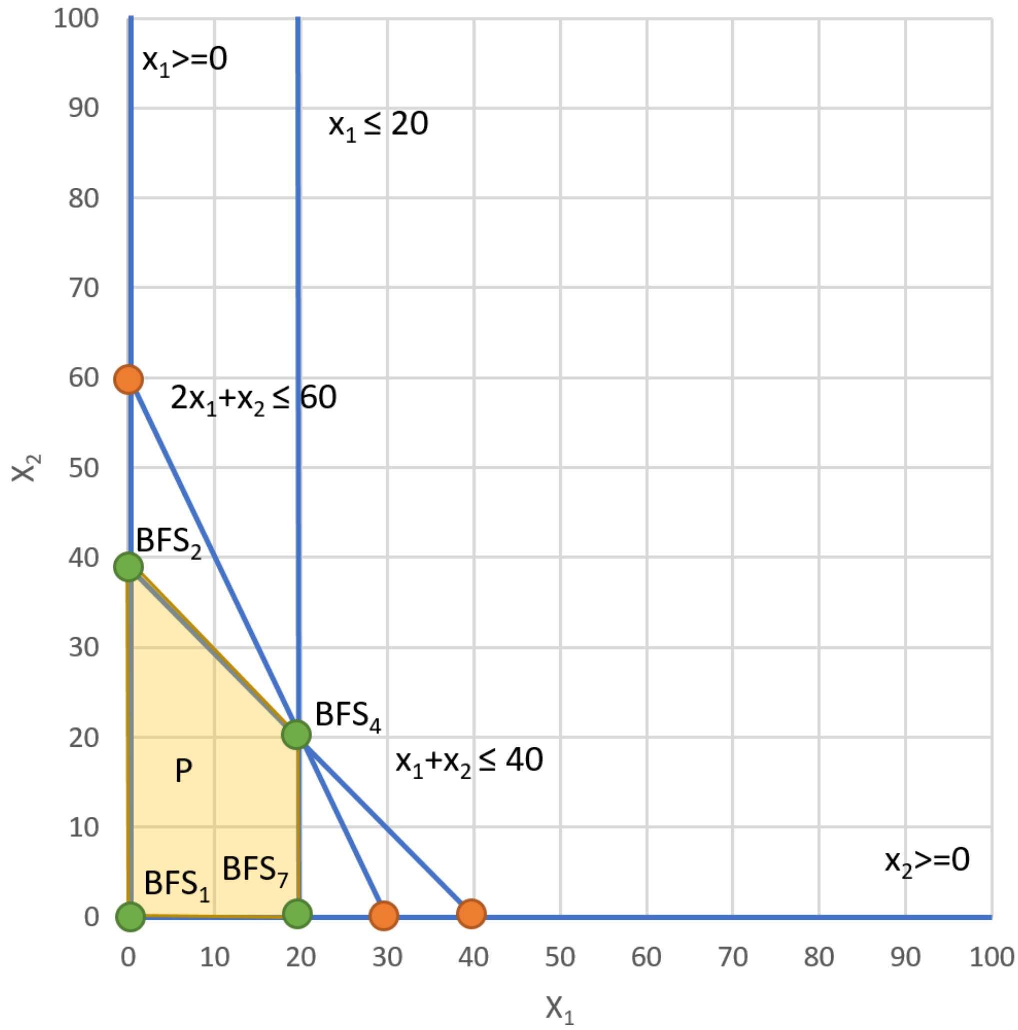
James Durst

ISE 505 – HW#4

2.7)

a) & e)



b)

$$\mathrm{P} = \{(x_1, x_2, x_3, x_4, x_5 \in R^5 | x_1 + x_2 + x_3 = 40\,; 2x_1 + x_2 + x_4 = 60\,;\ x_1 + x_5 = 20\,;\ x_1, x_2, x_3, x_4, x_5 \ \geq 0\}$$

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c) & d)

