

# EXPECTED VALUE EXPANSIONS IN ROOTED GRAPHS

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ABSTRACT. When  $G$  is a rooted graph where each edge may succeed with probability  $p$ , we consider the expected number of vertices in the operational component of  $G$  containing the root. This expected value  $EV(G; p)$  is a polynomial in  $p$ . We present several distinct equivalent formulations of  $EV(G; p)$ , unifying prior treatments of this topic. We use results on network resilience (introduced by Colbourn) to obtain complexity results for computing  $EV(G; p)$ . We use some of these formulations to derive closed form expressions for  $EV(G; p)$  for some specific classes of graphs. We conclude by considering ‘uniformly optimal’ rooted graphs, root placement and some counterexamples.

## 1. INTRODUCTION

On the evening of October 17, 1989, millions of baseball fans tuned in to watch the third game of the World Series between the San Francisco Giants and the Oakland Athletics. At 5:04 p.m. Pacific Daylight Time, just before the game was scheduled to begin, however, a strong earthquake rocked the San Francisco Bay area and the television signal from Candlestick Park was interrupted. The Bay bridge was badly damaged, and the structural damage to the stadium caused a 10 day delay in the series, which was eventually swept by the Athletics.

This dramatic example serves as the backdrop to our investigations. When some catastrophic event interrupts the flow in a rooted network, we are interested in how many vertices remain connected to the root. For the San Francisco earthquake of 1989, this question can be applied to a variety of networks: water supply, sewage treatment and drainage, electrical supply, natural gas supply, computer networks, and so on. In short, almost every standard application in network reliability was affected in this natural catastrophe.

Reliability theory has received copious attention recently, but the expected value polynomial is relatively less studied. A closely related invariant is Colbourn’s *network resilience* for an (unrooted) graph [8], which is the expected number of node pairs that remain connected. Amin, Siegrist and Slater have also explored this topic (using the term *pair-connected reliability*) in a series of papers; see [4, 5, 14, 13] for a sample of their results.

In section 2 we develop several equivalent ways to compute the expected value for the number of vertices that remain connected to the root vertex. When each edge has the same independent probability  $p$  of succeeding, the expected value is a polynomial in  $p$ . A deletion/contraction recursion (Proposition 2.3), a related subtree expansion (Proposition 2.4) and a probabilistic expansion (Proposition 2.6) are all given. The proof of the subtree expansion is deferred to section 3. For many rooted

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graphs, the probabilistic expansion is the most efficient, but complexity results (Proposition 2.8) adapted from Colbourn's work on resilience show that calculating the expected value (for fixed  $p$ ) is  $\#P$ -complete, even for planar networks.

Section 4 gives explicit, closed form expressions for  $EV(G; p)$  for rooted trees, circuits, fans and wheels. These results are based on calculations of Amin, Siegrist and Slater [5], but can be derived in other ways.

In section 5, we introduce a new measure  $u(G)$  for rooted graphs - the *uniform expected rank*:  $u(G) = \int_0^1 EV(G; p) dp$ . This measure corresponds to the situation when no information about the distribution of  $p$  exists. Then, among all graphs on  $n$  vertices and  $m$  edges, there is a unique value  $u_{n,m}$  corresponding to the maximum possible  $u(G)$ . A graph achieving this maximum is termed *uniformly optimal*. The main result of the section is Proposition 5.4, characterizing the optimal trees and the optimal unicyclic graphs. We also present an example to show how vertex placement can affect both  $EV(G; p)$  and  $u(G)$ .

We conclude with some counterexamples in section 6, showing that  $u(G)$  is not necessarily maximized when the root is adjacent to all other vertices. We briefly indicate the results of a computer search that finds all uniformly optimal graphs when  $n$  and  $m$  are manageable.

There are several possibilities for extending this work in the future.

- Application of the techniques developed here to rooted directed graphs could have implications for directed network design. Many of our results apply to the directed case.
- The assumption that  $p$  is uniformly distributed is not realistic for most applications. It would be of interest to use real-world data to apply our optimality definition to situations where the distribution is not uniform (although results in this situation would probably be heuristic).
- Characterization of other optimal rooted graphs should be possible. In particular, it is natural to extend the characterization of Proposition 5.4 to other classes, for example, graphs on  $n$  vertices and  $m$  edges for  $n + 1 < m < \binom{n}{2}$ . Presumably, a partial catalog of optimality results could be developed.

## 2. DEFINITIONS AND EXAMPLES

Let  $G$  be a rooted graph, let  $E$  denote the edges of  $G$  and suppose each edge of the graph has the same independent probability  $p$  of succeeding after some catastrophic event. Although it is not difficult to generalize so that each edge  $e$  has an independent probability  $p_e$  of success, we do not treat this situation here. We remark that nearly all of the formulas we derive have analogous expressions in the more general situation. For a subset of edges  $S \subseteq E$ , we let  $r(S)$  denote the number of vertices (besides the root) in the component of the subgraph  $S$  that contains the root. (We simultaneously refer to  $S$  as a subset of edges and as a *subgraph*.)

