ISE/OR/MA 505 Solution #10

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Textbook Chapter 5

Problem 5.1

Figure **??** shows the graphs of the five functions.

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Figure 1: Problem 5.1

(a) A quadratic algorithm does not necessarily always perform better than a cubic algorithm. It depends the problem size and the coefficient in the complexity. For example, f_4 is not always less than f_2.

(b) A polynomial algorithm does not always perform better than an exponential algorithm for the same reason. For example, f_4 is not always less than f_3.

Problem 5.2

Note that $C(n, m)$ is monotonically increasing as n increases. So we only need to prove $C(2m, m) \geq 2^m$ for any nonnegative integer m. When $m = 0$, $C(0, 0) = 1 = 2^0$. When $m \geq 1$, we have

$$C(2m, m) = \frac{(2m)!}{m!m!} = \frac{(2 \cdot 4 \cdots 2m) \cdot (1 \cdot 3 \cdots (2m-1))}{(1 \cdot 2 \cdots m) \cdot (1 \cdot 2 \cdots m)} = 2^m \cdot \frac{1 \cdot 3 \cdots (2m-1)}{1 \cdot 2 \cdots m} \geq 2^m.$$

Textbook Chapter 7

Problem 7.1

(a) W.l.o.g, we assume that the linear programming problem is in the standard form

$$\min\ c^T x \quad \text{s.t. } Ax = b,\ x \geq 0,$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Given a basis B, the corresponding basic solution can be obtained by solving the following system of linear equations:

$$\begin{pmatrix} B & N \\ 0 & I \end{pmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

Therefore, using Algorithm A, we are able to list all the basic solutions to this LP problem. Then, the LP problem can be solved by finding the basic feasible solution with minimal objective value.

(b) Given a system of linear equations, we construct a linear programming problem whose objective function is constant and feasible domain is defined by those linear equations. Using Algorithm B, we are able to solve the LP problem, i.e., obtain an optimal solution. This optimal solution is feasible, and thus solves the system of linear equations.

(c) From (a) and (b), it is of the same difficulty (computational complexity) to solve systems of linear equations and linear programming problems, respectively. Since the systems of linear equations can be solved in polynomial time, we know that linear programming problems are polynomial-time solvable.

Other problems

Exercise 4

Given an interior point x^k of $\mathbb{R}^n_+$, the affine scaling transformation T_k is defined by

$$T_k(x) = X_k^{-1}x,$$

where $X_k = diag(x^k) \in \mathbb{R}^{n\times n}$.

(1) Since x^k is an interior point of $\mathbb{R}^n_+$, $x_i^k > 0$ holds for $i = 1,\ldots,n$ and thus the matrix X_k^{-1} exists. From the definition of T_k, we have $T_k(x) \in \mathbb{R}^n_+$ as long as $x \in \mathbb{R}^n_+$. Moreover, if $y = T_k(x)$, $\bar{y} = T_k(\bar{x})$ and $y = \bar{y}$, then $x = \bar{x}$ is guaranteed. Therefore, T_k is a well-defined mapping from $\mathbb{R}^n_+$ to $\mathbb{R}^n_+$.

(2) $T_k(x^k) = X_k^{-1}x^k = e$.

(3) If x is a vertex of $\mathbb{R}^n_+$, then $x = 0$. Hence $T_k(x) = 0$ is a vertex of $\mathbb{R}^n_+$.

(4) If x is on the boundary of $\mathbb{R}^n_+$, then $x_i = 0$ for some i. Then $y = T_k(x)$ satisfies $y_i = x_i/x_i^k = 0$. Hence, $T_k(x)$ is on the boundary.

(5) If x is in the interior of $\mathbb{R}^n_+$, then $x_i > 0$ for $i = 1,\ldots,n$. Then $y = T_k(x)$ satisfies $y_i = x_i/x_i^k > 0$ for $i = 1,\ldots,n$. Hence, $T_k(x)$ is in the interior.

