The Multiplicative Weights Update Method:

Applications to Linear Programming and Semidefinite Programming

Zheming Gao and Missy Gaddy

Based on paper by Arora, Hazan, and Kale (2012)

November 29, 2017

Outline

1. Algorithm Overview
2. MW Algorithm for Linear Programming
3. MW Algorithm for Semidefinite Programming

Outline

Multiplicative Weights (MW) Algorithm Overview

- A decision maker has a set of n possible decisions
- Begin with equal probabilities of choosing each decision
- In each iteration:
  1. Decision maker chooses a decision
  2. Obtains a (possibly negative) payoff
  3. Adjusts the probabilites with a multiplicative factor based on payoff

Notation

- t: iteration number (aka - "round")
- T: Number of iterations in total.
- $p^{(t)} \in \mathbb{R}^n$: probability vector for round t
  $p_i^{(t)}$ is the probability of selecting decision i in round t
- $m^{(t)} \in \mathbb{R}^n$: gain vector "revealed by nature" after the decision in round t is made
  (assume $m_i^{(t)} \in [-1, 1]$, $\forall i$, $\forall t$)

General MW Algorithm

Fix $\eta \leq \frac{1}{2}$. Initialize $w_i^{(0)} = 1$, $\forall i = 1, \ldots, n$.

For $t = 1, \ldots, T$:

1. Choose decision i with probability $p_i^{(t)} = \frac{w_i^{(t)}}{\sum_i w_i^{(t)}}$
2. Observe the payoff $m^{(t)}$
3. Update the weights: for each i,

$$w_i^{(t+1)} = w_i^{(t)} \exp(\eta m_i^{(t)})$$

Performance

- Expected gain is "not too much less" than the gain of the best decisions

$$\sum_{t=1}^{T} m^{(t)} \cdot p^{(t)} \geq \sum_{t=1}^{T} m_i^{(t)} - \eta \sum_{t=1}^{T} (m_i^{(t)})^2 \cdot p^{(t)} - \frac{\ln n}{\eta}$$

$(\forall\, i = 1, \ldots, n)$.

- MW has many applications in machine learning, game theory, online decision making, etc.

Outline

1. Algorithm Overview
2. MW Algorithm for Linear Programming
3. MW Algorithm for Semidefinite Programming

Linear Classifier Problem

MW algorithm can be used to solve a linear classifier problem

- m labeled examples (ℓ_j, a_j), with $\ell_j \in \{-1, 1\}$ and $a_j \in \mathbb{R}^n$
- Want to find a normal vector x to a hyperplane so that

$$\operatorname{sgn}(a_j^T x) = \ell_j \quad \forall j = 1, \ldots, m$$

- Equivalently, we want to find x such that

$$\ell_j a_j^T x \geq 0 \quad \forall j = 1, \ldots, m$$

(Redefine $a_j \to \ell_j a_j$)

MW Algorithm to Solve an LP

MW Algorithm can be used to solve the following LP:

$$\begin{aligned} a_j^T x &\geq 0 \quad \forall j = 1, \ldots, m \\ \mathbb{1}^T x &= 1 \\ x_i &\geq 0 \quad \forall i = 1, \ldots, n \end{aligned}$$

(x is analogous to the distribution p)

MW Algorithm to Solve the Linear Classifier Problem

Define $\rho = \max_j ||a_j||_\infty$, $\eta = \frac{\epsilon}{2\rho}$, $\epsilon > 0$.

