1.) $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix} = a^2b^2 - a^2b^2 = \boxed{0}$

$\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha + \sin^2\alpha = \boxed{1}$

$$\begin{vmatrix} 1 & x & x \\ x & 2 & x \\ x & x & 3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & x \\ x & 3 \end{vmatrix} - x\begin{vmatrix} x & x \\ x & 3 \end{vmatrix} + x\begin{vmatrix} x & 2 \\ x & x \end{vmatrix} = 6 - x^2 - x(3x - x^2) + x(x^2 - 2x) = 6 - x^2 - 3x^2 + x^3 + x^3 - 2x^2$$

$\hookrightarrow \boxed{= 6 - 6x^2 + 2x^2}$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} b & c \\ b^2 & c^2 \end{vmatrix} - \begin{vmatrix} a & c \\ a^2 & c^2 \end{vmatrix} + \begin{vmatrix} a & b \\ a^2 & b^2 \end{vmatrix} = bc^2 - b^2c - ac^2 + a^2c + ab^2 - a^2b = \boxed{(a-c)(b-c)(b-a)}$$

$$\begin{vmatrix} 1^+ & 0 & 2 & a \\ 2^- & 0 & b & 0 \\ 3^+ & c & 4 & 5 \\ d^- & 0^+ & 0^- & 0^+ \end{vmatrix} = -d\begin{vmatrix} 0^+ & 2 & a \\ 0^- & b & 0 \\ c^+ & 4 & 5 \end{vmatrix} = -d\left(c\begin{vmatrix} 2 & a \\ b & 0 \end{vmatrix}\right) = -d(c(-ba)) = \boxed{abcd}$$

$$\begin{vmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix} = a\begin{vmatrix} b & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} = a\left(b\begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ -1 & d \end{vmatrix}\right) + \begin{vmatrix} c & 1 \\ -1 & d \end{vmatrix} = a(b(cd+1)+d) + cd + 1$$

$\hookrightarrow = \boxed{abcd + ab + ad + cd + 1}$

2.)
$$\left[\begin{array}{cccc|c} 2 & -\tfrac12 & -\tfrac12 & 0 & 0 \\ -\tfrac12 & 2 & 0 & -\tfrac12 & 3 \\ -\tfrac12 & 0 & 2 & -\tfrac12 & 3 \\ 0 & -\tfrac12 & -\tfrac12 & 2 & 0 \end{array}\right] \begin{array}{l} -2r_1 \\ -2r_2 \\ -2r_3 \\ -2r_4 \end{array} \rightarrow \left[\begin{array}{cccc|c} -4 & 1 & 1 & 0 & 0 \\ 1 & -4 & 0 & 1 & -6 \\ 1 & 0 & -4 & 1 & -6 \\ 0 & 1 & 1 & -4 & 0 \end{array}\right] \xrightarrow{\text{swap } r_1, r_2} \left[\begin{array}{cccc|c} 1 & -4 & 0 & 1 & -6 \\ -4 & 1 & 1 & 0 & 0 \\ 1 & 0 & -4 & 1 & -6 \\ 0 & 1 & 1 & -4 & 0 \end{array}\right] \xrightarrow[R_3 - r_1]{R_2 + 4r_1} \left[\begin{array}{cccc|c} 1 & -4 & 0 & 1 & -6 \\ 0 & -15 & 1 & 4 & -24 \\ 0 & 4 & -4 & 0 & 0 \\ 0 & 1 & 1 & -4 & 0 \end{array}\right]$$

swap r_2, r_4
$$\hookrightarrow \left[\begin{array}{cccc|c} 1 & -4 & 0 & 1 & -6 \\ 0 & 1 & 1 & -4 & 0 \\ 0 & 4 & -4 & 0 & 0 \\ 0 & -15 & 1 & 4 & -24 \end{array}\right] \begin{array}{l} R_1 + 4r_2 \\ R_3 - 4r_2 \\ R_4 + 15r_2 \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 4 & -15 & -6 \\ 0 & 1 & 1 & -4 & 0 \\ 0 & 0 & -8 & 16 & 0 \\ 0 & 0 & 16 & -56 & -24 \end{array}\right] \xrightarrow{-\frac18 r_3} \left[\begin{array}{cccc|c} 1 & 0 & 4 & -15 & -6 \\ 0 & 1 & 1 & -4 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 16 & -56 & -24 \end{array}\right] \begin{array}{l} R_1 - 4r_3 \\ R_2 - r_3 \\ R_4 - 16r_3 \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -7 & -6 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & -24 & -24 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{24} r_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -7 & -6 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 + 7r_4 \\ R_2 + 2r_4 \\ R_3 + 2r_4 \end{array} \rightarrow \boxed{\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} x_1 = 1 \\ x_2 = 2 \\ x_3 = 2 \\ x_4 = 1 \end{array}}$$

