Some diagnostic problems with solutions

Feel free to look up definitions you forgot. Also, feel free to look up various formulas (e.g., the expected value of the binomial random variable) and similar things. You are not expected to remember them.

1 (Pre)calculus

Problem 1.1. The shaded region in the Venn diagram corresponds to

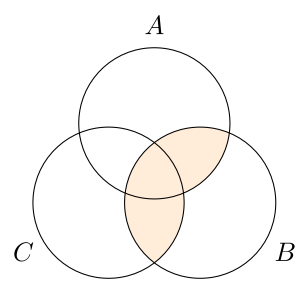
(a) $(A \cap B) \cup C$

(b) $(A \cap B) \cup (A \cap C) \cup (B \cap C)$

(c) $C \cap (A \cup B)$

(d) $(A \cup C) \cap (B \cup C)$

(e) none of the above



Problem 1.2. If $n!$ is the factorial function $n! = n \times (n-1) \times \cdots \times 2 \times 1$, then $\log(\sqrt[n]{n!}) =$

(a) $\sum_{i=1}^{n} \log(n/i)$ (b) $\frac{1}{n}\sum_{i=1}^{n} \log(i)$ (c) $\sqrt[n]{\prod_{i=1}^{n} \log(n)}$ (d) $\frac{1}{n}\prod_{i=1}^{n} \log(i)$ (e) none of the above

Problem 1.3. $\sum_{i=1}^{99} \log_{10}(\frac{i}{i+1}) =$

(a) $\prod_{i=1}^{99} \left(\frac{i}{i+1}\right)^{10}$ (b) $\log_{10}\left(\sum_{i=1}^{99} \prod_{n=1}^{i} \frac{n}{n+1}\right)$ (c) $1/2$ (d) -2 (e) none of the above

Problem 1.4. $\sum_{n=0}^{\infty} e^{-2n} = 1 + e^{-2} + e^{-4} + \cdots =$

(a) $1 - e^{-2}$ (b) $1 + e^{-2}(1 - e^{-2})$ (c) $(1 - e^{-2n})/(1 - e^{-n})$ (d) $\frac{e^2}{e^2-1}$ (e) none of the above

Problem 1.5. The Taylor expansion of the function $f(x) = e^x$ around $x = 0$ is

(a) $1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$

(b) $1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$

(c) $1 + x + x^2/2 + x^3/3! + x^4/4! + x^5/5! + \ldots$

(d) $1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \ldots$

(e) none of the above

Problem 1.6. $\sum_{n=0}^{\infty} \frac{1}{n!} e^{-2n} = 1 + e^{-2} + \frac{e^{-4}}{2} + \cdots =$

(a) e^{-2} (b) $e^{e^{-2}}$ (c) $e^{-e^{-2}}$ (d) $1 - e^{-2}$ (e) $1 - e^{e^{-2}}$

Problem 1.7. If $F(x) = \int_0^x f(y)\,dy$, $G(x) = F(x)^n$ and $n \in \{1, 2, 3, \dots\}$, then

(a) $G'(x) = f^n(x)$ (b) $G'(x) = f(x)^{n-1}F(x)$ (c) $G'(x) = nF(x)^{n-1}$ (d) $G'(x) = nf(x)F(x)^{n-1}$ (e) none of the above

2 Elementary probability

Problem 2.1. Twelve people enter the elevator on the ground floor of a ten-story building and start riding up. By the time the elevator reaches the top floor, all the people have exited the elevator. What is the probability that at least two people exited the elevator on the same floor? (*Hint:* Think!)

(a) $\binom{12}{0}(\frac{1}{10})^{11}(\frac{9}{10})^1$ (b) $\binom{12}{0}(\frac{1}{10})^{12} + \binom{12}{1}(\frac{1}{10})^{11}(\frac{9}{10})^1$ (c) $\frac{1}{12}$ (d) $\frac{1}{10}$ (e) 1

Problem 2.2. 4 fair coins are tossed. The probability that at least one for them came up *heads* is

(a) $\frac{15}{16}$ (b) $\frac{1}{16}$ (c) $\frac{1}{4}$ (d) $\frac{3}{4}$ (e) none of the above

Problem 2.3. Every possible combination of a letter in the alphabet (i.e., chosen from the 26-element set $\{A, B, C, \dots, X, Y, Z\}$) and a number from the set $\{1, 2, \dots, 19, 20\}$ is written on a card. The cards are otherwise identical, and well shuffled in a deck. If a single card is drawn from that deck, the probability that the number on it is odd or that the letter is a vowel (i.e., in the set $\{A, E, I, O, U\}$) is

(a) $\frac{21}{52}$ (b) $\frac{18}{26}$ (c) $\frac{5}{52}$ (d) $\frac{31}{52}$ (e) none of the above

Problem 2.4. A die is rolled 5 times; let the obtained numbers be given by $Y_1, \dots, Y_5$. The probability that the values of $Y_1, \dots, Y_5$ are all different from each other is

(a) $(5/6)^5$ (b) $(1/6)^5$ (c) $\frac{6!}{6^5}$ (d) $\frac{5!}{6^5}$ (e) none of the above

