· **matrix**: an array of real numbers with dimensions $m \times n$ (m rows and n columns)

- also written as $A = (a_{ij})$; a_{ij} is the number in the ith row and jth column.
- a vector with n components is an $n \times 1$ matrix
- if $m = n$, the matrix is a **square** of $n \times n$ dimensions. Entries from the top left to the bottom right are its **diagonal** (or diagonal entries) with the opposite being its cross diagonal.
- a **zero matrix** (0) has only zeroes as components.

diagonal (entries)

cross diagonal

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

2×2

2 rows, 2 columns

square matrix

☆ each row in any $m \times n$ matrix is essentially one vector.

$$\vec{v} = (a \ b \ c \ \ldots) \qquad \vec{v} \begin{pmatrix} a \\ b \\ \vdots \end{pmatrix}$$

$n \times 1$ vector

☆ two matrices are only equal if their entries are equal.

· to add matrices (with the same dimensions), add like components. If $A = (a_{ij})$, $B = (b_{ij}) \to A + B = (a_{ij} + b_{ij})$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \to A + B \to \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

- $A + 0 = 0 + A = A$
- multiplying a scalar to a matrix means multiplying all of the components by the scalar. If $A = (a_{ij})$, K = scalar $\rightarrow$ $K \cdot A = K(a_{ij}) = (Ka_{ij})$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow K \cdot A = K\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} Ka & Kb \\ Kc & Kd \end{pmatrix}$$

K = scalar

- $-1(A) = -A \rightarrow A + (-A) = -A + A = 0$
- to subtract matrices (with the same dimensions), subtract like components. If $A = (a_{ij})$, $B = (b_{ij}) \rightarrow A - B = (a_{ij} - b_{ij})$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \rightarrow A \cdot B \rightarrow \begin{pmatrix} a-e & b-f \\ c-g & d-h \end{pmatrix}$$

- matrices can be multiplied if the number of columns in one matrix equals the number of rows of another matrix.

if A is $m \times n$ and B is $n \times r$,
AB has dimensions $m \times r$

- analytically, each row of one matrix is being multiplied with each column of another to create a new matrix whose entries are the dot product of each row x column.

$$A = \begin{array}{cc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{array}{l} \text{row1 } (i=1) \\ \text{row 2 } (i=2) \end{array} \end{array}$$

$$\begin{array}{ccc} & \text{column1 } (j=1) & \text{column 2 } (j=2) \\ B = & \begin{pmatrix} e \\ g \end{pmatrix} & \begin{pmatrix} f \\ h \end{pmatrix} \end{array}$$

$$\longrightarrow \quad AB = \begin{pmatrix} ae+bg & af+bh \\ ce+bg & cf+dh \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 6 \end{pmatrix}$; find AB

$$AB = \begin{pmatrix} 1\cdot 0 + 2\cdot -1 & 1\cdot 1 + 2\cdot 6 \\ 3\cdot 0 + 4\cdot -1 & 3\cdot 1 + 4\cdot 6 \end{pmatrix} = \begin{pmatrix} -2 & 13 \\ -4 & 27 \end{pmatrix}$$

- matrix multiplication is not communative, $AB \neq BA$ (most cases) (unless the components happen to make it work).
- **identity matrix:** a square matrix whose entries on the diagonal are all 1 with the rest being 0; denoted with I or I_n.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- for any square matrix, $IA = AI = A$
- **trace:** the sum of all entries on the diagonal of a square matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow Tr(A) = a + d$$

- **determinant**: for a 2×2 matrix is the difference between the products of the diagonal entries and cross diagonal entries.
  - for larger square matrices, use a calculator.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \det(A) = ad - bc$$

- systems of linear equations can be written using matrices in two ways:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$

① $\underset{\substack{2\times 3 \\ \text{matrix}}}{\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}} \underset{\substack{3\times 1 \\ \text{matrix}}}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{\substack{2\times 1 \\ \text{matrix}}}{\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}}$

☆ if you multiply the two LHS matrices, you will get the RHS matrix.

☆ in either case, rows represent each linear equation

② $\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{array}\right)$

- ② is called an **augmented matrix**, the line helps delineate between coefficients on our variables (each column is for each variable) and the constants each equation is equal to.

you're finding the intersection of the lines.

