1.

i.

$$\det\begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}$$

$$= a^2b^2 - (ab)(ab)$$

$$= a^2b^2 - a^2b^2$$

$$= 0$$

ii.

$$\det\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \cos^2\alpha - (-\sin\alpha)(\sin\alpha)$$

$$= \cos^2\alpha + \sin^2\alpha$$

$$= 1$$

iii.

$$\det\begin{pmatrix} 1 & x & x \\ x & 2 & x \\ x & x & 3 \end{pmatrix}$$

$$= 1\begin{vmatrix} 2 & x \\ x & 3 \end{vmatrix} - x\begin{vmatrix} x & x \\ x & 3 \end{vmatrix} + x\begin{vmatrix} x & 2 \\ x & x \end{vmatrix}$$

$$= 1 * (2 * 3 - x * x) - x(x * 3 - x * x) + x(x * x - 2 * x)$$

$$= (6 - x^2) - x(3x - x^2) + x(x^2 - 2x)$$

$$= 6 - x^2 - 3x^2 + x^3 + x^3 - 2x^2$$

$$= 2x^3 - 6x^2 + 6$$

iv.

$$\det\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$$

$$= 1\begin{vmatrix} b & c \\ b^2 & c^2 \end{vmatrix} - 1\begin{vmatrix} a & c \\ a^2 & c^2 \end{vmatrix} + 1\begin{vmatrix} a & b \\ a^2 & b^2 \end{vmatrix}$$

$$= (bc^2 - b^2c) - (ac^2 - a^2c) + (ab^2 - a^2b)$$

$$= ab^2 - a^2b - ac^2 + a^2c + bc^2 - b^2c$$

v.

$$\det\begin{pmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{pmatrix}$$

$$= -0\begin{vmatrix} 2 & b & 0 \\ 3 & 4 & 5 \\ d & 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 & a \\ 3 & 4 & 5 \\ d & 0 & 0 \end{vmatrix} - c\begin{vmatrix} 1 & 2 & a \\ 2 & b & 0 \\ d & 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 & a \\ 2 & 0 & 0 \\ 3 & c & 5 \end{vmatrix}$$

$$= -c\begin{vmatrix} 1 & 2 & a \\ 2 & b & 0 \\ d & 0 & 0 \end{vmatrix}$$

$$= -c\left(a\begin{vmatrix} 2 & b \\ d & 0 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ d & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 2 & b \end{vmatrix}\right)$$

$$= -c\left(a\begin{vmatrix} 2 & b \\ d & 0 \end{vmatrix}\right)$$

$$= -c\big(a(2 * 0 - b * d)\big)$$

$$= -c * a * (-bd)$$

$$= abcd$$

vi.

$$\det\begin{pmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{pmatrix}$$

$$= -a\begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} + (-1)\begin{vmatrix} 1 & 0 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} - 0\begin{vmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & -1 & d \end{vmatrix} + 0\begin{vmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ -1 & c & 1 \end{vmatrix}$$

$$= -a\begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix}$$

$$= -a\begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} - \left(1\begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix} - 0\begin{vmatrix} -1 & 1 \\ 0 & d \end{vmatrix} + 0\begin{vmatrix} -1 & c \\ 0 & -1 \end{vmatrix}\right)$$

$$= -a\begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} - (cd + 1)$$

$$= -a\left(b\begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ 0 & d \end{vmatrix} + 0\begin{vmatrix} -1 & c \\ 0 & -1 \end{vmatrix}\right) - (cd + 1)$$

$$= -a(b(cd + 1) - (-d - 0)) - (cd + 1)$$

$$= -a(bcd + b + d) - (cd + 1)$$

$$= -abcd - ab - ad - cd - 1$$

2.

i.

$$\begin{cases} 2x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 = 0 \\ -\frac{1}{2}x_1 + 2x_2 - \frac{1}{2}x_4 = 3 \\ -\frac{1}{2}x_1 + 2x_3 - \frac{1}{2}x_4 = 3 \\ -\frac{1}{2}x_1 - \frac{1}{2}x_3 + 2x_4 = 0 \end{cases}$$

