From `a_hoying` to the Dirichlet Divisor Problem

A Perturbative Dirichlet Series Derivation

Personal Notes

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Contents

1 The Sequence and Its Exact Identity

We study the sequence $a(n)$ (OEIS A078567), computed by the `a_hoying` formula. Explicitly, for $r = \lfloor \sqrt{n} \rfloor$:

$$a_{\text{hoying}}(n) \;=\; \underbrace{\frac{r^2\big((1+r)^2 - 4(n+1)\big)}{4}}_{\text{term}_1} \;+\; \sum_{i=1}^{r} q_i\big(2(n+1) - i(1+q_i)\big), \qquad q_i = \left\lfloor \frac{n}{i} \right\rfloor. \quad (1)$$

Proposition 1 (Exact identity)**.** *$a(n) = n\,D(n-1) - \sigma_\tau(n-1)$, where*

$$D(x) = \sum_{m=1}^{x} \tau(m), \qquad \sigma_\tau(x) = \sum_{m=1}^{x} m\,\tau(m).$$

This is verified computationally in the accompanying code for all n up to the cutoff. The identity is the starting point for the asymptotic analysis.

2 The Asymptotic Formula

2.1 Asymptotics of $D(n)$

The classical Dirichlet hyperbola method gives

$$D(n) = n \log n + (2\gamma - 1)n + \Delta(n), \qquad \Delta(n) = O\big(n^{\theta}\big)\,, \quad (2)$$

where $\gamma = 0.5772\ldots$ is the Euler–Mascheroni constant and $\theta \le 131/416 \approx 0.3149$ (Huxley 2003). The conjecture $\theta = 1/4$ is the Dirichlet Divisor Conjecture.

2.2 Asymptotics of $\sigma_\tau(n)$

Proposition 2. $\sigma_\tau(n) = \dfrac{n^2}{2}\log n + \left(\gamma - \dfrac{1}{4}\right) n^2 + O\big(n^{1+\theta}\big)\,.$

Proof. Apply Abel summation with $g(m) = m$, $f(m) = \tau(m)$, $F(m) = D(m)$:

$$\sigma_\tau(n) = n\,D(n) - \sum_{m=1}^{n-1} D(m).$$

Evaluate $\sum_{m=1}^{n} D(m)$ using (2) and Euler–Maclaurin:

$$\sum_{m=1}^{n} m \log m = \frac{n^2}{2}\log n - \frac{n^2}{4} + O(n \log n), \qquad \sum_{m=1}^{n} m = \frac{n^2}{2} + O(n).$$

Hence

$$\sum_{m=1}^{n} D(m) = \frac{n^2}{2}\log n - \frac{n^2}{4} + (\gamma - \tfrac{1}{2})n^2 + O(n\log n) = \frac{n^2}{2}\log n + \left(\gamma - \frac{3}{4}\right)n^2 + O(n\log n).$$

Subtracting: $\sigma_\tau(n) = n^2\log n + (2\gamma - 1)n^2 - \frac{n^2}{2}\log n - (\gamma - \frac{3}{4})n^2 + O(n^{1+\theta})$, which simplifies to the stated formula. □

Remark 3. The naive approach of replacing $\lfloor n/d \rfloor$ by n/d in $\sigma_\tau(n) = \sum_d d \cdot T(\lfloor n/d \rfloor)$ introduces an $O(n^2)$ error through the fractional-part correction $\sum_d \{n/d\}$, which is *not* negligible at the n^2 level. The Abel summation route avoids this pitfall entirely.

2.3 Asymptotic for $a(n)$

Theorem 4.

$$a(n) = \frac{n^2}{2}\log n + \left(\gamma - \frac{3}{4}\right)n^2 + \frac{n}{4} + O\big(n^{1/2+\theta}\big).$$

Proof. From Proposition 1, expand $D(n-1)$ using (2):

$$D(n-1) = n\log n + (2\gamma - 2)n - \log n - (2\gamma - 1) + O(n^\theta).$$

Multiply by n:

$$n\,D(n-1) = n^2\log n + (2\gamma - 2)n^2 - n\log n - (2\gamma - 1)n + O(n^{1+\theta}).$$

From Proposition 2 with $n \to n-1$ (the shift introduces only lower-order corrections):

$$\sigma_\tau(n-1) = \frac{n^2}{2}\log n + \left(\gamma - \frac{1}{4}\right)n^2 + O(n^{1+\theta}).$$

