Tutorial

Solutions to linear system of equations using sparse QR decomposition

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Q.(1) (a) Consider a vector $\boldsymbol{v} = [v(1), v(2)]^T \in \mathbb{R}^2$ and an angle θ. Note that the matrix M given by $M = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$, where $c = \cos(\theta)$ and $s = \sin(\theta)$ rotates the vector $\boldsymbol{v}$ by θ radians in the clockwise direction. Use M to construct a function `rot(v,theta)` that rotates the vector `v` by an angle of `theta` radians in the clockwise direction. Use the following function `vec_rot(v,theta)` to plot the vectors `v` and `rot(v,theta)` in the same figure.

```
function out = vec_rot(v,theta)
     v_th = rot(v,theta);
    figure(1)
    quiver(0,0,v(1),v(2),'r');
      hold on;
    quiver(0,0,v_th(1),v_th(2),'b');
```

(b) Construct a function `TD_angle(v)` to find the angle between `v` and x-axis. Check `TD_angle(rot(v,theta))`. Plot `v` and `rot(v,TD_angle(v))` using `vec_rot`. Observe that the rotated vector is aligned with the x-axis.

(c) Compute $\|\mathbf{v}\|$ and $\|$`rot(v,theta)`$\|$. Check whether the norm is changed or not.

Q.(2) (a) Use the GNU-Octave (Scilab) command `v = [v(1);...;v(m)]` to input a vector $\boldsymbol{v} = [v(1), \ldots, v(m)]^{\mathrm{T}}$. Find the angle θ_j between the vector $[\boldsymbol{v}(1) \;\; \boldsymbol{v}(j)]^{\mathrm{T}}$ and positive x-axis using the function `TD_angle([v(1) v(j)])` from Q.(1)a.

(b) Use the 2×2 Givens rotation matrix $M_j = \begin{bmatrix} \cos(\theta_j) & \sin(\theta_j) \\ -\sin(\theta_j) & \cos(\theta_j) \end{bmatrix}$ to construct the $m \times m$ Givens rotation matrix Q_{1j}. You can employ the commands `Q_1j = eye(m); Q_1j((1,j],[1,j])) = M_j` for this. Verify that the transformed vector $Q_{1j}\boldsymbol{v}$ has the properties $(Q_{1j}\boldsymbol{v})(j) = 0$, $(Q_{1j}\boldsymbol{v})(1) = \|[v(1) \;\; v(j)]^T\|_2$ and $(Q_{1j}\boldsymbol{v})(i) = \boldsymbol{v}(i)$ for $i \neq 1, j$.

(c) Write a function `given_vec(v)` that transforms a vector $\boldsymbol{v} = [v(1), \ldots, v(m)]^{\mathrm{T}}$ into $[*, \; 0, \ldots, 0]^{\mathrm{T}}$ using Givens rotations.

Q.(3) (a) Consider a matrix $A_{m\times n}$. Each column of A is an $m \times 1$ vector. Let the j^{th} column of A be A_j. Construct a matrix Q_{ij} using the following commands for $i > j$.

```
Q_ij= eye(n,n)
theta_ij = TD_angle([A(j,j),A(i,j)])
M_ ij =  = [cos(theta) sin(theta); -sin(theta) cos(theta)]
Q_ij([i,j],[i,j])= M_ij
```

Observe the transformed matrix $Q_{ij}A$. What can you conclude from the values of $(Q_{ij}A)(j,j)$ and $(Q_{ij}A)(i,j)$.

(b) Run the the following loop

```
for j = 1:n
  for i = j+1:m
     Q = eye(m, m);
     theta = TD_angle([A(j,j), A(i,j)]);
     M = [cos(theta) sin(theta);-sin(theta) cos(theta)];
     Q([j,i],[j,i])= M;
     A = Q*A
  end
end
```

Note that each element A_{ij} with $i > j$ is zero now.

(c) A has the structure $A = \left[\frac{R}{0}\right]$. Check $A^T A$ and $R^T R$ are equal.

Q.(4) Consider the system of equations $A\boldsymbol{X} = \boldsymbol{b}$. Transform A into form $\left[\frac{R}{0}\right]$ using Givens rotations, where R is an upper triangular matrix and then use backward-forward substitution to find $\boldsymbol{X}$. Check $\|A\boldsymbol{X} - \boldsymbol{b}\|_2$. Is it zero? What if you take $\boldsymbol{b}$ in the column space of A? What can you conclude from these results?

Q.(5) (a) Consider m data points $\{(t_i, f(t_i))\}_{i=1,\dots,m}$ from an experiment. Use the command `M_t = [t_1,...,t_m]` and `M_ft = [f(t_1),...,f(t_m)]` to enter this information in an octave file `M_data.m`. Plot the data using the command `plot(M_t, M_ft, '*')`.

(b) Let $P_n(t)$ be the n^{th} $(m > n)$ degree polynomial such that the error $\frac{1}{2}\sum_{i=1}^{m} \|f(t_i) - P_n(t_i)\|_2$ is the least. Note that this error is equal to $\|A\boldsymbol{x} - \boldsymbol{b}\|_2$, where A is the Vandermonde matrix $A = \begin{bmatrix} 1 & t_1 & t_1^2 & \dots & t_1^n \\ 1 & t_2 & t_2^2 & \dots & t_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_m & t_m^2 & \dots & t_m^n \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1, & x_2, & \dots, & x_{n+1} \end{bmatrix}^{\mathrm{T}}$ and $\boldsymbol{b} = \begin{bmatrix} f(t_1), & f(t_2), & \dots, & f(t_m) \end{bmatrix}^T$.

(c) Use the Given's rotations and the QR decomposition to find the values $x_1, x_2, \dots, x_{n+1}$. Your code must be in a function file format with the name `best_fit(M_t, M_ft,n)`. Plot the polynomial $P_n(t)$ for $t \in [\min(t_i)_i, \max(t_i)_i]$ along with the plot in Q.(5a).