Computational ODEs: 4.3

- $y'' - 2y' + 5y = e^x \cos(2x)$ $\qquad\qquad$ $y_p = xe^x(a\cos(2x) + b\sin(2x))$

$m^2 - 2m + 5 = 0$

$(m-1)^2 = -4$

$m = 1 \pm 2i$

$y = e^x(c_1\cos(2x) + c_2\sin(2x))$

$y_p = xe^x(a\cos(2x) + b\sin(2x))$

$y_p' = xe^x \cdot \frac{d}{dx}[a\cos(2x) + b\sin(2x)] + \frac{d}{dx}[xe^x] \cdot (a\cos(2x) + b\sin(2x))$

$= xe^x(-2a\sin(2x) + 2b\cos(2x)) + (e^x + xe^x)(a\cos(2x) + b\sin(2x))$

$= e^x(a\cos(2x) + b\sin(2x)) + xe^x((b-2a)\sin(2x) + (a+2b)\cos(2x))$

$y_p'' = e^x \cdot \frac{d}{dx}[a\cos(2x) + b\sin(2x)] + \frac{d}{dx}[e^x](a\cos(2x) + b\sin(2x)) + xe^x \cdot \frac{d}{dx}[(b-2a)\sin(2x) + (a+2b)\cos(2x)]$
$+ \frac{d}{dx}[xe^x]((b-2a)\sin(2x) + (a+2b)\cos(2x))$

$= e^x(-2a\sin(2x) + 2b\cos(2x)) + e^x(a\cos(2x) + b\sin(2x)) + xe^x(2(b-2a)\cos(2x) - 2(a+2b)\sin(2x))$
$+ (e^x + xe^x)((b-2a)\sin(2x) + (a+2b)\cos(2x))$

$= e^x((-2a+b+b-2a)\sin(2x) + (2b+a+a+2b)\cos(2x)) + xe^x((2(b-2a) + (a+2b))\cos(2x) + (-2(a+2b) + (b-2a))\sin(2x))$

$= e^x((-4a+2b)\sin(2x) + (2a+4b)\cos(2x)) + xe^x((-3a+4b)\cos(2x) + (-4a-3b)\sin(2x))$

$e^x\sin(2x)$ coeffs: $\quad 0 = -4a + 2b - 2b + 5(0) = -4a \quad \rightarrow a = 0$

$e^x\cos(2x)$ coeffs: $\quad 1 = 2a + 4b - 2a + 5(0) = 4b \quad \rightarrow b = \frac{1}{4}$

$y_p = \frac{1}{4}xe^x\sin(2x)$

<u>check:</u> $y_p' = \frac{1}{4}(2xe^x\cos(2x) + (e^x + xe^x)\sin(2x)) = \frac{1}{2}xe^x\cos(2x) + \frac{1}{4}e^x\sin(2x) + \frac{1}{4}xe^x\sin(2x)$

$y_p'' = \frac{1}{2}((-2xe^x\sin(2x) + (e^x + xe^x)\cos(2x)) + \frac{1}{4}e^x\sin(2x) + \frac{1}{2}e^x\cos(2x) + \frac{1}{2}xe^x\cos(2x) + \frac{1}{4}(e^x + xe^x)\sin(2x))$

$= e^x\cos(2x) + \frac{1}{2}e^x\sin(2x) + xe^x\cos(2x) - \frac{3}{4}xe^x\sin(2x)$

$e^x\cos(2x) + \cancel{\frac{1}{2}e^x\sin(2x)} + \cancel{xe^x\cos(2x)} - \cancel{\frac{3}{4}xe^x\sin(2x)} - \cancel{xe^x\cos(2x)} - \cancel{\frac{1}{2}e^x\sin(2x)} - \cancel{\frac{1}{2}xe^x\sin(2x)} + \cancel{\frac{5}{4}xe^x\sin(2x)}$

$= e^x\cos(2x)$ ✓

$$\boxed{y = e^x(c_1\cos(2x) + c_2\sin(2x)) + \frac{1}{4}xe^x\sin(2x)}$$

• $y'' - 2y' + 5y = e^x\cos(2x) \quad \Rightarrow \quad y'' = 2y' - 5y + e^x\cos(2x)$

let $y_1 = y,\ y_2 = y'$ : $\quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ 2y_2 - 5y_1 + e^x\cos(2x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^x\cos(2x) \end{pmatrix}$

$y' = A\,y + F$

$0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -5 & 2-\lambda \end{vmatrix} = -\lambda(2-\lambda) + 5 = \lambda^2 - 2\lambda + 5 \qquad \lambda^2 - 2\lambda + 1 = -4$

$\lambda = 1 \pm 2i$

$\underline{\lambda = 1+2i}$ : $(A - \lambda I)k_1 = 0$

$\begin{pmatrix} -1-2i & 1 \\ -5 & 1-2i \end{pmatrix} \rightarrow \begin{pmatrix} 1+2i & -1 \\ 0 & 0 \end{pmatrix} \qquad k_1 = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix}$

$(-1-2i)(1-2i) = -1(1+2i)(1-2i) = -1(1+4) = -5$

$Y_{1c} = [B_1\cos(\beta t) - B_2\sin(\beta t)]e^{\alpha t} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(2x) - \begin{pmatrix} 0 \\ 2 \end{pmatrix}\sin(2x)\right]e^x$

$Y_{2c} = [B_2\cos(\beta t) + B_1\sin(\beta t)]e^{\alpha t} = \left[\begin{pmatrix} 0 \\ 2 \end{pmatrix}\cos(2x) + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin(2x)\right]e^x$

$B_1 = Re(k_1) \quad B_2 = Im(k_1)$

$Y_c = c_1Y_{1c} + c_2Y_{2c} = c_1\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(2x) - \begin{pmatrix} 0 \\ 2 \end{pmatrix}\sin(2x)\right]e^x + c_2\left[\begin{pmatrix} 0 \\ 2 \end{pmatrix}\cos(2x) + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin(2x)\right]e^x \Rightarrow Y_c = c_1\cos(2x)e^x + c_2\sin(2x)e^x$

$Y_p = xe^x(a\cos(2x) + b\sin(2x))$

$Y_p' = e^x(a\cos(2x) + b\sin(2x)) + xe^x((b-2a)\sin(2x) + (a+2b)\cos(2x))$

$Y_p'' = Y_2 = -5Y_p + 2Y_p' + e^x\cos(2x)$ ← this is exactly the particular sol we solted before.

$Y_p = \frac{1}{4}xe^x\sin(2x)$

$\boxed{Y = Y_c + Y_p = c_1\cos(2x)e^x + c_2\sin(2x)e^x + \frac{1}{4}xe^x\sin(2x)}$ ✓ this is the general sol we got earlier.