Module 2

Matrices and Linear Maps

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Problem 1: 20 points total

(1) (5 points) Prove that the column vectors of the matrix A_2 given by

$$A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{pmatrix}$$

are linearly independent.

$$A_2^{-1} = \begin{pmatrix} 2 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$A_2^{-1}A_2 = I$
$I_{11} = 1(2) + 1(0) + 1(-\frac{1}{2}) + 1(-\frac{1}{2}) = 1$
$I_{12} = 1(-1) + 1(1) + 1(0) + 1(0) = 0$
$I_{13} = 1(-1) + 1(0) + 1(1) + 1(0) = 0$
$I_{14} = 1(1) + 1(-1) + 1(-\frac{1}{2}) + 1(\frac{1}{2}) = 0$
$I_{21} = 1(2) + 2(0) + 1(-\frac{1}{2}) + 3(-\frac{1}{2}) = 0$
$I_{22} = 1(-1) + 2(1) + 1(0) + 3(0) = 1$
$I_{23} = 1(-1) + 2(0) + 1(1) + 3(0) = 0$
$I_{24} = 1(1) + 2(-1) + 1(-\frac{1}{2}) + 3(\frac{1}{2}) = 0$
$I_{31} = 1(2) + 1(0) + 2(-\frac{1}{2}) + 2(-\frac{1}{2}) = 0$
$I_{32} = 1(-1) + 1(1) + 2(0) + 2(0) = 0$
$I_{33} = 1(-1) + 1(0) + 2(1) + 2(0) = 1$
$I_{34} = 1(1) + 1(-1) + 2(-\frac{1}{2}) + 2(\frac{1}{2}) = 0$
$I_{41} = 1(2) + 1(0) + 1(-\frac{1}{2}) + 3(-\frac{1}{2}) = 0$
$I_{42} = 1(-1) + 1(1) + 1(0) + 3(0) = 0$
$I_{43} = 1(-1) + 1(0) + 1(1) + 3(0) = 0$
$I_{44} = 1(1) + 1(-1) + 1(-\frac{1}{2}) + 3(\frac{1}{2}) = 1$

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

ALTERNATE SOLUTION:

We can prove linear independence by solving a linear system of equations such that the only solution for a, b, c, d is 0.

$$a * \begin{pmatrix}1\\1\\1\\1\end{pmatrix} + b * \begin{pmatrix}1\\2\\1\\1\end{pmatrix} + c * \begin{pmatrix}1\\1\\2\\1\end{pmatrix} + d * \begin{pmatrix}1\\3\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$a + b + c + d = 0 \tag{1}$$
$$a + 2b + c + 3d = 0 \tag{2}$$
$$a + b + 2c + 2d = 0 \tag{3}$$
$$a + b + c + 3d = 0 \tag{4}$$

Subtracting (1) from (4) we get:

$$3d = 0$$

Therefore, $d = 0$. Subtracting (2) from (4), we get:

$$b = 0$$

We now know that $b = 0$. Using (1),

$$\begin{aligned} a + b + c + d &= 0 \\ a &= -b - c - d \end{aligned}$$

Substituting this into (3), we get:

$$\begin{aligned} -b - c - d + +b + 2c + 2d &= 0 \\ c + d &= 0 \\ c + 0 &= 0 \\ c &= 0 \end{aligned}$$

Finally, we can use (2) to solve for a:

$$\begin{aligned} a + 2b + c + 3d &= 0 \\ a + 2 * 0 + 0 + 3 * 0 &= 0 \\ a &= 0 \end{aligned}$$

Thus, we have proven that the only solution to the linear system of equations is 0, proving this matrix's columns are linearly independent.

