Jonathan Lovin
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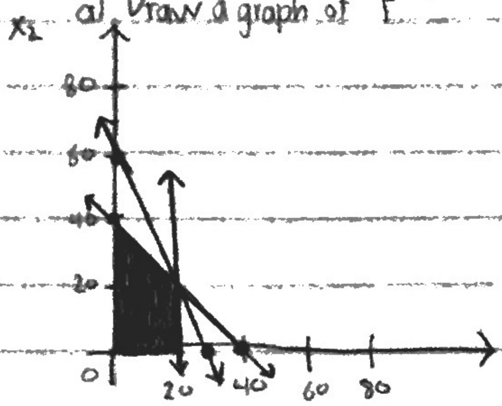
OR 505
Dr. Fang

2.7 Let $P = [(x_1, x_2) \in R^2 \mid x_1 + x_2 \leq 40,\ 2x_1 + x_2 \leq 60,\ x_1 \leq 20,\ x_1, x_2 \geq 0]$

a) Draw a graph of P:



b) Convert P to standard equality form:

$P = [(x_1, x_2) \in R^2 \mid x_1 + x_2 + s_1 = 40,\ 2x_1 + x_2 + s_2 = 60,\ x_1 + s_3 = 20,\ x_1, x_2, s_1, s_2, s_3 \geq 0]$

c) Generate all basic solutions:

$m = 3$ equations $\quad n = 5$ variables $\quad n - m = 2$

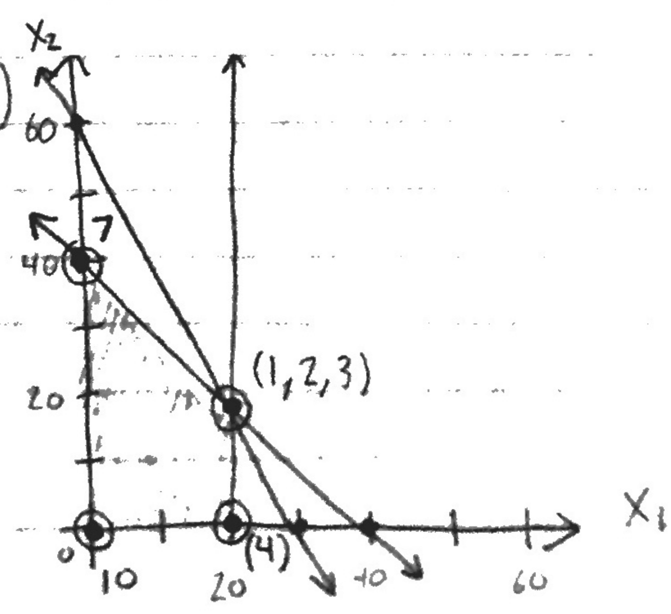
	X_1	X_2	S_1	S_2	S_3		
1	20	20	0	0	0	$X_B^T = \{X_1, X_2, S_1\}$	$X_N^T = \{s_1, s_2\}$
2	20	20	0	0	0	$X_B^T = \{X_1, X_2, S_2\}$	$X_N^T = \{s_1, s_3\}$
3	20	20	0	0	0	$X_B^T = \{X_1, X_2, S_3\}$	$X_N^T = \{s_2, s_3\}$
4	20	0	20	20	0	$X_B^T = \{x_1, s_1, s_2\}$	$X_N^T = \{x_2, s_3\}$
5	30	0	10	0	-10	$X_B^T = \{X_1, s_1, s_3\}$	$X_N^T = \{x_2, s_2\}$
6	40	0	0	-20	-20	$X_B^T = \{X_1, S_2, S_3\}$	$X_N^T = \{x_2, s_1\}$
7	0	40	0	20	20	$X_B^T \{X_2, S_2, S_3\}$	$X_N^T = \{x_1, s_1\}$
8	0	60	-20	0	20	$X_B^T = \{X_2, S_1, S_3\}$	$X_N^T = \{x_1, s_2\}$
~~9~~	~~0~~	[illegible]	[illegible]	[illegible]	~~0~~	[illegible]	[illegible]
10	0	0	40	60	20	$X_B^T \{s_1, s_2, s_3\}$	$X_N^T \{X_1, X_2\}$

d) Find all basic feasible solutions:

1. $x_B^T \{x_1, x_2, s_3\}$ $\quad x_N^T = \{s_1, s_2\}$
2. $x_B^T = \{x_1, x_2, s_2\}$ $\quad x_N^T = \{s_1, s_3\}$
3. $x_B^T = \{x_1, x_2, s_1\}$ $\quad x_N^T = \{s_2, s_3\}$
4. $x_B^T = \{x_1, s_1, s_2\}$ $\quad x_N^T = \{x_2, s_3\}$

7. $x_B^T \{x_2, s_2, s_3\}$ $\quad x_N^T = \{x_1, s_1\}$
10. $x_B^T \{s_1, s_2, s_3\}$ $\quad x_N^T = \{x_1, x_2\}$

e)



2.10 Prove that for a degenerate basic feasible solution with $p < m$ positive elements, its corresponding extreme point P may correspond to $C(n-p, n-m)$ different basic feasible solutions at the same time.

Since the degenerate basic feasible solution has $p < m$ positive elements, $n-p > n-m$. Out of $n-p$ variables, we can pick any $n-m$ of them to be our nonbasic variables. Since only p of m variables will be positive (others will be 0) we can pick any combination of $n-m$ elements from $n-p$ variables and represent the same solution. Therefore, extreme point P with a degenerate solution has $\binom{n-p}{n-m}$ basic feasible solutions where there is a set of $n-p$ elements that cycle between the basis and 0.

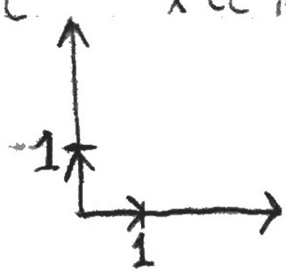
2.11 Let M be the 2×2 identity matrix. Show that

a) M_c, the convex cone generated by M, is the first orthant of R^2

$\lambda \geq 0$ $\lambda x \in C$ $x \in C$ is a cone

$x_1 = [1, 0]$

$x_2 = [0, 1]$

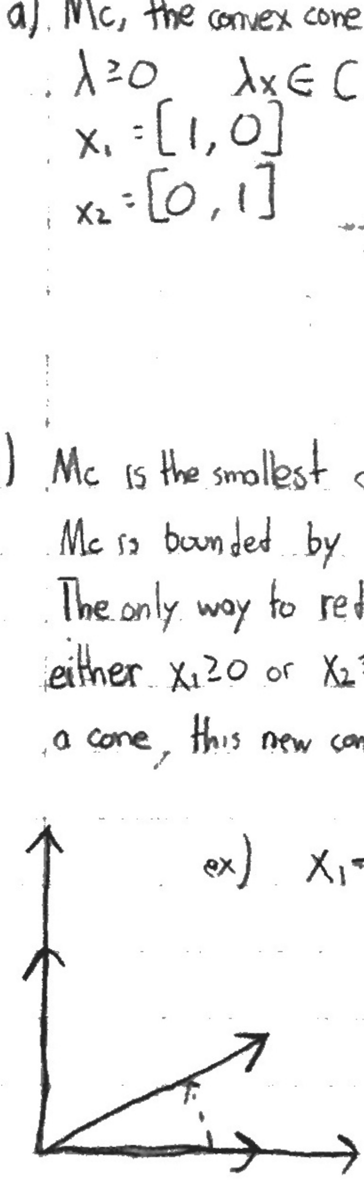


Since $\lambda x \in C$, and $\lambda \geq 0$, and positive scalar multiplied by the x_1 or x_2 will also be within M_c. Therefore, since x_1 and $x_2 \to +\infty$ within M_c and M_c is convex, M_c is the first orthant.

b) M_c is the smallest convex cone which contains the column vectors $(1,0)^T$ and $(0,1)^T$

M_c is bounded by $x_1, x_2 \geq 0$.

