Some diagnostic problems with solutions

Feel free to look up definitions you forgot. Also, feel free to look up various formulas (e.g., the expected value of the binomial random variable) and similar things. You are not expected to remember them.

1 (Pre)calculus

Problem 1.1. The shaded region in the Venn diagram corresponds to

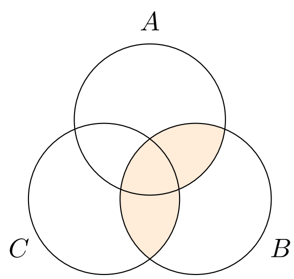
(a) $(A \cap B) \cup C$

(b) $(A \cap B) \cup (A \cap C) \cup (B \cap C)$

(c) $C \cap (A \cup B)$

(d) $(A \cup C) \cap (B \cup C)$

(e) none of the above



Solution: The correct answer is (e).

Problem 1.2. If $n!$ is the factorial function $n! = n \times (n-1) \times \cdots \times 2 \times 1$, then $\log(\sqrt[n]{n!}) =$

(a) $\sum_{i=1}^{n} \log(n/i)$ (b) $\frac{1}{n}\sum_{i=1}^{n} \log(i)$ (c) $\sqrt[n]{\prod_{i=1}^{n} \log(n)}$ (d) $\frac{1}{n}\prod_{i=1}^{n} \log(i)$ (e) none of the above

Solution: The correct answer is (b).

Problem 1.3. $\sum_{i=1}^{99} \log_{10}(\frac{i}{i+1}) =$

(a) $\prod_{i=1}^{99} \left(\frac{i}{i+1}\right)^{10}$ (b) $\log_{10}\left(\sum_{i=1}^{99}\prod_{n=1}^{i}\frac{n}{n+1}\right)$ (c) $1/2$ (d) -2 (e) none of the above

Solution: The correct answer is (d). The sum of logs is the log of the product, so the expression above equals $\log_{10}(\prod_{i=1}^{99}\frac{i}{i+1})$. The product inside is

$$\tfrac{1}{2} \times \tfrac{2}{3} \times \cdots \times \tfrac{98}{99} \times \tfrac{99}{100} = \tfrac{1}{100},$$

and $\log_{10}(1/100) = -2$.

Problem 1.4. $\sum_{n=0}^{\infty} e^{-2n} = 1 + e^{-2} + e^{-4} + \cdots =$

(a) $1 - e^{-2}$ (b) $1 + e^{-2}(1 - e^{-2})$ (c) $(1 - e^{-2n})/(1 - e^{-n})$ (d) $\frac{e^2}{e^2-1}$ (e) none of the above

Solution: The correct answer is (d). This is a sum of a geometric series $1 + q + q^2 + \ldots$ with $q = e^{-2}$, therefore, it is given by

$$\tfrac{1}{1-q} = \tfrac{1}{1-e^{-2}} = \tfrac{e^2}{e^2-1}.$$

Problem 1.5. The Taylor expansion of the function $f(x) = e^x$ around $x = 0$ is

(a) $1 + x + x^2 + x^3 + x^4 + x^5 + \dots$

(b) $1 - x + x^2 - x^3 + x^4 - x^5 + \dots$

(c) $1 + x + x^2/2 + x^3/3! + x^4/4! + x^5/5! + \dots$

(d) $1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \dots$

(e) none of the above

Solution: The correct answer is (c).

Problem 1.6. $\sum_{n=0}^{\infty} \frac{1}{n!} e^{-2n} = 1 + e^{-2} + \frac{e^{-4}}{2} + \dots =$

(a) e^{-2} (b) $e^{e^{-2}}$ (c) $e^{-e^{-2}}$ (d) $1 - e^{-2}$ (e) $1 - e^{e^{-2}}$

Solution: The correct answer is (b). The is the Taylor expansion of the function e^x around 0, evaluated at $x = e^{-2}$.

