OR 505

Homework 1

Blake Schwartz

1 (a) $a^2b^2 - a^2b^2 = \boxed{\emptyset}$ (b) $\cos^2\alpha + \sin^2\alpha = \boxed{1}$

(c) $\det\begin{pmatrix} 1 & x & x \\ x & 2 & x \\ x & x & 3 \end{pmatrix} = 6 - x^2 - x(3x - x^2) + x(x^2 - 2x)$

$\boxed{= 6 - 6x^2 + 2x^3}$

(d) $\det\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} = bc^2 - b^2c - ac^2 + a^2c + ab^2 - a^2b$

$\boxed{= -(a-b)(a-c)(b-c)}$

(e) $$\det\begin{pmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{pmatrix} = 1 \cdot \begin{vmatrix} 0 & b & 0 \\ c & 4 & 5 \\ 0 & 0 & 0 \end{vmatrix} - 0 + 2 \cdot \begin{vmatrix} 2 & 0 & 0 \\ 3 & 4 & 5 \\ d & 0 & 0 \end{vmatrix} - a \begin{vmatrix} 2 & 0 & b \\ 3 & c & 4 \\ d & 0 & 0 \end{vmatrix}$$

$= -ab \cdot -cd$

$= \boxed{abcd}$

(f)

$$\det\begin{pmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{pmatrix} = a\begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} - \begin{vmatrix} -1 & 1 & 0 \\ 0 & c & 1 \\ 0 & -1 & d \end{vmatrix}$$

$= a(b(cd+1+d)) + cd + 1 = \boxed{abcd + ba + da + cd + 1}$

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$$\left[\begin{array}{cccc|c} 2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 2 & 0 & -\frac{1}{2} & 3 \\ -\frac{1}{2} & 0 & 2 & -\frac{1}{2} & 3 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 2 & 0 \end{array}\right] \xrightarrow[-L_3]{\times 4 + L_1} \left[\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 0 & -\frac{1}{2} & 7\frac{1}{2} & -2 & 12 \\ 0 & 0 & -8 & 4 & -12 \end{array}\right] \xrightarrow[\times \frac{7}{2} + L_3]{\times 4 + L_2} \left[\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 28 & -8 & 48 \\ 0 & 0 & 0 & 6 & 6 \end{array}\right]$$

$$\boxed{\begin{aligned} X_4 &= 1 \\ X_3 &= (12 + 2X_4)/7 = 2 \\ X_2 &= X_3 = 2 \\ X_1 &= (X_2 + X_3)/4 = 1 \end{aligned}}$$

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$$\left[\begin{array}{cccc|c} 2 & 3 & 5 & 1 & 3 \\ 3 & 4 & 2 & 3 & -2 \\ 1 & 2 & 8 & -1 & 8 \\ 7 & 9 & 1 & 8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 7 & 9 & 1 & 8 & 0 \\ 0 & -1 & -11 & 3 & -13 \\ 0 & -2 & -22 & 6 & -26 \\ 7 & 9 & 1 & 8 & 0 \end{array}\right]$$

The converging equations mean no solution exists

③ $A = P \Lambda Q = \begin{bmatrix} 7 & -12 \\ 4 & -7 \end{bmatrix}$ $\qquad A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

So $A^8 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$A^9 = \begin{bmatrix} 7 & -12 \\ 4 & -7 \end{bmatrix}$

$\left.\begin{aligned} A^{2n} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ A^{2n+1} &= \begin{bmatrix} 7 & -12 \\ 4 & -7 \end{bmatrix} \end{aligned}\right\}$ for all n

(4) $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ $M^{-1} = \begin{bmatrix} 1 & -1/2 & 0 \\ 1 & -1/2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ $S = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$M^{-1} \times S = \begin{bmatrix} 1/2 \\ -3/2 \\ 2 \end{bmatrix}$ $\boxed{\begin{aligned} X_1 &= 1/2 \\ X_2 &= -3/2 \\ X_3 &= 2 \end{aligned}}$

