Discrete Mathematics 52 (1984) 51–65
North-Holland

MATROIDS OVER F_p WHICH ARE RATIONAL EXCLUDED MINORS

Gary GORDON

Department of Mathematical Sciences, Williams College, Williamstown, MA 01267, USA

Received 24 June 1983
Revised 3 April 1984

For a given prime p, we construct a collection of 2^p matroids $G_{p,\alpha}$ with (1) $\chi_{pf}(G_{p,\alpha}) = \{p\}$, and (2) $G_{p,\alpha}$ is an excluded minor for rational representability. The motivating construction (Section 2) disproves a conjecture of Reid [4], using relatively high-rank, high-cardinality matroids. The general construction (Section 3) makes use of ordered partitions ($\chi_{pf}(G)$ denotes the prime-field characteristic set of G, i.e., the set of prime fields over which G may be represented, while G can be represented over fields of no other characteristic.) Finally, Section 4 offers another construction with the same properties—a kind of projective dual to Section 2.

1. Introduction

We assume the reader is familiar with the definition of a matroid and some of the basic terminology associated with matroids. An introduction to matroid theory can be found in [1] or [5], e.g.

A matroid M is *representable* or *coordinatizable* over a field F if there is a map $\phi : M \to V$, V an F-vector space, such that I is independent in M iff $|I| = |\phi(I)|$ and $\phi(I)$ is linearly independent in V. We say that $\phi(M)$ is a *coördinatization* or a *representation* of M over F.

The characteristic set $\chi(M)$ of a matroid M is the (possibly empty) set of field characteristics over which M can be represented. This contrasts with the definition of a prime-field characteristic set. A set of field characteristics $\chi_{pf}(M)$ is the *prime-field characteristic set* of a matroid M if, for each characteristic in $\chi(M)$, M can actually be represented over the prime field of that characteristic. Hence every matroid has a characteristic set (i.e., $\chi(M)$ exists for any matroid M) but not every matroid has a prime-field characteristic set. Although this definition may seem unnatural, it has the advantage that general theorems which hold for $\chi(M)$ also hold for $\chi_{pf}(M)$ (since $\chi_{pf}(M)$ is a characteristic set). If we are to define a (weak) prime-field characteristic set of M as simply the set of primes p with M representable over F_p, then there are examples of matroids which would have infinite, but not cofinite such sets (see [1, Exercise 14′ on p. 99]). This weak set seems 'less interesting' than the definition given, since whether or not p is in the prime-field characteristic set of M should depend on intrinsic properties of the characteristic and not simply on the size of F_p. We add, however, that the distinction drawn here is not important in this paper.

The notion of sequential uniqueness will also be used. Two matrices M_1 and M_2 are *projectively equivalent* if $M_2 = N \cdot M_1 \cdot D$ for some nonsingular matrix N and some nonsingular diagonal matrix D. A matrix M representing a matroid G is *projectively unique* if any matrix representing G over the same field is projectively equivalent to M. Finally, a matroid G is *sequentially unique* if there is an ordering of the points of $G = \{p_1, p_2, \ldots, p_n\}$ such that for each $1 \leq i \leq n$, there is a projectively unique matrix M_i representing the submatroid $\{p_1, p_2, \ldots, p_i\}$. More information can be found in [2].

The problem of finding prime-field characteristic sets is closely tied to questions concerning excluded or forbidden minors. A matroid G is an *excluded* or *forbidden minor* for representation over a field F if G cannot be represented over F but any minor of G can be.

For p prime, consider the set of (non-isomorphic) matroids G satisfying:

(1) G is representable over F_p, and

(2) G is a forbidden minor for rational representability.

Reid [4] conjectures that any such G can have at most (approx.) $2p$ points. Our construction in Sections 2 and 4 disprove this conjecture, using higher-rank matroids (rank >3) in a nontrivial way.

2. Basic construction

In [1, pp. 108–110], Brylawski and Reid construct examples of matroids G with the following properties:

(1) $\chi_{pf}(G) = \{p\}$.

(2) G is a forbidden minor for rational representability.

These matroids all have at most $2p+3$ points. It was conjectured that any such matroid can have at most approximately $2p$ points. We disprove that here, giving a collection of examples of matroids satisfying both (1) and (2) with cardinality between $2p+2$ and $4p-4$ and rank between 3 and $p+1$.

Example 2.1. Let $p \geq 5$ be prime and consider the following matrix M_p:

$$\begin{array}{c}
\begin{array}{ccccccccccccccccc} c_1 & \cdots & c_{p+1} & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & & & & a_{3p-5} \end{array} \\
\left[\begin{array}{c|cccccccccccccc}
 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & & 1 & 0 & 1 \\
 & 1 & 0 & 1 & 2 & 0 & 1 & 3 & 0 & 1 & 4 & & 0 & 1 & p-1 \\
I_{(p+1)\times(p+1)} & 1 & 1 & 1 & 3 & 1 & 1 & 4 & 1 & 1 & 5 & \cdots & 1 & 1 & p \\
 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & & & & \\
 & & & & & 0 & 0 & 0 & 1 & 1 & 1 & & \vdots & \vdots & \vdots \\
 & \vdots & & & & & & & 0 & 0 & 0 & & & & \\
 & & & & & & & & & & & & & & \\
 & & & & & & & & & & \vdots & & 0 & 0 & 0 \\
 & 0 & & & & & & & & & 0 & & 1 & 1 & 1
\end{array}\right]
\end{array}$$

