From b1_sqrt to the Gauss Circle Problem

A Perturbative Dirichlet Series Derivation

Companion to: *From a_hoying to the Dirichlet Divisor Problem*

Personal Notes

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Contents

1 Setup: The Sequence and Its Exact Formula

1.1 The Gauss Circle Problem

Let $r_2(k)$ denote the number of ways to write k as a sum of two integer squares (counting signs and order):

$$r_2(k) = \#\{(x,y) \in \mathbb{Z}^2 : x^2 + y^2 = k\}.$$

Define the *lattice point count*:

$$A(N) = \sum_{k=1}^{N} r_2(k) = \#\{(x,y) \in \mathbb{Z}^2 : 1 \le x^2 + y^2 \le N\},$$

the number of non-origin lattice points in the disk of radius $\sqrt{N}$. The Gauss circle problem asks for the size of the error in the approximation $A(N) \approx \pi N$.

1.2 The Sequence $b_1(N)$

The code computes:

$$b_1(N) = \sum_{k=1}^{N-1} (N-k)\, r_2(k). \tag{1}$$

Via the character representation $r_2(k) = 4\sum_{d|k} \chi_{-4}(d)$ (where χ_{-4} is the non-principal Dirichlet character mod 4, given by $\chi_{-4}(n) = 1, 0, -1, 0$ for $n \equiv 1, 2, 3, 0 \pmod 4$), the code evaluates $b_1(N)$ in $O(\sqrt{N})$ time via hyperbola blocking over the prefix sums

$$C(x) = \sum_{n=1}^{x} \chi_{-4}(n), \qquad W(x) = \sum_{n=1}^{x} n\, \chi_{-4}(n).$$

The key exact formula used is:

$$b_1(N) = 4 \sum_{d=1}^{N-1} \chi_{-4}(d) \left[N \left\lfloor \frac{N-1}{d} \right\rfloor - d \cdot \frac{\lfloor \frac{N-1}{d} \rfloor \left(\lfloor \frac{N-1}{d} \rfloor + 1 \right)}{2} \right]. \tag{2}$$

1.3 The Abel Summation Identity

Proposition 1 (Exact identity)**.**

$$b_1(N) = \sum_{m=1}^{N-1} A(m).$$

Proof. By discrete Abel summation (summation by parts):

$$\sum_{k=1}^{N-1} (N-k)\, r_2(k) = \sum_{k=1}^{N-1} r_2(k) \sum_{m=k}^{N-1} 1 = \sum_{m=1}^{N-1} \sum_{k=1}^{m} r_2(k) = \sum_{m=1}^{N-1} A(m). \qquad \square$$

An immediate corollary is that $A(N) = b_1(N+1) - b_1(N)$, which the code uses in `--delta` mode to compute $A(N)$ exactly.

2 The Asymptotic Formula for $b_1(N)$

2.1 Gauss Circle Error

The classical result of Gauss gives:

$$A(N) = \pi N + \Delta_A(N), \qquad \Delta_A(N) = O(N^\theta), \tag{3}$$

where $\theta \le 131/416 \approx 0.3149$ (Huxley 2003) and the conjecture $\theta = 1/4$ is the Gauss circle conjecture (Hardy–Landau).

Note: since A excludes the origin, $\Delta_A(N) = \Delta_L(N) - 1$ where Δ_L is the standard Gauss circle error for the full lattice count including origin.

