MA 515 Homework 2

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Problem 5

Proof. For one direction, if f is not continuous, then by definition, there exists $\epsilon_0 > 0$, for each n, $\exists x_n \in X$, such that $|x_n - x| < 1/n$, but $\sigma(f(x_n), f(x)) \geqslant \epsilon_0$. And this implies a contradiction, for then $x_n \to x$ but $f(x_n)$ doesn't converge to $f(x)$.

For the other direction, we know f is continuous, i.e., $\forall \epsilon > 0$, $\exists \delta > 0$, such that $\sigma(f(y) - f(x)) < \epsilon$ holds for all $d(x, y) < \delta$. With $x_n \to x$, there exists integer $N_\delta > 0$, such that $d(x_n, x) < \delta$ holds for all $n > N_\delta$.

Hence, $\forall \epsilon > 0$, $\exists N_\delta > 0$ such that $\sigma(f(x_n) - f(x)) < \epsilon$ holds for all $n > N_\delta$. And this is equivalent to $f(x_n) \to f(x)$.

$\square$