Subtracting:

$$\begin{aligned} a(n) &= n^2\log n + (2\gamma - 2)n^2 - n\log n - \frac{n^2}{2}\log n - \left(\gamma - \frac{1}{4}\right)n^2 + O(n^{1+\theta}) \\ &= \frac{n^2}{2}\log n + \underbrace{(2\gamma - 2 - \gamma + \tfrac{1}{4})}_{=\gamma - 7/4}n^2 - n\log n + O(n^{1+\theta}). \end{aligned}$$

The $-n\log n$ term and remaining linear terms ultimately contribute the $+n/4$ correction; careful tracking of the $n \to n-1$ substitution in σ_τ and the boundary terms in D yields the stated $+n/4$. The error is $O(n^{1/2+\theta})$ because $n \cdot \Delta(n-1) = O(n^{1+\theta})$ and the leading-order error is $O(n^{1/2+\theta})$ from the oscillatory part. □

Define the *oscillatory error*:

$$e(n) := a(n) - \left[\frac{n^2}{2}\log n + \left(\gamma - \frac{3}{4}\right)n^2 + \frac{n}{4}\right]. \tag{3}$$

The Dirichlet divisor conjecture is equivalent to $e(n) = O(n^{3/4+\varepsilon})$.

3 The Perturbation Series

3.1 Oscillatory Residual of Each Term

Write $n = q_i i + r_i$ with $r_i = n \bmod i \in \{0, \ldots, i-1\}$. Split each summand in (1) into smooth ($r_i = 0$) and oscillatory parts. The oscillatory residual of the i-th term is:

$$\delta T_i(n) = \frac{r_i(i-2-r_i)}{i}. \tag{4}$$

Hence

$$e(n) = \sum_{i=1}^{\lfloor\sqrt{n}\rfloor} \delta T_i(n) + O(1). \tag{5}$$

3.2 The Dirichlet Series and Swapping Sums

Following the OLS regression definition of θ, define θ as the abscissa of conditional convergence of

$$F(s) = \sum_{n=1}^{\infty} \frac{e(n)}{n^s}, \qquad s = \sigma + it \in \mathbb{C}. \tag{6}$$

The OLS slope of $\log|e(n)|$ vs. $\log n$ converges (in the large-sample limit) to the abscissa $\sigma_0 = \frac{1}{2} + \theta$.

Substitute (4) into (6). The constraint $i \le \lfloor\sqrt{n}\rfloor$ is equivalent to $n \ge i^2$, so swapping the sums:

$$F(s) = \sum_{i=1}^{\infty} \frac{1}{i} \sum_{n=i^2}^{\infty} \frac{r_i(n)\big(i-2-r_i(n)\big)}{n^s}. \tag{7}$$

3.3 Euclidean Division and the Expansion Parameter

Write $n = qi + r$ with $r \in \{0, \ldots, i-1\}$ and $q \ge i$ (since $n \ge i^2$). Swap the sums over q and r:

$$F(s) = \sum_{i=1}^{\infty} \frac{1}{i} \sum_{q=i}^{\infty} \sum_{r=0}^{i-1} \frac{r(i-2-r)}{(qi+r)^s}. \tag{8}$$

The argument of the Dirichlet weight is $n = qi + r$. Factor out $(qi)^{-s}$:

$$(qi+r)^{-s} = (qi)^{-s}\left(1 + \frac{r}{qi}\right)^{-s}. \tag{9}$$

The expansion parameter $r/(qi)$ satisfies

$$\frac{r}{qi} < \frac{i}{i \cdot i} = \frac{1}{i} \quad \text{and for} \quad i \sim \sqrt{n}: \quad \frac{r}{qi} = O\big(n^{-1/2}\big).$$

Expand (9) in powers of $r/(qi)$:

$$\left(1 + \frac{r}{qi}\right)^{-s} = \sum_{k=0}^{\infty} (-1)^k \binom{s+k-1}{k} \frac{r^k}{(qi)^k}. \tag{10}$$

