Homework 6 Solutions

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Problem 1 (4.1)

Proof. We reform the primal problem into standard form.

$$\begin{aligned} \text{Minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} - \mathbf{s} = \mathbf{b}. \\ & \mathbf{x}, \mathbf{s} \geqslant \mathbf{0} \end{aligned} \tag{1}$$

Consider the dual of (1). We can rewrite the constraint,

$$\begin{bmatrix} A & -I \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} = \mathbf{b}, \qquad \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} \geqslant \mathbf{0}.$$

where A is m by n, $\mathbf{x} \in \mathbb{R}^n, c \in \mathbb{R}^n, \mathbf{s} \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^m$. Hence, the dual is

$$\begin{aligned} \text{Maximize} \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & \begin{bmatrix} A^T \\ -I \end{bmatrix} \mathbf{y} \leqslant \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \end{bmatrix}. \end{aligned} \tag{2}$$

(2) is equivalent to

$$\begin{aligned} \text{Maximize} \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & A^T\mathbf{y} \leqslant \mathbf{c} \\ & \mathbf{y} \geqslant \mathbf{0}. \end{aligned}$$

where A is m by n, $\mathbf{y} \in \mathbb{R}^m, \mathbf{b} \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

□

Problem 2 (4.2)

You will find (a) and (b) are primal-dual problems, i.e., (b) is the dual of (a) and the dual of (b) is (a).

Problem 3 (4.3)

1. We reform (a-1) into standard form. Let $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$.

$$\begin{aligned} \text{Minimize} \quad & \begin{bmatrix} \mathbf{c}^T & -\mathbf{c}^T \end{bmatrix} \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{bmatrix} \\ \text{subject to} \quad & \begin{bmatrix} A & -A \end{bmatrix} \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{bmatrix} = \mathbf{b} \\ & \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{bmatrix} \geqslant \mathbf{0}. \end{aligned} \tag{3}$$

The dual of (3) is

$$\begin{aligned} \text{Maximize} \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & \begin{bmatrix} A^T \\ -A^T \end{bmatrix} \mathbf{y} \leqslant \begin{bmatrix} \mathbf{c} \\ -\mathbf{c} \end{bmatrix}. \end{aligned} \tag{4}$$

The constrain of (4) is equivalent to $A^T\mathbf{x} = \mathbf{c}$.

Hence, (a-1) and (a-2) are primal-dual problems.

2. For (b-1), let $-\mathbf{x} = \mathbf{y}$, then $\mathbf{y} \geqslant \mathbf{0}$. Then (b-1) will be in the similar form of the primal problem in example 4.2 on the book.

$$\begin{aligned} \text{Minimize} \quad & -\mathbf{c}^T\mathbf{y} \\ \text{subject to} \quad & A\mathbf{y} \geqslant -\mathbf{b}. \\ & \mathbf{y} \geqslant \mathbf{0} \end{aligned} \tag{5}$$

and its dual is

$$\begin{aligned} \text{Maximize} \quad & -\mathbf{b}^T\mathbf{z} \\ \text{subject to} \quad & A^T\mathbf{z} \leqslant -\mathbf{c}. \\ & \mathbf{z} \geqslant \mathbf{0} \end{aligned} \tag{6}$$

Let $\mathbf{w} = -\mathbf{z}$ in (6) and we get exact same form of (b-2).

Problem 4 (4.4)

$$\begin{aligned} \text{Maximize} \quad & 14y_1 + 17y_2 + 19y_3 + 100 & \\ \text{subject to} \quad & 3y_1 + 5y_2 + 2y_3 & \geqslant 9 \\ & 8y_1 - 2y_2 + 4y_3 & \leqslant 6 \\ & -5y_1 + 6y_2 & = -4 \\ & y_1 \geqslant 0, y_3 \leqslant 0, y_2 \text{ unrestricted.} & \end{aligned} \tag{7}$$

Problem 5 (4.5)

(a) & (b) Actually, this is the same problem as example 4.2. The dual of (a) is (b) and the dual of (b) is (a).

