MA 515-001, Fall 2017, Homework 6

Due: Fri Dec 01, 2017, in-class.

Problem 1. *Let X be a normed space. Given $\bar{x} \in X$, show that there exists a bounded linear function $\Phi : X \to \mathbb{R}$ such that*

$$\|\Phi(\bar{x})\| \ = \ \|\bar{x}\|\,.$$

Problem 2. *Let X be a normed space. If $f(x) = f(y)$ for all $f \in X^*$, show that $x = y$.*

Problem 3. *Let Y be a subspace of a normed space of X and let $\Lambda : X \to \mathbb{R}^n$ be a bounded linear operator from Y into the Euclidean space $\mathbb{R}^n$.* Show that Λ can be extended to a linear operator $\tilde{\Lambda} : X \to \mathbb{R}^n$ such that

$$\|\tilde{\Lambda}\|_\infty \ \leq \ \sqrt{n} \cdot \|\Lambda\|_\infty\,.$$

Problem 4. *Given a Banach space X , let $\varphi \in X^*$ and $\{\varphi_n\}_{n\geq 1} \subset X^*$ be such that*

$$\|\varphi_n\|_\infty \ = \ \sup_{\|x\|\leq 1} \|\varphi_n(x)\| \ \leq \ M$$

for some constant $M > 0$. Assume that there exists a dense $S = \{y_1, y_2, ..., y_k, ...\}$ in X such that for every $y_k \in S$ it holds

$$\lim_{n\to\infty} \varphi_n(y_k) \ = \ \varphi(y_k)\,.$$

Show that φ_n is weakly star convergent to φ in X^* and $\|\varphi\|_\infty \leq M$.

Problem 5. *Let X be a finite dimensional normed space. Consider a weakly convergent sequence $x_n \rightharpoonup x$. Prove that the sequence converges strongly.*

Problem 6. *In the space l^p of sequences of real numbers such that consider unit vectors*

$$e_1 \ = \ (1,0,0,\dots), \quad e_2 \ = \ (0,1,0,\dots), \quad e_3 \ = \ (0,0,1,\cdots), \qquad \text{etc}\dots.$$

Prove that

- *(i) For $1 < p < +\infty$, then the sequence $\{e_n\}_{n\geq}$ weakly converges to 0 in l^p.*
- *(ii) In the space, the sequence $\{e_n\}_{n\geq}$ does not admit any weakly convergent subsequence.*

Problem 7. *Let $(X, \|\cdot\|)$ be a Banach space such that the parallelogram identity holds*

$$\|x-y\|^2 + \|x+y\|^2 \ = \ 2\cdot\|x\|^2 + 2\cdot\|y\|^2 \qquad \forall x, y \in X\,.$$

Show that

$$\langle x, y\rangle \ \doteq \ \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right)$$

is an inner product on X, which yields the same norm $\|\cdot\|$.

Problem 8. *Prove that*

(i) *In* $\mathbb{R}^2$, *the norm* $\|x\|_1 = |x_1| + |x_2|$ *is not generalized by an inner product.*

(ii) *On the space of real-valued continuous function* $C([0,1])$, *the norm*

$$\|f\|_\infty \;=\; \max_{x\in[0,1]} |f(x)|$$

is not generalized by an inner product.

Problem 9. *Given a sequence* $\{x_n\}_{n\geq 1}$ *in a Hilbert space* H, *show that the strong convergence* $\|x_n - x\| \to 0$ *if and only if*

$$x_n \rightharpoonup x \qquad \text{and} \qquad \lim_{n\to+\infty} \|x_n\| \;=\; \|x\|\,.$$

Problem 10. *Let* H *be a Hilbert space. Show that*

(i) *If* $\{e_n\}_{n\geq 1}$ *be an orthonormal sequence then* e_n *is weakly convergent to* 0.

(ii) *For a given* $x \in H$ *with* $\|x\| \leq 1$, *construct a sequence of* $\{x_n\}_{n\geq 1} \subset H$ *with* $\|x_n\| = 1$ *such that* x_n *is weakly convergent to* x.

Problem 11. *Let* $\{v_n\}_{n\geq 1}$ *be orthonormal and unit vectors. Show that*

$$\sum_{n\geq 1} \alpha_n v_n \text{ converges if and only if } \sum_{n=1}^{\infty} |\alpha_n|^2 \;<\; +\infty\,.$$

Problem 12. *Let* H *be a Hilbert space, and let* $\Omega \subset H$ *be closed, convex subset.*

(i) *Prove that* $x \in H$, *there exists a unique point* $y \in \Omega$ *such that*

$$\|x - y\| \;=\; d_\Omega(x) \;\doteq\; \min_{w\in\Omega} \|x - w\|\,.$$

This point of minimum distance $y = \pi_\Omega(x)$ *is called the perpendicular projection of* x *into* Ω.

(ii) *Show that* $y = \pi_\Omega(x)$ *if and only if* $\langle w - y, y - x\rangle \geq 0$ *for all* $w \in \Omega$.