The only way to reduce the size of M_c is to tighten in these constraints by changing either $x_1 \geq 0$ or $x_2 \geq 0$ a linear combination of the two variables. In order for M_c to remain a cone, this new constraint must be a ray that comes from $(0,0)$.



ex) $x_1 - 2x_2 \geq 0$

However, adding any such constraint will take either $(1,0)^T$ or $(0,1)^T$ out of M_c with certainty. Therefore, M_c is the smallest cone which contains the column vectors $(1,0)^T$ and $(0,1)^T$

4. E formed by all extremal directions of the nonempty feasible domain $P = \{x \in R^n \mid Ax = b, x \geq 0\}$ of a standard form linear program.

a) Show that, for any vector $d \in R^n$, $d \in E$ if and only if "$Ad = 0$ and $d \geq 0$".

i) Assume $d \in R^n$ such that $x_0 \in P$ the ray $\{x \in R^n \mid x = x_0 + \lambda d, \lambda \geq 0, x \geq 0\}$

$\rightarrow P = \{x \in R^n \mid A(x_0 + \lambda d) = b, x \geq 0, \lambda \geq 0\}$

$Ax_0 + A\lambda d = b$

$Ax_0 = b$ so $A\lambda d = 0$ where $\lambda \geq 0$ $\therefore$ $Ad = 0$ and $d \geq 0$ b/c $\lambda \geq 0$ and as $d \to \infty$ x would become negative an $\ngeq 0$

ii) Assume $d \in R^n$ $Ad = 0$ $d \geq 0$

$x_0 \in P \{x \mid x = x_0 + \lambda d\}$

$A(x_0 + \lambda d) = b$

$Ax_0 = b$ $Ad = 0, d \geq 0$

$Ax_0 + \lambda Ad = b$

$P = \{x \in R^n \mid A(x_0 + \lambda d) = b, x \geq 0, \lambda \geq 0, d \geq 0\}$

b) Prove E is a cone:

?

Need to show $\lambda x \in E$

$\lambda x \in C$ $x \in C$ $\lambda \geq 0$

4a) we know $\lambda \geq 0$ for any point $x \in E$ and $A(x + \lambda d) = b$ is also in E

$\therefore$ E is a cone.

c) Prove E is convex:

If $\sum \lambda_i = 1$ $\lambda_i \geq 0$

Let $x_1, x_2 \in E$ $Ax_1 = 0$ $Ax_2 = 0$, $x_1, x_2 \geq 0$

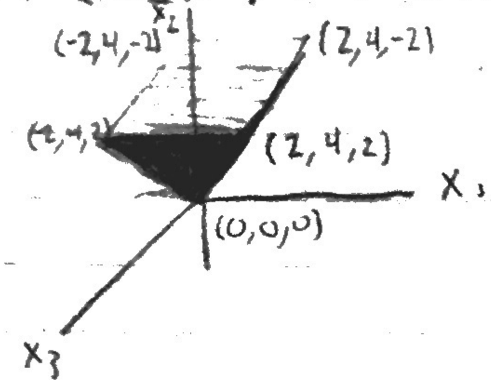
Since $x_1, x_2 \geq 0$, $\lambda x_1 + (1-\lambda)x_2 \geq 0$ for $\lambda \in [0,1]$

$A(\lambda x_1 + (1-\lambda)x_2) = 0$

$\lambda \underbrace{Ax_1}_{0} + (1-\lambda)\underbrace{Ax_2}_{0} = 0$ $\therefore$ E is convex

5. $F_3 = \{x \in R^3 \mid |x_1| + |x_3| \le x_2\}$

1)



Unbounded in the x_2^+ direction

2) $B = \{x \in R^3 \mid |x_1| + |x_3| = x_2\}$

3) $I = \{x \in R^3 \mid |x_1| + |x_3| < x_2\}$

4) By inspection, (0,0,0) is the only vertice and extreme point.

