Module 2

Matrices and Linear Maps

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Problem 1: 20 points total

(1) (5 points) Prove that the column vectors of the matrix A_2 given by

$$A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{pmatrix}$$

are linearly independent.

$$A_2^{-1} = \begin{pmatrix} 2 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$A_2^{-1}A_2 = I$
$I_{11} = 1(2) + 1(0) + 1(-\frac{1}{2}) + 1(-\frac{1}{2}) = 1$
$I_{12} = 1(-1) + 1(1) + 1(0) + 1(0) = 0$
$I_{13} = 1(-1) + 1(0) + 1(1) + 1(0) = 0$
$I_{14} = 1(1) + 1(-1) + 1(-\frac{1}{2}) + 1(\frac{1}{2}) = 0$
$I_{21} = 1(2) + 2(0) + 1(-\frac{1}{2}) + 3(-\frac{1}{2}) = 0$
$I_{22} = 1(-1) + 2(1) + 1(0) + 3(0) = 1$
$I_{23} = 1(-1) + 2(0) + 1(1) + 3(0) = 0$
$I_{24} = 1(1) + 2(-1) + 1(-\frac{1}{2}) + 3(\frac{1}{2}) = 0$
$I_{31} = 1(2) + 1(0) + 2(-\frac{1}{2}) + 2(-\frac{1}{2}) = 0$
$I_{32} = 1(-1) + 1(1) + 2(0) + 2(0) = 0$
$I_{33} = 1(-1) + 1(0) + 2(1) + 2(0) = 1$
$I_{34} = 1(1) + 1(-1) + 2(-\frac{1}{2}) + 2(\frac{1}{2}) = 0$
$I_{41} = 1(2) + 1(0) + 1(-\frac{1}{2}) + 3(-\frac{1}{2}) = 0$
$I_{42} = 1(-1) + 1(1) + 1(0) + 3(0) = 0$
$I_{43} = 1(-1) + 1(0) + 1(1) + 3(0) = 0$
$I_{44} = 1(1) + 1(-1) + 1(-\frac{1}{2}) + 3(\frac{1}{2}) = 1$

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(2) (5 points) Prove that the column vectors of the matrix B_2 given by

$$B_2 = \begin{pmatrix} 1 & -2 & 2 & -2 \\ 0 & -3 & 2 & -3 \\ 3 & -5 & 5 & -4 \\ 3 & -4 & 4 & -4 \end{pmatrix}$$

are linearly independent.

$$B_2^{-1} = \begin{pmatrix} -2 & 0 & 0 & 1 \\ 4\frac{1}{2} & -1 & -1 & -\frac{1}{2} \\ 4\frac{1}{2} & -1 & 0 & -1\frac{1}{2} \\ -1\frac{1}{2} & 0 & 1 & -\frac{1}{2} \end{pmatrix}$$

$B_2^{-1}B_2 = I$
$I_{11} = 1(-2) + -2(4\frac{1}{2}) + 2(4\frac{1}{2}) + -2(-1\frac{1}{2}) = 1$
$I_{12} = 1(0) + -2(-1) + 2(-1) + -2(0) = 0$
$I_{13} = 1(0) + -2(-1) + 2(0) + -2(1) = 0$
$I_{14} = 1(1) + -2(-\frac{1}{2}) + 2(-1\frac{1}{2}) + -2(-\frac{1}{2}) = 0$
$I_{21} = 0(-2) + -3(4\frac{1}{2}) + 2(4\frac{1}{2}) + -3(-1\frac{1}{2}) = 0$
$I_{22} = 0(0) + -3(-1) + 2(-1) + -3(0) = 1$
$I_{23} = 0(0) + -3(-1) + 2(0) + -3(1) = 0$
$I_{24} = 0(1) + -3(-\frac{1}{2}) + 2(-1\frac{1}{2}) + -3(-\frac{1}{2}) = 0$
$I_{31} = 3(-2) + -5(4\frac{1}{2}) + 5(4\frac{1}{2}) + -4(-1\frac{1}{2}) = 0$
$I_{32} = 3(0) + -5(-1) + 5(-1) + -4(0) = 0$
$I_{33} = 3(0) + -5(-1) + 5(0) + -4(1) = 1$
$I_{34} = 3(1) + -5(-\frac{1}{2}) + 5(-1\frac{1}{2}) + -4(-\frac{1}{2}) = 0$
$I_{41} = 3(-2) + -4(4\frac{1}{2}) + 4(4\frac{1}{2}) + -4(-1\frac{1}{2}) = 0$
$I_{42} = 3(0) + -4(-1) + 4(-1) + -4(0) = 0$
$I_{43} = 3(0) + -4(-1) + 4(0) + -4(1) = 0$
$I_{44} = 3(1) + -4(-\frac{1}{2}) + 4(-1\frac{1}{2}) + -4(-\frac{1}{2}) = 1$

