Homework 1 Solutions

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1 Problem 1

Solutions:

1.
$$det\begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix} = a^2b^2 - abab = 0.$$

2.
$$det\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \cos^2\alpha + \sin^2\alpha = 1.$$

3.
$$det\begin{pmatrix} 1 & x & x \\ x & 2 & x \\ x & x & 3 \end{pmatrix} = 1\times 2\times 3 + x^3 + x^3 - 2x^2 - x^2 - 3x^2 = 2x^3 - 6x^2 + 6.$$

4. Similar to the problem above,
$$\begin{aligned} det\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} &= bc^2 + a^2c + ab^2 - a^2b - ac^2 - b^2c \\ &= ab(b-a) + ac(a-c) + bc(c-b). \end{aligned}$$

5. Change the order of columns. And apply the operations on matrix blocks:
$$\begin{aligned} det\begin{pmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{pmatrix} &= -det\begin{pmatrix} 0 & 2 & a & 1 \\ 0 & b & 0 & 2 \\ c & 4 & 5 & 3 \\ 0 & 0 & 0 & d \end{pmatrix} \\ &= -det\begin{pmatrix} 0 & 2 & a \\ 0 & b & 0 \\ c & 4 & 5 \end{pmatrix}\cdot d \\ &= abcd. \end{aligned}$$

6. Same idea as the problem above. Note that it won't change the determinant of a matrix if we add one row multiplied by a constant number on another row.

   suppose $a \neq 0$. Then, add $\frac{1}{a}$row1 on row2. The determinant doesn't change.

$$det\begin{pmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{pmatrix} = det\begin{pmatrix} a & 1 & 0 & 0 \\ 0 & b+\frac{1}{a} & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{pmatrix}$$

   Then change orders of rows, and change orders of columns.

$$\begin{aligned} det\begin{pmatrix} a & 1 & 0 & 0 \\ 0 & b+\frac{1}{a} & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{pmatrix} &= -det\begin{pmatrix} 0 & b+\frac{1}{a} & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \\ a & 1 & 0 & 0 \end{pmatrix} \\ &= det\begin{pmatrix} b+\frac{1}{a} & 1 & 0 & 0 \\ -1 & c & 1 & 0 \\ 0 & -1 & d & 0 \\ 1 & 0 & 0 & a \end{pmatrix} \\ &= det\begin{pmatrix} b+\frac{1}{a} & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{pmatrix} \cdot a \\ &= 1 + ab + ad + cd + abcd. \end{aligned}$$

2 Problem 2

Solution: All linear systems can be formed as $Ax = b$.

1. In this problem, $x \in \mathbb{R}^4$, A is a 4 by 4 matrix and b is a 4 by 1 vector.

$$[A|b] = \begin{bmatrix} 2 & -1/2 & -1/2 & 0 & 0 \\ -1/2 & 2 & 0 & -1/2 & 3 \\ -1/2 & 0 & 2 & -1/2 & 3 \\ 0 & -1/2 & -1/2 & 2 & 0 \end{bmatrix} \xrightarrow{Gauss \quad elimination} \begin{bmatrix} 1 & -1/4 & -1/4 & 0 & 0 \\ 0 & 1 & -1/15 & -4/15 & 8/5 \\ 0 & 0 & 1 & -2/7 & 12/7 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

   Thus, the solution to the linear system is $x = (1, 2, 2, 1)^T$.

2.
$$[A|b] = \begin{bmatrix} 2 & 3 & 5 & 1 & 3 \\ 3 & 4 & 2 & 3 & -2 \\ 1 & 2 & 28 & -1 & 8 \\ 7 & 9 & 1 & 8 & 0 \end{bmatrix} \xrightarrow{Gauss \quad elimination} \begin{bmatrix} 1 & 3/2 & 5/2 & 1/2 & 3/2 \\ 0 & 1 & 11 & -3 & 13 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

   We see $[A|b]$ is not in full rank so there is no solution to the linear system.

3 Problem 3

$A = P\Lambda Q$, $A^2 = P\Lambda QP\Lambda Q$. Since $QP = I$, $A^2 = P\lambda^2 Q$. Hence, $A^k = P\Lambda^k Q$.

It is enough to calculate what Λ^k. Since Λ is a diagonal matrix,

- k is a odd number,

  $\Lambda^k = \Lambda$. Then
  $$A^k = P\Lambda Q = \begin{pmatrix} 7 & -12 \\ 6 & -7 \end{pmatrix}.$$

- k is an even number,

  $\Lambda^k = I$. Then
  $$A^k = PIQ = PQ = I.$$

4 Problem 4

The linear system can be formed as $Ax = b$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

It is clear that A is invertible.(You may check that A is of full rank) To find the inverse of A, i.e. A^{-1}, there is one method by Gauss elimination

$$[A|I] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{Gauss \quad elimination} \begin{bmatrix} 1 & 0 & 0 & 1 & -1/2 & 0 \\ 0 & 1 & 0 & 1 & -1/2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} = [I|A^{-1}]$$

The solution is $x = A^{-1}b = (1/2, -3/2, 2)^T$.

5 Problem 5

1. False Counter example:

   Let
   $$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

   Neither A nor B is invertible, but $A + B = I$, which is invertible.

2. True

   You may use different method to prove this claim. We will give two methods here. The second one is a more popular one. For those students who are also taking MA 523/723 courses, you may want to have a look at the first method.

