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1]

a] Para demonstrar que $f(q_1, P_1)$ es una cantidad conservada debemos monstrar que $\{f, H\} = 0$

$$\{f,H\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial P_i} - \frac{\partial f}{\partial P_i}\frac{\partial H}{\partial q_i}\right)$$

$$= \frac{\partial f}{\partial q_1}\frac{\partial H}{\partial P_1} - \frac{\partial f}{\partial P_1}\frac{\partial H}{\partial q_1}$$

$$= \frac{\partial f}{\partial q_1}\frac{\partial H}{\partial f}\frac{\partial f}{\partial P_1} - \frac{\partial f}{\partial P_1}\frac{\partial H}{\partial f}\frac{\partial f}{\partial q_1} = 0$$

b] Para el caso donde el pontencia $V = \frac{\vec{q}\cdot\vec{r}}{r^3}$ con $\vec{q} = q_z\hat{z}$ y $\vec{r} = r\hat{r}$

$$V = \frac{q_z \cos\theta}{r^2} \quad \text{porque} \quad \hat{r}\cdot\hat{z} = \cos\theta$$

Por ende nuestro Lagrangiano en coordenadas esfericas seria

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2\operatorname{sen}^2\theta\right) - \frac{q_z\cos\theta}{r^2}$$

Arora aplicando la trasformada de Legendre para hallar el Hamiltoniano

$$\frac{\partial L}{\partial\dot{\varphi}} = P_\varphi \qquad \frac{\partial L}{\partial\dot{\theta}} = P_\theta \qquad \frac{\partial L}{\partial\dot{r}} = P_r$$

$$mr^2\dot{\varphi}\operatorname{sen}^2\theta = P_\varphi \qquad mr^2\dot{\theta} = P_\theta \qquad m\dot{r} = P_r$$

$$\dot{\varphi} = \frac{P_\varphi}{mr^2\operatorname{sen}^2\theta} \qquad \dot{\theta} = \frac{P_\theta}{mr^2} \qquad \dot{r} = \frac{P_r}{m}$$

Por ende nuestro Hamiltoniano quedaría

$$H = \dot{r}P_r + \dot{\theta}P_\theta + \dot{\varphi}P_\varphi - L$$

$$H = \frac{1}{2m}\left(P_r^2 + \frac{P_\theta^2}{r^2} + \frac{P_\varphi^2}{r^2 \operatorname{sen}^2\theta}\right) + \frac{q_z \cos\theta}{r^2}$$

Note que φ es cíclica por ende las ecuaciones de Hamilton tenemos

$$\dot{P}_\varphi = -\frac{\partial H}{\partial \varphi}$$

$$P_\varphi = C$$

Por ende remplazando esto en nuestro Hamiltoniano

$$H = \frac{1}{2m}\left(P_r^2 + \frac{P_\theta^2}{r^2} + \frac{C}{r^2 \operatorname{sen}^2\theta}\right) + \frac{q_z \cos\theta}{r^2}$$

$$H = \frac{1}{2m}P_r^2 + \frac{1}{r^2}\underbrace{\left(\frac{P_\theta^2}{2m} + \frac{C}{2m\operatorname{sen}^2\theta} + q_z\cos\theta\right)}_{f(\theta, P_\theta)}$$

$$H = \frac{1}{2m}P_r^2 + \frac{1}{r^2}\left(f(\theta, P_\theta)\right)$$

Por el resutado anterior

$$f(\theta, P_\theta) = cte$$

tambien se conserva la Energia ya que $\frac{\partial H}{\partial t} = 0$

1)

a) Para demonstrar que $f_1 = pq - 2Ht$ es una constante debo probar que

$$\frac{df_1}{dt} = \{f_1, H\} + \frac{\partial f_1}{\partial t} = 0 \qquad \text{con } H = \frac{1}{2}p^2 - \frac{1}{2q^2}$$

$$\frac{df_1}{dt} = \frac{\partial f_1}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial f_1}{\partial t}$$

$$= \left(p - 2\frac{\partial H}{\partial q}t\right)p - \left(q - 2\frac{\partial H}{\partial p}t\right)\left(\frac{1}{q^3}\right) - 2H$$

$$= \left(p - 2\left(\frac{1}{q^3}\right)t\right)p - \left(q - 2pt\right)\left(\frac{1}{q^3}\right) - 2H$$

