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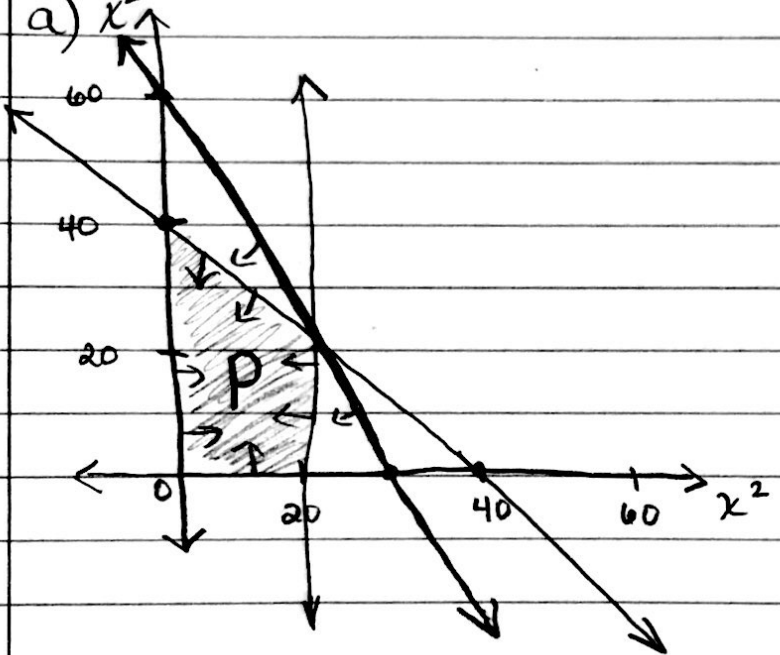
ISE 505
Fall 2017
Homework #4

① <u>Problem 2.7</u>

$$P = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 \le 40,\ 2x_1 + x_2 \le 60,\ x_1 \le 20,\ x_1, x_2 \ge 0\}$$

a)



b)

$$x_1 + x_2 + x_3 = 40$$
$$2x_1 + x_2 + x_4 = 60$$
$$x_1 + x_5 = 20$$
$$x_1, x_2, x_3, x_4, x_5 \ge 0$$

$$P = \{\mathbf{x} \in \mathbb{R}^5 \mid x_1 + x_2 + x_3 = 40,\ 2x_1 + x_2 + x_4 = 60,\ x_1 + x_5 = 20,\ x_1, x_2, x_3, x_4, x_5 \ge 0\}$$

① c)

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

<u>Basic Solutions</u> Rank = $\underline{\underline{3}}$ 2 Degrees of Freedom

<u>$x_1, x_2 = 0$:</u>

$$V_1 = \begin{pmatrix} 0 \\ 0 \\ 40 \\ 60 \\ 20 \end{pmatrix}$$

<u>$x_1, x_3 = 0$:</u>

$$V_2 = \begin{pmatrix} 0 \\ 40 \\ 0 \\ 20 \\ 20 \end{pmatrix}$$

<u>$x_1, x_4 = 0$:</u>

$$V_3 = \begin{pmatrix} 0 \\ 60 \\ -20 \\ 0 \\ 20 \end{pmatrix}$$

<u>$x_1, x_5 = 0$:</u>

Not Possible

<u>$x_2, x_3 = 0$:</u>

$$V_4 = \begin{pmatrix} 40 \\ 0 \\ 0 \\ -20 \\ -20 \end{pmatrix}$$

<u>$x_2, x_4 = 0$:</u>

$$V_5 = \begin{pmatrix} 30 \\ 0 \\ 10 \\ 0 \\ -10 \end{pmatrix}$$

<u>$x_2, x_5 = 0$:</u>

$$V_6 = \begin{pmatrix} 20 \\ 0 \\ 20 \\ 20 \\ 0 \end{pmatrix}$$

<u>$x_3, x_4 = 0$</u>

$$V_7 = \begin{pmatrix} 20 \\ 20 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

<u>$x_3, x_5 = 0$</u>

$$V_8 = \begin{pmatrix} 20 \\ 20 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

<u>$x_4, x_5 = 0$:</u>

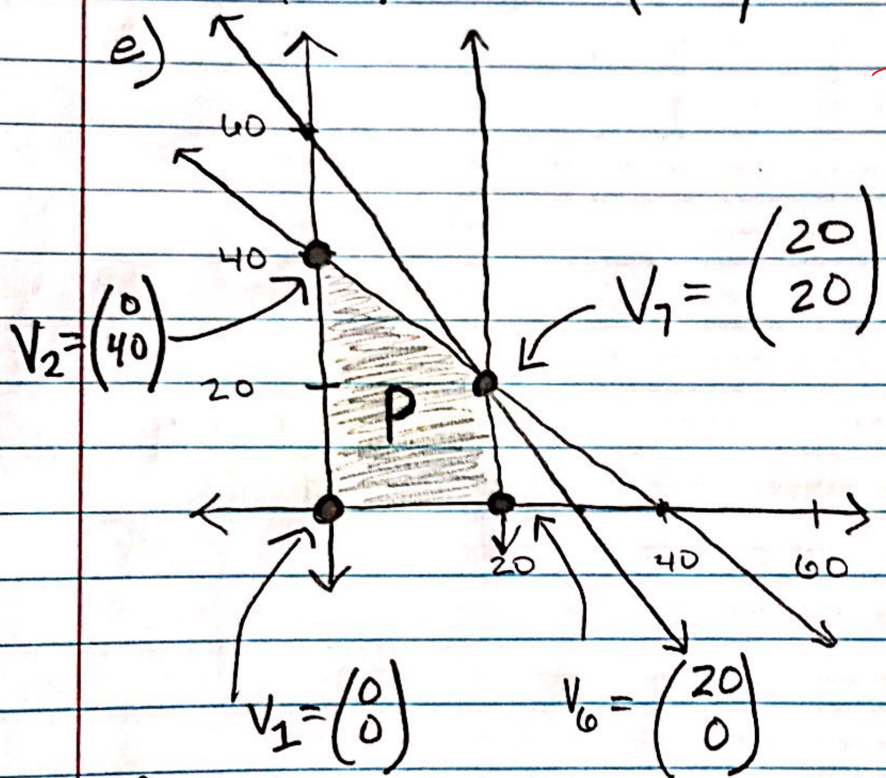
$$V_9 = \begin{pmatrix} 20 \\ 20 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

