MA 515 Homework 6

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Problem 1

Proof. Given $\bar{x} \in X$, let $V = span\{\bar{x}\}$ and define linear bounded operator $F : V \to \mathbb{R}$, such that $\forall x \in V$, $F(x) = \|x\|$. Hence, by Hahn-Banach theorem, there exists $\Phi : X \to \mathbb{R}$ such that $\Phi = F$ on V and $\|\Phi\|_\infty = \|F\|_\infty$. Since $\bar{x} \in V \subset X$, $\|\Phi(\bar{x})\| = \|F(\bar{x})\| = \|\bar{x}\|$. $\square$

Problem 2

Proof. Suppose $x \neq y$. Let $V = span\{x, y\}$ functional $f : V \to \mathbb{R}$ such that $\forall s, t \in \mathbb{R}$,

$$f(sx + ty) = s\|x\| - t\|y\|.$$

Hence, $f(x) = \|x\|$, $f(y) = -\|y\|$ and $f(x) \neq f(y)$. By the theorem, there exists a functional $F : X \to \mathbb{R}$ such that $F = f$ on V and $\|f\|_\infty = \|F\|_\infty$, which is a contradiction. $\square$

Problem 3

Proof. Since $\Lambda : Y \to \mathbb{R}^n$, $\forall y \in Y$, $\Lambda(y) = (\Lambda_1(y), \ldots, \Lambda_n(y))$, where linear bounded functionals $\Lambda_i : Y \to \mathbb{R}$, $\forall i = 1, \ldots, n$. By Hahn-Banach theorem, there exists a linear bounded functional for each i such that $\widetilde{\Lambda}_i : X \to \mathbb{R}$, and $\|\widetilde{\Lambda}_i\|_\infty = \|\Lambda_i\|_\infty$. Let operator $\widetilde{\Lambda} : X \to \mathbb{R}^n$ such that

$$\widetilde{\Lambda}(y) = (\widetilde{\Lambda}_1(y), \ldots, \widetilde{\Lambda}_n(y)), \quad \forall y \in X.$$

Also, we notice that $\|\Lambda_i\|_\infty \leqslant \|\Lambda\|_\infty$. Then, $\forall x \in X \cap S(0, 1)$

$$\begin{aligned}
\|\widetilde{\Lambda}(x)\| &= \sqrt{|\widetilde{\Lambda}_1(x)|^2 + \cdots + |\widetilde{\Lambda}_n(x)|^2} \\
&\leqslant \sqrt{(\|\widetilde{\Lambda}_1\|_\infty^2 + \cdots + \|\widetilde{\Lambda}_n\|_\infty^2)\|x\|_X^2} \\
&= \sqrt{(\|\Lambda_1\|_\infty^2 + \cdots + \|\Lambda_n\|_\infty^2)\|x\|_X^2} \\
&\leqslant \sqrt{n}\|\Lambda\|_\infty \|x\|_X.
\end{aligned}$$

Divided by $\|x\|_X$ on both sides and take supreme,we have

$$\|\widetilde{\Lambda}\|_\infty \leqslant \sqrt{n}\|\Lambda\|_\infty.$$

$\square$

Problem 4

Proof. We apply Banach-Alaoglu theorem directly, so there exists a subsequence $\{\varphi_{n_k}\}$ such that $\varphi_{n_k} \stackrel{*}{\rightharpoonup} \varphi$. Hence, for any $x \in X$, $\varphi_{n_k}(x) \to \varphi(x)$. Additionally, we know for any $y_k \in S$, $\varphi_n(y_k) \to \varphi(y_k)$ and $\overline{S} = X$. Hence, $\varphi_n(x) \to \varphi(x)$, $\forall x \in X$. And it is equivalent to $\varphi_n \stackrel{*}{\rightharpoonup} \varphi$ on X.

Next, we need to show φ is bounded. For any $x \in X \cap S(0,1)$,

$$\|\varphi(x)\| = \lim_{n\to+\infty} \|\varphi_n(x)\| \leqslant M.$$

Take supreme on left side of $\|x\| = 1$, we have $\|\varphi\|_\infty \leqslant M$.