The rank  $r(S)$  is the *greedoid* rank when the rooted graph is considered as an edge branching greedoid. Greedoids are generalizations of matroids, which, in turn, simultaneously generalize graphs and matrices. For more information on greedoids, the interested reader is referred to [7] and [11]. We will not explicitly refer to greedoids in this paper.

**Definition 2.1.** Let  $G$  be a rooted graph. The expected value  $EV(G; p)$  is

$$EV(G; p) = \sum_{S \subseteq E} r(S) p^{|S|} (1-p)^{|E-S|}.$$

We will consider simple rooted graphs (i.e., graphs without loops and multiple edges) throughout this paper. This restriction makes sense from the viewpoint of network reliability – there is no reason to build a network with loops and multiple edges. We remark, however, that all of the work here easily generalizes to graphs with loops and multiple edges, and we will relax this restriction when loops and multiple edges arise in certain computations, as in the deletion/contraction algorithm of Proposition 2.3.

Definition 2.1 is consistent with the usual interpretation for expected value. For example, if  $G$  is a rooted triangle, then the reader can check the rank of each of the 8 subsets of edges. Expanding the resulting polynomial gives  $EV(G; p) = 2p + 2p^2 - 2p^3$ . Note that  $EV(G; 1) = 2$ , corresponding to the situation in which each edge is certain to survive.

The next result collects some easy consequences of the definition. We omit the proofs.

**Proposition 2.2.** *Let  $G$  be a connected, simple, rooted graph with  $n$  vertices, and suppose the root vertex has degree  $d$ . Then*

1.  $EV(G; p)$  has degree at most  $n$ ;
2.  $EV(G; p) = dp + p^2 g(p)$  for some polynomial  $g(p)$ ;
3.  $EV(G; 0) = 0$ ;
4.  $EV(G; 1) = n - 1$ .

There are several equivalent ways to calculate  $EV(G; p)$ . We present a recursive procedure based on the familiar operations of deletion and contraction to compute  $EV(G; p)$ . Recall that if  $e$  is an edge in  $G$ , then the deletion  $G - e$  is the graph obtained from  $G$  by simply removing the edge  $e$ ; the contraction  $G/e$  is obtained by removing  $e$  and then identifying the two endpoints of  $e$ . (We will not contract loops in graphs that arise in this process.)

**Proposition 2.3** (Deletion-Contraction). *Let  $G$  be a rooted graph and let  $e$  ( $\neq$  loop) be an edge adjacent to the root. Then*

$$EV(G; p) = (1-p) \cdot EV(G - e; p) + p \cdot EV(G/e; p) + p.$$

*Proof.* We write

$$EV(G; p) = \sum_{S: e \in S} r_G(S) p^{|S|} (1-p)^{|E(G)-S|} + \sum_{S: e \notin S} r_G(S) p^{|S|} (1-p)^{|E(G)-S|}.$$

Now  $\sum_{S: e \notin S} r_G(S) p^{|S|} (1-p)^{|E(G)-S|} = (1-p) EV(G - e; p)$  because  $r_G(S) = r_{G-e}(S)$  whenever  $e \notin S$ , where  $r_H(S)$  denotes the rank of  $S$  in the graph  $H$ .

When  $e \in S$ , it is straightforward to show  $r_G(S) = r_{G/e}(S - e) + 1$  by considering spanning trees in  $S$ . Then

$$\begin{aligned} \sum_{S: e \in S} r_G(S) p^{|S|} (1-p)^{|E(G)-S|} &= p \sum_{S: e \in S} (r_{G/e}(S - e) + 1) p^{|S-e|} (1-p)^{|E(G)-S|} \\ &= p \cdot EV(G/e; p) + p \sum_{S: e \in S} p^{|S-e|} (1-p)^{|E(G)-S|} \\ &= p \cdot EV(G/e; p) + p. \end{aligned}$$

□

A similar deletion/contraction formula holds for unrooted graphs where the rank of a subset of edges  $S$  is the size of the largest acyclic subset of  $S$  – the matroid rank (Proposition 2.1 of [6]).

We can use Proposition 2.3 to collect terms from Definition 2.1 to reduce the number of terms in the expansion. The term grouping we describe corresponds to an interval partition of the power set  $2^E$  so that each interval gives rise to one term in the expansion. Further, there is a one-to-one correspondence between intervals in  $2^E$  and rooted subtrees of  $G$ . We explore this partition in the next section.

We give an algorithmic description of the recursive deletion/contraction procedure which leads to a rooted subtree expansion of  $EV(G; p)$ . Roughly, every subtree contributes to  $EV(G; p)$  in this expansion as the edges that survive, and for each subtree  $T$ , the edges that would increase the rank of  $T$  are the edges that must fail. In the algorithm, surviving edges are precisely those edges that are contracted, while deleted edges correspond to those that fail. Edges that are neither contracted nor deleted along the way do not affect the product.

