MA 515 Homework 6

Zheming Gao

December 3, 2017

Problem 1

Problem 2

Proof. Suppose $x \neq y$. Let $V = span\{x, y\}$ functional $f : V \to \mathbb{R}$ such that $\forall s, t \in \mathbb{R}$,

$$f(sx + ty) = s\|x\| - t\|y\|.$$

Hence, $f(x) = \|x\|$, $f(y) = -\|y\|$ and $f(x) \neq f(y)$. By the theorem, there exists a functional $F : X \to \mathbb{R}$ such that $F = f$ on V and $\|f\|_\infty = \|F\|_\infty$, which is a contradiction. $\square$

Problem 3

Problem 7

Proof.   1. $< x, x >= 1/2(\|x + x\|^2 - \|x\|^2 - \|x\|^2) = \|x\|^2 \geqslant 0$. And $< x, x >= 0$ if and only if $x = 0$.

2. It is also clear that $< x, y >=< y, x >, \forall x, y \in X$.

3. We will show $< x, y + z >=< x, y > + < x, z >, \forall x, y, z \in X$. By definition, we know

$$< x, y + z >= \frac{1}{2}(\|x + y + z\|^2 - \|x\|^2 - \|y + z\|^2).$$

and

$$\begin{aligned}\|x + y + z\|^2 &= 2\|x\|^2 + 2\|y + z\|^2 - \|x - y - z\|^2 \\ &= 2\|x + y\|^2 + 2\|z\|^2 - \|x + y - z\|^2\end{aligned} \tag{1}$$

Also, with Parallelogram theorem, we have

$$\|x - y - z\|^2 + \|x + y - z\|^2 = 2\|x - z\| + 2\|y\|^2.$$

Hence, plug it in (1) and have

$$\|x + y + z\|^2 = \|x\|^2 + \|y + z\|^2 + \|x + y\|^2 + \|z\|^2 - \|x - z\|^2 - \|y\|^2.$$

which implies

$$
\begin{aligned}
< x, y + z > &= \frac{1}{2}(\|x + y\|^2 - \|x - z\|^2 + \|z\|^2 - \|y\|^2) \\
&= \frac{1}{2}(\|x + y\|^2 - \|y\|^2 - \|x\|^2 - \|x - z\|^2 + \|z\|^2 + \|x\|^2) \\
&= < x, y > + < x, z >
\end{aligned}
$$

4. We need to show that $< \lambda x, y >= \lambda < x, y >$, $\forall x, y \in X, \lambda \in \mathbb{R}$.

   To show this, we need a few steps. Firstly, it holds for $\lambda \in \mathbb{N}$ and it can be proved by induction. Also,

$$
\begin{aligned}
< x, -y > &= \frac{1}{2}(\|x - y\|^2 - \|x\|^2 - \|y\|^2) \\
&= \frac{1}{2}(-\|x + y\|^2 + \|y\|^2 + \|x\|^2) \\
&= - < x, y >
\end{aligned}
$$

   Hence, it holds for $\lambda = -1$ and so holds for $\lambda \in \mathbb{Z}$.

   Next we will show that it holds for $\lambda \in \mathbb{Q}$. Let $\lambda = p/q, (q \neq 0), p, q \in \mathbb{Z}$. Hence,

$$q < x, \lambda y >= q < x, \frac{p}{q} y >=< x, py >= p < x, y > .$$

   Both sides divided by q and we have $< \lambda x, y >= \lambda < x, y >, \forall \lambda \in \mathbb{Q}$.

   Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\forall \lambda \in \mathbb{R}$, there exists a sequence of rational numbers $\{\lambda_n\}_{n \in \mathbb{N}}$ such that $\lambda_n \to \lambda$. Hence, $< \lambda x, y >= \lambda < x, y >, \forall \lambda \in \mathbb{R}$.

   In conclusion, $< \cdot, \cdot >$ is an inner product.

   $\square$

Problem 8

(i) *Proof.* Suppose $\| \cdot \|_1$ is induced by inner product, i.e., $\| \cdot \|_1 = \sqrt{< \cdot, \cdot >}$. However, if so, then $\| \cdot \|$ must satisfy parallelogram identity. For $a = (1, 1)^T, b = (-1, 2)^T$,

$$2\|a\|_1^2 + 2\|b\|_1^2 = 26 \neq 18 = \|a - b\|_1^2 + \|a + b\|_1^2.$$

   This is a contradiction.

   $\square$

(ii) Still, it breaks the parallelogram identity.

