

MA 515 Homework 3

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Problem 1

- (a) It is not a norm because it violates property (ii). Let $a = -1, x = 1$. Then $\|ax\| = \|-1\| = 2$. However, $|a|\|x\| = 1$.
- (b) It is a norm.

Proof. Check the first one. If $\|f\| = 0$, then $\sup_{t \geq 0} e^{\lambda t} |f(t)| = 0$. Since for any $t \geq 0$, $e^{\lambda t} |f(t)| \geq 0$ and $e^{\lambda t} > 0$, we have $|f(t)| = 0, \forall t \geq 0$. This is to say $f \equiv 0$ on its domain. For the other direction, $\|f\| = 0$ is true when $f = 0$.

For (ii), $\forall \alpha \in \mathbb{R}$,

$$\|\alpha x\| = \sup_{t \geq 0} e^{\lambda t} |\alpha f(t)| = \sup_{t \geq 0} e^{\lambda t} |\alpha| \cdot |f(t)| = |\alpha| \sup_{t \geq 0} e^{\lambda t} |f(t)| = |\alpha| \cdot \|x\|.$$

To check the triangle inequality, we pick $f, g \in X$,

$$\begin{aligned} \|f + g\| &= \sup_{t \geq 0} e^{\lambda t} |f(t) + g(t)| \leq \sup_{t \geq 0} e^{\lambda t} (|f(t)| + |g(t)|) \\ &\leq \sup_{t \geq 0} e^{\lambda t} |f(t)| + \sup_{t \geq 0} e^{\lambda t} |g(t)| = \|f\| + \|g\|. \end{aligned}$$

In conclusion, it is a norm.

□

- (c) It is a norm.

Proof. We have shown that for ℓ^p space, $\|x\|_p = (\sum_{i=1}^{+\infty} |x_i|^p)^{1/p}$ is a norm for $1 \leq p \leq +\infty$. Then, truncate it for only first two entries. Consider set $S = \{x \in \ell^p \mid x = (x_1, x_2, 0, \dots), x_1, x_2 \in \mathbb{R}\}$. We have $\|x\|_p = \|x\|, \forall x \in S$. Since $\|\cdot\|_p$ is a norm on ℓ^p , it must be a norm on $S \subset \ell^p$ as $0 \in S$. Then we conclude that $\|\cdot\|$ is a norm on \mathbb{R}^2 .

□

- (d) It is not a norm. Consider $x = (1, 0), y = (0, 1)$. Then $\|x + y\| = \|(1, 1)\| = 2^{1/p} > 2$. However, $\|x\| + \|y\| = 2 < \|x + y\|$. This breaks the triangle inequality.

Problem 2

Proof.

Step 1 We are going to show $\|(\cdot, \cdot)\|$ is a norm on $X \times Y$.

If $\|(x, y)\| = 0$, then $\max\{\|x\|_X, \|y\|_Y\} = 0$, which implies that $\|x\|_X = \|y\|_Y = 0$. Hence, $x = y = 0$. The other direction is obvious.

To check the second property, take arbitrarily $\alpha \in \mathbb{R}$ and we have

$$\begin{aligned}\|\alpha(x, y)\| &= \|(\alpha x, \alpha y)\| = \max\{\|\alpha x\|_X, \|\alpha y\|_Y\} \\ &= \max\{|\alpha| \cdot \|x\|_X, |\alpha| \cdot \|y\|_Y\} \\ &= |\alpha| \max\{\|x\|_X, \|y\|_Y\} = |\alpha| \cdot \|(x, y)\|.\end{aligned}$$

For triangle inequality, take arbitrarily $(x, y), (z, w) \in X \times Y$, we have

$$\begin{aligned}\|(x, y) + (z, w)\| &= \|(x + z, y + w)\| = \max\{\|x + z\|_X, \|y + w\|_Y\} \\ &\leq \max\{\|x\|_X + \|z\|_X, \|y\|_Y + \|w\|_Y\} \\ &= \max\{\|x\|_X, \|y\|_Y\} + \max\{\|z\|_X, \|w\|_Y\} \\ &= \|(x, y)\| + \|(z, w)\|.\end{aligned}$$

Hence, $\|(\cdot, \cdot)\|$ is a norm on $X \times Y$.

Step 2 We need to prove that $X \times Y$ is complete. Take any Cauchy sequence $\{z_n = (x_n, y_n)\}_{n \in \mathbb{N}} \subset X \times Y$. $\forall \epsilon > 0$, there exists $N > 0$ such that

$$\|z_n - z_m\| = \|(x_n, y_n) - (x_m, y_m)\| < \epsilon, \quad \forall n, m > N.$$

This is equivalent to $\max\{\|x_n - x_m\|_X, \|y_n - y_m\|_Y\} < \epsilon \forall n, m > N$ and also implies that

$$\|x_n - x_m\|_X < \epsilon, \quad \|y_n - y_m\|_Y < \epsilon, \quad \forall n, m > N.$$

Hence we conclude that $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences on X and Y respectively. Since $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, $\{x_n\}$ converges on X and $\{y_n\}$ converges on Y . Let $\lim_{n \rightarrow +\infty} x_n = x$, $\lim_{n \rightarrow +\infty} y_n = y$, and $z = (x, y) \in X \times Y$. Then,

$$\|z_n - z\| = \|(x_n, y_n) - (x, y)\| = \max\{\|x_n - x\|_X, \|y_n - y\|_Y\} < \epsilon, \quad \forall n > N.$$

Hence, $\{z_n\}$ converges to on $z \in X \times Y$. This proves the claim.