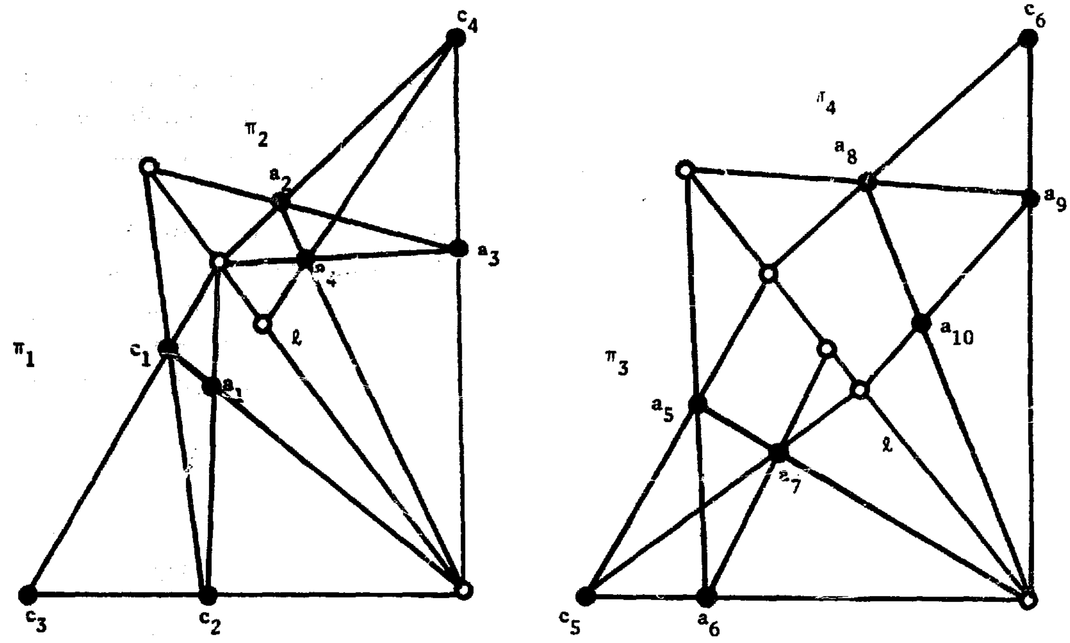

Fig. 1.

We take dependences over F_p, giving us our matroid G_p. This matroid has the following geometric interpretation. Let l be the line determined by the equations $x_1+x_2=x_3$, $x_4=\cdots=x_{p+1}=0$. Each time we add one in our matrix M_p we choose a new plane in a new dimension. There are a total of $p-1$ such planes, $\pi_1,\ldots,\pi_{p-1}$. (See Fig. 1.)

Theorem 2.2. (1) *G_p is a sequentially unique matroid.*
(2) $\chi_{pf}(G_p)=\{p\}$.
(3) *G_p is a forbidden minor for representability over the rationals.*

Before we prove Theorem 2.2, we consider a specific example. Let $v=5$. Then

$$\begin{array}{cc} & \begin{array}{cccccccccccccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \end{array} \\ M_5= & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 3 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 1 & 1 & 4 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

The planes $\pi_i (1\leq i\leq 4)$ are pictured in Fig. 1. The 'empty circles' appearing on l represent points of l in the ambient projective space which are not in the matroid G_5.

The dependence $\{c_1, c_2, c_6, a_{10}\}$ yields a subdeterminant equal to 5. The resulting matroid G_5 has rank 6 and cardinality 16 and $\chi_{pf}(G_5) = \{5\}$.

Part (3) of Theorem 2.2 says that any minor of G_5 can be coordinatized over Q. We demonstrate that fact for deletion by exhibiting a matrix that simultaneously represents $G_5 - x$ over F_5 and over Q. We choose $x = a_3$ for this example; the argument is similar for other choices for x.

Let $M_5 - x$ be the matrix representing $G_5 - x$ where $x = a_3$. We build $M_5 - x$ as follows: We may assume the columns preceding a_3 in M_5 receive the same coordinates in $M_5 - x$. To recoordinatize a_4, the dependences $\{c_2, c_3, a_2, a_4\}$ and $\{c_1, a_1, a_2, a_4\}$ give $a_4^{\mathrm{T}} = [1, y, y+1, 1, 0, 0]$ where y is temporarily indeterminate. No other dependence involving columns preceding (and including) a_4 will determine a value for y, so we turn our attention to π_3 and π_4.

Columns a_5, a_6, a_8 and a_9 will receive the same coordinates as they did in M_5 (their defining dependences did not depend on a_3 or a_4). However, $a_7^{\mathrm{T}} = [1, y+1, y+2, 0, 1, 0]$ in $M_5 - x$ (from the three dependences $\{c_2, c_3, a_5, a_7\}$, $\{c_1, a_1, a_5, a_7\}$ and $\{c_4, a_4, a_6, a_7\}$) and $a_{10}^{\mathrm{T}} = [1, y+2, y+3, 0, 0, 1]$. Thus,

$$
\begin{array}{c}
\begin{array}{cccccccccccccc}
c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & a_1 & a_2 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9
\end{array}\\
M_5 - x = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & y & 0 & 1 & y+1 & 0 & 1\\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & y+1 & 1 & 1 & y+2 & 1 & 1\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 &
\end{bmatrix}.
\end{array}
$$

Now the dependence $\{c_1, c_2, c_6, a_{10}\}$ gives $y+3=0$. Hence if we set $y=-3$ in $M_5 - x$, then we have a coordinatization of $G_5 - x$ over F_5. To see that this is also a coordinatization over Q, we must verify that 5 divides no nonzero subdeterminant in $M_5 - x$. A systematic check of these subdeterminants establishes this, and hence $M_5 - x$ simultaneously represents $G_5 - x$ over the fields F_5 and Q. (In fact, $M_5 - x$ represents $G_5 - x$ over F_p for any $p \geqslant 5$ and G_5 is a forbidden minor for representability over F_p for any $p > 5$.)

An alternate form for the matrix M_p, which is more geometrically suggestive, is given below:

$$
\begin{array}{cccc}
\pi_1 & \pi_2 & \pi_3 & \qquad\qquad \pi_{p-1}
\end{array}
$$

$$
\left[\begin{array}{cccc|cccc|cccc|c|cccc}
0&1&0&1 & 0&1&0&1 & 0&1&0&1 & & 0&1&0&1\\
0&0&1&1 & 0&0&1&2 & 0&0&1&3 & & 0&0&1&p-1\\
1&1&1&1 & 0&0&0&0 & 0&0&0&0 & & 0&0&0&0\\
\multicolumn{4}{c|}{0} & 1&1&1&1 & 0&0&0&0 & \cdots & &\vdots&&\\
 & & & & \multicolumn{4}{c|}{0} & 1&1&1&1 & & & & &\\
 & & & & & & & & \multicolumn{4}{c|}{0} & & & & &\\
 & & & & & & & & & & & & & 0&0&0&0\\
 & & & & & & & & & & & & & 1&1&1&1
\end{array}\right].
$$

A more detailed analysis of deletion and consideration of contraction are included in the proof of Theorem 2.2, which we now give.

