28/10/20
Monday, October 26, 2020
8:07 AM

Unit - 2

Derivative : Let $y = f(x)$ be a function of x, then derivative of y w.r.t x is $\frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\frac{dy}{dx}$ represents rate of change of y w.r.t. x

$\frac{dy}{dx}$ represents slope of tangent to curve $y = f(x)$

$\left(\frac{dy}{dx}\right)_{x=c} = f'(c) = [f'(x)]_{x=c} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

Q → $f(x) = \begin{cases} 1+x, & \text{if } x \le 2 \\ 5-x, & \text{if } x > 2 \end{cases}$

Show that f is not differentiable at $x = 2$

$f(x) = x^3$
$f'(x) = 3x^2$
$f'(2) = 12$

Sol. We shall calculate $f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$

Consider $\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$, Put $x = 2 - h$, $h \to 0$

$h = -0.5$
$2 - h$
$= 2.5$

$= \lim_{h \to 0} \frac{f(2-h) - f(2)}{\cancel{2} - h - \cancel{2}} = \lim_{h \to 0} \frac{1 + 2 - h - (1+2)}{-h}$

$(1+2)$

$= \lim_{h \to 0} \frac{-h}{-h} = \lim_{h \to 0} (1) = 1$

Again $\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$, Put $x = 2 + k$, where $k \to 0$

$= \lim_{k \to 0} \frac{f(2+k) - f(2)}{k + \cancel{2} - \cancel{2}} = \lim_{k \to 0} \frac{5 - 2 - k - 3}{k}$

$= \lim_{k \to 0} \frac{-k}{k} = \lim_{k \to 0} (-1) = -1$

⇒ $f'(2)$ does not exist ⇒ $f(x)$ is not differentiable at $x = 2$.

Q → $f(x) = |x - 4|$, show that $f(x)$ is continuous at $x = 4$, but is not differentiable at $x = 4$

Sol. → $f(x) = |x - 4| = \begin{cases} (x-4), & \text{if } x \ge 4 \\ -(x-4), & \text{if } x < 4 \end{cases}$

$f(x)$ will be continuous at $x = 4$ if $\lim f(x) = f(4)$

$f(x)$ will be Continuous at $x=4$ if $\underset{x\to 4}{Lt} f(x) = f(4)$

Now $f(4) = 4-4 = 0$

Now $\underset{x\to 4^-}{Lt} f(x)$, put $x = 4-h, h\to 0$

$$= \underset{h\to 0}{Lt} f(4-h) = \underset{h\to 0}{Lt} -(4-h-4) = \underset{h\to 0}{Lt} h = 0$$

Again

$\underset{x\to 4^+}{Lt} f(x)$, put $x = 4+k, k\to 0$

$$= \underset{k\to 0}{Lt} f(4+k) = \underset{k\to 0}{Lt} (4+k-4) = \underset{k\to 0}{Lt} k = 0$$

clearly $f(4) = \underset{x\to 4^-}{Lt} f(x) = \underset{x\to 4^+}{Lt} f(x) = 0$

$\Rightarrow$ $f(x)$ is continuous at $x=4$

Now, we shall calculate $f'(4) = \underset{x\to 4}{Lt} \frac{f(x)-f(4)}{x-4}$

Consider

$\underset{x\to 4^-}{Lt} \frac{f(x)-f(4)}{x-4}$, put $x = 4-h$, where $h\to 0$

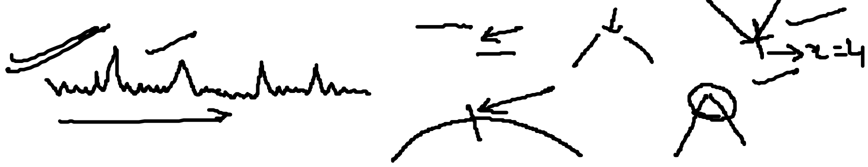
$$= \underset{h\to 0}{Lt} \frac{f(4-h)-f(4)}{4-h-4} = \underset{h\to 0}{Lt} \frac{h-0}{-h} = \underset{h\to 0}{Lt} (-1) = -1$$

Again $\underset{x\to 4^+}{Lt} \frac{f(x)-f(4)}{x-4}$, put $x = 4+k, k\to 0$

$$= \underset{k\to 0}{Lt} \frac{f(4+k)-f(4)}{4+k-4} = \underset{k\to 0}{Lt} \frac{k-0}{k} = \underset{k\to 0}{Lt} (1) = 1$$

$\Rightarrow$ $f'(4)$ does not exist & $f(x)$ is not differentiable at $x=4$

A function which is differentiable at a point is always Continuous at that point, but Converse is not true.



Formulae : (i) $\frac{d}{dx}[u(x) \pm v(x)] = \frac{du}{dx} \pm \frac{dv}{dx}$

(ii) $\frac{d}{dx}[k\, u(x)] = k \frac{d}{dx} u(x)$ ($k \to$ scalar)

(iii) $\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$

(iv) $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

(v) $\frac{d}{dx}[\text{Constant}] = 0$

Q. Find derivative of following functions

(i) $(3x+7)^2$ (ii) $\frac{3x-5}{3x+7}$ (iii) $\left(\frac{3x-5}{3x+7}\right)^2$

$\frac{d}{dx}(x^n) = nx^{n-1}$

Sol. (i) $y = (3x+7)^2 = 9x^2 + 49 + 42x$

$\Rightarrow \frac{dy}{dx} = 9\frac{d}{dx}(x^2) + \frac{d}{dx}(49) + 42\frac{d}{dx}(x)$

$= 9(2x) + 0 + 42(1) = 18x + 42$

(ii) $y = \frac{3x-5}{3x+7} \Rightarrow \frac{dy}{dx} = \frac{(3x+7)3 - (3x-5)3}{(3x+7)^2}$

$= \frac{36}{(3x+7)^2}$

(iii) $y = \left(\frac{3x-5}{3x+7}\right)^2 \Rightarrow \frac{dy}{dx} = 2\left(\frac{3x-5}{3x+7}\right)\frac{d}{dx}\left(\frac{3x-5}{3x+7}\right)$

$= 2\left(\frac{3x-5}{3x+7}\right)\frac{36}{(3x+7)^2} = \frac{72(3x-5)}{(3x+7)^3}$

Q. Find derivative of following functions

(i) e^{x^3} (ii) $\log[\log x]$

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}[\log x] = \frac{1}{x}$

Sol. (i) $y = e^{x^3} \Rightarrow \frac{dy}{dx} = e^{x^3}\cdot\frac{d}{dx}(x^3)$

$= 3x^2 e^{x^3}$

(ii) $y = \log(\log x) \Rightarrow \frac{dy}{dx}$ ...

(ii) $y = \log(\log x) \Rightarrow \frac{dy}{dx} = \frac{1}{\log x} \times \frac{d}{dx}(\log x)$

$$= \frac{1}{x \log x}$$

H.W. $y = \log_7(\log_7(\log_7(x)))$ find $\frac{dy}{dx}$

29/10/20 $y = \sqrt{a+\sqrt{a+\sqrt{a+x^2}}}$ find $\frac{dy}{dx}$

$\frac{d}{dx}(x^n) = nx^{n-1}$

Sol. $\frac{dy}{dx} = \frac{d}{dx}\left[a+\sqrt{a+\sqrt{a+x^2}}\right]^{1/2}$

$$= \frac{1}{2}\left[a+\sqrt{a+\sqrt{a+x^2}}\right]^{-1/2} \frac{d}{dx}\left[a+\sqrt{a+\sqrt{a+x^2}}\right]$$

$$= \frac{1}{2\underbrace{\sqrt{a+\sqrt{a+\sqrt{a+x^2}}}}_{A}}\left[0+\frac{d}{dx}\left(a+\sqrt{a+x^2}\right)^{1/2}\right]$$

$$= \frac{1}{2A}\cdot\frac{1}{2}\left[a+\sqrt{a+x^2}\right]^{-1/2}\frac{d}{dx}\left[a+\sqrt{a+x^2}\right]$$

$$= \frac{1}{4A\underbrace{\sqrt{a+\sqrt{a+x^2}}}_{B}}\left[0+\frac{d}{dx}(a+x^2)^{1/2}\right]$$

$$= \frac{1}{4AB}\frac{1}{2}(a+x^2)^{-1/2}\frac{d}{dx}\left[a+x^2\right]$$

$$= \frac{1}{4\cancel{8}AB\sqrt{a+x^2}}(\cancel{2}x) = \frac{x}{4\sqrt{a+\sqrt{a+\sqrt{a+x^2}}}\sqrt{a+\sqrt{a+x^2}}\sqrt{a+x^2}}$$

Q. $y = 5^{\sqrt{x^2+1}} + \left(\sqrt{x^2+1}\right)^5$ find $\frac{dy}{dx}$ | $\frac{d}{dx}(x^n) = nx^{n-1}$

Sol. $\frac{dy}{dx} = \frac{d}{dx}\left[5^{\sqrt{x^2+1}}\right] + \frac{d}{dx}(x^2+1)^{5/2}$

$\frac{d}{dx}(a^x) = a^x \log a$

$$= 5^{\sqrt{x^2+1}}\log 5 \frac{d}{dx}(x^2+1)^{1/2} + \frac{5}{2}(x^2+1)^{3/2} \frac{d}{dx}(x^2+1)$$

