Homework 7 Solutions

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Problem 1 (4.7)

Proof. Suppose that the dual is feasible and bounded. Then, it has a finite optimum. By strong duality theorem, the dual of dual, which is the primal, also has a finite optimum. But this is a contradiction to the infeasibility of primal problem.

□

Problem 2 (4.8)

Problem 4 (4.11)

Proof. If $Ax = b, x \geqslant 0$ has a solution x_0 and $A^T w \geqslant 0$, then $x_0^T A^T w \geqslant$. This implies that $b^T w \geqslant 0$.

Conversely, we need to show that if $Ax = b, x \geqslant 0$ has no solution, then $b^T w < 0$ as $A^w \geqslant 0$. Indeed, this is true due to Farkas Lemma.

□

Problem 5 (4.13)

Proof. If $Ax \leqslant b, x \geqslant 0$ has a solution x_0 and $A^T w \geqslant 0, w \geqslant 0$, then $x_0^T A^T \leqslant b^T$. Multiply w on both sides and we have

$$0 = x_0^T A^T w \leqslant b^T w.$$

which is $b^T w \geqslant 0$.

Conversely, we need to show that if $b^T w \geqslant 0$ when $A^T w \geqslant 0, w \geqslant 0$, then $Ax \leqslant b, x \geqslant 0$ has a solution. Consider the following primal dual problem,

$$\begin{aligned} \text{Min} \quad & 0^T x \\ \text{Subject to} \quad & Ax \leqslant b \\ & x \geqslant 0 \end{aligned} \tag{1}$$

$$\begin{aligned} \text{Max} \quad & b^T y \\ \text{Subject to} \quad & A^T y \leqslant 0 \\ & y \leqslant 0 \end{aligned} \tag{2}$$

(2) is equivalent to (3)

$$\begin{aligned} \text{Max} \quad & -b^T w \\ \text{Subject to} \quad & A^T w \geqslant 0 \\ & w \geqslant 0 \end{aligned} \tag{3}$$

In (3), it is obvious that $w = 0$ is a feasible solution. What's more, it is also an optimal solution due to the assumption that $b^T w \geqslant 0$ when $A^T w \geqslant 0, w \geqslant 0$. Hence, $\max -b^T w = 0$. By strong duality theorem, we know (1) is also feasible.

$\square$