Taller

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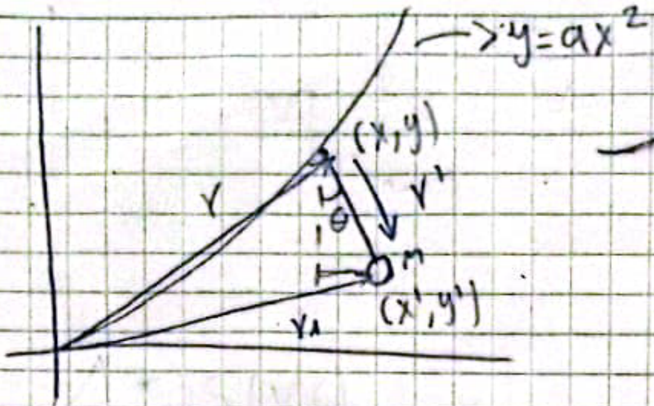
1)



Primero planteamos el Lagrangiano y con el hallamos el Hamiltoniano

$$\vec{r} + \vec{r'} = \vec{r_1}$$

$$(x,y) + (l\,sen\theta, -l\cos\theta) = (x', y')$$

$$(\dot{x}, \dot{y}) + \left(l\cos\theta\dot{\theta}, l\,sen\theta\dot{\theta}\right) = (\dot{x}', \dot{y}')$$

$$L = \frac{1}{2}m\left(\dot{r}_1 \cdot \dot{r}_1\right) - mg\left(y - l\cos\theta\right)$$

$$L = \frac{1}{2}m\left((\dot{x} + l\cos\theta\dot{\theta})^2 + (l\,sen\theta\dot{\theta} + \dot{y})^2\right) - mg\left(y - l\cos\theta\right)$$

Remplazando la ligadura

$$y = ax^2 \quad y \quad \dot{y} = 2ax\dot{x}$$

$$L = \frac{1}{2}m\left((\dot{x} + l\cos\theta\dot{\theta})^2 + (2ax\dot{x} + l\,sen\theta\dot{\theta})^2\right) - mg\left(ax^2 - l\cos\theta\right)$$

Norma

$$L = \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}l\cos\theta\dot{\theta} + l^2\cos^2\theta\dot{\theta}^2 + 4a^2x^2\dot{x}^2 + 4ax\dot{x}l\,\text{sen}\theta\dot{\theta}\right.$$

$$\left. + l^2\text{sen}^2\theta\dot{\theta}^2\right) - mg(ax^2 - l\cos\theta)$$

$$L = \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}l\cos\theta\dot{\theta} + l^2\dot{\theta}^2 + 4a^2x^2\dot{x}^2 + 4ax\dot{x}l\,\text{sen}\theta\dot{\theta}\right) - mg(ax^2 -$$

$$l\cos\theta)$$

Ahora para hallar H primero hallemos $\dot{x}(P_x)$ y $\dot{\theta}(P_\theta)$

$$\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}m\left(2\dot{x} + 2l\cos\theta\dot{\theta} + 8a^2x^2\dot{x} + 4axl\,\text{sen}\theta\dot{\theta}\right) = P_x$$

$$= m\left(\dot{x} + l\cos\theta\dot{\theta} + 4a^2x^2\dot{x} + 2axl\,\text{sen}\theta\dot{\theta}\right) = P_x \quad ①$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m\left(2\dot{x}l\cos\theta + 2l^2\dot{\theta} + 4ax\dot{x}l\,\text{sen}\theta\right) = P_\theta$$

$$= m\left(\dot{x}l\cos\theta + l^2\dot{\theta} + 2ax\dot{x}l\,\text{sen}\theta\right) = P_\theta \quad ②$$

despejando $\dot{\theta}$ de ②

$$\dot{\theta} = \frac{1}{l^2}\left(\frac{P_\theta}{m} - \dot{x}l\cos\theta - 2ax\dot{x}l\,\text{sen}\theta\right) \quad ③$$

Remplazando en ①

$$\dot{x} + \frac{1}{l}\cos\theta\left(\frac{P_\theta}{m} - \dot{x}l\cos\theta - 2ax\dot{x}l\,\text{sen}\theta\right) + 4a^2x^2\dot{x} +$$

$$2ax\,\text{sen}\theta\frac{1}{l}\left(\frac{P_\theta}{m} - \dot{x}l\cos\theta - 2ax\dot{x}l\,\text{sen}\theta\right) = \frac{P_x}{m}$$

$$\dot{x} + \frac{P_\theta \cos\theta}{lm} - \dot{x}\cos^2\theta - 2ax\dot{x}\,\text{sen}\theta\cos\theta + 4a^2x^2\dot{x} + \frac{2ax\,\text{sen}\theta\, P_\theta}{lm}$$

$$-2ax\dot{x}\cos\theta\,\text{sen}\theta - 4a^2x^2\dot{x}\,\text{sen}^2\theta = \frac{P_x}{m}$$

