Quantum Mechanics

Exam 2019

Section A

1) in Dirac Notation

a) $\psi(x) \rightarrow |\psi\rangle$

b) $\int \Psi^* \hat{A} \Phi \, d\tau = \int (\hat{A}^\dagger \Psi)^* \Phi \, d\tau = \left[\int \Phi^* \hat{A}^\dagger \Psi \, d\tau\right]^*$

$\rightarrow \langle \Psi | \hat{A} | \Phi \rangle = \langle \hat{A}^\dagger \Psi | \Phi \rangle = \langle \Phi | \hat{A}^\dagger | \Psi \rangle^*$

c) $\int \Psi^* \Phi \, d\tau = \langle \Psi | \Phi \rangle$

d)

$|\Psi\rangle = \sum_n c_n |\phi_n\rangle$ and $|\Phi\rangle = \sum_m d_m |\phi_m\rangle$

$c_n = \langle \phi_n | \Psi \rangle$ $\qquad d_m = \langle \phi_m | \Phi \rangle$

so then

$$\int \Psi^* \Phi \, d\tau = \langle \Psi | \Phi \rangle = \sum_n c_n^* \langle \phi_n | \sum_m d_m |\phi_m\rangle$$

$$= \sum_n \sum_m c_n^* d_m \langle \phi_n | \phi_m \rangle$$

$$= \sum_n c_n^* d_n$$

2)

a) The Hamiltonian transforms a state

$$|\psi\rangle = \hat{H}|\chi\rangle \quad (1)$$

when the states may be expressed in terms of their basis $|\psi\rangle = \sum_n c_n |\phi_n\rangle$ and $|\chi\rangle = \sum_m d_m |\phi_m\rangle$

so (1)

$$\langle\phi_n|\psi\rangle = \langle\phi_n|\hat{H}|\chi\rangle$$

$$c_n = \sum_m \langle\phi_n|\hat{H}|\phi_m\rangle d_m$$

so the c_n and d_m may be interpreted as vectors $\underline{c} = \underline{\underline{H}}\,\underline{\underline{d}}$

and $\langle\phi_n|\hat{H}|\phi_m\rangle = H_{mn}$ as the matrix element of the Hamiltonian matrix

so in its eigenbasis $\hat{H} = \begin{pmatrix} E_1 & 0 & \cdots \\ 0 & E_2 & \cdots \\ \vdots & & E_3 & \\ & & & \ddots \end{pmatrix}$

b) TISE $\quad H|\chi_j\rangle = E_n|\chi_j\rangle$

By Linear Algebra. - By solving the characteristic Equation $\det(\underline{H} - \lambda\underline{\underline{I}}) = 0$ for the Hamiltonian we can obtain the eigenvalues of the Matrix which correspond to the energy eigenvalues. The eigenstates / eigenvectors can then also be found by standard linear algebra.

The good thing is that the energy eigenvalues may be found without specifically knowing the eigenstates

3) $$(\Delta A)(\Delta B) \geq \frac{1}{2}\left|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle\right|$$

i) Recall $[\hat{x},\hat{p}] = i\hbar$

then $(\Delta x)(\Delta p) \geq \frac{1}{2}\left|\langle i\hbar\rangle\right|$

$$(\Delta x)(\Delta p) \geq \frac{\hbar}{2}$$

As the commutator of the operators is not 0. The uncertainty of the product of the two measurements is larger than $\frac{\hbar}{2}$.

ii) Recall $[L_x, L_y] = i\hbar L_z$

$$(\Delta L_x)(\Delta L_y) \geq \frac{1}{2}\left|\langle m_l\, l|\, i\hbar L_z |\, l\, m_l\rangle\right|$$

$$\geq \frac{1}{2}\left|\langle m_l, l|\, i\hbar\, m\hbar\, |l, m_l\rangle\right|$$

$$\geq \frac{1}{2} m\hbar^2$$

$$\geq \frac{m\hbar^2}{2}$$

4) $$\hat{H}_0|\psi_n^{(0)}\rangle = E_n^{(0)}|\psi_n^{(0)}\rangle$$

↳ introduce perturbation $\hat{H} = \hat{H}^{(0)} + \lambda\hat{H}'$

a) Assuming the corrections to the energy due to the perturbation are small we may apply perturbation theory.

The basic idea is to expand eigenvalues and eigenstates in terms of powerseries that converge quickly.

$$E_n = \sum_{j}^{\infty} \lambda^{(j)} E_n^{(j)} \quad \text{and} \quad |\psi_n\rangle = \sum_{j=0}^{\infty} \lambda^j |\psi_n^{(j)}\rangle$$

So that then the eigen value eqn

$$\hat{H} | \Psi_n \rangle = E_n | \Psi_n \rangle$$

$$(\hat{H}^{(0)} + \lambda \hat{H}') (| \Psi_n^{(0)} \rangle + \lambda | \Psi_n^{(1)} \rangle + \lambda^2 | \Psi_n^{(2)} \rangle + \dots)$$

$$= (E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots) (| \Psi_n^{(0)} \rangle + \lambda | \Psi_n^{(1)} \rangle + \lambda^2 | \Psi_n^{(2)} \rangle$$

By collecting terms of the same power λ we may obtain corrections to the energy as the eigenvalues.

b) 1st order energy correction

$$E_n^{(1)} = \langle \Psi_0 | \hat{H}' | \Psi_0 \rangle$$

5) $\frac{1}{2}$-spin particles

a) $\hat{S}_1^2 | \Psi \rangle = s_1 (s_1 + 1) \hbar^2 | \Psi \rangle = \underbrace{\frac{3}{4} \hbar^2}_{\text{Eigenvalue}} | \Psi \rangle \quad s_1 = \frac{1}{2}$