(2) (5 points) Prove that the column vectors of the matrix B_2 given by

$$B_2 = \begin{pmatrix} 1 & -2 & 2 & -2 \\ 0 & -3 & 2 & -3 \\ 3 & -5 & 5 & -4 \\ 3 & -4 & 4 & -4 \end{pmatrix}$$

are linearly independent.

$$B_2^{-1} = \begin{pmatrix} -2 & 0 & 0 & 1 \\ 4\frac{1}{2} & -1 & -1 & -\frac{1}{2} \\ 4\frac{1}{2} & -1 & 0 & -1\frac{1}{2} \\ -1\frac{1}{2} & 0 & 1 & -\frac{1}{2} \end{pmatrix}$$

$B_2^{-1}B_2 = I$
$I_{11} = 1(-2) + -2(4\frac{1}{2}) + 2(4\frac{1}{2}) + -2(-1\frac{1}{2}) = 1$
$I_{12} = 1(0) + -2(-1) + 2(-1) + -2(0) = 0$
$I_{13} = 1(0) + -2(-1) + 2(0) + -2(1) = 0$
$I_{14} = 1(1) + -2(-\frac{1}{2}) + 2(-1\frac{1}{2}) + -2(-\frac{1}{2}) = 0$
$I_{21} = 0(-2) + -3(4\frac{1}{2}) + 2(4\frac{1}{2}) + -3(-1\frac{1}{2}) = 0$
$I_{22} = 0(0) + -3(-1) + 2(-1) + -3(0) = 1$
$I_{23} = 0(0) + -3(-1) + 2(0) + -3(1) = 0$
$I_{24} = 0(1) + -3(-\frac{1}{2}) + 2(-1\frac{1}{2}) + -3(-\frac{1}{2}) = 0$
$I_{31} = 3(-2) + -5(4\frac{1}{2}) + 5(4\frac{1}{2}) + -4(-1\frac{1}{2}) = 0$
$I_{32} = 3(0) + -5(-1) + 5(-1) + -4(0) = 0$
$I_{33} = 3(0) + -5(-1) + 5(0) + -4(1) = 1$
$I_{34} = 3(1) + -5(-\frac{1}{2}) + 5(-1\frac{1}{2}) + -4(-\frac{1}{2}) = 0$
$I_{41} = 3(-2) + -4(4\frac{1}{2}) + 4(4\frac{1}{2}) + -4(-1\frac{1}{2}) = 0$
$I_{42} = 3(0) + -4(-1) + 4(-1) + -4(0) = 0$
$I_{43} = 3(0) + -4(-1) + 4(0) + -4(1) = 0$
$I_{44} = 3(1) + -4(-\frac{1}{2}) + 4(-1\frac{1}{2}) + -4(-\frac{1}{2}) = 1$

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

ALTERNATE SOLUTION

We can prove linear independence by solving a linear system of equations such that the only solution for a, b, c, d is 0.

$$a * \begin{pmatrix}1\\0\\3\\3\end{pmatrix} + b * \begin{pmatrix}-2\\-3\\-5\\-4\end{pmatrix} + c * \begin{pmatrix}2\\2\\5\\4\end{pmatrix} + d * \begin{pmatrix}-2\\-3\\-4\\-4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{align} a - 2b + 2c - 2d &= 0 \tag{1}\\ -3b + 2c - 3d &= 0 \tag{2}\\ 3a - 5b + 5c - 4d &= 0 \tag{3}\\ 3a - 4b + 4c - 4d &= 0 \tag{4} \end{align}$$

Using (1), we can solve for a as follows:

$$\begin{aligned} a - 2b + 2c - 2d = 0 \\ a = 2b - 2c + 2d \end{aligned}$$

We can substitute this into (3):

$$\begin{aligned} 3a - 5b + 5c - 4d &= 0 \\ 3 * (2b - 2c + 2d) - 5b + 5c - 4d &= 0 \\ 6b - 6c + 6d - 5b + 5c - 4d &= 0 \\ b - c + 2d &= 0 \\ b &= c - 2d \end{aligned}$$

Using (4), we can substitute a and b as follows:

$$\begin{aligned} 3a - 4b + 4c - 4d &= 0 \\ 3 * (2b - 2c + 2d) - 4 * (c - 2d) + 4c - 4d &= 0 \\ 3 * (2 * (c - 2d) - 2c + 2d) - 4c + 8d + 4c - 4d &= 0 \\ 3 * (2c - 4d - 2c + 2d) + 4d &= 0 \\ 6c - 12d - 6c + 6d + 4d &= 0 \\ -2d &= 0 \\ d &= 0 \end{aligned}$$

Plugging this into (2),

$$\begin{aligned}
-3b + 2c - 3d &= 0 \\
-3 * (c - 2d) + 2c - 3 * 0 &= 0 \\
-3c + 6d + 2c &= 0 \\
-c &= 0 \\
c &= 0
\end{aligned}$$

This implies that $b = c - 2d = 0 - 2 * 0 = 0$. This implies from the first solving for a: $a = 2b - 2c + 2d = 2 * 0 - 2 * 0 + 2 * 0 = 0$. Therefore we can say that the columns of B_2 are linearly independent.