Initialize $x_i^{(0)} = \frac{1}{n}, \forall i$

While $\exists\, j$ s.t. $a_j^T x^{(t)} < 0$

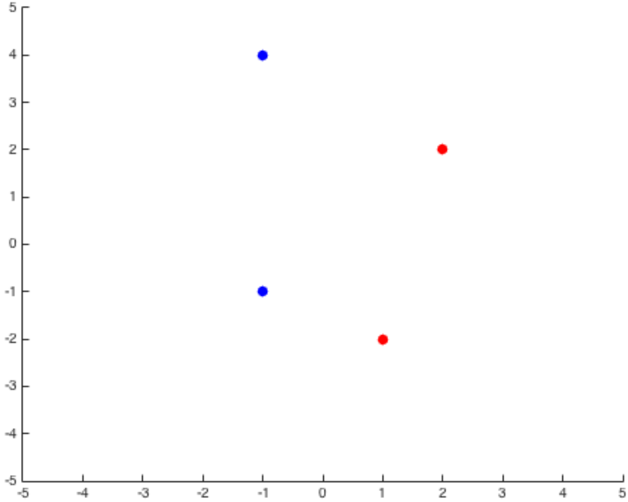
1. Find the index j such that $a_j^T x^{(t)} < 0$
2. Obtain payoff $m^{(t)} = \frac{a_j}{\rho}$
3. Update $x^{(t)}$: for each i,

$$x_i^{(t+1)} = x_i^{(t)} \exp(\eta m_i^{(t)})$$

,

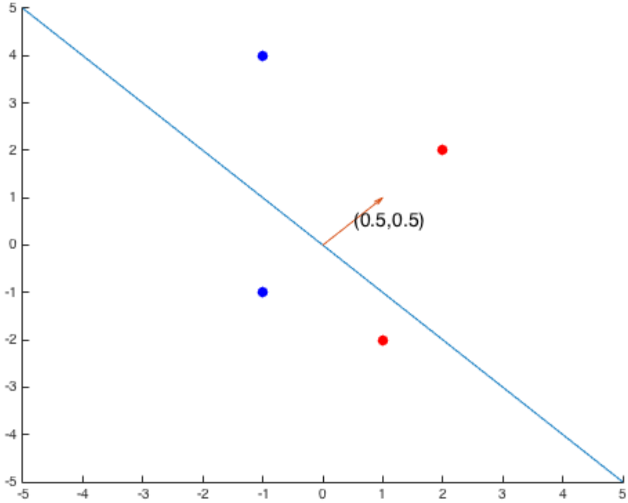
Linear Classifier: Toy Example

Find a separating hyperplane for the following points



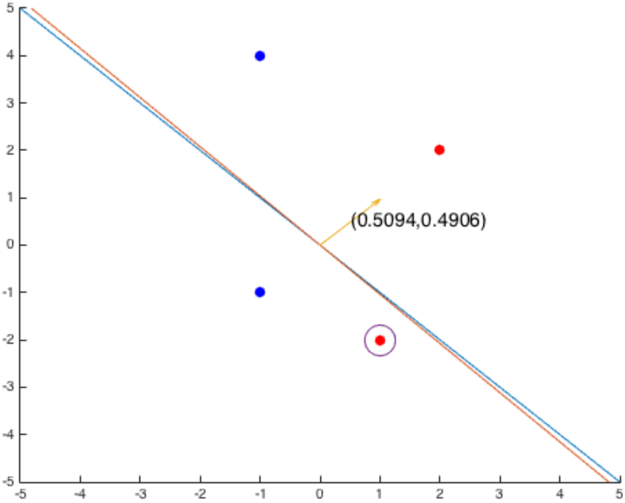
Linear Classifier: Toy Example

Initialize: $x^{(0)} = [\frac{1}{2}, \frac{1}{2}]$.



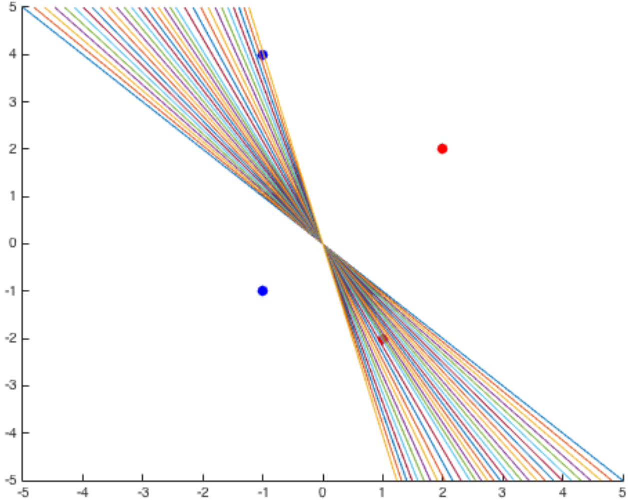
Linear Classifier: Toy Example

Misclassified example yields gain of $m^{(0)} = [\frac{1}{2}, -1]$.