- when solving systems, we could use the elimination method: we're allowed to scale equations with scalars, replace equations by adding a multiple of another to it, or swap the order of equations; ultimately, we want to add the equations so that variables get eliminated, making the system easier to solve.

Ex: $x+y=3$, $2x-y=9$

$$\begin{array}{r} x+y=3 \\ +\ 2x-y=9 \\ \hline ? \end{array} \longrightarrow \begin{array}{r} x+y=3 \\ +\ 2x-y=9 \\ \hline 3x+0=12 \end{array}$$

$y-y=0$ (y variable eliminated!)

$3x=12 \rightarrow x=4$

$2(4)-y=9 \rightarrow y=-1$

solution: $x=4$, $y=-1$

- we can write the solution to a system of linear equations as an augmented matrix; this is said to be the reduced row echelon form of the original system's augmented matrix.

☆ criteria to be in RREF:

1. All zero rows (if any) are at the bottom
2. The first nonzero entry of a row is to the right of the first nonzero entry in the row above
3. Below the first nonzero entry of a row, all entries are zero.

Ex:

$$\begin{pmatrix} * & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & * & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & * & \bullet \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

$\bullet$ = any #

$*$ = any nonzero #

↳ are also called pivots since they are the 1st nonzero number in the row; the final equation and the original equation in that row geometrically "pivot" around the solution point described by the row in RREF.

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 9 \end{array}\right) = \underbrace{\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \end{array}\right)}$$

☆ notice how the reduce row echelon form of the original system is an identity matrix.

$1x+0y=4 \longrightarrow x=4$

$0x+1y=-1 \qquad y=-1$

- **row reduction:** applying the elimination method of solving systems of linear equations to matrices that represent them.
  - the goal is to obtain a reduced row echelon form of a system of linear equations, which simply means make the LHS an identity matrix.
  - there are 3 operations we can use to make the LHS of a augmented matrix an identity matrix:

1. scale any row by a nonzero number
2. swap any two rows
3. add any two rows together, replacing one with their sum.

- the general process of making the identity matrix:

☆ before beginning row reduction, try to swap rows so that rows with 1s on the diagonal or zeroes below them are first.

1. make the 1st diagonal entry 1 using a valid operation.
2. make the entries underneath that 1 zeroes, specfically by adding the correct multiple of the first row to the rows below.
3. make the 2nd diagonal entry 1 using a valid operation.
4. repeat step 2 but use row 2 for the operations.
5. repeat 1-4 until you make the bottom right entry 1
6. repeat step 2 but for all zeroes above each diagonal entry.

the process terminates when the identity matrix has been formed or when we can no longer zero out entries.

Ex: $x+y=3$, $2x-y=9$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 9 \end{array}\right)$$

① make the 1st diagonal entry 1 $\longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 9 \end{array}\right)$

② zero out entries below using R1

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 9 \end{array}\right) \xrightarrow{-2\cdot R1+R2} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ -2(1)+2 & -2(1)+-1 & -2(3)+9 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -3 & 3 \end{array}\right)$$

③ make the 2nd diagonal entry 1

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -3 & 3 \end{array}\right) \xrightarrow{-\frac{1}{3}R2} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ -\frac{1}{3}(0) & -\frac{1}{3}(-3) & -\frac{1}{3}(3) \end{array}\right) = \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -1 \end{array}\right)$$

④ zero out the entry above the 2nd diagonal using R2

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -1 \end{array}\right) \xrightarrow{-R2+R1} \left(\begin{array}{cc|c} -(0)+1 & -(1)+1 & -(-1)+3 \\ 0 & 1 & -1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \end{array}\right)$$

$\underbrace{\hphantom{1\ 0}}_{\text{identity created}}$

reduced row echelon form: $\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \end{array}\right)$

- there are 3 possible outputs from the row reduction method:

① a proper idendity matrix (like above)

② an <u>inconsistent system</u> that has no solution (the lines never intersect because they are parallel).

$$\left(\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 0 & -2 \end{array}\right) \rightarrow \begin{array}{l} 1x + 3y = 7 \ \checkmark \\ 0x + 0y = -2 \rightarrow 0 \neq -2 \ \times \end{array}$$

③ rows that are completely zeroed out and thus have no info

$$\left(\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \begin{array}{l} x + 3y = 7 \\ 0 = 0 \end{array}$$

★ for 2x2 matrices, you stop the row reduction method here if this occurs.

