Homework 2 Solutions

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Problem 1.1

Solutions:

1. Let $x_2 = x_2^+ - x_2^-$ and $x_2^+, x_2^- \geqslant 0$. And add slack variables ξ_1, ξ_2 on the first and the second constraint respectively.

$$\begin{aligned} \text{Minimize} \quad & 4x_1 + \sqrt{2}x_2^+ - \sqrt{2}x_2^- - 0.35x_3 \\ \text{subject to} \quad & -0.001x_1 + 200x_2^+ - 200x_2^- - \xi_1 = 7\sqrt{261} \\ & 7.07x_2^+ - 7.07x_2^- - 2.62x_3 + \xi_2 = -4 \\ & x_1, x_2^+, x_2^-, x_3, \xi_1, \xi_2 \geqslant 0 \end{aligned}$$

2. Let $a_1 = x_1 - 20$, $a_3 = x_3 + 15$, then $a_1, a_3 \geqslant 0$, and $x_1 = a_1 + 20, x_3 = a_3 - 15$.

   Add a slack variable ξ_1 on the second constraint. The standard form is

$$\begin{aligned} \text{Minimize} \quad & 3.1a_1 - 2\sqrt{2}x_2 + a_3 + 47 \\ \text{subject to} \quad & 100a_1 - 20x_2 = -1993 \\ & -11a_1 - 7\pi x_2 - 2a_3 + \xi_1 = 590 \\ & a_1, x_2, a_3, \xi_1 \geqslant 0 \end{aligned}$$

3. Since $x_3 \leqslant 10$, $10 - x_3 \geqslant 0$. Let $a_3 = 10 - x_3$, then $a_3 \geqslant 0$ and $x_3 = 10 - a_3$. Let $x_1 = x_1^+ - x_1^-$, where $x_1^+, x_1^- \geqslant 0$. Add slack variables on each constraint and the standard form is the following,

$$\begin{aligned} \text{Minimize} \quad & -x_1^+ + x_1^- - 3x_2 - 2a_3 + 20 \\ \text{subject to} \quad & 3x_1^+ - 3x_1^- - 5x_2 - \xi_1 = -2 \\ & 3x_1^+ - 3x_1^- - 5x_2 + \xi_2 = 15 \\ & -5x_1^+ + 5x_1^- + 20x_2 - \xi_3 = 11 \\ & -5x_1^+ + 5x_1^- + 20x_2 + \xi_4 = 40 \\ & x_1^+, x_1^-, x_2, a_3, \xi_i (i = 1, \ldots, 4) \geqslant 0 \end{aligned}$$

Problem 1.2

a) Let $x_1 = x_1^+ - x_1^-$, where $x_1^+, x_1^- \geqslant 0$. The standard form is the following,

$$\begin{aligned} \text{Minimize} \quad & 2x_1^+ - 2x_1^- + 6x_2 + 8x_3 \\ \text{subject to} \quad & x_1^+ - x_1^- + 2x_2 + x_3 = 5 \\ & 4x_1^+ - 4x_1^- + 2x_3 = 12 \\ & x_1^+, x_1^-, x_2, x_3 \geqslant 0 \end{aligned}$$

b) From the first constraint, solve x_1 as $x_1 = 5 - 2x_2 - x_3$. Plug it into the objective function and also the second constraint, then reform the LP problem as the following,

$$\begin{aligned} \text{Minimize} \quad & 2x_2 + 6x_3 + 10 \\ \text{subject to} \quad & -2x_2 - 2x_3 = -8 \\ & x_2, x_3 \geqslant 0 \end{aligned}$$

c) It is already in the standard form.

d) Use graphic method to solve the problem. Optimal value is $z* = 10$ and the optimal solution is $x* = (x_2^*, x_3^*) = (0, 0)$.

Problem 1.3

a) No. Because there is a nonlinear term x_1^2 in the objective function and the first constraint.

b) Yes. Use the first constraint, solve x_1^2 and get $x_1^2 = x_2$. Plug it into the objective function and get

$$\begin{aligned} \text{Minimize} \quad & 2x_2 + 4x_3 \\ \text{subject to} \quad & 2x_2 + 4x_3 \geqslant 4 \\ & x_1, x_3 \geqslant 0, x_2 \geqslant 2 \end{aligned}$$

c) Use the similar technique in problem 1.1 and 1.2. Let $a_2 = x_2 - 2$. Then $a_2 \geqslant 0$, and $x_2 = a_2 + 2$. Add a slack variable on the constraint.

$$\begin{aligned} \text{Minimize} \quad & 2a_2 + 4x_3 + 4 \\ \text{subject to} \quad & 2a_2 + 4x_3 - \xi = 0 \\ & x_1, a_2, x_3, \xi \geqslant 0 \end{aligned}$$

d) Yes. To solve the LP problem, use graphic method.

To solve the original problem, we can graph the feasible region in 3-D and use graph method to solve it.

Problem 1.4

a) No. Because it has absolute-value functions in the objective function.

b) Let $x_i = x_i^+ - x_i^-$, $i = 1, 2, 3$, where $x_i^+, x_i^- \geqslant 0$. Then $|x_i| = x_i^+ + x_i^-$. Add one slack variable ξ_1 on the first constraint and the problem is reformed as

$$\begin{array}{rl} \text{Minimize} & x_1^+ + x_1^- + 2x_2^+ + 2x_2^- - x_3^+ - x_3^- \\ \text{subject to} & x_1^+ - x_1^- + x_2^+ - x_2^- - x_3^+ + x_3^- + \xi_1 = 10 \\ & x_1^+ - x_1^- - 3x_2^+ + 3x_2^- + 2x_3^+ - 2x_3^- = 12 \\ & x_i^+, x_i^- \geqslant 0, i = 1, 2, 3. \\ & \xi_1 \geqslant 0 \end{array}$$

c) Use the similar technique as in (b), let $a_1 = x_1 - 5$ and $a_2 = x_2 + 4$. Then, $|x_1 - 5| = a_1^+ + a_1^-$ and $|x_2 + 4| = a_2^+ + a_2^-$.

Also, it is clear to see that

$$x_1 = a_1 + 5 = a_1^+ - a_1^- + 5, \quad x_2 = a_2 - 4 = a_2^+ - a_2^- - 4.$$

Plug above into the problem and get the following standard form.

$$\begin{array}{rl} \text{Minimize} & a_1^+ + a_1^- + a_2^+ + a_2^- \\ \text{subject to} & a_1^+ - a_1^- + a_2^+ - a_2^- + \xi_1 = 10 \\ & a_1^+ - a_1^- - 3a_2^+ + 3a_2^- - \xi_2 = -15 \\ & a_i^+, a_i^-, \xi_i \geqslant 0, i = 1, 2. \end{array}$$

Problem 1.5