#### DELETION/CONTRACTION RESOLUTION ALGORITHM

1. Use deletion and contraction repeatedly to resolve your rooted graph  $G$  into a collection of rank 0 minors (in which the only surviving edges are loops or are disconnected from the root).
2. For each rank 0 minor, keep track of the elements contracted along the way – these will correspond to the rooted subtrees of  $G$ .
3. For each rooted subtree  $T$ , let  $L(T)$  be the edges which were neither deleted nor contracted in arriving at  $T$ .
4. Then the subtree  $T$  contributes the term  $|T|p^{|T|}(1-p)^{|E|-|T|-|L(T)|}$  to the polynomial  $EV(G; p)$ . The exponent  $|E| - |T| - |L(T)|$  represents the edges that were deleted in arriving at  $T$  (since every edge is either deleted, contracted or survives till the end as a loop.)

We postpone the proof of Proposition 2.4 to the next section.

**Proposition 2.4** (Subtree expansion). *Let  $G$  be a rooted graph,  $T$  a rooted subtree and  $L(T)$  be the leaves  $T$ . Then*

$$EV(G; p) = \sum_{T \in \mathcal{T}} |T|p^{|T|}(1-p)^{|E|-|T|-|L(T)|},$$

where  $\mathcal{T}$  is the collection of all rooted subtrees of  $G$ ,

Note that the value of  $L(T)$  depends not only on the subtree  $T$ , but also on the specific deletion/contraction resolution used. The fact that this ostensible dependence on the order in which we operate on the edges leads to a polynomial which is independent of this order is directly analogous to Tutte’s famous basis activities approach to the Tutte polynomial. More connections between order and Tutte polynomials can be found in [10] and [15]. An explicit connection between activities and reliability can be found in [16]. We give a small example to illustrate the definition and these two propositions.

**Example 2.5.** Let  $G$  be the rooted triangle at the top of Figure 1. In the picture, we use the convention that the left-hand child of a graph minor is obtained by contraction and the right-hand child is obtained by deletion.

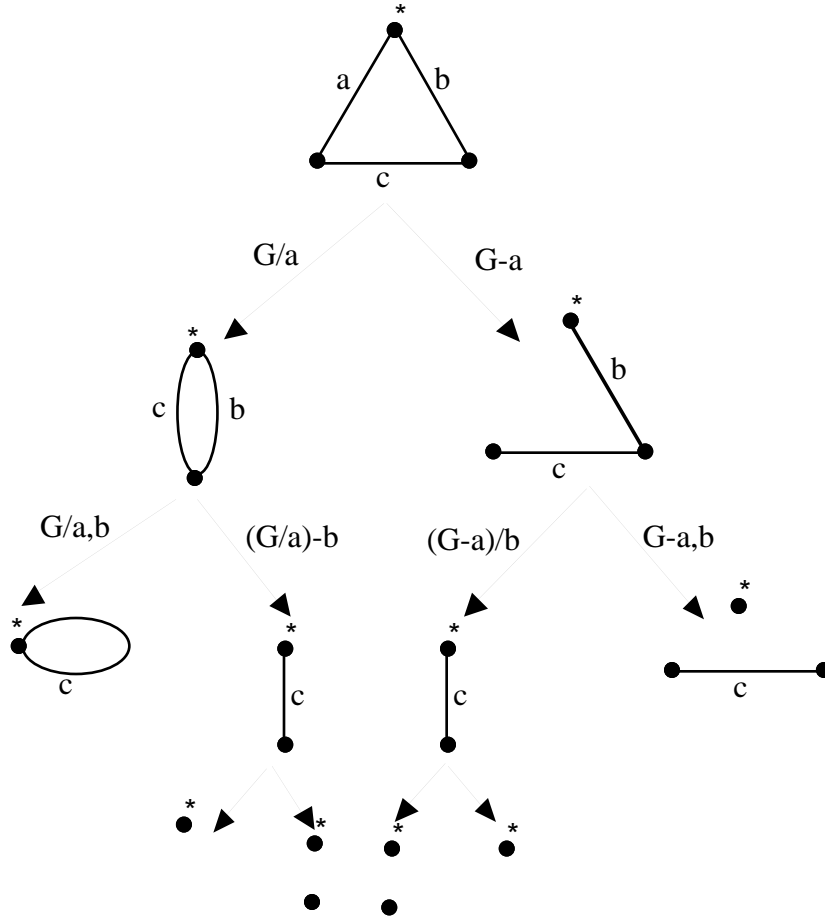


FIGURE 1. Deletion and contraction resolution

Using Definition 2.1 to calculate  $EV(G)$  requires 8 terms. The subtree expansion of Proposition 2.4 requires 6 terms. In Table 1, we give the term corresponding to each subtree. The resulting polynomial is  $EV(G) = 2p + 2p^2 - 2p^3$ .

$T$	$L(T)$	$ T p^{ T }(1-p)^{ E - T - L(T) }$
$\emptyset$	$c$	0
$a$	$\emptyset$	$p(1-p)^2$
$b$	$\emptyset$	$p(1-p)^2$
$a, b$	$c$	$2p^2$
$a, c$	$\emptyset$	$2p^2(1-p)$
$b, c$	$\emptyset$	$2p^2(1-p)$

TABLE 1.