$\frac{d}{dx}(e^x) = e^x$

$$= 5^{\sqrt{x^2+1}}\log 5 \frac{1}{2}(x^2+1)^{-1/2}\frac{d}{dx}(x^2+1) + \frac{5}{2}(x^2+1)^{3/2}[\cancel{2}x]$$

$$= 5^{\sqrt{x^2+1}} \log 5 \; \frac{1}{2}(x^2+1)^{-1/2} \frac{d}{dx}[x^2+1] + \frac{5}{\cancel{2}}(x^2+1)^{3/2}[\cancel{2}x]$$

$$= \frac{5^{\sqrt{x^2+1}} \log 5}{\cancel{2}\sqrt{x^2+1}}[\cancel{2}x] + 5x(x^2+1)^{3/2}$$

Q→ $y = \log_2[\log_3(\log_4(x))]$

Sol. $\frac{dy}{dx} = \frac{d}{dx}\left[\log_2(\log_3(\log_4 x))\right]$

$$= \frac{d}{dx}\left[\frac{\log_e(\log_3(\log_4 x))}{\log_e 2}\right]$$

$$= \frac{1}{\log_e 2} \times \frac{1}{\log_3(\log_4(x))} \frac{d}{dx}[\log_3(\log_4 x)]$$

$$= \frac{1}{\log_e 2} \times \frac{1}{\log_3(\log_4(x))} \frac{d}{dx}\left[\frac{\log_e(\log_4 x)}{\log_e 3}\right]$$

$\log_e x$ — Natural log ($\ln$)

$\log_{10} x$ — Common log

$\log_e e = 1$

$\log_{10} 10 = 1$

$\log_a a = 1$

Base-change formula

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$\frac{d}{dx}\log_e x = \frac{1}{x}$

$$= \frac{1}{\log_e 2} \frac{1}{\log_3(\log_4(x))} \frac{1}{\log_e 3} \frac{1}{\log_4 x} \frac{d}{dx}(\log_4 x)$$

$$= \frac{1}{\log_e 2 \log_e 3} \frac{1}{\log_3(\log_4 x)} \frac{1}{\log_4 x} \frac{d}{dx}\left[\frac{\log_e x}{\log_e 4}\right]$$

$$= \frac{1}{\log_e 2 \log_e 3 \log_e 4} \frac{1}{\log_3(\log_4 x)\log_4 x} \cdot \frac{1}{x}$$

Parametric Differentiation :

Consider $x = f(t)$, $y = g(t)$ called as parametric equations, so $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$x^2 + y^2 = 4$

$x = 2\cos\theta, y = 2\sin\theta$

$y^2 = 4ax$

$x = at^2, y = 2at$

Q→ $y = e^t \cdot \log t$, $x = t \cdot \log t$

equations, so $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$x^2+y^2=4$
$x = 2\cos\theta, y = 2\sin\theta$
$y^2 = 4ax$
$x = at^2, \ y = 2at$
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
$x = a\cos\theta, y = b\sin\theta$

Q→ $y = e^t \cdot \log t$, $x = t \cdot \log t$
find $\frac{dy}{dx}$

Sol. $\frac{dy}{dt} = \frac{d}{dt}\left[e^t \log t\right]$

$= e^t \frac{1}{t} + \log t \cdot e^t = e^t\left[\frac{1+t\log t}{t}\right]$

Now $\frac{dx}{dt} = \frac{d}{dt}\left[t \log t\right] = t \cdot \frac{1}{t} + \log t \cdot 1$

$= [1+\log t]$

Now $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^t[1+t\log t]}{t[1+\log t]}$

Q→ $x = e^{\theta}\left[\theta + \frac{1}{\theta}\right]$, $y = e^{-\theta}\left[\theta - \frac{1}{\theta}\right]$, find $\frac{dy}{dx}$

Sol→ $\frac{dx}{d\theta} = e^{\theta}\left[1 - \frac{1}{\theta^2}\right] + \left(\theta + \frac{1}{\theta}\right) e^{\theta}$

$= e^{\theta}\left[\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2}\right]$

Now $\frac{dy}{d\theta} = e^{-\theta}\left[1 + \frac{1}{\theta^2}\right] + \left(\theta - \frac{1}{\theta}\right) e^{-\theta}(-1)$

$= e^{-\theta}\left[\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right]$

Now $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{e^{-\theta}[-\theta^3 + \theta^2 + \theta + 1]}{e^{\theta}[\theta^3 + \theta^2 + \theta - 1]}$

Q→ Differentiate $(7x^5 - 11x^2)$ w.r.t. $(7x^2 - 15x)$

Sol→ $u = 7x^5 - 11x^2$ & $v = 7x^2 - 15x$

We need to find $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$\therefore \frac{du}{dx} = 35x^4 - 22x$ & $\frac{dv}{dx} = 14x - 15$

Now $\frac{du}{dx}$ ... $\frac{dv}{dx}$ = ...

$$\Rightarrow \frac{du}{dv} = \frac{35x^4 - 22x}{14x - 15} = \left(\frac{d[7x^5 - 11x^2]}{d[7x^2 - 15x]}\right)$$

H.W.

Q→ find derivative of x w.r.t x^{-7}

Implicit differentiation:

A function defined as $f(x,y) = c$ is called an implicit function.

\# $y = \cos x, \sin x, e^x, \tan x, (x^2 + 2x - 1), x^2 + \tan x + e^x + \log\left(\frac{x^2+1}{2}\right)$

Explicit function

\# $x^2 + y^2 = 4$

$\{ y = f(x)$ or $x = f(y)$

$u = f(v)$, or $v = f(u)$

$\{ f(x,y) = c, f(u,v) = c$

\# $x^2 + y^2 + 2xy = 0$ → Implicit

$\Rightarrow y = -x \pm \sqrt{x - 4x^2}$ → Explicit

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Q→ $x^3 + x^2y + xy^2 + y^3 = 81$

find $\frac{dy}{dx}$

Sol. diff w.r.t. x

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) + \frac{d}{dx}(y^3) = \frac{d}{dx}(81)$$

$$\Rightarrow 3x^2 + x^2\frac{dy}{dx} + y(2x) + x\frac{d}{dx}(y^2) + y^2 \cdot 1 + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 + x^2\frac{dy}{dx} + 2xy + x\,2y\frac{dy}{dx} + y^2 + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow [x^2 + 2xy + 3y^2]\frac{dy}{dx} = -(3x^2 + 2xy + y^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-[3x^2 + 2xy + y^2]}{x^2 + 2xy + 3y^2}$$

Q→ $xy\log(x+y) = 1$, $\frac{dy}{dx} = ?$

diff. w.r.t. x

$$\Rightarrow \frac{d}{dx}[xy\log(x+y)] = \frac{d}{dx}(1)$$

$\Rightarrow xy\frac{d}{}\log(x+y) + x\log(x+y)\frac{dy}{}$

\# $\frac{d}{dx}(uvw)$

$= uv\frac{dw}{dx}$

$+ uw\frac{dv}{dx}$

$+ vw\frac{du}{}$

$$+ y\log(x+y)\frac{dx}{dx} = 0$$

$$\Rightarrow xy\frac{1}{x+y}\frac{d}{dx}[x+y] + x\log(x+y)\frac{dy}{dx} + y\log(x+y) = 0$$

$$\Rightarrow \frac{xy}{x+y}\left[1+\frac{dy}{dx}\right] + x\log(x+y)\frac{dy}{dx} + y\log(x+y) = 0$$

$$\Rightarrow \left[\frac{xy}{x+y} + x\log(x+y)\right]\frac{dy}{dx} = -\left[y\log(x+y) + \frac{xy}{x+y}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\left[y\log(x+y) + \frac{xy}{x+y}\right]}{\left[x\log(x+y) + \frac{xy}{x+y}\right]}$$

Q → $y = \sqrt{5^x + \sqrt{5^x + \sqrt{5^x + \cdots \infty}}}$, show that

$$(2y-1)\frac{dy}{dx} = 5^x \log 5$$

Sol → $y = \sqrt{5^x + \left(\sqrt{5^x + \sqrt{5^x + \sqrt{5^x + \cdots \infty}}}\right)}$

$$\Rightarrow y = \sqrt{5^x + y}$$

$$\Rightarrow y^2 = 5^x + y$$

$$\Rightarrow y^2 - y - 5^x = 0$$

diff w.r.t x

$$\Rightarrow 2y\frac{dy}{dx} - \frac{dy}{dx} - 5^x\log 5 = 0$$

$$\Rightarrow (2y-1)\frac{dy}{dx} = 5^x\log 5$$

Logarithmic differentiation

x^n, a^x

{Variable}
{Constant}

$V^C \to x^n$

$C^V \to a^x$

$C^C \to C$

Q → If $y = x^x$, find $\frac{dy}{dx}$

Sol → Take log

$$\log y = x\log x$$

diff w.r.t x

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$

$\Rightarrow \frac{dy}{dx} = x^x[1 + \log x]$

$v^v \to ?$

$\frac{d(x^x)}{dx} \to ?$

Q→ $y = x^{x^x}$, find $\frac{dy}{dx}$

Take log

$\Rightarrow \log y = \log x^{x^x} = x^x \log x$

Take log

$\Rightarrow \log(\log y) = \log[x^x \log x] = \log x^x + \log(\log x)$

$\Rightarrow \log(\log y) = x\log x + \log(\log x)$

diff w.r.t x

$\Rightarrow \frac{d}{dx}[\log(\log y)] = \frac{d}{dx}[x\log x] + \frac{d}{dx}[\log(\log x)]$

$\Rightarrow \frac{1}{\log y}\frac{d(\log y)}{dx} = [1+\log x] + \frac{1}{\log x}\frac{d}{dx}(\log x)$

$\Rightarrow \frac{1}{\log y}\frac{1}{y}\frac{dy}{dx} = 1 + \log x + \frac{1}{x\log x}$