□

Problem 3

- (a) *Proof.* Firstly, we need to show that d_f is a metric on X . $\forall x, y \in X$, $d_f(x, y) = f(\|x - y\|_X) \geq 0$ holds due to the property of f . And $d_f(x, x) = f(0) = 0$. Also, $d_f(x, y) = d_f(y, x)$ is obvious since $\|x - y\|_X = \|y - x\|_X$. For triangle inequality, we need to use the facts that f is increasing and ii,

$$\begin{aligned} d_f(x, y) + d_f(y, z) &= f(\|x - y\|_X) + f(\|y - z\|_X) \geq f(\|x - y\|_X + \|y - z\|_X) \\ &\geq f(\|x - y + y - z\|_X) = f(\|x - z\|_X) = d_f(x, z). \end{aligned}$$

Hence, d_f is a metric well-defined on X .

Next, we need to show (X, d_f) is complete. Take any Cauchy sequence $\{x_n\}_n \in \mathbb{N}$ from (X, d_f) and it is enough to show it converges in X . $\forall \epsilon > 0$, there exists $N > 0$ such that $d_f(x_n, x_m) = f(\|x_n - x_m\|_X) < \epsilon$, for all $n, m > N$. Since f is an increasing continuous function from \mathbb{R}_+ to \mathbb{R}_+ , then there exists an increasing continuous inverse function f^{-1} of f such that $f^{-1}(d_f(x, y)) = \|x - y\|_X$. Hence,

$$\|x_m - x_n\|_X = f^{-1}(d_f(x_m, x_n)) < f^{-1}(\epsilon) \rightarrow 0, \text{ as } \epsilon \rightarrow 0, \forall n, m > N$$

Hence, $\{x_n\}$ is also a Cauchy sequence on $(X, \|\cdot\|_X)$ and so it converges in X in norm $\|\cdot\|_X$. Let $x_n \rightarrow x \in X$ in $\|\cdot\|_X$. Then we have $\lim_{n \rightarrow +\infty} \|x_n - x\|_X = 0$. With the continuity of f , we know $\lim_{n \rightarrow +\infty} f(\|x_n - x\|_X) = 0$. And this improves that $x_n \rightarrow x$ in distance d_f . Hence $\{x_n\}$ is convergent on (X, d_f) .

In conclusion, (X, d_f) is a complete metric space. □

- (b) *Proof.* Take arbitrarily $x, y \in B_f(0, 1)$ and $\alpha \in (0, 1)$. Then $z = \alpha x + (1 - \alpha)y \in X$. It is enough to show that $d_f(0, z) < 1$. Use property (ii) and the property given in (b), we have the following,

$$\begin{aligned} d_f(0, z) &= f(\|z\|_X) \leq f(\alpha\|x\|_X + (1 - \alpha)\|y\|_X) \\ &\leq f(\alpha\|x\|_X) + f((1 - \alpha)\|y\|_X) \\ &= \alpha f(\|x\|_X) + (1 - \alpha)f(\|y\|_X) \\ &= \alpha d_f(0, x) + (1 - \alpha)d_f(0, y) < \alpha + 1 - \alpha = 1. \end{aligned}$$

This proves that $z \in B_f(0, 1)$ and then $B_f(0, 1)$ is convex. □

Problem 4

Proof. " \Leftarrow ", if absolute convergence of any sequence $\{x_n\}$ implies the convergence of this sequence, then $(X, \|\cdot\|)$ is complete. Take $\{x_n\}_{n \in \mathbb{N}}$ as a Cauchy sequence on X , then from

the definition we can always find a subsequence $\{x_{n_k}\}$ such that $\|x_{n_k} - x_{n_{k+1}}\| < 2^{-k}$. Let $y_k = x_{n_k} - x_{n_{k+1}}$ and we know for each $N > 0$,

$$\sum_{k=1}^N \|y_k\| = \sum_{k=1}^N \|x_{n_k} - x_{n_{k+1}}\| < \sum_{k=1}^N \frac{1}{2^k}.$$

Since N is arbitrarily chosen, we take the limits on both sides such that $\sum_{k=1}^{+\infty} \|y_k\| < +\infty$. This implies the convergent of $\{y_k\}$, i.e, $\lim_{N \rightarrow +\infty} \sum_{k=1}^N y_k = \sum_{k=1}^{+\infty} y_k < +\infty$.

