MA 515 Homework 1

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Problem 1

Proof. To show that (X, d) is a metric space, we need to verify that $d(x, y)$ is well defined as a distance. Obviously, $\forall x, yinX$, $d(x, x) = 0$ and $d(x, y) = d(y, x)$ due to the properties of absolute value. Hence, we only need to verify the triangle inequality holds.

Let $a = \sqrt{|x - y|}$ and $b = \sqrt{|y - z|}$. Hence, a and b are non-negative, which promise the following relations,

$$(a + b)^2 - (a^2 + b^2) = 2ab \geqslant 0.$$

Therefore, we have

$$d(x, y) + d(y, z) = a + b \geqslant \sqrt{a^2 + b^2} = \sqrt{|x - y| + |y - z|} \geqslant \sqrt{|x - z|} = d(x, z).$$

which proves the triangle inequality. In conclusion, (X, d) is a metric space.

□

Problem 2

Yes. (X, d) is a metric space.

Proof. We will use the result from problem 1 to show this. Still, with the properties of absolute value and integration, we know that $\forall f, g \in X$, $d(f, f) = 0$ and $d(f, g) = d(g, f)$. Then, we only need to prove the triangle inequality. $\forall f, g, h \in X$, use the linearity of integration, triangle inequality for absolute value and result from problem 1, we have

$$\begin{aligned} d(f, g) + d(g, h) &= \int_a^b |g(t) - f(t)| + \sqrt{|g(t) - f(t)|} dt + \int_a^b |h(t) - g(t)| + \sqrt{|h(t) - g(t)|} dt \\ &= \int_a^b |g(t) - f(t)| + |h(t) - g(t)| + \sqrt{|g(t) - f(t)|} + \sqrt{|h(t) - g(t)|} dt \\ &\geqslant \int_a^b |h(t) - f(t)| + \sqrt{|h(t) - f(t)|} dt = d(f, h). \end{aligned}$$

In conclusion, (X, d) is a metric space.

□

Problem 3

We will prove (b) first and use it to prove (a).

1. (b)

   Proof. Since (X, d) is a metric space, so $\forall x, y, z \in X$ we apply triangle inequality on $d(x, z)$ and obtain $d(x, z) \leqslant d(x, y) + d(y, z)$, from where we know that $d(x, z) - d(y, z) \leqslant d(x, y)$. Similarly, from $d(y, z) \leqslant d(y, x) + d(x, z)$, we know $d(y, z) - d(x, z) \leqslant d(x, y)$.

   In conclusion, $\forall x, y, z \in X$,

   $$|d(x, z) - d(y, z)| \leqslant d(x, y).$$

   □

2. (a)

   Proof. Now we use (b) to show (a). Rewrite left hand side and use triangle inequality,

   $$\begin{aligned} LHS = |d(x, y) - d(z, w)| &= |d(x, y) - d(y, z) + d(y, z) - d(z, w)| \\ &\leqslant |d(x, y) - d(y, z)| + |d(y, z) - d(z, w)| \\ &= |d(y, x) - d(y, z)| + |d(z, y) - d(z, w)| \\ &\leqslant d(x, z) + d(y, w) \end{aligned}$$

   This proves the inequality in (a).

   □

Problem 4

1. (a)

   Example:

   We might let $x = \{x_n\}_{n \in \mathbb{N}}$ and $x_n = 1/\sqrt{n}$. This satisfies $x_n \to 0$ as $n \to +\infty$, but doesn't satisfy $x \in \ell^2$ since

   $$\sum_{n=1}^{+\infty} |x_n|^2 = \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty.$$

2. (b)

   Example:

   Use the same $x = \{x_n\}_{n \in \mathbb{N}} = \{1/\sqrt{n}\}$. $x \notin \ell^2$, but $x \in \ell^3$ since

   $$\sum_{n=1}^{+\infty} |x_n|^3 = \sum_{n=1}^{+\infty} \left(\frac{1}{n}\right)^{3/2} < +\infty.$$

Problem 5

Proof. We will prove this by contradiction. Let $x := \{x_n\}_{n=1}^{+\infty} \in \ell^p$, so x satisfies $\sum_{n=1}^{+\infty} |x_n|^p < +\infty$. $(p > 1)$

Suppose that x is not a bounded sequence, i.e., for any $M > 0$, $\exists N$ such that $\forall n > N$, $|x_n| > M$. We might let $M = 1$, then there exists N_0 such that $\forall n > N_0$,

$$\sum_{n=1}^{+\infty} |x_n|^p \geqslant \sum_{n=N_0+1}^{+\infty} |x_n|^p > \sum_{n=N_0+1}^{+\infty} M^p > +\infty.$$

which contradicts to the fact that $x \in \ell^p$. Hence, x is a bounded sequence. Equivalently, x is in $\ell^{+\infty}$.