While Proposition 2.4 is an improvement over the definition in the sense that it involves fewer terms, this improvement is not significant since an arbitrary graph may have an exponential number of rooted subtrees. A much more efficient expansion can be obtained by using methods of A. Amin, K. Siegrist and P. Slater [4], who have studied the *pair-connectivity* of a graph using probabilistic techniques. Their approach is directly applicable in this setting. For each non-root vertex  $v \in V$ , we let  $I(v)$  be an indicator function for whether  $v$  is reachable from the root. Thus  $I(v) = 0$  if there is no path connecting  $v$  to the root and  $I(v) = 1$  if there is such a path. Let  $Pr(v)$  denote the probability that  $v$  is reachable from the root. It is immediate that  $E(I(v)) = Pr(v)$ , where  $E$  is the expected value operator.

**Proposition 2.6** (Probabilistic vertex expansion). *Let  $G$  be a rooted graph with root vertex  $*$ , and let  $V$  denote the vertices of  $G$ . Then*

$$EV(G; p) = \sum_{* \neq v \in V} Pr(v).$$

*Proof.* Let  $E$  denote the expected value operator. Then

$$EV(G; p) = E \left( \sum_{* \neq v \in V} I(v) \right) = \sum_{* \neq v \in V} E(I(v)) = \sum_{* \neq v \in V} Pr(v).$$

□

We mention one corollary of this result. The *direct sum*  $G_1 \oplus G_2$  of the rooted graphs  $G_1$  and  $G_2$  is the rooted graph formed by identifying the two roots.

**Corollary 2.7** (Direct Sum).  $EV(G_1 \oplus G_2; p) = EV(G_1; p) + EV(G_2; p)$ .

While Proposition 2.6 gives this result immediately, we remark that an inductive proof which uses Proposition 2.3 is also routine. Proposition 2.6 also provides an inductive proof of the deletion/contraction formula given in Proposition 2.3.

Proposition 2.6 gives a much simpler way to calculate  $EV(G; p)$ , provided  $Pr(v)$  is easy to calculate. For example, if  $G$  is a rooted tree, then  $Pr(v) = p^{d(*, v)}$ , where  $d(*, v)$  is the distance from the root to  $v$ . Since determining the distance from a given vertex to any other vertex in a tree can be done in linear time, this computation is very efficient. Consequences of this are explored in [3] and [4].

While calculation of  $EV(G; p)$  is immediate for rooted trees, it is much harder for arbitrary planar graphs. The next proposition adapts two complexity results of Colbourn [8] to the computation of  $EV(G; p)$ . A graph is *series-parallel* if it contains no subgraph homeomorphic to  $K_4$ .

**Proposition 2.8.** 1. *Suppose  $G$  is a rooted series-parallel graph with  $n$  vertices. Then  $EV(G; p)$  can be computed in  $O(n)$  time.*

2. *For rooted planar graphs  $G$ , computation of  $EV(G; p)$  is  $\#P$ -complete.*

*Proof.*

1. This result is Theorem 3.4 of [8].
2. Let  $Res(G)$  be the network resilience of  $G$  as defined in [8], and let  $V$  be the vertex set. Then  $Res(G) = \frac{1}{2} \sum_{x \in V} EV(G_x; p)$ , where  $G_x$  is the graph  $G$  rooted at  $x$ . Hence, if there were a polynomial algorithm for computing  $EV(G_x; p)$ , there would be one for computing  $Res(G)$ . But resilience of planar networks is  $\#P$ -complete by Theorem 4.3 of [8].

□

### 3. AN INTERVAL PARTITION

In this section we give a structural result that will allow us to prove Proposition 2.4. We partition the Boolean lattice of subsets of edges into intervals, where each interval gives rise to one term in the related expansion. Although the partition is order dependent, it yields the an order independent polynomial.

For rooted subtrees of a graph  $G$ , we use the computation tree approach developed in [9]. As in Example 2.5, we repeatedly delete and contract edges adjacent to the root. The procedure stops when there are no (non-loop) edges available to delete or contract. For each terminal node  $s$  in this process, let  $T_s$  denote the edges that were contracted in reaching this node and let  $L_s$  denote the edges that remain (either as loops or edges not reachable from the root). The following proposition is proven in [9].

**Proposition 3.1** (Theorem 2.5 in [9]). *Let  $T_s$  be the edges that are contracted in any deletion-contraction resolution of the rooted graph  $G$ , and let  $L_s$  be the edges which were neither deleted nor contracted in arriving at the node  $s$ . Then  $\{T_s\}$  is the collection of all rooted subtrees of  $G$ . Further, the intervals  $[T_s, T_s \cup L_s]$  partition the Boolean lattice  $2^E$ .*

We now prove Proposition 2.4.