Proof of Theorem 2.2. (1) From [2], we may assume the first nonzero entry in each row of the submatrix $[\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{3p-5}]$ and the first non-zero entry in each column of M_p are all equal to one. For each a_i, $1 \leqslant i \leqslant 3p-5$, we list the (less than or equal to) three planes whose intersection determines the remaining coordinates. (If $k<3$ planes appear, then $4-k$ of the entries in column $\boldsymbol{a}_i$ are assumed to be 1 without loss of generality.)

$$\boldsymbol{a}_1: \quad \boldsymbol{c}_1\boldsymbol{c}_2\boldsymbol{c}_3.$$

For $1 \leqslant i \leqslant p-2$,

$$\boldsymbol{a}_{3i-1}: \quad \boldsymbol{c}_1\boldsymbol{c}_3\boldsymbol{c}_{i+3};\ \boldsymbol{c}_2\boldsymbol{c}_{i+3}\boldsymbol{a}_1,$$

$$\boldsymbol{a}_{3i}: \quad \boldsymbol{c}_1\boldsymbol{c}_2\boldsymbol{a}_{3i-1};\ \boldsymbol{c}_1\boldsymbol{c}_{i+3}\boldsymbol{a}_1;\ \boldsymbol{c}_2\boldsymbol{c}_3\boldsymbol{c}_{i+3},$$

$$\boldsymbol{a}_{3i+1}: \quad \boldsymbol{c}_1\boldsymbol{a}_1\boldsymbol{a}_{3i-1};\ \boldsymbol{c}_2\boldsymbol{c}_3\boldsymbol{a}_{3i-1};\ \boldsymbol{c}_{i+2}\boldsymbol{a}_{3i-2}\boldsymbol{a}_{3i}.$$

Hence each column is uniquely determined and (1) is proven.

(2) The dependence $\boldsymbol{c}_1$, $\boldsymbol{c}_2$, $\boldsymbol{c}_{p+1}$, $\boldsymbol{a}_{3p-5}$ gives a subdeterminant equal to p. Thus, these four columns will be dependent over a field F if and only if $\mathrm{char}(F)=p$. Since G_p is sequentially unique, $\chi_{pf}(G_p)=\{p\}$ and we are done.

(3) The proof of (3) is long and not particularly illuminating. It involves checking various subdeterminants of matrices representing G_p-x and G_p/x to verify rational representability. It is included here for completeness.

We must show that any minor of G_p is coordinatizable over Q. We separate into two cases.

Case 1. Deletion. It suffices to show that G_p-x is coordinatizable over Q for each $x \in G_p$. First we delete the column corresponding to x. Then we can coordinatize G_p-x 'as freely as possible' by leaving the columns preceding x unchanged, and recoordinatizing subsequent columns.

Let M_p-x be our general matrix (derived from M_p) representing G_p-x. We now give M_p-x explicitly for each $x \in G_p$. In Table 1, I gives each point x in turn to be deleted from G_p, II gives the form of the columns in M_p, and III gives the equations that the variables introduced in II must satisfy. As usual, the column corresponding to x is deleted from M_p in M_p-x. (In II, the nonzero entries appearing after position 3 are assumed to be in position $i+2$ or $j+2$, as appropriate.)

All the equations in III are satisfied over Q. For example, $x=\boldsymbol{c}_1$ implies $y=p-1$ and $z=1/(p-1)$.

We will show that the matrices M_p-x described above simultaneously represent G_p-x over F_p and over Q. To do this, we systematically examine subdeter-

Table 1

I	II	III
$x=e_1$	$a_1^T=[1,0,0,\dots,0]$ For $2\le i\le p-1$, $a_{3i-4}^T=[1,1,0,\dots,1,\dots,0]$ $a_{3i-3}^T=[0,1,1,0,\dots,y,\dots,0]$ $a_{3i-2}^T=[1,(i-1)z,(i-1)z-1,0,\dots,1,\dots,0]$	 $y-(p-1)=0$ $yz=1$
$x=e_2$	Same as e_1, except permute rows of M_p-x by (1 3 2).	Same as e_1
$x=e_3$	Same as a_1 with rows 1 and 2 switched (see below)	Same as a_1
For $2\le i\le p-1$, $x=e_{i+2}$	$a_{3i-4}^T=[0,0,\dots,1\dots,0]$ $a_{3i-3}^T=[1,-1,0,\dots,1,\dots,0]$ $a_{3i-2}^T=[0,i,i,0,\dots,-1,\dots,0]$ For $i<j\le p-1$, $a_{3j-2}^T=[1,y+j-i-1,y+j-i,0,\dots,1,\dots,0]$	$y-(i+1-p)=0$
$x=a_1$	Same as a_{3i-3} below, where $i=0$ in coordinates	$y-(1-p)=0$
For $2\le i\le p-1$, $x=a_{3i-3}$	For $i\le j\le p-1$, $a_{3j-2}^T=[1,y+j-i,y+j-i+1,0,\dots,1,\dots,0]$	$y-(i-p)=0$
For $2\le i\le p-1$, $x=a_{3i-2}$	Same as a_{3i-3}, where a_{3i-2} does not appear	Same as a_{3i-3}
For $2\le i\le p-1$, $x=a_{3i-4}$	For $i\le j\le p-1$, $a_{3j-2}^T=[1,y+j-1,y+j,0,\dots,1,\dots,0]$	$y-(1-p)=0$
$x=a_{3p-5}$	No change—just delete x from M_p	

minants of M_p-x. Let S be a collection of $p+1$ columns in M_p-x for some x.