$\hat{S}_2^2 | \Psi \rangle = \underline{\underline{\frac{3}{4} \hbar^2}} | \Psi \rangle \qquad s_2 = \frac{1}{2}$

b) $\hat{\underline{S}} = \hat{\underline{S}}_1 + \hat{\underline{S}}_2$

$\hat{S}^2 \rightarrow S = |\frac{1}{2} - \frac{1}{2}| \dots |\frac{1}{2} + \frac{1}{2}| = 0, 1$ //

then for $S = 1 \rightarrow 2\hbar^2$

$S = 0 \rightarrow 0$ //

c) $\hat{H}_{int} = -\frac{\varepsilon}{\hbar^2}\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$

recall $\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = \frac{1}{2}\left(\hat{\vec{S}}^2 - \hat{\vec{S}}_1^2 - \hat{\vec{S}}_2^2\right)$

$\hat{H}_{int} = -\frac{\varepsilon}{2\hbar^2}\left(\hat{\vec{S}}^2 - \hat{\vec{S}}_1^2 - \hat{\vec{S}}_2^2\right)$

so for Triplet $\quad \hat{H}_{int} = -\frac{\varepsilon}{2\hbar^2}\left(2\hbar^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2\right)$

$$= \frac{-\varepsilon}{2\hbar^2}\left(2\hbar^2 - \frac{3}{2}\hbar^2\right)$$

$$= \frac{-\varepsilon}{2}\left(+\frac{1}{2}\right) = +\frac{\varepsilon}{4}$$

for Singlet

$$\hat{H}_{int} = \frac{-\varepsilon}{2\hbar^2}\left(0 - \left(\frac{3}{4}\hbar\right) - \left(\frac{3}{4}\hbar\right)\right)$$

$$= \frac{-\varepsilon}{2\hbar^2}\left(-\frac{3}{2}\hbar^2\right) = \frac{3\varepsilon}{4}$$

so the difference is then

$$E_T - E_S = +\frac{\varepsilon}{4} - \frac{3}{4}\varepsilon = -\varepsilon$$

a)

Bosons are integer spin particles that have wavefcns that are symmetric with respect to exchange of two identical particles such that more than one identical bosons may occupy the same state.

$$\Psi(1,2) = \phi_a(1)\,\phi_b(2) = +\Psi(2,1) = +\phi_a(2)\,\phi(1)$$

Fermions are half integer spin particles that have wavefcns that are antisymmetric with respect to exchange of two identical particles such that no two or more identical fermions may occupy the same state.

$$\Psi(1,2) = \phi_a(1)\,\phi_b(2) = -\Psi(2,1) = -\phi_a(2)\,\phi_b(1)$$

b) "No two fermions may occupy the same quantum state." → Taking the above wavefn as a superposition

$$\Psi = \frac{1}{\sqrt{2}}\left(\phi_a(1)\,\phi_b(2) - \phi_a(2)\,\phi_b(1)\right)$$

If $a=b$ → $\Psi = 0$ → No identical particles may occupy the same state.

Section B

7) $|\Psi\rangle = \sum_n c_n |\Phi_n\rangle$ or $|\Psi\rangle = \sum_m d_m |\chi_m\rangle$

a)
with
$c_n = \langle \Phi_n | \Psi \rangle$ $\qquad d_m = \sum_n \langle \chi_m | \Psi \rangle$

Closure Relation

$$d_m = \sum_n \langle \chi_m | \overbrace{\Phi_n \rangle \langle \Phi_n}^{} | \Psi \rangle$$

$$= \sum_n S_{mn} \; c_n$$

where $S_{mn} = \langle \chi_m | \Phi_n \rangle$ is the element of the similarity transform, so then interprete d_m and c_n as vectors

$$\underline{d} = \underline{\underline{S}} \; \underline{c}$$

b) to transform a operator $\hat{A}$ from one basis $|\Phi_n\rangle$ to another $|\chi_m\rangle$

$$\langle \chi_l | \hat{A} | \chi_m \rangle = \sum_i \sum_j \langle \chi_l | \underbrace{\Phi_i \rangle \langle \Phi_i}_{=1} | \hat{A} | \underbrace{\Phi_j \rangle \langle \Phi_j}_{=1} | \chi_m \rangle$$

$$= \sum_i \sum_j S_{nm} \hat{A}_{ij} S^*_{mn}$$

$$A_\chi = S A_\Psi S^{-1}$$ is the similarity transform.

c) spin $\frac{1}{2}$ system

$$\hat{H} = \varepsilon\hbar\left(\cos(\eta)\hat{\sigma}_z + \sin(\eta)\,\hat{\sigma}_y\right)$$

$$= \varepsilon\hbar\left(\cos(\eta)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \sin(\eta)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right)$$

$$\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \hat{H} = \varepsilon\hbar\begin{pmatrix}\cos\eta & -i\sin\eta\\ i\sin\eta & -\cos\eta\end{pmatrix}$$

$$\hat{\sigma}_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$\hat{H} = \varepsilon\hbar \begin{array}{c|cc} & |\alpha\rangle & |\beta\rangle \\ \hline \langle\alpha| & \cos\eta & -i\sin\eta \\ \langle\beta| & i\sin\eta & -\cos\eta \end{array}$$

solve characteristic eqn

$$0 = \begin{vmatrix}\cos\eta - \lambda & -i\sin\eta\\ i\sin\eta & -\cos\eta - \lambda\end{vmatrix} = -\cos^2\eta - \cancel{\lambda\cos\eta} + \cancel{\lambda\cos\eta} + \lambda^2 + i^2\sin\eta$$

$$0 = \lambda^2 - 1$$

$$1 = \lambda^2 \rightarrow \lambda = \pm 1 \Rightarrow \boxed{E = \pm\varepsilon\hbar}$$

and the eigenvectors $\lambda = 1$

$$\begin{pmatrix}\cos\eta - 1 & -i\sin\eta\\ i\sin\eta & -\cos\eta - 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 1\begin{pmatrix}x\\ y\end{pmatrix} \quad \text{for } \lambda = 1$$

$$(\cos\eta - 1)\,x - y\,i\sin\eta = x \quad \text{I}$$

$$i\sin\eta\,x - (\cos\eta - 1)\,y = y \quad \text{II}$$

for I use $\sin\eta = 2\sin\frac{\eta}{2}\cos\frac{\eta}{2}$

$$\cos\eta\,x - y\,i\left(2\sin\frac{\eta}{2}\cos\frac{\eta}{2}\right) = 0$$

d)