① d)

Basic Feasible Solutions

$$V_1 = \begin{pmatrix} 0 \\ 0 \\ 40 \\ 60 \\ 20 \end{pmatrix} \quad V_2 = \begin{pmatrix} 0 \\ 40 \\ 0 \\ 20 \\ 20 \end{pmatrix} \quad V_6 = \begin{pmatrix} 20 \\ 0 \\ 20 \\ 20 \\ 0 \end{pmatrix} \quad \underset{(V_8, V_9)}{V_7} = \begin{pmatrix} 20 \\ 20 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

e)



f) $\begin{pmatrix} 20 \\ 20 \end{pmatrix}$ in $\mathbb{R}^2$ or $\begin{pmatrix} 20 \\ 20 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ in $\mathbb{R}^5$

corresponds to degenerate basic feasible solutions.

② Problem 2.10

Prove that for a degenerate basic feasible solution with $p<m$ positive elements, its corresponding extreme point P may correspond to $C(n-p, n-m)$ different basic feasible solutions at the same time.

n = # of variables
m = # of constraint equations
p = # of positive elements in degenerate bfs

$n-m$ degrees of freedom

${}_nC_{n-m}$ possible vertices (extreme points)

Each basic feasible solution in $\mathbb{R}^n$ will have at least $n-m$ zero values and at most m non-zero values.

The degenerate cases always have additional zero values. If we know for certain there are p positive elements, that leaves $n-p$ possible positions to arrange the zero values, and as noted above, every basic feasible solution has at least $n-m$ zero values.

$n-p$ possible positions to arrange $n-m$ zero values

${}_{n-p}C_{n-m}$ and since $p<m$ it will be degenerate

③ Problem 2.11

Let M be the 2×2 identity matrix.
Show that

a) M_c, the convex cone generated by M, is the first orthant of $\mathbb{R}^2$

b) M_c is the smallest convex cone that which contains the column vectors $(1,0)^T$ and $(0,1)^T$.

a)

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{let } e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$M = [e_1 \ e_2]$$

$$M_c = \{ v \in \mathbb{R}^2 \mid v = \lambda_1 e_1 + \lambda_2 e_2, \ \lambda_1, \lambda_2 \geq 0 \}$$

Let $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, $w_1, w_2 \geq 0$

represent any vector in the first orthant of $\mathbb{R}^2$.

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ w_2 \end{bmatrix} = w_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + w_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= w_1 e_1 + w_2 e_2 = v \in M_c$$

for $w_1, w_2 \geq 0$

④ Let the set E be formed by all extremal directions of the nonempty feasible domain $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0\}$ of a standard form linear program.

a) Show that, for any vector $d \in \mathbb{R}^n$, $d \in E$ if and only if "$Ad = 0$ and $d \geq 0$".

A→B: Let $d \in E$, then $\forall\, x_0 \in P$

$$\{x \in \mathbb{R}^n \mid x = x_0 + \lambda d,\ \lambda \geq 0\} \subset P$$

$$Ax = b$$
$$A(x_0 + \lambda d) = b$$

$$Ax_0 + \lambda Ad = b$$
$$b + \lambda Ad = b \qquad \Rightarrow \lambda Ad = 0$$

$$\therefore\ Ad = 0\ \checkmark$$

Since $x \geq 0$ and $x_0 \geq 0$ in

$$x = x_0 + \lambda d \qquad (\lambda d) \geq 0$$

so either $\lambda, d \leq 0$ or $\lambda, d \geq 0$
but we already know $\lambda \geq 0$ $\therefore\ d \geq 0\ \checkmark$

B→A: Let $d \in \mathbb{R}^n$, $Ad = 0$, and $d \geq 0$

$$(Ad = 0) \cdot \lambda \qquad \lambda Ad = 0$$
$$\qquad\qquad +b \quad +b$$

$$b + \lambda Ad = b \qquad Ax_0 + \lambda Ad = b$$

$A(x_0 + \lambda d) = b$ $\quad x_0 + \lambda d \in P$

since $x \geq 0$ and $d \geq 0$, $\therefore \lambda \geq 0$

Then if we restrict $\lambda \geq 0$ then we can guarantee that at least $\{x \in \mathbb{R}^n \mid x = x_0 + \lambda d, \lambda \geq 0\} \subset P$

making d, by definition, an extremal direction.

b)

Out of Time

I dropped my second class this week so this won't keep happening. I definitely bit off more than I can chew this semester. Hopefully, with the other class off my schedule, my performance in this class will improve.

Understand If you need help please let me know

Also it might be a good idea to talk with Dr Tang for some advice on this class

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