Fall 2017 ISE 505: Linear Programming

Assignment 1 by Mochen Liao

1. Compute the determinant of each of the following matrices:

$$A = \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix} \qquad \det A = a^2 \times b^2 - ab \times ab = 0$$

$$B = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad \det B = \cos^2\alpha + \sin^2\alpha = 1$$

$$C = \begin{pmatrix} 1 & x & x \\ x & 2 & x \\ x & x & x \end{pmatrix} \qquad \det C = 1 \cdot \det\begin{pmatrix} 2 & x \\ x & x \end{pmatrix} - x \cdot \det\begin{pmatrix} x & x \\ x & x \end{pmatrix} + x \cdot \det\begin{pmatrix} x & 2 \\ x & x \end{pmatrix}$$

$$= 2x - x^2 + x(x^2 - 2x) = x(x-1)(x-2)$$

$$D = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \det D = \det\begin{pmatrix} a & b \\ a^2 & b^2 \end{pmatrix} - \det\begin{pmatrix} a & c \\ a^2 & c^2 \end{pmatrix} + \det\begin{pmatrix} b & c \\ b^2 & c^2 \end{pmatrix}$$

$$\begin{aligned}
&= ab^2 - a^2b - ac^2 + a^2c + bc^2 - b^2c \\
&= ab^2 - a^2b - ac^2 + a^2c + bc(c-b) \\
&= a(b+c)(b-c) + a^2(c-b) + bc(c-b) \\
&= (b-c)(ab + ac - a^2 - bc) \\
&= -(b-c)[a(b-a) - c(b-a)] \\
&= (b-c)(a-c)(a-b)
\end{aligned}$$

$$E = \begin{pmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{pmatrix} \qquad \det E = (-1)^{4+1} \cdot d \cdot \det\begin{pmatrix} 0 & 2 & a \\ 0 & b & 0 \\ c & 4 & 5 \end{pmatrix}$$

$$= -d \cdot (-1)^{3+1} \cdot c \cdot (2 \times 0 - a \times b) = abcd$$

$$F = \begin{pmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{pmatrix} \qquad \det F = a \cdot \det\begin{pmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{pmatrix} - \det\begin{pmatrix} -1 & 1 & 0 \\ 0 & c & 1 \\ 0 & -1 & d \end{pmatrix}$$

$$\begin{aligned}
&= a \cdot [b(cd+1) + d] + (cd+1) \\
&= (ab+1)(cd+1) + ad
\end{aligned}$$

2. Solve the following systems of linear equations using the Gaussian elimination method:

$$\begin{cases} 2x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 = 0 \\ -\frac{1}{2}x_1 + 2x_2 - \frac{1}{2}x_4 = 3 \\ -\frac{1}{2}x_1 + 2x_3 - \frac{1}{2}x_4 = 3 \\ -\frac{1}{2}x_2 - \frac{1}{2}x_3 + 2x_4 = 0 \end{cases} \xrightarrow{\text{Transferred to matrix form}} \begin{pmatrix} 2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 2 & 0 & -\frac{1}{2} & 3 \\ -\frac{1}{2} & 0 & 2 & -\frac{1}{2} & 3 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 2 & 0 \end{pmatrix}$$

$$\xrightarrow[r_1+4r_3\to r_3]{r_1+4r_2\to r_2} \begin{pmatrix} 2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{15}{2} & -\frac{1}{2} & -2 & 12 \\ 0 & -\frac{1}{2} & \frac{15}{2} & -2 & 12 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 2 & 0 \end{pmatrix} \xrightarrow[r_2+15r_3\to r_3]{r_3-r_4\to r_4} \begin{pmatrix} 2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{15}{2} & -\frac{1}{2} & -2 & 12 \\ 0 & 0 & 112 & -32 & 192 \\ 0 & 0 & 8 & -4 & 12 \end{pmatrix}$$

$$\xrightarrow{\substack{r_3-14r_4\to r_4 \\ \frac{1}{24}r_4\to r_4 \\ 32r_4+r_3\to r_3 \\ \frac{1}{112}r_3\to r_3}} \begin{pmatrix} 2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{15}{2} & -\frac{1}{2} & -2 & 12 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{\substack{\frac{1}{2}r_3+r_2\to r_2 \\ 2r_4+r_2\to r_2 \\ 15r_2\to r_2}} \begin{pmatrix} 2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{\substack{\frac{1}{2}r_3+r_1\to r_1 \\ \frac{1}{2}r_2+r_1\to r_1 \\ \frac{1}{2}r_1\to r_1}} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

So the solution of this system of linear equations:

$(x_1, x_2, x_3, x_4) = (1, 2, 2, 1)$

$$\begin{cases} 2x_1+3x_2+5x_3+x_4=3 \\ 3x_1+4x_2+2x_3+3x_4=-2 \\ x_1+2x_2+8x_3-x_4=8 \\ 7x_1+9x_2+x_3+8x_4=0 \end{cases} \xrightarrow{\text{Transferred to matrix form}} \begin{pmatrix} 2 & 3 & 5 & 1 & 3 \\ 3 & 4 & 2 & 3 & -2 \\ 1 & 2 & 8 & -1 & 8 \\ 7 & 9 & 1 & 8 & 0 \end{pmatrix}$$

$$\xrightarrow[-\frac{2}{3}r_1+r_2\to r_2]{\substack{-7r_3+r_4\to r_4 \\ -r_2+3r_3\to r_3}} \begin{pmatrix} 2 & 3 & 5 & 1 & 3 \\ 0 & -\frac{1}{2} & -\frac{11}{2} & -\frac{3}{2} & -\frac{13}{2} \\ 0 & 2 & 22 & -6 & 26 \\ 0 & -5 & -55 & 15 & -56 \end{pmatrix} \xrightarrow{\substack{\frac{5}{2}r_3+r_4\to r_4 \\ 4r_2+r_3\to r_3}} \begin{pmatrix} 2 & 3 & 5 & 1 & 3 \\ 0 & -\frac{1}{2} & -\frac{11}{2} & -\frac{3}{2} & -\frac{13}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix}$$

So this is an inconsistent system, no solution could be found.

3. Given $A=P\wedge Q$, where

$$P=\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \quad \wedge=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q=\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

Notice that $QP=I_2$, compute A^8, A^9, and A^{2n}, A^{2n+1} for n being a positive integer.

Solution:

We could also found that $\wedge^2=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=I_2$

Therefore, $A^8=P(\wedge QP)^7\wedge Q=P(\wedge I_2)^7\wedge Q=P\wedge^8 Q=PI_2Q=PQ=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=I_2$

$$A^9=A^8\cdot A=I_2P\wedge Q=P\wedge Q=\begin{pmatrix} 7 & -12 \\ 4 & -7 \end{pmatrix}$$

$$A^{2n}=P(\wedge QP)^{2n-1}\wedge Q=P\wedge^{2n}Q=P(I_2)^nQ=PQ=I_2$$

$$A^{2n+1}=A^{2n}\cdot A=A=\begin{pmatrix} 7 & -12 \\ 4 & -7 \end{pmatrix}$$

4. Solve the following system of linear equations by using the inverse matrix: $\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_2 + 2x_3 = 1 \\ x_1 - x_2 = 2 \end{cases}$

Solution: This system is equivalent to Ax = b, where

$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Therefore, x = A^{-1}b. Because the determinant of A: det A = 2 + 0 = 2 > 0, so A is invertible. Thus the x could be calculated by using the inverse matrix.

$$\left(A\ I\right) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 1 & 1 & 0 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

So

$$x = A^{-1}b = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & -1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 2 \end{pmatrix}$$

The solution of this system of linear equations:

$(x_1, x_2, x_3) = (1/2, -3/2, 2)$

5. Let both A and B be n×n matrices. Are the following propositions true? If it is true, please provide a mathematical proof. Otherwise, give a counterexample.

(i) If both A and B are not invertible, then $(A + B)$ is not invertible. **False.**

Counterexample:

$$A=\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix},\quad B=\begin{pmatrix}4 & 6\\ 2 & 3\end{pmatrix},\quad A+B=\begin{pmatrix}5 & 8\\ 3 & 5\end{pmatrix}$$

In this situation, the determinant of A and B are both 0, when means A and B are not invertible. However, $(A + B)$ is invertible. It could be calculated that $(A+B)^{-1}=\begin{pmatrix}\frac{23}{5} & 8\\ 3 & 5\end{pmatrix}$.

Therefore this proposition is false.

(ii) If AB is invertible, then both A and B are invertible. **True.**

Proof:

According to the multiplicative property, det (AB) = (det A) (det B).

Therefore if AB is invertible, then det $(AB) \neq 0$, which means both det $A \neq 0$ and det $B \neq 0$.

Then according to det $A \neq 0$, it could be deducted that A is invertible; the B could also be deducted as an invertible matrix.

Finally, we could proof this proposition is true.

(iii) If AB is not invertible, then neither A nor B is invertible.

False.

Counterexample:

$$A=\begin{pmatrix}6 & 9\\ 2 & 3\end{pmatrix},\quad B=\begin{pmatrix}5 & 8\\ 3 & 5\end{pmatrix},\quad AB=\begin{pmatrix}57 & 93\\ 19 & 31\end{pmatrix}$$

We could see that det AB =det A = 0, AB and A are not invertible.

However, det B = 1, which means B is invertible.

Therefore this proposition is false.

(iv) If A is invertible, then kA is also invertible with k being a nonzero real number. **True.**

Proof:

While A is invertible, then det A = a $\neq$ 0.

$$\text{if } A=\begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{nn}\end{pmatrix}, \text{ then } kA=\begin{pmatrix}ka_{11} & \cdots & ka_{1n}\\ \vdots & \ddots & \vdots\\ ka_{n1} & \cdots & ka_{nn}\end{pmatrix}$$

Thus, according to the row operation property of matrix, det kA = k^na. Because a $\neq$ 0, and $k \neq 0$, det kA = k^na $\neq$ 0.