$$A = \begin{bmatrix} 2 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 2 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 2 \end{bmatrix} \; b = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix} \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 2 & 0 & -\frac{1}{2} & 3 \\ -\frac{1}{2} & 0 & 2 & -\frac{1}{2} & 3 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & -1 & -1 & 0 & 0 \\ -4 & 16 & 0 & -4 & 24 \\ -4 & 0 & 16 & -4 & 24 \\ -4 & 0 & -4 & 16 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 15 & -1 & -4 & 24 \\ 0 & -1 & 15 & -4 & 24 \\ 0 & -1 & -5 & 16 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 15 & -1 & -4 & 24 \\ 0 & -15 & 225 & -60 & 360 \\ 0 & -15 & -75 & 240 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 1 & -\frac{1}{15} & -\frac{4}{15} & \frac{8}{5} \\ 0 & 0 & 224 & -64 & 384 \\ 0 & 0 & -76 & 236 & 24 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 1 & -\frac{1}{15} & -\frac{4}{15} & \frac{8}{5} \\ 0 & 0 & 7 & -2 & 12 \\ 0 & 0 & -19 & 59 & 6 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 1 & -\frac{1}{15} & -\frac{4}{15} & \frac{8}{5} \\ 0 & 0 & 1 & -\frac{2}{7} & \frac{12}{7} \\ 0 & 0 & 0 & \frac{375}{133} & \frac{270}{133} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 1 & -\frac{1}{15} & -\frac{4}{15} & \frac{8}{5} \\ 0 & 0 & 1 & -\frac{2}{7} & \frac{12}{7} \\ 0 & 0 & 0 & 1 & \frac{54}{75} \end{bmatrix}$$

$$x = \begin{bmatrix} \frac{1}{4}x_2 + \frac{1}{4}x_3 \\ \frac{1}{15}x_3 + \frac{4}{15}(0.72) + \frac{8}{5} \\ \frac{2}{7}(0.72) + \frac{12}{7} \\ 0.72 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}x_2 + \frac{1}{4}(1.92) \\ \frac{1}{15}(1.92) + \frac{4}{15}(0.72) \\ 1.92 \\ 0.72 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(1.92) + \frac{1}{4}(1.92) \\ 1.92 \\ 1.92 \\ 0.72 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.96 \\ 1.92 \\ 1.92 \\ 0.72 \end{bmatrix}$$

ii.

$$\begin{cases} 2x_1 + 3x_2 + 5x_3 + x_4 = 3 \\ 3x_1 + 4x_2 + 2x_3 + 3x_4 = -2 \\ x_1 + 2x_2 + 8x_3 - x_4 = 8 \\ 7x_1 + 9x_2 + x_3 + 8x_4 = 0 \end{cases}$$

$$A = \begin{bmatrix} 2 & 3 & 5 & 1 \\ 3 & 4 & 2 & 3 \\ 1 & 2 & 8 & -1 \\ 7 & 9 & 1 & 8 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ -2 \\ 8 \\ 0 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 5 & 1 & 3 \\ 3 & 4 & 2 & 3 & -2 \\ 1 & 2 & 8 & -1 & 8 \\ 7 & 9 & 1 & 8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & \frac{1}{2} & \frac{3}{2} \\ 3-3 & \frac{8}{2}-\frac{9}{2} & \frac{4}{2}-\frac{15}{2} & \frac{6}{2}-\frac{3}{2} & -\frac{6}{2}-\frac{9}{2} \\ 1-1 & \frac{4}{2}-\frac{3}{2} & \frac{16}{2}-\frac{5}{2} & -\frac{2}{2}-\frac{1}{2} & \frac{16}{2}-\frac{3}{2} \\ 7-7 & \frac{18}{2}-\frac{21}{2} & \frac{2}{2}-\frac{35}{2} & \frac{16}{2}-\frac{7}{2} & 0-\frac{21}{2} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & -\frac{1}{2} & -\frac{11}{2} & \frac{3}{2} & -\frac{15}{2} \\ 0 & \frac{1}{2} & \frac{11}{2} & -\frac{3}{2} & \frac{13}{2} \\ 0 & -\frac{3}{2} & -\frac{33}{2} & \frac{9}{2} & -\frac{21}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 11 & -3 & 15 \\ 0 & 1 & 11 & -3 & 13 \\ 0 & 1 & 11 & -3 & 7 \end{bmatrix}$$

no solution exists for x that simultaneously satisfies the following:

$$\begin{cases} 0x_1 + x_2 + 11x_3 - 3x_4 = 15 \\ 0x_1 + x_2 + 11x_3 - 3x_4 = 13 \\ 0x_1 + x_2 + 11x_3 - 3x_4 = 7 \end{cases}$$

3.

