

Name: _____

Score: _____ /15

Linear Algebra Worksheet

1. Given the following matrices, evaluate the given expression. If it is not possible or does not make sense, explain why not.

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 5 \\ -2 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -3 \\ 2 & -3 & 0 \end{pmatrix}, E = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 3 & 5 \\ 1 & -1 & 0 \end{pmatrix}$$

- (a) $2A + 3B$ (b) $C + D$ (c) $\det(A)$ (d) $\det(CD)$ (e) DC
 (f) $\text{Tr}(E)$ (g) CE (h) CB (i) $ED + A$

a) $2A + 3B$

$$2 \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} + 3 \begin{pmatrix} 1 & 5 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 15 \\ -6 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 19 \\ -8 & 2 \end{pmatrix}$$

b) $C + D$

this operation isn't possible
since C and D have different dimensions.

c) $\det(A)$

$$\det(A) = (3)(1) - (2)(-1) = 4$$

d) $\det(CD)$

$$CD = \begin{pmatrix} 0 \cdot 1 + 1 \cdot -1 + -2 \cdot -2 & 0 \cdot 1 + 1 \cdot 2 + -2 \cdot -3 \\ -1 \cdot 1 + 0 \cdot -1 + 1 \cdot 2 & -1 \cdot 1 + 0 \cdot 2 + 1 \cdot -3 \end{pmatrix} = \begin{pmatrix} -5 & -5 \\ 1 & -4 \end{pmatrix}$$

$$\det(CD) = (-5)(-4) - (-5)(1) = 25$$

e) DC

$$DC = \begin{pmatrix} (1 \cdot 0) + (1 \cdot -1) & (1 \cdot 1) + (1 \cdot 0) & (1 \cdot -2) + (1 \cdot 1) \\ (1 \cdot 0) + (2 \cdot -1) & (-1 \cdot 1) + (2 \cdot 0) & (-1 \cdot -2) + (2 \cdot 1) \\ (0 \cdot 0) + (2 \cdot -1) & (2 \cdot 1) + (-3 \cdot 0) & (2 \cdot -2) + (-3 \cdot 1) \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ -2 & -1 & 4 \\ 4 & 2 & -7 \end{pmatrix}$$

f) $\text{Tr}(E)$

$$\text{Tr}(E) = 4 + 3 + 0 = 7$$

g) CE

$$CE = \begin{pmatrix} (0 \cdot 4) + (1 \cdot -2) + (-2 \cdot 1) & (0 \cdot 0) + (1 \cdot 3) + (-2 \cdot -1) & (0 \cdot 1) + (1 \cdot 5) + (-2 \cdot 0) \\ (-1 \cdot 4) + (0 \cdot -2) + (1 \cdot 1) & (-1 \cdot 0) + (0 \cdot 3) + (1 \cdot -1) & (-1 \cdot 1) + (0 \cdot 5) + (1 \cdot 0) \end{pmatrix} = \begin{pmatrix} -4 & 5 & 5 \\ -3 & -1 & -1 \end{pmatrix}$$

h) CB

this operation is not possible
of columns in C \neq # of rows of B
(3 \neq 2)

i) $ED + A$

this operation isn't possible
the dimensions of ED \neq the dimensions of A (3x2 \neq 2x2)

2. Find the solution to the following system by row-reducing an augmented matrix. Be sure to check your solution.

$$\begin{array}{rcl} 4x & + & 3y = 5 \\ 3x & - & 6y = 12 \end{array}$$

$$\left(\begin{array}{cc|c} 4 & 3 & 5 \\ 3 & -6 & 12 \end{array} \right) \xrightarrow{R1 \leftrightarrow R2} \left(\begin{array}{cc|c} 3 & -6 & 12 \\ 4 & 3 & 5 \end{array} \right)$$

$$\xrightarrow{\frac{1}{3}R1} \left(\begin{array}{cc|c} 1 & -2 & 4 \\ 4 & 3 & 5 \end{array} \right) \xrightarrow{-4R1+R2} \left(\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 11 & -11 \end{array} \right)$$