$\square$

Problem 5

Proof. If X is a finite dimensional normed space, then we take an orthonormal basis $\{e_1, \dots, e_n\} \subset X$, and linear functionals $\{f_1, \dots, f_n\} \subset X^*$, such that

$$f_i(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Hence, for each x_j, it can be expressed as $x_j = \sum_{i=1}^n \alpha_i^j e_i$ and $f_k(x_j) = \alpha_k^j$, $\forall k = 1, \dots, n$. Additionally, since $x_j \rightharpoonup x$, we have $f_k(x_j) \to f_k(x)$, $\forall k$. Hence,

$$\|x_j - x\|^2 = \|\sum_{i=1}^n \alpha_i^j e_i - \sum_{i=1}^n \beta_i e_i\|^2 = \|\sum_{i=1}^n (f_i(x_j) - f_i(x))e_i\|^2 \leqslant \sum_{i=1}^n |f_i(x_j) - f(x)|^2.$$

since it is a finite summation and for each i, $|f_i(x_j) - f(x)|^2$ goes to 0, as $j \to +\infty$. Hence, strongly convergence is proved.

$\square$

Problem 6

(i) *Proof.* $1 < p < +\infty$ and $1/p + 1/q = 1$. For any $f \in (\ell^p)^*$, and $x \in \ell^p$, there exists $\alpha = \{\alpha_n\}_{n\in\mathbb{N}} \in \ell^q$ such that $f(x) = \sum_{i=1}^\infty \alpha_i x_i$.

Hence, $\forall f \in (\ell^p)^*$, $f(e_j) = \sum_{i=1}^\infty \beta_i e_j^i$ and $\sum_{i=1}^\infty |\beta_i|^q < +\infty$.

$$\|f(e_j) - 0\| = \|\beta_j - 0\| = |\beta_j|.$$

Since $|\beta_i|^q \to 0$ as $i \to +\infty$, we know $\|f(e_j) - 0\| \to 0$, $\forall f \in (\ell^p)^*$ as $j \to +\infty$. Hence, $\{e_n\}_{n\geqslant 1}$ weakly converges to 0 in ℓ^p.

□

(ii) *Proof.* Any subseqence $\{e_{n_k}\}_{n\geqslant 1}$ is not Cauchy in ℓ^p and so it doesn't converge in ℓ^p. Indeed, for each $k > l$,

$$\|e_{n_k} - e_{n_l}\| = \left(\sum_{i=1}^{\infty} |e^i_{n_k} - e^i_{n_l}|^p\right)^{1/p} = 2^{1/p}.$$

□

Problem 7

Proof. 1. $< x, x >= 1/2(\|x + x\|^2 - \|x\|^2 - \|x\|^2) = \|x\|^2 \geqslant 0$. And $< x, x >= 0$ if and only if $x = 0$.

2. It is also clear that $< x, y >=< y, x >, \forall x, y \in X$.

3. We will show $< x, y + z >=< x, y > + < x, z >$, $\forall x, y, z \in X$. By definition, we know

$$< x, y + z >= \frac{1}{2}(\|x + y + z\|^2 - \|x\|^2 - \|y + z\|^2).$$

and

$$\begin{aligned} \|x + y + z\|^2 &= 2\|x\|^2 + 2\|y + z\|^2 - \|x - y - z\|^2 \\ &= 2\|x + y\|^2 + 2\|z\|^2 - \|x + y - z\|^2 \end{aligned} \tag{1}$$

Also, with Parallelogram theorem, we have

$$\|x - y - z\|^2 + \|x + y - z\|^2 = 2\|x - z\| + 2\|y\|^2.$$

Hence, plug it in (1) and have

$$\|x + y + z\|^2 = \|x\|^2 + \|y + z\|^2 + \|x + y\|^2 + \|z\|^2 - \|x - z\|^2 - \|y\|^2.$$

which implies

$$\begin{aligned} < x, y + z >&= \frac{1}{2}(\|x + y\|^2 - \|x - z\|^2 + \|z\|^2 - \|y\|^2) \\ &= \frac{1}{2}(\|x + y\|^2 - \|y\|^2 - \|x\|^2 - \|x - z\|^2 + \|z\|^2 + \|x\|^2) \\ &= < x, y > + < x, z > \end{aligned}$$

4. We need to show that $< \lambda x, y >= \lambda < x, y >, \forall x, y \in X, \lambda \in \mathbb{R}$.

   To show this, we need a few steps. Firstly, it holds for $\lambda \in \mathbb{N}$ and it can be proved by induction. Also,

$$
\begin{aligned}
< x, -y >= & \frac{1}{2}(\|x-y\|^2 - \|x\|^2 - \|y\|^2) \\
= & \frac{1}{2}(-\|x+y\|^2 + \|y\|^2 + \|x\|^2) \\
= & - < x, y >
\end{aligned}
$$

   Hence, it holds for $\lambda = -1$ and so holds for $\lambda \in \mathbb{Z}$.