$$= \left(p^2 - \cancel{\frac{2pt}{q^3}} - \frac{1}{q^2} + \cancel{\frac{2pt}{q^3}}\right) - 2H$$

$$= p^2 - \frac{1}{q^2} - p^2 + \frac{1}{q^2} = 0 \quad \blacksquare$$

b) Para mostrar que $Q = \lambda q$ y $P = \lambda^{-1}p$ es canonica aplicamos a parentesis de Poisson

$$\{Q, P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$$

$$= \lambda\lambda^{-1} - 0 = 1 \quad \blacksquare$$

Ahora para que la transformacion sea infinitesimal

$\lambda = (1+\varepsilon)$

$$\boxed{Q = q + \varepsilon q}$$

$$\frac{\partial G}{\partial P}$$

$$P = \frac{1}{1+\varepsilon} p$$

$$P = (1-\varepsilon)p$$

$$\boxed{P = p + \varepsilon(-p)}$$

$$-\frac{\partial G}{\partial q}$$

Noto que

$$q = \frac{\partial G(q,P)}{\partial P} \qquad -p = -\frac{\partial G(q,P)}{\partial q}$$

Ahora bien $f(q,p) = \frac{\partial G}{\partial P} \approx \frac{\partial G}{\partial p} + \cancel{O(\epsilon)}$

Por ende

$$q = \frac{\partial(q,p)}{\partial p} \qquad p = \frac{\partial G(q,p)}{\partial q}$$

Por ende

$$\boxed{G = pq = f_1 + 2Ht}$$

3)

a) Para el caso donde $V = \frac{q_z z}{\sqrt{(x^2+y^2+z^2)^3}}$ el Hamiltoniano

$$H = \frac{1}{2m}\left(P_x^2 + P_y^2 + P_z^2\right) + \frac{q_z z}{\sqrt{(x^2+y^2+z^2)^3}}$$

$$\{\overbrace{xP_x + yP_y + zP_z}^{f}, H\}$$

$$= \sum_{i=1}^{5}\left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial P_i} - \frac{\partial f}{\partial P_i}\frac{\partial H}{\partial q_i}\right)$$

$$= \frac{P_x^2}{m} - x(q_z z)\left(\frac{-3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) + \frac{P_y^2}{m} - y(q_z z)\left(\frac{-3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2y)$$

$$+ \frac{P_z^2}{m} - \frac{q_z\left((x^2+y^2+z^2)^{3/2}\right) - \frac{3}{2}(x^2+y^2+z^2)^{1/2}\cancel{2z}\cdot q_z z}{(x^2+y^2+z^2)^3}$$

$$= \frac{P_x^2}{m} + \frac{P_y^2}{m} + \frac{P_z^2}{m} + \frac{3q_z z x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{3q_z z y^2}{(x^2+y^2+z^2)^{5/2}} - \frac{q_z z}{(x^2+y^2+z^2)^{3/2}}$$

$$+ \frac{3q_z z^2 \cdot z}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{P_x^2}{m} + \frac{P_y^2}{m} + \frac{P_z^2}{m} + 3q_z z\left(\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}\right) - \frac{q_z z}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{P_x^2}{m} + \frac{P_y^2}{m} + \frac{P_z^2}{m} + \frac{2q_z z}{(x^2+y^2+z^2)^{3/2}} = 2H$$

En base a esto podemos proponer la siguiente cantidad

$$f_1 = \vec{P}\cdot\vec{r} - 2Ht$$

$$\frac{df_1}{dt} = \{\vec{P}\cdot\vec{r} - 2Ht, H\} + \frac{\partial f_1}{\partial t}$$

$$= \{\vec{P}\cdot\vec{r}, H\} - 2t\cancel{\{H, H\}}^{0} + \frac{\partial f_1}{\partial t}$$

$$= 2H - 2H = 0$$

Por ende f_1 es una cantidad conservada y la transformación infinitesimal que genera es

$$\boxed{Q_i = q_i + \epsilon \frac{\partial f_1}{\partial P_i}} \qquad \boxed{P_i = P_i - \epsilon \frac{\partial f_1}{\partial q}}$$

con $q_1, q_2, q_3 = x, y, z$

y $P_1, P_2, P_3 = P_x, P_y, P_z$