Basic Solutions:																	
1:	x_1	=	0	x_2	=	0	x_3	=	40	x_4	=	60	x_5	=	20	BFS	$\{x_3, x_4, x_5\}$
2:	x_1	=	0	x_2	=	40	x_3	=	0	x_4	=	20	x_5	=	20	BFS	$\{x_2, x_4, x_5\}$
3:	x_1	=	0	x_2	=	60	x_3	=	-20	x_4	=	0	x_5	=	20		
4:	x_1	=	20	x_2	=	20	x_3	=	0	x_4	=	0	x_5	=	0	BFS	$\{x_1, x_2, [x_3, x_4, x_5]\}$
5:	x_1	=	40	x_2	=	0	x_3	=	0	x_4	=	-10	x_5	=	-20		
6:	x_1	=	30	x_2	=	0	x_3	=	10	x_4	=	0	x_5	=	-10		
7:	x_1	=	20	x_2	=	0	x_3	=	20	x_4	=	10	x_5	=	0	BFS	$\{x_1, x_3, x_4\}$

f)

Extreme point (20, 20, 0, 0, 0) is degenerate as it contains 3 different BFS:
$\{x_1, x_2, x_3\}$; $\{x_1, x_2, x_4\}$; $\{x_1, x_2, x_5\}$

2.10)

In the above LP problem we have n=5 variables when it is converted to standard form with m=3 constraints. At the extreme point (20, 20, 0, 0, 0) we see that x_1 and x_2 must be in the basis. But x_1, x_1, and x_1 are all equal to zero at this point. Thus, any one of them can be in the basis without changing the extreme point. This means that the three bases: $\{x_1, x_2, x_3\}$; $\{x_1, x_2, x_4\}$; $\{x_1, x_2, x_5\}$ can all be represented by the extreme point (20, 20, 0, 0, 0). Furthermore, we can see if we take the number of positive elements for the given extreme point ($p = 2$) and find the number of combinations between non-positive elements choosing from the number of non-basic elements in the solution set, $C\binom{n-p}{n-m}$, we can identify the number of basic feasible solutions that will correspond to the particular point.

$$C\binom{5-2}{5-3} = C\binom{3}{2} = \frac{3!}{2!\,(3-2)!} = \frac{6}{2} = 3\ different\ bases\ (bfs)$$

2.11)

a)

b)

The first orthant of R^n is defined to be a subset of R^n where $x_1, x_1, \ldots, x_1 \geq 0$. So for $\{M \subset R^2 | x_1, x_2 \in R^2 | x_1, x_2 \geq 0\}$ the vectors of the matrix $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in R^2 can be defined:

<1 0> defines one edge $x_2 = 0$ of the first orthant
<0 1> defines the other edge $x_1 = 0$ of the first orthant

A convex cone is defined to be a subset of C when the sum of the products of all x's and corresponding scalars (λ) in a vector space fall within C. In this case, C is the vector space created by the matrix M which, as shown above, is defined by the edges of the first orthant in R^2.

$$Mx = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So, given the matrix M, and that x_1 and x_2 remain positive, the matrix M generates a convex cone that falls within the first orthant of R^2.

c)

For the vectors <1 0> and <0 1>, we can see that the matrix $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ forms the smallest convex cone that contains the vectors by simple algebra. Assume $\Delta \neq 0$:

$$M = \begin{bmatrix} 1 \pm \Delta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + \Delta x_1 \\ x_2 \end{bmatrix}$$

If $\Delta > 0$: Increases size of the subset by Δx_1 beyond vector <1 0>.
If $\Delta < 0$: Decreases size of the subset by Δx_1, which no longer contains the vector <1 0>.

$$M = \begin{bmatrix} 1 & 0 \pm \Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + \Delta x_2 \\ x_2 \end{bmatrix}$$

If $\Delta > 0$: Increases size of the subset by Δx_2 beyond vector <1 0>.
If $\Delta < 0$: Decreases size of the subset by Δx_2, which no longer contains the vector <1 0>.

$$M = \begin{bmatrix} 1 & 0 \\ 0 \pm \Delta & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \Delta x_1 + x_2 \end{bmatrix}$$

If $\Delta > 0$: Increases size of the subset by Δx_1 beyond vector <0 1>.
If $\Delta < 0$: Decreases size of the subset by Δx_1, which no longer contains the vector <0 1>.

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \pm \Delta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 + \Delta x_2 \end{bmatrix}$$

If $\Delta > 0$: Increases size of the subset by Δx_2 beyond vector <0 1>.
If $\Delta < 0$: Decreases size of the subset by Δx_2, which no longer contains the vector <0 1>.