5) For any two points x_1, x_2 and $y_1, y_2 \in F_3$

Prove F_3 is convex:

- By inspection, F_3 is convex because any combination of two points in F_3 fall in F_3

Prove F_3 is cone: any two points in F_3 $(x, y \in F_3)$ and $\lambda \ge 0$, $\lambda_1 + \lambda_2 = 1$

$|x_1| + |x_3| \ge x_2$, $|y_1| + |y_3| \le y_2$

$|\lambda_1 x_1 + \lambda_2 y_2| + |\lambda_1 x_2 + \lambda_1 y_1| \le |\lambda_1 x_1| + |\lambda_2 y_2| + |\lambda_1 x_2| + |\lambda_2 y_2|$

$= \lambda_1(|x_1| + |x_2|) + \lambda_2(|y_1| + |y_2|) \le \lambda_1 x_3 + \lambda_2 y_3$

Therefore, F_3 is a cone.

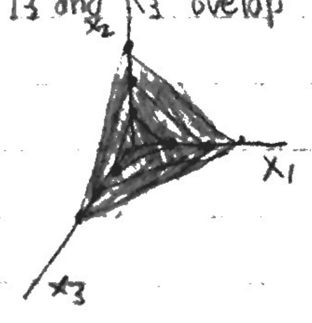
$\therefore$ F_3 is a convex cone

6) Relation between F_3 and the nonnegative orthant $R_+^3 = \{x \in R^3 \mid x_j \ge 0 \text{ for } j = 1,2,3\}$

F_3 is a cone bounded by 4 rays while R_+^3 is a cone bounded by 3 rays.

Both F_3 and R_3 are convex.

F_3 and R_3 overlap but both have a feasible region outside of the other.



6. $P_1 = \{x \in R^2 \mid 2x_1 - 4x_2 - 1 \leq 0, 3x_1 - x_2 + 3 \geq 0, x_1 \geq 0, x_2 \geq 0\}$

$-10x_1 - 12 = 1$ $\quad -10x_1 = 13$ $\quad x_1 = -13/10$

$x_2 = 3x_1 + 3$

$$\begin{array}{lll} 2x_1 - 4x_2 \leq 1 & \rightarrow & 2x_1 - 4x_2 + x_3 + 0 = 1 \\ 3x_1 - x_2 \geq -3 & & 3x_1 - x_2 + 0 - x_4 = -3 \\ x_1, x_2 \geq 0 & & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

a)

	x_1	x_2	x_3	x_4			x_1	x_2	x_3	x_4	
1.	$-13/10$	$-9/10$	0	0	×	4.	0	0	1	3	✓
2.	-1	0	3	0	×	5.	0	$-1/4$	0	$11/4$	×
3.	$1/2$	0	0	$9/2$	✓	6.	0	3	13	0	✓

b) BFS: $\{3, 4, 6\}$

c) $Ad = 0$.

$$\left[\begin{array}{cccc|c} 2 & -4 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \end{array}\right] /2 \rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 1/2 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \end{array}\right] -3R_1$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1/2 & 0 & 0 \\ 0 & 5 & -3/2 & 1 & 0 \end{array}\right] /5 \rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 1/2 & 0 & 0 \\ 0 & 1 & -3/10 & 1/5 & 0 \end{array}\right] +2R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1/10 & 2/5 & 0 \\ 0 & 1 & -3/10 & 1/5 & 0 \end{array}\right]$$

$d_1 = 1/10\, x_3 - 2/5\, x_4$

$d_2 = 3/10\, x_3 - 1/5\, x_4$

$\forall d \in R^2$ $\quad d = \begin{bmatrix} 1/10 \\ 3/10 \\ 1 \\ 0 \end{bmatrix} d_3 + \begin{bmatrix} -2/5 \\ -1/5 \\ 0 \\ 1 \end{bmatrix} d_4$ gives all extremal directions $\in P_2$

d) Start at $(0, 0, 1, 3)$

Find direction to adjacent vertices $(3, 6)$

3. $\begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 9/2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1 \\ 3/2 \end{bmatrix} \Rightarrow$ in R^2: $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \times 2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

6. $\begin{bmatrix} 0 \\ 3 \\ 13 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 12 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 1 \end{bmatrix} \Rightarrow$ in R^2. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