$$A = P\Lambda Q$$

$$QP = I_2$$

$$P = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Q = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

$$A^2 = AA = P\Lambda QP\Lambda \mathrm{Q} = \mathrm{P\Lambda I_2 \Lambda Q} = \mathrm{P\Lambda^2 Q}$$

$$A^3 = A^2 A = \mathrm{P\Lambda^2 Q}P\Lambda\mathrm{Q} = \mathrm{P\Lambda^2 I_2 \Lambda Q} = \mathrm{P\Lambda^3 Q}$$

$$\ldots$$

$$A^n = \mathrm{P}\Lambda^n\mathrm{Q}$$

$$\Lambda^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Lambda^2 = \Lambda\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Lambda^3 = \Lambda^2\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Lambda^1$$

thus for positive integer m:

$$\Lambda^{2m} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Lambda^{2m-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so

$$A^8 = \mathrm{P\Lambda^8 Q} = \mathrm{P}\Lambda^{2*4}Q = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$A^9 = \mathrm{P\Lambda^9 Q} = \mathrm{P}\Lambda^{2*5-1}Q = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & -12 \\ 4 & -7 \end{pmatrix}$$

so

$$A^{2n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$$A^{2n-1} = \begin{pmatrix} 7 & -12 \\ 4 & -7 \end{pmatrix}$$

4.

$$\text{Solve } Ax = b$$

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_2 + 2x_3 = 1 \\ x_1 - x_2 = 2 \end{cases}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A|I = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow Ix = A^{-1}b \Rightarrow x = A^{-1}b$$

$$A^{-1}b = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{2} + 0 \\ 1 - \frac{1}{2} - 2 \\ -1 + 1 + 2 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -3/2 \\ 2 \end{bmatrix}$$

5.

i.

$$\text{Consider } A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix}$$

Neither A nor B is invertible yet

$$A + B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

is invertible

thus the statement is false

ii.

Since AB is invertible, $\det AB \neq 0$

If either A or B is not invertible,

then its determinant would be zero.

but $\det(AB) = \det(A)\det(B)$

which is greater than zero.

thus the statement is true

iii.

$$\text{Consider } A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

A is invertible, B is not invertible yet

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is not invertible

thus the statement is false

iv.

Since A is invertible, $\det A \neq 0$

Moreover if A is an nxn matrix then

$$\det kA = k^n \det A$$

Since $\det A \neq 0$ and $k \neq 0$,

$$0 \neq k^n \det A = \det kA$$

so kA is invertible

thus the statement is true

6.

$$\mathrm{A} = \mathrm{B}^T$$

$$\text{simplify } \mathrm{A}^{\mathrm{T}}(B^{-1}A^{-1} + I)^T$$

$$\mathrm{A}^{\mathrm{T}}(B^{-1}A^{-1} + I)^T = \mathrm{A}^{\mathrm{T}}((A^{-1})^T(B^{-1})^T + I^T)$$

$$= \mathrm{A}^{\mathrm{T}}((A^T)^{-1}(B^{-1})^T + I) = (B^{-1})^T + A^T$$

$$= (B^T)^{-1} + A^T = A^{-1} + B$$

$$\text{thus (b)} = B + A^{-1}$$

7.

Express α linearly in terms of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$

$$\alpha = (1,2,1,1), \qquad \alpha^T = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\alpha_1 = (1,1,1,1), \qquad \alpha_1^T = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \alpha_2 = (1,1,-1,-1), \qquad \alpha_2^T = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\alpha_3 = (1,-1,1,-1), \qquad \alpha_3^T = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \alpha_4 = (1,-1,-1,1), \qquad \alpha_4^T = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + x_4\alpha_4$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \qquad A = (\alpha_1^T \quad \alpha_2^T \quad \alpha_3^T \quad \alpha_4^T) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \qquad \alpha^T = b = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

$$Ax = b \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

use Gauss

$$\begin{pmatrix}1&1&1&1&1\\1&1&-1&-1&2\\1&-1&1&-1&1\\1&-1&-1&1&1\end{pmatrix}\rightarrow\begin{pmatrix}1&1&1&1&1\\0&0&-2&-2&1\\0&-2&0&-2&0\\0&-2&-2&0&0\end{pmatrix}\rightarrow\begin{pmatrix}1&1&1&1&1\\0&0&-2&-2&1\\0&1&0&1&0\\0&1&1&0&0\end{pmatrix}$$

$$\rightarrow\begin{pmatrix}1&1&1&1&1\\0&1&0&1&0\\0&1&1&0&0\\0&0&-2&-2&1\end{pmatrix}\rightarrow\begin{pmatrix}1&1&1&1&1\\0&1&0&1&0\\0&0&1&-1&0\\0&0&-2&-2&1\end{pmatrix}\rightarrow\begin{pmatrix}1&1&1&1&1\\0&1&0&1&0\\0&0&1&-1&0\\0&0&0&-4&1\end{pmatrix}$$