Part 1. Suppose the $p+1$ columns of S are unchanged by deletion of x. We show that p divides no nonzero subdeterminant of this form. We also remark that this completely suffices for $x=a_{3p-5}$, since all subdeterminants of M_p-x have this form.

Suppose $d=\det(S)\neq 0$. The columns in M_p with nonzero entries in row $i+2$ ($i>1$) correspond precisely to the points of G_p lying in plane π_i. Since $d\neq 0$, S must contain columns corresponding to points in each plane π_i ($i>1$). Since there are $p-2$ such planes and $r(G_p)=p+1$, there are only 3 other columns in S.

We order the columns in S so that for $i>3$, b_i has a nonzero entry (namely, 1) in row i. If column b_1 has a nonzero entry in row i for $i>3$, then perform the elementary column operation b_1-b_i to replace b_1. Similarly, replace b_2 and b_3 by

b_2-b_j and b_3-b_k respectively if necessary. Now our submatrix looks like this:

$$\begin{array}{c}
\begin{array}{cccccc} b_1 & b_2 & b_3 & \cdots & & b_{p+1} \end{array} \\
\left|\begin{array}{cccccc}
e_1 & e_2 & e_3 & 0 & \cdots & 0 \\
f_1 & f_2 & f_3 & 0 & & \\
 & & & & & \vdots \\
g_1 & g_2 & g_3 & 0 & & \\
0 & 0 & 0 & 1 & & \\
\cdot & \cdot & \cdot & & \cdot & \\
\cdot & \cdot & \cdot & & \cdot & \\
\cdot & \cdot & \cdot & & \cdot & 0 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{array}\right|
\end{array}.$$

To determine the possible values for e_i, f_i and g_i $(1\leq i\leq 3)$, we consider the pairwise differences of appropriate columns in M_p. A computation reveals that the first 3 entries in b_i must be of one of the following forms:

$$\left|\begin{array}{cccccc}
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & k \\
0 & 0 & 1 & 1 & 1 & k+1
\end{array}\right|,$$

where $-1\leq k\leq p-2$. A check of the possible 3×3 subdeterminants generated gives us $|d|<p$.

Thus S is linearly dependent in M_p-x (over Q or over F_p) iff S is linearly dependent in the matrix obtained from M_p by simply deleting the column corresponding to x (over F_p). Since the latter matrix clearly represents G_p-x (over F_p), we have S is linearly dependent in M_p-x (over Q or over F_p) iff the corresponding points of G_p-x are dependent. This completes Part 1.

Part 2. Now suppose S contains columns whose coordinates are changed after x is delected. Then we may assume y or z (or both) appear in S. Again, we show that S is linearly dependent over Q iff the corresponding points in G_p-x are dependent. This amounts to showing that p divides no nonzero subdeterminant of M_p-x and that M_p-x represents G_p-x over F_p.

Let $\det(S)=d$. Since y was chosen to be a root of a certain subdeterminant *over* Q, we get $d\equiv 0 \pmod p$ implies $d=0$. A careful proof of this follows from consideration of the lengthy list of the subdeterminants of M_p-x. In each case, if $d\equiv 0 \pmod p$, then either $d=0$ for reasons having nothing to do with y or for the reason mentioned above. In either case, the column dependences of M_p-x are the same over Q as they are over F_p.

It remains to show that M_p-x represents G_p-x over F_p. To do this, we show that the rational equations which y satisfies are the only restrictions on y arising from the matroid dependences of G_p-x. We examine the k-point circuits in G_p for $k>4$ to show no other equation involving y will arise.

Let C be any 5 point circuit in G_p. Then $C\subseteq \pi_i\cup\pi_j$ for some pair $i\neq j$. To see this, suppose C contains points in 3 different planes π_i, π_j and π_k. Now the points of π_i correspond to the vectors in the matrix M_p with nonzero entries in row $i+2$. Furthermore, there are no vectors in M_p with nonzero entries in any 2 of the 3 rows $i+2$, $j+2$, $k+2$ (for $i, j, k \neq 1$). Hence, for a linear combination of the points of C to be zero, C must include at least 2 points each from π_i, π_j, and π_k (unless, i, j or k equals 1). This is impossible, since C has only 5 points. Now if $i=1$ (say), then C must still contain 2 points each from π_j and π_k. The reader may then verify that this cannot happen either by checking c_1, c_2, c_3, and a_1 in turn as potential members of C. Hence every 5-point circuit is contained in $\pi_i \cup \pi_j$ for some pair $i \neq j$.

But every 5-point subset of $\pi_1 \cup \pi_j$ is dependent (since $r(\pi_i \cup \pi_j) = 4$) and hence no new information can be obtained by examining 5-point circuits.

For even larger circuits, similar arguments prevail. In each case, the information obtained from examining these circuits will already be known.

Thus the matroid $G_p - x$ is represented by the matrix $M_p - x$ over both Q and F_p. This concludes our consideration of Case 1, deletion.

Case 2. Contraction. Again it suffices to show G_p/x is coordinatizable over Q for each $x \in G_p$. As in Case 1, let M_p/x be the general matrix representing G_p/x. We form M_p/x in much the same way we formed $M_p - x$ in Case 1. For full consideration, see Table 2.

For each $x \in G_p$, M_p/x will simultaneously represent G_p/x over F_p and Q. The proof is similar to the one given for deletion, and we omit it. This completes our consideration of Case 2.

Thus, any minor of G_p can be represented over Q. Hence G_p is a forbidden minor for rational representation and (3) is proven. □

Remark 2.3. If we examine the proof of Theorem 2.2 closely, then we see that we could replace the rationals with F_q for any prime $q > p$. Since all the subdeterminants occurring in $M_p - x$ and M_p/x (for any $x \in G_p$) are the products of primes less than p, we have $q \in \chi_{pf}(G_p - x)$ and $q \in \chi_{pf}(G_p/x)$ for any $x \in G_p$. Thus, we can replace (3) in Theorem 2.2 by

(3′) G_p is a forbidden minor for F_q-representability for any prime $q > p$.