$$\xrightarrow{\frac{1}{11}R2} \left(\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 1 & -1 \end{array} \right) \xrightarrow{2R2+R1} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array} \right)$$

$$x = 2$$

$$y = -1$$

3. Find the solution to the following system by row-reducing an augmented matrix. Be sure to check your solution.

$$\begin{array}{rcl} x - 3y + 5z & = & -14 \\ 2x + y - 6z & = & 20 \\ 3x - 2y + z & = & 0 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 5 & -14 \\ 2 & 1 & -6 & 20 \\ 3 & -2 & 1 & 0 \end{array} \right) \xrightarrow{\begin{array}{l} -2R1+R2 \\ -3R1+R3 \end{array}} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -14 \\ 0 & 7 & -16 & 48 \\ 0 & 7 & -14 & 42 \end{array} \right)$$

$$\xrightarrow{\frac{1}{7}R2} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -14 \\ 0 & 1 & -\frac{16}{7} & \frac{48}{7} \\ 0 & 7 & 14 & 42 \end{array} \right) \xrightarrow{-7R2+R3} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -14 \\ 0 & 1 & -\frac{16}{7} & \frac{48}{7} \\ 0 & 0 & 2 & -6 \end{array} \right)$$

$$\xrightarrow{\frac{1}{2}R3} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -14 \\ 0 & 1 & -\frac{16}{7} & \frac{48}{7} \\ 0 & 0 & 1 & -3 \end{array} \right) \xrightarrow{\begin{array}{l} \frac{16}{7}R3+R2 \\ -5R3+R1 \end{array}} \left(\begin{array}{ccc|c} 1 & -3 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{array} \right)$$

$$x - 3y = 1 \rightarrow x - 3(0) = 1 \rightarrow x = -1$$

$$y = 0$$

$$z = -3$$

$$\begin{array}{l} x = -1 \\ y = 0 \\ z = 3 \end{array}$$

4. Is $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ an eigenvector for the matrix $A = \begin{pmatrix} 5 & -1 \\ 2 & 4 \end{pmatrix}$? If it is, determine the eigenvalue to which it corresponds; if not, explain why not.

$$A\vec{k} = \lambda\vec{k}$$

$$\begin{pmatrix} 5 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} (5)(3) + (-1)(-2) \\ (2)(3) + (4)(-2) \end{pmatrix} = \begin{pmatrix} 15 + 2 \\ 6 + -8 \end{pmatrix} = \begin{pmatrix} 17 \\ -2 \end{pmatrix}$$

The vector $\langle 17, -2 \rangle$ is not a scalar multiple of $\langle 3, -2 \rangle$, so therefore $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ is not an eigenvector of A .

5. Find all eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$.

$$\begin{aligned} P(\lambda) &= \lambda^2 - (a+d) + (ad-bc) & \lambda_1 = 3 \\ p(\lambda) &= \lambda^2 - (1+0)\lambda + [0(0) - (3)(2)] & \left[\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \vec{k} = \vec{0} \rightarrow \left[\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \right] \vec{k} = \vec{0} \rightarrow \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} \vec{k} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ p(\lambda) &= \lambda^2 - \lambda - 6 & \xrightarrow{-\frac{1}{2}R1} \begin{pmatrix} 1 & -\frac{3}{2} & 0 \\ 2 & -3 & 0 \end{pmatrix} \xrightarrow{-2R1+R2} \begin{pmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{2R1} \begin{pmatrix} 2 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow 2x - 3y = 0 \rightarrow 2x = 3y \rightarrow \vec{k} = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ O &= (\lambda-3)(\lambda+2) & \lambda_2 = -2 \\ \lambda_1 = 3 & \lambda_2 = -2 & \left[\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} - (-2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \vec{k} = \vec{0} \rightarrow \left[\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right] \vec{k} = \vec{0} \rightarrow \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \vec{k} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ & \xrightarrow{\frac{1}{3}R1} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \end{pmatrix} \xrightarrow{-2R1+R2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow x+y=0 \rightarrow y=-x \rightarrow \vec{k} = c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

