MA 515 Prerequisites Test Solutions

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Problem 1

Proof. For all $\epsilon > 0$, we need to find a N such that $\forall n > N$, $|x_n - 2| < \epsilon$.

We may let $N = \lfloor 2/\epsilon - 1 \rfloor + 1$. Then, for all $n > N$,

$$|\frac{2n}{n+1} - 2| - \epsilon = \frac{2}{n+1} - \epsilon < \frac{2}{\lfloor 2/\epsilon - 1 \rfloor + 2} < \frac{2}{2/\epsilon} = \epsilon.$$

In conclusion, the sequence $\{x_n\}$ converges to 2.

□

Problem 2

During the proof, we will need the linearity of limit. i.e. $\alpha \in \mathbb{R}, \{x_n\} \to 3$ and $\{y_n\} \to 5$, then

- $$\lim_{n\to\infty} \alpha x_n = 3\alpha$$
- $$\lim_{n\to\infty} x_n + y_n = 3 + 5 = 8.$$

Proof. We need to show that $\forall \epsilon > 0, \exists N$, such that $\forall n > N$, $|x_n y_n - 15| < \epsilon$.

Since both $\{x_n\}$ and $\{y_n\}$ are convergent, we know,$\forall \epsilon_1 > 0, \exists N_1, N_2$, such that for all $m > N_1, n > N_2$,

$$|x_m - 3| < \sqrt{\epsilon_1}, \quad |y_n - 5| < \sqrt{\epsilon_1}.$$

We may let $N_3 = \max N_1, N_2$. Then for all $n > N_3$,

$$|(x_n - 3)(y_n - 5)| < \epsilon_1.$$

In last inequality, use triangle inequality and obtain the left-hand side:

$$
\begin{aligned}
LHS = |x_n y_n - 3y_n - 5x_n + 30 - 15| &\geqslant |x_n y_n - 15| - |3y_n + 5x_n - 30| \\
&\geqslant |x_n y_n - 15| - |3(y_n - 5)| - |5(x_n - 3)| \\
&\geqslant |x_n y_n - 15| - 8\sqrt{\epsilon_1}
\end{aligned}
$$

Hence, we have the following,

$$|x_n y_n - 15| \leqslant 8\sqrt{\epsilon_1} + \epsilon_1 \quad \to 0$$

as ϵ goes to 0.
Then the claim of $\lim_{n\to\infty} x_n y_n = 15$ is proved.

□

Problem 3

Proof. From $\{x_n\}$ is a Cauchy sequence we know that $\forall \epsilon > 0, \exists N$ such that $\forall n, m > N$, $|x_n - x_m| < \epsilon$. We may let $m = N + 1$ and then $\forall n > N, |x_n - x_{N+1}| < \epsilon$, which implies $|x_n| < \epsilon + |x_{N+1}|$.

For all $\epsilon > 0$, we may let $K_\epsilon = \max\{|x_1|, |x_2|, \ldots, |x_N|, |x_{N+1} + \epsilon|\}$. Then $\forall n \in \mathbb{N}$, $|x_n| < K_\epsilon$. Here $K_\epsilon < +\infty$.

Since ϵ is arbitrary chosen and we let it go to 0. Then $K_\epsilon \to K < +\infty$ and $\{x_n\}$ is bounded by K.

Hence, Cauchy sequence is bounded.

□

Problem 4

Recall the definition of continuous. A function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at c if f is defined at c and satisfies that $\forall \epsilon > 0$, there exists $\delta > 0$ such that $\forall |x_n - c| < \delta$, $|f(x_n) - f(c)| < \epsilon$.

Proof. Since the sequence $\{x_n\}$ converges to c, we know that $\forall \epsilon_0 > 0, \exists N_0$ such that $|x_n - c| \epsilon_0$. With the fact that f is continuous at c, $\forall \epsilon > 0$, there exists $\delta > 0$. And for this δ, there exists N_ϵ, such that $\forall n > N_\epsilon$,

$$|f(x_n) - f(c)| < \epsilon.$$

Rewrite above, we have $\forall \epsilon > 0, \exists N_\epsilon$, such that

$$\forall n > N_\epsilon, \quad |f(x_n) - f(c)| < \epsilon.$$

which proves the equivalent statement of $\lim_{n\to\infty} f(x_n) = f(c)$.

□

Problem 5

$\forall x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, let metric $d_1(x, y) = ||x - y||$, where $||x|| = \sqrt{x_1^2 + x_2^2}$ is the norm on X. If we show (X, d_1) is a metric space, then we can use this to show the triangle inequality for metric d.

$\forall x \in X$, let $\bar{x} = (2x_1, 3x_2)$, and it is clear that $d(x, y) = d_1(\bar{x}, \bar{y})$. Then $\forall x, y, z \in X$,

$$d(x, y) + d(y, z) = d_1(\bar{x}, \bar{y}) + d_1(\bar{y}, \bar{z}) \geqslant d_1(\bar{x}, \bar{z}) = d(x, z).$$

Moreover, from the definition we can see that d satisfies

1. $d(x, x) = 0$
2. $d(x, y) = d(y, x)$.

Hence, (X, d) is a metric space.

In the following we only need to show that (X, d_1) is a metric space.

Proof. We see that d_1 satisfies that $d(x, x) = 0$ and $d(x, y) = d(y, x)$. So the only thing we need to show is the triangle inequality of d_1. i.e., $\forall x, y, z \in X$,

$$||x - y|| + ||y - z|| \geqslant ||x - z||.$$

We may let $x - y = a, y - z = b$ and $x - z = a + b$. Then use Cauchy-Schwartz inequality,

$$\begin{aligned} ||a + b||^2 = (a + b)^T(a + b) &= ||a||^2 + ||b||^2 + 2a^T b \\ &\leqslant ||a||^2 + ||b||^2 + 2||a||||b|| = (||a|| + ||b||)^2 \end{aligned}$$

Take the square root on both sides and obtain $||a + b|| \leqslant ||a|| + ||b||$, which proves the triangle inequality of metric d_1.

□

Problem 6

Proof. Obviously, $\{x_n\}$ is a Cauchy sequence. Recall that in problem 3, we have already shown that $\{x_n\}$ is bounded. Use Bolzano-Weierstrass theorem and we know that there exists a convergent subsequence $\{x_{n_k}\}$. Let's say, the subsequence converges to c. i.e., $\forall \epsilon > 0$, $\exists N_1$, such that $\forall n_k > N_1$, $|x_{n_k} - c| < \epsilon/2$. what's more, since $\{x_n\}$ is Cauchy, we know $\exists N_2$, such that $\forall m, n > N_2$, $|x_m - x_n| < \epsilon/2$.

Combine those two conditions and take $N_3 = \max\{N_1, N_2\}$, we see that $\forall n, n_k > N_3$,

$$|x_n - c| = |x_n - x_{n_k} + x_{x_k} - c| \leqslant |x_n - x_{n_k}| + |x_{x_k} - c| < \epsilon/2 + \epsilon/2 = \epsilon.$$

which proves the claim that $\{x_n\}$ is convergent.

□