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ORIGINAL PAPER

Matroid automorphisms of the root system H_3

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Abstract Let H_3 be the root system associated with the icosahedron, and let $M(H_3)$ be the linear dependence matroid corresponding to this root system. We prove $\mathrm{Aut}(M(H_3)) \cong S_5$, and interpret these automorphisms geometrically.

Keywords Coxeter group · Matroid automorphism

Mathematics Subject Classification 05B35

1 Introduction

We study the combinatorial symmetry of the non-crystallographic irreducible root system H_3. This root system consists of 30 vectors in $\mathbb{R}^3$ which can be separated into 15 positive roots and their negatives. An alternate way to view this root system is related to the symmetries of an icosahedron. For each of the 15 planes of symmetry of a regular icosahedron (or, dually, a dodecahedron), take two normal vectors (pointed in opposite directions). This gives us a collection of 30 vectors which turn out to be parallel to the 30 edges of the icosahedron—see Fig. 1. (In fact, each plane of symmetry contains two parallel edges of the icosahedron, and is normal to two parallel edges.)

We use matroids to model the combinatorics of H_3. Given a finite set M of vectors in $\mathbb{R}^n$, the linear dependence matroid is the matroid whose independent sets are simply the linearly independent subsets of vectors of M. In particular, we consider the linear dependence matroid associated with the (positive) roots from the root system H_3, and we denote this matroid $M(H_3)$.

Dedicated to Thomas Brylawski.

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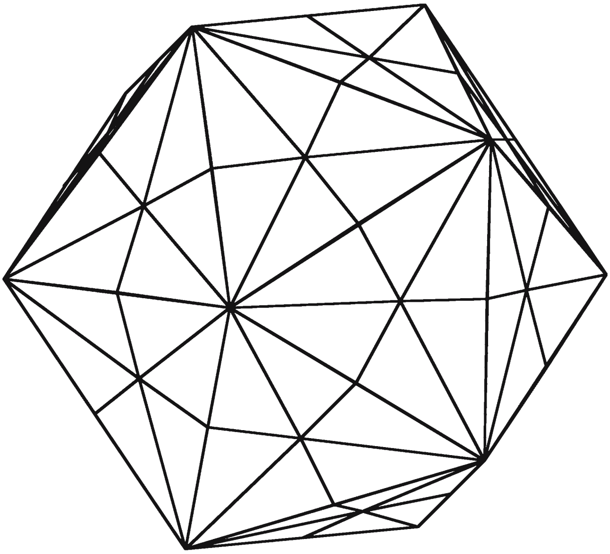

Fig. 1 The icosahedron with its reflections

We let W be the Coxeter group for the root system. W is just the symmetry group of an icosahdron, written $*532$ in Conway's orbifold notation to represent the three reflections that generate the group. The standard representation of this group is $W = \langle R_1, R_2, R_3\colon (R_1R_2)^5 = (R_2R_3)^3 = (R_1R_3)^2 = R_i^2 = 1\rangle$. It is well known that $W \cong A_5 \times \mathbb{Z}_2$. Geometrically, we can view the A_5 subgroup (with orbifold notation 532) as corresponding to the 60 rotations of the icosahedron which act on five embedded cubes in an icosahedron, and $\mathbb{Z}_2$ is generated by *central inversion*[1], i.e., the map that sends (x_1, x_2, x_3) to $(-x_1, -x_2, -x_3)$. This map commutes with every $w \in W$, and, with the identity element, forms the center of the group $Z(W)$. Much more information can be found in [4,5].

It is clear that any geometric symmetry of the icosahedron will preserve the dependence or independence of any subset of the 15 normal vectors that comprise the icosahedral matroid $M(H_3)$, i.e., if $w \in W$, then $w \in \mathrm{Aut}(M(H_3))$. In fact, since central inversion in W corresponds to the identity operation in $\mathrm{Aut}(M(H_3))$, we have a 2-to-1 map from W to $\mathrm{Aut}(M(H_3))$—this is made precise in Proposition 4.1. (This reflects a problem with the use of matroids as the model for combinatorial symmetry. We can overcome this difficulty by using *oriented matroids*, which is the approach we address briefly in Sect. 5—see Theorem 5.1.)

Our goal in this work is to explore the converse: Given any operation σ that preserves the matroid dependences, must σ correspond to some $w \in W$? When W is the Coxeter group corresponding to the root systems A_n or B_n, the answer is yes [3]: every matroid automorphism corresponds to a geometric symmetry of the root system. We interpret these results in the following way: combinatorial symmetry (preserving dependence) and geometric symmetry (via isometries of the ambient Euclidean space) are essentially the same.