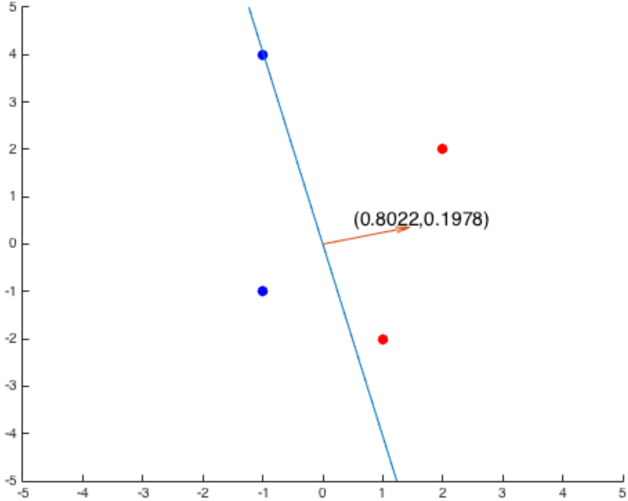
Linear Classifier: Toy Example

Continued iterations:



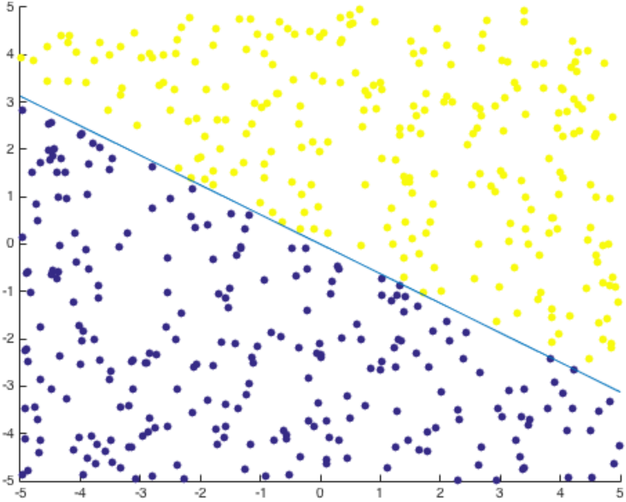
Linear Classifier: Toy Example

Final separating hyperplane, and solution to the LP:



Larger linear classifier problem

500 labeled examples



Outline

1. Algorithm Overview
2. MW Algorithm for Linear Programming
3. MW Algorithm for Semidefinite Programming

Matrix MW Algorithm

Solving an SDP using Multiplicative Weights algorithm:

Recall the LP:

$$\begin{aligned} a_j^T x &\geq 0 \quad \forall j \\ \mathbb{1}^T x &= 1 \\ x_i &\geq 0 \quad \forall i \end{aligned}$$

Now the SDP:

$$\begin{aligned} A_j \bullet X &\geq 0 \quad \forall j = 1, \ldots, m \\ \mathrm{Tr}(X) &= 1 \\ X &\in \mathbb{S}_+^n \end{aligned}$$

Matrix MW Algorithm

Define $\rho = \max_j ||A_j||$, $\eta = -\ln(1-\epsilon), \epsilon > 0$.

Initialize $W^{(0)} = I_n$ and $X^{(0)} = W^{(0)}/Tr(W^{(0)})$.

