4.1

$$A = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix}, \quad \text{and } C = \begin{pmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{pmatrix}.$$

1. By hand, find the eigenvalues and eigenvectors for A, B, and C.

(1)

Matrix A eigs

$$p(\lambda) = \det(A - \lambda I_3) = 0$$

$$p(\lambda) = \det\left[\begin{pmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \det\begin{pmatrix} 1-\lambda & 1 & 4 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)\left[(2-\lambda)(1-\lambda) - (1)(0)\right] - 1\left[(0)(1-\lambda) - (0)(1)\right] + 4\left[(0)(1) - (2-\lambda)(1)\right]$$

$$= -\lambda^3 + 4\lambda^2 - \lambda - 6 = 0$$

$$\begin{array}{r|rrrr} 1 & -1 & 4 & -1 & -6 \\ & & -1 & -3 & \\ \hline & -1 & -3 & -4 & \end{array} \times \qquad \begin{array}{r|rrrr} 2 & -1 & 4 & -1 & -6 \\ & & -2 & 4 & 6 \\ \hline & -1 & 2 & 3 & 0 \end{array} \checkmark \rightarrow (\lambda-2)(-\lambda^2+2\lambda+3) = (\lambda-2)(\lambda-3)(\lambda+1)$$

$$\boxed{\lambda = 3, -1, 2}$$

$$A - \lambda I_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda_1 = 3$

$$\left(\begin{array}{ccc|c} 1-3 & 1 & 4 & 0 \\ 0 & 2-3 & 0 & 0 \\ 1 & 1 & 1-3 & 0 \end{array}\right) \xrightarrow{R_1+3R_3} \left(\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \xrightarrow{R_3-R_1} \left(\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{array}\right) \xrightarrow{-1R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{array}\right) \xrightarrow{R_3+3R_2} \left(\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \begin{array}{l} x+4y-2z=0 \rightarrow x=2z \\ y=0 \end{array}$$

$$\vec{K}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda_2 = -1$

$$\left(\begin{array}{ccc|c} 1-^-1 & 1 & 4 & 0 \\ 0 & 2-^-1 & 0 & 0 \\ 1 & 1 & 1-^-1 & 0 \end{array}\right) \xrightarrow{R_1-R_3} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right) \xrightarrow{R_3-R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{3}R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \begin{array}{l} x+2z=0 \\ y=0 \end{array} \rightarrow \vec{K}_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = 2$

$$\left(\begin{array}{ccc|c} 1-2 & 1 & 4 & 0 \\ 0 & 2-2 & 0 & 0 \\ 1 & 1 & 1-2 & 0 \end{array}\right) \xrightarrow{-1R_1} \left(\begin{array}{ccc|c} 1 & -1 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c} 1 & -1 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -4 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & -1 & -4 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \begin{array}{l} x - y - 4z = 0 \\ y + \frac{3}{2}z = 0 \end{array} \rightarrow \vec{K}_3 = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$$

Matrix B eigs

$$p(\lambda) = \det(A - \lambda I_3) = 0$$

$$p(\lambda) = \det\left[\begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \det\begin{pmatrix} -1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 3 & -1-\lambda \end{pmatrix}$$

$$= (-1-\lambda)\left[(2-\lambda)(-1-\lambda) - (1)(3)\right] - (1)\left[(1)(-1-\lambda) - (1)(0)\right] + 0$$

$$= -\lambda^3 + 7\lambda + 6 = 0$$

$$\begin{array}{r|rrrr} 1 & -1 & 0 & 7 & 6 \\ & & -1 & -1 & 6 \\ \hline & -1 & -1 & 6 & 12 \end{array} \times \qquad \begin{array}{r|rrrr} -1 & -1 & 0 & 7 & 6 \\ & & 1 & -1 & -6 \\ \hline & -1 & 1 & 6 & 0 \end{array} \checkmark \rightarrow (\lambda+1)(-\lambda^2+\lambda+6) = (\lambda+1)(\lambda-3)(\lambda+2) \rightarrow \boxed{\lambda = 3, -1, -2}$$

$$A - \lambda I_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda_1 = 3$

$$\left(\begin{array}{ccc|c} -1-3 & 1 & 0 & 0 \\ 1 & 2-3 & 1 & 0 \\ 0 & 3 & -1-3 & 0 \end{array}\right) \xrightarrow{R_1 + 5R_2} \left(\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right) \xrightarrow{R_2 + -1R_1} \left(\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right) \xrightarrow{\frac{1}{3}R_2}$$

$$\left(\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ 0 & 1 & \frac{-4}{3} & 0 \\ 0 & 3 & -4 & 0 \end{array}\right) \xrightarrow{R_3 + -3R_2} \left(\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \begin{array}{l} x - 4y + 5z = 0 \\ y - \frac{4}{3}z = 0 \rightarrow y = \frac{4}{3}z \end{array} \rightarrow \vec{K}_1 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