For $\mathrm{Aut}(M(H_3))$, we show (Theorem 3.3) the answer to this question is no—there exist combinatorial symmetries of the root system H_3 that do not come from any isometry of $\mathbb{R}^3$. In spite of this difficulty, these non-geometric combinatorial symmetries can be understood geometrically. We include such an interpretation in Sect. 4. In Sect. 5, we conclude with two

[1] Chemists call this operation *improper reflection.*

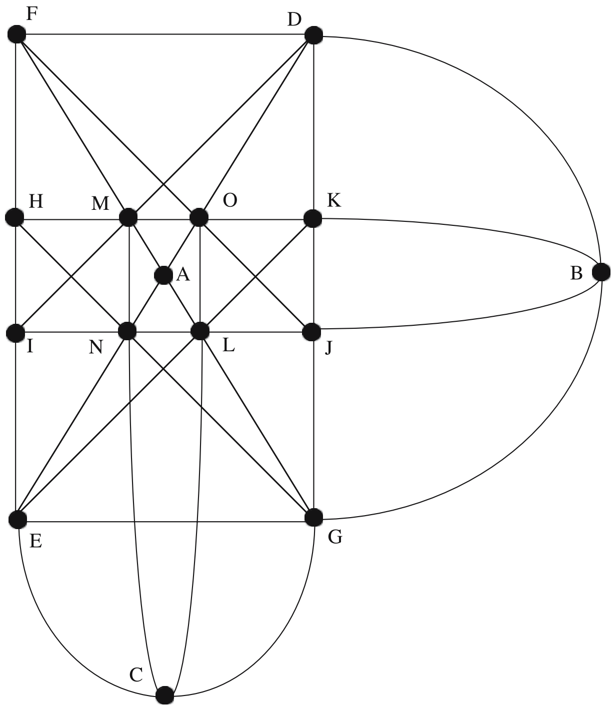

Fig. 2 An affine diagram of $M(H_3)$

alternative ways to approach the entire problem: we replace matroids with oriented matroids and we replace the icosahedron with a projective quotient.

We thank Laura Anderson and Thomas Zaslavsky for several useful suggestions. We also thank the anonymous referee for suggesting the affine diagram representing $M(H_3)$ given in Fig. 2.

2 Description of the matroid $M(H_3)$

We begin with a few definitions from matroid theory; we refer the reader to the first chapter of [6] for an introduction.

Definition 2.1 Let E be a finite set and $\mathcal{I}$ a family of subsets of E. Then a *matroid* is a pair $M = (E, \mathcal{I})$ satisfying:

(1) $\emptyset \in \mathcal{I}$;
(2) If $J \subseteq I$ and $I \in \mathcal{I}$, then $J \in \mathcal{I}$;
(3) If $I, J \in \mathcal{I}$ with $|I| < |J|$, then $I \cup \{x\} \in \mathcal{I}$ for some $x \in J - I$.

The family $\mathcal{I}$ is called the *independent* sets of the matroid. In particular, when E is a finite set of vectors over a field, $M = (E, \mathcal{I})$ forms a matroid, where $\mathcal{I}$ is the family of subsets of E that are linearly independent over the field.

Definition 2.2 The *icosahedral matroid* $M(H_3)$ is defined to be the linear dependence matroid on the set $\{A, B, \ldots, O\}$ over $\mathbb{Q}[\tau]$ of the 15 column vectors in the 3×15 matrix R:

$$\begin{array}{c} \begin{array}{ccccccccccccccc} A & B & C & D & E & F & G & H & I & J & K & L & M & N & O \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & 1 & 1 & \tau & -\tau & \tau & \tau & \tau^2 & \tau^2 & -\tau^2 & \tau^2 \\ 0 & 1 & 0 & \tau & \tau & -\tau & \tau & -\tau^2 & \tau^2 & \tau^2 & \tau^2 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & \tau^2 & \tau^2 & \tau^2 & -\tau^2 & 1 & 1 & -1 & 1 & -\tau & \tau & \tau & \tau \end{pmatrix} \end{array}$$

where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden mean (so $\tau^2 = \tau + 1$).

If $S \subseteq E$, define the *rank* of S by $r(S) = \max_{I \in \mathcal{I}}\{|I| : I \subseteq S\}$. A set $F \subseteq E$ is *closed* or a *flat* of the matroid if $r(F \cup \{x\}) = r(F) + 1$ for all $x \in E - F$. A k-point *line* l is a rank 2 flat of cardinality k.

2.1 Flats of the icosahedral matroid

The rank 3 matroid $M(H_3)$ can be described completely in terms of its flats. There are fifteen 2-point lines, ten 3-point lines, and six 5-point lines. We list these flats below. For convenience, we write $[ABC]$ to represent the three 2-point lines AB, AC, and BC. We also write AHJ for $\{A, H, J\}$, etc.

Fifteen 2-point lines: $[ABC]$, $[DHL]$, $[EJM]$, $[FKN]$, $[GIO]$;
Ten 3-point lines: AHJ, AIK, BDF, BEG, CLO, CMN, DIM, FJO, GHN, EKL;
Six 5-point lines: $ADENO$, $AFGLM$, $BHKMO$, $BIJLN$, $CDGJK$, $CEFHI$.