Problem 1.7. If $F(x) = \int_0^x f(y)\,dy$, $G(x) = F(x)^n$ and $n \in \{1, 2, 3, \dots\}$, then

(a) $G'(x) = f^n(x)$ (b) $G'(x) = f(x)^{n-1}F(x)$ (c) $G'(x) = nF(x)^{n-1}$ (d) $G'(x) = nf(x)F(x)^{n-1}$ (e) none of the above

Solution: The correct answer is (d).

2 Elementary probability

Problem 2.1. Twelve people enter the elevator on the ground floor of a ten-story building and start riding up. By the time the elevator reaches the top floor, all the people have exited the elevator. What is the probability that at least two people exited the elevator on the same floor? (*Hint:* Think!)

(a) $\binom{12}{0}(\frac{1}{10})^{11}(\frac{9}{10})^1$ (b) $\binom{12}{0}(\frac{1}{10})^{12} + \binom{12}{1}(\frac{1}{10})^{11}(\frac{9}{10})^1$ (c) $\frac{1}{12}$ (d) $\frac{1}{10}$ (e) 1

Solution: The correct answer is (e). If 12 people need to exit on ten floors, there must be at least one floor where at least two people exit.

Problem 2.2. 4 fair coins are tossed. The probability that at least one for them came up *heads* is

(a) $\frac{15}{16}$ (b) $\frac{1}{16}$ (c) $\frac{1}{4}$ (d) $\frac{3}{4}$ (e) none of the above

Solution: The correct answer is (a).

Problem 2.3. Every possible combination of a letter in the alphabet (i.e., chosen from the 26-element set $\{A, B, C, \dots, X, Y, Z\}$) and a number from the set $\{1, 2, \dots, 19, 20\}$ is written on a card. The cards are otherwise identical, and well shuffled in a deck. If a single card is drawn from that deck, the probability that the number on it is odd or that the letter is a vowel (i.e., in the set $\{A, E, I, O, U\}$) is

(a) $\frac{21}{52}$ (b) $\frac{18}{26}$ (c) $\frac{5}{52}$ (d) $\frac{31}{52}$ (e) none of the above

Solution: The correct answer is (d).

Problem 2.4. A die is rolled 5 times; let the obtained numbers be given by $Y_1, \dots, Y_5$. The probability that the values of $Y_1, \dots, Y_5$ are all different from each other is

(a) $(5/6)^5$ (b) $(1/6)^5$ (c) $\frac{6!}{6^5}$ (d) $\frac{5!}{6^5}$ (e) none of the above

Solution: The correct answer is (c). There are $6 \times 5 \times 4 \times 3 \times 2 = 6!$ way of arranging 5 different numbers from 1 to 6 in a row, and there are 6^5 such arrangements when they do not have to be different.

Alternatively, once the first die is thrown, the probability that the second one will not result in the same number is $5/6$. Then, given the first two, the third one will take a value different from the first two with probability $4/6$, etc. Therefore the probability is

$$\frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} \times \frac{2}{6} = \frac{5 \times 4 \times 4 \times 2}{6^4} = \frac{5!}{6^4} = \frac{6!}{6^5}$$

Problem 2.5. A coin is tossed, and, independently, a 6-sided die is rolled. Let

$$A = \{4 \text{ is obtained on the die}\} \text{ and}$$
$$B = \{\textit{Heads} \text{ is obtained on the coin and an even number is obtained on the die}\}.$$

Then

(a) A and B are mutually exclusive (b) A and B are independent (c) $A \subseteq B$ (d) $A \cap B = B$ (e) none of the above

Solution: The correct answer is (e).

Problem 2.6. Let A and B be two events, and the only thing we know about them is that $\mathbb{P}[A] = \mathbb{P}[B] = \frac{2}{3}$. Then, it is necessarily true that

(a) $A = B$

(b) $A \subseteq B$ or $B \subseteq A$

(c) A and B are independent

(d) A and B^c are mutually exclusive

(e) all of the above are possible, but not necessarily true

Solution: The correct answer is (e).