2. 2nd system

$$\begin{bmatrix} 2 & 3 & 5 & 1 & 3 \\ 3 & 4 & 2 & 3 & -2 \\ 1 & 2 & 8 & -1 & 8 \\ 7 & 9 & 1 & 8 & 0 \end{bmatrix} \xrightarrow[r_1, r_3]{\text{swap}} \begin{bmatrix} 1 & 2 & 8 & -1 & 8 \\ 3 & 4 & 2 & 3 & -2 \\ 2 & 3 & 5 & 1 & 3 \\ 7 & 9 & 1 & 8 & 0 \end{bmatrix} \begin{matrix} \\ R_2 - 3r_1 \\ R_3 - 2r_1 \\ R_4 - 7r_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 8 & -1 & 8 \\ 0 & -2 & -22 & 6 & -26 \\ 0 & -1 & -11 & 3 & -13 \\ 0 & -5 & -55 & 15 & -56 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2}$$

$$\hookrightarrow \begin{bmatrix} 1 & 2 & 8 & -1 & 8 \\ 0 & 1 & 11 & -3 & 13 \\ 0 & -1 & -11 & 3 & -13 \\ 0 & -5 & -55 & 15 & -56 \end{bmatrix} \begin{matrix} R_1 - 2r_2 \\ \\ R_3 + r_2 \\ R_4 + 5r_2 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 0 & -14 & 5 & -18 \\ 0 & 1 & 11 & -3 & 13 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix} \xrightarrow[\text{Then } \frac{1}{9}r_3]{\text{Swap } r_3, r_4} \begin{bmatrix} 1 & 0 & -14 & 5 & -18 \\ 0 & 1 & 11 & -3 & 13 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 - 13r_3]{R_1 + 18r_3}$$

$$\hookrightarrow \boxed{\begin{bmatrix} 1 & 0 & -14 & 5 & 0 \\ 0 & 1 & 11 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

3.)

$$P \wedge Q = A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\boxed{A^8 = A^{2n} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A^9 = A^{2n+1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

4.)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad [A \mid I] = [I \mid A^{-1}] \qquad \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - r_1]{}$$

$$\hookrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1/2 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_3 + 2r_2]{R_1 - r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/2 & 0 \\ 0 & 1 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow{R_2 - r_3}$$

$$\hookrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/2 & 0 \\ 0 & 1 & 0 & 1 & -1/2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & 0 \\ 1 & -1/2 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \rightarrow \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -3/2 \\ 2 \end{bmatrix}}$$

5i) False $A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 5 \\ 4 & 10 \end{bmatrix}$ $A + B = \begin{bmatrix} 3 & 8 \\ 7 & 19 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 8 \\ 7 & 19 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} 19 & -8 \\ -7 & 3 \end{bmatrix}}$

5ii) True

$$C = B(AB)^{-1} \quad D = (AB)^{-1}A \rightarrow \boxed{\begin{aligned} &AC = A(B(AB)^{-1}) = AB(AB)^{-1} = I \rightarrow C = A^{-1} \\ &DB = ((AB)^{-1}A)B = (AB)^{-1}AB = I \rightarrow D = B^{-1} \end{aligned} \quad \text{A and B are invertible}}$$

5iii) True $\det(AB) = \det(A)\det(B) = 0 \rightarrow \det(A) = 0$, $\det(B) = 0$

5.iv.) True $\left(\frac{1}{k}A^{-1}\right)\cdot(kA)=\left(\frac{1}{k}k\right)(A^{-1}A)=I$

6.) $A=B^T \to B=A^T \quad (B^{-1})^T=(B^T)^{-1}$ and $(A^{-1})^T=(A^T)^{-1}$

$$A^T(B^{-1}A^{-1}+I)^T=B((B^{-1})^T(A^{-1})^T+I)=B((B^T)^{-1}(A^T)^{-1}+I)\to$$

$$\hookrightarrow =B(A^{-1}B^{-1}+I)=A^{-1}BB^{-1}+B=A^{-1}+B=\boxed{B+A^{-1}}$$

$\boxed{\text{Answer B}}$

7.) $\alpha=a\alpha_1+b\alpha_2+c\alpha_3+d\alpha_4$

$$\begin{bmatrix}1\\2\\1\\1\end{bmatrix}=a\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+b\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}+c\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}+d\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$$