$$x=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}5/4\\1/4\\-1/4\\-1/4\end{pmatrix}$$

check

$$\alpha=x_1\alpha_1+x_2\alpha_2+x_3\alpha_3+x_4\alpha_4=\frac{5}{4}\alpha_1+\frac{1}{4}\alpha_2-\frac{1}{4}\alpha_3-\frac{1}{4}\alpha_4$$

$$=\frac{5}{4}(1,1,1,1)+\frac{1}{4}(1,1,-1,-1)-\frac{1}{4}(1,-1,1,-1)-\frac{1}{4}(1,-1,-1,1)$$

$$=\left(\frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4}\right)+\left(\frac{1}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4}\right)-\left(\frac{1}{4},-\frac{1}{4},\frac{1}{4},-\frac{1}{4}\right)-\left(\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4}\right)$$

$$=\left(\frac{5}{4}+\frac{1}{4}-\frac{1}{4}-\frac{1}{4},\frac{5}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4},\frac{5}{4}-\frac{1}{4}+\frac{1}{4}-\frac{1}{4},\frac{5}{4}-\frac{1}{4}+\frac{1}{4}-\frac{1}{4}\right)=\left(\frac{4}{4},\frac{8}{4},\frac{4}{4},\frac{4}{4}\right)=(1,2,1,1)$$

thus

$$\alpha=\frac{5}{4}\alpha_1+\frac{1}{4}\alpha_2-\frac{1}{4}\alpha_3-\frac{1}{4}\alpha_4$$

8.

Linear independence: $\alpha_1=(1,1,1)^T,\alpha_2=(0,2,5)^T,\alpha_3=(1,3,6)^T$

$$\alpha_1=(1,1,1)^T,\alpha_2=(0,2,5)^T,\alpha_3=(1,3,6)^T$$

No, these are not linearly independent since one vector

can be expressed as a combination of the others:

$$\alpha_1+\alpha_2=(1,1,1)^T+(0,2,5)^T=(1,3,6)^T=\alpha_3$$

9.

Prove that

$\delta_1 + \delta_2, \delta_2 + \delta_3, \delta_1 + \delta_3$ are linearly independent $\Leftrightarrow$ $\delta_1, \delta_2, \delta_3$ are linearly independent

Backward Direction

Assume that

$\delta_1, \delta_2, \delta_3$ are linearly independent

then for scalars a, b, c

$a\delta_1 + b\delta_2 + c\delta_3 = 0$ implies $a = b = c = 0$

so for scalars x, y, z

consider $x(\delta_1 + \delta_2) + y(\delta_2 + \delta_3) + z(\delta_1 + \delta_3\ ) = 0$

$xz\delta_1 + xy\delta_2 + yz\delta_3 = 0$

which is only true when $xz = xy = yz = 0$

thus

$\delta_1 + \delta_2, \delta_2 + \delta_3, \delta_1 + \delta_3$ are linearly independent

Forward Direction

Assume that

$\delta_1 + \delta_2, \delta_2 + \delta_3, \delta_1 + \delta_3$ are linearly independent

then for scalars a, b, c

if $a(\delta_1 + \delta_2) + b(\delta_2 + \delta_3) + c(\delta_1 + \delta_3\ ) = 0$

then $a = b = c = 0$

which reduces to

$ac\delta_1 + ab\delta_2 + bc\delta_3 = 0$

thus

$\delta_1, \delta_2, \delta_3$ are linearly independent

10.

i.

$$\text{Find rank}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank} = 3$$

ii.

$$\begin{pmatrix} 3 & 2 & -1 & -3 & -2 \\ 2 & -1 & 3 & 1 & -3 \\ 4 & 5 & -5 & -6 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & -1 & -\frac{2}{3} \\ 2 & -1 & 3 & 1 & -3 \\ 0 & 7 & -11 & -8 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & -1 & -\frac{2}{3} \\ 0 & -\frac{7}{3} & \frac{11}{3} & 3 & -\frac{5}{3} \\ 0 & 7 & -11 & -8 & 7 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & -1 & -\frac{2}{3} \\ 0 & -7 & 11 & 3 & -5 \\ 0 & 7 & -11 & -8 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & -1 & -\frac{2}{3} \\ 0 & -7 & 11 & 3 & -5 \\ 0 & 0 & 0 & -5 & 2 \end{pmatrix}$$

$$\text{rank} = 3$$