(6)
- Since T_k is well defined, it is a one-to-one mapping.
- For each $y \in \mathbb{R}^n_+$, $x = X_k y$ satisfies $x \in \mathbb{R}^n_+$ and $T_k(x) = y$. Hence, T_k is an onto mapping (every y can be mapped onto).
- T_k^{-1} is a well-defined mapping.
- $T_k^{-1} \circ T_k$ is the identity mapping from $\mathbb{R}^n_+$ to $\mathbb{R}^n_+$: $T_k^{-1}(T_k(x)) = T_k^{-1}(X_k^{-1}x) = X_k X_k^{-1}x = x$ for each $x \in \mathbb{R}^n_+$.

Exercise 5

(a) $x = (1,2)^T$. $X = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. $X^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ The affine scaling mapping is $y = X^{-1}x$ while the inverse mapping is $x = Xy$. Then the transformed linear program is

$$\begin{array}{rl} \min & y_1 + 4y_2 \\ \text{s.t.} & 2y_1 + 2y_2 = 4 \\ & y_1, y_2 \geq 0 \end{array}$$

(b)

$$d_y = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \end{pmatrix}\left[(2,2)\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right]^{-1}(2,2)\right]\begin{pmatrix} -1 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -\frac{3}{2} \end{pmatrix}.$$

$$d_x = Xd_y = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \frac{3}{2} \\ -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -3 \end{pmatrix}.$$

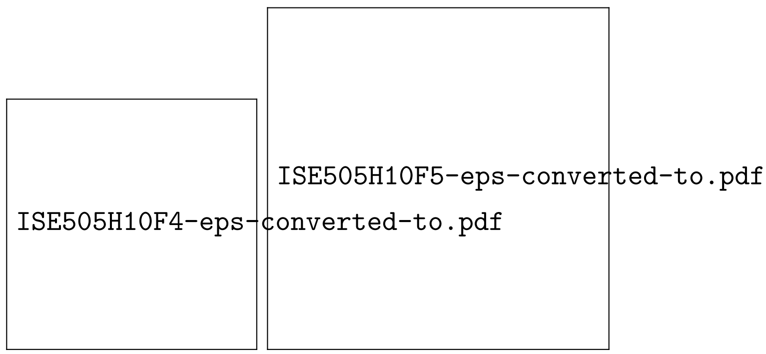

Figure 2: Problem 5.(a,b)

(c) $x = (\frac{1}{3}, \frac{10}{3})^T$. $X = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{10}{3} \end{bmatrix}$. $X^{-1} = \begin{bmatrix} 3 & 0 \\ 0 & \frac{3}{10} \end{bmatrix}$ The affine scaling mapping is $y = X^{-1}x$ while the inverse mapping is $x = Xy$. Then the transformed linear program is

$$\begin{array}{ll} \min & -\frac{1}{3}y_1 - \frac{20}{3}y_2 \\ \text{s.t.} & \frac{2}{3}y_1 + \frac{10}{3}y_2 = 4 \\ & y_1, y_2 \geq 0 \end{array}$$

$$d_y = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{10}{3} \end{pmatrix}\left[(\frac{2}{3}, \frac{10}{3})\begin{pmatrix} \frac{2}{3} \\ \frac{10}{3} \end{pmatrix}\right]^{-1}(\frac{2}{3}, \frac{10}{3})\right]\begin{pmatrix} -\frac{1}{3} \\ -\frac{20}{3} \end{pmatrix} = \begin{pmatrix} \frac{25}{26} \\ -\frac{5}{26} \end{pmatrix}.$$

$$d_x = Xd_y = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{10}{3} \end{pmatrix}\begin{pmatrix} \frac{25}{26} \\ -\frac{5}{26} \end{pmatrix} = \begin{pmatrix} \frac{25}{78} \\ -\frac{50}{78} \end{pmatrix}.$$

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Figure 3: Problem 5.(c)