(3) (10 points) Prove that the coordinates of the column vectors of the matrix B_2 over the basis consisting of the column vectors of A_2 are the columns of the matrix P_2 given by

$$P_2 = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -3 & 1 & -2 & 1 \\ 1 & -2 & 2 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

$B2_{11} = 2(1) + -3(1) + 1(1) + 1(1) = 1$
$B2_{21} = 2(1) + -3(2) + 1(1) + 1(3) = 0$
$B2_{31} = 2(1) + -3(1) + 1(2) + 1(2) = 3$
$B2_{41} = 2(1) + -3(1) + 1(1) + 1(3) = 3$
$B2_{12} = 0(1) + 1(1) + -2(1) + -1(1) = -2$
$B2_{22} = 0(1) + 1(2) + -2(1) + -1(3) = -3$
$B2_{32} = 0(1) + 1(1) + -2(2) + -1(2) = -5$
$B2_{42} = 0(1) + 1(1) + -2(1) + -1(3) = -4$
$B2_{13} = 1(1) + -2(1) + 2(1) + 1(1) = 2$
$B2_{23} = 1(1) + -2(2) + 2(1) + 1(3) = -2$
$B2_{33} = 1(1) + -2(1) + 2(2) + 1(2) = 5$
$B2_{43} = 1(1) + -2(1) + 2(1) + 1(3) = 4$
$B2_{14} = -1(1) + 1(1) + -1(1) + -1(1) = -2$
$B2_{24} = -1(1) + 1(2) + -1(1) + -1(3) = -3$
$B2_{34} = -1(1) + 1(1) + -1(2) + -1(2) = -4$
$B2_{44} = -1(1) + 1(1) + -1(1) + -1(3) = -4$
ALTERNATE SOLUTION

We need to solve:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{pmatrix} * \begin{pmatrix} 2 & 0 & 1 & -1 \\ -3 & 1 & -2 & 1 \\ 1 & -2 & 2 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 & -2 \\ 0 & -3 & 2 & -3 \\ 3 & -5 & 5 & -4 \\ 3 & -4 & 4 & -4 \end{pmatrix}$$

We can simplify this to be $a * p = c$, where the subscript represents, the row and then the column. We want to show that $b = c$.

$$c_{1,1} = 1*2 + 1*-3 + 1*1 + 1*1$$
$$c_{1,1} = 2 - 3 + 1 + 1$$
$$c_{1,1} = 1$$

$$c_{1,2} = 1*0 + 1*1 + 1*-2 + 1*-1$$
$$c_{1,2} = 0 + 1 - 2 - 1$$
$$c_{1,2} = -2$$

$$c_{1,3} = 1*1 + 1*-2 + 1*2 + 1*1$$
$$c_{1,3} = 1 - 2 + 2 + 1$$
$$c_{1,3} = 2$$

$$c_{1,4} = 1*-1 + 1*1 + 1*-1 + 1*-1$$
$$c_{1,4} = -1 + 1 - 1 - 1$$
$$c_{1,4} = -2$$

$$c_{2,1} = 1*2 + 2*-3 + 1*1 + 3*1$$
$$c_{2,1} = 2 - 6 + 1 + 3$$
$$c_{2,1} = 0$$

$$c_{2,2} = 1*0 + 2*1 + 1*-2 + 3*-1$$
$$c_{2,2} = 0 + 2 - 2 - 3$$
$$c_{2,2} = -3$$

$$c_{2,3} = 1 * 1 + 2 * -2 + 1 * 2 + 3 * 1$$
$$c_{2,3} = 1 - 4 + 2 + 3$$
$$c_{2,3} = 2$$

$$c_{2,4} = 1 * -1 + 2 * 1 + 1 * -1 + 3 * -1$$
$$c_{2,4} = -1 + 2 - 1 - 3$$
$$c_{2,4} = -3$$