2.2 Summing the Error

From Proposition 1:

$$\begin{aligned} b_1(N) &= \sum_{m=1}^{N-1} A(m) = \sum_{m=1}^{N-1} \big[\pi m + \Delta_A(m)\big] \\ &= \pi \cdot \frac{(N-1)N}{2} + \sum_{m=1}^{N-1} \Delta_A(m). \end{aligned} \tag{4}$$

2.3 The Mean of Δ_A

Proposition 2. $\sum_{m=1}^{N-1} \Delta_A(m) = \left(\frac{\pi}{2} - 1\right) N + E_3(N)$, *where* $E_3(N) = O(N^{1/2+\theta})$ *is the oscillatory residual.*

Sketch. Since $\Delta_A(m) = \Delta_L(m) - 1$, summing the -1 terms gives $-(N-1)$. The sum of $\Delta_L(m)$ over $m \le N$ has mean $\sim (\pi/2-1)N + O(N^{1/2+\theta})$ by the Hardy–Voronoï expansion and stationary phase (see Section 3). This is confirmed empirically: at $N = 10^{18}$ the coefficient of N is measured as $0.570785\ldots$ vs. the theoretical $\pi/2 - 1 = 0.570796\ldots$ (agreement to 5 significant figures). $\square$

2.4 The Full Three-Term Asymptotic

Substituting Proposition 2 into (4):

$$\begin{aligned} b_1(N) &= \frac{\pi N(N-1)}{2} + \left(\frac{\pi}{2} - 1\right) N + E_3(N) \\ &= \frac{\pi}{2} N^2 - \frac{\pi}{2} N + \frac{\pi}{2} N - N + E_3(N) \\ &\quad \text{[the } \pi N/2 \text{ terms combine]} \end{aligned} \tag{5}$$

$$\boxed{b_1(N) = \frac{\pi}{2} N^2 - N + E_3(N).} \tag{6}$$

Remark 3. The asymptotic $\frac{\pi}{2}N^2 - N$ is computed in the code as `asym_f128(N) = N*((pi/2)*N - 1)` using `__float128` arithmetic to avoid catastrophic cancellation: at $N = 10^{18}$, $b_1(N) \sim 1.57 \times 10^{36}$ while $E_3 \sim 10^{13}$, requiring ~ 23 digits of headroom.

Define the oscillatory error:

$$E_3(N) := b_1(N) - \frac{\pi}{2} N^2 + N. \tag{7}$$

The Gauss circle conjecture is equivalent to $E_3(N) = O(N^{3/4+\varepsilon})$.

3 The Dominant Oscillatory Mode and Stationary Phase

3.1 Hardy–Voronoï Expansion

The Hardy–Voronoï expansion of the Gauss circle error gives:

$$\Delta_L(m) \sim \frac{4}{\pi} m^{1/4} \sum_{j=1}^{\infty} \frac{r_2(j)}{j^{3/4}} \cos\left(2\pi\sqrt{jm} - \frac{\pi}{4}\right), \tag{8}$$

where the dominant ($j = 1$) term is:

$$\Delta_L(m) \underset{m\to\infty}{\sim} \frac{4}{\pi} m^{1/4} \cos\left(2\pi\sqrt{m} - \frac{\pi}{4}\right). \tag{9}$$

3.2 Summing by Stationary Phase

To evaluate $E_3(N) \approx \sum_{m=1}^{N-1} \Delta_L(m)$, approximate the sum by an integral with the substitution $u = \sqrt{m}$, $m = u^2$, $dm = 2u\,du$:

$$E_3(N) \approx \frac{4}{\pi} \int_1^{\sqrt{N}} u^{1/2} \cos\left(2\pi u - \frac{\pi}{4}\right) 2u\,du = \frac{8}{\pi} \int_1^{\sqrt{N}} u^{3/2} \cos\left(2\pi u - \frac{\pi}{4}\right) du. \tag{10}$$

Integrate by parts once (stationary phase):

$$\int^X u^{3/2}\cos(2\pi u+\phi)\,du = \frac{u^{3/2}}{2\pi}\sin(2\pi u+\phi)\Big|^X - \frac{3}{4\pi}\int^X u^{1/2}\sin(2\pi u+\phi)\,du. \quad (11)$$

The boundary term dominates:

$$\boxed{E_3(N) \;\sim\; \frac{4}{\pi^2}\,N^{3/4}\sin\!\Big(2\pi\sqrt{N}-\frac{5\pi}{4}\Big) + O\big(N^{1/4}\big)\,.} \quad (12)$$

The amplitude $\frac{4}{\pi^2}N^{3/4} \approx 1.28\times10^{13}$ at $N = 10^{18}$. The code observes a peak $|E_3| \approx 1.93\times10^{13}$ at $N \approx 9.33\times10^{17}$, consistent with multi-harmonic constructive interference from the $j \ge 2$ terms in (8).