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(3) (10 points) Prove that the coordinates of the column vectors of the matrix B_2 over the basis consisting of the column vectors of A_2 are the columns of the matrix P_2 given by

$$P_2 = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -3 & 1 & -2 & 1 \\ 1 & -2 & 2 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

$B2_{11} = 2(1) + -3(1) + 1(1) + 1(1) = 1$
$B2_{21} = 2(1) + -3(2) + 1(1) + 1(3) = 0$
$B2_{31} = 2(1) + -3(1) + 1(2) + 1(2) = 3$
$B2_{41} = 2(1) + -3(1) + 1(1) + 1(3) = 3$
$B2_{12} = 0(1) + 1(1) + -2(1) + -1(1) = -2$
$B2_{22} = 0(1) + 1(2) + -2(1) + -1(3) = -3$
$B2_{32} = 0(1) + 1(1) + -2(2) + -1(2) = -5$
$B2_{42} = 0(1) + 1(1) + -2(1) + -1(3) = -4$
$B2_{13} = 1(1) + -2(1) + 2(1) + 1(1) = 2$
$B2_{23} = 1(1) + -2(2) + 2(1) + 1(3) = -2$
$B2_{33} = 1(1) + -2(1) + 2(2) + 1(2) = 5$
$B2_{43} = 1(1) + -2(1) + 2(1) + 1(3) = 4$
$B2_{14} = -1(1) + 1(1) + -1(1) + -1(1) = -2$
$B2_{24} = -1(1) + 1(2) + -1(1) + -1(3) = -3$
$B2_{34} = -1(1) + 1(1) + -1(2) + -1(2) = -4$
$B2_{44} = -1(1) + 1(1) + -1(1) + -1(3) = -4$

Check that $A_2P_2 = B_2$.
Done.
Prove that

$$P_2^{-1} = \begin{pmatrix} -1 & -1 & -1 & 1 \\ 2 & 1 & 1 & -2 \\ 2 & 1 & 2 & -3 \\ -1 & -1 & 0 & -1 \end{pmatrix}.$$

$P_2^{-1}P_2 = I$
$I_{11} = 2(-1) + 0(2) + 1(2) + -1(-1) = 1$
$I_{12} = 2(-1) + 0(1) + 1(1) + -1(-1) = 0$
$I_{13} = 2(-1) + 0(1) + 1(2) + -1(0) = 0$
$I_{14} = 2(1) + 0(-2) + 1(-3) + -1(-1) = 0$
$I_{21} = -3(-1) + 1(2) + -2(2) + 1(-1) = 0$
$I_{22} = -3(-1) + 1(1) + -2(1) + 1(-1) = 1$
$I_{23} = -3(-1) + 1(1) + -2(2) + 1(0) = 0$
$I_{24} = -3(1) + 1(-2) + -2(-3) + 1(-1) = 0$
$I_{31} = 1(-1) + -2(2) + 2(2) + -1(-1) = 0$
$I_{32} = 1(-1) + -2(1) + 2(1) + -1(-1) = 0$
$I_{33} = 1(-1) + -2(1) + 2(2) + -1(0) = 1$
$I_{34} = 1(1) + -2(-2) + 2(-3) + -1(-1) = 0$
$I_{41} = 1(-1) + -1(2) + 1(2) + -1(-1) = 0$
$I_{42} = 1(-1) + -1(1) + 1(1) + -1(-1) = 0$
$I_{43} = 1(-1) + -1(1) + 1(2) + -1(0) = 0$

$I_{44} = 1(1) + -1(-2) + 1(-3) + -1(-1) = 1$

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

What are the coordinates over the basis consisting of the column vectors of B_2 of the vector whose coordinates over the basis consisting of the column vectors of A_2 are $(2, -3, 0, 0)$?