$$-\cancel{y}\, -y\, i\, 2\sin\frac{\eta}{2}\cos\frac{\eta}{2} = x(1-\cos\eta)$$

$$= x\, 2\sin^2\frac{\eta}{2} \qquad (1-\cos\theta) = 2\sin^2\frac{\theta}{2}$$

$$-\frac{y}{x} = \frac{2\sin^2 \eta/2}{2i\sin\frac{\eta}{2}\cos\frac{\eta}{2}}$$

$$\frac{y}{x} = \frac{-2\sin \eta/2}{i\cos\frac{\eta}{2}}$$

$$\frac{y}{x} = \frac{i\sin\eta/2}{\cos\eta/2}$$

so $$|+\rangle = \begin{pmatrix}\cos\eta/2 \\ i\sin\eta/2\end{pmatrix}$$

for $\lambda = -1$

$$(\cos\eta)\, x - y\, i\sin\eta = -x \quad \mathrm{I}$$

$$i\sin\eta\, x - \cos\eta\, y = -y$$

$$\mathrm{I} \quad \cos\eta\, x - y\, i\sin\frac{\eta}{2}\cos\frac{\eta}{2} = -x$$

$$-y2i\sin\frac{\eta}{2}\cos\frac{\eta}{2} = (-1-\cos x)\, x$$

$$1+\cos\theta = 2\cos\frac{\theta}{2} \qquad y2i\sin\frac{\eta}{2}\cos\frac{\eta}{2} = (1+\cos x)\, x$$

$$= 2\cos^2\frac{\theta}{2}\, x$$

so $$\frac{y}{x} = \frac{2\cos^2\frac{\theta}{2}}{i2\sin\frac{\eta}{2}\cos\frac{\eta}{2}} = \frac{\cos\eta/2}{i\sin\frac{\eta}{2}} = \frac{-i\cos\eta/2}{\sin\eta/2}$$

$$|-\rangle = \begin{pmatrix} \sin \eta/2 \\ -i\cos \eta/2 \end{pmatrix} \qquad |+\rangle = \begin{pmatrix} \cos \eta/2 \\ i\sin \eta/2 \end{pmatrix}$$

d) Recall the Similarity Transform $S_{mn} = \langle \pm | \alpha/\beta \rangle$

$$\underline{\underline{S}} = \begin{array}{c|c} & \begin{array}{cc} |\alpha\rangle & |\beta\rangle \end{array} \\ \begin{array}{c} \langle + | \\ \langle - | \end{array} & \begin{pmatrix} \cos \varsigma & -i\sin \xi \\ \sin \xi & +i\cos \xi \end{pmatrix} \end{array}$$

complex conjugate

$$\langle + | \alpha \rangle = \begin{pmatrix} \cos \eta/2 & -i\sin \eta/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\cos \eta/2$$

$$\langle + | \beta \rangle = \begin{pmatrix} \cos \eta/2 & -i\sin \eta/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -i\sin \eta/2$$

$$\langle - | \alpha \rangle = \begin{pmatrix} \sin \eta/2 & +i\cos \eta/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sin \eta/2$$

$$\langle - | \beta \rangle = \begin{pmatrix} \sin \eta/2 & +i\cos \eta/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = +i\cos \eta/2$$

let $\eta/2 = \varsigma$

$S^{-1} = S^{\dagger}$ as it is unitary

$$S^{-1} = \begin{pmatrix} \cos \varsigma & \sin \xi \\ +i\sin \varsigma & -i\sin \xi \end{pmatrix}$$

c)

$|\psi(t=0)\rangle = |\alpha\rangle$ $\qquad \gamma = \xi$

Recall time evolution $|\psi(t)\rangle = |\psi(0)\rangle e^{-iE_n t/\hbar}$

then as we know the eigenvalues in basis $|+\rangle$ and $|-\rangle$ transform $|\alpha\rangle$ to that basis

$$|\psi(t=0)\rangle_{\pm} = S^{-1}|\alpha\rangle = \begin{pmatrix}\cos\delta & -i\sin\delta \\ \sin\delta & i\cos\delta\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

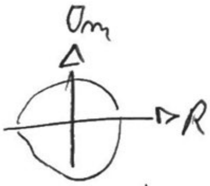
$$|\psi(t=0)\rangle_{\pm} = \cos\gamma\, |+\rangle + \sin\gamma\, |-\rangle$$

then at a later time

$$|\psi(t)\rangle = \cos\gamma\, e^{-i\varepsilon t}|+\rangle + \sin\gamma\, e^{+i\varepsilon t}$$

then at $t' = t + \frac{\pi}{\varepsilon}$

$$|\psi(t')\rangle = \cos\gamma\, e^{-i\varepsilon(t+\frac{\pi}{\varepsilon})}|+\rangle + \sin\gamma\, e^{+i\varepsilon(t+\frac{\pi}{\varepsilon})}$$



$$= e^{-i\pi}\cos\gamma\, e^{-i\varepsilon t}|+\rangle + e^{i\pi}\sin\gamma\, e^{+i\varepsilon t}$$

$$|\psi(t')\rangle = -\cos\gamma\, e^{-i\varepsilon t}|+\rangle - \sin\gamma\, e^{+i\varepsilon t}$$

Now eval

$$\langle\psi(t)|\psi(t')\rangle = \cos\gamma\, e^{+i\varepsilon t}(-\cos\gamma)\, e^{-i\varepsilon t}\langle +|+\rangle + \sin\gamma\, e^{-i\varepsilon t}(-\sin\gamma)\, e^{+i\varepsilon t}$$

$$= -\cos^2\gamma + \sin^2\gamma$$

$$= -(\cos^2\gamma + \sin^2\gamma)$$

$$= -1$$

8) 1-D QHO

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$$

with rasing and lowering operators

$$\hat{a}_\pm = \frac{1}{\sqrt{2}} \left(\gamma \hat{x} \mp \frac{i}{\hbar \gamma} \hat{p} \right) \qquad \gamma = \sqrt{\frac{m\omega}{\hbar}}$$