3.3 The Stationary-Phase Exponent Shift

Equation (12) shows that the exponent in E_3 is $3/4 = \theta + 1/2$ where $\theta = 1/4$. In general:

Proposition 4 (Stationary-phase shift). *If $\Delta_L(N) = O(N^\theta)$ then $E_3(N) = O(N^{\theta+1/2})$. Equivalently, the OLS slope α of $\log|E_3|$ vs. $\log N$ relates to the Gauss circle exponent by:*

$$\alpha = \theta + \frac{1}{2}, \qquad \theta = \alpha - \frac{1}{2}.$$

This is why the code's `--delta` mode (which measures Δ directly) reads θ without any correction, while the integral mode measures $\alpha = \theta + 1/2$ and must subtract $1/2$.

4 The Perturbation Series

4.1 The Character Structure

The analogue of the oscillatory residual from the divisor-problem notes is built from the χ_{-4} character. Write $n = qi + r$ with $r \in \{0, \ldots, i-1\}$. The oscillatory part of the i-th block in (2) is:

$$\delta B_i(N) = 4\chi_{-4}(r)\cdot\frac{r(i-r)}{i} \quad \text{(schematically).} \quad (13)$$

The crucial difference from the divisor case is the factor $\chi_{-4}(r)$: only $r \equiv 1, 3 \pmod 4$ contribute, with signs $+1, -1$ respectively.

4.2 The Dirichlet Series

Defining $e(N) = E_3(N)$ and $F_{\text{Gauss}}(s) = \sum_{N=1}^{\infty} e(N)/N^s$, the same sum-swap and Euclidean division steps as in the divisor problem give:

$$F_{\text{Gauss}}(s) = \sum_{i=1}^{\infty} \frac{1}{i} \sum_{q=i}^{\infty} \sum_{r=0}^{i-1} \frac{\chi_{-4}(r) \cdot r(i-r)}{(qi+r)^s}. \tag{14}$$

4.3 Perturbation Expansion

Factor out $(qi)^{-s}$ and expand $(1 + r/(qi))^{-s}$ exactly as before:

$$F_{\text{Gauss}}(s) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} (-1)^k \binom{s+k-1}{k} \frac{P_k^{\chi}(i)}{i^{s+k+1}} \sum_{q=i}^{\infty} q^{-(s+k)}, \tag{15}$$

where the *character perturbation polynomials* are:

$$P_k^{\chi}(i) = \sum_{r=0}^{i-1} \chi_{-4}(r) \cdot r(i-r) \cdot r^k = i\, S_{k+1}^{\chi}(i-1) - S_{k+2}^{\chi}(i-1), \tag{16}$$

and $S_m^{\chi}(n) = \sum_{r=0}^{n} r^m \chi_{-4}(r)$ are *character power sums.* Since χ_{-4} has period 4, these reduce to Bernoulli-type sums over a period:

$$\begin{aligned} S_1^{\chi}(4q + r_0) &= q\big[(4q+1) - (4q+3)\big] + \text{partial period} \\ &= -2q + (\text{explicit boundary terms}). \end{aligned} \tag{17}$$

Remark 5 (Quasi-polynomial structure)**.** Unlike the divisor case, $P_k^{\chi}(i)$ is *not* a polynomial in i with fixed rational coefficients. Because χ_{-4} has period 4, the character power sums $S_m^{\chi}(n) = \sum_{r=0}^{n} r^m \chi_{-4}(r)$ are *quasi-polynomials* in n: polynomials whose coefficients depend on $n \bmod 4$. Consequently $P_k^{\chi}(i)$ is a quasi-polynomial of degree $k+2$ in i, with coefficients that are piecewise-constant in $i \bmod 4$.

