MA 515-001, Fall 2017, Homework 5

Due: Mon Nov 06, 2017, in-class.

Problem 1. Let $T : X \to Y$ be a linear operator. Assume that the range of T

$$V \doteq T(X) \subset Y .$$

is a n-dimensional vector space. Show that

$$X = ker(T) \oplus Y_0$$

for some subspace Y_0 with $\dim(Y_0) = \dim(V) = n$.

Problem 2. Let T be a linear operator from a normed linear space X to a finite-dimensional normed linear space Y. Show that T is continuous if and only

$$Ker(T) = \{x \in X \mid T(x) = 0\}$$

is a closed subspace of X.

Problem 3. Let X and Y be Banach spaces. Prove that every continuous function (not necessarily linear) $f : X \to Y$ has closed graph.

Problem 4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function with closed graph. Show that f is continuous.
(b) Construct a function $g : \mathbb{R} \to \mathbb{R}$ is bijective, has closed graph, but is not continuous.

Problem 5. Let X, Y, Z be Banach spaces and let

$$T_1 : X \to Y \qquad \text{and} \qquad T_2 : Y \to Z$$

be linear operators. Assume that one of the two operators is continuous while the other is compact. Show that the composition $T_2 \circ T_1 : X \to Z$ is compact.

Problem 6. Let X, Y be Banach spaces, and let $T : X \to Y$ be a linear continuous bijection.

(i) Prove that there exists $\beta > 0$ such that

$$\|T(x)\| \geq \beta \cdot \|x\| \qquad \forall x \in X .$$

(ii) Let $\Psi \in B(X, Y)$ be any bounded linear operator with norm $\|\Psi\|_\infty\| < \beta$. Using the contraction mapping theorem, prove that, for any $y \in Y$, the equation

$$x = T^{-1}(y - \Psi(x))$$

has a unique solution.

Problem 7. Let $(X, \|\cdot\|_X), (Y, \|\cdot\|)$ and $(Z, \|\cdot\|_Z)$ be Banach spaces. Let $B : X \times Y \to Z$ be a bilinear map, i.e., $x \mapsto B(x, y)$ is a linear map for given $y \in Y$ and $y \mapsto B(x, y)$ is a linear map for given $x \in X$. Assume that B is continuous at the origin.

(i) Show that $x \mapsto B(x, y)$ is a linear bounded map for given $y \in Y$ and $y \mapsto B(x, y)$ is a linear bounded map for given $x \in X$

(ii) Using Banach-Steihaus theorem to show that there exists a constant $C > 0$ such that

$$\|B(x, y)\|_Z \;\leq\; C \cdot \|x\|_X \cdot \|y\|_Y \qquad \forall (x, y) \in X \times Y\,.$$

Problem 8. Let $X = C([0, 1])$ be a space of continuous real function on $[0, 1]$ and

$$\|f\|_\infty \;=\; \sup_{x \in [0,1]} |f(x)| \qquad \forall f \in C([0, 1]).$$

Consider the linear operator $T : X \to X$ defined by

$$T[f](0) \;=\; f(0) \qquad \text{and} \qquad T[f](t) \;=\; \frac{1}{t} \cdot \int_0^t f(s)\; ds \qquad \forall t \in (0, 1]\,.$$

Prove that

(i) T is continuous.

(ii) T is one-to-one but not onto.

(iii) T is not compact.