$$c_{3,1} = 1 * 2 + 1 * -3 + 2 * 1 + 2 * 1$$
$$c_{3,1} = 2 - 3 + 2 + 2$$
$$c_{3,1} = 3$$

$$c_{3,2} = 1 * 0 + 1 * 1 + 2 * -2 + 2 * -1$$
$$c_{3,2} = 0 + 1 - 4 - 2$$
$$c_{3,2} = -5$$

$$c_{3,3} = 1 * 1 + 1 * -2 + 2 * 2 + 2 * 1$$
$$c_{3,3} = 1 - 2 + 4 + 2$$
$$c_{3,3} = 5$$

$$c_{3,4} = 1 * -1 + 1 * 1 + 2 * -1 + 2 * -1$$
$$c_{3,4} = -1 + 1 - 2 - 2$$
$$c_{3,4} = -4$$

$$
\begin{aligned}
c_{4,1} &= 1*2+1*-3+1*1+3*1\\
c_{4,1} &= 2-3+1+3\\
c_{4,1} &= 3
\end{aligned}
$$

$$
\begin{aligned}
c_{4,2} &= 1*0+1*1+1*-2+3*-1\\
c_{4,2} &= 0+1-2-3\\
c_{4,2} &= -4
\end{aligned}
$$

$$
\begin{aligned}
c_{4,3} &= 1*1+1*-2+1*2+3*1\\
c_{4,3} &= 1-2+2+3\\
c_{4,3} &= 4
\end{aligned}
$$

$$
\begin{aligned}
c_{4,4} &= 1*-1+1*1+1*-1+3*-1\\
c_{4,4} &= -1+1-1-3\\
c_{4,4} &= -4
\end{aligned}
$$

Thus we have proven that the coordinates of the column vectors of the matrix B_2 over the basis consisting of the column vectors of A_2 are the columns of the matrix P_2 since $b = c$.

Check that $A_2P_2 = B_2$.
Done.
Prove that

$$P_2^{-1} = \begin{pmatrix} -1 & -1 & -1 & 1\\ 2 & 1 & 1 & -2\\ 2 & 1 & 2 & -3\\ -1 & -1 & 0 & -1 \end{pmatrix}.$$

$P_2^{-1}P_2 = I$
$I_{11} = 2(-1) + 0(2) + 1(2) + -1(-1) = 1$

$I_{12} = 2(-1) + 0(1) + 1(1) + -1(-1) = 0$
$I_{13} = 2(-1) + 0(1) + 1(2) + -1(0) = 0$
$I_{14} = 2(1) + 0(-2) + 1(-3) + -1(-1) = 0$
$I_{21} = -3(-1) + 1(2) + -2(2) + 1(-1) = 0$
$I_{22} = -3(-1) + 1(1) + -2(1) + 1(-1) = 1$
$I_{23} = -3(-1) + 1(1) + -2(2) + 1(0) = 0$
$I_{24} = -3(1) + 1(-2) + -2(-3) + 1(-1) = 0$
$I_{31} = 1(-1) + -2(2) + 2(2) + -1(-1) = 0$
$I_{32} = 1(-1) + -2(1) + 2(1) + -1(-1) = 0$
$I_{33} = 1(-1) + -2(1) + 2(2) + -1(0) = 1$
$I_{34} = 1(1) + -2(-2) + 2(-3) + -1(-1) = 0$
$I_{41} = 1(-1) + -1(2) + 1(2) + -1(-1) = 0$
$I_{42} = 1(-1) + -1(1) + 1(1) + -1(-1) = 0$
$I_{43} = 1(-1) + -1(1) + 1(2) + -1(0) = 0$
$I_{44} = 1(1) + -1(-2) + 1(-3) + -1(-1) = 1$

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

ALTERNATE SOLUTION

To prove that this is the inverse of P_2, we can multiply the two matrices together and if they equal the identity matrix, we can say that this is indeed the inverse of P_2.