(c) We reform the primal problem into a standard form.

$$\begin{aligned} \text{Minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} - \mathbf{s} = \mathbf{l} \\ & \mathbf{x} + \mathbf{t} = \mathbf{u} \\ & \mathbf{s}, \mathbf{t} \geqslant \mathbf{0}. \end{aligned} \tag{8}$$

Consider $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$. We can rewrite (8) as

$$\begin{aligned} \text{Minimize} \quad & \begin{bmatrix} \mathbf{c}^T & -\mathbf{c}^T & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{s} \\ \mathbf{t} \end{bmatrix} \\ \text{subject to} \quad & \begin{bmatrix} A & -A & 0 & 0 \\ I & -I & -I & 0 \\ I & -I & 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{s} \\ \mathbf{t} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{l} \\ \mathbf{u} \end{bmatrix} \\ & \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{s} \\ \mathbf{t} \end{bmatrix} \geqslant \mathbf{0}. \end{aligned} \tag{9}$$

where A is m by n, $\mathbf{x}^+, \mathbf{x}^-, \mathbf{s}, \mathbf{t} \in \mathbb{R}^n, c \in \mathbb{R}^n$ and $\mathbf{b}, \mathbf{l}, \mathbf{u} \in \mathbb{R}^m$. Hence, the dual is

$$\begin{aligned} \text{Maximize} \quad & \begin{bmatrix} \mathbf{b}^T & \mathbf{l}^T & \mathbf{u}^T \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} \\ \text{subject to} \quad & \begin{bmatrix} A^T & I & I \\ -A^T & -I & -I \\ 0 & -I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} \leqslant \begin{bmatrix} \mathbf{c} \\ -\mathbf{c} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \end{aligned} \tag{10}$$

where A is m by n, $\mathbf{w}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^m, c \in \mathbb{R}^n$ and $\mathbf{b}, \mathbf{l}, \mathbf{u} \in \mathbb{R}^m$.

Simplify it and get

$$\begin{aligned} \text{Maximize} \quad & \mathbf{b}^T\mathbf{w} + \mathbf{l}^T\mathbf{y} + \mathbf{u}^T\mathbf{z} \\ \text{subject to} \quad & A^T\mathbf{w} + \mathbf{y} + \mathbf{z} = \mathbf{c} \\ & \mathbf{y} \geqslant \mathbf{0}, \mathbf{z} \leqslant \mathbf{0}. \end{aligned} \tag{11}$$

Problem 6 (4.6)

Let primal problem be the following:

$$\begin{aligned} \text{Minimize} \quad & -x_1 + x_2 \\ \text{subject to} \quad & 2x_1 - x_2 \leqslant -2 \\ & -2x_1 + x_2 \leqslant 1. \end{aligned} \tag{12}$$

And the dual of (12) is

$$\begin{aligned} \text{Maximize} \quad & -2y_1 + y_2 \\ \text{subject to} \quad & 2y_1 - 2y_2 = -1 \\ & -y_1 + y_2 = 1 \\ & y_1, y_2 \geqslant 0. \end{aligned} \tag{13}$$

See both (12) and (13) are infeasible.

Problem 7

$$\begin{aligned} \text{Minimize} \quad & -40x_1 - 30x_2 \\ \text{subject to} \quad & x_1 + x_2 \leqslant 40 \\ & 2x_1 + x_2 \leqslant 60 \\ & x_1, x_2 \geqslant 0. \end{aligned} \tag{14}$$