This does not break the argument. Summing over i in (15) splits into the four residue classes $i \bmod 4$, each contributing a genuine polynomial piece; the result is a finite linear combination of L-functions twisted by those residue classes.

Each $P_k^{\chi}(i)$ is a quasi-polynomial of degree $k+2$ in i with coefficients depending on $i \bmod 4$. Explicitly for $k = 0$:

$$P_0^{\chi}(i) = \begin{cases} \frac{i^3}{8} + O(i^2) & i \equiv 0 \pmod 4, \\ \frac{i^3 - 4i}{8} & i \equiv 1 \pmod 4, \\ \frac{i^3}{8} + O(i^2) & i \equiv 2 \pmod 4, \\ \frac{i^3 - 4i}{8} & i \equiv 3 \pmod 4, \end{cases} \tag{18}$$

with uniform leading coefficient $i^3/8$.

4.4 The Dominant L-Function

Summing over i, the $k = 0$ term is:

$$F_0^{\text{Gauss}}(s) = L(s, \chi_{-4}) \cdot Q_0^{\chi}(s), \tag{19}$$

where $L(s, \chi_{-4}) = \sum_{n=1}^{\infty} \chi_{-4}(n)/n^s = 1 - 1/3^s + 1/5^s - \cdots$ is the Dirichlet L-function of χ_{-4}, and

$$Q_0^{\chi}(s) = \sum_{i=1}^{\infty} \frac{P_0^{\chi}(i)}{i^{s+1}} \tag{20}$$

is a finite linear combination of L-functions at nearby arguments, analytic and generically nonzero in the strip $3/4 < \text{Re}(s) < 2$.

5 Truncation: Which Terms Are Negligible?

The size argument is identical to the divisor problem. The k-th term contributes $e_k(N) = O(N^{1/2+\theta-k/2})$, giving:

k	Size of $e_k(N)$	Abscissa	Under Huxley ($\theta < 1/2$)
0	$O(N^{1/2+\theta})$	$1/2 + \theta$	≈ 0.815
1	$O(N^{\theta})$	θ	≈ 0.315
2	$O(N^{\theta-1/2})$	$\theta - 1/2$	< 0
$k \geq 2$	$O(N^{\theta-1/2})$	< 0	converges everywhere

Lemma 6 (Abscissa stability under analytic perturbation)**.** *Let $F(s) = F_0(s) + H(s)$ where $H(s)$ converges absolutely for $\text{Re}(s) > \alpha$. If $\sigma_c(F_0) > \alpha$, then $\sigma_c(F) = \sigma_c(F_0)$.*

Proof. On $\text{Re}(s) > \alpha$, H converges absolutely, hence conditionally. On $\text{Re}(s) > \sigma_c(F_0)$, F_0 converges conditionally. Their sum F therefore converges conditionally on $\text{Re}(s) > \max(\sigma_c(F_0), \alpha) = \sigma_c(F_0)$. For the lower bound: on $\alpha < \text{Re}(s) \leq \sigma_c(F_0)$, write $F_0 = F - H$; conditional convergence of F would force conditional convergence of F_0, contradicting the definition of $\sigma_c(F_0)$. □

Proposition 7. *Under Huxley's proven bound $\theta < 1/2$, the $k \geq 2$ tail $R(s) = \sum_{k \geq 2}(\cdots)$ is absolutely convergent for all $\text{Re}(s) > 0$. By Lemma 6 it does not affect the abscissa of F_{Gauss}, which is therefore determined by $F_0^{\text{Gauss}} + F_1^{\text{Gauss}}$ alone.*

Theorem 8 (Main reduction, Gauss circle)**.** *The series $F_{\text{Gauss}}(s)$ decomposes as $F_0^{\text{Gauss}} + F_1^{\text{Gauss}} + R(s)$, where:*

$$\boxed{F_0^{\text{Gauss}}(s) = L(s, \chi_{-4}) \cdot Q_0^{\chi}(s),} \tag{21}$$

$F_1^{\text{Gauss}}(s)$ is analytic for $\text{Re}(s) > 3/4$, and $R(s)$ is absolutely convergent for $\text{Re}(s) > 0$ (Proposition 7). Consequently $\sigma_0 \leq 1/2 + \theta_{\text{Gauss}}$, with the bound coming from F_0^{Gauss}.

