Module 1

Vector Spaces, Bases, Linear Maps

Problem 1: 10 points

Let A_4 be the following matrix:

$$A_4 = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \\ -1 & 0 & 1 & -1 \\ -2 & -1 & 4 & 0 \end{pmatrix}.$$

Prove that the columns of A_4 are linearly independent. Find the coordinates of the vector $x = (7, 14, -1, 2)$ over the basis consisting of the column vectors of A_4.

a(1,2,-1,-2)+b(2,3,0,-1)+c(1,2,1,4)+d(1,3,-1,0)=(0,0,0,0)
a(1,2,-1,-2)=0, iff a=0
b(2,3,0,-1)=0, iff b=0
c(1,2,1,4)=0 iff c=0
d(1,3,-1,0)=0 iff d=0
a(1,2,-1,-2)+b(2,3,0,-1)+c(1,2,1,4)+d(1,3,-1,0)=(7,14,-1,2)
a=0
b=2
c=1
d=2
(a,b,c,d)=(0,2,1,2)

Problem 2: 10 points

Consider the following Haar matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}.$$

Prove that the columns of H are linearly independent.

Hint. Compute the product $H^\top H$.

$$H^\top H = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix} = M$$

M is a diagonal matrix, so it is invertible (there exists a matrix M^{-1})
so $M^{-1} \cdot H^\top H = M \cdot M^{-1} = I_4$, the identity matrix
so $M^{-1} \cdot H^\top = H^{-1}$ (H is invertible)
so the columns of H are linearly independent

Problem 3: 10 points

Let $E = \mathbb{R} \times \mathbb{R}$, and define the addition operation

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \quad x_1, x_2, y_1, y_2 \in \mathbb{R},$$

and the multiplication operation $\cdot\colon \mathbb{R} \times E \to E$ by

$$\lambda \cdot (x, y) = (\lambda x, y), \quad \lambda, x, y \in \mathbb{R}.$$

Show that E with the above operations $+$ and $\cdot$ is not a vector space. Which of the axioms is violated?

associativity $u + (v + w) = (u + v) + w$ is violated because there is no w
axiom (V1) $\alpha(u + v) = (\alpha \cdot u) + (\alpha \cdot v)$ is violated because $\lambda \cdot (x, y) = (\lambda x, y) \neq (\lambda x, \lambda y)$

Problem 4: 15 points total

(1) (5 points) Let A be an $n \times n$ matrix. If A is invertible, prove that for any $x \in \mathbb{R}^n$, if $Ax = 0$, then $x = 0$.

$A^{-1}Ax = 0 \cdot A^{-1}$
so $x = 0 \cdot A^{-1} = 0$

(2) (10 points) Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix. Prove that $I_m - AB$ is invertible iff $I_n - BA$ is invertible.

Hint. If for all $x \in \mathbb{R}^n$, $Mx = 0$ implies that $x = 0$, then M is invertible.

$B(I_m - AB) = (I_n - BA)B$
$B^{-1}(I_m - AB)^{-1} = (I_n - BA)^{-1}B^{-1}$
$B \cdot B^{-1}(I_m - AB)^{-1} = B \cdot (I_n - BA)^{-1}B^{-1}$
$(I_m - AB)^{-1} = B \cdot (I_n - BA)^{-1}B^{-1}$

so an inverse exists if B and $(I_m - AB)$ are invertible
if $(I_n - BA)$ is not invertible, then $BAx \neq 0$
so $ABAx = Ax \neq 0$
so $(I_m - AB)$ is not invertible

Problem 5: 10 points

Let $f : E \to F$ be a linear map which is also a bijection (it is injective and surjective). Prove that the inverse function $f^{-1} : E \to F$ is linear.

$f^{-1}\lambda(x) = \lambda \cdot f^{-1}(x)$
$\lambda \cdot f^{-1}(x) = f^{-1}(f(\lambda \cdot f^{-1}(x)))$ because f is bijective
$f^{-1}(f(\lambda \cdot f^{-1}(x))) = f^{-1}\lambda(x)$
since λ is a scalar f^{-1} is linear
also, $f^{-1}(x+y) = f^{-1}(x) + f^{-1}(y)$
$f^{-1}(x) + f^{-1}(y) = f^{-1}(f(f^{-1}(x) + f^{-1}(y)))$
$f^{-1}(f(f^{-1}(x) + f^{-1}(y))) = f^{-1}(x+y)$

Total: 55 points