This is not the way to prove it Please check the solution

4.

$$P = \{x \in R^n | Ax = b | x \geq 0\}$$

1) By the Resolution Theorem, given a set *E* which is formed by all extremal directions of P, we can show that for any vector $d \in R^n, d \in E$ iff $Ad = 0 \; and \; d \geq 0$.

Let $V = \{v^i \in R^n | i \in I\}$ be the set of all extreme points of *P* with a finite index set *I*. Then for each $x \in P$, we have

-2

$x = \sum_{i \in I} \lambda_i v^i + d \;\; where \;\; \sum_{i \in I} \lambda_i = 1, \lambda_i \geq 0 \; for \; i \in I$

Let ***p*** be the number of positive elements of $x \in P$. When $p = 0$, ***x*** is a zero vector, which means it must be a vertex, thus $d = 0$. Consider when $(p = 0,1,2, \ldots, k)$ and ***x*** has k+1 positive elements. Since *P* is defined by a standard linear program, $\bar{A}$ must be linearly independent columns within ***A*** that correspond to the ***p*** positive elements of ***x*** and there must be a non-zero vector $w \in R^n \; where \; \bar{A}w = 0$. When ***x*** is not a vertex, it ~~must then hold~~ that there also exists a non-zero vector ***d*** that is in the extremal direction of *P* so as not to have ***x*** correspond to a vertex of *P*. Thus, $d = 0$ if *P* is bounded and $d > 0$ if *P* is unbounded.

X Why?

2) *E* must be a cone as long as all λ_i in the equation $x = \sum_{i \in I} \lambda_i v^i + d$ are greater than or equal to zero. By the above proof, we defined that $\lambda_i \geq 0 \; for \; i \in I$. Thus, since $E, x \in R^n$, *E* must be a cone within R^n.

This is an element in P, not in E

3) By the same argument, we showed in the above proof that $x = \sum_{i \in I} \lambda_i v^i + d \;\; where \; \sum_{i \in I} \lambda_i = 1, \lambda_i \geq 0 \; for \; i \in I$. This definition satisfies both requirements of a convex set,

$$\sum_{i \in I} \lambda_i = 1, \lambda_i \geq 0$$

Thus, since $E, x \in R^n$, *E* must also be a convex set within R^n.

please check the solution

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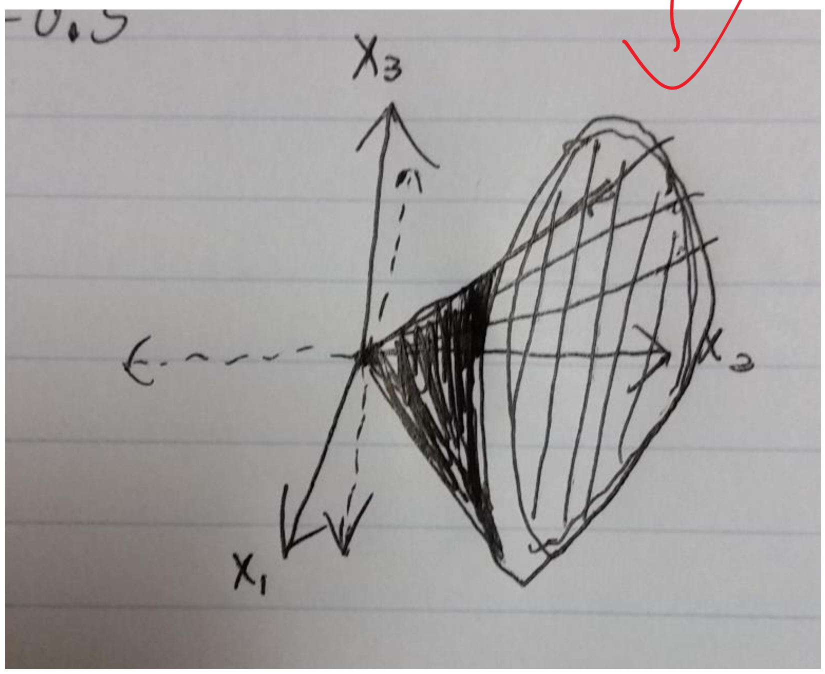
5.