6. Find all eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}$. (Be careful, this one's tricky!)

$$P(\lambda) = \lambda^2 - (2+2)\lambda + [(2)(2) - (0)(4)]$$

$$\lambda_1 = 2$$

$$P(\lambda) = \lambda^2 - 4\lambda + 4$$

$$0 = (\lambda - 2)^2$$

$$\lambda_1 = 2$$

$$\left[\begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix} - (2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \vec{K} = \vec{0} \rightarrow \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix} \vec{K} = \vec{0} \rightarrow \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix} \left| \begin{array}{l} 0 \\ 0 \end{array} \right.$$

$$\xrightarrow{R1 \leftrightarrow R2} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{4}R1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow x=0 \rightarrow \vec{K} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

y can be
ANY NUMBER!
The only condition
is $x=0$



7. Find all eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$. (This one's trickier!)

$$P(\lambda) = \lambda^2 - (1+1)\lambda + [(1)(1) - (-2)(2)]$$

$$\lambda_1 = 1+2i$$

$$\left[\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} - (1+2i) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \vec{K} = \vec{0}$$

$$P(\lambda) = \lambda^2 - 2\lambda + 5$$

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}$$

$$\lambda = \frac{2 \pm \sqrt{-16}}{2}$$

$$\lambda = \frac{2 \pm 4i}{2}$$

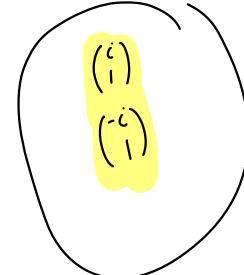
$$\lambda_1 = 1+2i \quad \lambda_2 = 1-2i$$

$$\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \vec{K} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \left| \begin{array}{l} 0 \\ 0 \end{array} \right.$$

$$\xrightarrow{R1 \leftrightarrow R2} \begin{pmatrix} 2 & -2i & 0 \\ -2i & -2 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2}R1} \begin{pmatrix} 1 & -i & 0 \\ -2i & -2 & 0 \end{pmatrix} \xrightarrow{2iR1 + R2} \begin{pmatrix} 1 & -i & 0 \\ 0 & -2-i & 0 \end{pmatrix}$$

$$\xrightarrow{(1+2i)R2} \begin{pmatrix} 1 & -i & 0 \\ 0 & 5 & 0 \end{pmatrix} \xrightarrow{\frac{1}{5}R2} \begin{pmatrix} 1 & -i & 0 \\ 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{array}{l} x - iy = 0 \\ y = 0 \end{array}$$

$$\xrightarrow{iR2 + R1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{array}{l} x = 0 \\ y = 0 \end{array}$$



- matrix: an array of real numbers with dimensions $m \times n$
 (m rows and n columns)
 - also written as $A = (a_{ij})$; a_{ij} is the number in the i th row and j th column.
 - a vector with n components is an $n \times 1$ matrix
 - if $m=n$, the matrix is a square of $n \times n$ dimensions.
 Entries from the top left to the bottom right are its diagonal (or diagonal entries) with the opposite being its cross diagonal.
 - a zero matrix (0) has only zeroes as components.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

2 × 2
2 rows 2 columns

→ diagonal (entries)
→ cross diagonal

$$\vec{v} = (a \ b \ c \ \dots) \quad \vec{v} \begin{pmatrix} a \\ b \\ \vdots \end{pmatrix}$$

$n \times 1$ vector

* two matrices are only equal if their entries are equal.