$\Rightarrow \frac{dy}{dx} = x^{x^x} x^x \log x\left[1 + \log x + \frac{1}{x\log x}\right]$

Q→ Differentiate $x^x + x^{\log x}$ w.r.t. x

Sol → $y = x^x + x^{\log x}$

$\Rightarrow y = u + v$, where $u = x^x$, $v = x^{\log x}$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ — (i)

$y = x^x + x^{\log x}$

$\log y = \log[x^x + x^{\log x}]$

$\log(a+b) \neq \log a + \log b$

Now $u = x^x \Rightarrow \frac{du}{dx} = x^x[1+\log x]$ — (ii)

Again $v = x^{\log x}$

Take log

$\Rightarrow \log v = \log x \log x = (\log x)^2$

diff w.r.t. x

$\Rightarrow \frac{1}{v}\frac{dv}{dx} = 2\log x \cdot \frac{1}{x}$

$$\Rightarrow \frac{dv}{dx} = \frac{2\log x}{x}\, x^{\log x} \quad \text{(iii)}$$

Put (ii) & (iii) in (i)

$$\Rightarrow \frac{dy}{dx} = x^{x}[1+\log x] + \frac{2\log x}{x}\, x^{\log x}$$

Q → $y = x^{x^{x\cdots\infty}}$ P.T. $x\frac{dy}{dx} = \frac{y^2}{1-y\log x}$

Sol. → $y = x^{x^{x\cdots\infty}} = x^{y}$ | $y - x^{y} = 0$

$$\Rightarrow y = x^{y}$$

Take log

$$\log y = y\log x$$

diff w.r.t x

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{y}{x} + \log x \cdot \frac{dy}{dx}$$

$$\Rightarrow (1 - y\log x)\frac{dy}{dx} = \frac{y^2}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x[1-y\log x]}$$

Q → $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$ find $\frac{dy}{dx}$

Sol. → Take log

$$\log y = \frac{1}{2}\left[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)\right]$$

Diff w.r.t x

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

Q → Differentiate following functions

(i) $\sin(x^2+5)$ (ii) $\cos(\sin x)$ (iii) $5^{\log\sin x}$

(iv) $e^{\sin\sqrt{x}}$ (v) $e^{x}\log\sqrt{x}\cdot\tan x$

Sol. → (i) $y = \sin(x^2+5)$ | $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\sin(x^2+5)$$
$$= \cos(x^2+5)\cdot(2x) = 2x\cos(x^2+5)$$

(ii) $y = \cos(\sin x)$ $\qquad \frac{d}{dx}\cos x = -\sin x$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\cos(\sin x) = -\sin(\sin x)\cos x$$

(iii) $y = 5^{\log \sin x} \Rightarrow \frac{dy}{dx} = \frac{d}{dx} 5^{\log(\sin x)}$

$$= 5^{\log(\sin x)} \log 5 \; \frac{d}{dx}\log(\sin x)$$
$$= 5^{\log(\sin x)} \log 5 \cdot \frac{1}{\sin x}\cdot \cos x$$
$$= \cot x \cdot 5^{\log \sin x} \cdot \log 5$$

(iv) $y = e^{\sin\sqrt{x}} \Rightarrow \frac{dy}{dx} = e^{\sin\sqrt{x}} \cdot \cos\sqrt{x}\; \frac{1}{2\sqrt{x}}$

$$= \frac{e^{\sin\sqrt{x}}\cos\sqrt{x}}{2\sqrt{x}}$$

(v) $y = e^x \cdot \log\sqrt{x} \cdot \tan x = e^x \, \frac{\log x}{2} \tan x$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[e^x \log x \sec^2 x + \frac{e^x \tan x}{x} + e^x \log x \tan x\right] \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Q→ $y = \log\sqrt{\frac{1-\cos x}{1+\cos x}}$, find $\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\log\sqrt{\frac{\cancel{2}\sin^2 x/2}{\cancel{2}\cos^2 x/2}} = \frac{d}{dx}\log\left[\tan(x/2)\right]$$
$$= \frac{1}{\tan(x/2)} \cdot \sec^2(x/2)\cdot\frac{1}{2}$$
$$= \frac{\cos(x/2)}{\sin(x/2)} \cdot \frac{1}{\cos^2 x/2} \times \frac{1}{2} = \frac{1}{2\sin x/2 \cos x/2}$$
$$= \frac{1}{\sin x} = \operatorname{cosec} x$$

H.W.

(i) $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots \infty}}}$ find $\frac{dy}{dx}$

(ii) $y = \sqrt{1 + \cot x}$, $\frac{dy}{dx} = ?$

(iii) find $\frac{dy}{dx}$, if $y = (\sin x)^{\cos x}$

(iv) Differentiate $x^{\sin x}$ w.r.t $(\sin x)^{x}$

02/11/20 Q. Differentiate $x^{\sin x} + (\sin x)^{x}$ w.r.t. x

Sol. $y = \underbrace{x^{\sin x}}_{u} + \underbrace{(\sin x)^{x}}_{v}$

$\Rightarrow y = u + v$, where $u = x^{\sin x}$

$v = (\sin x)^{x}$

$\log(a+b) \neq \log a + \log b$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ — (i)

Now $u = x^{\sin x}$

$\log u = \sin x \log x$

$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{\sin x}{x} + \log x \cos x$

$\Rightarrow \frac{du}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \log x \cos x\right]$ — (ii)

Now $v = (\sin x)^{x}$

$\log v = x \log \sin x$

$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{x}{\sin x}\cdot\cos x + \log \sin x \cdot 1$

$\Rightarrow \frac{dv}{dx} = (\sin x)^{x}\left[x\cot x + \log \sin x\right]$ — (iii)

Put (ii) & (iii) in (i)

$\therefore \frac{dy}{dx} =$ (i) + (ii)

$\sin[f(x)]$ w.r.t $f(x)$

Q. Differentiate $\sin(x^2)$ w.r.t. x^2

Sol. $u = \sin(x^2)$ & $v = x^2$

We need to find $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$ — (i)

Now $u = \sin(x^2) \Rightarrow \frac{du}{dx} = \cos(x^2)\cdot 2x$

$$\text{Now } v = x^2 \Rightarrow \frac{dv}{dx} = 2x$$

$$\text{Now } \frac{dy}{dv} = \frac{\cancel{2x}\cos(x^2)}{\cancel{2x}} = \cos(x^2)$$

H.W.

Q → Differentiate $\log[f(x)]$ w.r.t $f(x)$

Q → $\sin(xy) + \frac{x}{y} = x^2 - y$

find $\frac{dy}{dx}$

Sol. → $\underbrace{\sin(xy) + \frac{x}{y} - x^2 + y}_{\text{Diff w.r.t } x} = 0$

$$\Rightarrow \cos(xy)\cdot\left[x\frac{dy}{dx} + y\cdot 1\right] + \frac{y\cdot 1 - x\cdot\frac{dy}{dx}}{y^2} - 2x + \frac{dy}{dx} = 0$$

$f(x,y) = c$

$$\Rightarrow \left[x\cos(xy) - \frac{x}{y^2} + 1\right]\frac{dy}{dx} = -y\cos(xy) - \frac{1}{y} + 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x - y\cos(xy) - \frac{1}{y}}{x\cos(xy) - \frac{x}{y^2} + 1}$$

Diff $\tan[2x - 3x^2 + 2]$ w.r.t $2x - 3x^2 + 2$

(a) $\sec^2[2x - 3x^2 + 2]$

(b) $\sec^2[\;''\;][2 - 6x]$

(c) $\sec^2[\;''\;][2x - 3x^2 + 2]$

(d) None

Inverse trigonometric functions :

(i) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

(ii) $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

(iv) $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$

(v) $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$

(vi) $\frac{d}{dx}\operatorname{cosec}^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$

Q → Differentiate w.r.t x

(i) $\sin^{-1}(x\sqrt{x})$ (ii) $e^{\sin^{-1}x}$ (iii) $\sin(\tan^{-1}e^{-x})$

(iv) $e^{\cot^{-1}(x^2)}$ (v) $\cos^{-1}\left(\frac{x}{1+x}\right)$ (vi) $x\sec^{-1}x$ } H.W.