We note that (3′) implies (3). This follows from [3, Proposition 3.4], which creates an integer matrix simultaneously representing a matroid H over each prime $q > p$. Then H is represented over Q by this same matrix, and setting $H = G_p - x$ or G_p/x, we have (3).

3. A generalization of G_p

Let α be an ordered partition of $p-1$. We write $p-1 = b_1 + b_2 + \cdots + b_k$, where each $b_i > 0$ and $k > 1$. We associate a matroid with α in the following way. Each

Table 2. Recoordinatizing G_p/x[a]

I	II[b]	III[a]
$x=c_1$	For $1 \leq i \leq p-2$, $a^T_{3i+1}=[1, r_i+1, 0, \dots, r_i, \dots, 0]$	$(1+i-p)r_i=1$
For $1 \leq i \leq p-2$, $x=c_{i+3}$	For $i \leq j \leq p-2$, $a^T_{3i-2}=[1, y, y+1, 0, \dots, 0]$ $a^T_{3j+1}=[1, y+j-i, y+j+1-i, 0, \dots, 1, \dots, 0]$ If $i<p-2$, $a^T_{3i-1}=[1, 0, 1, 0, \dots, 0]$ $a^T_{3i}=[0, 1, 1, 0, \dots, 0]$	$y+(p-1-i)=0$
$x=c_2$	Same as $x=c_{i+2}$, with $i=2$, with 2nd entry of a_{3j-2} deleted	$y+(p-3)=0$
$x=c_3$	For $1 \leq i \leq p-2$, $a^T_{3i}=[0, 1, 0, \dots, y, \dots, 0]$ $a^T_{3i+1}=[1, 1+iz, 0, \dots, 1, \dots, 0]$	$y-(1-p)=0$ $yz=1$
$x=a_1$	For $1 \leq i \leq p-2$, $a^T_{3i-1}=[0, 1, 0, \dots, 1, \dots, 0]$ $a^T_{3i}=[1, 0, \dots, y, \dots, 0]$ $a^T_{3i+1}=[1+iz, 1, 0, \dots, 1, \dots, 0]$	$yz=1$ $y+(p-1)=0$
For $1 \leq i \leq p-2$, $x=a_{3i-1}$	For columns not in π_{i+1}, recoordinatize as in $x=c_{i+3}$ $c^T_{i+3}=[1, 0, 1, 0, \dots, 0]$ $a^T_{3i}=[1, -1, 0, \dots, 0]$ $a^T_{3i+1}=[0, 1, 1, 0, \dots, 0]$	Same as $x=c_{i+3}$
For $1 \leq i \leq p-2$, $x=a_{3i}$	For columns not in π_{i+1}, recoordinatize as in $x=c_{i+3}$ $c^T_{i+3}=[0, 1, 1, 0, \dots, 0]$ $a^T_{3i-1}=[1, -1, 0, \dots, 0]$ $a^T_{3i+1}=[1, y-1, y, \dots, 0]$	Same as $x=c_{i+3}$
For $1 \leq i \leq p-2$, $x=a_{3i+1}$	For columns not in π_{i+1}, recoordinatize as in $x=c_{i+3}$ $c^T_{i+3}=[1, y, y+1, 0, \dots, 0]$ $a^T_{3i-1}=[0, 1, 1, 0, \dots, 0]$ $a^T_{3i}=[1, i, i+1, 0, \dots, 0]$	Same as $x=c_{i+3}$
$x=a_{3p-5}$	$c^T_{p+1}=[1, -1, 0, \dots, 0]$ $a^T_{3p-7}=[0, 1, 1, 0, \dots, 0]$ $a^T_{3p-6}=[1, p-2, p-1, 0, \dots, 0]$	None

[a] As before, all equations in III are satisfied over Q.

[b] Any column which does not appear explicitly in II is recoordinatized by deleting entry j (where $x \in \pi_j$).

[c] Nonzero entries appearing in row $i>3$ are now in row $i-1$ if $i>j+2$ (where $x \in \pi_j$) and remain in row i, otherwise.

block b_i will correspond to a plane π_i (as in Example 2.1) which adds b_i to our running sum.

More precisely, let $d_i = b_1 + \cdots + b_i$, $r = k+2$, and consider the following matrix $M_{p,\alpha}$:

$$\left[\begin{array}{c|ccccccc|cccccccc|c|cccccc}
 & \multicolumn{7}{c}{\overbrace{\hphantom{1\;1\;1\;1\;1\;\cdots\;d_1}}^{\pi_1}} & \multicolumn{8}{c}{\overbrace{\hphantom{1\;0\;d_1+1\;d_1+1\;d_1+2\;d_1+2\;\cdots\;d_2}}^{\pi_2}} & & \multicolumn{6}{c}{\overbrace{\hphantom{1\;0\;d_{k-1}+1\;d_{k-1}+1\;\cdots\;d_k}}^{\pi_k}} \\
 & 1&1&1&1&1& &1 & 1&0&1&1&1&1& &1 & & 1&0&1&1& &1 \\
 & 1&1&2&2&3&\cdots&d_1 & 0&1&d_1+1&d_1+1&d_1+2&d_1+2& &d_2 & & 0&1&d_{k-1}+1&d_{k-1}+1& &d_k \\
I_{r\times r} & 1&2&2&3&3& &d_1 & 1&1&d_1+2&d_1+2&d_1+3&d_1+3&\cdots&d_2+1 & \cdots & 1&1&d_{k-1}+2&d_{k-1}+2&\cdots&d_k+1 \\
 & 0&0&0&0&0& &0 & 1&1&1&0&1&0& &1 & \cdots & 0&0&0&0& &0 \\
 & & & & & & & & 0&0&0&0&0&0& &0 & & \vdots&\vdots&\vdots&\vdots& &\vdots \\
 & & & & & & & & & & & & & & & & & 0&0&0&0& &0 \\
 & & & & & & & & & & & & & & & & & 1&1&1&0& &1
\end{array}\right]$$

Taking dependences of $M_{p,\alpha}$ over the prime p, we get a matroid $G_{p,\alpha}$. As in Section 2, we have nontrivial planes $\pi_1, \ldots, \pi_k$ (one for each block of α). In addition, each block b_i contributes $b_i - 1$ new points to l.