a) $[a_-, a_+] = a_- a_+ - a_+ a_-$

$$= \frac{1}{2} \left(\gamma \hat{x} + \frac{i}{\hbar \gamma} \hat{p} \right) \left(\gamma \hat{x} - \frac{i}{\hbar \gamma} \hat{p} \right) - \frac{1}{2} \left(\gamma \hat{x} - \frac{i}{\hbar \gamma} \hat{p} \right) \left(\gamma \hat{x} + \frac{i}{\hbar \gamma} \hat{p} \right)$$

$$= \frac{1}{2} \left\{ \cancel{\gamma^2 x^2} - \frac{i}{\hbar} \hat{x} p + \frac{i}{\hbar} \hat{p} x - \cancel{\frac{i^2}{\hbar^2 \gamma^2} \hat{p}^2} \right\}$$

$$- \frac{1}{2} \left\{ \cancel{\gamma^2 \hat{x}^2} + \frac{i}{\hbar} \hat{x} \hat{p} - \frac{i}{\hbar} \hat{p} \hat{x} - \cancel{\frac{i^2}{\hbar^2 \gamma^2} \hat{p}^2} \right\}$$

$$= \frac{1}{2} \left\{ - \frac{i}{\hbar} \hat{x} \hat{p} + \frac{i}{\hbar} \hat{p} x - \frac{i}{\hbar} \hat{x} \hat{p} - \frac{i}{\hbar} \hat{p} x \right\} \qquad [x, p] = i\hbar$$

$$= \frac{i}{2\hbar} \left\{ - \hat{x} \hat{p} + \hat{p} \hat{x} - \hat{x} \hat{p} - \hat{p} \hat{x} \right\}$$

$$= \frac{i}{2\hbar} \left\{ - \{ x \hat{p} - \hat{p} \hat{x} \} + [\hat{p}, \hat{x}] \right\}$$

$$= \frac{i}{2\hbar} \left\{ - [x, p] + [\hat{p}, \hat{x}] \right\}$$

$$= \frac{i}{2\hbar} \left\{ - i\hbar + (-i\hbar) \right\} = \frac{i}{2\hbar} \left\{ -2i\hbar \right\}$$

$= 1$ QED

Recall $N = a_+ a_-$

$$a_+ a_- = \frac{1}{2}\left(\gamma x - \frac{i}{\hbar\gamma}\hat{p}\right)\left(\gamma x + \frac{i}{\hbar\gamma}\hat{p}\right)$$

$$= \frac{1}{2}\left(\gamma^2 x^2 + \frac{i}{\hbar}x\hat{p} - \frac{i}{\hbar}\hat{p}x - \frac{i^2}{\hbar^2\gamma^2}\hat{p}^2\right)$$

$$= \frac{1}{2}\left(\frac{m\omega x^2}{\hbar} + \frac{i}{\hbar}[x,\hat{p}] + \frac{\hat{p}^2\hbar}{\hbar^2 m\omega}\right)$$

$$= \frac{1}{2}\left(\frac{m\omega x^2}{\hbar} + \frac{i}{\hbar}i\hbar + \frac{\hat{p}^2}{\hbar m\omega}\right)$$

$$= \frac{1}{2}\left(\frac{m\omega x^2}{\hbar} - 1 + \frac{\hat{p}^2}{\hbar m\omega}\right)$$

$$a_+ a_- + \frac{1}{2} = \frac{1}{2}\left(\frac{m\omega x^2}{\hbar} + \frac{\hat{p}^2}{\hbar m\omega}\right)$$

$$\hbar\omega\left(a_+ a_- + \frac{1}{2}\right) = \frac{m\omega^2 x^2}{2} + \frac{\hat{p}^2}{2m}$$

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

b) $a_-|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$$N|n\rangle = n\,|n\rangle$$

$$\hat{x}\hat{p} = \frac{1}{\sqrt{2}\gamma}[\hat{a}_+ + a_-]\,\frac{i\hbar\gamma}{\sqrt{2}}[\hat{a}_+ - \hat{a}_-]$$

$$= \frac{i\hbar}{2}\left\{\hat{a}_+^2 - \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ - \hat{a}_-^2\right\}$$

$$= \frac{i\hbar}{2}\left\{\hat{a}_+^2 + [a_-, a_+] - \hat{a}_-^2\right\}$$

$$= \frac{i\hbar}{2}\left\{\hat{a}_+^2 + 1 - \hat{a}_-^2\right\}$$

$$\hat{x}\hat{p}|n\rangle = \frac{i\hbar}{2}\left\{ \hat{a}_+^2|n\rangle + |n\rangle + \hat{a}_-^2|n\rangle \right\}$$

$$= \frac{i\hbar}{2}\left\{ \hat{a}_+\sqrt{n+1}\,|n+1\rangle + |n\rangle + \hat{a}_-\sqrt{n}\,|n-1\rangle \right\}$$

$$= \frac{i\hbar}{2}\left\{ \sqrt{n+2}\sqrt{n+1}\,|n+2\rangle + |n\rangle + \sqrt{n-1}\sqrt{n}\,|n-2\rangle \right\}$$

and

$$\hat{p}\hat{x} = \frac{i\hbar\gamma}{\sqrt{2}}\left[\hat{a}_+ - \hat{a}_-\right]\frac{1}{\sqrt{2}\gamma}\left[a_+ + a_-\right]$$

$$= \frac{i\hbar}{\sqrt{2}}\left[\hat{a}_+^2 + \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+ - \hat{a}_-^2\right]$$

$$= \frac{i\hbar}{2}\left[\hat{a}_+^2 + [a_+, a_-] - \hat{a}^{2}\right]$$

$$= \frac{i\hbar}{2}\left[\hat{a}_+^2 - 1 - \hat{a}^2\right]$$

so

$$\hat{p}\hat{x}|n\rangle = \frac{i\hbar}{2}\left\{ \sqrt{n+2}\sqrt{n+1}\,|n+2\rangle - |n\rangle + \sqrt{n-1}\sqrt{n}\,|n-2\rangle \right\}$$

c) $$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle - \sqrt{\frac{2}{3}}|2\rangle$$

