MA 515 Homework 5

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October 23, 2017

Problem 1

Proof. Let $V = \text{span}\{v_1, \ldots, v_n\}$, where $v_1, \ldots, v_n$ are linearly independent elements in V. Then there exists n linearly independent elements $x_1, \ldots, x_n \in X$ such that $T(x_i) = v_i, i = 1, \ldots, n$. The existence promised by the fact that T is a linear operator. Let $Y_0 = \text{span}\{x_1, \ldots, x_n\}$. Hence, $\dim(Y_0) = \dim(V) = n$.

Also, $\ker(T) \cap Y_0 = \{0\}$. Indeed, if $\exists y \neq 0, y \in Y_0$ such that $T(y) = 0$. Let $y = \sum_{i=1}^n \beta_i x_i$. Then there exists $\beta_i \neq 0$. By linearity of T, $T(y) = \sum_{i=1}^n \beta_i T(x_i) = \sum_{i=1}^n \beta_i v_i \neq 0$ and it yields a contradiction.

Next, we will show $\ker(T) + Y_0 = X$. For any $x \in X$, let $x = \left[x - \frac{T(x)}{T(x_1)} x_1\right] + \frac{T(x)}{T(x_1)} x_1$. Then notice that

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.... $\square$

Problem 2

Proof. First we need to show that Λ is bounded. Take arbitrarily $x = \{x_n\}_{n \in \mathbb{N}} \in \ell^\infty$, and there exists $M > 0$ such that $|x_i| \leqslant M$ for any $i \geqslant 0$. Therefore,

$$\begin{aligned} ||\Lambda(x)||_{\ell^\infty} = ||y||_{\ell^\infty} &= \sup_{i \geqslant 1} |y_i| \\ &= \sup_{i \geqslant 1} \left| \frac{x_1 + \cdots + x_i}{i} \right| \\ &\leqslant \sup_{i \geqslant 1} \left| \frac{\sum_{j=1}^i |x_j|}{i} \right| \leqslant M. \end{aligned}$$

Hence, $||\Lambda||_\infty = \sup_{||x||_{\ell^\infty}=1} ||\Lambda(x)||_{\ell^\infty} \leqslant M$.

Next we need to find the value of $||\Lambda||_\infty$. From the definition,

$$||\Lambda||_\infty = \sup_{||x||_{\ell^\infty}=1} ||\Lambda(x)||_{\ell^\infty} = \sup_{||x||_{\ell^\infty}=1} \sup_{i \geqslant 1} \left| \frac{x_1 + \cdots + x_i}{i} \right|.$$

Also, $||x||_{\ell^\infty} = 1$ implies $|x_j| \leqslant 1, \forall j \geqslant 1$. Hence,

$$\sup_{||x||_{\ell^\infty}=1} \sup_{i\geqslant 1} \left| \frac{x_1 + \cdots + x_i}{i} \right| = \sup_{i\geqslant 1} \frac{i \cdot 1}{i} = 1.$$

Hence, $||\Lambda||_\infty = 1$.

□

Problem 3

Proof. It is enough to show that $\mathcal{N}(T) = \{0\}$. If so, then T is invertible and is a linear operator. Indeed, supposed there exists $x \neq 0, x \in \mathcal{N}(T)$, then $0 = \|T(x)\| \geqslant \|x\| > 0$, which yields a contradiction.

Next we need to show T^{-1} is bounded. Since T is surjective and invertible, we know T must be a bijection. If so, for any $y \in Y$, we have $\|y\| \geqslant b\|T^{-1}(y)\|$. Hence,

$$\sup_{\|y\|\neq 0} \frac{\|T^{-1}(y)\|}{\|y\|} \leqslant \frac{1}{b} < +\infty.$$

i.e., T^{-1} is a bounded linear operator.

□

Problem 4

Proof. We want to show $\|T_n(x_n) - T(x)\|_Y \to 0$ as $n \to +\infty$. By triangle inequality,

$$\|T_n(x_n)-T(x)\|_Y \leqslant \|T_n(x_n)-T(x_n)\|_Y+\|T(x_n)-T(x)\|_Y \leqslant \|T_n(x_n)-T(x_n)\|_Y+\|T\|_\infty\|x_n-x\|_X.$$

and we know $\|x_n - x\| \to 0$ as $n \to +\infty$. Hence, it is enough to show that $\|T_n(x_n) - T(x_n)\|_Y \to 0$ as $n \to +\infty$.

$\forall x_m \in X, m = 1, 2, \cdots$, $\lim_{n\to+\infty} ||T_n(x_m) - T(x_m)|| = 0$ since $\{T_n\}$ converges to T. Hence, we have

$$\lim_{m\to+\infty} \lim_{n\to+\infty} ||T_n(x_m) - T(x_m)|| = 0.$$

This implies $\|T_n(x_n) - T(x_n)\|_Y \to 0$ because $\{\|T_n(x_n) - T(x_n)\|\}_{n\in\mathbb{N}}$ is a subsequence of $\{\|T_n(x_m) - T(x_m)\|\}_{m,n\in\mathbb{N}}$.

