Homework 5 Solutions

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Problem 1 (3.1)

Suppose $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$. Add slack variables$s_1, \dots, s_m \geqslant 0$ on each row and make it a standard form.

Let starting feasible solution be $(x_1, \dots, x_n, s_1, \dots, x_m)^T = (0, \dots, 0, b_1, \dots, b_m)^T$.

Problem 2 (3.2)

(a) *Proof.* Let d be a feasible direction at point $x \in P$. Then, there exists $\lambda > 0$ such that $x + \lambda d \in P$, which implies $A(x + \lambda d) = b$. Since $Ax = b$, we know that $\lambda Ad = 0$ and this implies $Ad = 0$.

□

(b) *Proof.* let $d = (d_1, \dots, d_n)^T$ be a feasible direction at x. Let $\alpha = \min\{\frac{x_i}{-d_i} | d_i < 0, i = 1, \dots, n\}$. If $d \geqslant 0$, then let $\alpha = 1$.

It is clear that $\alpha > 0$ and $x + \alpha d \geqslant 0$.

□

Problem 3 (3.3)

$$\begin{aligned} \text{Minimize} \quad & -2x_1 - x_2 + x_3 + x_4 + 2x_5 & \\ \text{subject to} \quad & -2x_1 + x_2 + x_3 + x_4 + x_5 & = 12 \\ & -x_1 + 2x_2 + x_4 - x_5 & = 5 \\ & x_1 - 3x_2 + x_3 + 4x_5 & = 11 \\ & x_1, x_2, x_3, x_4, x_5 & \geqslant 0 \end{aligned}$$

Here, $A = \begin{pmatrix} -2 & 1 & 1 & 1 & 1 \\ -1 & 2 & 0 & 1 & -1 \\ 1 & -3 & 1 & 0 & 4 \end{pmatrix}$ and $b = \begin{pmatrix} 12 \\ 5 \\ 11 \end{pmatrix}$

(a) $B = [A_3, A_4, A_5] = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 4 \end{pmatrix}$, and $N = [A_1, A_2] = \begin{pmatrix} -2 & 1 \\ -1 & 2 \\ 1 & -3 \end{pmatrix}$.

The fundamental matrix $M = \begin{bmatrix} B & N \\ \mathbf{0} & I \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & -1 & 2 \\ 1 & 0 & 4 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ and $M^{-1} = \begin{bmatrix} B^{-1} & -B^{-1}N \\ \mathbf{0} & I \end{bmatrix} =$
$\begin{bmatrix} 2 & -2 & -1 & 3 & -1 \\ -1/2 & 3/2 & 1/2 & 0 & -1 \\ -1/2 & 1/2 & 1/2 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) Apply Gaussian elimination on matrix $[B, N]$ and get reduced row rechelon form

$$\begin{bmatrix} 1 & 0 & 0 & -3 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

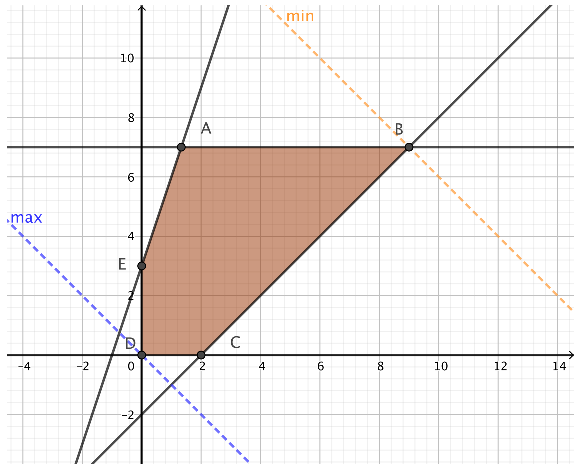
Hence, $x_3 = 3x_1 - x_2 + 3, x_4 = -x_2 + 7, x_5 = -x_1 + x_2 + 2$. Reform the LP problem using only two variables as the following:

$$\begin{array}{rll} \text{Minimize} & -x_1 - x_2 (+14) & \\ \text{subject to} & 3x_1 - x_2 + 3 & \geqslant 0 \\ & -x_2 + 7 & \geqslant 0 \\ & -x_1 + x_2 + 2 & \geqslant 0 \\ & x_1, x_2 & \geqslant 0 \end{array}$$

(c) The feasible domain is part of the intersection of three hyperplanes on $\mathbb{R}^5$, hence, its dimension is reduced by 3 and can be represented in $\mathbb{R}^2$.

We plot the region of P.

Figure 1: Region P.



(d) Basic feasible solution $\mathbf{x} = (0, 0, 3, 7, 2)^T$. And it is corresponding to point $D = (0, 0)^T$ on fig.1.