   Let $f(x) = x, g(x) = 2x$. Hence, $\|f\|_\infty = 1, \|g\|_\infty = 2$. But $\|f + g\|_\infty = 3, \|f - g\|_\infty = 1$. So

$$\|f + g\|_\infty + \|f - g\|_\infty \neq 2\|f\|_\infty + 2\|g\|_\infty.$$

Problem 9

Proof. "$\Rightarrow$", proved in class. $\langle x, x_n \rangle$ converges $\langle x, x \rangle = \|x\|^2$. Hence,

$$\lim_{n \to +\infty} \|x_n - x\|^2 = \lim_{n \to +\infty} \langle x_n - x, x_n - x \rangle = \lim_{n \to +\infty} \|x_n\|^2 - 2 \langle x, x_n \rangle + \|x\|^2 = 0$$

"$\Leftarrow$". If $x_n \to x$, then by Cauchy-Schwartz inequality,

$$0 \leqslant \lim_{n \to +\infty} |\langle x_n - x, x \rangle| \leqslant \lim_{n \to +\infty} \|x_n - x\| \|x\| = 0.$$

By squeeze theorem, $\lim_{n \to +\infty} \langle x_n - x, x \rangle = 0$. Hence, $x_n \rightharpoonup x$.

Also,
$$
\begin{aligned}
0 = \overline{\lim_{n \to +\infty}} \|x_n - x\|^2 &= \overline{\lim_{n \to +\infty}} \langle x_n - x, x_n - x \rangle \\
&= \overline{\lim_{n \to +\infty}} \|x_n\|^2 - 2 \overline{\lim_{n \to +\infty}} \langle x, x_n \rangle + \|x\|^2 \\
&= \overline{\lim_{n \to +\infty}} \|x_n\|^2 - 2 \lim_{n \to +\infty} \langle x, x_n \rangle + \|x\|^2 \\
&= \overline{\lim_{n \to +\infty}} \|x_n\|^2 - \|x\|^2
\end{aligned}
$$

Similarly, we have $\underline{\lim}_{n \to +\infty} \|x_n\|^2 - \|x\|^2 = 0$. Hence, $\lim_{n \to +\infty} \|x_n\| = \|x\|$.

$\square$

Problem 10

Proof. (i) (Shown in class) Since $\{e_n\}_{n \in \mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$, for any $x \in \mathcal{H}$, it can be expressed as

$$x = \sum_{i=1}^{\infty} \alpha_i e_i \qquad \alpha_i \in \mathbb{R}.$$

Hence,

$$\langle x, e_i \rangle = \alpha_i \quad \text{and} \quad \langle x, x \rangle = \|x\|^2 = \sum_{i=1}^{\infty} \alpha_i^2 < +\infty.$$

Hence,

$$\lim_{n \to +\infty} \langle x, e_i \rangle = \lim_{n \to +\infty} \alpha_i = 0.$$

i.e., $e_n \rightharpoonup 0$.

(ii) Let $\{e_n\}_{n \geqslant 1}$ be an orthonormal basis of H such that $e_n \perp x$, $\forall n \geqslant 2$. Hence, for each $n \in \mathbb{N}$ and $n \geqslant 2$, let $x_n = x + \lambda e_n (\lambda > 0)$. We want $\|x_n\| = 1$, i.e.,

$$\|x_n\|^2 = \|x + \lambda e_n\|^2 = \|x\|^2 + \lambda^2 = 1$$

which yields that $\lambda = \sqrt{1 - \|x\|^2}$.

Next, we need to show, $x_n = x + \sqrt{1 - \|x\|^2} e_n$ weakly converges to x as $n \to +\infty$. Indeed, for any arbitrarily taken $y \in H$, use result in (i),

$$\lim_{n \to +\infty} \langle y, x_n - x \rangle = \sqrt{1 - \|x\|^2} \langle y, en \rangle = 0$$

Hence, $x_n \rightharpoonup x$.

$\square$

Problem 11

Proof. If $\sum_{n=1}^{\infty} |\alpha_n|^2 < +\infty$. Let partial sum $S_n = \sum_{i=1}^{n} \alpha_i v_i$, we need to show that S_n converges as $n \to +\infty$. Since H is a Hilbert space, it is enough to show $\{S_n\}_{n \geqslant 1}$ is a Cauchy sequence. Indeed, for $m, n \in \mathbb{N}$, $m > n$,

$$\|S_n - S_m\|^2 = \|\sum_{i=n+1}^{m} \alpha_i v_i\|^2 \leqslant \sum_{i=n+1}^{m} |\alpha_n|^2.$$

Since $\sum_{n=1}^{\infty} |\alpha_n|^2 < +\infty$, $\sum_{i=n+1}^{m} |\alpha_n|^2 \to 0$ as $m, n \to +\infty$. Hence, $\{S_n\}$ is Cauchy and so it converges in H.

Conversely, if $\{S_n\}$ converges in H, and denote the limit as S. Hence,

$$+\infty \geqslant \|S\|^2 = \langle \sum_{i=1}^{\infty} \alpha_i v_i, \sum_{i=1}^{\infty} \alpha_i v_i \rangle = \sum_{n=1}^{\infty} |\alpha_n|^2.$$

$\square$

Problem 12

(i) *Proof.* Suppose both p and q are in Ω, such that $p = \pi_\Omega(x), q = \pi_\Omega(x)$. Then, let sequence $\{y_n\}$ be

$$y_n = \begin{cases} p & \text{if n odd} \\ q & \text{if n even} \end{cases}$$

Hence, $\{y_n\}$ converges since $\|y_n - x\| \to d_\Omega(x)$. And this forces $p = q$.

$\square$

(ii)