Eigenvalues and Eigenvectors

major question: given a square matrix A, is it possible to find a scalar λ such that:

$$\underbrace{A}_{\text{linear transformation}}\underbrace{\vec{K}}_{\text{nonzero vector}} = \lambda\vec{K} \rightarrow A = \lambda \text{ (for } \vec{K})$$

answer: it is always possible to find λ for any square matrix A, however, only certain vectors $\vec{K}$ can work.

- **eigenvalue:** a scalar λ associated with a square matrix A and vectors that satisfy $A\vec{K} = \lambda\vec{K}$ → a matrix transform using a specific vector = a scalar transform with the vector

  ☆ can be imaginary!

  - any $n \times n$ matrix could have $1 \leq \lambda \leq n$ distinct eigenvalues.

- **eigenvector:** a vector that has its direction unchanged or reversed by a given linear transformation applied to it; the linear transformation can thus be represented using a scalar value.
  - there are an infinite number of eigenvectors for every eigenvalue since any constant multiple of a known eigenvector is also an eigenvector of that linear transformation.

- **characteristic polynomial:** a polynomial for a specific square matrix whose roots are the eigenvalues of the matrix; denoted with $p(\lambda)$

1. $$p(\lambda) = \det(A - \lambda I_n)$$

☆ the inputs are eigenvalues
the outputs are determinants of $A - \lambda I$

n = dimensions of A, degree of p

- $p(\lambda)$ allows us to find the eigenvalues of A
- if we plug these values into the following vector equation, we can find the eigenvectors of A:

② $$(A-\lambda I_n)\vec{K}=\vec{0}$$

use these formulas

NOTE: $p(\lambda)$ for all 2×2 matrices has the following form:

$$p(\lambda)=\lambda^2-Tr(A)\lambda+\det(A)$$
$$=\lambda^2-(a+d)\lambda+(ad-bc)$$

Ex: find all eigenvalues and eigenvectors of $A=\begin{pmatrix}2&4\\3&6\end{pmatrix}$

① find $p(\lambda)$

$$p(\lambda)=\lambda^2-(2+6)\lambda+[(2)(6)-(4)(3)]$$

$$p(\lambda)=\lambda^2-8\lambda$$

② find the roots of $p(\lambda)$ to get eigenvalues of A

$$0=\lambda^2-8\lambda \rightarrow \lambda_1=0 \quad \lambda_2=8$$

③ sub eigenvalues into vector equation to get eigenvectors of A

↳ $A\vec{K} = \lambda\vec{K}$

$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\lambda_1 = 0$

$$\left[\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} - (0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\vec{K} = \vec{0} \longrightarrow \left[\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}\right]\vec{K} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underbrace{\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 3 & 6 & 0 \end{array}\right)} = \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 6 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow x + 2y = 0 \rightarrow x = -2y$$

$\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}\vec{K} = \vec{0}$ is the same as $\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\vec{K} = c_1\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

↓ accounts for the infinite $\vec{K}$ for one λ

☆ the inputs $x=-2$ and $y=1$ satisfy the equation $x=-2y$. They are the "simplest" points that do this.

☆ $A \cdot \vec{K} = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}\left[c_1\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

this is what it means for $c_1\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ to be an eigenvector to A with eigenvalue 0.

$\lambda_2 = 8$

$$\left[\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} - 8\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\vec{K} = \vec{0} \longrightarrow \left[\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} - \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}\right]\vec{K} = \vec{0}$$

$$\rightarrow \begin{pmatrix} 2-8 & 4-0 \\ 3-0 & 6-8 \end{pmatrix}\vec{K} = \vec{0} \rightarrow \begin{pmatrix} -6 & 4 \\ 3 & -2 \end{pmatrix}\vec{K} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -6 & 4 & 0 \\ 3 & -2 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -\frac{2}{3} & 0 \\ 3 & -2 & 0 \end{array}\right) = \underbrace{\left(\begin{array}{cc|c} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)}_{\text{multiplied R1 by -3 to get rid of fraction}} \rightarrow -3x + 2y = 0$$

$$3x = 2y$$

$$\vec{K} = c_2\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$