$\lambda_2 = -1$

$$\left(\begin{array}{ccc|c} -1-(-1) & 1 & 0 & 0 \\ 1 & 2-(-1) & 1 & 0 \\ 0 & 3 & -1-(-1) & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right) \xrightarrow{R_3 + -3R_2} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow y = 0$$

$$x + 3y + z = 0 \rightarrow z = -x \rightarrow \vec{K}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda_3 = -2$

$$\left(\begin{array}{ccc|c} -1-(-2) & 1 & 0 & 0 \\ 1 & 2-(-2) & 1 & 0 \\ 0 & 3 & -1-(-2) & 0 \end{array}\right) \xrightarrow{R_2 + -1R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right) \xrightarrow{\frac{1}{3}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 3 & 1 & 0 \end{array}\right) \xrightarrow{R_3 + -3R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \begin{array}{l} x + y = 0 \\ y + \frac{1}{3}z = 0 \end{array} \to \vec{K}_3 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

Matrix C eigs

$p(\lambda) = \det(A - \lambda I_3) = 0$

$$p(\lambda) = \det\left[\begin{pmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \det\begin{pmatrix} -1-\lambda & 4 & 2 \\ 4 & -1-\lambda & -2 \\ 0 & 0 & 6-\lambda \end{pmatrix}$$

$$= (-1-\lambda)\left[(-1-\lambda)(6-\lambda) - (-2)(0)\right] - (4)\left[(4)(6-\lambda) - (-2)(0)\right] + (2)\left[(4)(0) - (-1-\lambda)(0)\right]$$

$$= -\lambda^3 + 4\lambda^2 + 27\lambda - 90$$

$$\begin{array}{r|rrrr} 1 & -1 & 4 & 27 & -90 \\ & & -1 & 3 & \\ \hline & -1 & 3 & & \times \end{array} \qquad \begin{array}{r|rrrr} 3 & -1 & 4 & 27 & -90 \\ & & -3 & 3 & 90 \\ \hline & -1 & 1 & 30 & 0 \ \checkmark \end{array}$$

$$\to (\lambda-3)(-\lambda^2+\lambda+30) = (\lambda-3)(\lambda+5)(\lambda-6) = 0$$

$$\boxed{\lambda = 3, -5, 6}$$

$A - \lambda I_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_1 = 3$

$$\left(\begin{array}{ccc|c} -1-3 & 4 & 2 & 0 \\ 4 & -1-3 & -2 & 0 \\ 0 & 0 & 6-3 & 0 \end{array}\right) \xrightarrow{-\frac{1}{4}R_1} \left(\begin{array}{ccc|c} 1 & -1 & -\frac{1}{2} & 0 \\ 4 & -4 & -2 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right) \xrightarrow{R_2 + -4R_1} \left(\begin{array}{ccc|c} 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right) \to \begin{array}{l} x - y - \frac{1}{2}z = 0 \\ z = 0 \\ x = y \end{array}$$

$$\vec{K}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda_2 = -5$

$$\left(\begin{array}{ccc|c} -1-(-5) & 4 & 2 & 0 \\ 4 & -1-(-5) & -2 & 0 \\ 0 & 0 & 6-(-5) & 0 \end{array}\right) \xrightarrow{\frac{1}{4}R_1} \left(\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & 0 \\ 4 & 4 & -2 & 0 \\ 0 & 0 & 11 & 0 \end{array}\right) \to \begin{array}{l} x + y + \frac{1}{2}z = 0 \\ 4x + 4y - 2z = 0 \\ 11z = 0 \end{array} \to y = -x \to \vec{K}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda_3 = 6$

$$\left(\begin{array}{ccc|c} -1-6 & 4 & 2 & 0 \\ 4 & -1-6 & -2 & 0 \\ 0 & 0 & 6-6 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} -7 & 4 & 2 & 0 \\ 4 & -7 & -2 & 0 \end{array}\right) \xrightarrow{R_1 + R_2} \left(\begin{array}{ccc|c} -3 & -3 & 0 & 0 \\ 4 & -7 & -2 & 0 \end{array}\right) \to \begin{array}{l} -3x - 3y = 0 \to x = -y \\ 4x - 7y - 2z = 0 \to 11x = 2z \end{array}$$

$$\vec{K}_3 = \begin{pmatrix} 2 \\ -2 \\ 11 \end{pmatrix}$$

Note:

- computer outputs unit vectors, so they will differ from my eigenvectors by a scalar value equal to the magnitude of the eigenvector.
- computer values may also differ by the scalar value of -1.