Thus, each point of the matroid is in precisely two 2-point lines, two 3-point lines and two 5-point lines. These flats have a geometric interpretation in terms of the icosahedron. Recall that each of the vectors $A, B, \ldots, O$ corresponds to a pair of parallel edges of the icosahedron. Then each 3-point line corresponds to a pair of parallel triangular faces of the icosahedron and each 5-point line corresponds to a pair of parallel pentagons (which form the faces of a dodecahedron, and correspond to the 6 pairs of opposite vertices of the icosahedron). For the 2-point lines, recall that five cubes can be embedded in the icosahedron in a natural way (see [2]). The three mutually perpendicular edges of one of these cubes are parallel to three corresponding icosahedron edges (six edges, actually, occurring in three pairs); thus, the three mutually perpendicular vectors A, B and C give rise to three 2-point lines in the matroid: AB, AC, and BC. We call such a set of three vectors an *orthoframe* and note that there are five orthoframes in this matroid.

2.2 K_6 representation of the icosahedral matroid

A standard way to represent a rank 3 matroid is through affine diagrams. In such a diagram, each point of the matroid is represented by a point in the plane, and three points which are dependent are collinear. The resulting diagram will then have the geometrically satisfying property that any two lines will meet in at most one point, and the k-point lines of the matroid are represented by k-point lines in the diagram. See Fig. 2 for one representation (the points B and C can be thought of as points 'at infinity' along the x and y axes, respectively).

Unfortunately, such a diagram is not especially helpful in determining the automorphisms of the matroid. We introduce a new, ad hoc way to represent the dependences of the icosahedral matroid using the complete graph K_6. Any matroid automorphism induces three corresponding permutations: a permutation of the six 5-point lines, another permutation of

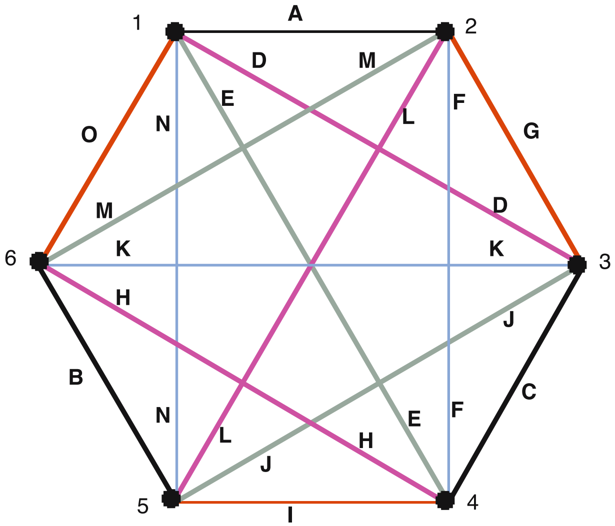

Fig. 3 Representing $M(H_3)$ using K_6

the ten 3-point lines and also a permutation of the five orthoframes. Our K_6 representation of the icosahedral matroid (see Fig. 3) displays each of these flats as follows:

- Each point of $M(H_3)$ corresponds to an edge of K_6.
- Each 5-point line corresponds to a vertex of K_6.
- Each of the five orthoframes corresponds to a colored maximum matching in K_6.
- Each of the ten 3-point lines corresponds to one of the remaining ten complete matchings in K_6.

In Fig. 3, for example, the 5-point line $ADENO$ corresponds to vertex 1, the orthoframe $[EJM]$ corresponds to the green maximum matching, and the 3-point line CLO corresponds to the (multi-colored) maximum matching with those edge labels. This representation will facilitate the computation of Aut($M(H_3)$), which we consider in the next section.

3 Automorphisms of the icosahedral matroid

We now determine the automorphism group Aut($M(H_3)$). We begin with two lemmas that reduce the problem from finding a permutation of S_{15} to S_5. Let e_n denote the identity permutation in S_n.

Lemma 3.1 *Let $\sigma \in S_{15}$ be an automorphism of the matroid $M(H_3)$, and let $\overline{\sigma} \in S_5$ be the induced permutation on orthoframes. If $\overline{\sigma} = e_5$, then $\sigma = e_{15}$.*

Proof We use the K_6 representation of the matroid. Any matroid automorphism must preserve all flats of the matroid, and so must also give a graph automorphism of K_6 (but not conversely). Suppose $\overline{\sigma}$ is the identity permutation on orthoframes. We must show $\sigma = e_{15}$, i.e., σ is the identity automorphism of K_6. (We abuse notation and write σ to represent both the automorphism of the matroid $M(H_3)$ and the induced graph automorphism of K_6.)

We show σ fixes all the vertices of K_6. We first show $\sigma(1) = 1$.