Remark 9 (The cancellation gap, Gauss version)**.** The identification $\sigma_0 = \sigma_c(F_0^{\mathrm{Gauss}})$ is morally correct but not automatic: F_1^{Gauss} converges only conditionally in the strip $3/4 < \mathrm{Re}(s) < 1/2 + \theta$, so Lemma 6 does not apply to it. Cancellation between the $k = 0$ and $k = 1$ layers could in principle shift σ_0 strictly left of $\sigma_c(F_0^{\mathrm{Gauss}})$.

Hardy's lower bound. This leftward shift is bounded below by a proven theorem: Hardy (1915) established the Ω-result $E_3(N) = \Omega(N^{1/4})$, which forces $\sigma_0 \geq 3/4$. Cancellation can therefore only place σ_0 inside the interval $[3/4, \sigma_c(F_0^{\mathrm{Gauss}}))$, not below $3/4$. The gap is solely about whether $\sigma_0 = \sigma_c(F_0^{\mathrm{Gauss}})$ exactly, or σ_0 lies strictly in that interval with $\theta_{\mathrm{Gauss}} < \mu_{\chi_{-4}}(3/4)$.

6 Reduction to a Contour Integral

6.1 Perron's Formula

Exactly as in the divisor problem, Perron's formula gives:

$$\sum_{N \leq x} E_3(N) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F_{\mathrm{Gauss}}(s) \, \frac{x^s}{s} \, ds. \tag{22}$$

Since Q_0^χ is analytic and bounded on the integration line, the integral reduces to a Fourier transform of $L(\sigma + it, \chi_{-4})/(\sigma + it)$ at frequency $\log x$.

6.2 The Final Statement

Definition 10 (Lindelöf μ-function for L-functions)**.** For a Dirichlet L-function $L(s, \chi)$, define $\mu_\chi(\sigma) := \inf\{\alpha \geq 0 : L(\sigma + it, \chi) = O(|t|^\alpha)\}$.
Theorem 11 (Gauss circle and L-function growth)**.** *The derivation establishes the upper bound*

$$\theta_{\mathrm{Gauss}} \;\leq\; \mu_{\chi_{-4}}\!\left(\tfrac{3}{4}\right), \tag{23}$$

where $\mu_{\chi_{-4}}$ is the Lindelöf μ-function for $L(s, \chi_{-4})$ (Definition 10). Explicitly: the Perron integral (22)*, dominated by $F_0^{\mathrm{Gauss}}(s)$, converges for* $\mathrm{Re}(s) > 1/2 + \mu_{\chi_{-4}}(3/4)$ *and yields $\sum_{N \leq x} E_3(N) = O(x^{1/2 + \mu_{\chi_{-4}}(3/4) + \varepsilon})$.*

The classical equality $\theta_{\mathrm{Gauss}} = \mu_{\chi_{-4}}(3/4)$ is known in the literature, but the reverse inequality $\theta_{\mathrm{Gauss}} \geq \mu_{\chi_{-4}}(3/4)$ requires a Tauberian argument controlling the cancellation between the $k = 0$ and $k = 1$ layers — the step identified as open in Remark 9. This derivation does not carry out that step. Accordingly, the correct statement derived here is:

$$\boxed{\theta_{\mathrm{Gauss}} \;\leq\; \mu_{\chi_{-4}}\!\left(\tfrac{3}{4}\right),}$$

with equality conditional on closing the cancellation gap.

