Homework 4 Solutions

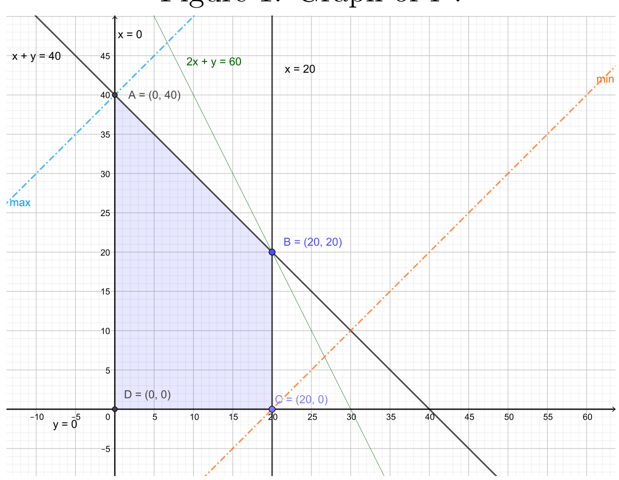
Zheming Gao

September 24, 2017

Problem 1

(a) The graph of P is the following figure.

Figure 1: Graph of P.



(b) Convert P to standard equality form.

$$\begin{cases} x_1 + x_2 + a_3 & & & = 40 \\ 2x_1 + x_2 & +a_4 & & = 60 \\ x_1 & & + a_5 & = 20 \\ x_1, x_2, a_3, a_4, a_5 & & & \geqslant 0. \end{cases}$$

(d)& (f) Basic solutions (in the form of $(x_1, x_2, a_3, a_4, a_5)$):

$$\begin{aligned} &(20, 20, 0, 0, 0)\star, \qquad (B) \\ &(20, 0, 20, 20, 0)\star, \qquad (C) \\ &(30, 0, 10, 0, -10) \\ &(40, 0, 0, -20, -20) \\ &(0, 60, -20, 0, 20) \\ &(0, 40, 0, 20, 20)\star, \qquad (A) \\ &(0, 0, 40, 60, 20)\star, \qquad (D) \end{aligned}$$

(c) Basic feasible solutions are those basic solutions with "$\star$".

(e) $(20, 20)^T$ in P is the extreme point that correspond to degenerate basic feasible solutions.

Problem 2

Proof. First we need to set up the problem. Let the origin LP problem have m variables in it. For its standard form, let the feasible region be $\{Ax = b, x \geqslant 0\}$ where $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$ after adding slack variables.

Now, we know that the number of positive elements in a degenerate basic feasible solution is p and $p < m$. Hence, the number of zero elements in it is $n - p$. It is possible that the corresponding extreme point has all positive entries. In other words, all original variables be positive and all zero entries are on the positions of those $n - m$ slack variables. Since there are $n - p$ zero entries on $n - m$ positions, there will be $C(n - p, n - m)$ different basic feasible solutions at the same time.

Hence, this situation may happen.

□

Problem 3

(a) *Proof.* Let M_c be the convex cone generated by M. Then, $\forall Y \in M_c$, there exists $W \in \mathbb{R}^2_+$, such that $Y = MW$. Since $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we know that $Y = W$. Hence, $M_c = \mathbb{R}^2_+$.

□

Attention: Here M_c is convex and it is easy to show. We will show it is convex in next part.

(b) *Proof.* We need to show that M_c is a convex cone and it is the smallest one that contains $(1, 0)^T$ and $(0, 1)^T$.

It is clear that $(1, 0)^T$ and $(0, 1)^T$ are in M_c, since we can pick $W_1 = (1, 0)^T$ and $W_2 = (0, 1)^T$ from $\mathbb{R}^2_+$ such that $Y_1 = MW_1, Y_2 = MW_2$.

