MA 515-001, Fall 2017, Practice Problems

Problem 1. Let K be a closed subset of metric space X. For any $x \in X$, the distance from x to K is defined as

$$d_K(x) \;=\; \inf_{w \in K} d(x, w)\,.$$

Show that x belongs to K if and only if $d_K(x) = 0$.

Problem 2. Given a metric space (X, d) and a nonempty subset B if X, *the distance from a point $x \in X$ to K* is defined as

$$d_K(x) \;=\; \inf_{w \in K} \; d(x, w)\,.$$

Show that d_K is a Lipschitz function with a Lipschitz constant 1, i.e.,

$$\Big| d_K(x) - d_K(y) \Big| \;\leq\; d(x, y) \qquad \forall x, y \in X\,.$$

Problem 3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Prove that the Cartersian product

$$X \times Y \;=\; \{(x, y) \mid x \in X, y \in Y\}$$

is also a Banach space, with norm

$$\|(x, y)\| \;=\; \max\{\|x\|_X, \|y\|_Y\} \qquad \forall (x, y) \in X \times Y\,.$$

Problem 4. Let (X, d) be a *compact* metric space and a map $T : X \to X$ such that

$$d(T(x), T(y)) \;<\; d(x, y) \qquad \forall x, y \in X\,.$$

Show that T has a fixed point.

Problem 5. Let $f : \mathbb{R} \to [0, 1]$ be a contractive map. Using Banach contraction principle to show that the equation

$$e^{f(x)} \;=\; 4x$$

has a unique solution.

Problem 6. If (X, d) is complete, show that $(X, \tilde{d})$, where

$$\tilde{d}(x, y) \;=\; \frac{d(x, y)}{1 + d(x, y)} \qquad \forall x, y \in X\,,$$

is also complete.

Problem 7. Let (X, d_X) and (Y, d_Y) be isometric, i.e., there exists a map $T : X \to Y$ such that $T(X) = Y$ and

$$d_Y(T(x_2), T(x_1)) \;=\; d_X(x_1, x_2) \qquad \forall x_1, x_2 \in X\,.$$

Show that if (X, d_X) is complete then (Y, d_Y) is complete.

Problem 8. Given X be a vector space, let $\|\cdot\|_1$ and $\|\cdot\|$ be equivalent norms in X, i.e., there exists a constant $C \geq 1$ such that

$$\frac{1}{C} \cdot \|x\| \;\leq\; \|x\|_1 \;\leq\; C \cdot \|x\| \qquad \forall x \in X\,.$$

Show that

(a) If $E \subset X$ is closed in $(X, \|\cdot\|)$ then E is closed in in $(X, \|\cdot\|_1)$.

(b) If $(X, \|\cdot\|)$ is complete then $(X, \|\cdot\|_1)$ is complete.

Problem 9. Given X be a vector space, let K be a compact subset of X. Show that

(a) If $E \subset X$ is open then $K\backslash E$ is compact in X

(b) If $E \subset X$ is compact then the set

$$H \;=\; \left\{ e^{\|x\|} \cdot x + 2 \cdot y \;\middle|\; x \in K, y \in E \right\}$$

is compact.