Square Matrix

- to add matrices (with the same dimensions), add like components. If $A = (a_{ij})$, $B = (b_{ij}) \rightarrow A + B = (a_{ij} + b_{ij})$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \rightarrow A + B \rightarrow \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

- $A + O = O + A = A$
- multiplying a scalar to a matrix x means multiplying all of the components by the scalar. If $A = (a_{ij})$, $K = \text{scalar} \rightarrow K \cdot A = K(a_{ij}) = (Ka_{ij})$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow K \cdot A = K \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} Ka & Kb \\ Kc & Kd \end{pmatrix}$$

$K = \text{scalar}$

- $-I(A) = -A \rightarrow A + (-A) = -A + A = O$

- to subtract matrices (with the same dimensions), subtract like components. If $A = (a_{ij})$, $B = (b_{ij}) \rightarrow A - B = (a_{ij} - b_{ij})$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow A - B \rightarrow \begin{pmatrix} a-e & b-f \\ c-g & d-h \end{pmatrix}$$

$$B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

- matrices can be multiplied if the number of columns in one matrix equals the number of rows of another matrix.

if A is $m \times n$ and B is $n \times r$,

AB has dimensions $M \times r$

- analytically, each row of one matrix is being multiplied with each column of another to create a new matrix whose entries are the dot product of each row \times column.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

row 1 ($i=1$)
row 2 ($i=2$)

column 1 ($j=1$) column 2 ($j=2$)

$$A \cdot B = \begin{pmatrix} a_{1j} \cdot b_{i1} & a_{1j} \cdot b_{i2} \\ a_{2j} \cdot b_{i1} & a_{2j} \cdot b_{i2} \end{pmatrix}$$

$$B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 6 \end{pmatrix}$; find AB

$$AB = \begin{pmatrix} 1 \cdot 0 + 2 \cdot -1 & 1 \cdot 1 + 2 \cdot 6 \\ 3 \cdot 0 + 4 \cdot -1 & 3 \cdot 1 + 4 \cdot 6 \end{pmatrix} = \begin{pmatrix} -2 & 13 \\ -4 & 27 \end{pmatrix}$$

- matrix multiplication is not commutative, $AB \neq BA$ (unless the components happen to make it work).

- identity matrix: a square matrix whose entries on the diagonal are all 1 with the rest being 0; denoted with I or I_n .

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- for any square matrix, $IA = AI = A$

- trace: the sum of all entries on the diagonal of a square matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \text{Tr}(A) = a + d$$

• determinant: for a 2×2 matrix is the difference between the products of the diagonal entries and cross diagonal entries.

- for larger square matrices, use a calculator.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \det(A) = ad - bc$$

• systems of linear equations can be written using matrices in two ways:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$\textcircled{1} \quad \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

* if you multiply the two LHS matrices, you will get the RHS matrix.

$$\textcircled{2} \quad \left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{array} \right)$$

* in either case, rows represent each linear equation

- ② is called an augmented matrix, the line helps delineate between coefficients on our variables (each column is for each variable) and the constants each equation is equal to.

- you're finding the intersection of the lines.
- when solving systems, we could use the elimination method: we're allowed to scale equations with scalars, replace equations by adding a multiple of another to it, or swap the order of equations; ultimately, we want to add the equations so that variables get eliminated, making the system easier to solve.

Ex: $x+y=3$, $2x-y=9$

$$\begin{array}{r}
 x + y = 3 \\
 + 2x - y = 9 \\
 \hline
 ? \quad \quad \quad
 \end{array}
 \rightarrow
 \begin{array}{r}
 x + y = 3 \\
 + 2x - y = 9 \\
 \hline
 3x + 0 = 12
 \end{array}
 \begin{array}{l}
 y - y = 0 \quad (\text{y variable eliminated!}) \\
 \uparrow
 \end{array}
 \begin{array}{l}
 3x = 12 \rightarrow x = 4 \\
 2(4) - y = 9 \rightarrow y = -1
 \end{array}$$

solution: $x = 4$, $y = -1$

- we can write the solution to a system of linear equations as an augmented matrix; this is said to be the reduced row echelon form of the original system's augmented matrix.

★ criteria to be in RREF:

- All zero rows (if any) are at the bottom
- The first nonzero entry of a row is to the right of the first nonzero entry in the row above
- Below the first nonzero entry of a row, all entries are zero.