Case 1 Suppose $\sigma(1) = 2$. Then, since orthoframes are fixed by $\overline{\sigma}$, we must have $\sigma(2) = 1$ (to preserve $[ABC]$) and $\sigma(3) = 6$ (to preserve $[GIO]$). But this forces the interchange of orthoframes $[DHL]$ and $[EJM]$, contradicting $\overline{\sigma} = e_5$.
Case 2 Suppose $\sigma(1) = 3$. Then, arguing as in Case 1, we get $\sigma(2) = 4$ and $\sigma(3) = 1$, which forces an interchange of the orthoframes $[GIO]$ and $[EJM]$, again contradicting $\overline{\sigma} = e_5$.
Case 3 Suppose $\sigma(1) = 4$. This induces the vertex transpositions (14) and (23), which forces an interchange of the orthoframes $[FKN]$ and $[DHL]$, again contradicting $\overline{\sigma} = e_5$.
Case 4 Suppose $\sigma(1) = 5$. This is similar to Case 2.
Case 5 Suppose $\sigma(1) = 6$. This is similar to Case 1.

Thus, $\sigma(1) = 1$. A similar argument shows $\sigma(i) = i$ for $2 \leq i \leq 6$, so σ is the identity automorphism of K_6. Since K_6 represents the matroid $M(H_3)$, we have $\sigma = e_{15}$, and we are done. □

The proof of the next lemma follows immediately from Lemma 3.1 by applying this result to the automorphism $\sigma_1\sigma_2^{-1}$.

Lemma 3.2 *Let $\sigma_1, \sigma_2 \in S_{15}$ be matroid automorphisms with induced orthoframe permutations $\overline{\sigma}_1, \overline{\sigma}_2 \in S_5$. If $\overline{\sigma}_1 = \overline{\sigma}_2$, then $\sigma_1 = \sigma_2$.*

We now prove the main theorem of this paper.

Theorem 3.3 *$Aut(M(H_3)) \cong S_5$.*

Proof By Lemma 3.2, the map $\phi : \mathrm{Aut}(M(H_3)) \to S_5$ defined by $\phi(\sigma) = \overline{\sigma}$ is injective. To show $\mathrm{Aut}(M(H_3)) \cong S_5$, we need to show that ϕ is an isomorphism.

Consider the graph K_6 geometrically, i.e., a regular hexagon with all of its diagonals drawn, and all of the edges colored as in Fig. 3. Then the reflection of this hexagon through a vertical line intersecting the midpoints of the edges labeled A and I induces a transposition on the set of five orthoframes: $[EJM] \leftrightarrow [DHL]$. But we can arrange to have any pair of orthoframes occupy these two positions in the hexagon, so every transposition of orthoframes occurs in the image of ϕ. Since S_5 is generated by transpositions, we have $\mathrm{Aut}(M(H_3)) \cong S_5$. □

Note that $\mathrm{Aut}(K_6) \cong S_6$, so there are many graph automorphisms of K_6 which do not preserve the icosahedral matroid. For example, the automorphism that switches vertices 1 and 2 of K_6 is not matroidal (for instance, this switch would send the orthoframe $[GIO]$ to the 3-point line DIM).

4 Geometric interpretations of matroid automorphisms

We now interpret the matroid automorphisms geometrically. For $w \in W$, let $\psi(w)$ be the matroid automorphism of $\mathrm{Aut}(M(H_3))$ that results from applying the isometry w to the column vectors of R. Formally, we have a group homomorphism $\psi : W \to \mathrm{Aut}(M(H_3))$. The next proposition shows precisely how these isometries act on the matroid.

Proposition 4.1 *Let $\psi : W \to Aut(M(H_3))$ be the group homomorphism obtained from applying the isometries of W to the column vectors of the matrix R. Then*

(1) $Im(\psi) \cong A_5$;
(2) $Ker(\psi) \cong \mathbb{Z}_2$.

Proof (1) Note that every isometry in W induces an even permutation of the five orthoframes. This follows because any reflection in W will fix one orthoframe (namely, the orthoframe whose edges are parallel to or normal to the plane of reflection) and will interchange the other four orthoframes in pairs. For example, in our K_6 representation, reflection through the line containing vertices 1 and 4 corresponds to the orthoframe permutation that fixes $[EJM]$ and interchanges $[ABC] \leftrightarrow [GIO]$ and $[DHL] \leftrightarrow [FKN]$. But this is a product of two transpositions of the orthoframes, and 15 such products are generated by the 15 reflections in W. These 15 disjoint products generate A_5, and, since the reflections generate W, we have that every isometry of W corresponds to some even permutation of the orthoframes. Thus $Im(\psi) \cong A_5$.
(2) Let $z \in W$ represent central inversion. Then $\psi(z)$ is the identity automorphism in $\mathrm{Aut}(M(H_3))$. Further, it is clear that $\psi(r) \neq 1$ for any rotation $r \in W$. Since $W \cong A_5 \times \mathbb{Z}_2$ is generated by the rotations (that generate A_5) and central inversion (that generates $\mathbb{Z}_2$), we have $Ker(\psi) \cong \mathbb{Z}_2$. □

An immediate consequence of Theorem 3.3 and Proposition 4.1 is that half of the 120 matroid automorphisms are not geometric—the half that induce odd permutations of the five orthoframes. We now examine all of these automorphisms in more detail.