☆ this difference ultimately doesnt matter since only the direction of eigenvectors are important.

ex: A eig-vectors: $\vec{K}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ $\vec{K}_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ $\vec{K}_3 = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$

$|\vec{K}_1| = \sqrt{5} \longrightarrow \vec{K}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \sqrt{5}\begin{pmatrix} 0.894 \\ 0 \\ 0.447 \end{pmatrix}$

$|\vec{K}_2| = \sqrt{5} \longrightarrow \vec{K}_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \sqrt{5}\begin{pmatrix} -0.894 \\ 0 \\ 0.447 \end{pmatrix}$

$|\vec{K}_3| = \sqrt{38} \longrightarrow \vec{K}_3 = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} = (-1)\sqrt{38}\begin{pmatrix} -0.811 \\ 0.486 \\ -0.324 \end{pmatrix}$

- For questions 3 and 4, I used my eigenvectors for simplicity

3. For the previous matrices, create a matrix V whose columns are the eigenvectors for these matrices, and a matrix D whose diagonal is the eigenvalues. That is, for eigenvectors v_1, v_2, and v_3 and eigenvalues λ_1, λ_2, and λ_3, let

$$V = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \quad \text{and } D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

4. Now, numerically compute V^{-1} for A, B, and C. Calculate the product VDV^{-1} for each of these matrices. What do you notice about this product?

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \rightarrow A^{-1} = \frac{1}{\det(A)} \left(C_{ij}\right)^T$$

↳ cofactor of entry x_{ij}

$C_{ij} = (-1)^{i+j} M_{ij}$

↳ det of matrix $(n-1)\times(n-1)$ obtained by "deleting" the ith row and jth column from A

$$A^{-1} = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

3-4

V_A^{-1}, D_A, VDV^{-1}

$$V = \begin{pmatrix} 2 & -2 & 5 \\ 0 & 0 & -3 \\ 1 & 1 & 2 \end{pmatrix} \rightarrow V^{-1} = \frac{1}{12}\begin{pmatrix} -3 & 9 & 6 \\ 3 & -1 & 6 \\ 0 & -4 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$V^{-1}DV = \frac{1}{12}\begin{pmatrix} 2 & -2 & 5 \\ 0 & 0 & -3 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} -3 & 9 & 6 \\ 3 & -1 & 6 \\ 0 & -4 & 0 \end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix} 6 & 2 & 10 \\ 0 & 0 & -6 \\ 3 & -1 & 4 \end{pmatrix}\begin{pmatrix} -3 & 9 & 6 \\ 3 & -1 & 6 \\ 0 & -4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 4 \\ 4 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \text{this matrix equals } A$$

V_B^{-1}, D_B, VDV^{-1}

$$V = \begin{pmatrix} 1 & -1 & 1 \\ 4 & 0 & -1 \\ 3 & 1 & 3 \end{pmatrix} \rightarrow V^{-1} = \frac{1}{20}\begin{pmatrix} 1 & 4 & 1 \\ -15 & 0 & -5 \\ 3 & 1 & 3 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$V^{-1}DV = \frac{1}{20}\begin{pmatrix} 1 & -1 & 1 \\ 4 & 0 & -1 \\ 3 & 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 4 & 1 \\ -15 & 0 & -5 \\ 3 & 1 & 3 \end{pmatrix}$$

$$= \frac{1}{20}\begin{pmatrix} 3 & 1 & -2 \\ 12 & 0 & 2 \\ 9 & -1 & -6 \end{pmatrix}\begin{pmatrix} 1 & 4 & 1 \\ -15 & 0 & -5 \\ 3 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix} \rightarrow \text{this matrix equals } B$$

V_c^{-1}, D_c, VDV^{-1}

$$V = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & -2 \\ 0 & 0 & 11 \end{pmatrix} \longrightarrow V^{-1} = \frac{1}{22}\begin{pmatrix} 11 & 11 & 0 \\ -11 & 11 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$VDV^{-1} = \frac{1}{22}\begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & -2 \\ 0 & 0 & 11 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} 11 & 11 & 0 \\ -11 & 11 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{22}\begin{pmatrix} 3 & 5 & 12 \\ 3 & -5 & -12 \\ 0 & 0 & 66 \end{pmatrix}\begin{pmatrix} 11 & 11 & 0 \\ -11 & 11 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 0 & 0 & 6 \end{pmatrix} \longrightarrow \text{this matrix equals } C$$