# MA 515-001, Fall 2017, Homework 1

Due: Mon Sep 11, 2017, in-class.

**Problem 1.** Let $X = \mathbb{R}$ be the set of all real number. For any $x, y \in \mathbb{R}$, define

$$d(x,y) \doteq \sqrt{|x-y|}\,.$$

Show that $(X, d)$ is a metric space.

**Problem 2.** Given $a, b \in \mathbb{R}$ and $a < b$, consider the set of all continuous real function on $[a, b]$

$$X \doteq C([a,b],\mathbb{R}) = \left\{ f : [a,b] \to \mathbb{R} \;\middle|\; f \text{ is continuous} \right\}.$$

Let $d : X \times X \to [0, +\infty)$ be such that

$$d(f,g) = \int_a^b |g(t) - f(t)| + \sqrt{|g(t) - f(t)|}\, dt \qquad \forall f, g \in X\,.$$

Is $(X, d)$ a metric space? If Yes, prove it.

**Problem 3.** Let $(X, d)$ be a metric space. Prove that

(a) For any $x, y, z, w \in X$, it holds

$$|d(x,y) - d(z,w)| \le d(x,z) + d(y,w)\,.$$

(b) For any $x, y, z \in X$, it holds

$$|d(x,z) - d(y,z)| \le d(x,y)\,.$$

**Problem 4.** Recalling that for any $p > 1$,

$$l^p = \left\{ x = \{x_i\}_{i\ge 1} \;\middle|\; \sum_{i=1}^{+\infty} |x_i|^p < +\infty \right\}.$$

(a). Find a sequence which converges to 0 but is not in $l^2$.

(b). Find a sequence $x = \{x_i\}_{i\ge1}$ such that $x \in l^3$ but $x \notin l^2$.

**Problem 5.** Recalling that

$$l^\infty = \left\{ x = \{x_i\}_{i\ge1} \;\middle|\; x \text{ is a bounded sequence} \right\}$$

and

$$d(x,y) \doteq \sup_{i\ge1} |x_i - y_i| \qquad \forall x, y \in l^\infty\,.$$

Show that $l^p \subset l^{+\infty}$ for all $p > 1$.

**Problem 6.** Let $(X, d)$ be a metric space. Consider an *increasing* function $f : [0, +\infty) \to [0, +\infty)$ be such that

(i) $f(0) = 0$ and $f(s) > 0$ for all $s > 0$

(ii) *(Supper-additivity)* $f(s+r) \ \leq \ f(s) + f(r)$ for all $s, r \in [0, +\infty)$.

Denote by

$$d_f(x,y) \ = \ f \circ (d(x,y)) \qquad \forall x, y \in X .$$

(a). Show that $(X, d_f)$ is a metric space.

(b). Using (a) to prove that

$$\tilde{d}(x,y) \ \doteq \ \frac{d(x,y)}{1 + d(x,y)} \qquad \forall x, y \in X$$

is a metric.

**Problem 7.** *(Product metric space)* Let $(X_1, d_1)$ and $(X_2, d_2)$ be metric spaces. Consider the Cartesian product

$$X \ = \ X_1 \times X_2 \ \doteq \ \{(x_1, x_2) \mid x_1 \in X_1 \text{ and } x_2 \in X_2\} .$$

Show that

(a) a metric $d$ is defined by

$$d(x,y) \ = \ d_1(x_1, y_1) + d_2(x_2, y_2) \qquad \forall x = (x_1, x_2), y = (y_1, y_2) \in X ;$$

(b) a metric $\tilde{d}$ is defined by

$$\tilde{d}(x,y) \ = \ \sqrt{d_1^2(x_1, y_1) + d_2^2(x_2, y_2)} \qquad \forall x = (x_1, x_2), y = (y_1, y_2) \in X .$$

**Problem 8.** Given a metric space $(X, d)$ and a nonempty subset $B$ if $X$, *the distance from a point $x \in X$ to $K$* is defined as

$$d_K(x) \ = \ \inf_{w \in K} \ d(x, w) .$$

Show that $d_K$ is a Lipschitz function with a Lipschitz constant 1, i.e.,

$$\big| d_K(x) - d_K(y) \big| \ \leq \ d(x,y) \qquad \forall x, y \in X .$$