MA 515 Homework 3

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Problem 1

(a) It is not a norm because it violates property (ii). Let $a = -1, x = 1$. Then $||ax|| = ||-1|| = 2$. However, $|a|||x|| = 1$.

(b) It is a norm.

Proof. Check the first one. If $||f|| = 0$, then $\sup_{t\geqslant 0} e^{\lambda t}|f(t)| = 0$. Since for any $t \geqslant 0$, $e^{\lambda t}|f(t)| \geqslant 0$ and $e^{\lambda t} > 0$, we have $|f(t)| = 0, \forall t \geqslant 0$. This is to say $f \equiv 0$ on its domain. For the other direction, $||f|| = 0$ is true when $f = 0$.

For (ii), $\forall \alpha \in \mathbb{R}$,

$$||\alpha x|| = \sup_{t\geqslant 0} e^{\lambda t}|\alpha f(t)| = \sup_{t\geqslant 0} e^{\lambda t}|\alpha| \cdot |f(t)| = |\alpha| \sup_{t\geqslant 0} e^{\lambda t}|f(t)| = |\alpha| \cdot ||x||.$$

To check the triangle inequality, we pick $f, g \in X$,

$$\begin{aligned} ||f+g|| = \sup_{t\geqslant 0} e^{\lambda t}|f(t)+g(t)| &\leqslant \sup_{t\geqslant 0} e^{\lambda t}\left(|f(t)| + |g(t)|\right) \\ &\leqslant \sup_{t\geqslant 0} e^{\lambda t}|f(t)| + \sup_{t\geqslant 0} e^{\lambda t}|g(t)| = ||f|| + ||g||. \end{aligned}$$

In conclusion, it is a norm.

□

(c) It is a norm.

Proof. We have shown that for ℓ^p space, $||x||_p = (\sum_{i=1}^{+\infty} |x_i|^p)^{1/p}$ is a norm for $1 \leqslant p \leqslant +\infty$. Then, truncate it for only first two entries. Consider set $S = \{x \in \ell^p | x = (x_1, x_2, 0, \dots), x_1, x_2 \in \mathbb{R}\}$. We have $||x||_p = ||x||, \forall x \in S$. Since $||\cdot||_p$ is a norm on ℓ^p, it must be a norm on $S \subset \ell^p$ as $0 \in S$. Then we conclude that $||\cdot||$ is a norm on $\mathbb{R}^2$.

□

(d) It is not a norm. Consider $x = (1, 0), y = (0, 1)$. Then $||x+y|| = ||(1,1)|| = 2^{1/p} > 2$. However, $||x|| + ||y|| = 2 < ||x+y||$. This breaks the triangle inequality.

Problem 2

Proof.

Step 1 We are going to show $||(\cdot,\cdot)||$ is a norm on $X \times Y$.

If $||(x,y)|| = 0$, then $\max\{||x||_X, ||y||_Y\} = 0$, which implies that $||x||_X = ||y||_Y = 0$. Hence, $x = y = 0$. The other direction is obvious.

To check the second property, take arbitrarily $\alpha \in \mathbb{R}$ and we have

$$\begin{aligned} ||\alpha(x,y)|| = ||(\alpha x, \alpha y)|| &= \max\{||\alpha x||_X, ||\alpha y||_Y\} \\ &= \max\{|\alpha| \cdot ||x||_X, |\alpha| \cdot ||y||_Y\} \\ &= |\alpha| \max\{||x||_X, ||y||_Y\} = |\alpha| \cdot ||(x,y)||. \end{aligned}$$

For triangle inequality, take arbitrarily $(x,y), (z,w) \in X \times Y$, we have

$$\begin{aligned} ||(x,y) + (z,w)|| = ||(x+z, y+w)|| &= \max\{||x+z||_X, ||y+w||_Y\} \\ &\leqslant \max\{||x||_X + ||z||_X, ||y||_Y + ||w||_Y\} \\ &= \max\{||x+||_X, ||y||_Y\} + \max\{||z||_X, ||w||_Y\} \\ &= ||(x,y)|| + ||(z,w)||. \end{aligned}$$

Hence, $||(\cdot,\cdot)||$ is a norm on $X \times Y$.