Problem 2.7. A class has 7 female and 13 male students. It is also known that there are 15 blue-eyed and 5 brown-eyed students in that class. The probability that a student picked at random is a brown-eyed female is

(a) $\frac{7}{80}$ (b) $\frac{13}{80}$ (c) $\frac{21}{80}$ (d) $\frac{39}{80}$ (e) not enough information is given

Solution: The correct answer is (e). We do not know how the eye color is distributed among male/female students, so we cannot compute the probability

$$\mathbb{P}[\{ \text{ brown-eyed } \} \cap \{ \text{ female } \}]$$

(*Note:* One thing we *do know* is that these two traits cannot be independent. If they were, $5/20 = 1/4$ of the female students would be brown-eyed, but that cannot be the case as there are 7 female students, and 7 is not divisible by 4.)

Problem 2.8. Which of the following pairs of events are *mutually exclusive*?

(a) A die is rolled; A = "the outcome is even", B = "the outcome is ≤ 4".

(b) A die is rolled; A = "the outcome is ≥ 4", B = "the outcome is ≤ 4".

(c) Two coins are tossed; A = "the outcome of the first coin is H", B = "the outcome of the second coin is H."

(d) Two dice are rolled; A = "the outcome of the the first die is even", B = "the outcome of the second die is odd."

(e) none of the above

Solution: The correct answer is (e). All of these pairs can happen at the same time.

Problem 2.9. Which of the following pairs of events are *independent*?

(a) A die is rolled; A = "the outcome is even", B = "the outcome is ≤ 3".

(b) A die is rolled; A = "the outcome is ≥ 4", B = "the outcome is ≤ 4".

(c) Three independent coins are tossed; A = "the outcome of the first coin is H", B = "the total number of Hs is 1. "

(d) A die is rolled; A = "the outcome is even", B = "the outcome is ≤ 2."

(e) none of the above

Solution: The correct answer is (d).

Problem 2.10. For each of the following statements, state whether it is true or false

1. Let A denote the event that a randomly chosen family has children of both genders, and let B denote the event that a family has at most one boy. Assume that the genders of the children are equally likely and mutually independent and that a family has **three** children. Then, the events A and B are independent.
2. Two dice are rolled. The events

   $$A = \{ \text{ the outcome of the first die is (strictly) smaller than the outcome on the second die.}\}$$

   and

   $$B = \{ \text{ the outcome of the first die is even}\}.$$

   Then A and B are independent.
3. If events E and F are independent, then their complements E^c and F^c are independent as well.

Solution:

1. True. The sample space for this problem is the set of all 8 possible combinations BBB, BBG, BGB, $\dots$, GGG. Since the genders are equally likely and independent, all 8 have the same probability $1/8$. The event A consists of 6 elementary events - all except BBB and GGG, so that $\mathbb{P}[A] = 3/4$. The event B contains the elementary events BGG, GBG, GGB and GGG, and, so $\mathbb{P}[B] = 1/2$.

   To check whether A and B are independent, we also need the probability of $A \cap B$. In this case, the elementary events that belong to both A and B are BGG, GBG and GGB, so $\mathbb{P}[A \cap B] = 3/8$. Since

   $$\mathbb{P}[A \cap B] = \tfrac{3}{8} = \tfrac{3}{4} \times \tfrac{1}{2} = \mathbb{P}[A]\mathbb{P}[B],$$

   the events A and B are independent, by definition.

2. False. Draw the table of all 36 combinations of possible outcomes of two dice to check the following statements:
   (a) A contains $5+4+3+2+1=15$ elementary events, so that $\mathbb{P}[A]=\frac{15}{36}$.
   (b) B contains $3\times 6=18$ elementary events, so that $\mathbb{P}[B]=\frac{1}{2}$
   (c) the elementary events in $A\cap B$ are $(2,3),(2,4),(2,5),(2,6),(4,5)$ and $(4,6)$, so that $\mathbb{P}[A\cap B]=\frac{6}{36}=\frac{1}{6}$.

