Homework 7 Solutions

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Problem 1 (4.7)

Proof. Suppose that the dual is feasible and bounded. Then, it has a finite optimum. By strong duality theorem, the dual of dual, which is the primal, also has a finite optimum. But this is a contradiction to the infeasibility of primal problem.

□

Problem 2 (4.8)

Problem 4 (4.11)

Proof. If $Ax = b, x \geqslant 0$ has a solution x_0 and $A^T w \geqslant 0$, then $x_0^T A^T w \geqslant$. This implies that $b^T w \geqslant 0$.

Conversely, we need to show that if $Ax = b, x \geqslant 0$ has no solution, then $b^T w < 0$ as $A^w \geqslant 0$. Indeed, this is true due to Farkas Lemma.

□

Problem 5 (4.13)

Proof. If $Ax \leqslant b, x \geqslant 0$ has a solution x_0 and $A^T w \geqslant 0, w \geqslant 0$, then $x_0^T A^T \leqslant b^T$. Multiply w on both sides and we have

$$0 = x_0^T A^T w \leqslant b^T w.$$

which is $b^T w \geqslant 0$.

Conversely, we need to show that if $b^T w \geqslant 0$ when $A^T w \geqslant 0, w \geqslant 0$, then $Ax \leqslant b, x \geqslant 0$ has a solution. Consider the following primal dual problem,

$$\begin{aligned} \text{Min} \quad & 0^T x \\ \text{Subject to} \quad & Ax \leqslant b \\ & x \geqslant 0 \end{aligned} \tag{1}$$

$$\begin{aligned} \text{Max} \quad & b^T y \\ \text{Subject to} \quad & A^T y \leqslant 0 \\ & y \leqslant 0 \end{aligned} \tag{2}$$

(2) is equivalent to (3)

$$\begin{aligned} \text{Max} \quad & -b^T w \\ \text{Subject to} \quad & A^T w \geqslant 0 \\ & w \geqslant 0 \end{aligned} \tag{3}$$

In (3), it is obvious that $w = 0$ is a feasible solution. What's more, it is also an optimal solution due to the assumption that $b^T w \geqslant 0$ when $A^T w \geqslant 0, w \geqslant 0$. Hence, $\max -b^T w = 0$. By strong duality theorem, we know (1) is also feasible.

□

Problem 6 (4.18)

Standard form of the primal:

$$\begin{aligned} \text{Minimize} \quad & 2x_1 + x_2 - x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 + a_1 && = 8 \\ & -x_1 + x_2 - 2x_3 + a_2 && = 4 \\ & x_1, x_2, x_3, a_1, a_2 && \geqslant 0 \end{aligned}$$

From point $x = [x_1, x_2, x_3, a_1, a_2]^T = [0, 0, 0, 8, 4]^T$, we know that $B = [A_4, A_5]$ and $N = [A_1, A_2, A_3]$. Compute reduced cost $r = c_N^T - c_B^T B^{-1} N = [2, 1, -1]$.

Note that r_3 is negative, so x_3 enter the basis. Construct

$$M^{-1} = \begin{bmatrix} B^{-1} & -B^{-1}N \\ 0 & I \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & -2 & -1 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

to figure out $\mathbf{d}^3 = [-1, 1, 1, 0, 0]^T$ and the step length $\alpha_3 = 8$. Hence, $x_{\text{new}} = x + \alpha_3 \mathbf{d}^3 = [0, 0, 8, 0, 20]^T$.

Next step:

From point $x = [x_1, x_2, x_3, x_4, x_5]^T = [0, 0, 8, 0, 20]^T$, we know that $B = [A_3, A_5]$ and $N = [A_1, A_2, A_4]$. Compute reduced cost $r = c_N^T - c_B^T B^{-1} N = [3, 3, 1]^T \geqslant 0$. Hence, this is the optimal solution. The optimal solution is $x^* = [0, 0, 8, 0, 20]^T$ and the optimal value $z^* = c^T x^* = -8$. The optimal dual is $w^{*T} = c_B^T B^{-1} = (-1, 0)$.

(a) x_2 is not a basic variable, so $c_B^T B^{-1}$ doesn't change. Also, to check optimality, consider

$$r^T = c_N^T - c_B^T B^{-1} N = [2, c_2, 0] - [-1, -2, -1] = [3, c_2 + 2, 1].$$

where c_2 now is 6. Hence, $r^T \geqslant 0$ so that the solution remains optimal.

(b) A_2 changed, but B and c_B^T don't change. So,

$$r^T = c_N^T - c_B^T B^{-1} N = [2, 1, 0] - [-1, 0] \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & a_{12} & 1 \\ -1 & 1 & 0 \end{bmatrix} = [3, 1 + a_{12}, 1].$$

where $a_{12} = 0.25$, $1 + a_{12} = 1.25 > 0$. Hence, the solution remains optimal.

(c) If we add new constraint $x_2 + x_3 = 3$, then substitute x_2 with $x_2 = 3 - x_3$ and we have a LP problem,

$$\begin{array}{lll} \text{Minimize} & 2x_1 - 2x_3 + 3 & \\ \text{subject to} & x_1 - x_3 & \leqslant 2 \\ & -x_1 - 3x_3 & \leqslant 1 \\ & x_3 & \leqslant 3 \\ & x_1, x_3 & \geqslant 0 \end{array}$$

Solve it and we get infinitely many optimal solutions. $x^* = [x_1, x_3]^T$ lies on the segment ($x_1 - x_3$ plane) between $[2, 0]^T$ and $[5, 3]^T$. Hence the optimal value is $z^* = 1$.

(d)