(e)

$$B^{-1}A = \begin{bmatrix} -3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}, \qquad B^{-1}b = \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix}$$

One explaination of this:

We see that $B^{-1}A$ is the same with the reduced row echelon form of A. And $B^{-1}b$ is exactly the basic feasible solution (positive entries). This is always true since

$$Ax = b \Leftrightarrow [B|N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b.$$

Since $x_N = 0$, we get $Bx_B = b$ so that $x_B = B^{-1}b$. This implies that $B^{-1}A$ is the reduced row echelon form of A for basic variable, since $B^{-1}Ax = B^{-1}b = x_B$.

(Other proper answers will be also acceptable).

(f) From M^{-1} we know that $\mathbf{d}^1 = (3, 0, -1, 1, 0)^T$ and $\mathbf{d}^2 = (-1, -1, 1, 0, 1)^T$. Reduced costs:

$$r^1 = [c_B^T | c_N^T]\mathbf{d}^1 = (1, 1, 2, -2, -1)\mathbf{d}^1 = -1, \qquad r^2 = [c_B^T | c_N^T]\mathbf{d}^2 = (1, 1, 2, -2, -1)\mathbf{d}^2 = -1$$

(g) From above, either direction leads to a potential reduction in the objective value, since r^1 and r^2 are both negative. Consider the nonnegativity constraint($x + \alpha\mathbf{d} \geqslant 0$), we get the step length for $\mathbf{d}^1$ is $\alpha_1 = 2$ and for $\mathbf{d}^2$ is $\alpha_2 = 3$.

(h) 1) If we take $\mathbf{d}^1$, then the new solution will be $\overline{x} = x + \alpha_1 \mathbf{d}^1 = (9,7,0,2,0)^T \geqslant 0$. The basis now is $\overline{B} = [A_3, A_4, A_1]$ and $\overline{N} = [A_5, A_2]$. It is easy to check that $\overline{B}\overline{x} = b$. Hence $\overline{x}$ is a basic feasible solution(BFS). $\overline{x}$ is also an adjacent extreme point of x. (On fig.1, $\overline{x}$ is the point C)

2) If we take $\mathbf{d}^2$, then the new solution will be $\overline{x} = x + \alpha_2 \mathbf{d}^2 = (0,4,5,0,3)^T \geqslant 0$. The basis now is $\overline{B} = [A_4, A_5, A_2]$ and $\overline{N} = [A_3, A_1]$. It is easy to check that $\overline{B}\overline{x} = b$. Hence $\overline{x}$ is a basic feasible solution(BFS). $\overline{x}$ is also an adjacent extreme point of x. (On fig.1, $\overline{x}$ is the point E)

(i) 1) If we take $\mathbf{d}^1$, update $\tilde{M} = \begin{bmatrix} \tilde{B} & \tilde{N} \\ \mathbf{0} & I \end{bmatrix}$.

$$\tilde{M} = \begin{bmatrix} 1 & 1 & -2 & 1 & 1 \\ 0 & 1 & -1 & -1 & 2 \\ 1 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \tilde{M}^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 & -3 & 2 \\ -1/2 & 3/2 & 1/2 & 0 & -1 \\ -1/2 & 1/2 & 1/2 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

From the last two columns of $\tilde{M}^{-1}$ we get $\tilde{\mathbf{d}}^5 = \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, $\tilde{\mathbf{d}}^2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Let $\tilde{c}^T = [\tilde{c}_B^T | \tilde{c}_N^T]$. We get $\tilde{c}^T \tilde{\mathbf{d}}^5 = 1 > 0$, but $\tilde{c}^T \tilde{\mathbf{d}}^2 = -2 < 0$. So $\tilde{x}$ is not an optimal solution since $\tilde{\mathbf{d}}^2$ is a good direction of translation.

2) If we take $\mathbf{d}^2$, update $\overline{M} = \begin{bmatrix} \overline{B} & \overline{N} \\ \mathbf{0} & I \end{bmatrix}$.

$$\overline{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & -2 \\ 1 & -1 & 2 & 0 & -1 \\ 0 & 4 & -3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \overline{M}^{-1} = \begin{bmatrix} -5/2 & 7/2 & 3/2 & 1 & -3 \\ 3/2 & -3/2 & -1/2 & -1 & 2 \\ 2 & -2 & -1 & -1 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

From the last two columns of $\overline{M}^{-1}$ we get $\overline{\mathbf{d}}^5 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, $\overline{\mathbf{d}}^2 = \begin{bmatrix} -3 \\ 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$.

Let $\overline{c}^T = [\overline{c}_B^T | \overline{c}_N^T]$. We get $\overline{c}^T \overline{\mathbf{d}}^3 = 1 > 0$, but $\overline{c}^T \overline{\mathbf{d}}^1 = -4 < 0$. So $\overline{x}$ is not an optimal solution since $\overline{\mathbf{d}}^1$ is a good direction of translation.