$$\begin{pmatrix} 2 & 0 & 1 & -1 \\ -3 & 1 & -2 & 1 \\ 1 & -2 & 2 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} * \begin{pmatrix} -1 & -1 & -1 & 1 \\ 2 & 1 & 1 & -2 \\ 2 & 1 & 2 & -3 \\ -1 & -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We will solve for the matrix, c as we did before and then will check to see if c is equal to I.

$$c_{1,1} = 2 * -1 + 0 * 2 + 1 * 2 + -1 * -1$$
$$c_{1,1} = -2 + 0 + 2 + 1$$
$$c_{1,1} = 1$$

$$c_{1,2} = 2 * -1 + 0 * 1 + 1 * 1 + -1 * -1$$
$$c_{1,2} = -2 + 0 + 1 + 1$$
$$c_{1,2} = 2$$

$$c_{1,3} = 2 * -1 + 0 * 1 + 1 * 2 + -1 * 0$$
$$c_{1,3} = -2 + 0 + 2 + 0$$
$$c_{1,3} = 0$$

$$c_{1,4} = 2 * 1 + 0 * -2 + 1 * -3 + -1 * -1$$
$$c_{1,4} = 2 + 0 - 3 + 1$$
$$c_{1,4} = 0$$

$$c_{2,1} = -3 * -1 + 1 * 2 + -2 * 2 + 1 * -1$$
$$c_{2,1} = 3 + 2 - 4 - 1$$
$$c_{2,1} = 0$$

$$c_{2,2} = -3 * -1 + 1 * 1 + -2 * 1 + 1 * -1$$
$$c_{2,2} = 3 + 1 - 2 - 1$$
$$c_{2,2} = 1$$

$$c_{2,3} = -3 * -1 + 1 * 1 + -2 * 2 + 1 * 0$$
$$c_{2,3} = 3 + 1 - 4 + 0$$
$$c_{2,3} = 0$$

$$c_{2,4} = -3 * 1 + 1 * -2 + -2 * -3 + 1 * -1$$
$$c_{2,4} = -3 - 2 + 6 - 1$$
$$c_{2,4} = 0$$

$$c_{3,1} = 1 * -1 + -2 * 2 + 2 * 2 + -1 * -1$$
$$c_{3,1} = -1 - 4 + 4 + 1$$
$$c_{3,1} = 0$$

$$c_{3,2} = 1 * -1 + -2 * 1 + 2 * 2 + -1 * -1$$
$$c_{3,2} = -1 - 2 + 4 + 1$$
$$c_{3,2} = 0$$

$$c_{3,3} = 1 * -1 + -2 * 1 + 2 * 2 + -1 * 0$$
$$c_{3,3} = -1 - 2 + 4 + 0$$
$$c_{3,3} = 1$$

$$c_{3,4} = 1 * 1 + -2 * -2 + 2 * -3 + -1 * -1$$
$$c_{3,4} = 1 + 4 - 6 + 1$$
$$c_{3,4} = 0$$

$$c_{4,1} = 1 * -1 + -1 * 2 + 1 * 2 + -1 * -1$$
$$c_{4,1} = -1 - 2 + 2 + 1$$
$$c_{4,1} = 0$$

$$\begin{aligned}
c_{4,1} &= 1 * -1 + -1 * 1 + 1 * 1 + -1 * -1 \\
c_{4,2} &= -1 - 1 + 1 + 1 \\
c_{4,2} &= 0
\end{aligned}$$

$$\begin{aligned}
c_{4,3} &= 1 * -1 + -1 * 1 + 1 * 2 + -1 * 0 \\
c_{4,3} &= -1 - 1 + 2 + 0 \\
c_{4,3} &= 0
\end{aligned}$$

$$\begin{aligned}
c_{4,4} &= 1 * 1 + -1 * -2 + 1 * -3 + -1 * -1 \\
c_{4,4} &= 1 + 2 - 3 + 1 \\
c_{4,4} &= 1
\end{aligned}$$

Therefore we can see that $c = I$ which is the identity matrix proving that P_2^{-1} is indeed the inverse of P_2.

What are the coordinates over the basis consisting of the column vectors of B_2 of the vector whose coordinates over the basis consisting of the column vectors of A_2 are $(2, -3, 0, 0)$?