Sol. → (i) $y = \sin^{-1}[x^{3/2}]$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^{3/2})^2}} \times \frac{d}{dx}(x^{3/2})$$

$$= \frac{3x^{1/2}}{2\sqrt{1-x^3}}$$

(ii) $y = e^{\sin^{-1} x} \Rightarrow \frac{dy}{dx} = e^{\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}}$

(iii) $y = \sin\left(\tan^{-1}\left(e^{-x}\right)\right)$

$$\Rightarrow \frac{dy}{dx} = \cos\left[\tan^{-1}\left(e^{-x}\right)\right] \frac{d}{dx}\tan^{-1}\left(e^{-x}\right)$$

$$= \cos\left[\tan^{-1}\left[e^{-x}\right]\right] \frac{1}{1+\left(e^{-x}\right)^2} e^{-x}(-)$$

(iv) $y = e^{\cot^{-1}(x^2)} \Rightarrow \frac{dy}{dx} = e^{\cot^{-1}(x^2)} \cdot \frac{(-1)}{\left[1+(x^2)^2\right]} 2x$

(v) $y = \cos^{-1}\left[\frac{x}{1+x}\right] \Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{x}{1+x}\right)^2}} \times \frac{d}{dx}\left[\frac{x}{1+x}\right]$

$$= \frac{-1}{\sqrt{1-\left(\frac{x}{1+x}\right)^2}} \quad \frac{(1+\cancel{x})\cdot 1 - \cancel{x}\cdot 1}{(1+x)^2}$$

$$= \frac{-1}{\sqrt{1-\left(\frac{x}{1+x}\right)^2}} \quad \frac{1}{(1+x)^2}$$

(vi) $y = x\sec^{-1}x$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[x\sec^{-1}x\right] = x\frac{1}{|x|\sqrt{x^2-1}} + \sec^{-1}x \cdot 1$$

$$= \frac{x}{|x|\sqrt{x^2-1}} + \sec^{-1}x$$

Q → Differentiate (i) $\tan^{-1}\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$

(ii) $\tan^{-1}\left[\frac{\cos x - \sin x}{\cos x + \sin x}\right]$ (iii) $\tan^{-1}\left[\frac{\sin x}{1+\cos x}\right]$

Sol. → (i) $y = \tan^{-1}\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \tan^{-1}\left[\frac{\cancel{2}\sin^2 x}{\cancel{2}\cos^2 x}\right]^{1/2}$

$\frac{dy}{dx}$ $\Rightarrow y = \tan^{-1}\left[\tan^2 x\right]^{1/2}$

$\Rightarrow y = \tan^{-1}[\tan x]$

$\Rightarrow y = x$

$\Rightarrow \frac{dy}{dx} = 1$

(ii) $y = \tan^{-1}\left[\frac{\cos x - \sin x}{\cos x + \sin x}\right] = \tan^{-1}\left[\frac{\cancel{\cos x}[1-\tan x]}{\cancel{\cos x}[1+\tan x]}\right]$

$\Rightarrow y = \tan^{-1}\left[\frac{1-\tan x}{1+\tan x}\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{4} - x\right)\right]$

$\Rightarrow y = \frac{\pi}{4} - x$

$\Rightarrow \frac{dy}{dx} = -1$

(iii) $y = \tan^{-1}\left[\frac{\sin x}{1+\cos x}\right] = \tan^{-1}\left[\frac{\cancel{2}\sin x/2\ \cancel{\cos x/2}}{\cancel{2}\cos^{2} x/2}\right]$

$\Rightarrow y = \tan^{-1}[\tan(x/2)]$

$\Rightarrow y = \frac{x}{2}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}$

Differentiate w.r.t x

(i) $\sin^{-1}\left[\frac{2x}{1+x^2}\right]$ (ii) $\tan^{-1}\left[\frac{2x}{1-x^2}\right]$

(iii) $\tan^{-1}\left[\frac{3x-x^3}{1-3x^2}\right]$

Sol (i) $y = \sin^{-1}\left[\frac{2x}{1+x^2}\right]$

Take $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\Rightarrow y = \sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right]$

$\Rightarrow y = \sin^{-1}[\sin 2\theta]$

$\Rightarrow y = 2\theta = 2\tan^{-1}x$

$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$

(ii) $y = \tan^{-1}\left[\frac{2x}{1-x^2}\right]$

Take $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\Rightarrow y = \tan^{-1}\left[\frac{2\tan\theta}{1-\tan^2\theta}\right] = \tan^{-1}\left[\tan 2\theta\right]$

$\Rightarrow y = 2\theta = 2\tan^{-1}x$

$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$

(iii) $y = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$

Take $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\Rightarrow y = \tan^{-1}\left[\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right] = \tan^{-1}\left[\tan 3\theta\right]$

$\Rightarrow y = 3\theta = 3\tan^{-1}x$

$\Rightarrow \frac{dy}{dx} = \frac{3}{1+x^2}$

(iv) Q $\rightarrow$ $\sin^{-1}(3x-4x^3)$ (v) $\cos^{-1}(2x^2-1)$ $\rightarrow$ $x = \cos\theta$

(vi) $\sin^{-1}\left[2x\sqrt{1-x^2}\right] \rightarrow \{x = \sin\theta$ $x = \cos\theta\}$

Sol. $\rightarrow$ $y = \sin^{-1}[3x - 4x^3]$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$\Rightarrow y = \sin^{-1}\left[3\sin\theta - 4\sin^3\theta\right]$

$\Rightarrow y = \sin^{-1}\left[\sin 3\theta\right] = 3\theta$

$\Rightarrow y = 3\sin^{-1}x$

$\Rightarrow \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$

05/11/20 Differentiate following

(i) $\cot^{-1}\left[\frac{1-x}{1+x}\right]$ (ii) $y = \tan^{-1}\left[\frac{x^{1/3} + a^{1/3}}{1 - x^{1/3}a^{1/3}}\right]$

(iii) $\cos^{-1}\left[2x\sqrt{1-x^2}\right]$

Sol. $\rightarrow$ $y = \cot^{-1}\left[\frac{1-x}{1+x}\right]$

$x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\Rightarrow y = \cot^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right] = \cot^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right]$

$\Rightarrow y = \cot^{-1}\left[\cot\left[\frac{\pi}{2} - \left(\frac{\pi}{4}-\theta\right)\right]\right]$

$= \cot^{-1}\left[\cot\left(\frac{\pi}{4}+\theta\right)\right]$

$y = \frac{\pi}{4} + \theta = \frac{\pi}{4} + \tan^{-1}x$

$\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$

& $\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$

$y = \frac{\pi}{4} + \theta = \frac{\pi}{4} + \tan^{-1} x$ $\qquad = \tan\theta$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$

Q → $y = \tan^{-1}\left[\frac{x^{1/3} + a^{1/3}}{1 - x^{1/3} a^{1/3}}\right]$ $\qquad$ & $B = \tan^{-1} a^{1/3}$

Take $x^{1/3} = \tan A$ & $a^{1/3} = \tan B \Rightarrow A = \tan^{-1}\left[x^{1/3}\right]$

$\Rightarrow y = \tan^{-1}\left[\frac{\tan A + \tan B}{1 - \tan A \tan B}\right] = \tan^{-1}\left[\tan(A+B)\right]$

$\Rightarrow y = A + B = \tan^{-1}\left(x^{1/3}\right) + \tan^{-1}\left(a^{1/3}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\left(x^{1/3}\right)^2} \frac{d}{dx}\left(x^{1/3}\right) + 0$

$= \frac{1}{1+x^{2/3}} \times \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}\left[1+x^{2/3}\right]}$

(iii) $y = \cos^{-1}\left[2x\sqrt{1-x^2}\right]$

Put $x = \sin\theta$ [OR $x = \cos\theta$] $\Rightarrow \theta = \sin^{-1} x$

$\Rightarrow y = \cos^{-1}\left[2\sin\theta\sqrt{1-\sin^2\theta}\right] = \cos^{-1}\left[2\sin\theta\cos\theta\right]$

$\Rightarrow y = \cos^{-1}\left[\sin 2\theta\right] = \cos^{-1}\left[\cos\left(\frac{\pi}{2} - 2\theta\right)\right]$ $\qquad \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow y = \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\sin^{-1} x$

$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$

Higher order derivatives : $\quad y = f(x) \Rightarrow \frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x)$ — 1st order derivative

$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x) \longrightarrow$ 2nd order derivative

$\Rightarrow \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{d^3y}{dx^3} \longrightarrow$ 3rd ______

Q → $y = \frac{\log x}{x}$, find $\frac{d^2y}{dx^2} \longrightarrow \frac{d}{dx}\left[\frac{dy}{dx}\right]$

Sol → $\frac{dy}{dx} = \frac{x \cdot \frac{1}{x} - \log x \cdot 1}{x^2} = \frac{1 - \log x}{x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{1 - \log x}{x^2}$

$\Rightarrow \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{1-\log x}{x^2}\right] = \quad x^2\left[0 - \frac{1}{x}\right] - (1 - \log x)\,2x$

$$\ldots \frac{\ldots}{(x^2)^2}$$

$$= \frac{x - 2x[1-\log x]}{x^4} = \frac{2x\log x - 3x}{x^4} = \frac{2\log x - 3}{x^3}$$

Q. If $x = a[\theta - \sin\theta]$, $y = a[1+\cos\theta]$

find $\frac{d^2y}{dx^2} = \rightarrow \frac{d}{dx}\left[\frac{dy}{dx}\right]$

Sol. $\frac{dy}{dx} = \left[\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right] = \frac{\cancel{a}[-\sin\theta]}{\cancel{a}[1-\cos\theta]}$

$$\Rightarrow \frac{dy}{dx} = \frac{-\cancel{2}\cancel{\sin\theta/2}\cos\theta/2}{\cancel{2}\sin^{\cancel{2}}\theta/2}$$

$$\Rightarrow \frac{dy}{dx} = -\cot(\theta/2)$$

y = displacement

$\frac{dy}{dt} \Rightarrow$ Velocity

$\frac{d^2y}{dt^2} \rightarrow$ Accⁿ

$\frac{d^3y}{dt^3} \rightarrow$ Jerk

(A) It has no physical significance

(B) Nothing

(C) Jaco

(D) Jerk

Now $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

$$= \frac{d}{dx}\left[-\cot(\theta/2)\right]$$

$$= -\left[-\operatorname{cosec}^2(\theta/2)\right]\frac{d}{dx}(\theta/2)$$

$$= \operatorname{cosec}^2(\theta/2)\cdot\frac{1}{2}\boxed{\frac{d\theta}{dx}}$$

$\frac{dx}{d\theta} = a[1-\cos\theta]$

$$= \frac{1}{2\sin^2(\theta/2)}\;\frac{1}{a[1-\cos\theta]}$$

$$= \frac{1}{2a\sin^2\theta/2 \cdot 2\sin^2\theta/2} = \frac{\operatorname{cosec}^4\theta/2}{4a}$$

