

Homework 2 Solutions

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Problem 1.1

Solutions:

1. Let $x_2 = x_2^+ - x_2^-$ and $x_2^+, x_2^- \geq 0$. And add slack variables ξ_1, ξ_2 on the first and the second constraint respectively.

$$\begin{aligned} \text{Minimize} \quad & 4x_1 + \sqrt{2}x_2^+ - \sqrt{2}x_2^- - 0.35x_3 \\ \text{subject to} \quad & -0.001x_1 + 200x_2^+ - 200x_2^- - \xi_1 = 7\sqrt{261} \\ & 7.07x_2^+ - 7.07x_2^- - 2.62x_3 + \xi_2 = -4 \\ & x_1, x_2^+, x_2^-, x_3, \xi_1, \xi_2 \geq 0 \end{aligned}$$

2. Let $a_1 = x_1 - 20$, $a_3 = x_3 + 15$, then $a_1, a_3 \geq 0$, and $x_1 = a_1 + 20$, $x_3 = a_3 - 15$.

Add a slack variable ξ_1 on the second constraint. The standard form is

$$\begin{aligned} \text{Minimize} \quad & 3.1a_1 - 2\sqrt{2}x_2 + a_3 \\ \text{subject to} \quad & 100a_1 - 20x_2 = -1993 \\ & -11a_1 - 7\pi x_2 - 2a_3 + \xi_1 = 590 \\ & a_1, x_2, a_3, \xi_1 \geq 0 \end{aligned}$$

Notice that the objective function should have been $3.1a_1 - 2\sqrt{2}x_2 + a_3 + 47$, which is an affine function. But to make it a linear function, we eliminate $+47$. We will obtain the same optimal solutions but will need to add 47 on the optimal value.

3. Since $x_3 \leq 10$, $10 - x_3 \geq 0$. Let $a_3 = 10 - x_3$, then $a_3 \geq 0$ and $x_3 = 10 - a_3$. Let $x_1 = x_1^+ - x_1^-$, where $x_1^+, x_1^- \geq 0$. Add slack variables on each constraint and the standard form is the following,

$$\begin{array}{ll}
\text{Minimize} & -x_1^+ + x_1^- - 3x_2 - 2a_3(+20) \\
\text{subject to} & 3x_1^+ - 3x_1^- - 5x_2 - \xi_1 = -2 \\
& 3x_1^+ - 3x_1^- - 5x_2 + \xi_2 = 15 \\
& -5x_1^+ + 5x_1^- + 20x_2 - \xi_3 = 11 \\
& -5x_1^+ + 5x_1^- + 20x_2 + \xi_4 = 40 \\
& x_1^+, x_1^-, x_2, a_3, \xi_i (i = 1, \dots, 4) \geq 0
\end{array}$$

Problem 1.2

a) Let $x_1 = x_1^+ - x_1^-$, where $x_1^+, x_1^- \geq 0$. The standard form is the following,

$$\begin{array}{ll}
\text{Minimize} & 2x_1^+ - 2x_1^- + 6x_2 + 8x_3 \\
\text{subject to} & x_1^+ - x_1^- + 2x_2 + x_3 = 5 \\
& 4x_1^+ - 4x_1^- + 2x_3 = 12 \\
& x_1^+, x_1^-, x_2, x_3 \geq 0
\end{array}$$

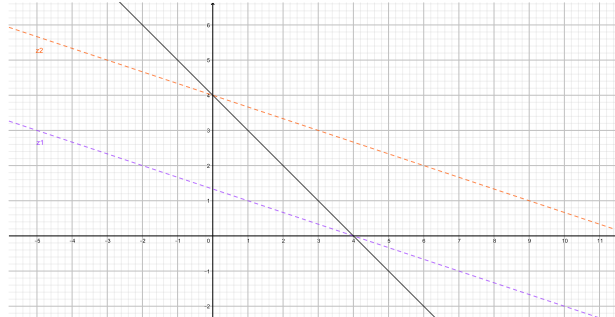
b) From the first constraint, solve x_1 as $x_1 = 5 - 2x_2 - x_3$. Plug it into the objective function and also the second constraint, then reform the LP problem as the following,

$$\begin{array}{ll}
\text{Minimize} & 2x_2 + 6x_3(+10) \\
\text{subject to} & -2x_2 - 2x_3 = -8 \\
& x_2, x_3 \geq 0
\end{array}$$

c) It is already in the standard form.

d) Use graphic method to solve the problem. Optimal value is $z^* = 18$ and the optimal solution is $x^* = (x_2^*, x_3^*) = (4, 0)$.