*Proof.* By Proposition 3.1, we have a partition of all edge subsets into intervals. Let  $S \subseteq E$  be a subset of edges. Then,  $r(S) = |T_s|$  for all  $S$  such that  $T_s \subseteq S \subseteq T_s \cup L_s$ . Thus

$$\begin{aligned} EV(G; p) &= \sum_{S \subseteq E} r(S) p^{|S|} (1-p)^{|E-S|} \\ &= \sum_{T_s \in \mathcal{T}} \sum_{S: T_s \subseteq S \subseteq T_s \cup L_s} r(S) p^{|S|} (1-p)^{|E-S|} \\ &= \sum_{T_s \in \mathcal{T}} |T_s| p^{|T_s|} (1-p)^{|E|-|T_s|-|L(T)|} \sum_{k=0}^{|L_s|} \binom{|L_s|}{k} p^k (1-p)^{|L_s|-k} \\ &= \sum_{T_s \in \mathcal{T}} |T_s| p^{|T_s|} (1-p)^{|E|-|T_s|-|L(T)|}. \end{aligned}$$

□

In Example 2.5, there are 6 subtrees that correspond to 6 intervals in the Boolean lattice. Four of these intervals are trivial, and the remaining two ( $[\emptyset, c]$  and  $[ab, abc]$ ) have length one.

### 4. SOME CLASSES

We now give explicit formulas for  $EV(G; p)$  for many classes of graphs. A *rooted fan*  $F_n$  is the graph obtained from joining one vertex (the root) to every vertex on the path  $P_n$  with  $n$  vertices. Thus,  $F_n$  has  $n+1$  vertices and  $2n-1$  edges. The *rooted wheel*  $W_n$  is obtained by joining one vertex (again, the root) to every vertex of the cycle  $C_n$ . Thus,  $W_n$  has  $n+1$  vertices and  $2n$  edges. (Equivalently,  $W_n$  is obtained by adding one edge to  $F_n$ .)

Rooted fans and wheels are useful models for network configurations since they have a relatively small number of edges, but allow every vertex direct access to the root (which might be a server in the application). For example, suppose a satellite needs to communicate to a series of ground stations configured in a path. Then the rooted fan models this situation. Rooted wheels model situations in which stations are linked in a ring, with a central server having direct access to each station.

In the next proposition, we list  $EV(G; p)$  for rooted trees, cycles, fans and wheels. For simplicity, we set  $q = 1 - p$  in the formulas for the fan and wheel.

**Proposition 4.1.** *Let  $T$  be a tree,  $C_n$  be a rooted cycle on  $n$  edges,  $F_n$  a rooted fan with  $n + 1$  vertices and  $W_n$  a rooted wheel with  $n + 1$  vertices. Then*

1. **Rooted trees:**  $EV(T; p) = \sum_{v \in V} p^{d(*, v)}$ , where  $d(*, v)$  is the distance from  $v$  to the root;
2. **Rooted cycles:**

$$EV(C_n; p) = \frac{2p - (n + 1)p^n + (n - 1)p^{n+1}}{1 - p};$$

3. **Rooted fans:**

$$EV(F_n; p) = \frac{np(1 - pq)(1 + p^3 - p^2(1 + (pq)^n)) - 2p^2q^2(1 - (pq)^n)}{(1 - pq)^3}$$

4. **Rooted wheels:**

$$EV(W_n; p) = n \left( \frac{p(1 - (pq)^n)}{1 - pq} + \frac{p^2q^2(1 - n(pq)^{n-1} + (n - 1)(pq)^n)}{(1 - pq)^2} \right).$$

*Proof.*

1. This appears as Theorem 1 in [4]. Also see Corollary 2.6 of [3].
2. First note that  $Pr(v) = p^a + p^b - p^n$ , where  $a + b = n$  and  $a$  and  $b$  are the distances from the root to  $v$  in the two different directions along the cycle. This immediately gives  $EV(C_n; p) = 2p + \dots + 2p^{n-1} - (n - 1)p^n$ . The formula given is a closed form version of this.
3. Let  $v_1, \dots, v_n$  be the vertices of the path, each of which is joined to the root. By equation (10) of [5],

$$Pr(v_i) = 1 - \frac{q(q + p^2(pq)^{i-1})(q + p^2(pq)^{n-i})}{(1 - pq)^2}.$$

To complete the proof, we need only compute  $\sum_{i=1}^n Pr(v_i)$ . The formula given is the result.

4. If  $v$  is any vertex besides the root, then  $Pr(v)$  is given in (7) of [5]. The formula for  $EV(W_n; p)$  is a closed form for  $nPr(v)$ .

□

We remark that it is also quite easy to verify the formulas for  $EV(C_n; p)$  and  $EV(T; p)$  (for a tree  $T$ ) using induction and Proposition 2.3. The derivations for rooted fans and wheels can also be done this way, but the details are a bit messy.

## 5. UNIFORM OPTIMAL ROOTED GRAPHS, CROSSINGS AND VERTEX PLACEMENT

Within reliability theory, polynomials are evaluated frequently at various values of  $p$ . For example, if it is known that network connections are very reliable, then a high value for  $p$  can be assumed in computing the reliability or the expected value.



Our approach differs from the standard one in that we do not assume any specific value for  $p$ , but specify a *distribution* of values. For example, if no information about the reliability of edges is available, then it is reasonable to assume that  $p$  is a random variable with uniform distribution. Then calculating the expected number of vertices that remain joined to the root amounts to computing an integral. This motivates the next definition.