(j) 1) If basic varialbes are x_3, x_4, x_1, then the reduced row echelon form (RREF) of $[\tilde{B}, \tilde{N}, -b]$ is

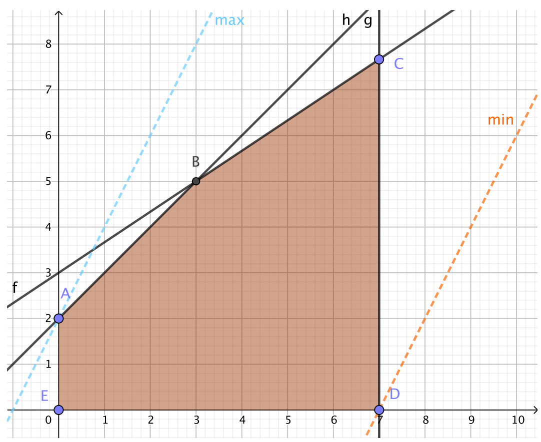
$$\begin{bmatrix} 1 & 0 & 0 & 3 & -2 & -9 \\ 0 & 1 & 0 & 0 & 1 & -7 \\ 0 & 0 & 1 & 1 & -1 & -2 \end{bmatrix}$$

and $x_3 = 2x_2 - 3x_5 + 9, x_4 = -x_2 + 7, x_1 = x_2 - x_5 + 2$. Reform the LP problem using only two variables as the following:

$$\begin{array}{rll} \text{Minimize} & -2x_2 + x_5 (+12) & \\ \text{subject to} & 2x_2 - 3x_5 + 9 & \geqslant 0 \\ & -x_2 + 7 & \geqslant 0 \\ & x_2 - x_5 + 2 & \geqslant 0 \\ & x_2, x_5 & \geqslant 0 \end{array}$$

We plot the region.

Figure 2: Region on x_2, x_5.



2) If basic varialbes are x_4, x_5, x_2, then the reduced row echelon form (RREF) of $[\overline{B}, \overline{N}, -b]$ is

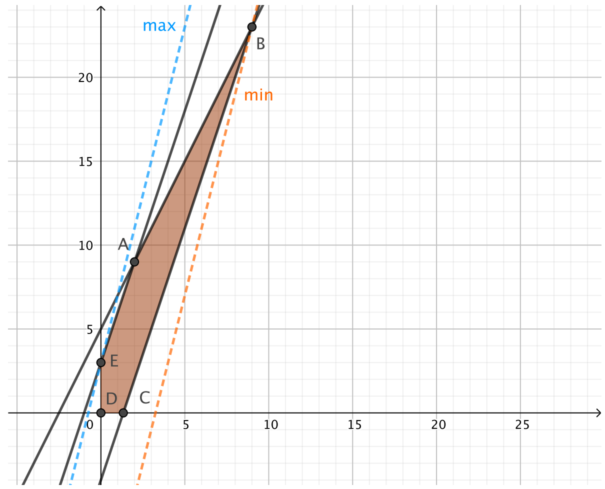
$$\begin{bmatrix} 1 & 0 & 0 & -1 & 3 & -4 \\ 0 & 1 & 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & 1 & -3 & -3 \end{bmatrix}$$

and $x_4 = x_3 - 3x_1 + 4, x_5 = -x_3 + 2x_1 + 5, x_2 = -x_3 + 3x_1 + 3$. Reform the LP problem using only two variables as the following:

$$\begin{array}{rll} \text{Minimize} & -4x_1 + x_3 (+11) & \\ \text{subject to} & x_3 - 3x_1 + 4 & \geqslant 0 \\ & -x_3 + 2x_1 + 5 & \geqslant 0 \\ & -x_3 + 3x_1 + 3 & \geqslant 0 \\ & x_1, x_3 & \geqslant 0 \end{array}$$

We plot the region.

Figure 3: Region on x_1, x_3.



(k) (Any proper answer will be acceptable).

(l) We can always express $n-2$ variables by using the rest 2 variables. Then, the LP problem can be reformed as an LP problem on $\mathbb{R}^2$.

Problem 4 (3.4)

From point $x = [x_B|x_N]^T = [x_5, x_6, x_1|x_2, x_3, x_4]^T$, we know that $B = [A_5, A_6, A_1]$ and $N = [A_2, A_3, A_4]$. Construct matrix $M = \begin{bmatrix} B & N \\ \mathbf{0} & I \end{bmatrix}$ and reduced cost $r = c_N^T - c_B^T B^{-1} N = [-1, -1, 1/2]$.

Note that r_2, r_3 are negative, and figure out the step length $\alpha_2 = \alpha_3 = 1/2$. Hence, we can pick either one from $\mathbf{d}^2$ or $\mathbf{d}^3$. Let's pick $\mathbf{d}^2$. $x_{\text{new}} = x + \alpha_2 \mathbf{d}^2 = (0, 1, 1/2, 1/2, 0, 0)^T$.