i) $$\langle \hat{p}\hat{x}\rangle = \langle\psi|\hat{p}\hat{x}|\psi\rangle$$

$$= \frac{i\hbar}{2}\Big(-\langle 0|0\rangle\left(\frac{1}{\sqrt{3}}\right)^2 - \sqrt{\frac{2}{3}}\langle 0|2-2\rangle\sqrt{2-1}\sqrt{2}\,\frac{1}{\sqrt{3}} - \left(\sqrt{\frac{2}{3}}\right)^2\langle 2|2\rangle(-1) - \sqrt{\frac{2}{3}}\langle 2|0+2\rangle\sqrt{0+2}\sqrt{0+1}\,\frac{1}{\sqrt{3}}\Big)$$

$$= \frac{i\hbar}{2}\left(-\frac{1}{3} - \frac{2}{3}\cancel{\sqrt{1}} \cancel{+\frac{2}{3}} - \cancel{\frac{2}{3}}\right) = \left(-\frac{1}{3} - \frac{2}{3}\right)\frac{i\hbar}{2} = -\frac{i\hbar}{2}$$

ii) $\langle [\hat{x}, \hat{p}] \rangle = \langle i\hbar \rangle = \langle \psi | i\hbar | \psi \rangle$

$= i\hbar$ //

or Eval Explicitly

iii) Using the above results we showed the well known uncertainty principle.

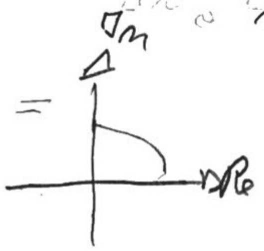
$$(\Delta x)(\Delta p) \geq \frac{1}{2} \left| \langle [x, p] \rangle \right|$$

$\langle n | x p | n \rangle \longrightarrow -\frac{i\hbar}{2} \geq \frac{1}{2} |i\hbar|$

↳ Consistent ?

d) $\gamma = 1$ $\quad \hat{x}_\theta = \frac{1}{\sqrt{2}} [\hat{a}_- e^{+i\theta} + \hat{a}_+ e^{-i\theta}]$

$$\hat{x}_{\theta + \frac{\pi}{2}} = \frac{1}{\sqrt{2}} \left[\hat{a}_- e^{+i(\theta + \frac{\pi}{2})} + \hat{a}_+ e^{-i(\theta + \frac{\pi}{2})} \right]$$



$$= \frac{1}{\sqrt{2}} \left[e^{+i\frac{\pi}{2}} \hat{a}_- e^{-i\theta} + e^{-i\frac{\pi}{2}} \hat{a}_+ e^{-i\theta} \right]$$

$$= \frac{i}{\sqrt{2}} \left[i \hat{a}_- e^{+i\theta} - \hat{a}_+ e^{-i\theta} \right]$$

then Recall that uncertainty in an observable $\hat{A}$:

$$\Delta A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$$

$$\langle \hat{x}_\theta \rangle = \langle 2 | \frac{1}{\sqrt{2}} [\hat{a}_- e^{+i\theta} + \hat{a}_+ e^{-i\theta}] | 2 \rangle$$

$$= \langle 2 | \frac{1}{\sqrt{2}} \left\{ \hat{a}_- | 2 \rangle e^{+i\theta} + \hat{a}_+ | 2 \rangle e^{-i\theta} \right\}$$

$$\langle \hat{x}_\theta \rangle = \langle 2| \frac{1}{\sqrt{2}} \left\{ \sqrt{2}\,|1\rangle e^{+i\theta} + \sqrt{3}\,|3\rangle e^{-i\theta} \right\}$$

$$= 0$$

$$\langle x_{\theta+\frac{\pi}{2}} \rangle = 0$$

$$\langle \hat{x}_\theta^2 \rangle = \langle 2| \hat{x}_\theta^2 |2\rangle$$

$$= \frac{1}{2} \langle 2| \left[a_- e^{i\theta} + a_+ e^{-i\theta} \right]^2 |2\rangle$$

$$= \frac{1}{2} \langle 2| \Big(\underbrace{a_-^2 e^{+2i\theta}}_{\text{yield Nothing}} + \hat{a}_-\hat{a}_+ + \hat{a}_+\hat{a}_- + \underbrace{a_+^2 e^{-2i\theta}}_{\text{yield Nothing}} \Big) |2\rangle$$

$$= \frac{1}{2} \langle 2| \left\{ \hat{a}_-\hat{a}_+ |2\rangle + \hat{a}_+\hat{a}_- |2\rangle \right\}$$

$$= \frac{1}{2} \langle 2| \left\{ \hat{a}_- \sqrt{3}\,|3\rangle + \hat{a}_+ \sqrt{2}\,|1\rangle \right\}$$

$$= \frac{1}{2} \langle 2| \left\{ \sqrt{3}\sqrt{3}\,|2\rangle + \sqrt{2}\sqrt{2}\,|2\rangle \right\}$$

$$= \frac{1}{2} \left\{ 3 + 2 \right\} = \frac{5}{2}$$

$$\langle \hat{x}_{\theta+\frac{\pi}{2}} \rangle = \langle 2| \hat{x}_{\theta+\frac{\pi}{2}} |2\rangle$$

$$= \frac{+i^2}{2} \langle 2| \left[\hat{a}_- e^{+i\theta} - \hat{a}_+ e^{-i\theta} \right]^2 |2\rangle$$

$$= -\frac{1}{2} \langle 2| \Big\{ \underbrace{\hat{a}_-^2 e^{+i\theta}}_{=0} + \hat{a}_-\hat{a}_+ - \hat{a}_-\hat{a}_+ + \underbrace{\hat{a}_+^2 e^{-i\theta}}_{=0} \Big\} |2\rangle$$

$$= -\frac{1}{2} \langle 2| \left\{ \hat{a}_-\hat{a}_+ - \hat{a}_-\hat{a}_+ \right\} |2\rangle$$

$$= \frac{5}{2}$$

so then

$$\Delta x_\theta = \sqrt{\langle x_\theta^2 \rangle - \langle x_\theta \rangle^2}$$

$$= \sqrt{\frac{5}{2} - 0^2} = \sqrt{\frac{5}{2}}$$

$$\Delta x_{\theta+\frac{\pi}{2}} = \sqrt{\langle x_{\theta+\frac{\pi}{2}}^2 \rangle - \langle x_{\theta+\frac{\pi}{2}} \rangle^2}$$

$$= \sqrt{+\frac{5}{2} - 0}$$

$$= \sqrt{+\frac{5}{2}}$$

$$\Delta x_\theta \, \Delta x_{\theta+\frac{\pi}{2}} = \sqrt{\frac{5}{2}} \cdot \sqrt{\frac{5}{2}} = \frac{5}{2}$$