**Definition 5.1.** Let  $G$  be a rooted graph with expected value polynomial  $EV(G; p)$ . Then the *uniform expected rank*  $u(G)$  is defined by

$$u(G) = \int_0^1 EV(G; p) dp.$$

Of course, if the distribution of  $p$  is known as some density function  $\delta(p)$ , then we could compute  $\int_0^1 EV(G; p) \delta(p) dp$  to give a more accurate measure of the expected number of vertices that remain joined to the root. It would be an interesting exercise with potentially wide application to apply this definition to real-world problems in which good data exist to estimate  $\delta(p)$ . We concentrate exclusively on the uniform case here, however.

One fundamental practical problem in network design concerns the location of the root. More specifically, given a graph  $G$ , find the vertex (or vertices)  $v$  such that  $u(G)$  is maximized when the root is placed at  $v$ . The next example shows how our reliability measures  $EV(G; p)$  and  $u(G)$  can vary depending on the root placement.

**Example 5.2.** Let  $G$  be the graph of Figure 2. Write  $G_v$  for the graph rooted at the vertex  $v$ . Then  $EV(G_5, p) \geq EV(G_v, p)$  for  $p$  in the range  $0 \leq p \leq 0.355$  (approx.) and all vertices  $v$ . We say vertex 5 is *locally optimal* for the range  $0 \leq p \leq 0.355$ . For this graph, we find vertex 6 is locally optimal for  $0.355 \leq p \leq 0.906$ , and vertex 7 is optimal for  $.906 \leq p \leq 1$ .

We also compute  $u(G_v)$  for vertices 5, 6, 7, and 8. We find

$$\begin{aligned} u(G_5) &= 4.68254\dots \\ u(G_6) &= 4.86706\dots \\ u(G_7) &= 4.81825\dots \\ u(G_8) &= 4.78373\dots \end{aligned}$$

Thus vertex 6 is the optimal location for the root when using  $u(G_v)$  as our criterion. We call such a vertex *uniformly optimal* for the location of the root.

The dependence of vertex location on the value of  $p$  is analogous to finding ‘crossings’ of the graphs of two reliability polynomials. See [12] for examples that show there is no most reliable network for specified parameters.

It follows from Proposition 2.2(2) that for small values of  $p$ , the locally optimal vertex is the vertex with the largest degree. For larger values of  $p$ , other factors having to do with the general idea of ‘centrality’ of a vertex play a role in optimality. For example, the vertex (or vertices) with the minimum average distance to other vertices is frequently the globally optimal vertex, i.e., the vertex that maximizes  $u(G_v)$ .

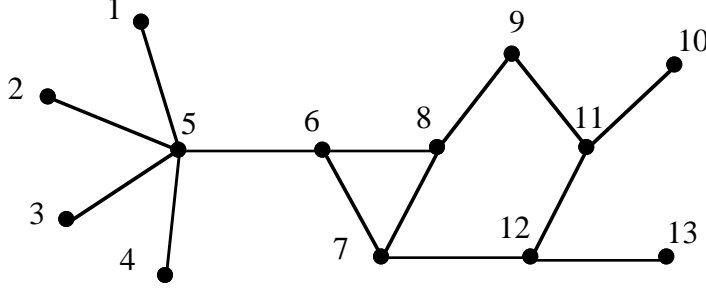


FIGURE 2. Dependence on root location.

Among all rooted graphs on  $n$  vertices and  $m$  edges, there is a unique maximum value for  $u(G)$ . We call a graph that achieves this maximum *uniformly optimal*. More formally, we make the following definition.

**Definition 5.3.** Let  $\mathcal{G}_{n,m}$  denote the class of all simple graphs with  $n$  vertices and  $m$  edges. Then  $G \in \mathcal{G}_{n,m}$  is *uniformly optimal* if  $u(G) \geq u(H)$  for all  $H \in \mathcal{G}_{n,m}$ . If  $G$  is uniformly optimal, we write  $u_{n,m}$  for  $u(G)$ .

The next proposition collects several consequences of this definition. Recall that a *rooted star* is a rooted tree in which every vertex is adjacent to the root.

**Proposition 5.4.** 1. If  $m < \frac{n(n-1)}{2}$ , then  $u_{n,m} < u_{n,m+1}$ .  
 2. If  $G \in \mathcal{G}_{n,n-1}$ , then  $u(G) \leq \frac{n-1}{2} = u_{n-1,n}$ , with equality iff  $G$  is a rooted star.  
 3. If  $G \in \mathcal{G}_{n,n}$  with no isolated vertices, then  $u(G) \leq \frac{3n-2}{6} = u_{n,n}$ , with equality iff  $G$  is a rooted star with one additional edge.  
 4. Suppose  $G_1, G_2 \in \mathcal{G}_{n,m}$ , and let  $d_i$  be the degree of the root in  $G_i$  ( $i = 1, 2$ ). If  $d_1 < d_2$ , then  $EV(G_1; p) < EV(G_2; p)$  if  $p$  is sufficiently small.