   Next we will show that it holds for $\lambda \in \mathbb{Q}$. Let $\lambda = p/q, (q \neq 0), p, q \in \mathbb{Z}$. Hence,

$$
q < x, \lambda y >= q < x, \frac{p}{q} y >=< x, py >= p < x, y > .
$$

   Both sides divided by q and we have $< \lambda x, y >= \lambda < x, y >, \forall \lambda \in \mathbb{Q}$.

   Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\forall \lambda \in \mathbb{R}$, there exists a sequence of rational numbers $\{\lambda_n\}_{n \in \mathbb{N}}$ such that $\lambda_n \to \lambda$. Hence, $< \lambda x, y >= \lambda < x, y >, \forall \lambda \in \mathbb{R}$.

   In conclusion, $< \cdot, \cdot >$ is an inner product.

□

Problem 8

(i) *Proof.* Suppose $\| \cdot \|_1$ is induced by inner product, i.e., $\| \cdot \|_1 = \sqrt{< \cdot, \cdot >}$. However, if so, then $\| \cdot \|$ must satisfy parallelogram identity. For $a = (1, 1)^T, b = (-1, 2)^T$,

$$
2\|a\|_1^2 + 2\|b\|_1^2 = 26 \neq 18 = \|a-b\|_1^2 + \|a+b\|_1^2.
$$

This is a contradiction.

□

(ii) Still, it breaks the parallelogram identity.

Let $f(x) = x, g(x) = 2x$. Hence, $\|f\|_\infty = 1, \|g\|_\infty = 2$. But $\|f+g\|_\infty = 3, \|f-g\|_\infty = 1$. So

$$
\|f+g\|_\infty + \|f-g\|_\infty \neq 2\|f\|_\infty + 2\|g\|_\infty.
$$

Problem 9

Proof. "⇒" , proved in class. $\langle x, x_n \rangle$ converges $\langle x, x \rangle = \|x\|^2$. Hence,

$$
\lim_{n \to +\infty} \|x_n - x\|^2 = \lim_{n \to +\infty} \langle x_n - x, x_n - x \rangle = \lim_{n \to +\infty} \|x_n\|^2 - 2\langle x, x_n \rangle + \|x\|^2 = 0
$$

"$\Leftarrow$" . If $x_n \to x$, then by Cauchy-Schwartz inequality,

$$0 \leqslant \lim_{n\to+\infty} |\langle x_n - x, x\rangle| \leqslant \lim_{n\to+\infty} \|x_n - x\|\|x\| = 0.$$

By squeeze theorem, $\lim_{n\to+\infty} \langle x_n - x, x\rangle = 0$. Hence, $x_n \rightharpoonup x$.

Also,

$$\begin{aligned} 0 = \overline{\lim_{n\to+\infty}} \|x_n - x\|^2 &= \overline{\lim_{n\to+\infty}} \langle x_n - x, x_n - x\rangle \\ &= \overline{\lim_{n\to+\infty}} \|x_n\|^2 - 2 \overline{\lim_{n\to+\infty}} \langle x, x_n\rangle + \|x\|^2 \\ &= \overline{\lim_{n\to+\infty}} \|x_n\|^2 - 2 \lim_{n\to+\infty} \langle x, x_n\rangle + \|x\|^2 \\ &= \overline{\lim_{n\to+\infty}} \|x_n\|^2 - \|x\|^2 \end{aligned}$$

Similarly, we have $\underline{\lim}_{n\to+\infty} \|x_n\|^2 - \|x\|^2 = 0$. Hence, $\lim_{n\to+\infty} \|x_n\| = \|x\|$.

□

Problem 10

Proof. (i) (Shown in class) Since $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$, for any $x \in \mathcal{H}$, it can be expressed as

$$x = \sum_{i=1}^{\infty} \alpha_i e_i \qquad \alpha_i \in \mathbb{R}.$$

Hence,

$$\langle x, e_i\rangle = \alpha_i \quad \text{and} \quad \langle x, x\rangle = \|x\|^2 = \sum_{i=1}^{\infty} \alpha_i^2 < +\infty.$$

Hence,

$$\lim_{n\to+\infty} \langle x, e_i\rangle = \lim_{n\to+\infty} \alpha_i = 0.$$

i.e., $e_n \rightharpoonup 0$.