Problem 2.5. A coin is tossed, and, independently, a 6-sided die is rolled. Let

$$A = \{4 \text{ is obtained on the die}\} \text{ and}$$
$$B = \{\textit{Heads} \text{ is obtained on the coin and an even number is obtained on the die}\}.$$

Then

(a) A and B are mutually exclusive (b) A and B are independent (c) $A \subseteq B$ (d) $A \cap B = B$ (e) none of the above

Problem 2.6. Let A and B be two events, and the only thing we know about them is that $\mathbb{P}[A] = \mathbb{P}[B] = \frac{2}{3}$. Then, it is necessarily true that

(a) $A = B$

(b) $A \subseteq B$ or $B \subseteq A$

(c) A and B are independent

(d) A and B^c are mutually exclusive

(e) all of the above are possible, but not necessarily true

Problem 2.7. A class has 7 female and 13 male students. It is also known that there are 15 blue-eyed and 5 brown-eyed students in that class. The probability that a student picked at random is a brown-eyed female is

(a) $\frac{7}{80}$ (b) $\frac{13}{80}$ (c) $\frac{21}{80}$ (d) $\frac{39}{80}$ (e) not enough information is given

Problem 2.8. Which of the following pairs of events are *mutually exclusive*?

(a) A die is rolled; $A =$ "the outcome is even", $B =$ "the outcome is ≤ 4".

(b) A die is rolled; $A =$ "the outcome is ≥ 4", $B =$ "the outcome is ≤ 4".

(c) Two coins are tossed; $A =$ "the outcome of the first coin is H", $B =$ "the outcome of the second coin is H."

(d) Two dice are rolled; $A =$ "the outcome of the the first die is even", $B =$ "the outcome of the second die is odd."

(e) none of the above

Problem 2.9. Which of the following pairs of events are *independent*?

(a) A die is rolled; $A =$ "the outcome is even", $B =$ "the outcome is ≤ 3".

(b) A die is rolled; $A =$ "the outcome is ≥ 4", $B =$ "the outcome is ≤ 4".

(c) Three independent coins are tossed; $A =$ "the outcome of the first coin is H", $B =$ "the total number of Hs is 1. "

(d) A die is rolled; $A =$ "the outcome is even", $B =$ "the outcome is ≤ 2."

(e) none of the above

Problem 2.10. For each of the following statements, state whether it is true or false

1. Let A denote the event that a randomly chosen family has children of both genders, and let B denote the event that a family has at most one boy. Assume that the genders of the children are equally likely and mutually independent and that a family has **three** children. Then, the events A and B are independent.
2. Two dice are rolled. The events

   $$A = \{ \text{ the outcome of the first die is (strictly) smaller than the outcome on the second die.}\}$$

   and

   $$B = \{ \text{ the outcome of the first die is even}\}.$$

   Then A and B are independent.
3. If events E and F are independent, then their complements E^c and F^c are independent as well.

3 Discrete random variables

Problem 3.1. Let Y be a random variable such that

$$\mathbb{P}[Y = 1] = 1/2, \mathbb{P}[Y = 3] = 1/3 \text{ and } \mathbb{P}[Y = 6] = 1/6.$$

With $|\cdot|$ denoting the absolute value, $\mathbb{E}[|Y - 2|] =$

(a) 1/2 (b) 1 (c) 3/2 (d) 5 (e) none of the above

Problem 3.2. Let X and Y be two independent rolls of a die. Then $\mathbb{P}[X - Y \geq 1 \text{ and } Y - X \leq 3]$ is

(a) 1/6 (b) 1/3 (c) 1/2 (d) 2/3 (e) none of the above

Problem 3.3. Three (independent) coins are tossed. Two of them are nickels, and the third is a quarter. Let X denote the total number of heads (H) obtained, and let Y denote the number of heads on the two nickels. (Note: For 2., 3. and 4. write down the distribution tables.)

1. Write down the joint-distribution table of (X, Y).
2. What is the conditional distribution of X given that $Y = 1$?
3. What is the distribution of $Z = X + Y$?
4. What is the distribution of $g(Z)$, where $g(z) = (z - 3)^2$ and $Z = X - Y$.

Problem 3.4. 9 coins are minted so that each of them has a different probability of coming up heads. These probabilities are 0.1, 0.2, . . . , and 0.9. We toss all 9 coins at the same time and count the total number of heads. The expected value of this quantity is:

(a) 4.5 (b) 5 (c) 5.5 (d) 6 (e) none of the above

Problem 3.5. A die is rolled and a coin tossed. The random variable X is defined as the number on the die if the outcome of the coin is H, and 2 otherwise. The distribution of X is

(a)

1	2	3	4	5	6
1/6	1/6	1/6	1/6	1/6	1/6

(b)

1	2	3	4	5	6
2/15	1/3	2/15	2/15	2/15	2/15

(c)

1	2	3	4	5	6
1/10	1/2	1/10	1/10	1/10	1/10

(d)