*Proof.*

1. Let  $G$  be a rooted graph on  $n$  vertices and  $m < \frac{n(n-1)}{2}$  edges, and let  $H$  be a rooted graph obtained from  $G$  by adding an edge. Then  $EV(G; p) \leq EV(H; p)$  for all  $0 \leq p \leq 1$ , and  $EV(G; p) < EV(H; p)$  for some  $p$ . The result now follows from the definition of  $u_{n,m}$ .
2. Let  $S_n$  be the rooted star on  $n$  vertices (including the root) and suppose  $G \in \mathcal{G}_{n,n-1}$  is not a rooted star. It is immediate that  $u(S_n) = \frac{n-1}{2}$ . Let  $e_1, \dots, e_m$  be the edges of  $G$  and  $e'_1, \dots, e'_m$  be the edges of  $S_n$ . If  $A$  is any subset of edges of  $G$  with corresponding subset  $A'$  of edges of  $S_n$ , then  $r_G(A) \leq r_{S_n}(A')$ . Further, this inequality will be strict whenever  $A$  does not form a rooted subtree in  $G$ . Thus, by Definitions 2.1 and 5.1,  $u(G) < u(S_n)$ .
3. We let  $G, H_n \in \mathcal{G}_{n,n}$ , where  $G$  has no isolated vertices and  $H_n$  is the rooted star with one additional edge. ( $H_n$  is formed as the direct sum of  $n-3$  edges and a single rooted triangle.) It is straightforward to check that  $EV(H_n; p) = (n-1)p + 2p^2 - 2p^3$ , and so  $u(H_n) = \frac{3n-2}{6}$ . We show  $u(G) \leq \frac{3n-2}{6}$ .

If  $G$  is disconnected, then  $G$  must contain an edge  $e$  which cannot be reached from the root. Let  $e'$  be the unique edge of  $H_n$  that is not adjacent to the root and set up a one-to-one correspondence between the edges of  $G$

and the edges of  $H_n$  so that  $e$  corresponds to  $e'$  (and the remainder of the correspondence is completed arbitrarily). Then, as in the proof of part (2) above, if  $A$  is any subset of edges of  $G$  with corresponding subset  $A'$  of edges of  $H_n$ , we again have  $r_G(A) \leq r_{H_n}(A')$ , and the inequality is strict for  $A = \{e\}$  and  $A' = \{e'\}$ . Thus  $u(G) < u(H_n)$  when  $G$  is disconnected.

Now suppose  $G$  is connected. Then we may assume  $G = G_1 \oplus \cdots \oplus G_k$ , where  $G_1$  has a (unique) cycle and  $G_2, \dots, G_k$  are rooted trees. By part 2 above (and Corollary 2.7), replacing  $G_2 \oplus \cdots \oplus G_k$  by the rooted star  $S_j$  (where  $j$  is the number of vertices of  $G_2 \oplus \cdots \oplus G_k$ ) increases  $u(G)$ . Thus, we may assume each  $G_i$  is a single edge for all  $i \geq 2$ .

We now show that we may further assume  $G_1$  is a rooted cycle. If this is not the case, let  $e_1, \dots, e_l$  are the edges of  $G_1$  which are not in the unique cycle. Then form a graph  $K_1$  as the direct sum of a cycle (the same size as the cycle in  $G_1$ ) and  $l$  single edges. Then the same rank comparison argument as before ( $r_{G_1}(A) \leq r_{K_1}(A')$  for all corresponding subsets of edges  $A$  and  $A'$ ) shows  $u(G_1) < u(K_1)$ .

Finally, it remains to show that  $u(C_k) < u(H_k)$  when  $k > 3$ . A direct calculation shows that  $u(C_4) = \frac{47}{30} < \frac{5}{3} = u(H_4)$ , and  $u(C_{k+1}) - u(C_k) = \frac{2}{k+2} < \frac{1}{2} = u(H_{k+1}) - u(H_k)$  when  $k > 4$ . Thus  $u(C_k) < u(H_k)$  for all  $k > 3$ . If  $u(G) = u(H_n)$ , then equality is forced at each step, and  $G \cong H_n$ .

4. Write  $EV(G_i) = d_i p + p^2 g_i(p)$  for  $i = 1, 2$ , where  $g_i(p)$  are polynomials in  $p$ , as in Proposition 2.2(2). The result is then immediate.

□

Proposition 5.4(2) characterizes uniformly optimal trees. Similar extremal results hold for pair-connected reliability (Theorem 3 of [4]) and unrooted trees (Proposition 4.3 of [3]). Proposition 5.4(3) characterizes uniformly optimal unicyclic graphs. For pair-connected reliability, Siegrist, Amin and Slater (Theorem 4.1 of [14]) show that the optimal unicyclic graphs are *star cycles* in which the root is in a cycle of length  $j$ , where  $j$  varies depending on  $p$ .

