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OR/ISE 505
Hw # 1

1. $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix} = a^2b^2 - ab \cdot ab = 0$

$\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha + \sin^2\alpha = \underline{1}$

$\begin{vmatrix} 1 & x & x \\ x & 2 & x \\ x & x & 3 \end{vmatrix} = 1[(6-x^2) - x(3x-x^2) + x(x^2-2x)$

$= 6 - x^2 - 3x^2 + x^3 + x^3 - 2x^2 = \underline{2x^3 - 6x^2 + 6}$

$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 1(bc^2 - b^2c) - (ac^2 - a^2c) + (ab^2 - a^2b)$

$= \underline{bc^2 - b^2c + a^2c - ac^2 + ab^2 - a^2b}$

$\begin{vmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{vmatrix} = -d \begin{vmatrix} 0 & 2 & a \\ 0 & b & 0 \\ c & 4 & 5 \end{vmatrix} = -d \cdot c(-ab)$

$= \underline{abcd}$

$\begin{vmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix} = a \begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} - \begin{vmatrix} -1 & 1 & 0 \\ 0 & c & 1 \\ 0 & -1 & d \end{vmatrix}$

$= a[b(cd+1) - 1(-d-0)] - [-1(cd+1)]$

$= \underline{abcd + ab + ad + cd + 1}$

2) $2x_1 - \frac{1}{2}x_2$

$4x_1 - x_2 - x_3 + 0x_4 = 0$ $-R4$

$-x_1 + 4x_2 - x_4 = 6$ $-R3$

a) $-x_1 - + 4x_3 - x_4 = 6$ $+4R1$

$-x_2 - x_3 + 4x_4 = 0$ $+4R3$

$4x_1 - 4x_4 = 0$ $x_1 = x_4$

$4x_2 - 4x_3 = 0$ $x_2 = x_3$

$15x_1 - 4x_2 - x_4 = 6$

$-4x_1 - x_2 + 15x_3 = 24$

$4x_1 - 4x_4 = 0$

$4x_2 - 4x_3 = 0$ $+4x_4$

$4x_1 + 4x_4 = 0$

$-x_2 - x_3 + 4x_4 = 0$

$-4x_3 + 15x_4 = 6$

$x_1 + x_4 = 0$

$x_2 + x_3 - 4x_4 = 0$

$x_3 - \frac{15}{4}x_4 = -\frac{6}{4} = -\frac{3}{2}$

$$\begin{bmatrix} 4 & -1 & -1 & 0 & 0 \\ -1 & 4 & 0 & -1 & 6 \\ -1 & 0 & 4 & -1 & 6 \\ 0 & -1 & -1 & 4 & 0 \end{bmatrix} -R3$$

$$\begin{bmatrix} 4 & -1 & -1 & 0 & 0 \\ 0 & 4 & -4 & 0 & 0 \\ -1 & 0 & 4 & -1 & 6 \\ 0 & -1 & -1 & 4 & 0 \end{bmatrix} +R2$$

$$\begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{15}{4} & -1 & 6 \\ 0 & 0 & -2 & 4 & 0 \end{bmatrix} +\frac{1}{4}R2$$

$$\begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & \frac{14}{4} & -1 & 6 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & \frac{14}{4} & -1 & 6 \end{bmatrix} -\frac{14}{4}\cdot R3$$

$$\begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 6 & 6 \end{bmatrix}$$

$\boxed{x_4 = 1 \quad x_3 = 2 \quad x_2 = 2 \quad x_1 = 1}$

2b)