Ex:

$$\left(\begin{array}{ccc|c}
 * & \dots & \dots & 3 \\
 0 & * & \dots & \\
 0 & 0 & 0 & * \\
 0 & 0 & 0 & 0 \\
 \end{array} \right)$$

* = any #
 ↳ are also called pivots since they are the 1st nonzero number in the row; the final equation and the original equation in that row geometrically "pivot" around the solution point described by the row in RREF.

$$\left(\begin{array}{ccc|c}
 1 & 1 & 3 \\
 0 & 1 & -1 \\
 \end{array} \right) = \left(\begin{array}{ccc|c}
 1 & 0 & 4 \\
 0 & 1 & -1 \\
 \end{array} \right)$$

★ notice how the reduce row echelon form of the original system is an identity matrix.

$$\begin{array}{l}
 1x + 0y = 4 \rightarrow x = 4 \\
 0x + 1y = -1 \rightarrow y = -1
 \end{array}$$

- row reduction: applying the elimination method of solving systems of linear equations to matrices that represent them.
 - the goal is to obtain a reduced row echelon form of a system of linear equations, which simply means make the LHS an identity matrix.
 - there are 3 operations we can use to make the LHS of a augmented matrix an identity matrix:

1. scale any row by a nonzero number
2. swap any two rows
3. add any two rows together, replacing one with their sum.

- the general process of making the identity matrix:
 1. make the 1st diagonal entry 1 using a valid operation.
 2. make the entries underneath that 1 zeroes, specifically by adding the correct multiple of the first row to the rows below.
 3. make the 2nd diagonal entry 1 using a valid operation.
 4. repeat step 2 but use row 2 for the operations.
 5. repeat 1-4 until you make the bottom right entry 1
 6. repeat step 2 but for all zeroes above each diagonal entry.

the process terminates when the identity matrix has been formed or when we can no longer zero out entries.

* before beginning row reduction, try to swap rows so that rows with 1s on the diagonal or zeroes below them are first.

$$Ex: x+y=3, 2x-y=9$$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 9 \end{array} \right)$$

① make the 1st diagonal entry 1 \rightarrow $\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 9 \end{array} \right)$

② zero out entries below using R1

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 9 \end{array} \right) \xrightarrow{-2 \cdot R1 + R2} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ -2(1)+2 & -2(1)+-1 & -2(3)+9 \end{array} \right)$$

$$= \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -3 & 3 \end{array} \right)$$

③ make the 2nd diagonal entry 1

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -3 & 3 \end{array} \right) \xrightarrow{-\frac{1}{3}R2} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ -\frac{1}{3}(0) & -\frac{1}{3}(-3) & -\frac{1}{3}(3) \end{array} \right) = \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -1 \end{array} \right)$$

④ zero out the entry above the 2nd diagonal using R2

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -1 \end{array} \right) \xrightarrow{-R2+R1} \left(\begin{array}{cc|c} -(0)+1 & -(1)+1 & -(-1)+3 \\ 0 & 1 & -1 \end{array} \right) = \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \end{array} \right)$$

identity created

reduced row echelon form: $\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \end{array} \right)$

- there are 3 possible outputs from the row reduction method:
 - ① a proper identity matrix (like above)
 - ② an inconsistent system that has no solution (the lines never intersect because they are parallel).

$$\left(\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 0 & -2 \end{array} \right) \rightarrow \begin{array}{l} 1x + 3y = 7 \quad \checkmark \\ 0x + 0y = -2 \rightarrow 0 \neq -2 \quad \times \end{array}$$

- ③ rows that are completely zeroed out and thus have no info

$$\left(\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 0 & 0 \end{array} \right) \rightarrow \begin{array}{l} x + 3y = 7 \\ 0 = 0 \end{array}$$

* for 2x2 matrices,
you stop the row
reduction method here
if this occurs.

