Homework 3 Solutions

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Problem 4

(1) *Proof.* If $d \in E$, then for any $x^0 \in P$, $x^0 + \lambda d \in P$, for all $\lambda \geqslant 0$. This implies that $Ax^0 = b, x^0 \geqslant 0$, and $A(x^0 + \lambda d) = b, x^0 + \lambda d \geqslant 0$. Eliminate Ax^0 from the last equality and get $\lambda Ad = 0$. Since $\lambda \geqslant$, $Ad = 0$ is proved. Also, $x^0 \geqslant 0$ and $\forall \lambda \geqslant 0$, hence $d \geqslant 0$.

Conversely, we need to show that for $d \in \mathbb{R}^n$, if $d \geqslant 0$ and $Ad = 0$, then $d \in E$. Take d that satisfies the given condition. For any $y \in P$, $Ay = b, y \geqslant 0$. Hence, we will have $A(y + \lambda d) = b$, $\forall \lambda \geqslant 0$. Also, it is true that $y + \lambda d \geqslant 0$, due to the positiveness of y, λ and d. In conclusion, d is a extremal direction of P. □

(2) *Proof.* We need to express E in a form of set.

$$E = \{d \in \mathbb{R}^n | y + \lambda d \in P, \forall \lambda \geqslant 0, y \in P\}.$$

Take any $d \in E$, need to check if $\alpha d \in E$, $\forall \alpha \geqslant 0$. Actually, this is true. For any $y \in P$, $y + \lambda(\alpha d) = y + (\lambda\alpha)d \in P$ because $\lambda\alpha \geqslant 0$. Thus, $\lambda d \in E$, which proves that E is a cone.

□

(3) *Proof.* Take two points $d_1, d_2 \in E$ and $\beta \in (0, 1)$. For any $y \in P, \lambda \geqslant 0$,

$$y + \lambda(\beta d_1 + (1 - \beta)d_2) = \beta(y + \lambda d_1) + (1 - \beta)(y + \lambda d_2).$$

Let $x^1 = y + \lambda d_1$ and $x^2 = y + \lambda d_2$. Hence, the convex combination of x^1 and x^2 are in P since P is a convex polyhedron. This implies that E is convex.

□

Problem 5

(1) We plot the graph of F_3.

(2)

$$B = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 | |x_1| + |x_3| = x_2\}.$$

(3)
$$I = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 | |x_1| + |x_3| < x_2\}.$$

(4) Extreme point: $(0,0,0)^T$.

Vertex: $(0,0,0)^T$.

(5) *Proof.* First, we show that F_3 is a cone. Take $x = (x_1, x_2, x_3)^T \in F_3$ and $\forall \lambda \geqslant 0$, check if $\lambda x \in F_3$.

$$|\lambda x_1| + |\lambda x_3| = \lambda|x_1| + \lambda|x_3| = \lambda(|x_1| + |x_3|) \leqslant \lambda x_2.$$

Hence, λx is in F_3. In conclusion, F_3 is a cone.

Next, we need to show F_3 is convex. Take $x, y \in F_3$ and $\forall \eta \in (0,1)$, $\eta x + (1-\eta)y = (\eta x_1 + (1-\eta)y_1, \eta x_2 + (1-\eta)y_2, \eta x_3 + (1-\eta)y_3)$. Use triangle inequality of absolute value and we get

$$\begin{aligned} |\eta x_1 + (1-\eta)y_1| + |\eta x_3 + (1-\eta)y_3| &\leqslant \eta|x_1| + (1-\eta)|y_1| + \eta|x_3| + (1-\eta)|y_3| \\ &= \eta(|x_1| + |x_3|) + (1-\eta)(|y_1| + |y_3|) \\ &\leqslant \eta x_2 + (1-\eta)y_2. \end{aligned}$$

Hence, $\eta x + (1-\eta)y \in F_3$. This implies that F_3 is convex.

□

(6) (Any reasonable answers will be fine for this question.)