$old = (2, -3, 0, 0)$
$new = A_2^{-1} \cdot old$
$new_1 = (2)(2) + (-1)(-3) + (-1)(0) + (-1)(0) = 7$
$new_2 = (0)(2) + (1)(-3) + (0)(0) + (-1)(0) = -3$
$new_2 = (-\frac{1}{2})(2) + (0)(-3) + (1)(0) + (-\frac{1}{2})(0) = -1$
$new_4 = (-\frac{1}{2})(2) + (0)(-3) + (0)(0) + (\frac{1}{2})(0) = -1$

$$A_2^{-1} \cdot old = \begin{pmatrix} 2 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ -1 \\ -1 \end{pmatrix} = new$$

Problem 2: 30 points total

Consider the polynomials

$$B_0^2(t) = (1-t)^2 \qquad B_1^2(t) = 2(1-t)t \qquad B_2^2(t) = t^2$$
$$B_0^3(t) = (1-t)^3 \qquad B_1^3(t) = 3(1-t)^2 t \qquad B_2^3(t) = 3(1-t)t^2 \qquad B_3^3(t) = t^3,$$

known as the *Bernstein polynomials* of degree 2 and 3.

(1) (10 points) Show that the Bernstein polynomials $B_0^2(t), B_1^2(t), B_2^2(t)$ are expressed as linear combinations of the basis $(1, t, t^2)$ of the vector space of polynomials of degree at most 2 as follows:

$$\begin{pmatrix} B_0^2(t) \\ B_1^2(t) \\ B_2^2(t) \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}.$$

Prove that

$$B_0^2(t) + B_1^2(t) + B_2^2(t) = 1.$$

(2) (10 points) Show that the Bernstein polynomials $B_0^3(t), B_1^3(t), B_2^3(t), B_3^3(t)$ are expressed as linear combinations of the basis $(1, t, t^2, t^3)$ of the vector space of polynomials of degree at most 3 as follows:

$$\begin{pmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{pmatrix} = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}.$$

Prove that

$$B_0^3(t) + B_1^3(t) + B_2^3(t) + B_3^3(t) = 1.$$

(3) (10 points) Prove that the Bernstein polynomials of degree 2 are linearly independent, and that the Bernstein polynomials of degree 3 are linearly independent.

Problem 3: 10 points

Prove that for every vector space E, if $f\colon E \to E$ is an idempotent linear map, i.e., $f \circ f = f$, then we have a direct sum

$$E = \operatorname{Ker} f \oplus \operatorname{Im} f,$$

so that f is the projection onto its image $\operatorname{Im} f$.
$u \in E$
$u = f(u) + (u - f(u))$
$f(u) = \operatorname{Im} f$
$f(u - f(u)) = f(u) - f(f(u)) = f(u) - f(u) = 0 = \operatorname{Ker} f$
WTS $f(u) \cap f(u - f(u)) = 0$
since $0 \in f(u)$ and $f(u - f(u)) = 0 \implies f(u) \cap f(u - f(u)) = f(u) \cap 0 = 0$

Problem 4: 20 points plus 15 points Extra Credit

Given any vector space E, a linear map $f\colon E \to E$ is an *involution* if $f \circ f = \mathrm{id}$.

(1) (10 points) Prove that an involution f is invertible. What is its inverse?
f is invertible if $f \circ f^{-1} = \mathrm{id}$
$f \circ f = \mathrm{id} \implies f \circ f^{-1} = \mathrm{id} \implies f = f^{-1}$
so f is the inverse of f

(2) (10 points) Let E_1 and E_{-1} be the subspaces of E defined as follows:

$$E_1 = \{u \in E \mid f(u) = u\}$$
$$E_{-1} = \{u \in E \mid f(u) = -u\}.$$

Prove that we have a direct sum

$$E = E_1 \oplus E_{-1}.$$

Hint. For every $u \in E$, write

$$u = \frac{u + f(u)}{2} + \frac{u - f(u)}{2}.$$

$f(\frac{u+f(u)}{2}) = \frac{1}{2} \cdot (f(u) + f(f(u))) = \frac{1}{2} \cdot (f(u) + (u)) = \frac{1}{2} \cdot (u + u) = \frac{1}{2} \cdot (2u) = u = E_1$
$f(\frac{u-f(u)}{2}) = \frac{1}{2} \cdot (f(u) - f(f(u))) = \frac{1}{2} \cdot (f(u) - (u)) = \frac{1}{2} \cdot (-u - u) = \frac{1}{2} \cdot (-2u) - u = E_{-1}$

WTS $E_1 \cap E_{-1} = (0)$

since $u \in E$ and $u \in E_{-1}$
then $f(u) = u = f(u) = -u$
so $u = -u \implies u = 0 \implies E_1 \cap E_{-1} = (0)$

(3) **Extra credit** (15 points) If E is finite-dimensional and f is an involution, prove that there is some basis of E with respect to which the matrix of f is of the form

$$I_{k,n-k} = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix (similarly for I_{n-k}) and $k = \dim(E_1)$. Can you give a geometric interpretation of the action of f (especially when $k = n - 1$)?

Total: 70 points   Extra Credit: 15 points