Lemma 3.1. *For α and $G_{p,\alpha}$ described above, we have $r(G_{p,\alpha}) = k+2$ and $|G_{p,\alpha}| = 2(p+k-1)$.*

Proof. By construction of $G_{p,\alpha}$, the number of planes π_i equals k (the number of blocks in the partition α). By construction of $M_{p,\alpha}$, rank equals (number of planes π_i)$+2$. So $r(G_{p,\alpha}) = k+2$.

To compute $|G_{p,\alpha}|$, we just count the columns of the matrix $M_{p,\alpha}$. Each plane π_i contains at least 4 points. Furthermore, each block b_i in α contributes $2(b_i - 1)$ more points to $G_{p,\alpha}$. So

$$\begin{aligned}
|G_{p,\alpha}| &= 4k + 2[(b_1-1) + (b_2-1) + \cdots + (b_k-1)] \\
&= 4k + 2[p-1-k] = 2(p+k-1). \quad \square
\end{aligned}$$

Theorem 3.2. *Let α and $G_{p,\alpha}$ be as above. Then*

(1) *$G_{p,\alpha}$ is a sequentially unique matroid.*

(2) *$\chi_{pf}(G_{p,\alpha})=\{p\}$.*

(3) *$G_{p,\alpha}$ is a forbidden minor for rational representability.*

The proof of Theorem 3.2 is virtually the same as the proof of Theorem 2.2. Verifying (1) and (2) is straightforward, while (3) requires checking subdeterminants of the matrices representing one point deletions and contractions of $G_{p,\alpha}$. We omit the poof.

As an example, let $p=7$ and let $\alpha=(2,1,3)$. Then

$$\begin{array}{rl} & \begin{array}{ccc} \qquad\qquad\pi_1 & \qquad\pi_2 & \qquad\qquad\pi_3 \end{array} \\ M_{7,\alpha}= & \left[\begin{array}{c|ccc|ccc|ccccccc} & 1&1&1 & 1&0&1 & 1&0&1&1&1&1&1\\ & 1&1&2 & 0&1&3 & 0&1&4&4&5&5&6\\ I_{5\times 5} & 1&2&2 & 1&1&4 & 1&1&5&5&6&6&7\\ & 0&0&0 & 1&1&1 & 0&0&0&0&0&0&0\\ & 0&0&0 & 0&0&0 & 1&1&1&0&1&0&1 \end{array}\right] \end{array}$$

Remark 3.3. (a) Let $\alpha=(1,1,\ldots,1)$, the partition of all ones. Then $G_{p,\alpha}=G_p$ from Section 2.

(b) Let $\alpha=(p-1)$, the partition with just one block. Then we add 3 columns to $M_{p,\alpha}$, giving a matroid isomorphic to $L_2(p)$ [1, p. 108], where $L_2(p)$ is the dependence matroid over F_p of the matrix

$$\begin{bmatrix} 1&0&0&1&1&1&0&1&0&1&0& & 1 & 0\\ 0&1&0&1&1&0&1&1&1&1&1&\cdots & 1 & 1\\ 0&0&0&1&0&1&1&2&2&3&3& & p-1 & p-1 \end{bmatrix}.$$

In this case, $r(G_{p,\alpha})=3$ and $|G_{p,\alpha}|=2p+3$.

To see that the matroids that arise from different ordered partitions are not isomorphic, we prove a proposition.

Proposition 3.4. *For each prime p, we have constructed 2^{p-2} pairwise non-isomorphic matroids $G_{p,\alpha}$ (indexed by ordered partitions) satisfying:*

(1) $3\leqslant r(G_{p,\alpha})\leqslant p+1$;

(2) $2p+2\leqslant|G_{p,\alpha}|\leqslant 4p-4$;

(3) $\chi_{pf}(G_{p,\alpha})=\{p\}$; *and*

(4) *$G_{p,\alpha}$ is a forbidden minor for Q-representability.*

Proof. Since there are 2^{p-2} ordered partitions of $p-1$, we need only show that different ordered parititions give different matroids.

Now suppose that $G_{p,\alpha}\sim G_{p,\beta}$ (where '$\sim$' means 'is isomorphic to'). Then α and β must have the same number of blocks, or else $r(G_{p,\alpha})\neq r(G_{p,\beta})$. Suppose

$\alpha = (a_1, a_2, \ldots, a_k)$ and $\beta = (b_1, b_2, \ldots, b_k)$. Further, let $\pi_1, \ldots, \pi_k$ be the nontrivial planes in $G_{p,\alpha}$ and $\tau_1, \ldots, \tau_k$ the nontrivial planes in $G_{p,\beta}$. Then these planes must correspond to each other (in some order) under the isomorphism, since these are the only rank 3 planes in $G_{p,\alpha}$ and $G_{p,\beta}$ containing l (and thus having high enough cardinality).

Claim. $\pi_i \sim \tau_i$ for all $1 \leq i \leq k$.

We first show this is true for $i = 1$. Since both π_1 and τ_1 have two pairs of lines meeting on l and this is not found in any other plane (except π_k and τ_k), we have no choice but $\pi_1 \sim \tau_1$. ($\pi_1 \sim \tau_k$ is not possible since it implies $\pi_2 \sim \tau_1$ which contradicts the previous sentence.)

Now assume $i > 1$ and $\pi_j \sim \tau_j$ for all $j < i$. We show $\pi_i \sim \tau_i$. Since $\pi_{i-1} \sim \tau_{i-1}$, we know that any line in π_{i-1} must correspond to a line in τ_{i-1}. Now consider $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3, y_4\}$ dependent planes in $G_{p,\alpha}$ and $G_{p,\beta}$ respectively, such that $x_1 \sim y_1$, $x_2 \sim y_2$, $x_1x_2 \subseteq \pi_{i-1}$, $y_1y_2 \subseteq \tau_{i-1}$, $x_3x_4 \subseteq \pi_i$, and $y_3y_4 \subseteq \tau_i$ (see Fig. 2). Then $x_3x_4 \sim y_3y_4$ (as dependences are preserved under matroid isomorphism). Hence $\pi_i \sim \tau_i$ (since $\pi_i \sim \tau_j$ for some j) and the claim is proven.