Allowing multiple edges in our rooted graphs can allow more flexibility in designing uniformly optimal networks. In particular, let  $R_n$  be the direct sum of one pair of parallel edges with  $n-2$  single edges (so  $R_n$  has  $n$  vertices and edges). Then  $EV(R_n; p) = np - p^2$ , and so  $u(R_n) = \frac{3n-2}{6}$ , as in part (3) of Proposition 5.4. We give the following interpretation for such a graph as a network model. Rather than physically building parallel edges in a network, increase the reliability of a single edge from  $p$  to  $2p-p^2$  (the value for the reliability of a pair of edges). Then expected value calculations in the network will match the calculations in the multigraph. A similar adjustment to other values can be used to extend this interpretation to an arbitrary multigraph.

The restriction of having no isolated vertices in Proposition 5.4(3) is rather mild from a physical viewpoint; it makes no sense at all to build networks with vertices isolated from the root. Nevertheless, it is easy to construct examples of rooted graphs  $G, H \in \mathcal{G}_{n,m}$  in which  $G$  is connected,  $H$  is not connected, but  $u(G) < u(H)$ .

## 6. EDGE FLIPPING AND COUNTEREXAMPLES

Given two connected, rooted graphs  $G, H \in \mathcal{G}_{n,m}$ , how can we quickly decide which graph is ‘better’? The answer depends, of course, on the application under consideration, and what we mean by ‘better’. If  $p$  is small, then the rooted graph

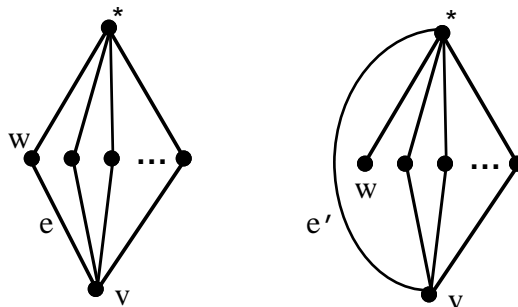


FIGURE 3. Edge flipping

whose root has higher degree is ‘better’ (Proposition 5.4(4)). If we compare  $u(G)$  and  $u(H)$ , we may not wish to select the rooted graph whose vertex has the higher degree.

As a simple example, let  $T_n$  be the rooted star with the root placed at a vertex of degree one, and let  $C_n$  denote the rooted cycle, as before. Then  $C_n$  is better than  $T_n$  when  $p$  is small, but  $u(C_n) \sim 2 \log n$ , while  $u(T_n) \sim \frac{n}{3}$ .

A more interesting set of examples can be constructed as follows. Suppose  $e$  is an edge in the rooted graph  $G$  with endpoints  $v$  and  $w$ , and assume  $e$  is not adjacent to the root. Then replace the edge  $e$  by an edge  $e'$  joining the root to the vertex  $v$ . We call this operation *flipping  $e$  at the vertex  $v$*  and denote the resulting graph  $G(e_v)$ . (Similarly, we also define  $G(e_w)$ .) Edge flipping increases the degree of the root, and so it obviously increases  $EV(G; p)$  when  $p$  is small. Does it also increase  $u(G)$ ? More generally, must any (connected) uniformly optimal graph be such that every vertex is adjacent to the root?

We answer both of these questions negatively in the next example.

**Example 6.1.** Let  $D_n$  be the rooted diamond graph of Figure 3.  $D_n$  has  $n$  vertices and  $2n - 4$  edges. ( $D_n$  is isomorphic to the complete bipartite graph  $K_{2,n-2}$ , with the root at one of the two vertices comprising one of the color classes.) Let  $v$  denote the unique vertex of distance 2 from the root (i.e.,  $v$  is the other vertex in the same color class as the root) and consider the graph  $D_n(e_v)$  obtained from  $D_n$  by flipping any edge  $e$  adjacent to  $v$  at  $v$ . (See Figure 3.)

Table 2 gives the (rounded) values of  $u(D_n)$  and  $u(D_n(e_v))$  for several small values of  $n$ . Note that edge flipping increases the integral for  $n < 9$ , but then edge flipping no longer increases  $u(G)$ . This pattern continues:  $u(D_n) > u(D_n(e_v))$  for all  $n \geq 9$ .

$n$	$u(D_n)$	$u(D_n(e_v))$
4	1.57	1.67
5	2.27	2.33
6	2.96	3
7	3.65	3.67
8	4.32	4.33
9	5.01	5
10	5.68	5.67

TABLE 2.

The preceding example shows that edge flipping does not always increase the value of  $u(G)$ . In fact, more is true:  $D_9$  is the unique uniformly optimal graph on 9 vertices and 14 edges, so  $u_{9,14} = \frac{64.447}{12.870} = 5.00754\dots$ . Thus uniformly optimal graphs need not have the root joined to every other vertex.

A C++ program was written to catalog the uniformly optimal graphs for small values of  $m$  and  $n$  and the values  $u_{m,n}$ . See the web site

<http://www.cs.lafayette.edu/~pattonm/reu/reu.html>

for the optimal graphs and details on the implementation of the program. (We generate all of the optimal graphs on 4, 5, 6, and 7 vertices, the optimal graphs with from 9 to 15 edges and 23 to 27 edges on 8 vertices, and the optimal graphs with between 9 and 13 edges on 9 vertices.)

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