Integrals: # $\boxed{\frac{d}{dx}[f(x)] = g(x)}$ $\boxed{\frac{d}{dx}(x^2) = 2x}$

$$\Rightarrow \int g(x)\,dx = f(x) + C$$

$\Rightarrow \int (2x)\,dx = x^2+1$ or x^2+2 or $x^2 - 1/2$ or $x^2 - \sqrt{3}$

$f(x) \pm C$

Formulae of integrals:

① $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

② $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C$

③ $\int \frac{1}{x}\,dx = \log x + C$

③ $\int \frac{1}{x}dx = \log x + c$

④ $\int \frac{1}{(ax+b)}dx = \frac{\log(ax+b)}{a} + c$

⑤ $\int e^x dx = e^x + c$

⑥ $\int e^{mx} dx = \frac{e^{mx}}{m} + c$

⑦ $\int \sin x \, dx = -\cos x + c$

⑧ $\int \cos x \, dx = \sin x + c$

⑨ $\int \tan x \, dx = -\log|\cos x| + c$ $\rightarrow$ $+ \frac{1}{\cos(x)} (+\sin x)$ $= \tan x$

⑩ $\int \cot x \, dx = \log|\sin x| + c$

⑪ $\int \operatorname{cosec} x \, dx = -\log|\operatorname{cosec} x + \cot x| + c$

⑫ $\int \sec x \, dx = \log|\sec x + \tan x| + c$

⑬ $\int \sec^2 x \, dx = \tan x + c$

⑭ $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

⑮ $\int \sec x \tan x \, dx = \sec x + c$

⑯ $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

Q→ $I = \int (4e^{3x}+1)\,dx = \int 4e^{3x}dx + \int dx = 4\frac{e^{3x}}{3} + x + c$

Q→ $I = \int x^2\left[1 - \frac{1}{x^2}\right]^2 dx = \int x^2\left[1 + \frac{1}{x^4} - \frac{2}{x^2}\right]dx$

$= \int\left[x^2 + \frac{1}{x^2} - 2\right]dx = \int x^2 dx + \int x^{-2}dx - 2\int dx$

$= \frac{x^3}{3} + \frac{x^{-1}}{-1} - 2x + c = \frac{x^3}{3} - \frac{1}{x} - 2x + c$

Q→ $I = \int \sec x[\sec x + \tan x]\,dx = \int \sec^2 x \, dx + \int \sec x \tan x \, dx$

$= \tan x + \sec x + c$

Q→ $I = \int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx = \int \frac{1}{\cos^2 x} \times \sin^2 x \, dx = \int \tan^2 x \, dx$

$= \int [\sec^2 x - 1] dx = \tan x - x + C$

Q→ $I = \int \frac{2 - 3\sin x}{\cos^2 x} dx = \int 2\sec^2 x \, dx - 3\int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx$

$= 2\int \sec^2 x \, dx - 3\int \tan x \cdot \sec x \, dx$

$= 2\tan x - 3\sec x + C$

Q→ $I = \int \frac{2x}{1+x^2} dx = \log|1+x^2| + C$

⇒ Put $1 + x^2 = t$

Taking differential

⇒ $2x \, dx = dt$

⇒ $I = \int \frac{dt}{t} = \log|t| + C$

$= \log|1+x^2| + C$

$\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + C$

$\int x^n \cdot 1 \, dx = \frac{x^{n+1}}{n+1} + c$

$\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$

$\int \frac{1}{x} dx = \log|x| + C$

06/11/20

Integrate following functions

① $\int \frac{(\log x)^2}{x} dx$ ② $\int \frac{1}{x + x\log x} dx$ ③ $\int \sin x \cos(\sin x) \, dx$

④ $\int \frac{1}{x - \sqrt{x}} dx$ ⑤ $\int x\sqrt{1+2x^2} \, dx$ ⑥ $\int (x^3 - 1)^{1/3} x^5 \, dx$

⑦ $\int \frac{x}{e^{x^2}} dx$

Sol. ① $I = \int (\log x)^2 \cdot \frac{1}{x} dx = \frac{(\log x)^3}{3} + C$

$\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + C$

$\boxed{\text{Otherwise } \log x = t}$

② $I = \int \frac{1}{[x + x\log x]} dx = \int \frac{1}{[1 + \log x]} \frac{1}{x} dx$

$1 + \log x = t$

Taking differential

$$\frac{1}{x}dx = dt$$

$$\Rightarrow I = \int \frac{1}{t}dt = \log t + C = \log|1+\log x| + \underline{\underline{C}}$$

③ $I = \int \sin x \sin(\cos x)\,dx$

Put $\cos x = t \Rightarrow -\sin x\,dx = dt$

$$\Rightarrow I = \int \sin(t)(-dt) = -[-\cos t] + C = \cos[\cos x] + C$$

④ $I = \int \frac{1}{x-\sqrt{x}}dx = \int \frac{1}{\sqrt{x}[\underbrace{\sqrt{x}-1}]}dx$

Take $\sqrt{x}-1 = t \Rightarrow \frac{1}{2}x^{-1/2}dx = dt$

$$\Rightarrow \frac{1}{\sqrt{x}}dx = 2dt$$

$$\Rightarrow I = \int \frac{1}{t}2dt = 2\log t + C = 2\log(\sqrt{x}-1) + \underline{\underline{C}}$$

⑤ $I = \int x\sqrt{1+2x^2}\,dx = \frac{1}{4}\int (\underbrace{1+2x^2})^{1/2}(4x)\,dx$

$$= \frac{1}{4}\frac{(1+2x^2)^{3/2}}{3/2} + C = \frac{1}{6}(1+2x^2)^{3/2} + \underline{\underline{C}}$$

$$\left\{ \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C \right.$$

⑥ $I = \int (x^3-1)^{1/3}x^5\,dx = \int (x^3-1)^{1/3}\underbrace{x^3}\cdot\underbrace{x^2\,dx}$

Put $x^3-1 = t \Rightarrow 3\underbrace{x^2\,dx} = dt$

$$\Rightarrow I = \int t^{1/3}[t+1]\frac{dt}{3} = \frac{1}{\cancel{3}}\left[\frac{t^{7/3}}{7/\cancel{3}} + \frac{t^{4/3}}{4/\cancel{3}}\right] + C$$

$$= \frac{(x^3-1)^{7/3}}{7} + \frac{(x^3-1)^{4/3}}{4} + \underline{\underline{C}}$$

⑦ $I = \int \frac{x}{\underbrace{e^{x^2}}}dx$ Put $x^2 = t \Rightarrow 2x\,dx = dt$

$$\Rightarrow I = \int \frac{1}{e^t}\frac{dt}{2} = \frac{1}{2}\int e^{-t}dt = \frac{1}{2}\frac{e^{-t}}{(-1)} + C$$

$$= -\frac{1}{2}e^{-x^2} + \underline{\underline{C}}$$

⑧ $I = \int \frac{e^{\tan^{-1}x}}{1+x^2}dx$ Put $\tan^{-1}x = t \Rightarrow \frac{1}{1+x^2}dx = dt$

$$\Rightarrow I = \int e^t dt = e^t + C = e^{\tan^{-1}x} + C$$

(8) $I = \int \frac{e^{\tan^{-1} x}}{1+x^2} dx$ Put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

$\Rightarrow I = \int e^t dt = e^t + C = e^{\tan^{-1} x} + C$

☆ (9) $I = \int \frac{e^{2x}-1}{e^{2x}+1} dx = \int \frac{\frac{e^{2x}-1}{e^x}}{\frac{e^{2x}+1}{e^x}} dx$

$= \int \frac{(e^x - e^{-x})}{(e^x + e^{-x})} dx = \log|e^x + e^{-x}| + C$

$\boxed{\text{\# otherwise } e^x + e^{-x} = t}$

$\int \frac{f'(x)}{f(x)} dx$

$= \log|f(x)| + C$

(10) $I = \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx =$

$= \frac{1}{2} \int \frac{2(e^{2x} - e^{-2x})}{(e^{2x} + e^{-2x})} dx$

$= \frac{\log|e^{2x} + e^{-2x}|}{2} + C$

log

(A) $(e^{2x} + e^{-2x})$

log

(B) $2(\left[e^{2x} + e^{-2x}\right])$

(C) $\log\left(\frac{e^{2x} + e^{-2x}}{2}\right)$

(D) None

(11) $I = \int \frac{(\sin^{-1} x)^1}{\sqrt{1-x^2}} \, dx = \frac{(\sin^{-1} x)^2}{2} + C$

$\int [f(x)]^n f'(x) dx$

$= \frac{[f(x)]^{n+1}}{n+1} + C$

☆ (12) $I = \int \frac{1}{1+\cot x} dx = \int \frac{1}{1 + \frac{\cos x}{\sin x}} dx = \int \frac{\sin x}{\sin x + \cos x} dx$

$= \frac{1}{2} \int \frac{2 \sin x}{\sin x + \cos x} dx = \frac{1}{2} \int \frac{[(\sin x + \cos x) + (\sin x - \cos x)] dx}{(\sin x + \cos x)}$

$= \frac{1}{2} \int dx + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx$

$= \frac{1}{2} x - \frac{1}{2} \int \frac{\cos x - \sin x}{(\sin x + \cos x)} dx$