$old = (2, -3, 0, 0)$
$new = A_2^{-1} \cdot old$
$new_1 = (2)(2) + (-1)(-3) + (-1)(0) + (-1)(0) = 7$
$new_2 = (0)(2) + (1)(-3) + (0)(0) + (-1)(0) = -3$
$new_2 = (-\frac{1}{2})(2) + (0)(-3) + (1)(0) + (-\frac{1}{2})(0) = -1$
$new_4 = (-\frac{1}{2})(2) + (0)(-3) + (0)(0) + (\frac{1}{2})(0) = -1$

$$A_2^{-1} \cdot old = \begin{pmatrix} 2 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ -1 \\ -1 \end{pmatrix} = new$$

ALTERNATE SOLUTION

To find these new coordinates, we can use the equation from the lectures where U represents the old basis (A) and V represents the new basis (B). We know the following equation:

$$x_U = P_{v,u} x_V$$

We are given X_U and want to find x_V. Solving for x_V, we get:

$$x_V = P_{V,U}^{-} 1 x_U$$

We can substitute the given vector, x_U, and the inverse matrix to get:

$$x_V = \begin{pmatrix} -1 & -1 & -1 & 1 \\ 2 & 1 & 1 & -2 \\ 2 & 1 & 2 & -3 \\ -1 & -1 & 0 & -1 \end{pmatrix} * \begin{pmatrix} 2 \\ -3 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} x_{V_1} &= -1 * 2 + -1 * -3 + -1 * 0 + -1 * 0 \\ x_{V_1} &= -2 + 3 + 0 + 0 \\ x_{V_1} &= 1 \end{aligned}$$

$$\begin{aligned} x_{V_2} &= 2 * 2 + 1 * -3 + 1 * 0 + -2 * 0 \\ x_{V_2} &= 4 - 3 + 0 + 0 \\ x_{V_2} &= 1 \end{aligned}$$

$$\begin{aligned} x_{V_3} &= 2 * 2 + 1 * -3 + 2 * 0 + -3 * 0 \\ x_{V_3} &= 1 \end{aligned}$$

$$\begin{aligned} x_{V_4} &= -1 * 2 + -1 * -3 + 0 * 0 + -1 * 0 \\ x_{V_4} &= 1 \end{aligned}$$

So the new coordinates are $(1, 1, 1, 1)$.

Problem 2: 30 points total

Consider the polynomials

$$B_0^2(t) = (1-t)^2 \qquad B_1^2(t) = 2(1-t)t \qquad B_2^2(t) = t^2$$
$$B_0^3(t) = (1-t)^3 \qquad B_1^3(t) = 3(1-t)^2 t \qquad B_2^3(t) = 3(1-t)t^2 \qquad B_3^3(t) = t^3,$$

known as the *Bernstein polynomials* of degree 2 and 3.

(1) (10 points) Show that the Bernstein polynomials $B_0^2(t), B_1^2(t), B_2^2(t)$ are expressed as linear combinations of the basis $(1, t, t^2)$ of the vector space of polynomials of degree at most 2 as follows:

$$\begin{pmatrix} B_0^2(t) \\ B_1^2(t) \\ B_2^2(t) \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}.$$

Multiplying this out, we get that the matrix represented by the Bernstein polynomials is as follows:

$$\begin{pmatrix} B_0^2(t) = 1*1 + -2*t + 1*t^2 \\ B_1^2(t) = 0*1 + 2*t + -2*t^2 \\ B_2^2(t) = 0*1 + 0*t + 1*t^2 \end{pmatrix} = \begin{pmatrix} B_0^2(t) = 1 + -2t + t^2 \\ B_1^2(t) = 0 + 2t + -2t^2 \\ B_2^2(t) = 0 + 0t + t^2 \end{pmatrix} = \begin{pmatrix} B_0^2(t) = (1-t)^2 \\ B_1^2(t) = 2(1-t)t \\ B_2^2(t) = t^2 \end{pmatrix}$$

Prove that

$$B_0^2(t) + B_1^2(t) + B_2^2(t) = 1.$$

Substituting the definitions of the Bernstein polynomials, we get:

$$\begin{aligned} B_0^2(t) + B_1^2(t) + B_2^2(t) &= 1 \\ (1-t)^2 + 2(1-t)t + t^2 &= 1 \\ t^2 - 2t + 1 + 2t - 2t^2 + t^2 &= 1 \\ 1 &= 1 \end{aligned}$$

Therefore we can show that the Bernstein polynomials of degree 2 combine to equal 1.