Step 2 We need to prove that $X \times Y$ is complete. Take any Cauchy sequence $\{z_n = (x_n, y_n)\}_{n\in\mathbb{N}} \subset X \times Y$. $\forall \epsilon > 0$, there exists $N > 0$ such that

$$||z_n - z_m|| = ||(x_n, y_n) - (x_m, y_m)|| < \epsilon, \quad \forall n, m > N.$$

This is equivalent to $\max\{||x_n - x_m||_X, ||y_n - y_m||_Y\} < \epsilon \forall n, m > N$ and also implies that

$$||x_n - x_m||_X < \epsilon, \quad ||y_n - y_m||_Y < \epsilon, \forall n, m > N.$$

Hence we conclude that $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences on X and Y respectively. Since $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ are Banach spaces, $\{x_n\}$ converges on X and $\{y_n\}$ converges on Y. Let $\lim_{n\to+\infty} x_n = x$, $\lim_{n\to+\infty} y_n = y$, and $z = (x,y) \in X \times Y$. Then,

$$||z_n - z|| = ||(x_n, y_n) - (x,y)|| = \max\{||x_n - x||_X, ||y_n - y||_Y\} < \epsilon, \quad \forall n > N.$$

Hence, $\{z_n\}$ converges to on $z \in X \times Y$. This proves the claim.

$\square$

Problem 3

(a) *Proof.* Firstly, we need to show that d_f is a metric on X. $\forall x, y \in X$, $d_f(x,y) = f(||x-y||_X) \geqslant 0$ holds due to the property of f. And $d_f(x,x) = f(0) = 0$. Also, $d_f(x,y) = d_f(y,x)$ is obvious since $||x-y||_X = ||y-x||_X$. For triangle inequality, we need to use the facts that f is increasing and ii,

$$\begin{aligned} d_f(x,y) + d_f(y,z) &= f(||x-y||_X) + f(||y-z||_X) \geqslant f(||x-y||_X + ||y-z||_X) \\ &\geqslant f(||x-y+y-z||_X) = f(||x-z||_X) = d_f(x,z). \end{aligned}$$

Hence, d_f is a metric well-defined on X.

Next, we need to show (X, d_f) is complete. Take any Cauchy sequence $\{x_n\}_n \in \mathbb{N}$ from (X, d_f) and it is enough to show it converges in X. $\forall \epsilon > 0$, there exists $N > 0$ such that $d_f(x_n, x_m) = f(||x_n - x_m||_X) < \epsilon$, for all $n, m > N$. Since f is an incr easing continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$, then there exists an increasing continuous inverse function f^{-1} of f such that $f^{-1}(d_f(x,y)) = ||x-y||_X$. Hence,

$$||x_m - x_n||_X = f^{-1}(d_f(x_m, x_n)) < f^{-1}(\epsilon) \to 0, \text{ as } \quad \epsilon \to 0, \forall n, m > N$$

Hence, $\{x_n\}$ is also a Cauchy sequence on $(X, ||\cdot||_X)$ and so it converges in X in norm $||\cdot||_X$. Let $x_n \to x \in X$ in $||\cdot||_X$. Then we have $\lim_{n\to+\infty} ||x_n - x||_X = 0$. With the continuity of f, we know $\lim_{n\to+\infty} f(||x_n - x||_X) = 0$. And this improves that $x_n \to x$ in distance d_f. Hence $\{x_n\}$ is convergent on (X, d_f).

In conclusion, (X, d_f) is a complete metric space.

