MA 515-001, Fall 2017, Homework 2

Due: Mon Sep 25, 2017, in-class.

Problem 1. Given any metric space (Y, d), let (X, d) be a subspace of Y, i.e., $X \subseteq Y$. Let E a subset of X, show that

(a) E is open in X if and only if $E = \mathcal{O} \cap X$ where $\mathcal{O}$ is open in Y.

(b) E is closed in X if and only if $E = F \cap X$ where F is closed in Y.

Problem 2. Let U and V be subsets of metric space X. Show that

(a) $\overline{U \cup V} = \overline{U} \cup \overline{V}$.

(b) $\overline{U \cap V} = \overline{U} \cap \overline{V}$ generally fails.

(c) if $U \subseteq V$ then $\overline{U} \subseteq \overline{V}$.

Problem 3. Let K be a closed subset of metric space X. For any $x \in X$, the distance from x to K is defined as

$$d_K(x) = \inf_{w \in K} d(x, w) .$$

Show that x belongs to K if and only if $d_K(x) = 0$.

Problem 4. Let (X, d) and (Y, σ) be metric spaces. Show that a map $f : X \to Y$ is continuous on X if and only if for every closed subset $F \subset Y$, the set $f^{-1}(F)$ is closed in X.

Problem 5. Given two metric spaces (X, d) and (Y, σ), consider a map $f : X \to Y$. Show that f is continuous at x if and only if for any sequence $\{x_n\}_{n\geq 1}$ which converges to x, then

$$\lim_{n\to\infty} f(x_n) = f(x) .$$

Problem 6. Let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence in a metric space (X, d). Assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to $x \in X$. Show that the sequence $\{x_n\}_{n\geq 1}$ converges to x.

Problem 7. Recall that

$$l^2 = \left\{ x = \{x_n\}_{n\geq 1} \;\middle|\; \sum_{n=1}^{\infty} |x_n|^2 < +\infty \right\}$$

and

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2 \right)^{\frac{1}{2}} \qquad \forall x, y \in l^2 .$$

Show that the metric space (l^2, d) is complete.

Problem 8. If (X, d) is complete, show that $(X, \tilde{d})$, where

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} \qquad \forall x, y \in X ,$$

is also complete.

Problem 9. Let (X, d_X) and (Y, d_Y) be isometric, i.e., there exists a map $T : X \to Y$ such that $T(X) = Y$ and

$$d_Y(T(x_2), T(x_1)) \;=\; d_X(x_1, x_2) \qquad \forall x_1, x_2 \in X\,.$$

Show that if (X, d_X) is complete then (Y, d_Y) is complete.

Problem 10. Recall that

$$l^\infty \;=\; \left\{x = \{x_n\}_{n\geq 1} \;\middle|\; \{x_n\}_{n\geq 1} \text{ is bounded}\right\}\,,$$

and

$$d(x, y) \;=\; \sup_{n\geq 1}\, |x_n - y_n|\,.$$

Show that the unit ball $B(0, 1)$ is not totally bounded.

Problem 11. For a subset E of a metric space X, show that E is totally bounded if and only if its closure $\overline{E}$ is totally bounded.

Problem 12. Let (X, d) be a metric space and a continuous real-valued map $f : X \to \mathbb{R}$. Given a compact subset $K \subset X$, show that there exists $x_{\max} \in K$ such that

$$f(x_{\max}) \;=\; \max_{y\in K} f(y)\,.$$

Problem 13*. Let K be a compact subset of a metric space X and $\mathcal{O}$ an open set containing K. Show that there is an open set $U$for which

$$K \;\subseteq\; U \;\subseteq\; \overline{U} \;\subseteq\; \mathcal{O}\,.$$