Homework 2 Solutions

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Problem 1

Since there is an absolute value in the objective function and x_3 is unrestricted, we may divide the feasible domain into two parts: $x_3 \geqslant 0$ and $x_3 < 0$.

For $x_3 \geqslant 0$, $|x_3| = x_3$. Hence, we may construct a LP problem:

$$\begin{aligned} \text{Manximize} \quad & 3x_1 - 2x_2 + 4x_3 \\ \text{subject to} \quad & -x_1 + 2x_2 \leqslant -5 \\ & 3x_2 - x_3 \geqslant 6 \\ & x_1, x_2, x_3 \geqslant 0 \end{aligned}$$

Similarly, for $x_3 < 0$,$|x_3| = -x_3$, and we can also construct a LP problem:

$$\begin{aligned} \text{Manximize} \quad & 3x_1 - 2x_2 - 4x_3 \\ \text{subject to} \quad & -x_1 + 2x_2 \leqslant -5 \\ & 3x_2 - x_3 \geqslant 6 \\ & x_1, x_2 \geqslant 0, x_3 \leqslant 0. \end{aligned}$$

Then convert those two LP problems above into standard forms:

$$\begin{aligned} \text{Minimize} \quad & -3x_1 + 2x_2 - 4x_3 \\ \text{subject to} \quad & -x_1 + 2x_2 + \xi_1 = -5 \\ & 3x_2 - x_3 - \xi_2 = 6 \\ & x_1, x_2, x_3, \xi_1, \xi_2 \geqslant 0 \end{aligned} \tag{1}$$

and

$$\begin{aligned} \text{Minimize} \quad & -3x_1 + 2x_2 - 4x_3 \\ \text{subject to} \quad & -x_1 + 2x_2 + \xi_1 = -5 \\ & 3x_2 + x_3 - \xi_2 = 6 \\ & x_1, x_2, x_3, \xi_1, \xi_2 \geqslant 0 \end{aligned} \tag{2}$$

Take the optimal solution as the one that solves (1) or (3) with a smaller optimal value.

Problem 2.1

Solutions:

1. We need to show that if the feasible domain is bounded, then the LP problem has a bounded optimal value.

   Proof. Consider the standard form of a LP problem.

$$\begin{aligned} \text{Minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geqslant 0 \end{aligned}$$

   where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$.

   Denote its feasible domain as $P := \{x \in \mathbb{R}^n | Ax = b, x \geqslant 0\}$. If P is bounded, then $\exists M \geqslant 0$ such that $||x|| \leqslant M$ for all $x \in P$.

   Recall Cauchy-Schwartz inequality, $\forall x, y \in \mathbb{R}^n$,

$$|x^T y| \leqslant ||x|| \cdot ||y||$$

   Hence, we have

$$c^T x \geqslant -||c|| \cdot ||x|| = ||c||(-||x||) \geqslant ||c|| \cdot (-M).$$

   Since c is a constant vector, M exists as a constant, we know $c^T x$ has a lower bound, which proves that the LP problem is bounded.

   □

2. Next we give a counterexample to show that the opposite direction is not ture.

   Consider the following LP problem:

$$\begin{aligned} \text{Minimize} \quad & 2x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \geqslant 1 \\ & x_1, x_2 \geqslant 0 \end{aligned} \tag{3}$$

   It is obvious that the feasible domain is not bounded but the problem is bounded.

Problem 2.3

Proof. To prove H is affine, take x, y from H arbitrarily and we need to show the affine combination of x and y is also in H. This is true, since for any $\alpha_1, \alpha_2 \in \mathbb{R}$ that satisfy $\alpha_1 + \alpha_2 = 1$,

$$a^T(\alpha_1 x + \alpha_2 y) = \alpha_1 a^T x + \alpha_2 a^T y = \alpha_1 \beta + \alpha_2 \beta = \beta.$$

which shows $\alpha_1 x + \alpha_2 y$ is in H.

For convexity, it follows from the fact that H is affine because the convex combination of two points is a special case of their affine combination.

□

Problem 2.4

1. *Proof.* We want to show $\forall x, y \in \cap_{i=1}^p C_i$, the convex combination of x, y is also in it.

   Since $x, y \in \cap_{i=1}^p C_i$, we know $x, y \in C_i$, $\forall i = 1, \ldots, p$. with the fact that C_i is convex, for any $\alpha \in (0, 1)$, $\alpha x + (1 - \alpha) y \in C_i$ holds for each index i. Hence, $\alpha x + (1 - \alpha) y \in \cap_{i=1}^p C_i$.

   The claim is then proved.

   □

2. $\cup_{i=1}^p$ may not be convex. A counterexample is to let $C_1 = \{(x, y) | x = 0, y \in \mathbb{R}\}$, and $C_1 = \{(x, y) | y = 0, x \in \mathbb{R}\}$. It is clear that C_1, C_2 are convex and they are x and y axis in $\mathbb{R}^2$. However, $C_1 \cup C_2$ is not convex.

   Let $a = (1, 0) \in C_2$, $b = (0, 1) \in C_1$. $a, b \in C_1 \cup C_2$, but $\frac{1}{2}a + \frac{1}{2}b = (1/2, 1/2) \notin C_1 \cup C_2$.

Problem 2.5

(It will be easy to use the results from the problems we just solved. But it is fine if you use other methods to prove this claim.)

Proof. Let $A \in \mathbb{R}^{m \times n}$ be $\begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}$, where a_i^T is the ith row of A. and $b = [b_1, \cdots, b_m]^T$. Then the feasible domain P is equivalent to

$$\cap_{i=1}^m \{x \in \mathbb{R}^n | a_i^T x = b_i\} \cap \{x \in \mathbb{R}^n | x \geqslant 0\}.$$

Let $P_i := \{x \in \mathbb{R}^n | a_i^T x = b_i\}$ and each P_i is a hyperplane. Also, it is obvious that $\{x \in \mathbb{R}^n | x \geqslant 0\}$ is convex(use the definition and easy to prove). Use the results from 2.3 and 2.4, and we know that the P_i is convex and intersection of convex sets is also convex. Hence, P is convex.

□

Problem 2.6

Proof. Consider a LP problem in standard form.

$$\begin{aligned} \text{Minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geqslant 0 \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$.

Denote its feasible domain as $P := \{x \in \mathbb{R}^n | Ax = b, x \geqslant 0\}$. Let the supporting hyperplane H of feasible domain P be the following,

$$H := \left\{ x \in \mathbb{R}^n | - c^T x = \beta \right\}.$$

and $\forall x \in P$, $-c^T x \leqslant \beta$. since $H \cap P \neq \phi$, take any $x^* \in H \cap P$, we have $c^T x^* = -\beta$. Hence, $\forall x \in P$, $c^T x \geqslant c^T x^* = -\beta$, which proves that x^* is an optimal solution to the LP problem.

$\square$