Example 4.2 We analyze the six conjugacy classes of matroid automorphisms—see Table 1. The dihedral group D_6 acts on our K_6 representation of the matroid to generate four kinds of graph automorphisms: half-turns, third turns, sixth turns and reflections through opposite vertices. For each kind of dihedral action, we give the induced cycle structure on the elements of $M(H_3)$ and on the various classes of flats. (For the 5-point lines, we use shorthand notation obtained from Fig. 3 by identifying vertex 1 with the flat $ADENO$, vertex 2 with $AFGLM$, and so on). We also consider induced 4- and 5-cycles on orthoframes, which are not generated by dihedral actions. In the table, we give the cycle index that the automorphism induces on the entire matroid and on each of the classes of flats.

We examine the six conjugacy classes of matroid automorphisms of Table 1 in more detail. When possible (in Cases 1–4), we give a dihedral operation to represent a typical automorphism of the class.

(1) Dihedral half-turn r_2: Let $\sigma \in \mathrm{Aut}(M(H_3))$ be the automorphism obtained from our K_6 representation by the dihedral operation of rotation of K_6 through an angle of 180°. There are 10 automorphisms in this conjugacy class. Then we get the following induced cycle structure on the matroid flats:

Table 1 Induced cycle structure of matroid automorphisms on flats

	No.	Group D_6	S_5	$M(H_3)$	Cycle index on flats Orthoframes	3-point lines	5-point lines
1	10	r_2	(ab)	$w_1^3 w_2^6$	$x_1^3 x_2$	$y_1^4 y_2^3$	z_2^3
2	20	r_3	(abc)	w_3^5	$x_1^2 x_3$	$y_1 y_3^3$	z_3^2
3	20	r_6	$(ab)(cde)$	$w_3 w_6^2$	$x_2 x_3$	$y_1 y_3 y_6$	z_6
4	15	R_v	$(ab)(cd)$	$w_1^3 w_2^6$	$x_1 x_2^2$	$y_1^2 y_4^2$	$z_1^2 z_2^2$
5	30	f_4	$(abcd)$	$w_1 w_2 w_4^3$	$x_1 x_4$	$y_2 y_4^2$	$z_1^2 z_4$
6	24	f_5	$(abcde)$	w_5^3	x_5	y_5^2	$z_1 z_5$

Matroid elements:	$(AI)(BG)(CO)(DH)(FN)(JM)$
Orthoframes:	$([ABC], [GIO])$
3-point lines:	$(AHJ, DIM)(BDF, GHN)(CMN, FJO)$
5-point lines:	$(14)(25)(36)$

(2) Dihedral third-turn r_3: Let σ correspond to clockwise rotation of K_6 through an angle of 120°. There are 20 automorphisms in this conjugacy class, and we get:

Matroid elements:	$(ACB)(GIO)(EKL)(FHM)(DJN)$
Orthoframes:	$([DHL], [EJM], [FKN])$
3-point lines:	$(AIK, CLO, BEG)(AHJ, CMN, BDF)$
	(DIM, FJO, GHN)
5-point lines:	$(135)(246)$

(3) Dihedral sixth-turn r_6: Let σ correspond to clockwise rotation of K_6 through an angle of 60°. There are 20 automorphisms in this class, and this time we get:

Matroid elements:	$(EKL)(AGIBO)(DFJHNM)$
Orthoframes:	$([ABC], [GIO])([DHL], [FKN], [EJM])$
3-point lines:	$(AHJ, GHN, CMN, DIM, BDF, FJO)$
	(AIK, BEG, CLO)
5-point lines:	(123456)

(4) Dihedral reflection through opposite vertices R_v: reflecting K_6 through vertices 1 and 4. Note that this operation fixes elements E, K and L pointwise. There are 15 such automorphisms up to conjugacy.

Matroid elements:	$(AO)(BG)(CI)(KL)(FH)(DN)$
Orthoframes:	$([ABC], [GIO])([DHL], [FKN])$
3-point lines:	$(AHJ, FJO)(AIK, CLO)(BDF, GHN)(DIM, CMN)$
5-point lines:	$(35)(26)$

(5) 4-cycle f_4 in S_5: This operation is not dihedral—there is no element of D_6 that will achieve this matroid automorphism. There are 30 automorphisms in this conjugacy class.

Matroid elements:	$(ANJG)(BKMO)(CFEI)(DL)$
Orthoframes:	$([ABC], [FNK], [EJM], [GIO])$
3-point lines:	$(AHJ, GHN)(AIK, CMN, FJO, BEG)$
	(BDF, EKL, DIM, CLO)
5-point lines:	(1532)

(6) 5-cycle f_5 in S_5: This operation is also not dihedral, and there are 24 automorphisms in this conjugacy class.