9)

a) $[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$ and $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$

$$\hat{S}_\pm \hat{S}_\mp = (\hat{S}_x \pm i\hat{S}_y)(\hat{S}_x \mp i\hat{S}_y)$$

$$= \hat{S}_x^2 \mp i\hat{S}_x\hat{S}_y \pm i\hat{S}_y\hat{S}_x \mp i^2\hat{S}_y^2$$

$$= \hat{S}_x^2 \mp i[\hat{S}_x, \hat{S}_y] \pm \hat{S}_y^2$$

$$= \hat{S}_x^2 \mp i i\hbar \hat{S}_z \pm \hat{S}_y^2$$

$$= \hat{S}_z^2 \pm \hbar\hat{S}_z \pm S_y^2$$ drop double signs

$$= \hat{S}^2 - \hat{S}_z^2 \pm \hbar\hat{S}_z$$

b)

$$[\hat{S}_z, \hat{S}_\pm] = [\hat{S}_z, \hat{S}_x \pm i\hat{S}_y]$$

$$= [\hat{S}_z, \hat{S}_{\cancel{x}}] \pm i[\hat{S}_z, S_y]$$

$$= i\hbar \hat{S}_{\cancel{y}} \pm i(+ i\hbar S_{\cancel{x}})$$

$$= i\hbar S_{\cancel{y}} \mp i\hbar S_{\cancel{x}}$$

$$= \pm\hbar(\hat{S}_x \mp i\hbar S_y)$$

$$= \pm\hbar\hat{S}_\pm$$

c) $\hat{S}^2 |s, m\rangle = \alpha |s\ m\rangle$

$S_z |s, m\rangle = \beta |s\ m\rangle$

$\hat{S}_+ \hat{S}_z |s\ m\rangle = \beta + n\hbar |s\ m\rangle$ to prove

$$\hat{S}_z \hat{S}_+ |s, m\rangle = (\hbar \hat{S}_+ - \hat{S}_+ S_z) |s\ m\rangle \qquad [S_z, \hat{S}_+] = +\hbar \hat{S}_+$$

$$S_z S_+ - S_+ S_z = \hbar S_+$$

$$= \hbar \hat{S}_+ |s, m\rangle + S_+ \beta |s\ m\rangle \qquad S_z S_+ = \hbar S_+ + S_+ S_z$$

$$= (\hbar + \beta) \hat{S}_+ |s\ m\rangle$$

By Repeated application we would get $+\hbar$ for each time we apply it, leading to $(n\hbar + \beta)$ on the n-th application.

d) $S = 1$

$$\hat{H} = D \hat{S}_z^2 - \mu B \hat{S}_z + C_s (\hat{S}_x^2 - \hat{S}_y^2)$$

we have the states $S = 1 \to m_s = -1, 0, 1$

so $|m_s\rangle \to |-1\rangle, |1\rangle$ $\quad |-1\rangle\ |0\rangle\ |1\rangle$

then

$$\hat{H} = \begin{array}{c|ccc} m_s & & & \\ \langle -1| & & & \\ \langle 0| & & \text{see next page} & \\ \langle 1| & & & \end{array}$$

$\langle m_s' | \hat{H} = D\hat{S}_z^2 - \gamma B \hat{S}_z + C_s(\hat{S}_x^2 - \hat{S}_y^2)$

$$S_x^2 - S_y^2 = \frac{1}{4}\left(S_+ + S_-\right)^2 + \frac{1}{4}\left(S_+ - S_-\right)^2$$
$$= \frac{1}{4}\left\{S_+^2 + \cancel{S_+S_-} + \cancel{S_-S_+} + S_-^2\right\} + \frac{1}{4}\left\{S_+^2 - \cancel{S_+S_-} - \cancel{S_-S_+} + S_-^2\right\}$$
$$= \frac{1}{2}\left\{S_+^2 + S_-^2\right\}$$

So the Hamiltonian becomes

$$\hat{H} = D\hat{S}_z^2 - \gamma B \hat{S}_z + \frac{C_s}{2}\left(S_+^2 + S_-^2\right)$$

Recall

$$S_\pm | s, m_s\rangle = \left[s(s+1) - m_s(m_s \pm 1)\right]^{1/2} | s, m \pm 1\rangle$$

then eval the matrix element as

$$\langle m_s' | \hat{H} | m_s\rangle = \langle m_s' | D S_z^2 - \gamma B S_z + \frac{C_s}{2} S_+^2 + \frac{C_s}{2} S_-^2 | m_s\rangle$$
$$= \langle m_s' | D S_z^2 | m_s\rangle - \langle m_s' | \gamma B S_z | m_s\rangle + \frac{C_s}{2}\langle m_s' | S_+^2 | m_s\rangle + \frac{C_s}{2}\langle m_s' | S_-^2 | m_s\rangle$$
$$= D m_s^2 \hbar^2 \delta_{m_s, m_s'} - \gamma B m_s \hbar \, \delta_{m_s, m_s'} + \frac{C_s}{2} \text{ some factor } \delta_{m_s+2, m_s'} + \frac{C_s}{2} \text{ some factor } \delta_{m_s-2, m_s'}$$

so

$$\hat{H} = \begin{array}{c|ccc} m_s' \backslash m_s & |-1\rangle & |0\rangle & |+1\rangle \\ \hline \langle -1| & & & \\ \langle 0| & & & \\ \langle +1| & & & \end{array} \begin{pmatrix} D\hbar^2 + \gamma B\hbar & 0 & C_s\hbar^2 \\ 0 & 0 & 0 \\ C_s\hbar^2 & 0 & D\hbar^2 - \gamma B\hbar \end{pmatrix}$$

for S_+^2 and S_-^2 recall that by the eign eqn

$$S_+ = \hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow S_+^2 = \hbar^2\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \hbar^2 2$$

also $S_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \rightarrow S_-^2 = \hbar^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \Rightarrow \hbar^2 2$

so the above in the Eqn (some factor)

e)