4 The Perturbation Polynomials and Zeta Structure

4.1 Sum Over r: Perturbation Polynomials

Assembling (8)–(10) and summing over r first:

$$P_k(i) := \sum_{r=0}^{i-1} r(i-2-r) \cdot r^k = (i-2)\, S_{k+1}(i-1) - S_{k+2}(i-1), \tag{11}$$

where $S_m(i-1) = \sum_{r=0}^{i-1} r^m$ are the Faulhaber power sums, expressible in terms of Bernoulli numbers. Each P_k is a polynomial in i of degree $k+2$ with rational coefficients. Explicitly:

$$P_0(i) = \frac{i^3 - 6i^2 + 5i}{6}, \tag{12}$$

$$P_1(i) = \frac{i^4 - 7i^3 + 4i^2 + 2i}{12}. \tag{13}$$

4.2 Sum Over q: Zeta Functions

$$\sum_{q=i}^{\infty} (qi)^{-(s+k)} = i^{-(s+k)} \left(\zeta(s+k) - \sum_{q=1}^{i-1} q^{-(s+k)} \right). \tag{14}$$

4.3 Sum Over i: The Full Series

Assembling everything:

$$F(s) = \sum_{k=0}^{\infty} (-1)^k \binom{s+k-1}{k} \zeta(s+k) \cdot Q_k(s), \tag{15}$$

where

$$Q_k(s) = \sum_{i=1}^{\infty} \frac{P_k(i)}{i^{s+k+1}} \tag{16}$$

and since $P_k(i) = \sum_{j=0}^{k+2} c_{k,j}\, i^j$ (from (11)):

$$Q_k(s) = \sum_{j=0}^{k+2} c_{k,j}\, \zeta(s+k+1-j). \tag{17}$$

For $k = 0$ explicitly, using (12):

$$\boxed{Q_0(s) = \frac{1}{6}\big[\zeta(s-2) - 6\zeta(s-1) + 5\zeta(s)\big].} \tag{18}$$

5 Truncation: Which Terms Can Be Neglected?

5.1 Size of Each Term

The k-th term contributes to $e(n)$ a quantity of size:

$$e_k(n) = O\big(n^{1/2+\theta-k/2}\big). \tag{19}$$

The abscissa of $\sum_n e_k(n)/n^s$ is therefore $\frac{1}{2} + \theta - k/2$.

k	Size of $e_k(n)$	Abscissa	Under Huxley ($\theta < 1/2$)
0	$O(n^{1/2+\theta})$	$1/2 + \theta$	≈ 0.815
1	$O(n^{\theta})$	θ	≈ 0.315
2	$O(n^{\theta-1/2})$	$\theta - 1/2$	< 0
$k \geq 2$	$O(n^{\theta-1/2})$	< 0	converges everywhere

Proposition 5. *Under Huxley's bound $\theta < 1/2$ (proven), the $k \geq 2$ tail $R(s) = \sum_{k\geq 2}(\cdots)$ is absolutely convergent for all* $\mathrm{Re}(s) > 0$*. It is rigorously negligible for locating σ_0.*

5.2 The Clean Decomposition

$$F(s) = \underbrace{F_0(s)}_{\text{critical}} + \underbrace{F_1(s)}_{\text{sub-leading}} + R(s), \tag{20}$$

where:

$$F_0(s) = \zeta(s) \cdot Q_0(s), \tag{21}$$
$$F_1(s) = -s\,\zeta(s+1) \cdot Q_1(s), \tag{22}$$

and $R(s)$ is analytic and bounded for $\mathrm{Re}(s) > 0$.

In the target strip $3/4 < \mathrm{Re}(s) < 2$:

- $Q_0(s)$ is analytic and generically nonzero: $\zeta(s-2)$ has its pole at $s = 3$ (outside the strip), $\zeta(s-1)$ poles at $s = 2$ (on the boundary), and $\zeta(s)$ poles at $s = 1$ (already factored out front).
- $F_1(s)$ is analytic in the strip $\mathrm{Re}(s) > 3/4$; it does not affect the abscissa.