Now $\pi_i \sim \tau_i$ for all $i \Rightarrow a_i = b_i$ for all i. Thus, $\alpha = \beta$ and we are done. $\square$

Recall that the dual G^* of a matroid G will have the same prime-field characteristic set as G and will be a forbidden minor for coordinatization iff G is. Hence, for each $G_{p,\alpha}$ in Proposition 3.4, $G^*_{p,\alpha}$ will also work. So we get

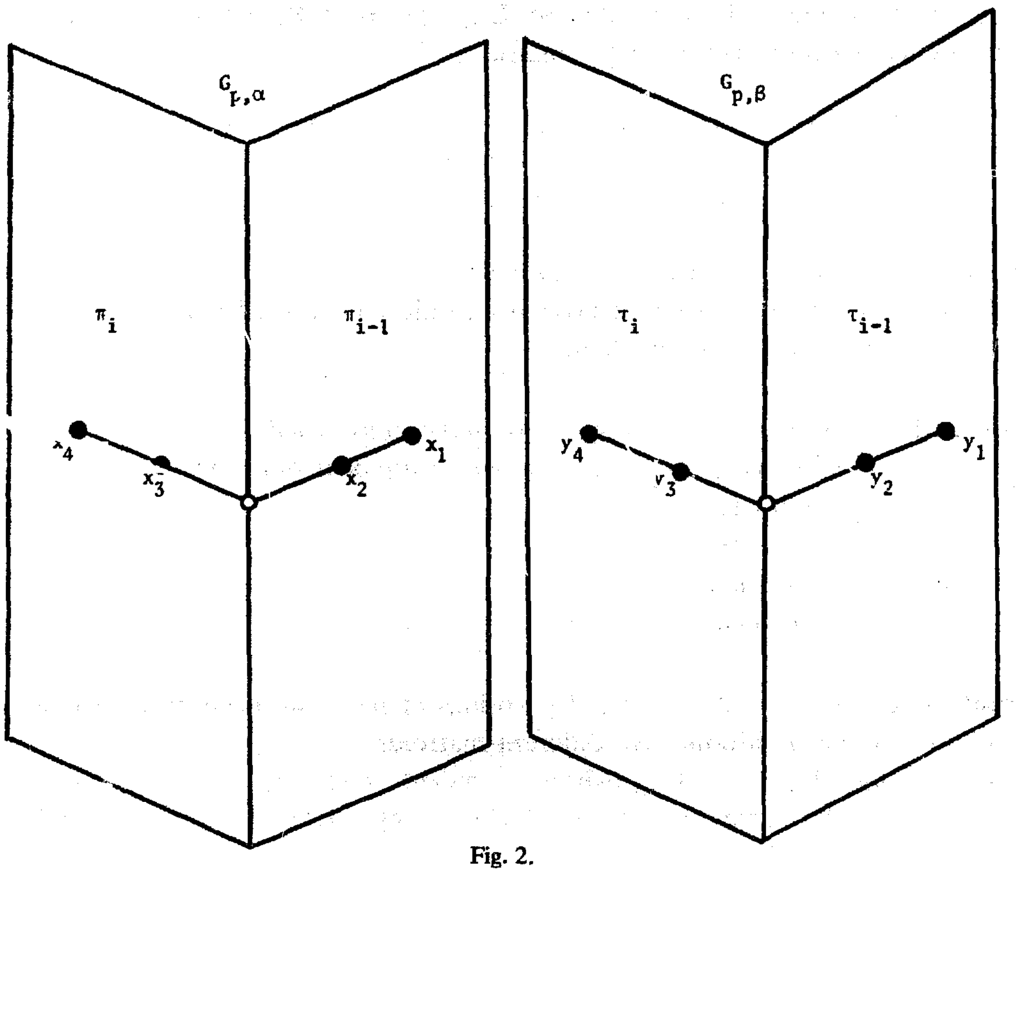

Fig. 2.

Corollary 3.5. *Let $p \geq 5$. Then there are at least 2^{p-1} pairwise non-isomorphic matroids with prime-field set $\{p\}$ that are all forbidden minors for rational representability.*

Proof. We need only check that $G^*_{p,\alpha} \nsim G_{p,\beta}$ for any ordered partitions α and β. Suppose $G^*_{p,\alpha} \sim G_{p,\beta}$. Then $|G^*_{p,\alpha}| = |G_{p,\beta}|$, so α and β must have the same number of blocks, say k. Now $r(G^*_{p,\alpha}) = |G_{p,\alpha}| - r(G_{p,\alpha}) = 2p + k - 4$ from Lemma 3.1. But $r(G^*_{p,\alpha}) = r(G_{p,\beta}) \Rightarrow 2p + k - 4 = k + 2 \Rightarrow p = 3$, which is a contradiction. Hence $G^*_{p,\alpha} \nsim G_{p,\beta}$ and the corollary is proven. $\square$

We note that we can consider the matrix $M_{p,\alpha}$ over characteristic 0. Then we obtain 2^{p-1} matroids $G'_{p,\alpha}$ with

(1) $\chi_{pf}(G'_{p,\alpha}) = \{q \text{ prime} : q > p\} \cup \{0\}$.

(2) $G'_{p,\alpha}$ is an excluded minor for characteristic p.

We note that $G'_{p,\alpha}$ is *not* an excluded minor for F_p, however. For example, $G'_{\alpha,p} - x$ for $x = a_1$ is representable over characteristic p, but when we try to assign a prime-field value to the indeterminate y, we are forced into recreating a subdeterminant equal to p. This problem does not arise over $F_p(y)$, where y is transcendental, since we need never assign y any value.

4. A projective dual construction

We can construct a kind of projective dual to the matroid G_p defined in Section 2. We will describe this matroid here, but we leave the proofs of all the theorems to the interested reader.

In the matroid G_p, we had a collection of planes π_i which shared a common line l. We now define a matroid H_p in which the planes π_i all share a common point Q. (See Fig. 3.)

