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MA 505 HW #1

1. A. $(a^2)(b^2) - (ab)(ab) = 0$

B. $[cos(\alpha)][cos(\alpha)] - [-sin(\alpha)][sin(\alpha)] = cos^2\alpha + sin^2\alpha = 1$

C. $(1)(6 - x^2) - (x)(3x - x^2) + (x)(x^2 - 2x) = 2x^3 - 6x^2 + 6$

D. $(1)(bc^2 - b^2c) - (1)(ac^2 - a^2c) + (1)(b^2a - a^2b)$

E. Start at the bottom row: $(-d)(c)(-ab) = abcd$

F. $(a)((b)(cd+1) + (-1)(-d)) - (1)((-1)(cd+1) + 0) = abcd + ab + ad + cd + 1$

2. A. $R_1/2, R_2 + \frac{1}{2}R_1, R_3 + \frac{1}{2}R_1, 8R_2/15, R_3 + \frac{1}{8}R_2, R_4 + \frac{1}{2}R_2, 15R_3/28, R_4 + \frac{8}{15}R_3,$ $7R_4/12$. This results in the matrix: $\begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{15} & -\frac{4}{15} \\ 0 & 0 & 1 & -\frac{2}{7} \\ 0 & 0 & 0 & 1 \end{pmatrix}$ with $b = [0, 8/15, 12/7, 1]^T$. The result is $x_1 = 1, x_2 = 2, x_3 = 2, x_4 = 1$.

B. $R_1/2, R_2 - 3R_1, R_3 - R_1, R_4 - 7R_1, -2R_2, R_3 - \frac{1}{2}R_2, R_4 + \frac{3}{2}R_2$. At this step, the 4th row of the matrix is zeros, but the 4th row of b is 9. The system is inconsistent.

3. $A^8 = (P \wedge Q)^8 = P \wedge QP \wedge ...P \wedge Q = P\wedge^8 Q$. Note that $\wedge^2 = I_2$. Therefore, $A^8 = A^{2n} = PQ = \begin{bmatrix} 1 & -12 \\ 0 & -7 \end{bmatrix}$. Similarly, $A^9 = A^{2n+1} = P \wedge Q = \begin{bmatrix} 7 & 0 \\ 4 & 1 \end{bmatrix}$.

4. Make a partitioned matrix with this matrix on the left and the identity matrix on the right. Perform the following operations: $\frac{1}{2}R_2, R_3 - R_1, R_3 + 2R_2, R_2 - R_3, R_1 - R_3, R_1 - R_2$. The right side becomes: $\begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & -1 \\ -1 & 1 & 1 \end{pmatrix}$. Multiplying the inverse by $[1, 1, 2]^T$, we get $[\frac{1}{2}\ -\frac{3}{2}\ 2]^T$.

5. (i) False. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = I_2$, which is invertible with $(I_2)^{-1} = I_2$.

(ii) True. $det(AB) = det(A)det(B) \neq 0 \Rightarrow det(A) \neq 0, det(B) \neq 0$

(iii) False. Let $A = I_2$ and B be an arbitrary singular 2x2 matrix. $AB = B$ is nonsingular, yet A is invertible.

(iv) True. $(kA)(\frac{1}{k}A^{-1}) = I$. Note that A^{-1} exists from the problem statement.

6. Note that $(A+B)^T = A^T + B^T$. We also need the Socks and Shoes principle: $(AB)^T = B^TA^T$. With these, we simplify: $(B^{-1}A^{-1} + I)^T = (A^{-1})^T(B^{-1})^T + I$. Also, $(A^{-1})^T = (A^T)^{-1}$ since $A^T(A^{-1})^T = (A^{-1}A)^T = I$. Therefore, $A^T(B^{-1}A^{-1} + I)^T = (B^{-1})^T + A^T = B + (B^T)^{-1} = B + A^{-1}$. The answer is (b).

7. $\alpha = \frac{5}{4}\alpha_1 + \frac{1}{4}\alpha_2 - \frac{1}{4}\alpha_3 - \frac{1}{4}\alpha_4$

8. No. $\alpha_1 + \alpha_2 = \alpha_3$. Hence, $\alpha_1 + \alpha_2 - \alpha_3 = 0$. For them to be linearly independent, all coefficients would have to be zero for a linear combination to equal 0.

9. $\Rightarrow$

Take $\delta_1 + \delta_2 = a_1$, $\delta_1 + \delta_3 = a_2, \delta_2 + \delta_3 = a_3$ are linearly independent. Then no linear combination with nonzero coefficients that sums to zero. Note that $\delta_1 = \frac{1}{2}(a_1 + a_2 - a_3)$.

Similarly, $\delta_2 = \frac{1}{2}(a_1 + a_3 - a_2)$ and $\delta_3 = \frac{1}{2}(a_3 + a_2 - a_1)$. Since $\delta_1, \delta_2, \delta_3$ can all be written as a linear combination of a_1, a_2, a_3, a linear combination of $\delta_1, \delta_2, \delta_3$ with nonzero coefficients summing to zero would suggest a linear combination of a_1, a_2, a_3 sums to zero. This is a contradiction, so $\delta_1, \delta_2, \delta_3$ are linearly independent.

$\Leftarrow$

It follows from the exact same reasoning as above, though writing $\delta_1 + \delta_2,\ \delta_1 + \delta_3, \delta_2 + \delta_3$ as linear combinations of $\delta_1, \delta_2, \delta_3$ is more obvious.

10. A. 3. Adding the 4th row to the second, the second row becomes [0,0,0,-4]. Operations between rows 2 and 3 produce a row of zeros. No further reductions to rows of zeros can be made. Hence, the rank is 3.

B. 3. Subtract twice the second row from the third. It becomes $[0, 7, -11, -8, 5]$ Subtract the second row from the first. It becomes $[1, 3, -4, -4, 1]$. Subtract twice the first row from the second. It becomes $[0, -7, 11, 9, -5]$ Adding the second line to the third reveals no rows can be filled with zeros since the leading column of each row is in a different spot. Therefore, the rank is 3.