MA 515 Homework 5

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Problem 1

Proof. Let $V = \text{span}\{v_1, \ldots, v_n\}$, where $v_1, \ldots, v_n$ are linearly independent elements in V. Then there exists n linearly independent elements $x_1, \ldots, x_n \in X$ such that $T(x_i) = v_i, i = 1, \ldots, n$. The existence promised by the fact that T is a linear operator. Let $Y_0 = \text{span}\{x_1, \ldots, x_n\}$. Hence, $\dim(Y_0) = \dim(V) = n$.

Also, $\ker(T) \cap Y_0 = \{0\}$. Indeed, if $\exists y \neq 0, y \in Y_0$ such that $T(y) = 0$. Let $y = \sum_{i=1}^n \beta_i x_i$. Then there exists $\beta_i \neq 0$. By linearity of T, $T(y) = \sum_{i=1}^n \beta_i T(x_i) = \sum_{i=1}^n \beta_i v_i \neq 0$ and it yields a contradiction.

Next, we will show $\ker(T) + Y_0 = X$. Suppose not, for any $x \in X$, there exists $z \notin \ker(T) + Y_0, w \in \ker(T), r \in Y_0$ such that $x = z + w + r$. Let $r = \sum_{i=1}^n t_i x_i$, and $T(x) = \sum_{i=1}^n \alpha v_i$. Hence,

$$\sum_{i=1}^n \alpha_i v_i = T(x) = T(z) + T(w) + T(r) = T(z) + \sum_{i=1}^n t_i v_i.$$

, which implies that $T(z) = \sum_{i=1}^n (\alpha_i - t_i) v_i$.

However, $z \notin \ker(T) + Y_0$ and so $T(z) \notin \text{span}\{v_1, \ldots, v_n\} \subset (\ker(T) + Y_0)$. Hence, it is a contradiction.

In conclusion, $\ker(T) + Y_0 = X$ and $\ker(T) \cap Y_0 = \{0\}$, and it implies that $X = \ker(T) \oplus Y_0$. □

Problem 2

Proof. If T is continuous, then the preimage(i.e., $\ker(T)$) of $\{0\}$ is closed since $\{0\}$ is closed. Also, $\ker(T)$ is a subspace due to the linearity of T.

If $\ker(T)$ is a closed subspace in X, we need to show that T is continuous, or equivalently, bounded. Since Y is a finite-dimensional space, from the result of problem 1, we know there exists a finite-dimensional subspace $Y_0 \subset X$ such that $X = \ker(T) \oplus Y_0$. Hence, for any $x \in X$, there exists $y \in \ker(T), z \in Y_0$ such that $x = y + z$.

Also, from $X = \ker(T) \oplus Y_0$, $\dim(Y_0)$ is finite and $\ker(T)$ is closed, we know that both projection maps $\Pi_{\ker}$ and Π_{Y_0} are bounded. $y = \Pi_{\ker}(x)$, $z = \Pi_{Y_0}(x)$.

Consider the norm of T. For any $x \in X$, by linearity, $T(x) = T(y) + T(z) = T(z) = T \circ \Pi_{Y_0}(x)$.

$$\|T\|_\infty = \sup_{x\in X\backslash\{0\}} \frac{\|T(x)\|_Y}{\|x\|_X} = \sup_{x\in X\backslash\{0\}} \frac{\|T\circ\Pi_{Y_0}(x)\|_Y}{\|\Pi_{Y_0}(x)\|_X}\frac{\|\Pi_{Y_0}(x)\|_X}{\|x\|_X}$$

Since Π_{Y_0} is bounded, $\frac{\|\Pi_{Y_0}(x)\|_X}{\|x\|_X} < +\infty$ for any $x \in X\backslash\{0\}$. Consider linear operator $T|_{Y_0} : Y_0 \to Y$ and it is bounded since $\dim(Y_0) < +\infty$.

$$\sup_{x\in X\backslash\{0\}} \frac{\|T\circ\Pi_{Y_0}(x)\|_Y}{\|\Pi_{Y_0}(x)\|_X} \leqslant \sup_{y\in Y_0\backslash\{0\}} \frac{\|T|_{Y_0}(y)\|_Y}{\|y\|_X} < +\infty.$$

Hence, $\|T\|_\infty < +\infty$ and so T is continuous.