(2) (10 points) Show that the Bernstein polynomials $B_0^3(t), B_1^3(t), B_2^3(t), B_3^3(t)$ are expressed as linear combinations of the basis $(1, t, t^2, t^3)$ of the vector space of polynomials of degree at most 3 as follows:

$$\begin{pmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{pmatrix} = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}.$$

Multiplying this out, we can solve for the Bernstein polynomials of degree 3 and check to see if they are equal to the definition of the Bernstein polynomials.

$$\begin{pmatrix} B_0^3(t) = 1 * 1 + -3 * t + 3 * t^2 + -1 * t^3 \\ B_1^3(t) = 0 * 1 + 3 * t + -6 * t^2 + 3 \qquad t^3 \\ B_2^3(t) = 0 * 1 + 0 * t + 3 * t^2 + -3 * t^3 \\ B_3^3(t) = 0 * 1 + 0 * t + 0 * t^2 + 1 * t^3 \end{pmatrix} =$$

$$\begin{pmatrix} B_0^3(t) = 1 - 3t + 3t^2 - t^3 \\ B_1^3(t) = 0 + 3t - 6t^2 + 3t^3 \\ B_2^3(t) = 0 + 0 + 3t^2 - 3t^3 \\ B_3^3(t) = t^3 \end{pmatrix} =$$

$$\begin{pmatrix} B_0^3(t) = -t^3 + 3t^2 - 3t + 1 \\ B_1^3(t) = 3(1 - 2t + t^2)t \\ B_2^3(t) = 3(1 - t)t^2 \\ B_3^3(t) = t^3 \end{pmatrix} =$$

$$\begin{pmatrix} B_0^3(t) = (1 - t)^3 \\ B_1^3(t) = 3(1 - t)^2 t \\ B_2^3(t) = 3(1 - t)t^2 \\ B_3^3(t) = t^3 \end{pmatrix}$$

Prove that

$$B_0^3(t) + B_1^3(t) + B_2^3(t) + B_3^3(t) = 1.$$

Substituting the definition for the Bernstein polynomials of degree 3 we get:

$$\begin{aligned} B_0^3(t) + B_1^3(t) + B_2^3(t) + B_3^3(t) &= 1 \\ (1 - t)^3 + 3(1 - t)^2 t + 3(1 - t)t^2 + t^3 &= 1 \\ -t^3 + 3t^2 - 3t + 1 + 3t - 6t^2 + 3t^3 + 3t^2 - 3t^3 + t^3 &= 1 \\ 1 &= 1 \end{aligned}$$

(3) (10 points) Prove that the Bernstein polynomials of degree 2 are linearly independent, and that the Bernstein polynomials of degree 3 are linearly independent.

Bernstein Polynomials of degree 2 proof:

$$B2 = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B2^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$B2^{-1}B2 = I$
$I_{11} = 1(1) + -2(0) + 1(0) = 1$
$I_{12} = 1(1) + -2(\frac{1}{2}) + 1(0) = 0$
$I_{13} = 1(1) + -2(1) + 1(1) = 0$
$I_{21} = 0(1) + -2(0) + 1(0) = 0$
$I_{22} = 0(1) + 2(\frac{1}{2}) + -2(0) = 1$
$I_{23} = 0(1) + 2(1) + -2(1) = 0$
$I_{31} = 0(1) + 0(0) + 1(0) = 0$
$I_{32} = 0(1) + 0(\frac{1}{2}) + 1(0) = 0$
$I_{33} = 0(1) + 0(1) + 1(1) = 1$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Bernstein Polynomials of degree 3 proof:

$$B3 = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B3^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$B3^{-1}B3 = I$
$I_{11} = 1(1) + -3(0) + 3(0) + -1(0) = 1$
$I_{12} = 1(1) + -3(\frac{1}{3}) + 3(0) + -1(0) = 0$
$I_{13} = 1(1) + -3(\frac{2}{3}) + 3(\frac{1}{3}) + -1(0) = 0$
$I_{14} = 1(1) + -3(1) + 3(1) + -1(1) = 0$
$I_{21} = 0(1) + 3(0) + -6(0) + 3(0) = 0$
$I_{22} = 0(1) + 3(\frac{1}{3}) + -6(0) + 3(0) = 1$