This means kA is also invertible.

Finally, we could proof this proposition is true.

6. Suppose that A and B are n×n matrices and $A = B^T$. Which of the following expressions is a simplified form of $A^T(B^{-1}A^{-1}+I)^T$? Why?

(a) $A + B$

(b) $B + A^{-1}$

(c) A^TB

(d) $A + A^{-1}$

(e) $A + B^{-1}$

(f) AA^T

Solution: The deduction process is listed below:

$$
\begin{aligned}
A^T(B^{-1}A^{-1}+I)^T &= A^T(B^{-1}A^{-1})T + A^T \\
&= A^T(A^{-1})^T(B^{-1})^T + B \\
&= (A^{-1}A)^T(B^{-1})^T + B \\
&= (B^{-1})^T + B \\
&= (B^T)^{-1} + B \\
&= B + A^{-1}
\end{aligned}
$$

Therefore, the (b) is the simplified form.

7. Express α linearly in terms of α_1, α_2, α_3 and α_4,

Where $\alpha=(1, 2, 1, 1)$, $\alpha_1=(1, 1, 1, 1)$, $\alpha_2=(1, 1, -1, -1)$, $\alpha_3=(1, -1, 1, -1)$, $\alpha_4=(1, -1, -1, 1)$

Solution: Let $A=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}=\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}$, and assume $Ax=\alpha$,

therefore we just need to solve x, each element of x will be the linear coefficient of α_1, α_2, α_3 and α_4.

At first, we need to solve A^{-1}:

It is easy to get that $A^{-1}=\begin{pmatrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}\\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4}\\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4}\end{pmatrix}$

Therefore $x=A^{-1}\alpha=\begin{pmatrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}\\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4}\\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4}\end{pmatrix}\begin{pmatrix}1\\ 2\\ 1\\ 1\end{pmatrix}=\begin{pmatrix}\frac{5}{4}\\ \frac{1}{4}\\ -\frac{1}{4}\\ -\frac{1}{4}\end{pmatrix}$

Finally, we could express α linearly in terms of α_1, α_2, α_3 and α_4 with the following type: $\alpha=\frac{5}{4}\alpha_1+\frac{1}{4}\alpha_2-\frac{1}{4}\alpha_3-\frac{1}{4}\alpha_4$

8. Are the following vectors linearly independent? Why?

$\alpha_1 = (1, 1, 1)^T$, $\alpha_2 = (0, 2, 5)^T$, $\alpha_3 = (1, 3, 6)^T$.

Solution: To judge the dependency of this three vectors, we need to get the rank of $(\alpha_1, \alpha_2, \alpha_3)$

$$(\alpha_1 \quad \alpha_2 \quad \alpha_3) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 5 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

So $R(\alpha_1, \alpha_2, \alpha_3) = 2 < 3$, therefore $\alpha_1, \alpha_2, \alpha_3$ are not linearly independent.

9. Prove that vectors $\delta_1 + \delta_2$, $\delta_2 + \delta_3$ and $\delta_3 + \delta_1$ are linearly independent, if and only if the δ_1, δ_2 are linearly independent.

Proof: (Assume that δ_1, δ_2 and δ_3 are n-dimensional vectors)

Let $A = (\delta_1, \delta_2, \delta_3)$, $B = (\delta_1 + \delta_2, \delta_2 + \delta_3, \delta_3 + \delta_1)$

Then it is easy to deduce that $B = A\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

Let $K = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Therefore $B = AK$.

Because $\det K = 1 \neq 0$, so $R(A) = R(B)$.

If the vectors group A is linearly dependent, then $R(A) < n$.

Therefore $R(B) = R(A) < n$, which means B is also a linearly dependent vectors group.

Also, K is an invertible matrix, thus only if the δ_1, δ_2 are linearly independent, vectors $\delta_1 + \delta_2$, $\delta_2 + \delta_3$ and $\delta_3 + \delta_1$ are linearly independent.

10. Find the rank of each of the following matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There are three nonzero rows in the transformed A.

So the rank of A is 3.

$$B = \begin{pmatrix} 3 & 2 & -1 & -3 & -2 \\ 2 & -1 & 3 & 1 & -3 \\ 4 & 5 & -5 & -6 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & -1 & -3 & -2 \\ 2 & -1 & 3 & 1 & -3 \\ 4 & 5 & -5 & -6 & 1 \end{pmatrix} \sim \begin{pmatrix} 3 & 2 & -1 & -3 & -2 \\ 0 & -\frac{7}{2} & \frac{11}{2} & \frac{9}{2} & -\frac{5}{2} \\ 0 & 7 & -11 & -8 & 7 \end{pmatrix} \sim \begin{pmatrix} 3 & 2 & -1 & -3 & -2 \\ 0 & -\frac{7}{2} & \frac{11}{2} & \frac{9}{2} & -\frac{5}{2} \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

There are three nonzero rows in the transformed B.

So the rank of B is 3.