$\int \frac{f'(x)}{f(x)} dx$

$= \log|f(x)| + C$

$= \frac{1}{2} x - \frac{1}{2} \log|\sin x + \cos x| + C$

(13) $I = \int x^3 \sin[\tan^{-1} x^4) \, dx$

$\tan^{-1} x \leftrightarrow \frac{1}{1+x^2}$

$1+x^8$

Put $\tan^{-1}x^4 = t$

$\Rightarrow \frac{1}{1+x^8}\cdot 4x^3\,dx = dt$

Take $\tan^{-1}x = t$

$\tan^{-1}(x^4) \to \frac{1}{1+(x^4)^2}$

$$\Rightarrow I = \int \sin(t)\,\frac{dt}{4} = -\frac{1}{4}\cos t + C = -\frac{1}{4}\cos(\tan^{-1}x^4) + C$$

(14) $I = \int \sin^2(2x+5)\,dx$

$= \int \frac{1-\cos 2[2x+5]}{2}\,dx$

$= \frac{1}{2}\left[x - \frac{\sin[4x+10]}{4}\right] + C$

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

$\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + C$

(15) $I = \int \sin^3(2x+1)\,dx$

$= \int \frac{3\sin[2x+1] - \sin 3(2x+1)}{4}\,dx$

$= \frac{1}{4}\left[3\left(-\frac{\cos(2x+1)}{2}\right) - \left(-\frac{\cos(6x+3)}{6}\right)\right] + C$

$\sin 3A = 3\sin A - 4\sin^3 A$

$4\sin^3 A = 3\sin A - \sin 3A$

(16) $I = \int \sin 3x \cos 4x\,dx = \frac{1}{2}\int 2\cos 4x \sin 3x\,dx$

$= \frac{1}{2}\int [\sin 7x - \sin x]\,dx$

$= \frac{1}{2}\left[-\frac{\cos 7x}{7} + \cos x\right] + C$

$2\cos A \sin B = \sin(A+B) - \sin(A-B)$

(17) $I = \int \cos^4(2x)\,dx = \int [\cos^2(2x)]^2\,dx$

$= \int \left(\frac{1+\cos 4x}{2}\right)^2 dx = \int \frac{1+\cos^2 4x + 2\cos 4x}{4}\,dx$

$\cos^2 A = \frac{1+\cos 2A}{2}$

$\sin^2 A = \frac{1-\cos 2A}{2}$

$= \frac{1}{4}\int\left[1 + \frac{1+\cos 8x}{2} + 2\cos 4x\right]dx$

$= \frac{1}{4}\left[\frac{3}{2}x + \frac{1}{2}\left(\frac{\sin 8x}{8}\right) + \cancel{2}\,\frac{\sin(4x)}{\cancel{4}\,2}\right] + C$

09/11/20

Q → Integrate :

(1) $I = \int \tan^4 x\,dx$ (2) $\int \sin^3 x + \cos^3 x\,dx$

① $I = \int \tan^4 x\,dx$ ② $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx$

③ $I = \int \frac{1}{\sin x \cos^3 x} dx$ ④ $\int \frac{\cos 2x}{(\cos x + \sin x)^2} dx$

<u>Sol.</u> ① $I = \int \tan^4 x\,dx = \int \tan^2 x \cdot \tan^2 x\,dx = \int \tan^2 x[\sec^2 x - 1]\,dx$

$= \int (\tan x)^2 \sec^2 x\,dx - \int \tan^2 x\,dx$

$= \int \underbrace{(\tan x)}_{f(x)}{}^{2} \underbrace{\sec^2 x}_{f'(x)}\,dx - \int (\sec^2 x - 1)\,dx$

$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$

$= \frac{\tan^3 x}{3} - \tan x + x + C$

② $I = \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^{\cancel{3}} x}{\cancel{\sin^2 x} \cos^2 x} dx + \int \frac{\cos^{\cancel{3}} x}{\sin^2 x \cancel{\cos^2 x}} dx$

$= \int \tan x \cdot \sec x\,dx + \int \cot x \cdot \operatorname{cosec} x\,dx$

$= \sec x - \operatorname{cosec} x + C$

③ $I = \int \frac{1}{\sin x \cos^3 x} dx = \int \frac{1}{\frac{\sin x}{\cos x} \cdot \cos^4 x} dx$

$= \int \frac{1}{\tan x} \cdot \sec^2 x \cdot \sec^2 x\,dx = \int \frac{1 + \tan^2 x}{\tan x} \sec^2 x\,dx$

Take $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$\Rightarrow I = \int \frac{1 + t^2}{t} dt = \int \left[\frac{1}{t} + t\right] dt = \log|t| + \frac{t^2}{2} + C$

$= \log|\tan x| + \frac{\tan^2 x}{2} + C$

④ $I = \int \frac{\cos 2x}{(\cos x + \sin x)^2} dx = \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx$

$= \int \frac{(\cos x - \sin x)\cancel{(\cos x + \sin x)}}{(\cos x + \sin x)^{\cancel{2}}} dx$

$= \int \frac{(-\sin x + \cos x)\,dx}{(\cos x + \sin x)}$, with $-\sin x + \cos x \to f'(x)$ and $\cos x + \sin x \to f(x)$

$\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$

$= \log|\cos x + \sin x| + C$

\# ① $\int \frac{dx}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$

(2) $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(3) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(4) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x + \sqrt{x^2-a^2}\right| + C$

(5) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

(6) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x + \sqrt{x^2+a^2}\right| + C$

Q → (1) $\int \frac{3x^2}{\sqrt{x^6+1}}\,dx$ (2) $\int \frac{1}{\sqrt{1+4x^2}}\,dx$ (3) $\int \frac{1}{\sqrt{9-25x^2}}\,dx$

(4) $\int \frac{x-1}{\sqrt{x^2-1}}$

Sol → (1) $I = \int \frac{3x^2}{\sqrt{(x^3)^2+1}}\,dx$

Put $x^3 = t \Rightarrow 3x^2\,dx = dt$

$\Rightarrow I = \int \frac{dt}{\sqrt{t^2+1^2}} = \log\left|t + \sqrt{t^2+1}\right| + C$

$= \log\left|x^3 + \sqrt{x^6+1}\right| + C$

(2) $I = \int \frac{1}{\sqrt{1+4x^2}}\,dx = \int \frac{1}{\sqrt{4\left[x^2+1/4\right]}}\,dx = \frac{1}{2}\int \frac{dx}{\sqrt{x^2+(1/2)^2}}$

$= \frac{1}{2}\log\left|x + \sqrt{x^2+1/4}\right| + C$

$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x + \sqrt{x^2+a^2}\right| + C$

(3) $I = \int \frac{dx}{\sqrt{9-25x^2}} = \int \frac{dx}{\sqrt{25\left[\frac{9}{25} - x^2\right]}}$

$= \frac{1}{5}\int \frac{dx}{\sqrt{(3/5)^2 - x^2}}$

$= \frac{1}{5}\sin^{-1}\left[\frac{x}{3/5}\right] + C$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

$$= \frac{1}{5} \sin^{-1}\left[\frac{5x}{3}\right] + C$$

(4) $I = \int \frac{x-1}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\int \frac{(2x-2)}{\sqrt{x^2-1}}\,dx$

$$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}\,dx - \frac{1}{\cancel{2}}\int \frac{\cancel{2}}{\sqrt{x^2-1}}\,dx$$

$$= \frac{1}{2}\int \underbrace{(x^2-1)}_{f(x)}^{-1/2 \to n} \underbrace{(2x)}_{f'(x)}\,dx - \int \frac{1}{\sqrt{x^2-1^2}}\,dx$$

Put $x^2-1=t$

$$= \frac{1}{\cancel{2}} \frac{(x^2-1)^{1/2}}{\cancel{1/2}} - \log\left|x+\sqrt{x^2-1}\right| + C$$

$$= \sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right| + C$$

$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right| + C$

Q → (5) $I = \int \frac{1}{\sqrt{x^2+2x+2}}\,dx$ (6) $\int \frac{1}{9x^2+6x+5}\,dx$

Sol → (5) $I = \int \frac{1}{\sqrt{x^2+2x+2}}\,dx = \int \frac{1}{\sqrt{x^2+2x+1+2-1}}\,dx$

$$= \int \frac{1}{\sqrt{(x+1)^2+1^2}}\,dx$$

$$= \log\left|(x+1)+\sqrt{(x+1)^2+1^2}\right| + C$$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right| + C$

(6) $I = \int \frac{1}{9x^2+6x+5}\,dx = \int \frac{1}{9\left[x^2+\frac{6}{9}x+\frac{5}{9}\right]}\,dx$

$$= \frac{1}{9}\int \frac{dx}{\left[x^2+\frac{2}{3}x+\frac{1}{9}+\frac{5}{9}-\frac{1}{9}\right]}$$

$$= \frac{1}{9}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2} = \frac{1}{\cancel{9}_3} \times \frac{1}{2/\cancel{3}} \tan^{-1}\left[\frac{x+1/3}{2/3}\right]$$

$$= \frac{1}{6} \tan^{-1}\left[\frac{3x+1}{2}\right] + C$$

H.W.

Q → $I = \int \frac{1}{\sqrt{7-6x-x^2}}\,dx$, Q → $I = \int \frac{1}{\sqrt{(x-1)(x-2)}}\,dx$

Q. $I = \int \frac{1}{x^2+3x+4}\,dx$

Q. $I = \int \frac{5x+3}{\sqrt{x^2+4x+10}}\,dx$

Sol:

$$I = \int \frac{5\left[x+3/5\right]}{\sqrt{x^2+4x+10}}\,dx$$

$$\Rightarrow I = \int \frac{\frac{5}{2}\left[2x+6/5\right]}{\sqrt{x^2+4x+10}}\,dx = \int \frac{\frac{5}{2}\left[2x+4+\frac{6}{5}-4\right]}{\sqrt{x^2+4x+10}}\,dx$$

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}\,dx + \frac{5}{2}\left(\frac{6}{5}-4\right)\int \frac{dx}{\sqrt{x^2+4x+10}}$$

$$= \frac{5}{2}\int \underbrace{(x^2+4x+10)}_{f(x)}^{-1/2 \to n}\underbrace{(2x+4)}_{f'(x)}dx - 7\int \frac{dx}{\sqrt{x^2+4x+4+10-4}}$$

$=$

$$= \frac{5}{2}\,\frac{(x^2+4x+10)^{1/2}}{1/2} - 7\int \frac{dx}{\sqrt{(x+2)^2+(\sqrt{6})^2}}$$

$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{(x+2)^2+(\sqrt{6})^2}\right| + C$$

Partial Fractions:

→ # [Proper] [Deg. of Num. < Deg. of Deno.]