Matroid elements:	$(AJFDI)(BMKHO)(CENLG)$
Orthoframes:	$([ABC], [JME], [FKN], [DHL], [GIO])$
3-point lines:	$(AHJ, FJO, BDF, DIM, AIK)$
	$(BEG, CMN, EKL, GHN, CLO)$
5-point lines:	(15234)

We examine two odd orthoframe permutations in more detail. Transpositions of orthoframes can be induced by a half-turn of K_6—the two orthoframes that label the six outer edges of the hexagon are swapped, and the remaining three orthoframes are fixed. To get a different transposition, we can conjugate the half-turn by a color-swapping operation that will bring the two orthoframes to be swapped to the outside of the hexagon. There are ten ways to choose the two colors, and there is a unique half-turn for any labeling of the hexagon, yielding the ten transpositions.

For example, to swap $[EJM]$ and $[DHL]$, first redraw K_6 with the purple and green edges forming the boundary of the hexagon, then perform the half-turn, then put the purple and green edges back in the interior of the hexagon. This can be achieved from the labeling in Fig. 3 by conjugating the half-turn (14)(25)(36) by the color swapping permutation (24653). The reader can check the product is the vertex permutation (12)(36)(45). This permutation of the vertices of K_6 preserves the flats of the matroid (and corresponds to the desired transposition of orthoframes).

We remark that swapping $[EJM]$ and $[DHL]$ can also be achieved directly from the labeling in Fig. 3 by a vertical reflection through the midpoints of edges A and I. We don't include vertical reflection in Table 1 since any such reflection induces the same action on the matroid as the half-turn does, and we can conjugate by the color-swaps to achieve the desired transposition. This also indicates that we have some freedom in describing the dihedral operation corresponding to these matroid automorphisms, however.

Generating 4- and 5-cycles of orthoframes is impossible with a single dihedral operation on K_6. For both of these operations, there are geometric connections to the icosahedron (although the connection is much more direct for the 5-cycles — see below). We sketch briefly how to use transpositions to create the induced 4-cycle $(\alpha\beta\gamma\delta)$, where $\alpha = [GIO]$, $\beta = [EJM]$, $\gamma = [FNK]$ and $\delta = [DHL]$. First, we write $(\alpha\beta\gamma\delta) = (\alpha\delta)(\alpha\gamma)(\alpha\beta)$. Then, conjugating the half-turn (14)(25)(36) by three different color-swaps gives:

- $(\alpha\beta) = (14)(25)(36)^{(1465)} = (15)(26)(34)$
- $(\alpha\gamma) = (14)(25)(36)^{(1564)} = (12)(35)(46)$
- $(\alpha\delta) = (14)(25)(36)^{(16)(45)} = (13)(24)(56)$

Thus, $(\alpha\beta\gamma\delta) = ((15)(26)(34))\,((12)(35)(46))\,((13)(24)(56)) = (3645)$. We leave consideration of the other cases in Table 1 to the interested reader.

We conclude this section by revisiting the problem that motivated this work: Interpret the matroid automorphisms geometrically. We know that half of the matroid automorphisms (those that correspond to even permutations of the orthoframes) correspond to isometries of W. In fact, by Proposition 4.1, if $\sigma \in \mathrm{Aut}(M(H_3))$ induces an even permutation of the orthoframes, then $\psi(w) = \psi(wz) = \sigma$ for some direct isometry $w \in W$ (where z is central inversion and $\psi : W \to \mathrm{Aut}(M(H_3))$ from Proposition 4.1).

Since matroid automorphisms preserve flats, and since 3-point lines correspond to (pairs of parallel) faces of the icosahedron, elements of the matroid correspond to (pairs of parallel) edges of the icosahedron and 5-point lines correspond to (opposite) vertices, matroid automorphims induce geometric maps on the vertices, edges and faces of the icosahedron. (One geometric alternative is to replace the icosahedron with an icosadodecahedron, where the 3-point lines would correspond to triangles and the 5-point lines would correspond to pentagons.) Thus, if $\sigma \in \mathrm{Aut}(M(H_3))$ induces an odd permutation of the orthoframes, we can still attempt to interpret the action of σ on the icosahedron.

Proposition 4.3 *Let $\sigma \in Aut(M(H_3))$. Then σ induces permutations σ_v, σ_e and σ_t of the vertices, edges and triangles (resp.) of the icosahedron.*