7. $B = \{x \in \mathbb{R}^2 \mid 2x_1 - 2x_2 - 3 \le 0,\ 8x_1 - x_2 + 4 \ge 0,\ x_1 \ge 0\}$

$2x_1 - 2x_2 \le 3$ • add slack variable x_3, $x_3 \ge 0$

$8x_1 - x_2 \ge -4$. add surplus variable x_4 $x_4 \ge 0$

$x_1 \ge 0$ • replace x_2 w/ x_2^+ and x_2^- $(x_2 = x_2^+ - x_2^-)$, add $x_2^+, x_2^- \ge 0$

$\Downarrow$

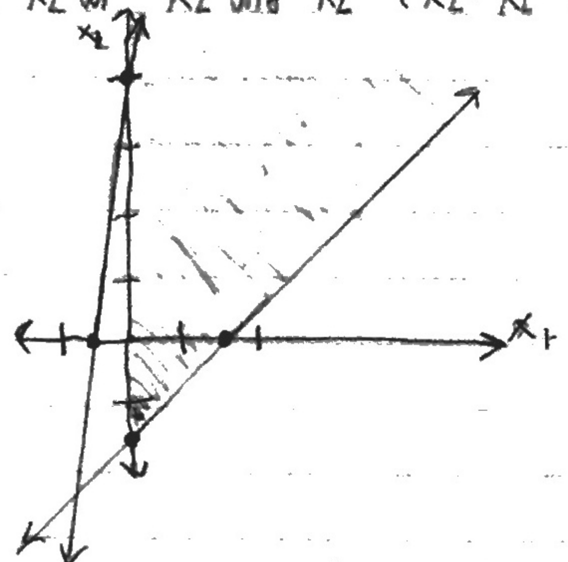
$2x_1 - 2x_2^+ + 2x_2^- + x_3 + 0 = 3$

$8x_1 - x_2^+ + x_2^- + 0 - x_4 = -4$

$x_1, x_2^+, x_2^-, x_3, x_4 \ge 0$

$-x_2^+ - x_2^- = 3/2$

$-x_2^+ + x_2^- \quad -4$



a)

	x_1	x_2^+	x_2^-	x_3	x_4	
1.	$-14/11$	$-16/7$	0	0	0	×
2.	$-11/14$	0	$16/7$	0	0	×
3.	$-1/2$	0	0	4	0	×
4.	$3/2$	0	0	0	16	✓
5.	0	$-3/2$	0	0	$-11/2$	×

	x_1	x_2^+	x_2^-	x_3	x_4	
6.	0	?	?	0	0	×
7.	0	4	0	11	0	✓
8.	0	0	0	3	4	✓
9.	0	0	$3/2$	0	$11/2$	✓
10.	0	0	-4	11	0	×

b) 4, 7, 8, 9

c) $Ad = 0$

$$\left[\begin{array}{ccccc|c} 2 & -2 & 2 & 1 & 0 & 0 \\ 8 & -1 & 1 & 0 & -1 & 0 \end{array}\right] \begin{array}{l} /2 \\ -4R_1 \end{array} \Rightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 1/2 & 0 & 0 \\ 0 & 7 & -7 & -4 & -1 & 0 \end{array}\right] /7$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & -4/7 & -1/7 & 0 \end{array}\right] +R_2 \Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1/14 & -1/7 & 0 \\ 0 & 1 & -1 & -4/7 & -1/7 & 0 \end{array}\right]$$

$d_1 = 1/14\, d_4 + 1/7\, d_5 \qquad d_2 = d_3 + 4/7\, d_4 + 1/7\, d_5$

$$\forall d \in \mathbb{R}^3,\ d = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} d_{x_2^-} + \begin{bmatrix} 1/14 \\ 4/7 \\ 0 \\ 1 \\ 0 \end{bmatrix} d_{x_3} + \begin{bmatrix} 1/7 \\ 1/7 \\ 0 \\ 0 \\ 1 \end{bmatrix} d_{x_4}$$

Gives all extremal directions $\in P_2$

d) 7. to 8

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 0 \\ 11 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 0 \\ -8 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$, in $\mathbb{R}^2$ we get $[0, -1]$

Good job

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