$$\begin{bmatrix}1&1&1&1&1\\1&1&-1&-1&2\\1&-1&1&-1&1\\1&-1&-1&1&1\end{bmatrix}\xrightarrow{R_2-R_3}\begin{bmatrix}1&1&1&1&1\\0&2&-2&0&1\\1&-1&1&-1&1\\1&-1&-1&1&1\end{bmatrix}\xrightarrow[R_3-R_4]{\frac{1}{2}R_2}\begin{bmatrix}1&1&1&1&1\\0&1&-1&0&1\\0&0&2&-2&0\\1&-1&-1&1&1\end{bmatrix}\xrightarrow{\frac{1}{2}R_3}$$

$$\hookrightarrow\begin{bmatrix}1&1&1&1&1\\0&-1&-1&0&1\\0&0&1&-1&0\\1&-1&-1&1&1\end{bmatrix}\xrightarrow[R_4-R_3]{}\begin{bmatrix}1&1&1&1&1\\0&1&-1&0&1\\0&0&1&-1&0\\1&-1&-2&2&1\end{bmatrix}\xrightarrow[R_4=R_1-R_4]{}\begin{bmatrix}1&1&1&1&1\\0&1&-1&0&1\\0&0&1&-1&0\\0&2&3&-1&0\end{bmatrix}\xrightarrow[R_4-2R_2]{}\begin{bmatrix}1&1&1&1&1\\0&1&-1&0&1\\0&0&1&-1&0\\0&0&5&-1&-2\end{bmatrix}$$

$$\hookrightarrow\begin{bmatrix}1&1&1&1&1\\0&1&-1&0&1\\0&0&1&-1&0\\0&0&0&4&-2\end{bmatrix}_{R_4-5R_3}\xrightarrow{\frac{1}{4}R_4}\boxed{\begin{bmatrix}1&1&1&1&1\\0&1&-1&0&1\\0&0&1&-1&0\\0&0&0&1&-1/2\end{bmatrix}}$$

$$\boxed{d=-\frac{1}{2}\quad b=\frac{1}{2}\quad c=-\frac{1}{2}\quad a=\frac{3}{2}}$$

8.) $\begin{vmatrix}1&1&1\\0&2&5\\1&3&6\end{vmatrix}=\begin{vmatrix}2&5\\3&6\end{vmatrix}+\begin{vmatrix}1&1\\2&5\end{vmatrix}=-3+3=0$

No, determinant of the matrix is zero, thus they are not linearly independent.

9.) Iff δ_1, δ_2, and δ_3 are L.I.: $a\delta_1 + b\delta_2 + c\delta_3 = 0 \rightarrow a=b=c=0$

and $x(\delta_1+\delta_2)+y(\delta_2+\delta_3)+z(\delta_3+\delta_1)=0 \rightarrow$

$\hookrightarrow (x+z)\delta_1 + (x+y)\delta_2 + (y+z)\delta_3 = 0$

$\hookrightarrow x+z = x+y = y+z = 0$

$\hookrightarrow x = -z$

$\hookrightarrow x+y = y-z \rightarrow y = z$

$\hookrightarrow y+z = 2y \rightarrow 2y = 0 \rightarrow y = 0 \rightarrow y = z = 0 \rightarrow -z = x = 0$

$\hookrightarrow x = y = z = 0$

10.)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & -1 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & -1 \end{bmatrix} \xrightarrow{R_4 - r_2} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix} \xrightarrow{\frac{1}{4}R_3} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix} \xrightarrow{R_4 + 4r_3}$$

$$\hookrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{Rank} = 3}$

2nd Matrix)

$$\begin{bmatrix} 3 & 2 & -1 & -3 & -2 \\ 2 & -1 & 3 & 1 & -3 \\ 4 & 5 & -5 & -6 & 1 \end{bmatrix} \xrightarrow{R_1 - r_2} \begin{bmatrix} 1 & 3 & -4 & -4 & 1 \\ 2 & -1 & 3 & 1 & -3 \\ 4 & 5 & -5 & -6 & 1 \end{bmatrix} \xrightarrow[R_3 - 4r_1]{R_2 - 2r_1} \begin{bmatrix} 1 & 3 & -4 & -4 & 1 \\ 0 & -7 & 11 & 9 & -3 \\ 0 & -7 & 11 & 10 & -3 \end{bmatrix} \xrightarrow{-\frac{1}{7}R_2} \begin{bmatrix} 1 & 3 & -4 & -4 & 1 \\ 0 & 1 & -11/7 & -9/7 & 5/7 \\ 0 & -7 & 11 & 10 & -3 \end{bmatrix} \rightarrow$$

$$\xrightarrow{R_3 + 7r_2} \begin{bmatrix} 1 & 3 & -4 & -4 & 1 \\ 0 & 1 & -11/7 & -9/7 & 5/7 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$\boxed{\text{Rank} = 3}$