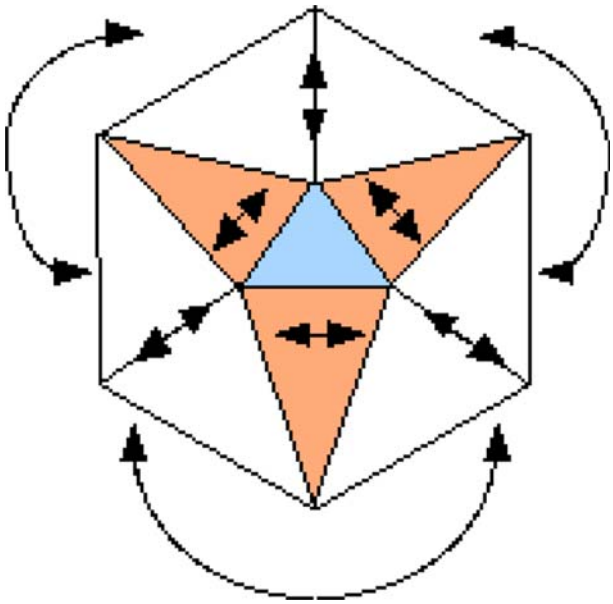

Fig. 4 Non-geometric action of orthoframe transposition $(\alpha\beta)$

Note 1. As noted above, transpositions of orthoframes can be induced by a half-turn of K_6. From Example 4.2(1), we know that this matroid automorphism induces a product of three disjoint transpositions of 3-point lines and three disjoint transpositions of 5-point lines. To visualize what this does to the icosahedron, we examine the triangle whose edges belong to the three ortho-frames which are fixed. This face is fixed, as are the three adjacent triangles. Finally, the three pairs of triangles surrounding these four triangles are swapped, and the vertices of the original triangle are swapped. See Fig. 4 for a diagram of this action. Although triangles are mapped to triangles and vertices are mapped to vertices by this action, the vertex-face incidence of the icosahedron is not preserved.

Note 2. This automorphism is induced by a sixth-turn of K_6, as in Example 4.2(3). From the example (or line 3 of Table 1), the induced map on 3-point lines is a product of a 3-cycle and a 6-cycle (with one line fixed), and the map on 5-point lines is simply a 6-cycle. In terms of the icosahedron, this means one face is fixed, with the three adjacent faces permuted in a 3-cycle (as in the geometric case with a third-turn through the center of the fixed face). The remaining faces are permuted in a 6-cycle, and the 12 vertices are permuted in two 6-cycles. Again, the vertex-face incidence is broken. In Fig. 5, each matroid element (and each flat) appears twice. The innermost and outermost faces are fixed, while the remaining faces belong to either a 3- or a 6-cycle.

Note 3. This is the automorphism examined in Example 4.2(5). Four-cycles do not arise from dihedral actions on K_6. To interpret this matroid automorphism geometrically, note that the induced permutation of 3-point lines is the product of a transposition and two 4-cycles, while the 5-point lines are permuted via a 4-cycle (with two 5-point lines fixed). The non-geometric action on the icosahedron is depicted in Fig. 6.

In Table 2, we remark that the 15 matroid automorphisms with orthoframe cycle structure $(ab)(cd)$ can be produced by either reflections or half-turns in the icosahedron. This follows from Proposition 4.1 and the fact that composing a half-turn of the icosahedron with

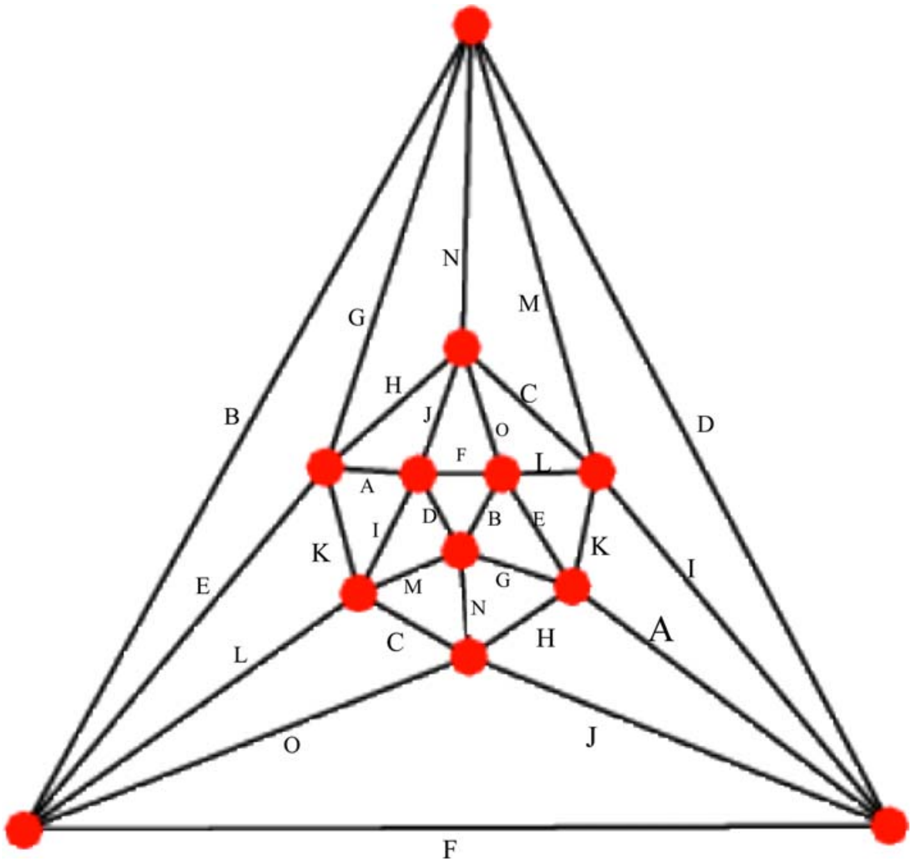

Fig. 5 Non-geometric action of orthoframe permutation $(\alpha\beta)(\gamma\delta\epsilon)$

