# MA 515 Homework 4

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## Problem 1

*Proof.* $\{T_n\}_{n\in\mathbb{N}}$ is a sequence of uniformly bounded linear operators and it satisfies

$$\lim_{n\to+\infty} T_n(x) := T(x)$$

for any $x \in X$. Now we want to show $T$ is a bounded linear operator. First, $T$ is linear because the limit operation on $T_n$ preserves the linearity of $T_n$. Also, $\mathcal{D}(T)$ is $X$, which yields that $T$ is a linear operator from $X$ to $Y$.

Next we need to show $T$ is bounded, or more precisely, $||T||_\infty \leqslant M$. Since $||T_n||_\infty < M$, we know for any $||x||_X = 1$, $||T_n(x)||_Y < M$. Hence,

$$\lim_{n\to+\infty} \|T_n(x)\| = ||T(x)|| \leqslant M.$$

which implies $\sup_{||x||_X=1} ||T(x)|| \leqslant M$, i.e., $||T||_\infty \leqslant M$.

□

## Problem 2

*Proof.* First we need to show that $\Lambda$ is bounded. Take arbitrarily $x = \{x_n\}_{n\in\mathbb{N}} \in \ell^\infty$, and there exists $M > 0$ such that $|x_i| \leqslant M$ for any $i \geqslant 0$. Therefore,

$$\begin{aligned} ||\Lambda(x)||_{\ell^\infty} = ||y||_{\ell^\infty} &= \sup_{i\geqslant 1} |y_i| \\ &= \sup_{i\geqslant 1} \left| \frac{x_1 + \cdots + x_i}{i} \right| \\ &\leqslant \sup_{i\geqslant 1} \left| \frac{\sum_{j=1}^{i} |x_j|}{i} \right| \leqslant M. \end{aligned}$$

Hence, $||\Lambda||_\infty = \sup_{||x||_{\ell^\infty}=1} ||\Lambda(x)||_{\ell^\infty} \leqslant M$.

Next we need to find the value of $||\Lambda||_\infty$. From the definition,

$$||\Lambda||_\infty = \sup_{||x||_{\ell^\infty}=1} ||\Lambda(x)||_{\ell^\infty} = \sup_{||x||_{\ell^\infty}=1} \sup_{i\geqslant 1} \left| \frac{x_1 + \cdots + x_i}{i} \right|.$$

Also, $||x||_{\ell^\infty} = 1$ implies $|x_j| \leqslant 1, \forall j \geqslant 1$. Hence,

$$\sup_{||x||_{\ell^\infty}=1} \sup_{i\geqslant 1} \left| \frac{x_1 + \cdots + x_i}{i} \right| = \sup_{i \geqslant 1} \frac{i \cdot 1}{i} = 1.$$

Hence, $||\Lambda||_\infty = 1$.

□

# Problem 3

# Problem 4

*Proof.* We want to show $\|T_n(x_n) - T(x)\|_Y \to 0$ as $n \to +\infty$. By triangle inequality,

$$\|T_n(x_n) - T(x)\|_Y \leqslant \|T_n(x_n) - T(x_n)\|_Y + \|T(x_n) - T(x)\|_Y \leqslant \|T_n(x_n) - T(x_n)\|_Y + \|T\|_\infty \|x_n - x\|_X.$$

and we know $\|x_n - x\| \to 0$ as $n \to +\infty$. Hence, it is enough to show that $\|T_n(x_n) - T(x_n)\|_Y \to 0$ as $n \to +\infty$.

$\forall x_m \in X, m = 1, 2, \cdots$, $\lim_{n\to+\infty} ||T_n(x_m) - T(x_m)|| = 0$ since $\{T_n\}$ converges to $T$. Hence, we have

$$\lim_{m\to+\infty} \lim_{n\to+\infty} ||T_n(x_m) - T(x_m)|| = 0.$$

This implies $\|T_n(x_n) - T(x_n)\|_Y \to 0$ because $\{\|T_n(x_n) - T(x_n)\|\}_{n\in\mathbb{N}}$ is a subsequence of $\{\|T_n(x_m) - T(x_m)\|\}_{m,n\in\mathbb{N}}$.

