MA 515 Homework 4

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Problem 1

Proof. $\{T_n\}_{n\in\mathbb{N}}$ is a sequence of uniformly bounded linear operators and it satisfies

$$\lim_{n\to+\infty} T_n(x) := T(x)$$

for any $x \in X$. Now we want to show T is a bounded linear operator. First, T is linear because the limit operation on T_n preserves the linearity of T_n. Also, $\mathcal{D}(T)$ is X, which yields that T is a linear operator from X to Y.

Next we need to show T is bounded, or more precisely, $||T||_\infty \leqslant M$. Since $||T_n||_\infty < M$, we know for any $||x||_X = 1$, $||T_n(x)||_Y < M$. Hence,

$$\lim_{n\to+\infty} \|T_n(x)\| = ||T(x)|| \leqslant M.$$

which implies $\sup_{||x||_X=1} ||T(x)|| \leqslant M$, i.e., $||T||_\infty \leqslant M$.

□

Problem 2

Proof. First we need to show that Λ is bounded. Take arbitrarily $x = \{x_n\}_{n\in\mathbb{N}} \in \ell^\infty$, and there exists $M > 0$ such that $|x_i| \leqslant M$ for any $i \geqslant 0$. Therefore,

$$\begin{aligned} ||\Lambda(x)||_{\ell^\infty} = ||y||_{\ell^\infty} &= \sup_{i\geqslant 1} |y_i| \\ &= \sup_{i\geqslant 1} \left| \frac{x_1 + \cdots + x_i}{i} \right| \\ &\leqslant \sup_{i\geqslant 1} \left| \frac{\sum_{j=1}^{i} |x_j|}{i} \right| \leqslant M. \end{aligned}$$

Hence, $||\Lambda||_\infty = \sup_{||x||_{\ell^\infty}=1} ||\Lambda(x)||_{\ell^\infty} \leqslant M$.

Next we need to find the value of $||\Lambda||_\infty$. From the definition,

$$||\Lambda||_\infty = \sup_{||x||_{\ell^\infty}=1} ||\Lambda(x)||_{\ell^\infty} = \sup_{||x||_{\ell^\infty}=1} \sup_{i\geqslant 1} \left| \frac{x_1 + \cdots + x_i}{i} \right|.$$

Also, $||x||_{\ell^\infty} = 1$ implies $|x_j| \leqslant 1, \forall j \geqslant 1$. Hence,

$$\sup_{||x||_{\ell^\infty}=1} \sup_{i\geqslant 1} \left| \frac{x_1 + \cdots + x_i}{i} \right| = \sup_{i\geqslant 1} \frac{i \cdot 1}{i} = 1.$$

Hence, $||\Lambda||_\infty = 1$.

□

Problem 3

Problem 4

Proof. We want to show $\|T_n(x_n) - T(x)\|_Y \to 0$ as $n \to +\infty$. By triangle inequality,

$$\|T_n(x_n) - T(x)\|_Y \leqslant \|T_n(x_n) - T(x_n)\|_Y + \|T(x_n) - T(x)\|_Y \leqslant \|T_n(x_n) - T(x_n)\|_Y + \|T\|_\infty \|x_n - x\|_X.$$

and we know $\|x_n - x\| \to 0$ as $n \to +\infty$. Hence, it is enough to show that $\|T_n(x_n) - T(x_n)\|_Y \to 0$ as $n \to +\infty$.

$\forall x_m \in X, m = 1, 2, \cdots$, $\lim_{n\to+\infty} ||T_n(x_m) - T(x_m)|| = 0$ since $\{T_n\}$ converges to T. Hence, we have

$$\lim_{m\to+\infty} \lim_{n\to+\infty} ||T_n(x_m) - T(x_m)|| = 0.$$

This implies $\|T_n(x_n) - T(x_n)\|_Y \to 0$ because $\{\|T_n(x_n) - T(x_n)\|\}_{n\in\mathbb{N}}$ is a subsequence of $\{\|T_n(x_m) - T(x_m)\|\}_{m,n\in\mathbb{N}}$.

In conclusion, $\lim_{n\to+\infty} ||T_n(x_n) - T(x_n)|| = 0$.

□