   It follows that A and B are not independent, since $\mathbb{P}[A]\times\mathbb{P}[B]=\frac{15}{72}\neq\frac{1}{6}=\mathbb{P}[A\cap B]$.
3. True. We need to check whether $\mathbb{P}[E^c\cap F^c]=\mathbb{P}[E^c]\times\mathbb{P}[F^c]$. Use the fact that $\mathbb{P}[E\cap F]=\mathbb{P}[E]\times\mathbb{P}[F]$ and draw a Venn diagram to convince yourself all the equalities below are correct

$$\begin{aligned}\mathbb{P}[E^c\cap F^c]&=\mathbb{P}[(E\cup F)^c]=1-\mathbb{P}[E\cup F]=1-\mathbb{P}[E]-\mathbb{P}[F]+\mathbb{P}[E\cap F]\\&=1-\mathbb{P}[E]-\mathbb{P}[F]+\mathbb{P}[E]\times\mathbb{P}[F]=(1-\mathbb{P}[E])\times(1-\mathbb{P}[F])=\mathbb{P}[E^c]\times\mathbb{P}[F^c]\end{aligned}$$

3 Discrete random variables

Problem 3.1. Let Y be a random variable such that

$$\mathbb{P}[Y=1]=1/2, \mathbb{P}[Y=3]=1/3 \text{ and } \mathbb{P}[Y=6]=1/6.$$

With $|\cdot|$ denoting the absolute value, $\mathbb{E}[|Y-2|]=$

(a) 1/2 (b) 1 (c) 3/2 (d) 5 (e) none of the above

Solution: The correct answer is (c). $\mathbb{E}[|Y-2|]=|1-2|\times\frac{1}{2}+|3-2|\times\frac{1}{3}+|6-2|\times\frac{1}{6}=\frac{1}{2}+\frac{1}{3}+\frac{4}{6}=3/2$.

Problem 3.2. Let X and Y be two independent rolls of a die. Then $\mathbb{P}[X-Y\geq 1 \text{ and } Y-X\leq 3]$ is

(a) 1/6 (b) 1/3 (c) 1/2 (d) 2/3 (e) none of the above

Solution: The correct answer is (e). Not that if $X-Y\geq 1$ then $Y-X\leq -1\leq 3$. Therefore, the probability above is the same as $\mathbb{P}[X-Y\geq 1]$. We need to count all pairs of (x,y) with $x,y\in\{1,2,3,4,5,6\}$ such that $x\geq y+1$ and add their probabilities (which are all equal to 1/36). This corresponds to the boxed squares in the joint distribution table below:

	1	2	3	4	5	6
1	1/36	$\boxed{1/36}$	$\boxed{1/36}$	$\boxed{1/36}$	$\boxed{1/36}$	$\boxed{1/36}$
2	1/36	1/36	$\boxed{1/36}$	$\boxed{1/36}$	$\boxed{1/36}$	$\boxed{1/36}$
3	1/36	1/36	1/36	$\boxed{1/36}$	$\boxed{1/36}$	$\boxed{1/36}$
4	1/36	1/36	1/36	1/36	$\boxed{1/36}$	$\boxed{1/36}$
5	1/36	1/36	1/36	1/36	1/36	$\boxed{1/36}$
6	1/36	1/36	1/36	1/36	1/36	1/36

There are 15 boxed squares, so the probability is 15/36.

Problem 3.3. Three (independent) coins are tossed. Two of them are nickels, and the third is a quarter. Let X denote the total number of heads (H) obtained, and let Y denote the number of heads on the two nickels. (Note: For 2., 3. and 4. write down the distribution tables.)