□

(b) *Proof.* Take arbitrarily $x, y \in B_f(0,1)$ and $\alpha \in (0,1)$. Then $z = \alpha x + (1-\alpha)y \in X$. It is enough to show that $d_f(0,z) < 1$. Use property (ii) and the property given in (b), we have the following,

$$\begin{aligned} d_f(0,z) = f(||z||_X) &\leqslant f(\alpha||x||_X + (1-\alpha)||y||_X) \\ &\leqslant f(\alpha||x||_X) + f((1-\alpha)||y||_X) \\ &\alpha f(||x||_X) + (1-\alpha) f(||y||_X) \\ &= \alpha d_f(0,x) + (1-\alpha) d_f(0,y) < \alpha + 1 - \alpha = 1. \end{aligned}$$

This proves that $z \in B_f(0,1)$ and then $B_f(0,1)$ is convex.

□

Problem 4

Proof. "$\Leftarrow$", if absolute convergence of any sequence $\{x_n\}$ implies the convergence of this sequence, then $(X, ||\cdot||)$ is complete. Take $\{x_n\}_{n\in\mathbb{N}}$ as a Cauchy sequence on X, then from the definition we can always find a subsequence $\{x_{n_k}\}$ such that $||x_{n_k} - x_{n_{k+1}}|| < 2^{-k}$. Let $y_k = x_{n_k} - x_{n_{k+1}}$ and we know for each $N > 0$,

$$\sum_{k=1}^{N} ||y_k|| = \sum_{k=1}^{N} ||x_{n_k} - x_{n_{k+1}}|| < \sum_{k=1}^{N} \frac{1}{2^k}.$$

Since N is arbitrarily chosen, we take the limits on both sides such that $\sum_{k=1}^{+\infty} ||y_k|| < +\infty$. This implies the convergent of $\{y_k\}$, i.e, $\lim_{N\to+\infty} \sum_{k=1}^{N} y_k = \sum_{k=1}^{+\infty} y_k < +\infty$.

Suppose $S_y = \lim_{N\to+\infty} \sum_{k=1}^{N} y_k$, then

$$S_y = \lim_{N\to+\infty} \sum_{k=1}^{N} y_k = \lim_{N\to+\infty} x_{n_1} - x_{n_{N+1}}.$$

which implies that $\{x_{n_k}\}$ converges to $x_{n_1} - S_y$. Since it is a subsequence of Cauchy sequence $\{x_n\}$, $\{x_n\}$ also converges. This proves the completeness of X.

"$\Rightarrow$". Conversely, with Banach space $(X, ||\cdot||)$, and the absolutely convergent $\{x_n\}$ has sum $\sum_{n=1}^{+\infty} ||x_n|| < +\infty$, we need to prove the infinite sum of $\{x_n\}$ is also finite.

Since $\sum_{n=1}^{+\infty} ||x_n|| < +\infty$, for any $\epsilon > 0$, there exists $N_0 > 0$ such that $\sum_{n=N_0+1}^{+\infty} ||x_n|| < \epsilon$. Let $z_k = \sum_{n=1}^{k} x_n \in X$, $\{z_k\}$ is a Cauchy sequence because for any $\epsilon > 0$, take $N = N_0$, then

$$||z_p, z_q|| \leqslant ||\sum_{n=q+1}^{p} x_n|| \leqslant \sum_{n=q+1}^{p} ||x_n|| \leqslant \sum_{n=N+1}^{+\infty} ||x_n|| < \epsilon, \quad \forall p, q > N.$$

Then $\{z_k\}$ converges in Banach space $(X, ||\cdot||)$.

In conclusion, we proved the claim.

$\square$

Problem 5

Proof. For any $\epsilon > 0$, and $\forall x \in \ell^p$, we need to find $e \in V$ such that $||e - x||_p < \epsilon$.