$$\dot{x}\left(1-\cos^2\theta - 4ax\,\text{sen}\theta\cos\theta + 4a^2x^2 - 4a^2x^2\text{sen}^2\theta\right) = \frac{P_x}{m} - \left(\frac{2ax\,\text{sen}\theta\, P_\theta + P_\theta\cos\theta}{lm}\right)$$

$$\boxed{\dot{x} = \frac{\frac{P_x}{m} - \left(\frac{2ax\,\text{sen}\theta\, P_\theta + P_\theta \cos\theta}{lm}\right)}{\left(1-\cos^2\theta - 4ax\,\text{sen}\theta\cos\theta + 4a^2x^2 - 4a^2x^2\text{sen}^2\theta\right)}}$$

Remplazando esto en (3) obtendremos $\dot{\theta}(P_\theta, P_x, x, \theta)$ y podremos aplicar nuestra transformada de Legendre

$$H(P_\theta, P_x, x, \theta) = P_\theta\dot{\theta} + P_x\dot{x} - L$$

2)

$$H = \frac{p^2}{2m} - A\left(\frac{p}{m}\cos rt + rq\,\text{sen}\,rt\right) + \frac{1}{2}Kq^2$$

a) Para hallar el Lagrangiano debemos hallar la transformada de legendre del Hamiltoniano

$$L = p(\dot{q})\dot{q} - H(q, p(\dot{q}), t) \quad \text{donde} \quad \frac{\partial H}{\partial p} = \dot{q}$$

$$\frac{\partial H}{\partial p} = \frac{p}{m} - \frac{A}{m}\cos rt = \dot{q}$$

$$\frac{p}{m} = \dot{q} + \frac{A}{m}\cos rt$$

$$p = m\dot{q} + A\cos rt$$

$$p^2 = (m^2\dot{q}^2 + 2m\dot{q}A\cos(rt) + A^2\cos^2(rt))$$

Remplazando en el L

$$L = (m\dot{q} + A\cos rt)\dot{q} - \frac{1}{2}\left(m\dot{q}^2 + 2\dot{q}A\cos(rt) + \frac{A^2\cos^2(rt)}{m}\right)$$
$$+ A\left(\left(\dot{q} + \frac{A\cos rt}{m}\right)(\cos rt) + rq\,\text{sen}(rt)\right) - \frac{1}{2}Kq^2$$

$$L = m\dot{q}^2 + A\dot{q}\cos(rt) - \frac{1}{2}m\dot{q}^2 - \dot{q}A\cos(rt) - \frac{1}{2}\frac{A^2\cos^2(rt)}{m}$$
$$+ A\dot{q}\cos(rt) + \frac{A^2\cos^2(rt)}{m} + Arq\,\text{sen}(rt) - \frac{1}{2}Kq^2$$

$$L = \frac{1}{2}m\dot{q}^2 + A\dot{q}\cos(rt) + \frac{1}{2}\frac{A^2\cos^2(rt)}{m} + Arq\,\text{sen}(rt) - \frac{1}{2}Kq^2$$

b)

$$L' = L \underbrace{- A\dot{q}\cos(\nu t) - \frac{1}{2}\frac{A^2\cos^2(\nu t)}{M} - A\nu q\,\mathrm{sen}(\nu t)}$$

$$L' = L + \frac{d}{dt}(F)$$

Por ende $L' = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2$

c) El Hamiltoniano asociado a L' esta dado por

$$H' = p\dot{q} - \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2 \qquad \text{con} \quad m\dot{q} = p$$

$$\dot{q} = \frac{p}{m}$$

$$H' = \frac{p^2}{2m} + \frac{1}{2}kq^2$$

No existe una relacion clara entre los Hamiltonianos en este caso, debido a que no es posible determinar la funcion F'

3. $H = q + te^{p}$ ¿La transformación $Q = q + e^{p}$ y $P = p$ es canónica?

$q = Q - e^{p} = Q - e^{P}$

$\tilde{H} = Q - e^{P} + te^{P}$

Para que la transformación sea canónica, hay que verificar que las ec. de Hamilton para el nuevo hamiltoniano se cumplan

$\frac{\partial \tilde{H}}{\partial Q} = -\dot{P}$ $\quad \frac{\partial \tilde{H}}{\partial P} = \dot{Q}$ $\Rightarrow$ Ec de Hamilton

Se verifica

$-\frac{\partial \tilde{H}}{\partial q} = \dot{p} = \dot{P}$

Pero esto a su vez es igual a

$-\frac{\partial \tilde{H}}{\partial Q}\frac{\partial Q}{\partial q} - \frac{\partial \tilde{H}}{\partial P}\frac{\partial P}{\partial q} = -\frac{\partial \tilde{H}}{\partial Q}(1) - \frac{\partial \tilde{H}}{\partial P}\cancel{\frac{\partial P}{\partial q}}^{0} = -\frac{\partial \tilde{H}}{\partial Q}$