1. Write down the joint-distribution table of (X,Y).
2. What is the conditional distribution of X given that $Y=1$?
3. What is the distribution of $Z=X+Y$?
4. What is the distribution of $g(Z)$, where $g(z)=(z-3)^2$ and $Z=X-Y$.

Solution:

1. Here is the distribution table for (X, Y). Marginal distributions for X and Y are in the top row and the right-most column.

	1/8	3/8	3/8	1/8	
2	0	0	1/8	1/8	1/4
1	0	1/4	1/4	0	1/2
0	1/8	1/8	0	0	1/4
Y					
X	**0**	**1**	**2**	**3**	

2. (This is the second row of the joint distribution table, normalized by the marginal probability $\mathbb{P}[Y = 1]$.)

k	1	2
$\mathbb{P}[X = k \vert Y = 1]$	1/2	1/2

.

3. To find the distribution of Z, we first observe that the values it can take are $0, 1, 2, 3, 4$ and 5. To find the probabilities of those values, we need to look into the joint distribution table and add up the probabilities in all squares the contribute towards those values (the diagonals):

k	0	1	2	3	4	5
$\mathbb{P}[Z = k]$	1/8	1/8	1/4	1/4	1/8	1/8

.

4. The values $g(Z)$ can take are $\{(-3)^2, (-2)^2, (-1)^2, 0^2, 1^2, 2^2\} = \{0, 1, 4, 9\}$. From the distribution table for Z, we easily get that the distribution table for $g(Z)$ is given by

k	0	1	4	9
$\mathbb{P}[g(Z) = k]$	1/4	3/8	1/4	1/8

.

Problem 3.4. 9 coins are minted so that each of them has a different probability of coming up heads. These probabilities are 0.1, 0.2, . . . , and 0.9. We toss all 9 coins at the same time and count the total number of heads. The expected value of this quantity is:

(a) 4.5 (b) 5 (c) 5.5 (d) 6 (e) none of the above

Solution: The correct answer is (a). Let the outcomes of the coins be denoted by $Y_1, \dots, Y_0$, where $Y_k = 1$ if the k-the coin came up heads, and 0 otherwise. The expected total number of heads is $\mathbb{E}[Y_1 + \dots + Y_9] = \mathbb{E}[Y_1] + \dots + \mathbb{E}[Y_9] = 0.1 + 0.2 + \dots + 0.9 = 4.5$.

Problem 3.5. A die is rolled and a coin tossed. The random variable X is defined as the number on the die if the outcome of the coin is H, and 2 otherwise. The distribution of X is

(a)

1	2	3	4	5	6
1/6	1/6	1/6	1/6	1/6	1/6

(b)

1	2	3	4	5	6
2/15	1/3	2/15	2/15	2/15	2/15

(c)

1	2	3	4	5	6
1/10	1/2	1/10	1/10	1/10	1/10

(d)

	1	2	3	4	5	6
	1/12	7/12	1/12	1/12	1/12	1/12

(e) none of the above

Solution: The correct answer is (d). $X = 2$ if we either get T on the coin (probability $1/2$) or if we get H on the coin and 2 on the die (probability $1/2 \times 1/6 = 1/12$). Therefore $\mathbb{P}[X = 2] = 1/2 + 1/12 = 7/12$. The other outcomes are all equally likely, with probability $1/12$.

Problem 3.6. n people vote in a general election, with only two candidates running. The vote of person i is denoted by Y_i and it can take values 0 and 1, depending which candidate they voted for (we encode one of them as 0 and the other as 1). We assume that votes are independent of each other and that each person votes for candidate 1 with probability p. If the total number of votes for candidate 1 is denoted by Y, then

(a) Y is a geometric random variable

(b) Y^2 is a binomial random variable

(c) Y is uniform on $\{0, 1, \dots, n\}$

(d) $\mathrm{Var}[Y] \leq \mathbb{E}[Y]$

(e) none of the above

Solution: The correct answer is (d). Y is a binomial random variable with parameters n and p, and, so, $\mathbb{E}[Y] = np$, $\mathrm{Var}[Y] = np(1-p) \leq np$. The Y is clearly not geometric, as it takes only finitely many possible values. Y^2 is not binomial, because the set of possible values is not contiguous (some values are skipped), and Y is not uniform since the binomial probabilities are not all the same.