Also, use the definition and easy to show M_c is a cone. $\forall \lambda \geqslant 0$, $Y \in M_c(Y = MW, W \in \mathbb{R}^2_+)$, λY is in M_c since $\lambda Y = M(\lambda W)$ and $\lambda W \in \mathbb{R}^2_+$.

For convexity, take $Y, Z \in M_c$, $\eta \in (0, 1)$. Let $Y = MW$, $Z = MQ$, where $W, Q \in \mathbb{R}^2_+$. Hence, $\eta Y + (1 - \eta)Z = M(\eta W + (1 - \eta)Z) \in M_c$ because $\eta W + (1 - \eta)Z \in \mathbb{R}^2_+$.

How to prove it is smallest? Suppose there is a convex cone $S \subset \mathbb{R}^2$ such that contains $(1, 0)^T$ and $(0, 1)^T$ but $S \subset M_c$.

Then for any $X = (x_1, x_2)^T \in M_c \setminus S$, $X = MX$(recall M is the identity matrix and X is also in $\mathbb{R}^2_+$). Since S is a convex cone, $x_1(1, 0)^T + x_2(0, 1)^T = (x_1, x_2)^T \in S$. This leads to a contradiction. In conclusion, M is the smallest convex cone that contains $(1,0)^T$ and $(0,1)^T$.

$\square$

Problem 4

(1) *Proof.* If $d \in E$, then for any $x^0 \in P$, $x^0 + \lambda d \in P$, for all $\lambda \geqslant 0$. This implies that $Ax^0 = b, x^0 \geqslant 0$, and $A(x^0 + \lambda d) = b, x^0 + \lambda d \geqslant 0$. Eliminate Ax^0 from the last equality and get $\lambda Ad = 0$. Since $\lambda \geqslant$, $Ad = 0$ is proved. Also, $x^0 \geqslant 0$ and $\forall \lambda \geqslant 0$, hence $d \geqslant 0$.

Conversely, we need to show that for $d \in \mathbb{R}^n$, if $d \geqslant 0$ and $Ad = 0$, then $d \in E$. Take d that satisfies the given condition. For any $y \in P$, $Ay = b, y \geqslant 0$. Hence, we will have $A(y + \lambda d) = b$, $\forall \lambda \geqslant 0$. Also, it is true that $y + \lambda d \geqslant 0$, due to the positiveness of y, λ and d. In conclusion, d is a extremal direction of P. $\square$

(2) *Proof.* We need to express E in a form of set.

$$E = \{d \in \mathbb{R}^n | y + \lambda d \in P, \forall \lambda \geqslant 0, y \in P\}.$$

Take any $d \in E$, need to check if $\alpha d \in E$, $\forall \alpha \geqslant 0$. Actually, this is true. For any $y \in P$, $y + \lambda(\alpha d) = y + (\lambda\alpha)d \in P$ because $\lambda\alpha \geqslant 0$. Thus, $\lambda d \in E$, which proves that E is a cone.

$\square$

(3) *Proof.* Take two points $d_1, d_2 \in E$ and $\beta \in (0,1)$. For any $y \in P, \lambda \geqslant 0$,

$$y + \lambda(\beta d_1 + (1-\beta)d_2) = \beta(y + \lambda d_1) + (1-\beta)(y + \lambda d_2).$$

Let $x^1 = y + \lambda d_1$ and $x^2 = y + \lambda d_2$. Hence, the convex combination of x^1 and x^2 are in P since P is a convex polyhedron. This implies that E is convex.

$\square$

Problem 5

(1) We plot the graph of F_3.

(2)
$$B = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 | |x_1| + |x_3| = x_2\}.$$

(3)
$$I = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 | |x_1| + |x_3| < x_2\}.$$

(4) Extreme point: $(0,0,0)^T$.

Vertex: $(0,0,0)^T$.