Since $x \in \ell^p$, we know $\sum_{i=1}^{+\infty} |x_i|^p < +\infty$ and this implies $\exists N$ such that $\sum_{i=N+1}^{+\infty} |x_i|^p < \epsilon^p$. Hence, we may let

$$e = \sum_{i=1}^{N} x_i e_i.$$

It is clear that $e \in V$ and check the distance between e and x.

$$||e - x||_p = \left(\sum_{i=1}^{N} |x_i - x_i|^p + \sum_{i=N+1}^{+\infty} |x_i|^p \right)^{1/p} = \left(\sum_{i=N+1}^{+\infty} |x_i|^p \right)^{1/p} < \epsilon.$$

Hence, V is dense in ℓ^p.

$\square$

Problem 9

Proof. let $g(x) = \frac{1}{4}e^{f(x)}$. We need to show that $g : \mathbb{R} \to \mathbb{R}$ is a contraction map. If so, then there exists a unique $x_0 \in \mathbb{R}$ such that $g(x_0) = x_0$, which is the unique solution to the equation.

For any $x \in \mathbb{R}$, $f(x) \in [0, 1]$. And also, let $h(z) = e^z, z \in [0, 1]$ and the derivative of h is in $[1, e]$. Thus, there exists $0 < c < 1$, such that

$$|g(x) - g(y)| = \frac{1}{4}|e^{f(x)} - e^{f(y)}| \leqslant \frac{1}{4}\max|h'| \cdot |f(x) - f(y)| \leqslant \frac{e}{4}c|x - y|.$$

Let $c' = \frac{e}{4}c$ and note that $c' \in (0, 1)$. Hence, g is a contraction map.

□

Problem 10

1. *Proof.* If $\alpha_1 = \alpha_2 = 0$, then the claim is trivial. If not, then we only need to show that there exists a positive β such that

$$\frac{||\alpha_1 e_1 + \alpha_2 e_2||}{|\alpha_1| + |\alpha_2|} \geqslant \beta.$$

Let function $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(c_1, c_2) = ||c_1 e_1 + c_2 e_2||$ and $\mathrm{dom} f = \{(c_1, c_2) \in \mathbb{R}^2 | |c_1| + |c_2| = 1\}$. It is clear that f is continuous on $\mathbb{R}^2$ and $\mathrm{dom} f$ is compact. So f reaches its minimum $K \geqslant 0$ on $\mathrm{dom} f$. Also, $K > 0$, because if $K = 0$ then $c_1 = c_2 = 0$, which leads to a contradiction that $(c_1, c_2) \notin \mathrm{dom} f$. Hence, we have

$$\frac{||\alpha_1 e_1 + \alpha_2 e_2||}{|\alpha_1| + |\alpha_2|} = \left\| \frac{\alpha_1}{|\alpha_1| + |\alpha_2|} e_1 + \frac{\alpha_2}{|\alpha_1| + |\alpha_2|} e_2 \right\| \geqslant K.$$

And this proves the claim.

□

2. *Proof.* Similar to the previous one, only need to show that there exists $\beta > 0$ such that

$$\frac{|| \sum_{i=1}^n \alpha_i x_i ||}{\sum_{i=1}^n |\alpha_i|} \geqslant \beta.$$

Let function $g : \mathbb{R}^n \to \mathbb{R}$ be $g(c) = || \sum_{i=1}^n c_i e_i ||$ and $\mathrm{dom} g = \{c \in \mathbb{R}^n | \sum_{i=1}^n |c_i| = 1\}$. It is clear that g is continuous on $\mathbb{R}^n$ and dom g is compact. So g reaches its minimum $\kappa \geqslant 0$ on dom g. Also, $\kappa \notin 0$ with the same reason above. Hence, we have

$$\frac{|| \sum_{i=1}^n \alpha_i e_i ||}{\sum_{i=1}^n |\alpha_i|} = \left\| \frac{\sum_{i=1}^n \alpha_i}{\sum_{i=1}^n |\alpha_i|} e_i \right\| \geqslant \kappa.$$

And this proves the claim.

□