Problem 3.7. A die is rolled 100 times and X denotes the total number of 6s obtained. The probability that X is at most 48 is given by

(a) $\sum_{k=1}^{48} \binom{100}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{100-k}$

(b) $\sum_{k=0}^{48} \binom{100}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{100-k}$

(c) $\sum_{k=48}^{100} \binom{100}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{100-k}$

(d) $\sum_{k=49}^{100} \binom{100}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{100-k}$

(e) none of the above

Solution: The correct answer is (b). The others have wrong limits of summation.

Problem 3.8. Let X and Y be independent with $\mathrm{Var}[X] = 5$ and $\mathrm{Var}[Y] = 4$. Then $\mathrm{sd}[X - Y]$ is

(a) 1 (b) $\sqrt{5} + 2$ (c) $\sqrt{5} - 2$ (d) 3 (e) not enough information is given

Solution: The correct answer is (d). Since X and Y are independent, we have $\mathrm{Var}[X - Y] = \mathrm{Var}[X + (-Y)] = \mathrm{Var}[X] + \mathrm{Var}[-Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] = 9$. Therefore, $\mathrm{sd}[X - Y] = 3$.

Problem 3.9. A new addition of Kafka's "Metamorphosis" has 72 pages. The printing press often malfunctions and introduces typos. The number of typos on each page has a Poisson distribution with mean log(3) and is independent of the number of typos on other pages (or other books). A book is thrown away if it contains typos on more than 32 pages. Use the normal approximation to estimate the proportion of books that get thrown away. (*Note:* log denotes the logarithm with the base e.)

Solution: By the formula for the Poisson distribution, the probability that a page contains no typos is $p = e^{-\ln(3)} = \frac{1}{3}$. Hence, the probability of finding a typo on a given page is $\frac{2}{3}$. The number of typos per book is therefore binomially distributed with the mean $\mu = 72 \times \frac{2}{3} = 48$ and standard deviation $\sigma = \sqrt{72 \times \frac{2}{3} \times \frac{1}{3}} = 4$.

In the normal approximation, we are looking for the probability

$$\mathbb{P}\left[Z \geq \frac{32\frac{1}{2} - 48}{4}\right] = \mathbb{P}[Z \geq -3.875] = \Phi(3.875) \cong 0.99995$$

This is a terrible printing press. We would, therefore, expect, approximately and on average, only one out of 18575 books *not* to be thrown away.

4 Continuous random variables

Problem 4.1. The pdf (probability density function) of the random variable Y is $f(y) = c\exp(-2y)$ for $y > 0$ and $f(y) = 0$ for $y \leq 0$. The constant c is

(a) 1/2 (b) 1 (c) 2 (d) 4 (e) none of the above

Solution: The correct answer is (c). We have $1 = \int_0^\infty ce^{-2y}\,dy = c \times \frac{1}{2}$.

Problem 4.2. The pdf (probability density function) of the random variable Y is given by

$$f(x) = \tfrac{1}{\pi(1+x^2)}.$$

The probability that Y is at most 5 is

(a) $\int_0^5 \frac{1}{\pi(1+x^2)}\,dx$ (b) $\int_{-\infty}^5 \frac{1}{\pi(1+x^2)}\,dx$ (c) $\int_{-5}^5 \frac{1}{\pi(1+x^2)}\,dx$ (d) $\int_5^\infty \frac{1}{\pi(1+x^2)}\,dx$ (e) none of the above

Solution: The correct answer is (b).