$\det(\underline{\underline{H}} - \lambda \underline{\underline{I}}) = 0$

$$0 = \begin{vmatrix} D\hbar^2 + \mu B\hbar - \lambda & 0 & C_s\hbar^2 \\ 0 & 0-\lambda & 0 \\ C_s\hbar^2 & 0 & D\hbar^2 - \mu B\hbar - \lambda \end{vmatrix} = (D\hbar^2 + \mu B\hbar - \lambda)\cdot(-\lambda\cdot[D\hbar^2 - \mu B\hbar - \lambda]) + 0 + C_s\hbar^2(0 - (-\lambda)[C_s\hbar^2])$$

$$0 = (D\hbar^2 + \mu B\hbar - \lambda)\cdot(-\lambda D\hbar^2 + \lambda\mu B\hbar + \lambda^2) + C_s^2\hbar^4\lambda$$

$$= -\lambda D^2\hbar^4 + \cancel{\lambda\mu B\hbar^3 D} + \lambda^2 D\hbar^2 - \cancel{\lambda D\hbar^3\mu B} + \lambda\mu^2B^2\hbar^2 + \cancel{\lambda^2\mu B\hbar}$$
$$+ \lambda^2 D\hbar^2 - \cancel{\lambda^2\mu B\hbar} - \lambda^3 + C_s^2\hbar^4\lambda$$

$$= -\lambda D^2\hbar^4 + \lambda^2 D\hbar^2 + \lambda\mu^2B^2\hbar^2 + \lambda^2 D\hbar^2 - \lambda^3 + C_s^2\hbar^4\lambda$$

$$0 = \lambda\hbar^4(C_s^2 - D^2) + 2\lambda^2 D\hbar^2 + \lambda\mu^2B^2\hbar^2 - \lambda^3$$

$$0 = \lambda\left(-\lambda^2 + 2D\hbar\lambda + (\hbar^4(C_s^2 - D^2) + \mu^2B^2\hbar^2)\right)$$

$\boxed{\lambda_0 = 0}$ $\quad 0 = -\lambda^2 + (2D\hbar)\lambda + (\hbar^4(C_s^2 - D^2) + \mu^2B^2\hbar^2)$

$\underline{\underline{E_0 = 0}}$

$$\lambda_{1,2} = \frac{-(2D\hbar^2) \pm \sqrt{(2D\hbar^2)^2 - 4(-1)(\hbar^4(C_s^2 - D^2) + \mu^2B^2\hbar^2)}}{2\cdot(-1)}$$

$$= \frac{-2D\hbar^2 \pm \sqrt{4D^2\hbar^4 + 4\hbar^4(C_s^2 - D^2) + 4\mu^2B^2\hbar^2}}{-2}$$

$$E_{1,2} = \lambda_{1,2} = +D\hbar^2 \pm \sqrt{\hbar^4 C_s^2 + \mu^2B^2\hbar^2}$$

Sketch

$\bar{B}$ when levels equally spaced ?

Recall Eigenvalues

$$E_0 = 0 \qquad E_{1,2} = D\hbar^2 \pm \sqrt{\mu^2 B^2 \hbar^2 + C_s^2 \hbar^4}$$

$$\overset{x_1}{\Delta E_{01}} = \overset{x_2}{\Delta E_{12}}$$

$$D\hbar^2 - \sqrt{\mu^2 B^2 \hbar^2 + C_s^2 \hbar^4} - 0 = D\hbar^2 + \sqrt{\mu^2 B^2 \hbar^2 + C_s^2 \hbar^4} - \left(D\hbar^2 - \sqrt{\mu^2 B^2 \hbar^2 + C_s^2 \hbar^4}\right)$$

$$D\hbar^2 - \sqrt{\mu^2 B^2 \hbar^2 + C_s^2 \hbar^4} = 2\sqrt{\mu^2 B^2 \hbar^2 + C_s^2 \hbar^4}$$

$$\frac{D\hbar^2}{3} = \sqrt{\mu^2 B^2 \hbar^2 + C_s^2 \hbar^4}$$

$$\frac{D^2\hbar^4}{9} = \mu^2 B^2 \hbar^2 + C_s^2 \hbar^4$$

$$\frac{D^2\hbar^4}{9} - C_s^2 \hbar^4 = \mu^2 \hbar^2 B^2$$

$$B^2 = \frac{1}{\mu^2 \hbar^2}\left(\frac{D^2\hbar^4}{9} - C_s^2 \hbar^4\right)$$

$$= \left(\frac{D^2\hbar^2}{9\mu^2} - \frac{C_s^2\hbar^2}{\mu^2}\right)$$

Soln's Wrong

$$B = \frac{\sqrt{D^2\hbar^2 - 9 C_s^2 \hbar^2}}{3\mu}$$

10)

a) i)

"The variational Principle states that the expectation value of the Hamiltonian evaluated in an arbitrary state is allways greater or equal to the ground state energy"

$$\frac{\langle \psi | \hat{H} | \psi \rangle}{\langle \psi | \psi \rangle} \geq E_0$$

ii)

One writes down a "trial wavefn, a wavefn which incoaporates the symmetry and functional parameters of the system dependent on a variational parameter.

The expectation value of this state may be then found and then minimized wrt the variational parameter. This yields the best estimate of the ground state energy for this family of trial wavefn.