In conclusion, $\lim_{n\to+\infty} ||T_n(x_n) - T(x_n)|| = 0$.

□

# Problem 5

*Proof.* $S \circ T$ is a linear operator. Indeed, the domain of $S \circ T$ is $X$, which is a subspace of itself. Also, function composition preserves linearity.

Next we need to show $S \circ T$ is bounded. Consider norm $\|S \circ T\|_\infty$,

$$\begin{aligned} \|S \circ T\|_\infty = \sup_{\|x\|_X=1} \|(S \circ T)(x)\|_Z &= \sup_{\|x\|_X=1} \|S(T(x))\|_Z \\ &\leqslant \sup_{\|x\|_X=1} \|S\|_\infty \|T(x)\|_Y \\ &= \|S\|_\infty \sup_{\|x\|_X=1} \|T(x)\|_Y \\ &= \|S\|_\infty \|T\|_\infty. \end{aligned}$$

Since both $S$ and $T$ are bounded linear operators, $\|S\|_\infty$ and $\|T\|_\infty$ are less than positive infinity and this leads to the conclusion that $\|S \circ T|_\infty < +\infty$.

In conclusion, $S \circ T$ is a bounded linear operator.

□

## Problem 6

*Proof.* (a) $T$ is a contraction mapping. Indeed, let $c = \|T\|_\infty < 1$. Hence, for any $x_1, x_2 \in X (x_1 \neq x_2)$,

$$\|T(x_1) - T(x_2)\| = \|T(x_1 - x_2)\| \leqslant \|T\|_\infty \|x_1 - x_2\| = c\|x_1 - x_2\|.$$

where $0 < c < 1$. Hence, there exists a unique $x_0 \in X$ such that $T(x_0) = x_0$. And this is equivalent to say that linear operator (which is easy to check) $\mathcal{N}(I - T) = \{x_0\}$. However, $\{0\} \in \mathcal{N}(I - T)$ always holds. Thus, $x_0 = 0$.

Next we need to show that $I - T$ is a one-to-one mapping. Suppose not, then there exists $y \in X$ and distinct $y_1, y_2 \in X$ such that $(I - T)(y_1) = (I - T)(y_2) = y$. By linearity, $(I - T)(y_1 - y_2) = 0$ and it yields that $y_1 = y_2$, which is a contradiction.

What's more, $I - T$ is surjective. (............)

(b) Let $S = \sum_{n=0}^{\infty} T^n$ and consider $\|S\|_\infty$.

$$\begin{aligned}
\|S\|_\infty &\leqslant \| \lim_{m\to+\infty} \sum_{n=0}^{m} T^n \|_\infty \\
&= \lim_{m\to+\infty} \| \sum_{n=0}^{m} T^n \|_\infty \\
&\leqslant \lim_{m\to+\infty} \sum_{n=0}^{m} \|T^n\|_\infty \\
&\leqslant \lim_{m\to+\infty} \sum_{n=0}^{m} \|T\|_\infty^n \\
&= \lim_{m\to+\infty} \sum_{n=0}^{m} c^n \\
&= \frac{1}{1-c} < +\infty.
\end{aligned}$$

We use triangle inequality above. Also, the limit and norm can exchange due to the continuity of norm.

Hence, $S$ is bounded in $\|\cdot\|_\infty$. And it is obvious that $S$ is a linear operator, so $S \in (B(X, X), \|\cdot\|_\infty)$.

(c) It is enough to check that $S \circ (I - T) = (I - T) \circ S = I$.

$$\begin{aligned}
S \circ (I - T) = S - S \circ T &= \sum_{n=0}^{\infty} T^n - \sum_{n=1}^{\infty} T^n \\
&= T^0 = I
\end{aligned}$$

$$
\begin{aligned}
(I - T) \circ S = S - T \circ S &= \sum_{n=0}^{\infty} T^n - \sum_{n=1}^{\infty} T^n \\
&= T^0 = I
\end{aligned}
$$

Hence, $S = (I - T)^{-1}$.

$\square$

# Problem 7