□

Problem 6

1. (a)

Proof. Need to show that $d_f(x, y)$ is a metric. First of all, $\forall x \in X$, since (X, d) is a metric space, we have

$$d_f(x, x) = f \circ d(x, x) = f(0) = 0.$$

Also, $\forall x, y \in X$,

$$d_f(x, y) = f \circ d(x, y) = f \circ d(y, x) = d_f(y, x).$$

For triangle inequality, $\forall x, y, z \in X$, use supper-additivity of f and obtain

$$\begin{aligned} d_f(x, y) + d_f(y, z) &= f \circ d(x, y) + f \circ d(y, z) \\ &\geqslant f \circ (d(x, y) + d(y, z)) \geqslant f \circ d(x, z) = d_f(x, z). \end{aligned}$$

In conclusion, (X, d_f) is a metric space.

□

2. (b)

Proof. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$. $f(x) = x/(1 + x)$. It is clear that

(i)

$$f(0) = 0 \text{ and } f(s) > 0 \text{ for all } s > 0,$$

(ii) f has supper-additivity, because $\forall x, y \geqslant 0$,

$$\begin{aligned} f(x+y) - f(x) - f(y) &= \frac{x+y}{1+x+y} - \frac{x}{1+x} - \frac{y}{1+y} \\ &= \left(\frac{x}{1+x+y} - \frac{x}{1+x}\right) + \left(\frac{y}{1+x+y} - \frac{y}{1+y}\right) \leqslant 0. \end{aligned}$$

With the result in part (a), $d_f(x, y) = \tilde{d}(x, y)$ is a metric.

□

Problem 7

1. (a)

Proof. To prove d is a metric, we need to verify three conditions. First of all, $\forall x \in X$, $d(x, x) = d_1(x_1, x_1) + d_2(x_2, x_2) = 0$.

Also, $\forall x, y \in X$,

$$d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2) = d_1(y_1, x_1) + d_2(y_2, x_2) = d(y, x).$$

For triangle inequality, $\forall x, y, z \in X$, we have

$$\begin{aligned} d(x, y) + d(y, z) &= d_1(x_1, y_1) + d_2(x_2, y_2) + d_1(y_1, z_1) + d_2(y_2, z_2) \\ &= [d_1(x_1, y_1) + d_1(y_1, z_1)] + [d_2(x_2, y_2) + d_2(y_2, z_2)] \\ &\geqslant d_1(x_1, z_1) + d_2(x_2, z_2) = d(x, z). \end{aligned}$$

Hence, d is a metric.

□

2. (b)

Proof. Similarly to part (a), it is only need to show the triangle inequality for $\tilde{d}$.

$\forall x, y \in X$, let $\tilde{d}(x, y) = ||x - y||$. Actually, $|| \cdot ||$ is the Euclidean norm defined on $\mathbb{R}^2$. We would like to show that the triangle inequality holds for $|| \cdot ||$. And it is enough to show that $\forall a, b \in X$, $||a + b|| \leqslant ||a|| + ||b||$. Use Cauchy-Schwartz inequality,

$$\begin{aligned} ||a + b||^2 = (a + b)^T (a + b) &= ||a||^2 + ||b||^2 + 2a^T b \\ &\leqslant ||a||^2 + ||b||^2 + 2||a||||b|| = (||a|| + ||b||)^2 \end{aligned}$$

Hence, $\tilde{d}$ is a metric.

□

Problem 8

Proof. $\forall x, y \in X$ and $K \subset X, K \neq \phi$, consider $d(x,y) + d_K(x)$.

$$\begin{aligned} d(x,y) + d_K(x) &= d(x,y) + \inf_{w \in K} d(x,w) \\ &= \inf_{w \in K} \{d(x,y) + d(x,w)\} \\ &\geqslant \inf_{w \in K} d(y,w) = d_K(y). \end{aligned}$$

Hence, $d_K(y) - d_K(x) \leqslant d(x,y)$.

Similarly, consider $d(x,y) + d_K(y)$ and with similar argument, we will get $d(x,y) \geqslant d_K(y) - d_K(x)$.

In all, d_K is 1-Lipschitz continuous. i.e.,

$$|d_K(y) - d_K(x)| \leqslant d(x,y) \quad \forall x, y \in X.$$

□