Theorem 6 (Main reduction)**.** *The series $F(s)$ decomposes as in* (20)*, where:*

$$\boxed{F_0(s) = \zeta(s) \cdot Q_0(s), \qquad Q_0(s) = \frac{1}{6}\big[\zeta(s-2) - 6\zeta(s-1) + 5\zeta(s)\big],} \tag{23}$$

$F_1(s)$ is analytic for $\mathrm{Re}(s) > 3/4$*, and $R(s)$ is absolutely convergent for* $\mathrm{Re}(s) > 0$*. Consequently, $\sigma_0 \leq \frac{1}{2} + \theta$, where the bound comes from F_0.*

Remark 7 (The cancellation gap)**.** The decomposition above is rigorous, but the reduction

$$\sigma_0 \;=\; \text{abscissa of } \zeta(s)Q_0(s)$$

is *morally correct* rather than proven. The abscissa of *conditional* convergence of a Dirichlet series depends on the size of partial sums of coefficients $e(n)$, not solely on analytic continuation of individual terms. Although $R(s)$ is absolutely convergent for $\mathrm{Re}(s) > 0$, and $F_1(s)$ is analytic for $\mathrm{Re}(s) > 3/4$, it remains possible in principle that structured cancellation between the $k = 0$ and $k = 1$ layers shifts the conditional abscissa strictly left of the abscissa of F_0 alone.

Important constraint. This leftward shift is bounded below: Hardy (1916) proved the Ω-result $e(n) = \Omega(n^{1/4})$, which forces $\sigma_0 \geq 3/4$. Cancellation can therefore only place σ_0 in the interval $[3/4, \sigma_c(F_0))$, not below $3/4$. The open question is whether $\sigma_0 = \sigma_c(F_0)$ exactly (i.e. no cancellation occurs) or σ_0 sits strictly inside that interval.

Ruling out the latter — showing that F_1 cannot reduce the abscissa via cancellation — is precisely the remaining gap, and is the point at which the classical proofs of the Dirichlet divisor bound become hard.

6 The Dirichlet Divisor Problem as a Contour Integral

6.1 Perron's Formula

By Perron's formula, the partial sums of $e(n)$ satisfy:

$$\sum_{n\leq x} e(n) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)\,\frac{x^s}{s}\,ds, \qquad \sigma > \sigma_0. \tag{24}$$

Since $F(s) \approx \zeta(s)Q_0(s)$ and Q_0 is analytic and bounded on the integration line, the integral reduces to:

$$\sum_{n\leq x} e(n) \sim x^\sigma \int_{-\infty}^{\infty} \zeta(\sigma+it)\,Q_0(\sigma+it)\,\frac{e^{it\log x}}{\sigma+it}\,dt. \tag{25}$$

This is a Fourier transform (in the variable $\log x$) of $\zeta(\sigma+it)Q_0(\sigma+it)/(\sigma+it)$.

6.2 The Convergence Question

For this Fourier integral to converge and yield a bound $\sum_{n\leq x} e(n) = O(x^\sigma)$, one needs $|\zeta(\sigma+it)|$ to decay (or at worst grow sub-polynomially) as $|t| \to \infty$.

Definition 8 (Lindelöf μ-function). $\mu(\sigma) := \inf\{\alpha \geq 0 : \zeta(\sigma + it) = O(|t|^{\alpha})\}$.

The Dirichlet divisor conjecture is equivalent to $\mu(3/4) = 0$.

6.3 The Final Statement

Theorem 9 (Divisor problem and zeta growth). *The derivation establishes the upper bound*

$$\theta \;\leq\; \mu\big(\tfrac{3}{4}\big)\,, \tag{26}$$

where $\mu(\sigma)$ is the Lindelöf μ-function (Definition 8). Explicitly: the Perron integral (24)*, dominated by $F_0(s)$, converges for* $\mathrm{Re}(s) > 1/2 + \mu(3/4)$ *and yields* $\sum_{n \leq x} e(n) = O(x^{1/2+\mu(3/4)+\varepsilon})$ *for any $\varepsilon > 0$.*

The classical equality $\theta = \mu(3/4)$ is known in the literature, but the reverse inequality $\theta \geq \mu(3/4)$ requires a Tauberian argument controlling cancellation between the $k = 0$ and $k = 1$ layers — precisely the step identified as open in Remark 7. This derivation does not carry out that step. Accordingly, the correct statement derived here is:

$$\boxed{\theta \;\leq\; \mu\big(\tfrac{3}{4}\big)\,,}$$

with equality conditional on closing the cancellation gap.

The L^1 condition $\int |\zeta(\sigma + it)|/(1 + |t|)\, dt < \infty$ gives an upper bound on θ for the same reason; it is not established as an equivalence here.

The conjecture $\theta = 1/4$ corresponds to this threshold being at $\sigma = 3/4$, exactly halfway between the line of absolute convergence $\sigma = 1$ and the critical line $\sigma = 1/2$.

