us make argumented matrix, then apply gauss elimination method.

$$C = [A/B]$$

$$C = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 1 & -1 & 2 & -2 \\ -1 & 2 & -1 & 2 \end{array}\right]$$

$$R_2 \longrightarrow R_2 - R_1/2$$

$$R_3 \longrightarrow R_3 + R_1/2$$

$$= \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3/2 & 5/2 & -4 \\ 0 & 5/2 & -3/2 & 4 \end{array}\right]$$

$$R_3 \longrightarrow R_3 + 5/3\, R_2$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3/2 & 5/2 & -4 \\ 0 & 0 & 8/3 & -8/3 \end{array}\right]$$

$$\delta(A) = 3 \qquad \delta(C) = 3$$

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$\rho(A) = \rho(C) = n$

$\frac{8}{3} z = \frac{-8}{3}$

$\boxed{z = -1}$

$\frac{-3}{2} y + \frac{5}{2} \cancel{z} = -4$

$\frac{-3}{2} y + = -4 + \frac{5}{2} \Rightarrow \frac{-3}{2} y = \frac{-8+5}{2}$

$\boxed{y = 1}$

$2x + y - z = 4$

$\cancel{2+1}$ $2x + 1 + 2 = 4$

$\boxed{x = 1}$.

H.W

(1) $2x + z = 3$

$x - y + z = 1$

$4x - 2y + 3z = 3$

(ii) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 5 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

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* Nilpotent Matrix :

A square matrix A is said to be nilpotent if there exist a possite integar K, such that
$\boxed{A^K = 0}$

The least ~~of~~ value of K is known as Index of nilpotency.

→ If $A \neq 0$
$A^2 = A \cdot A$

* Idempotent Matrix :

A square matrix A is said to be idempotent if $A^2 = A$

eg. : $A^3 = A^2 \cdot A$
$= A^2 \cdot A$
$= A^2 = A.$

* Involutory Matrix :

A matrix A is said to be
If $\boxed{A^2 = I}$

eg $A^3 = A^2 \cdot A = I \cdot A = A$
$A^4 = A^2 \cdot A^2 = I \cdot I = I$

$$A^n = \begin{cases} A, & n \text{ is odd} \\ I, & n \text{ is even.} \end{cases}$$

* Hermitian Matrix:

A square matrix A is said to be Hermitian matrix if $A^\theta = A$ or $\bar{A} = A^T$ or $A = (\bar{A})^T$

Conjugate

⇒ Sometimes, a Hermitian matrix is denoted by A^H or A^θ or A^*

* Skew-Hermitian Matrix:

A square matrix A is said to be skew Hermitian if $A^\theta = -A$ or $\bar{A} = -A^T$ or $A = -(\bar{A})^T$

* Remark:

(i) In an Hermitian matrix all the diagonal elements are real.

(ii) In a skew Hermitian matrix, all the diagonal elements are either zero or pure imaginary number.

* Gauss Jordan Method:

We are using this method to find Inverse of a matrix. In this method, we perform elementary transformation on the augmented matrix $[A/I]$ and reduce it to the form $[I/C]$.

$$[A/I] \xrightarrow[\text{transformation}]{\text{elementary}} [I/C]$$

Inverse

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H.W

(i) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

(i) $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$

$R_1 \to 2R_1 - R_2.$ $R_2 \to -R_2 + R_3$

$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$

$R_3 \to R_3 - R_1$

$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 3 & 3 & -2 & 1 & 1 \end{array}\right]$

$R_3 \to R_3 - 3R_2$

$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 & 4 & -2 \end{array}\right]$

$R_3 \to R_3/2$

$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$

$R_1 \to R_1 + R_3$ & $R_2 \to R_2 - R_3$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$

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* Eigen values and Eigen vectors :

Let $A = [a_{ij}]$ be a square matrix of order n. The matrix A may be singular or non-singular. Consider the homogenious system of eqⁿ

$$AX = \lambda X$$
$$AX - \lambda X = 0$$
$$(A - \lambda) X = 0$$
$$\boxed{(A - \lambda I) X = 0} \quad \text{--- (1)}$$

(i) The homogenious system of eqⁿ [eq. (1)] always has a trivial solⁿ.

$$x = y = z = 0 \text{ (trivial sol}^n\text{)}$$

But we need to find values of 'λ' for which the hom system (eq) has no trivial solⁿ.

(ii) The values of 'λ' for which non-trivial solⁿ of the homogenious system [eq. (1)] exist are called Eign values or characteristic values of A and the corresponding non-trivial solⁿ vector 'n' are called the eigen vector or characteristic vector of A.

* Propcotion :

(i) If A is diagonal (triangular) or an upper triangular a lower triangular matrix then the diagonal element of the matrix A are the eigen value of A.

(ii) If λ is an Eigen value of A, then $1/\lambda$ is " " " " A^{-1}.

Proof: $AX = \lambda X$ — (1)

Pre multiplying by A^{-1}

$A^{-1}AX = \lambda(A^{-1}X)$

$IX = \lambda(A^{-1}X)$

$X = \lambda(A^{-1}X)$

$\frac{1}{\lambda}X = A^{-1}X$

$$\boxed{A^{-1}X = \frac{1}{\lambda}X}$$

(iii) A and A' have the same eigen values but different eigen vector.

(iv) A has eigen value λ then A^m has eigen value λ^m.

Proof: $AX = \lambda X$ — (1)

Pre-multiply A.

$A(AX) = \lambda(AX)$

$A^2X = \lambda(AX)$

$A^2X = \lambda(\lambda X)$

$A^2X = \lambda^2 X$

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(v) The eign values of an ideoptent matrix are eithu zero or unity.

Proof:

$$Ax = \lambda x \quad -(I)$$
$$A(Ax) = \lambda(Ax)$$
$$A^2x = \lambda(\lambda x)$$
$$Ax = \lambda^2 x \quad -(II)$$

from (I) & (II)

$$\lambda^2 x = \lambda x \quad \text{or} \quad (\lambda^2 - \lambda)x = 0$$
$$\lambda^2 - \lambda = 0$$
$$\lambda = 0 \text{ or } 1.$$

(VI) The sum of the eign values of a matrix is the sum of the elements of principal diagnal.

(VII) The product of the eign values of a matrix of A is equal to its determinant.

(VIII) If 'λ' is an eign value of an orthoногonal matrix, then $\frac{1}{\lambda}$ is also its eign value.

$$\boxed{A \cdot A^T = I}$$