$$F_3 = \{x \in R^3 || x_1| + |x_3| \leq x_2\}$$

Not a "round" one

a)



b)

$$B = \{x \in R^3 | x_1^2 + x_3^2 = x_2^2\}$$

c)

$$I = \{x \in R^3 | x_1^2 + x_3^2 < x_2^2\}$$

d) All extremal points of F_3 also correspond to the vertices of F_3. The extremal points can be represented by the coordinates for the boundary points that satisfy the following:

$$B = \{x \in R^3 | x_1^2 + x_3^2 = x_2^2\}$$

−2

e) If we take any combination of vectors from F_3, then it will be a convex cone if the combination of x_1, x_2, and x_3 will satisfy the following:

$$\boldsymbol{x} \text{ is a convex combination of } \{x_1, x_2, x_3\} \text{ if } \sum_{i=1}^{3} \lambda_i = 1\,, \lambda_i \geq 0$$

-3

$$AND$$

$$\boldsymbol{x} \text{ is a conical combination of } \{x_1, x_2, x_3\} \text{ if } \lambda_i \geq 0$$

If we apply these to F_3, we see that for whatever values of x_i, the sum of $\lambda_i x_i$ will be within F_3.

$$\frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 = z \rightarrow x_1 + x_2 + x_3 = 3z \in F_3$$

Thus, $F_3 + F_3 \subset F_3$ and $\lambda F_3 \subset F_3$. This means that F_3 is both convex and a cone based upon the definitions above.

f) F_3 must be a convex cone split in quarter by the nonnegative orthant. We can base this upon the previous proof that F_3 is a convex cone and x_1, x_2, and x_3 are all greater or equal to zero. x_2 is already nonnegative due to the absolute values of x_1 and x_3 in F_3. Thus, the nonnegativity of x_1 and x_3 in R_+^3 are the variables that are split by halfspaces within F_3. Dividing the set into quarters.

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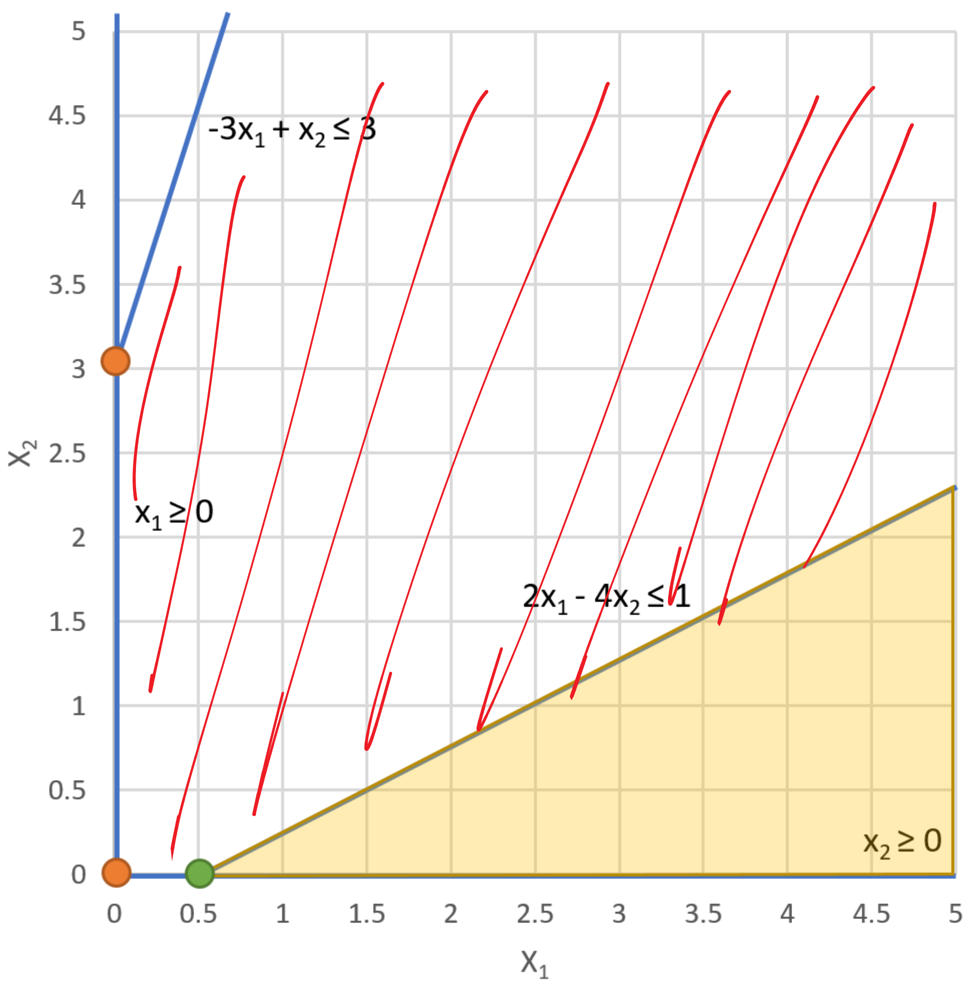
6. -3