$\frac{\partial \tilde{H}}{\partial Q} = \dot{P}$ se cumple

$\frac{\partial \tilde{H}}{\partial p} = \dot{q} = \dot{Q} - e^{p}\dot{p}$

Pero esto a su vez es igual a

$\frac{\partial \tilde{H}}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial \tilde{H}}{\partial P}\frac{\partial P}{\partial p} = \frac{\partial \tilde{H}}{\partial Q}(e^{p}) + \frac{\partial \tilde{H}}{\partial P}(1)$

$= -\cancel{\dot{P}}e^{P} + \frac{\partial \tilde{H}}{\partial P} = \dot{Q} - \cancel{\dot{P}}e^{P}$

$\frac{\partial \tilde{H}}{\partial P} = \dot{Q}$ Se cumple.

Para hallar la función generatriz, se puede plantear una de las 4 funciones clásicas porque únicamente se quieren mantener las ec. de Hamilton para el nuevo hamiltoniano

Se toma $F_2 = F_2(q, P, t)$, de tal manera que se cumpla que:

$$p = \frac{\partial F_2}{\partial q} \qquad Q = \frac{\partial F_2}{\partial P}$$

Pero Q es conocida, $Q = q + e^{P}$ y $P = P$

$$q + e^{P} = \frac{\partial F_2}{\partial P} \quad \text{integrando a cada lado}$$

$$qP + e^{P} = F_2 + f(q) \quad \text{Ahora derivando con respecto a } q$$

$$P = F_2' + f'(q) \qquad f'(q) = 0$$

$$\Rightarrow F_2 = qP + e^{P} \Rightarrow F_{generadora} = \boxed{qP + e^{P} - QP}$$

$$\tilde{H}(Q, P, t) = Q - e^{P} + te^{P}$$

$$\dot{Q} = \frac{\partial \tilde{H}}{\partial P} = -e^{P} + te^{P}$$

$$\dot{P} = -\frac{\partial \tilde{H}}{\partial Q} = -1 \Rightarrow P = = t + cte = t + k$$

$$\dot{Q} = -e^{t+k} + e^{t+k}t$$

$$Q = -\int e^{t+k}\,dt + \int e^{t+k}\,t\,dt \qquad \begin{matrix} u = t+k \\ du = dt \end{matrix}$$

$$\begin{matrix} u = t & du = dt \\ dv = e^{t+k} & v = e^{t+k} \end{matrix}$$

$$Q = -e^{t+k} + te^{t+k} - \int e^{t+k}$$

$$Q = -e^{t+k} + te^{t+k} - e^{t+k} + cte.$$

$$Q = te^{t+k} - 2e^{t+k} + cte$$

4) considere la siguiente transformacion

$$\rho = \frac{\sqrt{P}}{\sqrt{Q}} \quad ① \qquad q = \frac{\sqrt{P}Q^2}{\sqrt{Q}} \quad ②$$

Al remplazarla en el $H(q,\rho,t)$ este queda

$$H = \frac{1}{2}\left(q\rho^3 + \frac{q}{\rho}\right)$$

$$H = \frac{1}{2}\left(\frac{\sqrt{P}Q^2}{\sqrt{Q}}\frac{(\sqrt{P})^3}{(\sqrt{Q})^3} + \frac{\frac{\sqrt{P}Q^2}{\sqrt{Q}}}{\frac{\sqrt{P}}{\sqrt{Q}}}\right)$$

$$H = \frac{1}{2}\left(P^2 + Q^2\right)$$

Ahora debemos probar que la transformacion es canonica

De ② tenemos que

$$q^2 = PQ^3$$

$$③ \quad \boxed{\frac{q^2}{Q^3} = P} \longrightarrow \text{Remplazando en ①}$$

$$\rho^2 = \frac{\frac{q^2}{Q^3}}{\frac{Q}{1}}$$

$$\rho^2 = \frac{q^2}{Q^4}$$

$$\boxed{Q = \pm\frac{q^{1/2}}{\rho^{1/2}}}$$

Remplazando en 3

$$P = \frac{\frac{q^2}{1}}{\pm\frac{q^{3/2}}{\rho^{3/2}}}$$

$$\boxed{P = \pm q^{1/2}\rho^{3/2}}$$

Para que la transformacion sea canonica debe cumplir

$$\{Q,P\}=1$$

$$\{Q,P\}=\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p}-\frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$$

$$=\left(\frac{1}{2}\cancel{q^{-1/2}}\cancel{p^{-1/2}}\right)\left(\frac{3}{2}\cancel{p^{1/2}}\cancel{q^{1/2}}\right)-\left(-\frac{1}{2}\cancel{p^{-3/2}}\cancel{q^{1/2}}\right)\left(\frac{1}{2}\cancel{q^{-1/2}}\cancel{p^{3/2}}\right)$$

$$=\frac{3}{4}+\frac{1}{4}=1$$

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