(ii) Let $\{e_n\}_{n\geqslant 1}$ be an orthonormal basis of H such that $e_n \perp x$, $\forall n \geqslant 2$. Hence, for each $n \in \mathbb{N}$ and $n \geqslant 2$, let $x_n = x + \lambda e_n (\lambda > 0)$. We want $\|x_n\| = 1$, i.e.,

$$\|x_n\|^2 = \|x + \lambda e_n\|^2 = \|x\|^2 + \lambda^2 = 1$$

which yields that $\lambda = \sqrt{1 - \|x\|^2}$.

Next, we need to show, $x_n = x + \sqrt{1 - \|x\|^2} e_n$ weakly converges to x as $n \to +\infty$. Indeed, for any arbitrarily taken $y \in H$, use result in (i),

$$\lim_{n\to+\infty} \langle y, x_n - x\rangle = \sqrt{1 - \|x\|^2} \langle y, en\rangle = 0$$

Hence, $x_n \rightharpoonup x$.

$\square$

Problem 11

Proof. If $\sum_{n=1}^{\infty} |\alpha_n|^2 < +\infty$. Let partial sum $S_n = \sum_{i=1}^{n} \alpha_i v_i$, we need to show that S_n converges as $n \to +\infty$. Since H is a Hilbert space, it is enough to show $\{S_n\}_{n\geqslant 1}$ is a Cauchy sequence. Indeed, for $m, n \in \mathbb{N}$, $m > n$,

$$\|S_n - S_m\|^2 = \|\sum_{i=n+1}^{m} \alpha_i v_i\|^2 \leqslant \sum_{i=n+1}^{m} |\alpha_n|^2.$$

Since $\sum_{n=1}^{\infty} |\alpha_n|^2 < +\infty$, $\sum_{i=n+1}^{m} |\alpha_n|^2 \to 0$ as $m, n \to +\infty$. Hence, $\{S_n\}$ is Cauchy and so it converges in H.

Conversely, if $\{S_n\}$ converges in H, and denote the limit as S. Hence,

$$+\infty \geqslant \|S\|^2 = \langle \sum_{i=1}^{\infty} \alpha_i v_i, \sum_{i=1}^{\infty} \alpha_i v_i \rangle = \sum_{n=1}^{\infty} |\alpha_n|^2.$$

$\square$

Problem 12

(i) *Proof.* Suppose both p and q are in Ω, such that $p = \pi_\Omega(x), q = \pi_\Omega(x)$. Then, let sequence $\{y_n\}$ be

$$y_n = \begin{cases} p & \text{if n odd} \\ q & \text{if n even} \end{cases}$$

Hence, $\{y_n\}$ converges since $\|y_n - x\| \to d_\Omega(x)$. And this forces $p = q$.

$\square$

(ii) To show this, we would like to prove a lemma first.

Lemma 0.1. *For any $x, y \in H$, $\alpha > 0$,*

$$\langle x, y \rangle \leqslant 0 \iff \|x - \alpha y\| \geqslant \|x\|$$

Proof of Lemma. If $\langle x, y \rangle$, $\forall \alpha > 0$,

$$\langle x - \alpha y, x - \alpha y \rangle = \|x\|^2 + \alpha^2 \|y\|^2 - 2\alpha \langle x, y \rangle \geqslant \|x\|^2.$$

Conversely, if $\|x - \alpha y\| \geqslant \|x\|$, $\forall \alpha > 0$, i.e.,

$$\langle x - \alpha y, x - \alpha y \rangle = \|x\|^2 + \alpha^2 \|y\|^2 - 2\alpha \langle x, y \rangle \geqslant \|x\|^2.$$

which implies $\alpha^2\|y\|^2 - 2\alpha\langle x, y\rangle \geqslant 0$, and so $\frac{\alpha}{2}\|y\|^2 \geqslant \langle x, y\rangle$. Let $\alpha \to 0^+$ and obtain $\langle x, y\rangle \leqslant 0$.

$\square$

Next we will show the claim. $y = \pi_\Omega(x)$ is equivalent to $\|x - y\| = \min_{w\in\Omega}\|x - w\|$, which is

$$(\forall w \in \Omega), \|x - y\| \leqslant \|x - w\| = \|x - y - (w - y)\|.$$

And use the lemma, it is equivalent to $\langle x - y, w - y\rangle \leqslant 0$.