For helium one would choose a trial wavefn representing two single-electron wavefns of the form

$$\psi(\underline{r}_1, \underline{r}_2) = \frac{\lambda^3}{\pi} e^{-\lambda(r_1 + r_2)}$$

where λ is the variational parameter particle of m

$V(x) = c|x|$ and $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + c|x|$

trial wavefn $\psi(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{1}{2}\alpha x^2}$ $\alpha > 0$

b)

$$\langle T \rangle = \langle \psi | T | \psi \rangle$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\alpha x^2} \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right) e^{-\frac{1}{2}\alpha x^2}\, dx$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\alpha x^2} \left(-\frac{\hbar^2}{2m}\frac{d}{dx}\right)\left(-\frac{1}{2}\alpha \cdot 2x\right) e^{-\frac{1}{2}\alpha x^2}\, dx$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2} \left(-\frac{\hbar^2}{2m}\right)(-\alpha) \int_{-\infty}^{\infty} e^{-\frac{1}{2}\alpha x^2} \cdot \frac{d}{dx}\, x e^{-\frac{1}{2}\alpha x^2}\, dx$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2} \frac{\alpha\hbar^2}{2m} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\alpha x^2} \left(e^{-\frac{1}{2}\alpha x^2} + \left(-\frac{1}{2}\alpha x \cdot 2\right) x e^{-\frac{1}{2}\alpha x^2}\right) dx$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2} \frac{\alpha\hbar^2}{2m} \int_{-\infty}^{\infty} e^{-\alpha x^2} - \alpha x^2 e^{-\alpha x^2}\, dx$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2} \frac{\alpha\hbar^2}{2m} \int_{-\infty}^{\infty} (1 - \alpha x^2)\, e^{-\alpha x^2}\, dx$$

$$= \cancel{\left(\frac{\alpha}{\pi}\right)^{1/2}} \frac{\alpha\hbar^2}{2m} \frac{1}{2} \cancel{\left(\frac{\pi}{\alpha}\right)^{1/2}}$$

$$= \frac{\alpha\hbar^2}{4m}$$

QED

$$\langle V \rangle = \langle \psi | V | \psi \rangle$$

$$= \int_{-\infty}^{\infty} \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{1}{2}\alpha x^2} c|x| \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{1}{2}\alpha x^2} dx$$

$$= \sqrt{\frac{\alpha}{\pi}}\, c \int_{-\infty}^{\infty} |x|\, e^{-\frac{1}{2}\alpha x^2} dx$$

$$= \sqrt{\frac{\alpha}{\pi}}\, c\, 2 \int_{0}^{\infty} x\, e^{-\frac{1}{2}\alpha x^2} dx$$

$$= \sqrt{\frac{\alpha}{\pi}}\, c\, 2\, \frac{1}{2\alpha} = \frac{c}{\sqrt{\alpha\pi}} \quad \text{QED}$$

c) Apply variational method

The full hamiltonian is

$$\hat{H} = T + V$$

its expectation value in the arbitrary state of the trial wavefn $|\psi\rangle$

$$\langle \hat{H} \rangle = \langle T \rangle + \langle V \rangle$$

$$= \frac{\hbar^2 \alpha}{4m} + \frac{c}{\sqrt{\alpha\pi}}$$

then find value α at which $\langle \hat{H} \rangle$ min

as we minimize

$$\rightarrow 0 = \frac{\partial \langle \hat{H} \rangle}{\partial \alpha} = \frac{\hbar^2}{4m} + \frac{c}{\sqrt{\pi}}\left(\frac{-1}{2}\right)\alpha^{-3/2}$$

$$0 = \frac{\hbar^2}{4m} - \frac{c}{2\sqrt{\pi}}\alpha^{-3/2}$$

$$\frac{c}{2\sqrt{\pi}}\alpha^{-3/2} = \frac{\hbar^2}{4m}$$

$$\alpha^{-3/2} = \frac{2\sqrt{\pi}\hbar^2}{4mc} \rightarrow \alpha^{+3/2} = \frac{4mc}{2\sqrt{\pi}\hbar^2} \Rightarrow \frac{2mc}{\sqrt{\pi}\hbar^2}$$

$$\alpha = \left(\frac{2mc}{\sqrt{\pi}\,\hbar^2}\right)^{2/3} \qquad \text{Alpha at } \langle H \rangle_{min}$$

find $\langle H \rangle_{min}$

$$\langle H \rangle_{min} = \frac{\hbar^2}{4m}\left(\frac{2mc}{\sqrt{\pi}\,\hbar^2}\right)^{2/3} + \frac{c}{\sqrt{\pi}}\left(\left(\frac{2mc}{\sqrt{\pi}\,\hbar^2}\right)^{2/3}\right)^{-1/2}$$

$$= \left(\frac{\hbar^6}{4^3 m^3}\right)^{1/3}\left(\frac{2mc}{\sqrt{\pi}\,\hbar^2}\right)^{2/3} + \frac{c}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}\,\hbar^2}{2mc}\right)^{1/3}$$

$$= \left(\frac{\hbar^3}{8\,m^{3/2}}\frac{2mc}{\sqrt{\pi}\,\hbar^2}\right)^{2/3} + \frac{c}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}\,\hbar^2}{2mc}\right)^{1/3}$$

$$= \left(\frac{\hbar\,c}{4\sqrt{m}\sqrt{\pi}}\right)^{2/3} + \left(\frac{c^3\sqrt{\pi}\,\hbar^2}{\pi^{3/2}\,2mc}\right)^{1/3}$$

$$= \left(\frac{\hbar^2 c^2}{16\,m\,\cancel{\pi}}\right)^{1/3} + \left(\frac{c^2\hbar^2}{\pi\,2m}\right)^{1/3}$$

$$= \left(\frac{\hbar^2 c^2}{2m\pi}\right)^{1/3}\left(\frac{1}{16} + \frac{1}{2}\right)^{1/3} = \left(\frac{\hbar^2 c^2}{m}\right)^{1/3}\left(\frac{1}{2}\left(\frac{1}{2}\right)^{1/3} + \left(\frac{1}{2}\right)^{1/3}\right)$$

$$= \frac{3}{2}\left(\frac{1}{2}\frac{\hbar^2 c^2}{m}\right)^{1/3} = \frac{3}{2}\left(\frac{\hbar^2 c^2}{2m}\right)^{1/3}$$

Sol$\underline{n}$s Wrong

d) Further improvements to find better estimation of Energy?

The success of the variational method resides on the ability of the trial wavefn to incorporate the correct features of the ground state.
The trial wave-fn is rather simple as

$$\psi(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{1}{2}\alpha x^2}$$

Higher power terms and additional variational parameter may improve the result.

More ?

- May 2020