Figure 1: Problem 1.2 (d)



Problem 1.3

- a) No. Because there is a nonlinear term x_1^2 in the objective function and the first constraint.
- b) Yes. Use the first constraint, solve x_1^2 and get $x_1^2 = x_2$. Plug it into the objective function and get

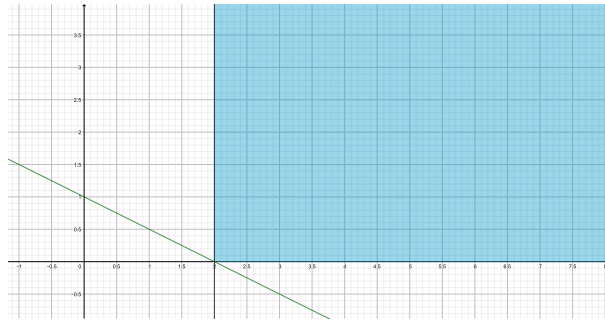
$$\begin{aligned} &\text{Minimize} && 2x_2 + 4x_3 \\ &\text{subject to} && 2x_2 + 4x_3 \geq 4 \\ &&& x_1, x_3 \geq 0, x_2 \geq 2 \end{aligned}$$

- c) Use the similar technique in problem 1.1 and 1.2. Let $a_2 = x_2 - 2$. Then $a_2 \geq 0$, and $x_2 = a_2 + 2$. Add a slack variable on the constraint.

$$\begin{aligned} &\text{Minimize} && 2a_2 + 4x_3(+4) \\ &\text{subject to} && 2a_2 + 4x_3 - \xi = 0 \\ &&& x_1, a_2, x_3, \xi \geq 0 \end{aligned}$$

- d) Yes. To solve the LP problem, use graphic method. The optimal solution is $x^* = (x_1^*, x_2^*, x_3^*, x_4^*)^T = (\sqrt{2}, 2, 0, 0)^T$. Optimal value is $z^* = 4$.

Figure 2: Problem 1.3 (d)



To solve the original problem, we can graph the feasible region in 3-D and use graph method to solve it.

Problem 1.4

- a) No. Because it has absolute-value functions in the objective function.
- b) Let $x_i = x_i^+ - x_i^-$, $i = 1, 2, 3$, where $x_i^+, x_i^- \geq 0$. Then $|x_i| = x_i^+ + x_i^-$. Add one slack variable ξ_1 on the first constraint and the problem is reformed as

$$\begin{aligned}
& \text{Minimize} && x_1^+ + x_1^- + 2x_2^+ + 2x_2^- - x_3^+ - x_3^- \\
& \text{subject to} && x_1^+ - x_1^- + x_2^+ - x_2^- - x_3^+ + x_3^- + \xi_1 = 10 \\
& && x_1^+ - x_1^- - 3x_2^+ + 3x_2^- + 2x_3^+ - 2x_3^- = 12 \\
& && x_i^+, x_i^- \geq 0, i = 1, 2, 3. \\
& && \xi_1 \geq 0
\end{aligned}$$

- c) Use the similar technique as in (b), let $a_1 = x_1 - 5$ and $a_2 = x_2 + 4$. Then, $|x_1 - 5| = a_1^+ + a_1^-$ and $|x_2 + 4| = a_2^+ + a_2^-$.

Also, it is clear to see that

$$x_1 = a_1 + 5 = a_1^+ - a_1^- + 5, \quad x_2 = a_2 - 4 = a_2^+ - a_2^- - 4.$$

Plug above into the problem and get the following standard form.

$$\begin{aligned}
& \text{Minimize} && a_1^+ + a_1^- + a_2^+ + a_2^- \\
& \text{subject to} && a_1^+ - a_1^- + a_2^+ - a_2^- + \xi_1 = 10 \\
& && a_1^+ - a_1^- - 3a_2^+ + 3a_2^- - \xi_2 = -15 \\
& && a_i^+, a_i^-, \xi_i \geq 0, i = 1, 2.
\end{aligned}$$

Problem 1.5

Usually, the goal for this type of problems is either to minimize the total cost or to maximize the total profit. For this problem, we will consider maximize the price. Let the objective function be $f(x) = 15x_1 + 25x_2$. Here x_1 and x_2 are the amount of Chip-1 and Chip-2 produced.

Consider the skilled and unskilled labor hours, and the available raw materials, the constraints are

$$3x_1 + 4x_2 \leq 100, \quad 2x_1 + 3x_2 \leq 70, \quad x_1 + 2x_2 \leq 30.$$

Also, the amount of product should be non-negative. With the demand on the sales contract, we have

$$x_1 \geq 0, \quad x_2 \geq 3.$$

In conclusion, we can form the problem as a LP problem,

$$\begin{aligned}
& \text{Maximize} && 15x_1 + 25x_2 \\
& \text{subject to} && 3x_1 + 4x_2 \leq 100 \\
& && 2x_1 + 3x_2 \leq 70 \\
& && x_1 + 2x_2 \leq 30 \\
& && x_1 \geq 0, x_2 \geq 3.
\end{aligned}$$

Problem 1.6

For this problem, our goal is to minimize the time(days) used for completing these 5 projects.