Table 2 Geometric interpretation of matroid automorphisms

	No.	Matroid auto.	Isometry	Explanation
1	10	(ab)	None	Note 1.
2	20	(abc)	r_3	1/3 or 2/3-turn in icosahedron
3	20	$(ab)(cde)$	None	Note 2
4	15	$(ab)(cd)$	R	Reflection in icosahedron
5	30	$(abcd)$	None	Note 3
6	24	$(abcde)$	r_5	1/5 or 2/5-turn in icosahedron

central inversion gives a reflection. Thus, we could replace 'Reflection in icosahedron' with 'Half-turn in icosahedron' in the table.

5 Oriented matroids and a projective icosahedron

A fundamental problem with our approach is that each matroid element corresponds to two distinct roots: v and $-v$. In this section, we outline briefly two ways to address this problem.

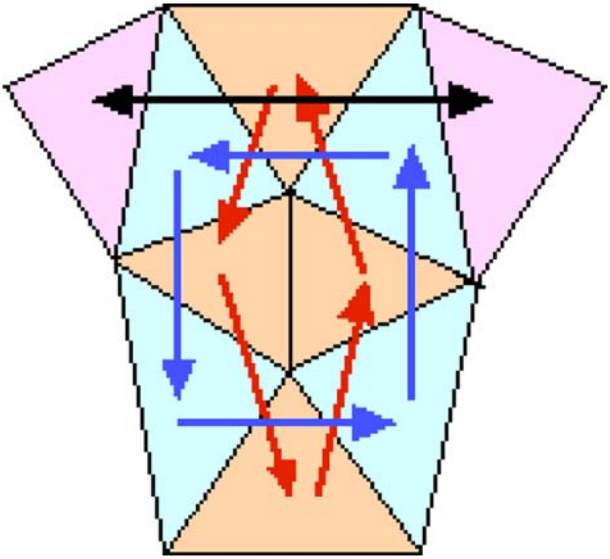

Fig. 6 Non-geometric action of orthoframe permutation $(\alpha\beta\gamma\delta)$

Oriented matroids

We do not include an introduction to oriented matroids here — we refer the reader to the first chapter of [1]. Let M be the oriented matroid on the 30 column vectors of the matrix R. Then Theorem 3.3 can easily be modified to prove the following.

Theorem 5.1 *Let M be the oriented matroid associated with the root system H_3. Then $Aut(M) \cong S_5 \times \mathbb{Z}_2$.*

Note that the map $\psi : W \to \mathrm{Aut}(M)$ that results from applying the isometry w to the column vectors of R is now injective, so the Coxeter group W is an index 2 subgroup of $\mathrm{Aut}(M)$. Thus, oriented matroids are a more natural setting for studying the combinatorics of H_3. All of the matroid results carry over to the oriented case—we leave detailed consideration of how to analyze specific oriented matroid automorphisms to the reader.

Projective icosahedron

Let I be the regular icosahedron and let $P(I) = I/\sim$, where $\sim$ is the equivalence relation on $\mathbb{R}^3$ given by $v \sim -v$. Then $P(I)$ has 6 vertices, 15 edges and 10 faces, so there is now a one-to-one correspondence between matroid flats and the various faces of $P(I)$:

- Vertices of $P(I)$ $\leftrightarrow$ 5-point lines of $M(H_3)$,
- Edges of $P(I)$ $\leftrightarrow$ elements of $M(H_3)$,
- Triangles of $P(I)$ $\leftrightarrow$ 3-point lines of $M(H_3)$,
- Orthoframes of $P(I)$ $\leftrightarrow$ 2-point lines (taken 3 at a time) in $M(H_3)$.

We view $P(I)$ as a projective version of the icosahedron. The advantage of using $P(I)$ instead of the icosahedron is that the isometry group $W(P(I)) \cong A_5$, so we have an injection $\psi : W(P(I)) \to \mathrm{Aut}(M(H_3))$. As in the oriented matroid case, the isometry group is an index 2 subgroup of the combinatorial automorphism group.

We conclude with the comment that it we believe it would be worthwhile to study Aut($M(S)$) for other root systems S, both in the oriented and non-oriented case. In particular, we conjecture that the Coxeter group of a root system is either isomorphic to the oriented matroid automorphism group (this occurs in the hyperoctahedral case [3]) or an index 2 subgroup of this group (Theorem 5.1).

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