$I_{23} = 0(1) + 3(\frac{2}{3}) + -6(\frac{1}{3}) + 3(0) = 0$
$I_{24} = 0(1) + 3(1) + -6(1) + 3(1) = 0$
$I_{31} = 0(1) + 0(0) + 3(0) + -3(0) = 0$
$I_{32} = 0(1) + 0(\frac{1}{3}) + 3(0) + -3(0) = 0$
$I_{33} = 0(1) + 0(\frac{2}{3}) + 3(\frac{1}{3}) + -3(0) = 1$
$I_{34} = 0(1) + 0(1) + 3(1) + -3(1) = 0$
$I_{41} = 0(1) + 0(0) + 0(0) + 1(0) = 0$
$I_{42} = 0(1) + 0(\frac{1}{3}1) + 0(0) + 1(0) = 0$
$I_{43} = 0(1) + 0(\frac{2}{3}) + 0(\frac{1}{3}) + 1(0) = 0$
$I_{44} = 0(1) + 0(1) + 0(1) + 1(1) = 1$

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 3: 10 points

Prove that for every vector space E, if $f\colon E \to E$ is an idempotent linear map, i.e., $f \circ f = f$, then we have a direct sum

$$E = \operatorname{Ker} f \oplus \operatorname{Im} f,$$

so that f is the projection onto its image $\operatorname{Im} f$.
$u \in E$
$u = f(u) + (u - f(u))$
$f(u) = \operatorname{Im} f$
$f(u - f(u)) = f(u) - f(f(u)) = f(u) - f(u) = 0 = \operatorname{Ker} f$
WTS $f(u) \cap f(u - f(u)) = 0$
since $0 \in f(u)$ and $f(u - f(u)) = 0 \implies f(u) \cap f(u - f(u)) = f(u) \cap 0 = 0$

Problem 4: 20 points plus 15 points Extra Credit

Given any vector space E, a linear map $f\colon E \to E$ is an *involution* if $f \circ f = \mathrm{id}$.

(1) (10 points) Prove that an involution f is invertible. What is its inverse?
f is invertible if $f \circ f^{-1} = \mathrm{id}$
$f \circ f = \mathrm{id} \implies f \circ f^{-1} = \mathrm{id} \implies f = f^{-1}$
so f is the inverse of f

(2) (10 points) Let E_1 and E_{-1} be the subspaces of E defined as follows:

$$E_1 = \{u \in E \mid f(u) = u\}$$
$$E_{-1} = \{u \in E \mid f(u) = -u\}.$$

Prove that we have a direct sum

$$E = E_1 \oplus E_{-1}.$$

Hint. For every $u \in E$, write

$$u = \frac{u + f(u)}{2} + \frac{u - f(u)}{2}.$$

$f(\frac{u+f(u)}{2}) = \frac{1}{2} \cdot (f(u) + f(f(u))) = \frac{1}{2} \cdot (f(u) + (u)) = \frac{1}{2} \cdot (u + u) = \frac{1}{2} \cdot (2u) = u = E_1$
$f(\frac{u-f(u)}{2}) = \frac{1}{2} \cdot (f(u) - f(f(u))) = \frac{1}{2} \cdot (f(u) - (u)) = \frac{1}{2} \cdot (-u - u) = \frac{1}{2} \cdot (-2u) - u = E_{-1}$

WTS $E_1 \cap E_{-1} = (0)$

since $u \in E$ and $u \in E_{-1}$
then $f(u) = u = f(u) = -u$
so $u = -u \implies u = 0 \implies E_1 \cap E_{-1} = (0)$

(3) **Extra credit** (15 points) If E is finite-dimensional and f is an involution, prove that there is some basis of E with respect to which the matrix of f is of the form

$$I_{k,n-k} = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix (similarly for I_{n-k}) and $k = \dim(E_1)$. Can you give a geometric interpretation of the action of f (especially when $k = n - 1$)?

Total: 70 points Extra Credit: 15 points