Tutte polynomial - Let M be a **matroid** with rank function r on the ground set E. The Tutte polynomial $t(M;x,y)$ of M is defined by

$$t(M;x,y) = \sum_{S \subseteq E} (x-1)^{r(M)-r(S)}(y-1)^{|S|-r(S)}.$$

We write $t(M)$ for $t(M;x,y)$ when no confusion can arise about the values of x and y. There are several equivalent reformulations of $t(M)$; the most useful such expression is recursive, using the matroid operations of deletion and contraction.

T0. If $E = \emptyset$, then $t(M) = 1$;
T1. If e is an isthmus, then $t(M) = xt(M/e)$;
T2. If e is a loop, then $t(M) = yt(M-e)$;
T3. If e is neither an isthmus nor a loop, then $t(M) = t(M/e) + t(M-e)$.

Some standard evaluations of the Tutte polynomial are:

(a) $t(M;1,1)$ is the number of bases of M;
(b) $t(M;2,1)$ is the number of independent subsets of M;
(c) $t(M;1,2)$ is the number of spanning subsets of M;
(d) $t(M;2,2) = 2^{|E|}$.

When a matroid M is constructed from other matroids $M_1, M_2, \ldots$, it is frequently possible to compute $t(M)$ from the $t(M_i)$. The most fundamental structural result of this kind concerns the direct sum of matroids: $t(M_1 \oplus M_2) = t(M_1)t(M_2)$. Another fundamental structural property of the Tutte polynomial is its transparent relationship with matroid duality: $t(M^*;x,y) = t(M;y,x)$. Many related results can be found in [3].

The Tutte polynomial has many significant connections with invariants in **graph theory**. Of central importance is the relationship between $t(M)$ and the **chromatic polynomial** $\chi(G;\lambda)$ of a graph G. If G is a graph having $v(G)$ vertices and $c(G)$ connected components, then

$$\chi(G;\lambda) = \lambda^{c(G)}(-1)^{v(G)-c(G)}t(M_G, 1-\lambda, 0),$$

where M_G is the cycle matroid of the graph G. Thus, when G is planar, $t(M_G;x,y)$ simultaneously carries information concerning proper colorings of G along with proper colorings of its dual graph G^*. This is the reason W. T. Tutte referred to it as a *dichromatic* polynomial [14] in his foundational work in this area. The polynomial was generalized from graphs to matroids by H. H. Crapo [5] and T. H. Brylawski [2].

Other invariants which can be obtained from the Tutte polynomial include the beta invariant $\beta(M)$, the Möbius function $\mu(M)$, and the characteristic polynomial $p(M;\lambda)$ of a matroid M. See [16] for more information about these invariants.

We give a very brief indication of the diversity of applications of the Tutte polynomial. An extensive introduction can be found in [4].

1. Acyclic orientations: Let $a(G)$ denote the number of acyclic orientations of a graph G. Then R. Stanley [13] proved $a(G) = t(M_G;2,0)$. A variety of related evaluations appear in [9].
2. The critical problem of H. Crapo and G.-C. Rota [7]: Let M be a rank r matroid which is represented over $GF(q)$ by $\phi : E \to GF(q)^n$ and let k be a positive integer. Then the number of k-tuples of linear functionals on $GF(q)^n$ which distinguish $\phi(E)$ equals $(-1)^r q^{k(n-r)} t(M, 1-q^k, 0)$. See [12] for extensions of the critical problem.
3. **Coding Theory**: Let C be a linear code over $GF(q)$ with codeweight polynomial $A(C;q,z) = \sum_{\mathbf{v} \in C} z^{w(\mathbf{v})}$, where $w(\mathbf{v})$ is the weight of the codeword $\mathbf{v}$. Then

$$A(C;q,z) = (1-z)^r z^{n-r} t(M_C; \frac{1+(q-1)z}{1-z}, \frac{1}{z}),$$

where M_C is the matroid associated with the code C. Many standard results in coding theory have interpretations using the Tutte polynomial [4].

4. **Hyperplane arrangements**: The number of regions in a central arrangement of hyperplanes is given by $t(M_H;2,0)$, where M_H is the matroid associated with the arrangement. Many generalizations of this result appear in [15].

5. **Combinatorial Topology**: The independent subsets of a matroid form a **shellable simplicial complex**. If $h_M(x)$ is the shelling polynomial associated with this complex, then $h_M(x) = t(x, 1)$ and $h_{M^*}(y) = t(1, y)$. This result is analogous to the dichromatic interpretation for planar graphs. See [1] for more information on how the Tutte polynomial is associated to simplicial complexes.
6. Statistical Mechanics, Network Reliability and Knot Theory: Suppose M is a probabilistic matroid, i.e., each $e \in E$ has an independent probability $p(e)$ of successful operation. The formula

$$t(M; x, y) = \sum_{S \subseteq E} \left(\prod_{e \in S} p(e) \right) \left(\prod_{e \notin S} (1 - p(e)) \right) (x-1)^{r(M)-r(S)} (y-1)^{|S|-r(S)}$$

then produces a probabilistic Tutte polynomial. In this more general stuation, $t(M; 1, 2)$ is the *reliability* of M, i.e., the probability that a randomly chosen subset of E spans. Related Tutte polynomials have applications in statistical mechanics and network reliability [5] and knot theory [11].
7. **Greedoids**: When G is a greedoid, the expansion $t(G; x, y) = \sum_{S \subseteq E} (x-1)^{r(G)-r(S)} (y-1)^{|S|-r(S)}$ remains valid. The recursion of T3 takes a slightly different form, however: If e is feasible, then $t(G) = t(G/e) + (x-1)^{r(G)-r(G-e)} t(G-e)$. There are many combinatorial structures which admit a greedoid rank function in a natural way, but do not possess a meaningful matroid structure. For example, if T_1 and T_2 are rooted trees, then computing the Tutte polynomial in this way gives $t(T_1) = t(T_2)$ if and only if T_1 and T_2 are isomorphic as rooted trees [8].

In general, it is $\#P$-hard to compute the Tutte polynomial of a graph or a matroid [10]. Certain evaluations are computable in polynomial-time for certain classes of matroids, however. For example, the number of spanning trees of a graph can be calculated in polynomial-time by the **matrix-tree theorem**; since this number equals $t(M; 1, 1)$, this evaluation is tractable.

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