Problem 4.3. The pdf (probability density function) of the random variable Y is given by

$$f(x) = \tfrac{1}{\pi(1+x^2)}.$$

The probability that Y is equal to 5 is

(a) $\int_0^5 \frac{1}{\pi(1+x^2)}\,dx$ (b) $\int_{-\infty}^5 \frac{1}{\pi(1+x^2)}\,dx$ (c) $\int_{-5}^5 \frac{1}{\pi(1+x^2)}\,dx$ (d) $\int_5^\infty \frac{1}{\pi(1+x^2)}\,dx$ (e) none of the above

Solution: The correct answer is (e). This probability is 0 because Y is a continuous random variable.

Problem 4.4. The CDF (cumulative distribution function) of the random variable Y is

$$F_Y(y) = e^y/(1+e^y).$$

The pdf $f_Y(y)$ is

(a) $\frac{e^y}{(1+e^y)^2}$ (b) $\frac{e^y}{1+e^y}$ (c) $\frac{1}{1+e^y}$ (d) 1 (e) none of the above

Solution: The correct answer is (a).

5 Conditional probability

Problem 5.1. A box contains 2 red and 2 black balls. Two balls are drawn from it (with any pair being equally likely). If it is known that there is at least one red ball remaining in the box, the probability that both balls drawn were black is

(a) 1/5 (b) 2/5 (c) 3/5 (d) 4/5 (e) none of the above

Solution: The event A="at least one red ball remained in the box" is exactly the same as A="at least one picked ball is black". We are looking for the probability of B="both picked balls are black", given A. We have $\mathbb{P}[B] = 1/2 \times 1/3 = 1/6$ and $\mathbb{P}[A] = 1 - \mathbb{P}[A^c] = 1 - 1/2 \times 1/3 = 5/6$. Since $B \subseteq A$, we have

$$\mathbb{P}[B|A] = \mathbb{P}[B \cap A]/\mathbb{P}[A] = \mathbb{P}[B]/\mathbb{P}[A] = \tfrac{1}{5}.$$

Problem 5.2. Four balls are drawn (without replacement) from a box which contains 4 black and 5 red balls. Given that the four drawn balls were *not* all of the same color, what is the probability that there were exactly two balls of each color among the four.

Solution: Let A denote the event that the colors of the balls drawn are not all the same, and let B denote the event that there are exactly two black balls and two red balls. We are looking for $\mathbb{P}[B|A]$. Since $B \subseteq A$, we have

$$\mathbb{P}[B|A] = \mathbb{P}[B \cap A]/\mathbb{P}[A] = \mathbb{P}[B]/\mathbb{P}[A].$$

To compute $\mathbb{P}[A]$, we note that the event A^c consists of only two elementary outcomes: either all balls are black or all balls are red. The probability of picking all red balls is

$$\frac{\binom{5}{4}}{\binom{9}{4}}$$

while the probability of picking all black balls is

$$\frac{\binom{4}{4}}{\binom{9}{4}} = \frac{1}{\binom{9}{4}}.$$

Putting these two together, we obtain

$$\mathbb{P}[A] = 1 - \frac{\binom{5}{4} + 1}{\binom{9}{4}}.$$

To compute $\mathbb{P}[B]$ we note that we can choose 2 red balls out of 5 in $\binom{5}{2}$ ways and, then, for each such choice, we have $\binom{4}{2}$ ways of choosing 2 black balls from the set of 4 black balls. Therefore,

$$\mathbb{P}[B] = \left(\binom{5}{2} \times \binom{4}{2}\right) \Big/ \binom{9}{4}.$$

Finally,

$$\mathbb{P}[B|A] = \frac{\binom{5}{2}\binom{4}{2}}{\binom{9}{4} - \binom{5}{4} - 1} = \frac{10 \times 6}{126 - 5 - 1} = \frac{1}{2}.$$

Problem 5.3. Two boxes are given. The first one contains 3 blue balls, 4 red balls and 2 green balls. The second one is empty. We start by picking a ball from the first box (all balls are equally likely to be picked):

- If its color is red, we return it to the first box, pick again, and place the picked ball into the second box.
- If its color is blue or green, we place it directly into the second box.