While $\exists\, j$ s.t. $A_j \bullet X^{(t)} < 0$:

1. Find the index j such that $A_j \bullet X^{(t)} < 0$
2. Obtain the payoff $M^{(t)} = \frac{A_j}{\rho}$
3. Update the weight matrix $W^{(t)}$ and $X^{(t)}$:

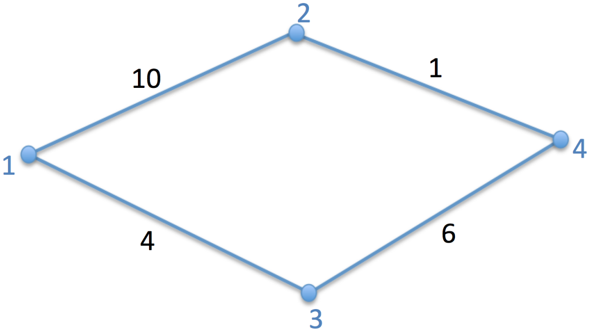
$$W^{(t+1)} = \exp\left(\eta \sum_{\tau=1}^{t} M^{(\tau)}\right)$$

$$X^{(t+1)} = \frac{W^{(t+1)}}{Tr(W^{(t+1)})}$$

Recall the MAXCUT problem

Given a weighted graph $G = (E, V)$, find a partition of the vertices into two vertex sets V_1 and V_2, i.e., $V = V_1 \dot{\cup} V_2$, that maximizes the "cut" edge weights.

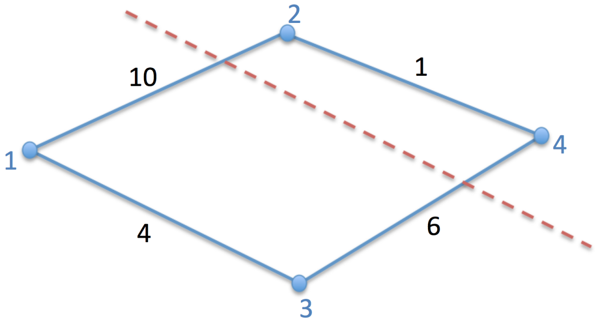
Figure 1: Toy example



Recall the MAXCUT problem

Given a weighted graph $G = (E, V)$, find a partition of the vertices into two vertex sets V_1 and V_2, i.e., $V = V_1 \dot{\cup} V_2$, that maximizes the "cut" edge weights.

Figure 2: Toy example



Recall the MAXCUT problem

For a graph $G = (E, V)$ with $|V| = n$ and $V = V_1 \dot\cup V_2$, the vector $x \in \mathbb{R}^n$ gives the partition:

$$x_i = \begin{cases} 1, & \text{if } v_i \in V_1 \\ -1, & \text{if } v_i \in V_2 \end{cases} \qquad i = 1, \ldots, n$$

Let w_{ij} := the weight on edge (i, j)

$$\max_{x \in \{-1,1\}^n} \sum_{(i,j) \in E} w_{ij} \frac{1 - x_i x_j}{2}$$

The MAXCUT relaxation

Introduce the auxiliary matrix variable $X = xx^T$:

$$\max_{x,X} \sum_{(i,j)\in E} w_{ij} \frac{1 - x_i x_j}{2}$$
$$\text{s.t. } X = xx^T$$
$$x_i^2 = 1$$

Then drop the rank 1 constraint for the relaxation:

$$\max_{X} \frac{1}{4} L \bullet X$$
$$\text{s.t. } X_{ii} = 1 \quad \forall\ i = 1 \ldots, n$$
$$X \succcurlyeq 0$$

$$L_{ij} = \begin{cases} \sum_{k\in V} w_{ik}, & i = j \\ -w_{ij}, & i \neq j \end{cases}$$

Reformulating the MAXCUT relaxation

Decision form: Does there exist a b^* such that

$$\frac{1}{4}L \bullet X \geq b^*$$
$$X_{ii} = 1 \quad \forall\ i = 1 \dots, n$$
$$X \succcurlyeq 0$$