(5) *Proof.* First, we show that F_3 is a cone. Take $x = (x_1, x_2, x_3)^T \in F_3$ and $\forall \lambda \geqslant 0$, check if $\lambda x \in F_3$.

$$|\lambda x_1| + |\lambda x_3| = \lambda|x_1| + \lambda|x_3| = \lambda(|x_1| + |x_3|) \leqslant \lambda x_2.$$

Hence, λx is in F_3. In conclusion, F_3 is a cone.

Next, we need to show F_3 is convex. Take $x, y \in F_3$ and $\forall \eta \in (0,1)$, $\eta x + (1-\eta)y = (\eta x_1 + (1-\eta)y_1, \eta x_2 + (1-\eta)y_2, \eta x_3 + (1-\eta)y_3)$. Use triangle inequality of absolute value and we get

$$\begin{aligned}
|\eta x_1 + (1-\eta)y_1| + |\eta x_3 + (1-\eta)y_3| &\leqslant \eta|x_1| + (1-\eta)|y_1| + \eta|x_3| + (1-\eta)|y_3| \\
&= \eta(|x_1| + |x_3|) + (1-\eta)(|y_1| + |y_3|) \\
&\leqslant \eta x_2 + (1-\eta)y_2.
\end{aligned}$$

Hence, $\eta x + (1-\eta)y \in F_3$. This implies that F_3 is convex.

□

(6) (Any reasonable answers will be fine for this question.)

Example answer:

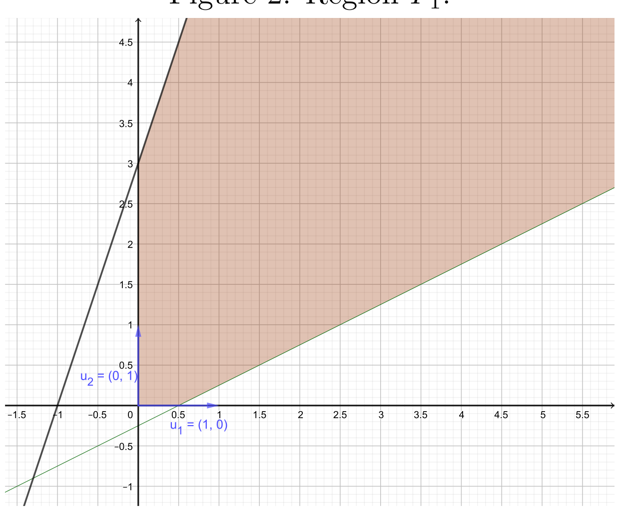
$\mathbb{R}^3_+$ are different from F_3 and $\mathbb{R}^3_+ \cap F_3 \neq \phi$.

$(-1, 2, -1)^T \notin \mathbb{R}^3_+$ but in F_3. $(2, 1, 2)^T \notin F_3$, but in $\mathbb{R}^3_+$.

Problem 6

We plot the region of P_1.

Figure 2: Region P_1.



(a) Convert P_1 to standard equality form.

$$\begin{cases} 2x_1 - 4x_2 + a_1 & = 1 \\ 3x_1 - x_2 \qquad -a_2 & = -3 \\ x_1, x_2, a_1, a_2 & \geqslant 0. \end{cases}$$

(b) Basic solutions (in the form of (x_1, x_2, a_1, a_2)):

$$\begin{aligned} &(-13/10, -9/10, 0, 0) \\ &(-1, 0, 3, 0) \\ &(1/2, 0, 0, 9/2)\star \\ &(0, 3, 13, 0)\star \\ &(0, -1/4, 0, 13/4) \\ &(0, 0, 1, 3)\star \end{aligned}$$

(c) Basic feasible solutions are those basic solutions with "$\star$".

(d) Let $A = \begin{bmatrix} 2 & -4 & 1 & 0 \\ 3 & -1 & 0 & -1 \end{bmatrix}$ and $d = (d_1, d_2, d_3, d_4)^T$. Solve $Ad = 0, d \geqslant 0$,

$$d = w_1 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 3 \end{pmatrix} + w_2 \begin{pmatrix} 0 \\ 1 \\ 4 \\ -1 \end{pmatrix}, \qquad w_1, w_2 \geqslant 0.$$

i.e., the set of all extremal directions is $V = \{d | 0.5d_1 \leqslant d_2 \leqslant 3d_1, d_i \geqslant 0, i = 1, 2, 3, 4\}$.