Next step:

From point $x = [x_2, x_6, x_1|x_5, x_3, x_4] = (1/2, 1, 1/2, 0, 0, 0)^T$, we know $B = [A_2, A_6, A_1]$ and $N = [A_5, A_3, A_4]$. Construct matrix M and reduced cost $r = c_N^T - c_B^T B^{-1} N = [-1/2, -1, -1/2]$. Find the most negative direction, pick $\mathbf{d}^3$ with $\alpha_3 = 1/2$. $x_{\text{new}} = x + \alpha_3 \mathbf{d}^3 = (1/2, 0, 1/2, 1/2, 0, 0)^T$.

Next step:

From point $x = [x_1, x_2, x_3|x_4, x_5, x_6] = (1/2, 1, 1/2, 0, 0, 0)^T$, we know $B = [A_1, A_2, A_3]$ and $N = [A_4, A_5, A_6]$. Compute the reduced cost $r = c_N^T - c_B^T B^{-1} N = [1/2, 1/2, 1/2] \geqslant 0$. Hence, $x = [x_1, x_2, x_3|x_4, x_5, x_6] = (1/2, 1, 1/2, 0, 0, 0)^T$ is the optimal solution.

Problem 5 (3.8)

Proof. For a degenerate BFS x with $p(< m)$ positive components, we have $n-p$ zero components in it. And also, $n-m$ of $n-p$ will be nonbasic variables and there will be at most $C(n-p, n-m)$ situations.

$\square$

Problem 6 (3.9)

Proof. We know $\overline{M} = \begin{bmatrix} B & N \\ \mathbf{0} & I \end{bmatrix}$ and let $W = \begin{bmatrix} B^{-1} & -B^{-1}N \\ \mathbf{0} & I \end{bmatrix} \in \mathbb{R}^{n\times n}$. Thus, it is enough to check $W\overline{M} = I$ and $\overline{M}W = I$.

Those are true since

$$W\overline{M} = \begin{bmatrix} B^{-1}B - B^{-1}N\mathbf{0} & B^{-1}N - B^{-1}NI \\ \mathbf{0}B + I\mathbf{0} & \mathbf{0}N + I * I \end{bmatrix} \quad = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$$

.

Similarly, $\overline{M}W = I$.

In conclusion, $\overline{M}^{-1} = W$.

$\square$

Problem 7 (3.13)

From point $x = [x_1, x_2, x_3, x_4]^T = [30, 0, 10, 0]^T$, we know that $B = [A_3, A_1]$ and $N = [A_2, A_4]$. Compute reduced cost $r = c_N^T - c_B^T B^{-1} N = [-1/2, 3/2]$.

Note that r_2 is negative, so x_2 enter the basis. Construct $M^{-1} = \begin{bmatrix} 1 & -1/2 & -1/2 & 1/2 \\ 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ to figure out $\mathbf{d}^2 = [-1/2, 1, -1/2, 0]^T$ and the step length $\alpha_2 = 20$. Hence, $x_{\text{new}} = x + \alpha_2 \mathbf{d}^2 = [20, 20, 0, 0]^T$.

Next step:

From point $x = [x_1, x_2, x_3, x_4]^T = [20, 20, 0, 0]^T$, we know that $B = [A_1, A_2]$ and $N = [A_3, A_4]$. Compute reduced cost $r = c_N^T - c_B^T B^{-1} N \geqslant 0$. Hence, this is the optimal solution. The optimal value $z^* = c^T x^* = -100$.

Problem 8 (3.14)

From point $x = [x_1, x_2, x_3, x_4, x_5]^T = [10, 0, 30, 0, 0]^T$, we know that $B = [A_3, A_2, A_1]$ and $N = [A_4, A_5]$. Compute reduced cost $r = c_N^T - c_B^T B^{-1} N = [2, -3]$.

Note that r_5 is negative, so x_5 enter the basis. Construct $M^{-1} = \begin{bmatrix} 1 & -1 & 1 & 1 & -1 \\ 0 & 1 & -2 & -1 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ to figure out $\mathbf{d}^5 = [-1, 2, -1, 0, 1]^T$ and the step length $\alpha_5 = 10$. Hence, $x_{\text{new}} = x + \alpha_5 \mathbf{d}^5 = [0, 20, 20, 0, 10]^T$.

Next step:

From point $x = [x_1, x_2, x_3, x_4, x_5]^T = [0, 20, 20, 0, 10]^T$, we know that $B = [A_2, A_3, A_5]$ and $N = [A_1, A_4]$. Compute reduced cost $r = c_N^T - c_B^T B^{-1} N \geqslant 0$. Hence, this is the optimal solution. The optimal value $z^* = c^T x^* = -100$.