Exam 1 Solutions

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Problem 1

(a) False. Counterexample:
$S_1 = \{x = (x_1, x_2) \in \mathbb{R}^2 | x_2 = 0, x_1 \geqslant 0\}$ and $S_2 = \{x = (x_1, x_2) \in \mathbb{R}^2 | x_1 = 0, x_2 \geqslant 0\}$.
Both S_1 and S_2 are convex but $S_1 \cup S_2$ is not convex.

(b) True. $\text{int}(S)$ is open and the interior of an open set is itself.

(c) False. Counterexample:
Let $P = \mathbb{R}^2_+$, i.e. the first quadrant. It is a unbounded polyhedron, but it is not affine.

(d) False. We have shown in homework that P is a convex set. So if it has two points, then any convex combination of them will be in P.

(e) True. The objective function $c^T x$ is countinuous on P, which is a nonempty closed convex set. So P is a nonempty compact set and $c^T x$ will reach its maximum and minimum. This is supported by Weierstrass theorem.

(f) True. We may reform P as a subset on $\mathbb{R}^2$. So the LP problem will have one equality constraint on $\mathbb{R}^2$ with nonnegativeness. Hence, it has at most two positive BFS.

(g) True. If x is an optimal solution, it must be a BFS, so its nonbasic variables must be zeros. Since it has $n - m$ nonbasic variables, it has at least $n - m$ zero entries. Thus, it has at most m positive components.

(h) False. A BS has $n - m$ nonbasic variables so it has at least $n - m$ zeros components.

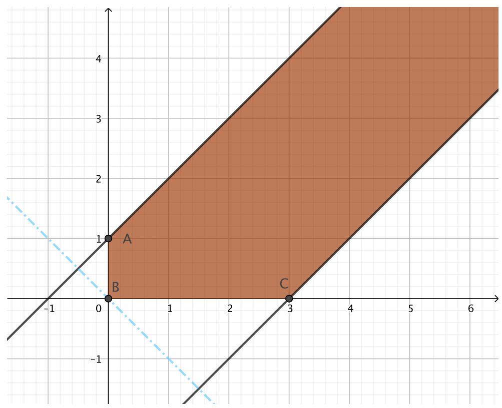
Problem 2

(a) The problem can be reformed as the following,

$$\begin{aligned} \text{Minimize} \quad & -x_1 - x_2 \\ \text{Subject to} \quad & x_1 - x_2 \leqslant 3 \\ & x_1 - x_2 \geqslant -1 \\ & x_1, x_2 \geqslant 0. \end{aligned}$$

We plot the figure.

Figure 1: Problem 2



From the figure we see that the problem is unbounded. This is to say that the objective value will goes to $-\infty$.

(b) There are 5 BS, 3 BFS(with '*').

$$\begin{gathered}[-1,0,4,0]^T \\ [3,0,0,4]^T\ * \\ [0,1,4,0]^T\ * \\ [0,-3,0,4]^T \\ [0,0,3,1]^T\ * \end{gathered}$$

(c) Start from BFS $x=[0,1,4,0]^T$. The basic matrix is $B=\begin{bmatrix} A2 & A_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$ and so $B=\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$. Hence the fundamental matrix

$$M=\begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad M^{-1}=\begin{bmatrix} 0 & -1 & 1 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Compute reduced cost

$$r_1 = c^T\mathbf{d}^1 = [-1,0,-1,0]\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = -2 < 0, \quad r_2 = c^T\mathbf{d}^4 = [-1,0,-1,0]\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = 1 > 0.$$

Hence we take $\mathbf{d}^1$. However, since $\mathbf{d}^1 > 0$ and $[B|N]d = 0$ (this is easy to check). This is to say that $\mathbf{d}^1$ is a extremal direction. Thus, $\forall \alpha > 0$ and $x + \alpha\mathbf{d}^1$ is feasible and $c^T(x + \alpha\mathbf{d}^1) \to -\infty$, as $\alpha \to +\infty$. Hence, the LP problem is unbounded.

Problem 3