Suppose $S_y = \lim_{N \rightarrow +\infty} \sum_{k=1}^N y_k$, then

$$S_y = \lim_{N \rightarrow +\infty} \sum_{k=1}^N y_k = \lim_{N \rightarrow +\infty} x_{n_1} - x_{n_{N+1}}.$$

which implies that $\{x_{n_k}\}$ converges to $x_{n_1} - S_y$. Since it is a subsequence of Cauchy sequence $\{x_n\}$, $\{x_n\}$ also converges. This proves the completeness of X .

" \Rightarrow ". Conversely, with Banach space $(X, \|\cdot\|)$, and the absolutely convergent $\{x_n\}$ has sum $\sum_{n=1}^{+\infty} \|x_n\| < +\infty$, we need to prove the infinite sum of $\{x_n\}$ is also finite.

Since $\sum_{n=1}^{+\infty} \|x_n\| < +\infty$, for any $\epsilon > 0$, there exists $N_0 > 0$ such that $\sum_{n=N_0+1}^{+\infty} \|x_n\| < \epsilon$. Let $z_k = \sum_{n=1}^k x_n \in X$, $\{z_k\}$ is a Cauchy sequence because for any $\epsilon > 0$, take $N = N_0$, then

$$\|z_p, z_q\| \leq \left\| \sum_{n=q+1}^p x_n \right\| \leq \sum_{n=q+1}^p \|x_n\| \leq \sum_{n=N+1}^{+\infty} \|x_n\| < \epsilon, \quad \forall p, q > N.$$

Then $\{z_k\}$ converges in Banach space $(X, \|\cdot\|)$.

In conclusion, we proved the claim. □

Problem 9

Proof. let $g(x) = \frac{1}{4}e^{f(x)}$. We need to show that $g : \mathbb{R} \rightarrow \mathbb{R}$ is a contraction map. If so, then there exists a unique $x_0 \in \mathbb{R}$ such that $g(x_0) = x_0$, which is the unique solution to the equation.

For any $x \in \mathbb{R}$, $f(x) \in [0, 1]$. And also, let $h(z) = e^z$, $z \in [0, 1]$ and the derivative of h is in $[1, e]$. Thus, there exists $0 < c < 1$, such that

$$|g(x) - g(y)| = \frac{1}{4}|e^{f(x)} - e^{f(y)}| \leq \frac{1}{4} \max |h'| \cdot |f(x) - f(y)| \leq \frac{e}{4}c|x - y|.$$

Let $c' = \frac{e}{4}c$ and note that $c' \in (0, 1)$. Hence, g is a contraction map. □

Problem 10

1. *Proof.* If $\alpha_1 = \alpha_2 = 0$, then the claim is trivial. If not, then we only need to show that there exists a positive β such that

$$\frac{||\alpha_1 e_1 + \alpha_2 e_2||}{|\alpha_1| + |\alpha_2|} \geq \beta.$$

Let function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be $f(c_1, c_2) = ||c_1 e_1 + c_2 e_2||$ and $\text{dom} f = \{(c_1, c_2) \in \mathbb{R}^2 | |c_1| + |c_2| = 1\}$. It is clear that f is continuous on \mathbb{R}^2 and $\text{dom} f$ is compact. So f reaches its minimum $K \geq 0$ on $\text{dom} f$. Also, $K > 0$, because if $K = 0$ then $c_1 = c_2 = 0$, which leads to a contradiction that $(c_1, c_2) \notin \text{dom} f$. Hence, we have

$$\frac{||\alpha_1 e_1 + \alpha_2 e_2||}{|\alpha_1| + |\alpha_2|} = \left\| \frac{\alpha_1}{|\alpha_1| + |\alpha_2|} e_1 + \frac{\alpha_2}{|\alpha_1| + |\alpha_2|} e_2 \right\| \geq K.$$

And this proves the claim. □

2. *Proof.* Similar to the previous one, only need to show that there exists $\beta > 0$ such that

$$\frac{||\sum_{i=1}^n \alpha_i x_i||}{\sum_{i=1}^n |\alpha_i|} \geq \beta.$$

Let function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be $g(c) = ||\sum_{i=1}^n c_i e_i||$ and $\text{dom} g = \{c \in \mathbb{R}^n | \sum_{i=1}^n |c_i| = 1\}$. It is clear that g is continuous on \mathbb{R}^n and $\text{dom} g$ is compact. So g reaches its minimum $\kappa \geq 0$ on $\text{dom} g$. Also, $\kappa \neq 0$ with the same reason above. Hence, we have

$$\frac{||\sum_{i=1}^n \alpha_i e_i||}{\sum_{i=1}^n |\alpha_i|} = \left\| \frac{\sum_{i=1}^n \alpha_i}{\sum_{i=1}^n |\alpha_i|} e_i \right\| \geq \kappa.$$

And this proves the claim. □