Note: Hardy's Ω-theorem ($E_3(N) = \Omega(N^{1/4})$) gives the independent lower bound $\theta_{\text{Gauss}} \geq 1/4$, hence also $\mu_{\chi_{-4}}(3/4) \geq 1/4$. This bounds σ_0 below by 3/4 but does not resolve the question of whether $\sigma_0 = \sigma_c(F_0^{\text{Gauss}})$.

The conjecture $\theta_{\text{Gauss}} = 1/4$ is equivalent to:

> $L(3/4 + it, \chi_{-4})$ *remains sub-polynomially bounded in* $|t|$.

7 The Sibling Relationship with the Divisor Problem

The two problems are exact siblings:

	Divisor problem	**Gauss circle**
Exact sequence	$a(n) = n\,D(n-1) - \sigma_\tau(n-1)$	$b_1(N) = \sum_{m<N} A(m)$
Building block	$\tau(n) = \sum_{d\mid n} 1$	$r_2(n) = 4\sum_{d\mid n} \chi_{-4}(d)$
Character	trivial (1)	χ_{-4} (period 4)
Main term	$\frac{n^2}{2}\log n + (\gamma - \frac{3}{4})n^2$	$\frac{\pi}{2}N^2 - N$
Error	$e(n) = O(n^{1/2+\theta})$	$E_3(N) = O(N^{1/2+\theta})$
L-function	$\zeta(s)$	$L(s, \chi_{-4})$
Critical line	$\mathrm{Re}(s) = 1/2$	$\mathrm{Re}(s) = 1/2$
Conjecture	$\theta = 1/4$ ($\sigma_0 = 3/4$)	$\theta = 1/4$ ($\sigma_0 = 3/4$)
Huxley bound	$\theta \leq 131/416$	$\theta \leq 131/416$

Both problems reduce to the same question — how far left into the critical strip does the relevant L-function remain well-behaved in the vertical direction — and both are controlled by the same van der Corput exponential sum technology. The identical Huxley bound is not a coincidence: the exponent pair machinery does not distinguish between ζ and $L(\cdot, \chi_{-4})$ at this level of precision.

7.1 The Generalisation

For any primitive Dirichlet character χ, define

$$b_\chi(N) = \sum_{k=1}^{N-1} (N-k) \left(\sum_{d\mid k} \chi(d) \right),$$

and the analogous error $E_\chi(N)$. The same derivation gives $F_\chi(s) \approx L(s, \chi) \cdot Q_0^\chi(s)$, and θ_χ satisfies $\theta_\chi \leq \mu_\chi(3/4)$ (upper bound from the Perron integral), with $\theta_\chi \geq 1/4$

from the Ω-theorem for each such character problem. The equality $\theta_\chi = \mu_\chi(3/4)$ holds modulo the same cancellation gap. There is one such problem for each L-function; the obstruction in every case is sub-polynomial vertical growth of $L(\cdot, \chi)$ near $\mathrm{Re}(s) = 3/4$, bounded below by the same Hardy floor.

8 The Hierarchy of Implications

$$\text{RH for } L(\cdot, \chi_{-4}) \implies \text{Lindelöf: } \mu(1/2) = 0 \implies \mu(3/4) = 0 \implies \theta_{\text{Gauss}} = \tfrac{1}{4}.$$

The Lindelöf hypothesis for $L(s, \chi_{-4})$ states $L(1/2 + it, \chi_{-4}) = O(|t|^\varepsilon)$ for all $\varepsilon > 0$. This would imply $\theta_{\text{Gauss}} = 1/4$ immediately. Current best (Huxley): $\theta_{\text{Gauss}} \leq 131/416 \approx 0.315$, empirically ≈ 0.263 at $N = 10^{18}$.

Both walls are the same wall. One is labelled ζ; the other is labelled $L(\cdot, \chi_{-4})$. Neither is coming down today.