□

Problem 3

Proof. Denote the graph of f as $G(f) := \{(x, f(x))|x \in X\} \subset X \times Y$. Let $\{z_n\}_{n\in\mathbb{N}} = \{(x_n, f(x_n))\}_{n\in\mathbb{N}} \subset G(f)$ that converges to $z = (x, y)$. It is enough to show that $y = f(x)$.

Indeed, $x = \lim_{n\to+\infty} x_n$ and so $\lim_{n\to+\infty} f(x_n) = f(x)$ due to the continuity of f. Also, $y = \lim_{n\to+\infty} f(x_n)$. Hence, $y = f(x)$.

□

Question: Here we only need X, Y to be metric spaces. We didn't really need completeness. Is it correct?

Problem 4

(a) *Proof.* Prove by contradiction.

Suppose that f is not continuous on $\mathbb{R}$. Hence, there exists one sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ that converges to x, such that a subsequence $\{x_{n_k}\}_{k\geqslant 1} \subset \{x_n\}_{n\geqslant 1}$, from which $\{f(x_{n_k})\}$ doesn't converge to $f(x)$.

Since f is bounded, we know that $\{f(x_{n_k})\}$ must have a convergent subsequence, denote as $\{f(x_{n_{k_l}})\}_{l\geqslant 1} \to y$. Also, we know $\{x_{n_{k_l}}\} \to x$ and with closeness of $G(f)$, we know $y = f(x)$. This is equivalent to say that $\{f(x_{n_{k_l}})\}_{l\geqslant 1} \to f(x)$. It leads to a contradiction to the assumption that $\{f(x_{n_k})\}$ doesn't converge to $f(x)$.

In conclusion, f is a continuous function.

□

(b) Let f be the following function,

$$f(x) = \begin{cases} \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Problem 5

Proof. $\Rightarrow$ If T_1 is compact and T_2 is continuous, then, for any bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{y_{n_k}\} := \{T_1(x_{n_k})\}$ converges in Y. Denote $\lim_{k\to+\infty} y_{n_k} = y$. Since T_2 is continuous, $T_2(y_{n_k}) \to T_2(y) \in Z$. i.e., for any bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{T_2 \circ T_1(x_{n_k})\}$ converges in Z. In conclusion, $T_2 \circ T_1$ is compact.

$\Leftarrow$ Conversely, suppose T_2 is compact and T_1 is continuous. Then, for any bounded set $U \in X$, $T_1(U) = V$ is also a bounded set in Y since T_1 is bounded. By compactness, $\overline{T_2(V)}$ is compact. Hence, for any bounded set $U \in X$, $T_2 \circ T_1(U)$ is compact in Z. Hence, $T_2 \circ T_1$ is compact.

□

Problem 6

(i) *Proof.* T is linear, bounded and bijective so that T^{-1} exists and bounded. Note that $\|x\| = \|T^{-1} \circ T(x)\|$. Hence, there exists $M > 0$, such that

$$\frac{\|x\|}{\|T(x)\|} = \frac{\|T^{-1} \circ T(x)\|}{\|T(x)\|} \leqslant \sup_{y\in X\setminus\{0\}} \frac{\|T^{-1}(y)\|}{\|y\|} \leqslant M.$$

Let $\beta = 1/M$, then $\|T(x)\| \geqslant \beta\|x\|, \forall x \in X$.

□

(ii) *Proof.* From (i) we know that T^{-1} exists as a linear bounded operator. $\|T^{-1}\| \leqslant M = 1/\beta$.

For each $y \in Y$, we define a map $Q_y : X \to X$ such that $Q_y(x) := T^{-1}(y - \Psi(x))$, where Ψ is a bounded linear operator with norm $\|\Psi\|_\infty < \beta$. It is enough to show that Q_y is a contraction mapping.