$$\begin{bmatrix} 2 & 3 & 5 & 1 & 3 \\ 3 & 4 & 2 & 3 & -2 \\ 1 & 2 & 8 & -1 & 8 \\ 7 & 9 & 1 & 8 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 8 & -1 & 8 \\ 3 & 4 & 2 & 3 & -2 \\ 2 & 3 & 5 & 1 & 3 \\ 7 & 9 & 1 & 8 & 0 \end{bmatrix} \begin{matrix} \\ -3R1 \\ -2R1 \\ -7R1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 8 & -1 & 8 \\ 0 & -2 & -22 & 6 & -26 \\ 0 & -1 & -11 & 3 & -13 \\ 0 & -5 & -55 & 15 & -56 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 8 & -1 & 8 \\ 0 & 1 & 11 & -3 & 13 \\ 0 & -2 & -22 & 6 & -26 \\ 0 & -5 & -55 & 15 & -56 \end{bmatrix} \begin{matrix} \\ \\ +2R2 \\ +5R2 \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 8 & -1 & 8 \\ 0 & 1 & 11 & -3 & 13 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

System doesn't reduce / inconsistent

3) $A = P \Lambda Q$

$P = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $Q = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

$\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{bmatrix} 7 & -12 \\ 4 & -7 \end{bmatrix} = A$

$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A^3 = \begin{bmatrix} 7 & -12 \\ 4 & -7 \end{bmatrix}$

$A^8 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A^9 = A = \begin{bmatrix} 7 & -12 \\ 4 & -7 \end{bmatrix}$ $A^{2n} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A^{2n+1} = A = \begin{bmatrix} 7 & -12 \\ 4 & -7 \end{bmatrix}$

4)

$$x_1 + x_2 + x_3 = 1$$
$$2x_2 + 2x_3 = 1$$
$$x_1 - x_2 = 2$$

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 1 & -1 & 0 & 2 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ \times\frac{1}{2} \\ -R1 \end{matrix} \rightarrow \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \begin{matrix} -R2 \\ \\ \\ \end{matrix}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \begin{matrix} \\ \\ +2R2 \end{matrix} \rightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \begin{matrix} \\ -R3 \\ \\ \end{matrix}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{matrix} \frac{1}{2} \\ -\frac{3}{2} \\ 2 \end{matrix}$$

$$x_1 = \frac{1}{2} \qquad x_2 = \frac{-3}{2} \qquad x_3 = 2$$

6) $A = B^T \qquad A^T = B$

$A^T(B^{-1}A^{-1} + I)^T$

$A^T[(B^{-1}A^{-1})^T + I^T]$

$= A^T[(A^{-1T}B^{-1T}) + I]$

$= B(B^{T^{-1}T}B^{-1T}) + B$

$= B(B^{-1} \cdot A^{-1}) + B$

$= B + A^{-1}$

5) i) false $\qquad A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \to$ not invertible $\det = 0 \leftarrow B = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$(A+B) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \to$ invertible

ii) false $\qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \qquad AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

AB is invertible but B is not

iii) false $\qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \qquad AB = \begin{bmatrix} 0 & 0 \\ -1 & -2 \end{bmatrix}$

~~Bis~~ AB is not invertibe but A is invertible
B is not invertible

iv) $(KA)^{-1} = k^{-1}A^{-1}$

true $\quad$ if A is invertible, A^{-1} exists
and if k is a non zero, real number, then an k^{-1} exists
and is a scalar so $k^{-1}A^{-1}$ is the
true inverse of KA

7) $\alpha = (1,2,1,1)$ $\quad \alpha_1 = (1,1,1,1)$ $\quad \alpha_2 = (1,1,-1,-1)$

$\alpha_3 = (1,-1,1,-1)$ $\quad \alpha_4 = (1,-1,-1,1)$

$$\begin{array}{cccc|c} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 2 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 2 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{bmatrix} \begin{matrix} \\ -R1 \\ -R1 \\ -R1 \end{matrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & 1 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & -2 & -2 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \cdot -\frac{1}{2} \\ -R3 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} \\ 0 & 0 & -2 & +2 & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ +2R3 \end{matrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 3 & -1 \end{bmatrix}$$

$x_4 = -\frac{1}{3}$

$x_3 + x_4 = -\frac{1}{2}$

$x_3 = -\frac{1}{6}$

$x_2 = \frac{1}{3}$

$x_1 = \frac{7}{6}$

$\alpha = \frac{7}{6}\alpha_1 + \frac{1}{3}\alpha_2 - \frac{1}{6}\alpha_3 - \frac{1}{3}\alpha_4$