① $\frac{\text{Polynomial}}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$

② $\frac{\text{Polynomial}}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{(x-c)}$

③ $\frac{\text{Polynomial}}{(x-a)^2(x-b)(x-c)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{B}{x-b} + \frac{C}{x-c}$

④ $\frac{\text{Polynomial}}{(x-a)^3(x-b)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \frac{B}{x-b}$

(5) $$\frac{\text{Polynomial}}{(x^2+bx+c)(x-a)} = \frac{Bx+C}{x^2+bx+c} + \frac{A}{x-a}$$

Q → Integrate (i) $\frac{x}{(x+1)(x+2)}$

Sol → $$I = \frac{x}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{(x+3)}$$

$$I = \frac{x}{(x+1)(x+3)} = +\frac{1}{2(x+1)} - \frac{1}{2(x+3)}$$

$$\Rightarrow \int \frac{x}{(x+1)(x+3)}dx = \frac{1}{2}\int \frac{1}{x+1}dx - \frac{1}{2}\int \frac{1}{x+3}dx$$

$$= \frac{1}{2}\log|x+1| - \frac{1}{2}\log|x+3| + c$$

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Q → $I = \int \frac{3x-1}{(x^2-9)}dx$

Sol → $$\frac{3x-1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$$

$$\Rightarrow 3x-1 = A(x+3) + B(x-3)$$

$$\Rightarrow 3x-1 = Ax + 3A + Bx - 3B$$

$$\Rightarrow A+B = 3 \text{ — (i)} \quad \& \quad 3A-3B = -1 \text{ — (ii)}$$

$$3\times(i) + (ii) \Rightarrow 6A = 8 \Rightarrow \boxed{A = 4/3} \; \& \; \boxed{B = 3 - 4/3 = 5/3}$$

$$\Rightarrow \frac{3x-1}{(x-3)(x+3)} = \frac{4}{3(x-3)} + \frac{5}{3(x+3)}$$

$$\Rightarrow \int \frac{3x-1}{(x-3)(x+3)}dx = \frac{4}{3}\int \frac{1}{x-3}dx + \frac{5}{3}\int \frac{1}{x+3}dx$$

$$= \frac{4}{3}\log|x-3| + \frac{5}{3}\log|x+3| + c$$

OR $$\frac{3x-1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$$

$$= \frac{4}{3(x-3)} + \frac{5}{3(x+3)}$$

$x-3=0 \Rightarrow \boxed{x=3}$

$$\frac{\cancel{8}\,4}{(x-3)\cancel{6}\,3}$$

$x+3=0$

$\Rightarrow x=-3$

$\cancel{-10}\,5$

$\frac{-4}{3}\log(x+3)$

Q → $I = \int \frac{3x-1}{(x-1)(x-2)(x-3)}\,dx$

Sol → $\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} + \frac{(-5)}{x-2} + \frac{7}{x-3}$

$\Rightarrow \int \frac{3x-1}{(x-1)(x-2)(x-3)}\,dx = \int \frac{1}{(x-1)}\,dx - 5\int \frac{1}{x-2}\,dx + 4\int \frac{1}{x-3}\,dx$

$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + \underline{c}$

$x-1=0 \Rightarrow \boxed{x=1}$

$\frac{2}{(x-1)(-1)(-2)}$

$x-3=0 \Rightarrow x=3$ $\frac{8}{2\times1\times(x-3)}$

$x-2=0 \Rightarrow x=2$ $\frac{5}{(1)(x-2)(-1)}$

Q → $I = \int \frac{x}{(x-1)^2(x+2)}\,dx$

Sol → $\frac{x}{(x-1)^2(x+2)} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$\Rightarrow x = A(x^2-2x+1) + B(x-1)(x+2) + C(x+2)$

$\Rightarrow x = A[x^2-2x+1] + B[x^2+x-2] + C[x+2]$

$\Rightarrow A+B=0$ — (i) & $-2A+B+C=1$ — (ii) & $A-2B+2C=0$ — (iii)

$2\times$(ii) − (iii) $\Rightarrow -5A+4B=2$ — (iv)

$5\times$(i) + (iv) $\Rightarrow 9B=2 \Rightarrow \boxed{B=2/9} \Rightarrow \boxed{A=-2/9}$

& $C = 1+2A-B = 1-\frac{4}{9}-\frac{2}{9} = \frac{3}{9} = \frac{1}{3}$

$\Rightarrow \boxed{C=1/3}$

$\Rightarrow \int \frac{x}{(x-1)^2(x+2)}\,dx = A\int \frac{1}{x+2}\,dx + B\int \frac{1}{x-1}\,dx + C\int \frac{1}{(x-1)^2}\,dx$

$= -\frac{2}{9}\log|x+2| + \frac{2}{9}\log|x-1| + \frac{1}{3}\left[-\frac{1}{x-1}\right] + \underline{c}$

Q → $I = \int \frac{x}{(x^2+1)(x-1)}\,dx$

Sol → $\frac{x}{\qquad} = \frac{A}{\qquad} + \frac{Bx+C}{\qquad}$

$(x+i)(x-i) \leftarrow (x^2+1)(x-1) \quad x-1 \quad x^2+1$

$$\Rightarrow x = A(x^2+1) + (Bx+C)(x-1)$$

$$\Rightarrow x = A[x^2+1] + [Bx^2 - Bx + Cx - C]$$

$A+B=0$ — (i) & $-B+C=1$ — (ii) & $A-C=0$ — (iii)

(i) − (iii) $\Rightarrow B+C=0 \Rightarrow \boxed{B=-C}$ $\therefore$ (ii) $\Rightarrow 2C=1 \Rightarrow \boxed{C=1/2}$

$\Rightarrow \boxed{B=-1/2}$ & $\boxed{A=1/2}$

$$\therefore \int \frac{x}{(x^2+1)(x-1)}dx = A\int \frac{1}{x-1}dx + \int \frac{Bx+C}{x^2+1}dx$$

$$= A\log|x-1| + \frac{B}{2}\int \frac{2x}{x^2+1}dx + C\int \frac{1}{x^2+1^2}dx$$

$2x \to f'(x)$, $x^2+1 \to f(x)$

$$= A\log|x-1| + \frac{B}{2}\log|x^2+1| + C\tan^{-1}x + C_1$$

Integration by Parts :

$$\int (uv) = u\int v - \int\left(\frac{du}{dx}\int v\right)$$

u → Ist, v → 2nd

I → L → A → T → E

I → Inverse → $\tan^{-1}x$

L → Log → $\log(x^2+1)$

A → Algeb → x^2+2x+1

T → Trig. → $\sin x$

E → Expo. → e^x

Q → $I = \int \underbrace{x}_{A}\,\underbrace{\sin x}_{T}\,dx = x\int \sin x\,dx - \int\left(\frac{dx}{dx}\cdot\int \sin x\,dx\right)dx$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

Q → $I = \int \underbrace{x^2}_{A}\,\underbrace{e^x}_{E}\,dx = x^2 e^x - \int (2x \cdot e^x)dx$

$$= x^2 e^x - 2\left[x e^x - \int (1\cdot e^x)dx\right]$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

Q → $I = \int \underbrace{\tan^{-1}x}_{u} \cdot \underbrace{1}_{v}\,dx = (\tan^{-1}x)x - \int\left(\frac{1}{1+x^2}\cdot x\right)dx$

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}dx$$

$$= x\tan^{-1}x - \frac{1}{2}\log|1+x^2| + c$$

$\int e^x\,[f(x)+f'(x)]\,dx = e^x f(x) + c$

Q→ $I = \int e^x\,[\underbrace{\sin x}_{f(x)} + \underbrace{\cos x}_{f'(x)}]\,dx = e^x \sin x + c$

$$= \int e^x \sin x\,dx + \int e^x \cos x\,dx$$

$$= \sin x\, e^x - \cancel{\int (\cos x \cdot e^x)\,dx} + \cancel{\int e^x \cos x\,dx}$$

$$= \sin x \cdot e^x + c$$

Q→ $I = \int e^x \left[\frac{1+\sin x}{1+\cos x}\right] dx = \int e^x \left[\frac{1}{1+\cos x} + \frac{\sin x}{1+\cos x}\right] dx$

$$= \int e^x \left[\frac{1}{2\cos^2 x/2} + \frac{\cancel{2}\sin x/2\,\cancel{\cos x/2}}{\cancel{2}\cos^{\cancel{2}} x/2}\right] dx$$

$$= \int e^x \left[\underbrace{\tan x/2}_{f(x)} + \underbrace{\tfrac{1}{2}\sec^2 x/2}_{f'(x)}\right] dx$$

$$= e^x \tan x/2 + c$$

① $\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + c$

② $\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + c$

③ $\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

Q→ $I = \int \sqrt{x^2+4x+5}\,dx = \int \sqrt{x^2+4x+4+1}\,dx$

$$= \int \sqrt{(x+2)^2+1^2}\,dx = \frac{(x+2)\sqrt{(x+2)^2+1^2}}{2} + \frac{1}{2}\log|x+2+\sqrt{(x+2)^2+1^2}| + c$$