Indeed, for any $x_1, x_2 \in X$, we have

$$\begin{aligned}\|Q_y(x_1) - Q_y(x_2)\| = \|T^{-1}(y - \Psi(x_1)) - T^{-1}(y - \Psi(x_2))\| &\leqslant \|T^{-1}\|_\infty \|y - \Psi(x_1) - (y - \Psi(x_2))\| \\ &= \|T^{-1}\| \|\Psi(x_1) - \Psi(x_2)\| \\ &\leqslant \|T^{-1}\|_\infty \|\Psi\|_\infty \|x_1 - x_2\| \\ &= c\|x_1 - x_2\|.\end{aligned}$$

where $c = \|T^{-1}\|_\infty \|\Psi\|_\infty \in (0, 1)$. This yields that Q_y is a contraction mapping for each $y \in Y$.

Hence, equation $x = Q_y(x)$ has a unique solution for each $y \in Y$.

□

Problem 7

(i) *Proof.* For given $y \in Y$, define $f_y : X \to G(B) \cap (X \times \{y\}) \subset (X \times Y)$, $f_y(x) = B(x, y)$. Hence, f_y is linear since B is bilinear. Also, we know that $X \times Y$ is a Banach space since both X and Y are Banach. By closed graph theorem, it is enough to show that $G(f_y)$ is closed.

Take a convergent sequence $\{a_n\} := \{(x_n, f_y(x_n))\} \subset G(f_y)$ and $a_n \to a = (a_1, a_2)$. i.e., $\lim_{n\to+\infty} x_n = a_1, \lim_{n\to+\infty} f_y(x_n) = a_2$. We want to show that $f_y(a_1) = a_2$. Indeed, since B is continuous at the origin, f_y is continuous at 0. Also, with the linearity of f_y, f_y is continuous on X.

then the sequence $\{(x_n, y)\} \to (a_1, y)$. Hence, $B(x_n, y) = f_y(x_n)$ converges to $B(a_1, y)$, which is $f_y(a_1)$. Hence, $a_2 = f_y(a_1)$, and so $G(f_y)$ is closed.

In conclusion, f_y is bounded for each $y \in Y$. Similarly, $g_x : y \mapsto B(x, y)$ is also bounded for each $x \in X$.

□

(ii) *Proof.* We will show that $\exists C > 0$ such that

$$\|B(x, y)\|_Z \leqslant C, \qquad \forall \|x\|_X, \|y\|_Y \leqslant 1.$$

If this is true, then

$$\|B(x, y)\|_Z = \|x\|_X \cdot \|y\|_Y \cdot \left\| B(\frac{x}{\|x\|_X}, \frac{y}{\|y\|_Y}) \right\| \leqslant C\|x\|_X \cdot \|y\|_Y$$

Indeed, consider a collection of linear operators: $\mathfrak{X} := \{B(x, \cdot) : Y \to Z \mid \|x\|_X \leqslant 1\}$.

Since for any $y \in Y$, $B(\cdot, y)$ is a linear bounded operator, there exists $C_y > 0$ such that

$$\sup_{\|x\|_X \leqslant 1} \|B(x, y)\|_Z \leqslant C_y.$$

Since $\sup_{\|x\|_X \leqslant 1} \|B(x, y)\|_Z = \sup_{h_x \in \mathfrak{X}} \|h_x(y)\|_Z$, we apply Banach-Steihaus theorem so that there exists $C > 0$ such that

$$\sup_{\|x\|_X \leqslant 1} \|B(x, y)\|_Z = \sup_{h_x \in \mathfrak{X}} \|h_x(y)\|_Z \leqslant C, \qquad \forall y \in Y.$$

Hence,

$$\sup_{\|x\|_X \leqslant 1, \|y\|_Y \leqslant 1} \|B(x, y)\|_Z \leqslant \sup_{\|x\|_X \leqslant 1, y \in Y} \|B(x, y)\|_Z \leqslant C.$$

which is $\|B(x, y)\|_Z \leqslant C, \forall \|x\|_X, \|y\|_Y \leqslant 1$.

