MA 515-001, Fall 2017, Homework 4

Due: Mon Oct 23, 2017, in-class.

Problem 1. Given $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ two normed spaces, let $T_n : X \to Y$ be bounded linear operators with

$$\|T_n\|_\infty \ < \ M \qquad \forall n \in \mathbb{N}$$

for some constant $M > 0$. Assume that

$$\lim_{n\to\infty} T_n(x) \ = \ T(x) \in Y \qquad \forall x \in X\,.$$

Show that $T : X \to Y$ is a bounded linear operator and

$$\|T\|_\infty \ \leq \ M\,.$$

Problem 2. Recalling that

$$l^\infty \ \left\{x = \{x_i\}_{i\geq 1} \ \middle| \ \{x_i\}_{i\geq 1} \text{ is bounded}\right\}$$

and

$$\|x\|_{l^\infty} \ \doteq \ \sup_{i\geq 1} |x_i|\,.$$

Define the operator $\Lambda : l^\infty \to l^\infty$ such that for any $x \in l^\infty$,

$$\Lambda(x) \ = \ y \ = \ \{y_n\}_{n\geq 1} \qquad \text{with} \qquad y_n \ = \ \frac{x_1 + x_2 + ... + x_n}{n}\,.$$

Show that Λ is bounded and compute $\|\Lambda\|_\infty$.

Problem 3. Let $T : X \to Y$ be a bounded linear operators such that

- T is surjective, i.e., $T(X) = Y$;
- There is a positive constant b such that

$$\|T(x)\| \ \geq \ b \cdot \|x\| \qquad \forall x \in X\,.$$

Show that $T^{-1} : Y \to X$ exists and is a linear bounded operator.

Problem 4. Given $(X, \|\cdot\|_X)$ and $(Y, \|Y\|_Y)$ two normed spaces, let $\{T_n\}_{n\geq 1} \subset B(x, Y)$ converge T in $B(X, Y)$ and let $\{x_n\}_{n\geq 1} \subset X$ converges to x in X. Show that $\{T_n(x_n)\}_{n\geq 1}$ converges to $T(x)$ in Y.

Problem 5. Given $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ normed spaces, let $T : X \to Y$ and $S : Y \to S$ be bounded linear operators. Show that the composition $S \circ T : X \to Z$, denoted by

$$(S \circ T)(x) \ = \ S(T(x)) \qquad \forall x \in X\,,$$

is also be bounded linear operator.

Problem 6. Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a bounded linear operator such that $\|T\|_\infty < 1$. Recalling that $I : X \to X$ such that

$$I(x) \;=\; x \qquad \forall x \in X\,.$$

(a) Use the Contraction Mapping Principle to show that $I - T$ is one-to-one and onto.

(b) Denote by $T^0 = I$ and $T^n = T^{n-1} \circ T$, show that the series $\sum_{n\to 0}^{\infty} T^n$ converges in $(B(X,X), \|\cdot\|_\infty)$.

(c) Show that $\sum_{n=0}^{\infty} T^n = (I-T)^{-1}$.

Problem 7. Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a bounded linear operator.

(a) Show that the series

$$\sum_{n=0}^{+\infty} \frac{T^n}{n!}$$

converges in $(B(X,X), \|\cdot\|_\infty)$

(b) Denote by

$$S \;\doteq\; \sum_{n=0}^{+\infty} \frac{T^n}{n!}\,.$$

Show that

$$\|S\|_\infty \;\le\; e^{\|T\|_\infty}\,.$$