Eigenvalues and Eigenvectors

Major question: given a square matrix A , is it possible to find a scalar λ such that:

$$A\vec{K} = \lambda\vec{K} \rightarrow A = \lambda \text{ (for } \vec{K})$$

↗ linear transformation ↗ nonzero vector

Answer: it is always possible to find λ for any square matrix A , however, only certain vectors \vec{K} can work.

- **eigenvalue:** a scalar λ associated with a square matrix A and vectors that satisfy $A\vec{K} = \lambda\vec{K}$
- any $n \times n$ matrix could have $1 \leq \lambda \leq n$ distinct eigenvalues.
 \star can be imaginary!
- **eigenvector:** a vector that has its direction unchanged or reversed by a given linear transformation applied to it; the linear transformation can thus be represented using a scalar value.
- there are an infinite number of eigenvectors for every eigenvalue Since any constant multiple of a known eigenvector is also an eigenvector of that linear transformation.
- **characteristic polynomial:** a polynomial for a specific square matrix whose roots are the eigenvalues of the matrix; denoted with $p(\lambda)$

①

$$p(\lambda) = \det(A - \lambda I_n)$$

★ the inputs are eigenvalues
the outputs are determinants
of $A - \lambda I$

$n =$ dimensions of A , degree of p



- $p(\lambda)$ allows us to find the eigenvalues of A
- if we plug these values into the following vector equation, we can find the eigenvectors of A :

(2)

$$(A - \lambda I_n) \vec{K} = \vec{0}$$

use these formulas

NOTE: $p(\lambda)$ for all 2×2 matrices has the following form:

$$\begin{aligned} p(\lambda) &= \lambda^2 - \text{Tr}(A)\lambda + \det(A) \\ &= \lambda^2 - (a+d)\lambda + (ad - bc) \end{aligned}$$

Ex: find all eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$

① find $p(\lambda)$

$$p(\lambda) = \lambda^2 - (2+6)\lambda + [(2)(6) - (4)(3)]$$

$$p(\lambda) = \lambda^2 - 8\lambda$$

② find the roots of $p(\lambda)$ to get eigenvalues of A

$$0 = \lambda^2 - 8\lambda \rightarrow \lambda_1 = 0 \quad \lambda_2 = 8$$

③ sub eigenvalues into vector equation to get eigenvectors of A

$$\lambda_1 = 0$$

$$\left[\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} - (0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \vec{K} = \vec{0} \rightarrow \left[\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} \right] \vec{K} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 3 & 6 & 0 \end{array} \right) = \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 6 & 0 \end{array} \right) = \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right) \rightarrow x + 2y = 0 \rightarrow x = -2y$$

$\left(\begin{array}{cc|c} 2 & 4 \\ 3 & 6 \end{array} \right) \vec{K} = \vec{0}$ is the same as $\left(\begin{array}{cc|c} 2 & 4 \\ 3 & 6 \end{array} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\vec{K} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

↓
accounts for the infinite \vec{K} for one λ

the inputs $x=-2$ and $y=1$ satisfy the equation $x=-2y$. They are the "simplest" points that do this.

* $A \cdot \vec{K} = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} c_1(-2) \\ c_1(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
this is what it means for $c_1(1)$ to be an eigenvector to A with eigenvalue 0.

$$\lambda_2 = 8$$

$$\left[\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} - 8 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \vec{K} = \vec{0} \rightarrow \left[\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} - \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \right] \vec{K} = \vec{0}$$

$$\rightarrow \begin{pmatrix} 2-8 & 4-0 \\ 3-0 & 6-8 \end{pmatrix} \vec{K} = \vec{0} \rightarrow \begin{pmatrix} -6 & 4 \\ 3 & -2 \end{pmatrix} \vec{K} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -6 & 4 & 0 \\ 3 & -2 & 0 \end{array} \right) = \left(\begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ 3 & -2 & 0 \end{array} \right) = \left(\begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array} \right) = \left(\begin{array}{cc|c} -3 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right) \rightarrow -3x + 2y = 0$$

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multiplied R1 by -3 to get rid of fraction

$$\vec{K} = c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$