(5) i) Not true. Ex. $A = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$, not invertible
$B = \begin{bmatrix} 2 & 8 \\ 1 & 4 \end{bmatrix}$, not invertible. $A+B = \begin{bmatrix} 4 & 11 \\ 3 & 7 \end{bmatrix}$, which is invertible. $(A+B)^{-1} = \begin{bmatrix} -1.4 & 2.2 \\ .6 & -.8 \end{bmatrix}$

ii) True. Let $C = AB$
Then $B = A^{-1}AB = A^{-1}C$ and $A = ABB^{-1} = CB^{-1}$
then $C = (CB^{-1})(A^{-1}C) = CB^{-1}A^{-1}C$
So $CB^{-1}A^{-1} = I$ (identity matrix)
and $B^{-1}A^{-1} = C^{-1} = (AB)^{-1}$
Since $(AB)^{-1}$ can be expressed as $B^{-1}A^{-1}$, A and B must be invertible if AB is invertible, Q.E.D

iii) Not true. Ex: $A = \begin{bmatrix} 2 & 8 \\ 1 & 4 \end{bmatrix}$, not invertible
$B = \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix}$, invertible. $AB = \begin{bmatrix} 36 & 12 \\ 18 & 6 \end{bmatrix}$, not invertible.

AB is not invertible, but B is.

iv) True. $(kA)^{-1} = k^{-1}A^{-1}$, $k \neq 0$
So if A is invertible and $k \neq 0$, (kA) is invertible, Q.E.D.

(6) $A^T(B^{-1}A^{-1} + I)^T = B(B^{-1}A^{-1} + I)^T$

$B(B^{-1})^T(A^{-1})^T + B = B(B^{-1})(A^{-1}) + B = A^{-1} + B$

So (b) is the simplified form

$$\text{①}\quad \begin{array}{c} \begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 2 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{array}{c} +L3 \\ -L1 \\ +L4 \\ -L1 \end{array} \begin{bmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 0 & -2 & -2 & 1 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & -2 & -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{array}{c} +L2 \\ L4/-2 \\ \\ +L2 \end{array} \begin{bmatrix} 1 & 0 & 0 & -2 & 3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -.5 \\ 0 & 0 & 0 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{array}{c} -L4/2 \\ -L3 \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 2.5 \\ 0 & 1 & 0 & -1 & .5 \\ 0 & 0 & 1 & 1 & -.5 \\ 0 & 0 & 0 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{array}{c} \\ -L4/4 \\ -L4/-4 \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 2.5 \\ 0 & 1 & 0 & 0 & .25 \\ 0 & 0 & 1 & 0 & -.25 \\ 0 & 0 & 0 & 1 & -.25 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\text{So } -\alpha \begin{bmatrix} 2.5 \\ .25 \\ -.25 \\ -.25 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}}$$

~~and~~ $\cancel{\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}}$

(8) $\alpha_{1,2,3} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 5 & 6 \end{bmatrix}$ $\quad Det(\alpha_{1,2,3}) = 12-15+5-2 = 0$

Since the determinant $= 0$, the vectors are linearly dependent.

(9) Iff $\delta_1, \delta_2, \delta_3$ are independent, $x_1\delta_1 + x_2\delta_2 + x_3\delta_3 =$ and $x_1 = x_2 = x_3 = 0$.

Iff $[\delta_1 + \delta_2], [\delta_2 + \delta_3], [\delta_1 + \delta_3]$ are linearly independent, then

$$y_1(\delta_1 + \delta_2) + y_2(\delta_2 + \delta_3) + y_3(\delta_1 + \delta_3) = 0$$

$$= (y_1 + y_3)\delta_1 + (y_1 + y_2)\delta_2 + (y_2 + y_3)\delta_3 = 0$$

Thus $(y_1 + y_3), (y_1 + y_2), (y_2 + y_3) = 0$

Since $y_1 = y_2 = y_3 = 0$ is the only solution, $\delta_1 + \delta_2, \delta_2 + \delta_3, \delta_1 + \delta_3$ are linearly independent Iff $\delta_1, \delta_2, \delta_3$ are linearly independent.

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$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 rows are linearly independent,
So matrix rank = 3

$$\begin{bmatrix} 3 & 2 & -1 & -3 & -2 \\ 2 & -1 & 3 & 1 & -3 \\ 4 & 5 & -5 & -6 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 2 & -1 & -3 & -2 \\ 0 & -7/3 & 11/3 & 3 & -5/3 \\ 0 & 7/3 & -11/3 & -2 & 11/3 \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{bmatrix} 3 & 2 & -1 & -3 & -2 \\ 0 & -7/3 & 11/3 & 3 & -5/3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

All 3 rows are linearly independent, so the
matrix rank = 3.