□

Problem 8

(i) *Proof.* We need to show that T is bounded. Consider norm $\|T\|_\infty$.

$$\|T\|_\infty = \sup_{\|f\|_\infty=1} \|T[f]\|_\infty = \max\{ \sup_{\substack{\|f\|_\infty=1,\\ t\in(0,1]}} |T[f](t)|, \sup_{\|f\|_\infty=1} |f(0)|\}.$$

It is clear that $|\sup_{\|f\|_\infty=1} f(0)| < +\infty$. For the other one,

$$\begin{aligned}\sup_{\substack{\|f\|_\infty=1,\\ t\in(0,1]}} |T[f](t)| &= \sup_{\substack{\|f\|_\infty=1,\\ t\in(0,1]}} \left|\frac{1}{t}\int_0^t f(s)ds\right| \\ &\leqslant \sup_{\substack{\|f\|_\infty=1,\\ t\in(0,1]}} \frac{\|f\|_\infty}{t}\int_0^t ds = 1 < +\infty.\end{aligned}$$

Hence, $\|T\|_\infty < +\infty$.

□

(ii) *Proof.* First, we will show T is a one-to-one map. Suppose there exists $f_1 \neq f_2$, $f_1, f_2 \in X$ such that $T[f_1] = T[f_2]$. This is to say that $\exists \|f_1 - f_2\|_\infty > 0$ such that $\|T[f_1] - T[f_2]\|_\infty = 0$. However,

$$\begin{aligned}\|T[f_1] - T[f_2]\|_\infty &= \sup_{t\in[0,1]} |T[f_1](t) - T[f_2](t)| \\ &= \max\left\{ |f_1(0) - f_2(0)|, \sup_{t\in(0,1]} \left|\frac{1}{t}\int_0^t f_1(s) - f_2(s)ds\right| \right\} = 0.\end{aligned}$$

which implies that $|f_1(0) - f_2(0)| = 0$, i.e. $f_1(0) = f_2(0)$. It also yields that $\forall t \in (0,1]$, $\left|\frac{1}{t}\int_0^t f_1(s) - f_2(s)ds\right| = 0$, i.e., $\forall t \in (0,1]$, $\int_0^t f_1(s)ds = \int_0^t f_2(s)ds$. Take derivative of t on both sides and get

$$\frac{d}{dt}\int_0^t f_1(s)ds = \frac{d}{dt}\int_0^t f_2(s)ds \quad \Rightarrow \quad f_1(t) = f_2(t), \forall t \in (0,1].$$

and this yields a contradiction to the assumption that $f_1 \neq f_2$. Hence, T is a one-to-one map.

Next, we will show that T is not an onto map.

Let $g = T[f], \forall f \in X = C([0,1])$. Since $T : X \to X$, $g \in X$. We compute $\frac{d}{dt}g(t)$,

$$\frac{d}{dt}g(t) = -\frac{1}{t^2}\int_0^t f(s)ds + \frac{1}{t}f(t) = \frac{f(t) - g(t)}{t}. \forall t \in (0,1).$$

Since f, g are well-defined on $[0,1]$, we know $\frac{d}{dt}g(t)$ exists on $(0,1)$. Also, notice that $\frac{f(t)-g(t)}{t}$ is continuous on $(0,1)$. If we take $g(t) = |t-1/2|, t \in [0,1]$. $g \in X$, but $\frac{dg}{dt}$ doesn't exist at $t = 1/2$, so there won't be any $f \in X$ such that $T[f] = g$.

Hence, T is not a onto map.

□

(iii) *Proof.* To show that T is not compact, we only need to find a sequence of function $\{f_n\}_{n\mathbb{N}} \subset X$ such that $\exists \epsilon > 0$, such that

$$\|T[f_m] - f[f_n]\|_\infty > \epsilon. \quad \forall m, n > 0.$$

Let us consider $f_n = -nx + n$ and it it obvious that $f_n \in C([0,1]), \forall n \in \mathbb{N}$.

$$T[f](t) = \begin{cases} n & t = 0 \\ -\dfrac{n}{2}t + n & t \in (0,1] \end{cases}$$

Without generality, assume that $m > n > 0$. Then we may let $\epsilon = 1/3$, then we have

$$\|T[f_m] - f[f_n]\|_\infty = \frac{m-n}{2} \geqslant \frac{1}{2} > \epsilon.$$

Since m, n are arbitrarily chosen from $\mathbb{N}$, it yields that T is not compact.

□