Theorem 1.6.8.

Suppose $X_n \to X$ a.s. Let g, h be continuous functions with
(i) $g \geq 0$ and $g(x) \to \infty$ as $|x| \to \infty$,
(ii) $|h(x)|/g(x) \to 0$ as $|x| \to \infty$,
and (iii) $\mathrm{Eg}\left(X_n\right) \leq K < \infty$ for all n.
Then $\mathrm{Eh}\left(X_n\right) \to Eh(X)$.
Proof. By subtracting a constant from h, we can suppose without loss of generality that $h(0)=0$. Pick M large so that $P(|X|=M)=0$ and $g(x)>0$ when $|x| \geq M$. Given a random variable Y, let $\bar{Y}=Y 1_{(|Y| \leq M)}$. Since $P(|X|=M)=0, \bar{X}_n \to \bar{X}$ a.s. Since $h\left(\bar{X}_n\right)$ is bounded and h is continuous, it follows from the bounded convergence theorem that
(a)

$$E h\left(\bar{X}_n\right) \to \mathrm{Eh}(\bar{X})$$

To control the effect of the truncation, we use the following:
(b) $|E h(\bar{Y})-E h(Y)| \leq E|h(\bar{Y})-h(Y)| \leq E(|h(Y)| ;|Y|>M) \leq \epsilon_M E g(Y)$ where $\epsilon_M=\sup \{|h(x)| / g(x):|x| \geq M\}$. To check the second inequality, note that when $|Y| \leq M, \bar{Y}=Y$, and we have supposed $h(0)=0$. The third inequality follows from the definition of ϵ_M. Taking $Y=X_n$ in (b) and using (iii), it follows that
(c)

$$\left|E h\left(\bar{X}_n\right)-E h\left(X_n\right)\right| \leq K \epsilon_M$$

To estimate $|E h(\bar{X})-E h(X)|$, we observe that $g \geq 0$ and g is continuous, so Fatou's lemma implies

$$E g(X) \leq \liminf_{n \to \infty} E g\left(X_n\right) \leq K$$

Taking $Y=X$ in (b) gives
(d)

$$|E h(\bar{X})-E h(X)| \leq K \epsilon_M$$

The triangle inequality implies

$$\begin{aligned} \left|E h\left(X_n\right)-E h(X)\right| \leq & \left|E h\left(X_n\right)-E h\left(\bar{X}_n\right)\right| \\ & +\left|E h\left(\bar{X}_n\right)-E h(\bar{X})\right|+|E h(\bar{X})-E h(X)| \end{aligned}$$

Taking limits and using (a), (c), (d), we have

$$\limsup_{n\to\infty} |Eh\left(X_n\right) - Eh(X)| \leq 2K\epsilon_M$$

which proves the desired result since $K < \infty$ and $\epsilon_M \to 0$ as $M \to \infty$.