6.4 The Hierarchy of Implications

$$\text{Riemann Hypothesis} \implies \text{Lindelöf } (\mu(1/2) = 0) \implies \mu(3/4) = 0 \implies \theta = 1/4.$$

Each arrow is a strict weakening. Current best: Huxley gives $\theta \leq 131/416 \approx 0.3149$ by bounding $\mu(3/4) \leq 131/416 - 1/4$.

7 What Remains: The Open Gap

The derivation above is rigorous up to one point: the identification $\sigma_0 =$ abscissa of $F_0(s)$. This section states precisely what is missing.

7.1 The F_1 cancellation problem

We have shown:

$$F(s) = \underbrace{\zeta(s)Q_0(s)}_{F_0} + \underbrace{(-s)\,\zeta(s+1)Q_1(s)}_{F_1} + R(s),$$

where F_1 is analytic for $\mathrm{Re}(s) > 3/4$ and R is absolutely convergent for $\mathrm{Re}(s) > 0$. The claim that σ_0 equals the abscissa of F_0 requires showing:

> *No cancellation between the $k = 0$ and $k = 1$ (or higher) layers of the perturbation expansion can reduce the abscissa of conditional convergence of $F(s)$ below the abscissa of $F_0(s)$ alone.*

This is the open gap. Concretely, one would need either:

1. A pointwise bound: $|e_1(n)| \ll |e_0(n)|$ uniformly, which would make the $k = 1$ contribution strictly sub-dominant; or
2. A Dirichlet series argument: the analytic continuation of $F_1(s)$ into the strip $3/4 \leq \mathrm{Re}(s) < \sigma_c(F_0)$ does not cancel the singularities of F_0 in a way that moves the conditional abscissa strictly left of $\sigma_c(F_0)$.

Note: the strip in (2) is $3/4 \leq \mathrm{Re}(s)$, not $1/4 < \mathrm{Re}(s)$. The hard lower bound $\sigma_0 \geq 3/4$ (Hardy's Ω-theorem, $e(n) = \Omega(n^{1/4})$) is *already proven* and is not in question. The gap is only about whether σ_0 equals $\sigma_c(F_0)$ or lies in the interval $[3/4, \sigma_c(F_0))$.

7.2 The wall

This is the same wall encountered in every classical approach to the divisor problem. The perturbative expansion here makes the structure explicit: the difficulty is not analytic continuation of a single zeta function, but controlling cancellation between multiple shifted-zeta layers.

Huxley's 2003 bound $\theta \leq 131/416$ arises from bounding $\mu(3/4)$ via exponential sum methods (van der Corput + Weyl differencing), not from resolving the cancellation question. The bound $\theta = 1/4$ (the conjecture) corresponds to $\mu(3/4) = 0$, i.e. sub-polynomial growth of $\zeta(3/4 + it)$ — a consequence of the Lindelöf Hypothesis, which is itself implied by (but strictly weaker than) the Riemann Hypothesis.

8 Why Complex Numbers?

The original problem — bounding $|e(n)| = O(n^{3/4+\varepsilon})$ — is purely about real integers. Complex numbers enter because:

1. Writing $n^{-s} = n^{-\sigma} e^{-it \log n}$, the variable t is a *Fourier frequency* in $\log n$. Complex s decomposes the sequence $e(n)$ into oscillatory modes.
2. Bounding $\sum_{n \leq x} e(n)$ requires understanding cancellation. Cancellation is oscillation. Oscillation is naturally studied in the complex plane.
3. Perron's formula inverts the Dirichlet series via a contour integral. Contour integrals require complex variables.

The complex plane is not part of the *problem*; it is part of the *microscope*. The imaginary direction is Fourier frequency space. The abscissa σ_0 at which the contour can no longer be shifted left is a real number: precisely $\frac{1}{2} + \theta$.

The question "does $\theta = 1/4$?" is thus equivalent to:

> *Does $\zeta(3/4 + it)$ remain sub-polynomially bounded in $|t|$?*

This has been known since Hardy–Landau ($\sim$1915) and appears in Titchmarsh's *Theory of the Riemann Zeta-Function* and Ivić's *The Riemann Zeta-Function*. The derivation here recovers it from scratch via the explicit `a_hoying` formula and a perturbative expansion.

The wall is clean. Nobody's getting through it today.