Use matrix T to express the given table. i.e.,

$$T = \begin{pmatrix} 5 & 5 & 7 & 4 & 8 \\ 6 & 5 & 8 & 3 & 7 \\ 6 & 8 & 9 & 5 & 10 \\ 7 & 6 & 6 & 3 & 6 \\ 6 & 7 & 10 & 6 & 11 \end{pmatrix} = \begin{pmatrix} t_A \\ t_B \\ t_C \\ t_D \\ t_E \end{pmatrix}$$

where t_A, \dots, t_E are rows of T . Let $a, b, c, d, e \in \mathbb{R}^5$ (i.e. $a = (a_1, \dots, a_5)^T$) be variables and each of them shows the index of the project assigned to that person. For example, if $d = (0, 1, 0, 0, 0)^T$, it means person D will work on project #2.

Then our goal is to minimize the time spent on these projects.

$$\text{Minimize } t_A a + t_B b + t_C c + t_D d + t_E e.$$

Consider that one person is assigned to do one project, we have

$$\sum_{i=1}^5 x_i = 1, \quad x \in \{a, b, c, d, e\}.$$

What's more, every project has to be covered by one person and it leads to the following constraint,

$$a_i + b_i + c_i + d_i + e_i = 1, \quad i = 1, \dots, 5.$$

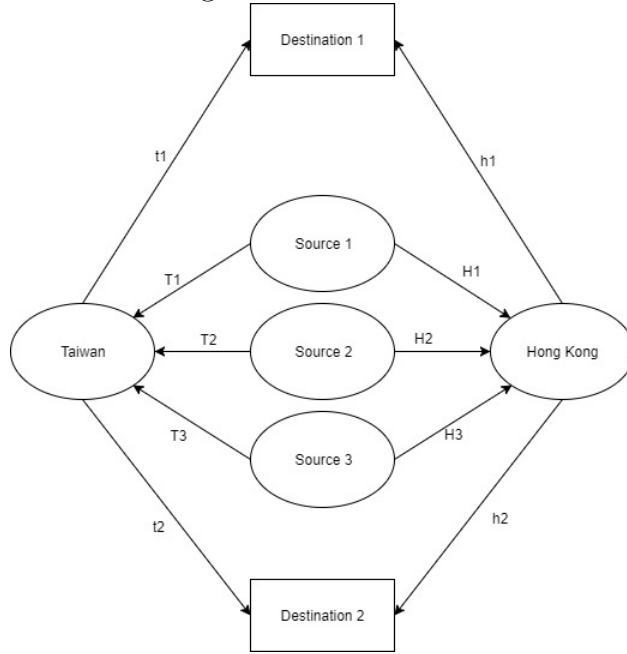
In conclusion, the integer linear programming problem is,

$$\begin{aligned} &\text{Minimize } t_A a + t_B b + t_C c + t_D d + t_E e \\ &\text{subject to } \sum_{i=1}^5 a_i = 1, \quad \sum_{i=1}^5 b_i = 1, \quad \sum_{i=1}^5 c_i = 1 \\ &\quad \sum_{i=1}^5 d_i = 1, \quad \sum_{i=1}^5 e_i = 1 \\ &\quad a_i + b_i + c_i + d_i + e_i = 1, \quad i = 1, \dots, 5 \\ &\quad a_i, b_i, c_i, d_i, e_i \geq 0, i = 1, \dots, 5. \end{aligned}$$

Problem 1.7

- a) Let H_1, H_2 and H_3 be the amount ship to HK from those three sources respectively. And let h_1, h_2 be output from HK to destinations respectively. Similar setting up for Taiwan.

Figure 3: Problem 1.7



Consider the no-blends from each sources, and demands from destinations, the LP problem is,

$$\begin{aligned}
 &\text{Minimize} && 50H_1 + 90H_2 + 70H_3 + 150h_1 + 180h_2 + 60T_1 + 95T_2 + 50T_3 + 130t_1 + 200t_2 \\
 &\text{subject to} && T_1 + H_1 = 60 \\
 &&& T_2 + H_2 = 45 \\
 &&& T_3 + H_3 = 30 \\
 &&& h_1 + t_1 = 80 \\
 &&& h_2 + t_2 = 55 \\
 &&& T_i, H_i \geq 0, i = 1, 2, 3 \\
 &&& t_j, h_j \geq 0, j = 1, 2.
 \end{aligned}$$

b) Add a constraint in the LP problem of a).

$$\sum_{i=1}^3 H_i \leq 60.$$

c) We may want to try to add one more constraint based on b).

$$\sum_{i=1}^3 T_i \leq 50.$$

This is fine. But after this we will see that there is no feasible solutions to the problem. The reason is that the output from HK and Taiwan will not meet the demands from destinations.

d) (There might be different answers for this part.)