Observe that $X_{ii} = 1 \implies (e_i e_i^T) \bullet X = 1$. Rescale X by $\frac{1}{n}$. Arrive at the formulation:

$$\left(\frac{n}{4b^*}L - I\right) \bullet X \geq 0$$
$$\left(n(e_i e_i^T) - I\right) \bullet X \geq 0 \quad \forall\ i = 1, \dots, n$$
$$\mathrm{Tr}(X) = 1$$
$$X \succcurlyeq 0$$

Reformulating the MAXCUT relaxation

Let $A_j = n(e_j e_j^T) - I$, $j = 1, \ldots, n$ and $A_{n+1} = \frac{n}{4b^*} L - I$.

$$\begin{aligned} A_j \bullet X &\geq 0 \quad \forall\, j = 1, \ldots, n \\ A_{n+1} \bullet X &\geq 0 \\ \mathrm{Tr}(X) &= 1 \\ X &\succcurlyeq 0 \end{aligned}$$

Matrix MW algorithm for MAXCUT (Large-Margin)

Define $\rho = \max_j ||A_j||$, $\eta = -\ln(1-\epsilon), \epsilon > 0$, $\delta > 0$.

Initialize $W^{(0)} = I_n$ and $X^{(0)} = W^{(0)} / Tr(W^{(0)})$.

While $\exists\, j$ s.t. $A_j \bullet X^{(t)} < -\delta$:

1. Find the index j such that $A_j \bullet X^{(t)} < -\delta$
2. Obtain the payoff $M^{(t)} = \frac{A_j}{\rho}$
3. Update the weight matrix $W^{(t)}$ and $X^{(t)}$:

$$W^{(t+1)} = \exp\left(\eta \sum_{\tau=1}^{t} M^{(\tau)}\right), \quad X^{(t+1)} = \frac{W^{(t+1)}}{Tr(W^{(t+1)})}$$

Toy MAXCUT problem

Number of nodes: 4. $b^* = 16$.

ϵ, δ	0.1	0.01	0.001	0.0001
T	20	328	3465	34827
$\lvert b^* - b^{(T)} \rvert$	0.4595	0.0482	0.0039	0.0002
$\frac{\lvert b^* - b^{(T)} \rvert}{\lvert b^* \rvert}$	2.88×10^{-2}	3.01×10^{-3}	2.44×10^{-4}	1.25×10^{-5}

Table 1: toy example

MAXCUT problem with 10 vertices

Number of nodes: 10. $b^* = 80$.

ϵ, δ	0.1	0.01	0.001	0.0001
T	19	633	7230	73228
$\lvert b^* - b^{(T)} \rvert$	1.9107	0.1908	0.0204	0.0011
$\frac{\lvert b^* - b^{(T)} \rvert}{\lvert b^* \rvert}$	2.39×10^{-2}	2.39×10^{-3}	2.55×10^{-4}	1.38×10^{-5}

Table 2: 10 nodes example

MAXCUT problem with 100 vertices

Number of nodes: 100. $b^* = 8190$.

ϵ, δ	0.1	0.01	0.001	0.0001
T	2777	49971		
$\lvert b^* - b^{(T)}\rvert$	3.2430	0.7318		
$\frac{\lvert b^* - b^{(T)}\rvert}{\lvert b^*\rvert}$	3.96×10^{-4}	8.94×10^{-5}		

Table 3: 100 nodes example

Conclusions

- The MW algorithm can be used to solve LP and SDP feasibility problems of certain form.
- Use the MW algorithm to solve Linear Classification problems.
- Matrix MW algorithm performance depends on ϵ and δ. (Parameter sensitive).

References

S. Arora, E. Hazan, and S. Kale. The multiplicative weights update method: A meta-algorithm and applications, *Theory of Computing*, 8 (2012), pp 121-164.

S. Arora and S. Kale. A combinatorial, primal-dual approach to semidefinite programs. In *Journal of the ACM*, 63 (2016).