In conclusion, $\lim_{n\to+\infty} ||T_n(x_n) - T(x_n)|| = 0$.

□

Problem 5

Proof. $S \circ T$ is a linear operator. Indeed, the domain of $S \circ T$ is X, which is a subspace of itself. Also, function composition preserves linearity.

Next we need to show $S \circ T$ is bounded. Consider norm $\|S \circ T\|_\infty$,

$$
\begin{aligned}
\|S \circ T\|_\infty = \sup_{\|x\|_X=1} \|(S \circ T)(x)\|_Z &= \sup_{\|x\|_X=1} \|S(T(x))\|_Z \\
&\leqslant \sup_{\|x\|_X=1} \|S\|_\infty \|T(x)\|_Y \\
&= \|S\|_\infty \sup_{\|x\|_X=1} \|T(x)\|_Y \\
&= \|S\|_\infty \|T\|_\infty.
\end{aligned}
$$

Since both S and T are bounded linear operators, $\|S\|_\infty$ and $\|T\|_\infty$ are less than positive infinity and this leads to the conclusion that $\|S \circ T|_\infty < +\infty$.

In conclusion, $S \circ T$ is a bounded linear operator.

□

Problem 6

Proof. (a) T is a contraction mapping. Indeed, let $c = \|T\|_\infty < 1$. Hence, for any $x_1, x_2 \in X(x_1 \neq x_2)$,

$$
\|T(x_1) - T(x_2)\| = \|T(x_1 - x_2)\| \leqslant \|T\|_\infty \|x_1 - x_2\| = c\|x_1 - x_2\|.
$$

where $0 < c < 1$. Hence, there exists a unique $x_0 \in X$ such that $T(x_0) = x_0$. And this is equivalent to say that linear operator (which is easy to check) $\mathcal{N}(I - T) = \{x_0\}$. However, $\{0\} \in \mathcal{N}(I - T)$ always holds. Thus, $x_0 = 0$.

Next we need to show that $I - T$ is a one-to-one mapping. Suppose not, then there exists $y \in X$ and distinct $y_1, y_2 \in X$ such that $(I - T)(y_1) = (I - T)(y_2) = y$. By linearity, $(I - T)(y_1 - y_2) = 0$ and it yields that $y_1 = y_2$, which is a contradiction.

What's more, $I - T$ is surjective. Indeed, $I - T$ maps from X to X. For any element $x \in X$, $\exists$ unique $y \in X$ such that $y = (I - T)(x)$. Hence, volume of the range of $I - T$ equals to the volume of domain. i.e., $|\mathcal{R}(I - T)| = |\mathcal{D}(I - T)| = |X|$. Also, $\mathcal{R}(I - T) \subset X$, which yields that $\mathcal{R}(I - T) = X$. In conclusion, $I - T$ is bijective.

(b) Let $S = \sum_{n=0}^{\infty} T^n$ and consider $\|S\|_\infty$.

$$
\begin{aligned}
\|S\|_\infty &\leqslant \|\lim_{m\to+\infty}\sum_{n=0}^{m} T^n\|_\infty \\
&= \lim_{m\to+\infty}\|\sum_{n=0}^{m} T^n\|_\infty \\
&\leqslant \lim_{m\to+\infty}\sum_{n=0}^{m} \|T^n\|_\infty \\
&\leqslant \lim_{m\to+\infty}\sum_{n=0}^{m} \|T\|_\infty^n \\
&= \lim_{m\to+\infty}\sum_{n=0}^{m} c^n \\
&= \frac{1}{1-c} < +\infty.
\end{aligned}
$$

We use triangle inequality above. Also, the limit and norm can exchange due to the continuity of norm.

Hence, S is bounded in $\|\cdot\|_\infty$. And it is obvious that S is a linear operator, so $S \in (B(X,X), \|\cdot\|_\infty)$.

(c) It is enough to check that $S \circ (I-T) = (I-T)\circ S = I$.

$$
\begin{aligned}
S\circ(I-T) = S - S\circ T &= \sum_{n=0}^{\infty} T^n - \sum_{n=1}^{\infty} T^n \\
&= T^0 = I
\end{aligned}
$$

$$
\begin{aligned}
(I-T)\circ S = S - T\circ S &= \sum_{n=0}^{\infty} T^n - \sum_{n=1}^{\infty} T^n \\
&= T^0 = I
\end{aligned}
$$

Hence, $S = (I-T)^{-1}$.

$\square$

Problem 7

Proof. For each $n \in \mathbb{N}$, $\|T^n\|_\infty \leqslant \|T\|_\infty^n$. Since T is a bounded linear operator, $\|T\|_\infty < +\infty$. By triangle inequality and Taylor theorem,

$$
\|S\|_\infty \leqslant \sum_{n=0}^{+\infty} \frac{\|T^n\|_\infty}{n!} \leqslant \sum_{n=0}^{+\infty} \frac{\|T\|_\infty^n}{n!} = e^{\|T\|_\infty}.
$$

$\square$