(e) From the figure, we can see that there are two moving directions to the adjacent points. u_1 is to point $(1/2, 0, 0, 9/2)^T$ and u_2 is to point $(0, 3, 13, 0)^T$.

Method 1:

$$u_1 = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 9/2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} = 1/2 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 3 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 3 \\ 13 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 1 \\ 4 \\ -1 \end{pmatrix}$$

Method 2: Use the inverse of fundamental matrix at $(0, 0, 1, 3)^T$,

$$M^{-1} = \begin{pmatrix} B^{-1} & -B^{-1}N \\ \mathbf{0} & I \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 & 4 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here we need to pay attention to the order of variables. And we will get the two directions to adjacent points by the last two columns of M^{-1}.

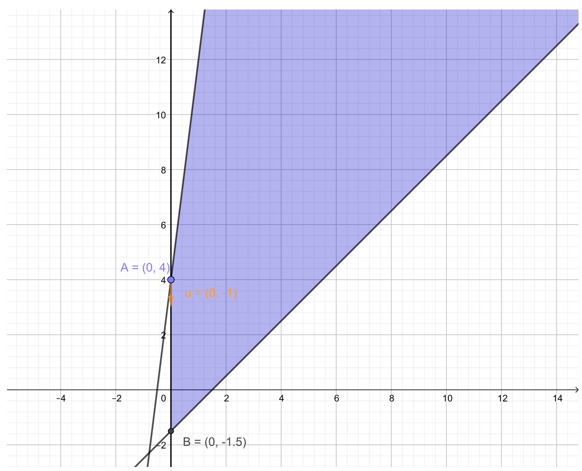
$$u_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 3 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ 4 \\ -1 \end{pmatrix}$$

.

Problem 7

We plot the region of P_2.

Figure 3: Region P_2.



(a) Convert P_2 to standard equality form.

$$\begin{cases} 2x_1 - 2x_2^+ + 2x_2^- + a_1 & = 3 \\ 8x_1 - x_2^+ + x_2^- \qquad\qquad -a_2 & = -4 \\ x_1, x_2^+, x_2^-, a_1, a_2 & \geqslant 0. \end{cases}$$

(b) Basic solutions (in the form of $(x_1, x_2^+, x_2^-, a_1, a_2)$):

$$\begin{aligned} &(-11/14, -16/7, 0, 0, 0) \\ &(-11/14, 0, 16/7, 0, 0) \\ &(-1/2, 0, 0, 4, 0) \\ &(3/2, 0, 0, 0, 16)\star \\ &(0, 4, 0, 11, 0)\star \\ &(0, -3/2, 0, 0, 11/2) \\ &(0, 0, -4, 11, 0) \\ &(0, 0, 3/2, 0, 11/2)\star \\ &(0, 0, 0, 3, 4)\star \end{aligned}$$

(c) Basic feasible solutions are those basic solutions with "$\star$".

(d) Let $A = \begin{bmatrix} 2 & -2 & 2 & 1 & 0 \\ 8 & -1 & 1 & 0 & -1 \end{bmatrix}$ and $d = (d_1, d_2^+, d_2^-, d_3, d_4)^T$. Solve $Ad = 0, d \geqslant 0$, and get

the set of all extremal directions is $V = \{d | d_1 \leqslant d_2^+ - d_2^- \leqslant 8d_1, d_i \geqslant 0\}$.

(e) From the figure or the BFS we got, we can see that there is only one moving directions to the adjacent point $(0, 0, 0, 3, 4)^T$.

$$u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \\ 0 \\ 11 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

Also, we may use M^{-1} to find it.