(a) Method 1

Proof. We use the notation $\mathcal{R}(A)$ on be half of "Range of matrix A". And there is a fact that $\mathcal{R}(AB) \subset \mathcal{R}(A)$. Hence, the dimensions of $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are related in the following inequality chain:

$$\dim(\mathcal{R}(AB)) \leq \dim(\mathcal{R}(A)) \leq n.$$

Since AB is invertible, $\dim(\mathcal{R}(AB)) = n$. Hence, $\dim(\mathcal{R}(AB)) = \dim(\mathcal{R}(A)) = n$, which implies two facts that

i. $\mathcal{R}(A)$ is of full rank.

ii. $\mathcal{R}(A) = \mathcal{R}(AB)$.

From (2(a)i) we know A is invertible and from (2(a)ii) we know that B is of full rank, which also implies that B is invertible. □

(b) Method 2

Proof. Since AB is invertible, we know $det(AB) \neq 0$. With $det(AB) = det(A) \cdot det(B)$, it implies $det(A) \cdot det(B) \neq 0$, which means both $det(A) \neq 0$ and $det(B) \neq 0$. This proves that both A and B are invertible.

□

3. False Counter example: We might let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence,

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Obviously, AB is not invertible, but A is invertible.

4. True.

Proof. Since A is invertible, we know that the kernel of A, denoted as $\mathcal{N}(A) = \{0\}$. i.e., $Ax = 0$ implies $x = 0$.

Let $kAx = 0$. Change the order of factors on left side and we obtain $A \cdot kx = 0$, which implies that $kx = 0$. With $k \neq 0$, we know that $x = 0$. In all, we conclude that $\mathcal{N}(kA) = \{0\}$, which is an equivalent to the fact that kA is invertible. And this proves the claim.

□

6 Problem 6

Solution: (b)

Proof. Since $A = B^T$, we know $A^{-1} = B^{-T}$. Then,

$$\begin{aligned} A^T(B^{-1}A^{-1} + I)^T &= A^T(A^{-T}B^{-T} + I) \\ &= A^T A^{-T} B^{-T} + A^T \\ &= B^{-T} + A^T \\ &= (B^T)^{-1} + A^T = A^{-1} + B \end{aligned}$$

□

7 Problem 7

Solution:

Let $A = [\alpha_1^T, \alpha_2^T, \alpha_3^T, \alpha_4^T]$, and $x = (x_1, x_2, x_3, x_4)^T$. Solve linear system $Ax = \alpha^T$ and we obtain $x = (\frac{5}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})^T$.

8 Problem 8

Solution:

Build matrix $A = [\alpha_1, \alpha_2, \alpha_3]$. vectors α_1, α_2 and α_3 are linearly independent if and only if A is of full rank. Hence, we only need to check the rank of A by Gauss elimination.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 5 & 6 \end{pmatrix} \xrightarrow{Gauss \quad elimination} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

This implies that $Rank(A) = 2 < 3$. In conclusion, α_1, α_2 and α_3 are NOT linearly independent.

9 Problem 9

Proof. First step, we'd like to show $\delta_1 + \delta_2$, $\delta_2 + \delta_3$ and $\delta_3 + \delta_1$ are linearly independent based on δ_1, δ_2 and δ_3 are linearly independent.

Recall the definition of linearly dependence δ_1, δ_2 and δ_3 are linearly independent if

$$\alpha_1\delta_1 + \alpha_2\delta_2 + \alpha_3\delta_3 = 0 \iff \alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Hence suppose $\beta_1(\delta_1 + \delta_2) + \beta_2(\delta_2 + \delta_3) + \beta_3(\delta_3 + \delta_1) = 0$, we only need to show that $\beta_1 = \beta_2 = \beta_3 = 0$.

Rearrange the left side of it and obtain $(\beta_1 + \beta_3)\delta_1 + (\beta_1 + \beta_2)\delta_2 + (\beta_2 + \beta_3)\delta_3 = 0$. From the independence of δ_1, δ_2 and δ_3, we have

$$\beta_1 + \beta_3 = 0, \beta_1 + \beta_2 = 0, \beta_2 + \beta_3 = 0.$$

which implies $\beta_1 = \beta_2 = \beta_3 = 0$.

For the opposite direction of this claim, use the similar idea above and we only need to show that

$$\eta_1\delta_1 + \eta_2\delta_2 + \eta_3\delta_3 = 0 \Rightarrow \eta_1 = \eta_2 = \eta_3 = 0$$

based on the linear independence of $\delta_1 + \delta_2$, $\delta_2 + \delta_3$ and $\delta_3 + \delta_1$.

Suppose $\eta_1\delta_1 + \eta_2\delta_2 + \eta_3\delta_3 = 0$. And construct coefficient $\gamma_1 = \frac{1}{2}(\eta_2 + \eta_1 - \eta_3)$, $\gamma_1 = \frac{1}{2}(\eta_2 - \eta_1 + \eta_3)$ and $\gamma_1 = \frac{1}{2}(\eta_3 - \eta_2 + \eta_1)$. It is easy to verify that

$$\gamma_1(\delta_1 + \delta_2) + \gamma_2(\delta_2 + \delta_3) + \gamma_3(\delta_3 + \delta_1) = \eta_1\delta_1 + \eta_2\delta_2 + \eta_3\delta_3 = 0.$$

By the independence of $\delta_1 + \delta_2$, $\delta_2 + \delta_3$ and $\delta_3 + \delta_1$, we obtain $\gamma_1 = \gamma_2 = \gamma_3 = 0$, which implies $\eta_1 = \eta_2 = \eta_3 = 0$.

Hence, the claim is proved. $\square$

10 Problem 10

Solution:

1. $$A \xrightarrow{Gauss \quad elimination} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   From this we know that $Rank(A) = 3$.

2. $$A \xrightarrow{Gauss \quad elimination} \begin{pmatrix} 3 & 2 & -1 & -3 & -2 \\ 0 & -7/3 & 11/3 & 3 & -5/3 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

   From this we know that $Rank(A) = 3$.