Example answer:

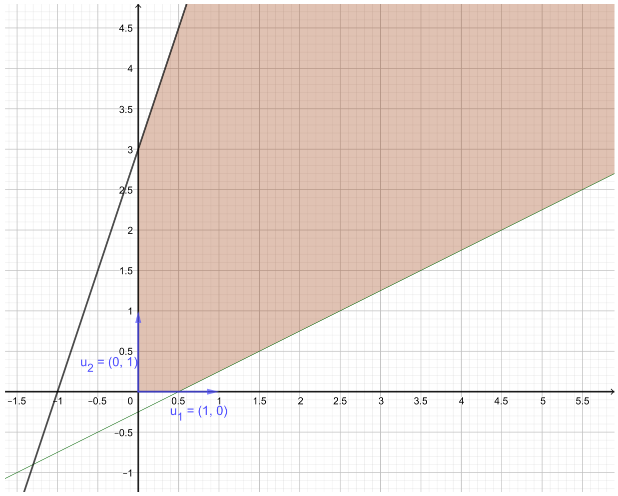
$\mathbb{R}^3_+$ are different from F_3 and $\mathbb{R}^3_+ \cap F_3 \neq \phi$.

$(-1,2,-1)^T \notin \mathbb{R}^3_+$ but in F_3. $(2,1,2)^T \notin F_3$, but in $\mathbb{R}^3_+$.

Problem 6

We plot the region of P_1.

Figure 1: Region P_1.



(a) Convert P_1 to standard equality form.

$$\begin{cases} 2x_1 - 4x_2 + a_1 & = 1 \\ 3x_1 - x_2 \quad\quad - a_2 & = -3 \\ x_1, x_2, a_1, a_2 & \geqslant 0. \end{cases}$$

(b) Basic solutions (in the form of (x_1, x_2, a_1, a_2)):

$$(-13/10, -9/10, 0, 0)$$
$$(-1, 0, 3, 0)$$
$$(1/2, 0, 0, 9/2)\star$$
$$(0, 3, 13, 0)\star$$
$$(0, -1/4, 0, 13/4)$$
$$(0, 0, 1, 3)\star$$

(c) Basic feasible solutions are those basic solutions with "$\star$".

(d) Let V be the set of all extremal directions.

$$V = \{v \in \mathbb{R}^2 | v = (1, d)^T, d \in [1/2, 3]\}.$$

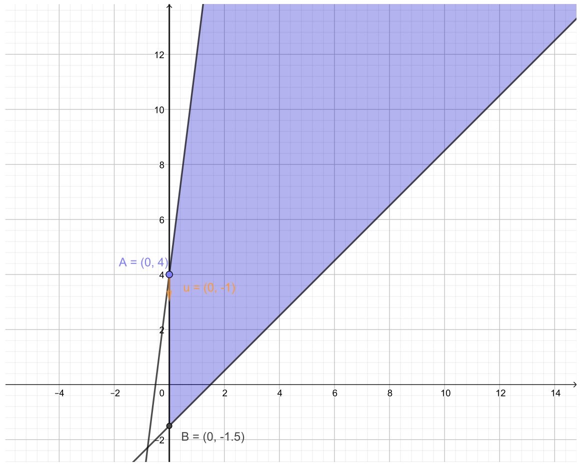
(e) From the figure, we can see that there are two moving directions to the adjacent points. u_1 is to point $(1/2, 0)^T$ and u_2 is to point $(0, 3)^T$.

$$u_1 = (1, 0)^T$$
$$u_2 = (0, 1)^T$$

Problem 7

We plot the region of P_2.

Figure 2: Region P_2.



(a) Convert P_2 to standard equality form.

$$\begin{cases} 2x_1 - 2x_2^+ + 2x_2^- + a_1 & = 3 \\ 8x_1 - x_2^+ + x_2^- \qquad\qquad -a_2 & = -4 \\ x_1, x_2^+, x_2^-, a_1, a_2 & \geqslant 0. \end{cases}$$

(b) Basic solutions (in the form of (x_1, x_2, a_1, a_2)):

$$\begin{aligned} &(-1/2, -1, 0, 0, 0) \\ &(-1/2, 1, 0, 0, 0) \\ &(-1/2, 0, 0, 4, 0) \\ &(3/2, 0, 0, 0, 16)\star \\ &(0, 4, 0, 11, 0)\star \\ &(0, -3/2, 0, 0, 11/2) \\ &(0, 0, -4, 11, 0) \\ &(0, 0, 3/2, 0, 11/2)\star \\ &(0, 0, 0, 3, 4)\star \end{aligned}$$

(c) Basic feasible solutions are those basic solutions with "$\star$".

(d) Let V be the set of all extremal directions.

$$V = \{v \in \mathbb{R}^2 | v = (1, d)^T, d \in [1, 8]\}.$$

(e) From the figure, we can see that there is only one moving directions to the adjacent point $(0, -3/2)^T$.

$$u = (0, -1)^T$$