$\alpha = \frac{7}{6}\alpha_1 + \frac{1}{3}\alpha_2 - \frac{1}{6}\alpha_3 - \frac{1}{3}\alpha_4$

$x_1 = \frac{7}{6}$ $\quad x_2 = \frac{1}{3}$

$x_3 = -\frac{1}{6}$ $\quad x_4 = -\frac{1}{3}$

8) $\alpha_1 = (1,1,1)^T$ $\alpha_2 = (0,2,5)^T$ $\alpha_3 = (1,3,6)^T$

$\det\begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 5 & 6 \end{vmatrix}$ $(12-15) - 0 + (5-2)$

$-3 + 3 = 0$ $\therefore$ not linearly independent

$\alpha_3 = \alpha_1 + \alpha_2$

9) if δ_1, δ_2 and δ_3 are linearly independent,

$x_1\delta_1 + x_2\delta_2 + x_3\delta_3 = 0$ means $x_1 = x_2 = x_3 = 0$

if $\delta_1+\delta_2$, $\delta_2+\delta_3$ and $\delta_3+\delta_1$ are linearly independent

then $x_1(\delta_1+\delta_2) + x_2(\delta_2+\delta_3) + x_3(\delta_3+\delta_1)$ means $x_1 = x_2 = x_3 = 0$

$\delta_1(x_1+x_3) + \delta_2(x_1+x_2) + \delta_3(x_2+x_3)$ means $x_1+x_3 = x_2+x_1 = x_3+x_2 = 0$

$\downarrow$ y_1, $\downarrow$ y_2, $\downarrow$ y_3

$y_1\delta_1 + y_2\delta_2 + y_3\delta_3 = 0$ $y_1 = y_2 = y_3 = 0$

therefore $\delta_1+\delta_2$, $\delta_2+\delta_3$, $\delta_3+\delta_1$ are linearly independent

• iff $\delta_1, \delta_2, \delta_3$ are linearly independent.

10)

a) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{matrix} \\ \\ +R2 \\ \\ \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{matrix} \\ \times -1 \\ \\ +2R3 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\underline{\text{Rank} = 3}$

b) $\begin{bmatrix} 3 & 2 & -1 & -3 & -2 \\ 2 & -1 & 3 & 1 & -3 \\ 4 & 5 & -5 & -6 & 1 \end{bmatrix} \begin{matrix} \\ \\ -2R2 \end{matrix} \begin{bmatrix} 3 & 2 & -1 & -3 & -2 \\ 2 & -1 & 3 & 1 & -3 \\ 0 & 7 & -11 & -8 & -7 \end{bmatrix} \begin{matrix} \\ \cdot\frac{3}{2} \\ \\ \end{matrix} \begin{bmatrix} 3 & 2 & -1 & -3 & -2 \\ 3 & -\frac{3}{2} & \frac{9}{2} & \frac{3}{2} & -\frac{9}{2} \\ 0 & 7 & -11 & -8 & -7 \end{bmatrix} \begin{matrix} \\ -R1 \\ \\ \end{matrix}$

$\begin{bmatrix} 3 & 2 & -1 & -3 & 2 \\ 0 & -\frac{7}{2} & \frac{7}{2} & \frac{9}{2} & -\frac{5}{2} \\ 0 & 7 & -11 & -8 & -7 \end{bmatrix} \begin{matrix} \\ \cdot 2 \\ \\ \end{matrix} \begin{bmatrix} 3 & 2 & -1 & -3 & -2 \\ 0 & -7 & 7 & 9 & -5 \\ 0 & 7 & -11 & -8 & -7 \end{bmatrix} \begin{matrix} \\ \\ +R3 \end{matrix} \begin{bmatrix} 3 & 2 & -1 & -3 & -2 \\ 0 & -7 & 7 & 9 & -5 \\ 0 & 0 & -4 & 1 & -12 \end{bmatrix}$ $\underline{\text{Rank} = 3}$