$$P_1 = \{x \in R^2 | 2x_1 - 4x_2 - 1 \leq 0\,; 3x_1 - x_2 + 3 \geq 0\,;\ x_1, x_2 \geq 0\}$$

$$P_1 = \{x \in R^2 | 2x_1 - 4x_2 + x_3 = 1\,; -3x_1 + x_2 + x_4 = 3\,;\ x_1, x_2, x_3, x_4 \geq 0\}$$

a)

b)



wrong feasible domain

$$C\binom{4}{2} = 6 \textit{ possible basic solutions}$$

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$$\{x_1 \quad x_2\}\{x_1 \quad x_3\}\{x_1 \quad x_4\}\{x_2 \quad x_3\}\{x_2 \quad x_4\}\{x_3 \quad x_4\}$$

$\{x_1 \quad x_2\}$ *Not Basic Solution* $2x_1 - 4x_2 = 1$ $-3x_1 + x_2 = 3$ $-10x_1 = 13 \rightarrow x_1 = -1.3$	$\{x_1 \quad x_3\}$ *Not Basic Solution* $2x_1 + x_3 = 1$ $-3x_1 = 3$ $x_1 = -1$
$\{x_1 \quad x_4\} \rightarrow \{0.5 \quad 4.5\}$ *Basic Solution* $2x_1 = 1$ $-3x_1 + x_4 = 3$ $x_1 = \frac{1}{2}$ $x_4 = 4.5$	$\{x_2 \quad x_3\}\{3 \quad 13\}$ *Basic Solution* $-4x_2 + x_3 = 1$ $x_2 = 3$ $x_2 = 3$ $x_3 = 13$
$\{x_2 \quad x_4\}$ *Not Basic Solution* $-4x_2 = 1$ $x_2 + x_4 = 3$ $x_2 = -0.25$	$\{x_3 \quad x_4\}\{1 \quad 3\}$ *Basic Solution* $x_3 = 1$ $x_4 = 3$ $x_3 = 1$ $x_4 = 3$

The method is correct

Basic Solutions:

1:	x_1	=	0	x_2	=	0	x_3	=	1	x_4	=	3	
2:	x_1	=	0	x_2	=	3	x_3	=	13	x_4	=	0	
3:	x_1	=	0.5	x_2	=	0	x_3	=	0	x_4	=	4.5	BFS

c)

$$A = \begin{bmatrix} 2 & -4 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{bmatrix}$$

$$2v_1 - 4v_2 + v_3 = 0$$
$$-3v_1 + v_2 + v_4 = 0$$
$$v_1 = 2\,;\; v_2 = 1$$
$$4 - 4 + v_3 = 0 \rightarrow v_3 = 0$$
$$-6 + 1 + v_4 = 0 \rightarrow v_4 = 5$$

$$d_1 = \{2\ 1\ 0\ 5\}$$

d)

$$< 0.5\ 0\ 0\ 4.5 >$$
$$< 0\ 3\ 13\ 0 >$$

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7. −6

$$P_2 = \{x \in R^2 | 2x_1 - 2x_2 - 3 \leq 0\,; 8x_1 - x_2 + 4 \geq 0\,;\; x_1 \geq 0\}$$

$$P_2 = \{x \in R^2 | 2x_1 - 4(x_2^+ - x_2^-) + x_3 = 3\,; -8x_1 + (x_2^+ - x_2^-) + x_4 = 4\,;\; x_1, x_2^+, x_2^-, x_3, x_4 \geq 0\}$$

$$C\binom{5}{2} = 30\, possible\ basic\ solutions$$

? Then ?