MA 515-001, Fall 2017, Practice Problems, Test 2

Problem 1. Recalling that

$$l^{\infty} \left\{ x = \{x_i\}_{i\geq 1} \;\middle|\; \{x_i\}_{i\geq 1} \text{ is bounded} \right\}$$

and

$$\|x\|_{l^{\infty}} \doteq \sup_{i\geq 1} |x_i| .$$

Define the operator $\Lambda : l^{\infty} \to l^{\infty}$ such that for any $x \in l^{\infty}$,

$$\Lambda(x) = y = \{y_n\}_{n\geq 1} \qquad \text{with} \qquad y_n = \frac{x_1 + 2x_2 + ... + nx_n}{n^2} .$$

Show that Λ is bounded and compute $\|\Lambda\|_{\infty}$.

Problem 2. Recalling that

$$l^{1} = \left\{ x = \{x_i\}_{i\geq 1} \;\middle|\; \sum_{i=1}^{\infty} |x_i| < +\infty \right\}$$

and

$$\|x\|_{l^1} \doteq \sum_{i=1}^{\infty} |x_i| \qquad \forall x \in l^1 .$$

Define the operator $\Lambda : l^1 \to l^1$ such that for any $x \in l^1$,

$$\Lambda(x) = y = \{y_n\}_{n\geq 1} \qquad \text{with} \qquad y_n = \frac{x_n + x_{n+1}}{2} .$$

Show that Λ is bounded and compute $\|\Lambda\|_{\infty}$.

Problem 3. *Given $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ two normed spaces, let $T_n : X \to Y$ be bounded linear operators with*

$$\|T_n\|_{\infty} < M \qquad \forall n \in \mathbb{N}$$

for some constant $M > 0$. Assume that

$$\lim_{n\to\infty} \frac{T_1(x) + ... + T_n(x)}{n} = T(x) \in Y \qquad \forall x \in X .$$

Show that $T : X \to Y$ is a bounded linear operator and

$$\|T\|_{\infty} \leq M .$$

Problem 4. Given $(X, \|\cdot\|_X)$ a normed space, let $\{T_n\}_{n\geq 1} \subset B(X, X)$ converge T in $B(X, X)$. Denote by

$$\Lambda_n(x) \doteq (T_n \circ T_n)(x) \qquad \text{and} \qquad \Lambda(x) \doteq (T \circ T)(x) \qquad \forall x \in X .$$

Show that

(i) The sequence $\{\Lambda_n\}_{n\geq 1}$ converges to Λ in $B(X,X)$.

(ii) If $\{x_n\}_{n\geq 1} \subset X$ converges to x in X then the sequence $\{\Lambda_n(x_n)\}_{n\geq 1}$ converges to $\Lambda(x)$ in X.

Problem 5. Let $T : X \to X$ be a bounded linear operators such that

- T is surjective, i.e., $T(X) = X$;
- There is a positive constant M such that

$$\|T(x)\| \;\geq\; M \cdot \|x\| \qquad \forall x \in X\,.$$

Denote by

$$T_2(x) \;=\; (T \circ T)(x) \;=\; T(T(x))\,.$$

Show that $(T_2)^{-1} : X \to X$ exists and is a linear bounded operator.

Problem 6. Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a bounded linear operator such that $\|T\|_\infty < 1$. Denote by

$$\Lambda(x) \;=\; \frac{x + T(x)}{2} \qquad \forall x \in X\,.$$

Show that the linear operator bounded $\Lambda : X \to X$ is one-to-one and onto.

Problem 7. Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a bounded linear operator such that $\|T\|_\infty < 3$. Denote by

$$\Lambda(x) \;=\; \frac{3x + T(x)}{4} \qquad \forall x \in X\,.$$

Show that the linear operator bounded $\Lambda : X \to X$ is one-to-one and onto.

Problem 8. Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a bounded linear operator with $\|T\|_\infty < M$. Denote by

$$T^0 \;=\; \mathbb{I} \qquad \text{and} \qquad T^n \;=\; T^{n-1} \circ T.$$

Show that the series

$$\sum_{n=0}^{+\infty} \frac{1}{2^n} \cdot \left(\mathbb{I} + \frac{T}{M}\right)^n \text{ converges in } B(X,X)\,.$$

Moreover,

$$\sum_{n=0}^{+\infty} \frac{1}{2^n} \cdot \left(\mathbb{I} + \frac{T}{M}\right)^n \;=\; \left(\frac{M \cdot \mathbb{I} - T}{2M}\right)^{-1}\,.$$

Problem 9. Let X be a Banach space, and let $T : X \to X$ be a linear continuous bijection. Show that there exists a constant $\beta > 1$ such that

$$\frac{1}{\beta} \cdot \|x\| \;\leq\; \|(T \circ T)(x)\| \;\leq\; \beta \cdot \|x\| \qquad \forall x \in X\,.$$