Let N be the following matrix:

$$
\begin{array}{c}
\begin{array}{ccccccccccccccccccccc}
a_1 & b_1 & c_1 & d_1 & a_2 & b_2 & c_2 & d_2 & a_3 & b_3 & c_3 & d_3 & & a_{p-2} & b_{p-2} & c_{p-2} & d_{p-2} & a_{p-1} & b_{p-1} & c_{p-1} & d_{p-1}
\end{array} \\
\begin{bmatrix}
0 & 1 & 1 & 2 & 0 & 1 & 1 & 3 & 0 & 1 & 1 & 4 & & 0 & 1 & 1 & p-1 & 0 & 1 & 1 & p \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & & & \vdots & & & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & & & & & & \vdots & & \\
 & \vdots & & & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & & & & & & & & & \\
 & & & & & \vdots & & & 0 & 0 & 0 & 0 & & & & & & & & & \\
 & & & & & & & & & \vdots & & & & 0 & 0 & 0 & 0 & & & & \\
 & & & & & & & & & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{bmatrix}
\end{array}
$$

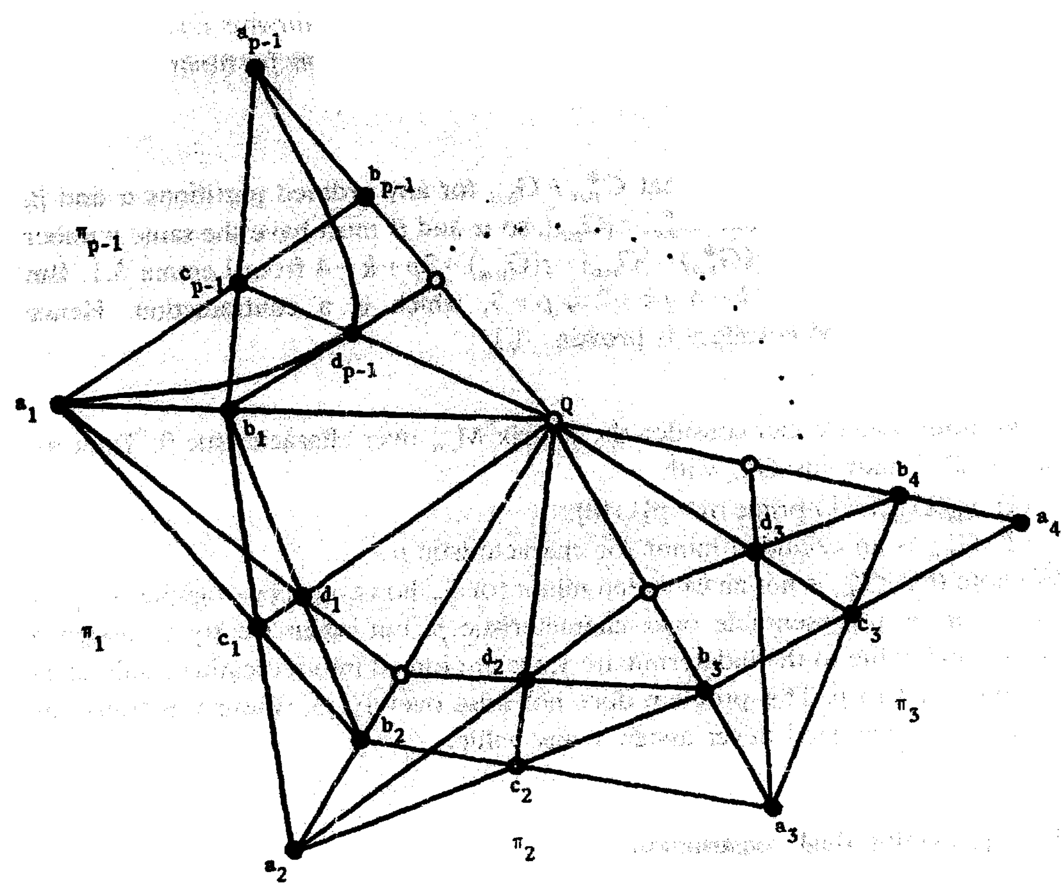

Fig. 3.

Note the dependence $\{a_1, a_{p-1}, d_{p-1}\}$ gives a subdeterminant equal to p. The implied point Q, which has coordinates $[1, 0, \ldots, 0]^T$, is not in H_p.

Theorem 4.1. *Let $p > 3$ be prime.*

(1) $\chi_{pf}(H_p) = \{p\}$.

(2) *H_p is a forbidden minor for rational representation.*

We can imitate the generalization in Section 3 to define $H_{p,\alpha}$ for any ordered partition α of $p-1$. If $\alpha = (b_1, \ldots, b_k)$ each block b_i will add $b_i - 1$ points to the line l_i $(= \pi_i \cap \pi_{i+1})$. We leave the explicit formulation of $H_{p,\alpha}$ to the reader.

Theorem 4.2. *Let $p > 3$ be prime.*

(1) $r(H_{p,\alpha}) = k + 1$.

(2) $|H_{p,\alpha}| = 2(p + k - 1)$.

(3) $\chi_{pf}(H_{p,\alpha}) = \{p\}$.

(4) *$H_{p,\alpha}$ is a forbidden minor for rational representation.*

Finally, combining this construction with Corollary 3.5, we can replace 2^{p-1} by 2^p in Corollary 3.5.

References

[1] T. Brylawski and D. Kelly, Matroids and Combinatorial Geometries, U.N.C. Lecture Note Series (Univ. of North Carolina, Chapel Hill, NC, 1980).

[2] T. Brylawski and D. Lucas, Uniquely representable combinatorial geometries, Proc. Internat. Colloquim in Combin Theory, Rome, Italy (1973).

[3] G. Gordon, Represenations of matroids over prime fields, Ph.D. Thesis, University of North Carolina, Chapel Hill, NC, 1983.

[4] R. Reid, Obstructions to representations of combinatorial geometries (unpublished; appears as Appendix in [1]).

[5] D.J.A. Welsh, Matroid Theory (Academic Press, London, 1976).