Q→ $I = \int \sqrt{4-x^2}\,dx = \int \sqrt{2^2-x^2}\,dx$

$$= \frac{x\sqrt{2^2-x^2}}{2} + \frac{2^2}{2}\sin^{-1}\left(\frac{x}{2}\right) + c$$

12/11/20 **Definite integral**, $\int_a^b f(x)\,dx = \left[g(x)+c\right]_{x=a}^{b} = \left[g(x)+c\right]_a^b$

$$= \left(g(b)+c\right) - \left(g(a)+c\right) = g(b)-g(a)$$

Properties: ① $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ $\qquad \int_1^2 x^2\,dx \quad \int_1^2 t^2\,dt$

② $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

③ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a<c<b$

⑤ $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

④ $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

⑥ $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

⑦ $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

⑧ $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even function, that is } f(-x) = f(x) \\ 0, & \text{if } f(x) \text{ is odd function, that is } f(-x) = -f(x) \end{cases}$

Q. ① $I = \int_0^{\pi/2} \cos^2 x\,dx$ $\qquad$ ② $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

Sol. $I = \int_0^{\pi/2 \leftarrow a} \underbrace{\cos^2 x}_{f(x)}\,dx$ — ① $\qquad \Big| \quad \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$\Rightarrow \quad I = \int_0^{\pi/2} \cos^2\left[\frac{\pi}{2}-x\right]dx$$

$$\Rightarrow I = \int_0^{\pi/2} \sin^2 x\,dx \quad \text{— (ii)}$$

$$\int_0^{\pi/2} \cos^2 x\,dx + \int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} (\cos^2 x + \sin^2 x)\,dx$$

$$\text{(i)}+\text{(ii)} \Rightarrow 2I = \int_0^{\pi/2} \left[\cos^2 x + \sin^2 x\right]dx = \int_0^{\pi/2} dx = (x)_0^{\pi/2} = \frac{\pi}{2} - 0$$

$$\Rightarrow \boxed{I = \pi/4}$$

Sin

(2) $$I = \int_0^{a \leftarrow \pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx \quad \text{— (i)} \qquad \rightarrow f(x)$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\frac{\pi}{2}-x)} + \sqrt{\cos(\frac{\pi}{2}-x)}}\,dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx \quad \text{— (ii)}$$

$$\text{(i)}+\text{(ii)} \Rightarrow 2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx = x\Big]_0^{\pi/2} = \pi/2$$

$$\Rightarrow \boxed{I = \pi/4}$$

(3) $$I = \int_{-\pi/2}^{\pi/2} \sin^7 x\,dx$$

Here $f(x) = \sin^7 x$ is an odd function

$$\Rightarrow I = \int_{-\pi/2}^{\pi/2} \sin^7 x\,dx = 0$$

$f(x) = \sin^7 x$

$f(-x) = \sin^7(-x)$

$= [-\sin(x)]^7$

$= -\sin^7 x$

$= -f(x)$

$\Rightarrow f(-x) = -f(x)$

$\Rightarrow f(x)$ is an odd function

$\int_{-a}^{a} f(x)\,dx = 0$, if $f(x)$ is odd

Q → $I = \int_0^{\pi/2} \log \sin x \, dx$ — (i)

$a \leftarrow \pi/2$

$\Rightarrow I = \int_0^{\pi/2} \log \sin\left[\frac{\pi}{2} - x\right] dx$

$\Rightarrow I = \int_0^{\pi/2} \log \cos x \, dx$ — (ii)

(i) + (ii) $\Rightarrow$ $2I = \int_0^{\pi/2} \left[\log \sin x + \log \cos x\right] dx$

$\Rightarrow 2I = \int_0^{\pi/2} \log(\sin x \cos x) dx = \int_0^{\pi/2} \log\left[\frac{2 \sin x \cos x}{2}\right] dx$

$\Rightarrow 2I = \int_0^{\pi/2} \left[\log(\sin 2x) - \log 2\right] dx$

$= \int_0^{\pi/2} \log \sin 2x \, dx - \log 2 \int_0^{\pi/2} dx$

Take $2x = t \Rightarrow 2dx = dt \Rightarrow dx = \frac{dt}{2}$

when $x = 0 \Rightarrow t = 0$ & $x = \pi/2 \Rightarrow t = \pi$

$\Rightarrow 2I = \int_{t=0}^{\pi} \log \sin t \frac{dt}{2} - (\log 2) \frac{\pi}{2}$

$\Rightarrow 2I = \frac{1}{2} \int_0^{\pi} \log \sin t \, dt - \frac{\pi}{2} \log 2$

$\Rightarrow 2I = \frac{1}{\cancel{2}} \times \cancel{2} \int_0^{\pi/2} \log \sin t \, dt - \frac{\pi}{2} \log 2$

$\Rightarrow 2I = \int_0^{\pi/2} \log \sin x \, dx - \frac{\pi}{2} \log 2$

$\int_0^{2a} f(x) dx = 2 \int_0^{a} f(x) dx$ if $f(2a - x) = f(x)$

$f(t) = \log \sin t, \quad 2a = \pi$

$f(2a - t) = \log \sin[2a - t]$

$$= \log \sin[\pi - t]$$
$$= \log \sin t$$
$$= f(t)$$

$$\Rightarrow 2I = I - \frac{\pi}{2}\log 2$$

$$\Rightarrow I = -\frac{\pi}{2}\log 2$$

$$\Rightarrow \boxed{\int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2}\log 2}$$

$$\# \quad \boxed{\int_0^{\pi/2} \log \cos x \, dx = -\frac{\pi}{2}\log 2}$$

Q→ $$I = \int_0^{\pi} \log[1+\cos x]\,dx \quad \text{— (i)}$$

$$\Rightarrow I = \int_0^{\pi} \log[1+\cos(\pi - x)]\,dx$$

$$I = \int_0^{\pi} \log[1-\cos x]\,dx \quad \text{— (ii)}$$

$$\therefore \text{(i)} + \text{(ii)} \Rightarrow 2I = \int_0^{\pi} [\log(1+\cos x) + \log(1-\cos x)]\,dx$$

$$\Rightarrow 2I = \int_0^{\pi} \log[(1+\cos x)(1-\cos x)]\,dx$$

$$= \int_0^{\pi} \log(1-\cos^2 x)\,dx = \int_0^{\pi} \log \sin^2 x \, dx$$

$$= 2\int_0^{\pi \to 2a} \log \underbrace{\sin x}_{\to f(x)} \, dx$$

$$\because f(\pi - x) = f(x)$$
$$\int_0^{\pi} f(x) = 2\int_0^{\pi/2} f(x)\,dx$$

$$\cancel{2}I = \cancel{2} \times 2 \boxed{\int_0^{\pi/2} \log \sin x \, dx}$$

$$\Rightarrow I = 2\left[-\frac{\pi}{\cancel{2}}\log 2\right] = -\pi \log 2$$

Q→ $$I = \int_0^{\pi/2} \sqrt{\sin\phi}\,\cos^5\phi\,d\phi = \int_0^{\pi/2} \sqrt{\sin\phi}\,(\cos^2\phi)^2 \cos\phi\,d\phi$$

$$= \int_0^{\pi/2} \sqrt{\sin\phi}\,[1-\sin^2\phi]^2 \cos\phi\,d\phi$$

$$= \int_{0} \sqrt{\sin\phi}\,\left[1-\sin^2\phi\right]^2 \underbrace{\cos\phi\, d\phi}$$

Take $\underline{\underline{\sin\phi}} = t \Rightarrow \underbrace{\cos\phi\, d\phi} = dt$

$\phi = 0 \Rightarrow t = 0$

$\phi = \pi/_2 \Rightarrow t = 1$

$$\Rightarrow I = \int_{t=0}^{1} \sqrt{t}\,\left[1-t^2\right]^2 dt$$

$$= \int_{t=0}^{1} \sqrt{t}\left[1 + t^4 - 2t^2\right] dt$$

$$= \int_{t=0}^{1} \left[t^{1/2} + t^{9/2} - 2\,t^{5/2}\right] dt$$

$$= \left.\frac{t^{3/2}}{3/2} + \frac{t^{11/2}}{11/2} - 2\frac{t^{7/2}}{7/2}\right]_0^1 = \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$

Q. $I = \int_0^1 \sin^{-1}\left[\frac{2x}{1+x^2}\right] dx$ $\quad \theta = \tan^{-1}x$

Put $x = \tan\theta \Rightarrow dx = \underbrace{\sec^2\theta\, d\theta}$

when $x = 0 \Rightarrow \theta = \tan^{-1}0 = 0$ | when $x = 1 \Rightarrow \theta = \tan^{-1}(1) = \pi/_4$

$$\Rightarrow I = \int_{\theta=0}^{\pi/4} \sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right] \sec^2\theta\, d\theta$$

$$= \int_{\theta=0}^{\pi/4} \sin^{-1}\left[\sin 2\theta\right] . \sec^2\theta\, d\theta = \int_{\theta=0}^{\pi/4} \underset{u}{2\theta}\ \underset{v}{\sec^2\theta}\, d\theta$$

$$= 2\left[\left(\theta . \tan\theta\right)_{\theta=0}^{\pi/4} - \int_{\theta=0}^{\pi/4} \left(1 . \tan\theta\right) d\theta\right]$$

$$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} + \left(\log|\cos\theta|\right)_{\theta=0}^{\pi/4}\right]$$