its standard form is

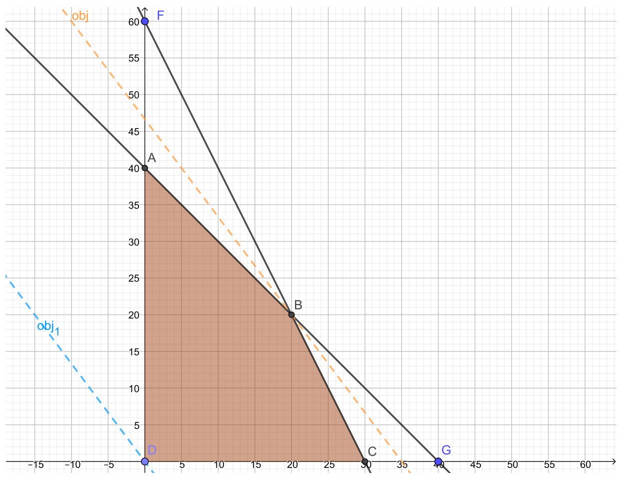
$$\begin{aligned} \text{Minimize} \quad & -40x_1 - 30x_2 \\ \text{subject to} \quad & x_1 + x_2 + s_1 = 40 \\ & 2x_1 + x_2 + s_2 = 60 \\ & x_1, x_2, s_1, s_2 \geqslant 0. \end{aligned} \tag{15}$$

And the dual is

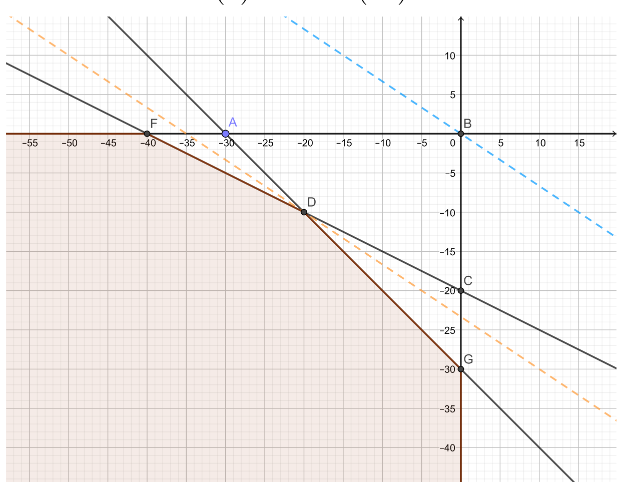
$$\begin{aligned} \text{Maximize} \quad & 40y_1 + 60y_2 \\ \text{subject to} \quad & y_1 + 2y_2 \leqslant -40 \\ & y_1 + y_2 \leqslant -30 \\ & y_1, y_2 \leqslant 0. \end{aligned} \tag{16}$$

the standard form of dual is

$$\begin{aligned} \text{Maximize} \quad & -40z_1 - 60z_2 \\ \text{subject to} \quad & -z_1 - 2z_2 + t_1 = -40 \\ & -z_1 - z_2 + t_2 = -30 \\ & z_1, z_2, t_1, t_2 \geqslant 0. \end{aligned} \tag{17}$$



(a) Primal (14)



(b) Dual (16)

Basic solutions of (15):

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 40 \\ 60 \end{bmatrix}$$

Table 1: Figure 1a

Point	Basis	BS of (15) $[x_1, x_2, x_3, x_4]^T$
A	$[A_2, A_4]$	$[0, 40, 0, 20]^T$
B	$[A_1, A_2]$	$[20, 20, 0, 0]^T$ *
C	$[A_1, A_3]$	$[30, 0, 10, 0]^T$
D	$[A_3, A_4]$	$[0, 0, 40, 60]^T$
F	$[A_2, A_3]$	$[0, 60, -20, 0]^T$
G	$[A_1, A_4]$	$[40, 0, 0, -20]^T$

Basic solutions of (15):

$$A = \begin{bmatrix} -1 & -2 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} -40 \\ -30 \end{bmatrix}$$

Table 2: Figure 1b

Point	Basis	BS of (17) $[z_1, z_2, z_3, z_4]^T$	Related to Primal
A	$[A_1, A_3]$	$[30, 0, -10, 0]^T$	C
B	$[A_3, A_4]$	$[0, 0, -40, -30]^T$	D
C	$[A_2, A_4]$	$[0, 20, 0, -10]^T$	A
D	$[A_1, A_2]$	$[20, 10, 0, 0]^T$ Δ	B
F	$[A_1, A_4]$	$[40, 0, 0, 10]^T$ Δ	G
G	$[A_2, A_3]$	$[0, 30, 20, 0]^T$ Δ	F

Primal BFS has star "*". It is point B on Fig 1a.
Dual feasible has "Δ". It is point D on Fig 1b.