1	2	3	4	5	6
1/12	7/12	1/12	1/12	1/12	1/12

(e) none of the above

Problem 3.6. n people vote in a general election, with only two candidates running. The vote of person i is denoted by Y_i and it can take values 0 and 1, depending which candidate they voted for (we encode one of them as 0 and the other as 1). We assume that votes are independent of each other and that each person votes for candidate 1 with probability p. If the total number of votes for candidate 1 is denoted by Y, then

(a) Y is a geometric random variable

(b) Y^2 is a binomial random variable

(c) Y is uniform on $\{0, 1, \ldots, n\}$

(d) $\mathrm{Var}[Y] \leq \mathbb{E}[Y]$

(e) none of the above

Problem 3.7. A die is rolled 100 times and X denotes the total number of 6s obtained. The probability that X is at most 48 is given by

(a) $\sum_{k=1}^{48} \binom{100}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{100-k}$

(b) $\sum_{k=0}^{48} \binom{100}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{100-k}$

(c) $\sum_{k=48}^{100} \binom{100}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{100-k}$

(d) $\sum_{k=49}^{100} \binom{100}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{100-k}$

(e) none of the above

Problem 3.8. Let X and Y be independent with $\mathrm{Var}[X] = 5$ and $\mathrm{Var}[Y] = 4$. Then $\mathrm{sd}[X - Y]$ is

(a) 1 (b) $\sqrt{5} + 2$ (c) $\sqrt{5} - 2$ (d) 3 (e) not enough information is given

Problem 3.9. A new addition of Kafka's "Metamorphosis" has 72 pages. The printing press often malfunctions and introduces typos. The number of typos on each page has a Poisson distribution with mean $\log(3)$ and is independent of the number of typos on other pages (or other books). A book is thrown away if it contains typos on more than 32 pages. Use the normal approximation to estimate the proportion of books that get thrown away. (*Note:* log denotes the logarithm with the base e.)

4 Continuous random variables

Problem 4.1. The pdf (probability density function) of the random variable Y is $f(y) = c\exp(-2y)$ for $y > 0$ and $f(y) = 0$ for $y \leq 0$. The constant c is

(a) 1/2 (b) 1 (c) 2 (d) 4 (e) none of the above

Problem 4.2. The pdf (probability density function) of the random variable Y is given by

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

The probability that Y is at most 5 is

(a) $\int_0^5 \frac{1}{\pi(1+x^2)}\, dx$ (b) $\int_{-\infty}^5 \frac{1}{\pi(1+x^2)}\, dx$ (c) $\int_{-5}^5 \frac{1}{\pi(1+x^2)}\, dx$ (d) $\int_5^\infty \frac{1}{\pi(1+x^2)}\, dx$ (e) none of the above

Problem 4.3. The pdf (probability density function) of the random variable Y is given by

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

The probability that Y <u>is equal to</u> 5 is

(a) $\int_0^5 \frac{1}{\pi(1+x^2)}\,dx$ (b) $\int_{-\infty}^5 \frac{1}{\pi(1+x^2)}\,dx$ (c) $\int_{-5}^5 \frac{1}{\pi(1+x^2)}\,dx$ (d) $\int_5^\infty \frac{1}{\pi(1+x^2)}\,dx$ (e) none of the above

Problem 4.4. The CDF (cumulative distribution function) of the random variable Y is

$$F_Y(y) = e^y/(1+e^y).$$

The pdf $f_Y(y)$ is

(a) $\frac{e^y}{(1+e^y)^2}$ (b) $\frac{e^y}{1+e^y}$ (c) $\frac{1}{1+e^y}$ (d) 1 (e) none of the above

5 Conditional probability

Problem 5.1. A box contains 2 red and 2 black balls. Two balls are drawn from it (with any pair being equally likely). If it is known that there is at least one red ball remaining in the box, the probability that both balls drawn were black is

(a) 1/5 (b) 2/5 (c) 3/5 (d) 4/5 (e) none of the above

Problem 5.2. Four balls are drawn (without replacement) from a box which contains 4 black and 5 red balls. Given that the four drawn balls were *not* all of the same color, what is the probability that there were exactly two balls of each color among the four.

Problem 5.3. Two boxes are given. The first one contains 3 blue balls, 4 red balls and 2 green balls. The second one is empty. We start by picking a ball from the first box (all balls are equally likely to be picked):

- If its color is red, we return it to the first box, pick again, and place the picked ball into the second box.
- If its color is blue or green, we place it directly into the second box.

Given that the ball in the second box is blue, what is the probability that the first ball we picked from the first box was red?

Problem 5.4. A fair (6-sided) die is thrown and its outcome is denoted by X. After that, X independent fair coins are tossed and the number of *heads* obtained is denoted by Y. Then $\mathbb{P}[X = 6|Y = 5]$ equals:

(a) 0 (b) 3/32 (c) 1/48 (d) 3/4 (e) none of the above

Problem 5.5. There are 10 green balls and 20 red balls in a box, well mixed together. We pick a ball at random, and then roll a die. If the ball was green, we write down the outcome of the die, but if the ball is red, we write down the number 3.

If the number 3 was written down, the probability that the ball was green is:

(a) 1/3 (b) 1/2 (c) 5/6 (d) 1 (e) none of the above