Given that the ball in the second box is blue, what is the probability that the first ball we picked from the first box was red?

Solution: Let R_1, B_1 and G_1 denote the events where the first ball we draw is red, blue and green (respectively). Similarly, let R_2, B_2, G_2 denote the evens where the ball placed in the second box is red, blue and green (respectively).

We are looking for $\mathbb{P}[R_1|B_2]$. By the Bayes formula, we have

$$\begin{aligned}\mathbb{P}[R_1|B_2] &= \frac{\mathbb{P}[B_2|R_1] \times \mathbb{P}[R_1]}{\mathbb{P}[B_2|R_1] \times \mathbb{P}[R_1] + \mathbb{P}[B_2|B_1] \times \mathbb{P}[B_1] + \mathbb{P}[B_2|G_1] \times \mathbb{P}[G_1]} \\ &= \frac{\frac{1}{3} \times \frac{4}{9}}{\frac{1}{3} \times \frac{4}{9} + 1 \times \frac{1}{3} + 0 \times \frac{2}{9}} = \frac{4}{13}.\end{aligned}$$

Problem 5.4. A fair (6-sided) die is thrown and its outcome is denoted by X. After that, X independent fair coins are tossed and the number of *heads* obtained is denoted by Y. Then $\mathbb{P}[X = 6|Y = 5]$ equals:

(a) 0 (b) 3/32 (c) 1/48 (d) 3/4 (e) none of the above

Solution: The correct answer is (d). By the law of total probability, we have

$$\begin{aligned}\mathbb{P}[Y = 5] = \mathbb{P}[Y = 5|X = 5]\,\mathbb{P}[X = 5] \\ + \mathbb{P}[Y = 5|X = 6]\,\mathbb{P}[X = 6] = (2^{-5} + 6 \times 2^{-6})\tfrac{1}{6} = \tfrac{1}{48}.\end{aligned}$$

The Bayes formula then implies that

$$\begin{aligned}\mathbb{P}[X = 6|Y = 5] &= \frac{\mathbb{P}[X = 6, Y = 5]}{\mathbb{P}[Y = 5]} = \frac{\mathbb{P}[Y = 5|X = 6]\,\mathbb{P}[X = 6]}{\mathbb{P}[Y = 5]} \\ &= \frac{\frac{3}{32} \times \frac{1}{6}}{\frac{1}{48}} = \frac{3}{4}.\end{aligned}$$

Problem 5.5. There are 10 green balls and 20 red balls in a box, well mixed together. We pick a ball at random, and then roll a die. If the ball was green, we write down the outcome of the die, but if the ball is red, we write down the number 3.

If the number 3 was written down, the probability that the ball was green is:

(a) 1/3 (b) 1/2 (c) 5/6 (d) 1 (e) none of the above

Solution: The correct answer is (e). Let R denote the event when the ball drawn was red, and $G = R^c$ the event corresponding to drawing a green ball, so that $\mathbb{P}[R] = 2/3$ and $\mathbb{P}[G] = 1/3$. If X denotes the number written down, we have

$$\mathbb{P}[X = 3|G] = 1/6 \text{ and } \mathbb{P}[X = 3|R] = 1.$$

Using Bayes formula,

$$\begin{aligned}\mathbb{P}[G|X = 3] &= \frac{\mathbb{P}[X = 3|G] \times \mathbb{P}[G]}{\mathbb{P}[X = 3|G] \times \mathbb{P}[G] + \mathbb{P}[X = 3|R] \times \mathbb{P}[R]} \\ &= \frac{1/6 \times 1/3}{1/6 \times 1/3 + 1 \times 2/3} = \frac{1}{13}.\end{aligned}$$