

# Homework 5 Solutions

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## Problem 1 (3.1)

Suppose  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$ . Add slack variables  $s_1, \dots, s_m \geq 0$  on each row and make it a standard form.

Let starting feasible solution be  $(x_1, \dots, x_n, s_1, \dots, s_m)^T = (0, \dots, 0, b_1, \dots, b_m)^T$ .

## Problem 2 (3.2)

- (a) *Proof.* Let  $d$  be a feasible direction at point  $x \in P$ . Then, there exists  $\lambda > 0$  such that  $x + \lambda d \in P$ , which implies  $A(x + \lambda d) = b$ . Since  $Ax = b$ , we know that  $\lambda Ad = 0$  and this implies  $Ad = 0$ . □

- (b) *Proof.* let  $d = (d_1, \dots, d_n)^T$  be a feasible direction at  $x$ . Let  $\alpha = \min\{\frac{x_i}{-d_i} | d_i < 0, i = 1, \dots, n\}$ . If  $d \geq 0$ , then let  $\alpha = 1$ .

It is clear that  $\alpha > 0$  and  $x + \alpha d \geq 0$ . □

## Problem 3 (3.3)

$$\begin{array}{ll} \text{Minimize} & -2x_1 - x_2 + x_3 + x_4 + 2x_5 \\ \text{subject to} & -2x_1 + x_2 + x_3 + x_4 + x_5 = 12 \\ & -x_1 + 2x_2 + x_4 - x_5 = 5 \\ & x_1 - 3x_2 + x_3 + 4x_5 = 11 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

$$\text{Here, } A = \begin{pmatrix} -2 & 1 & 1 & 1 & 1 \\ -1 & 2 & 0 & 1 & -1 \\ 1 & -3 & 1 & 0 & 4 \end{pmatrix} \text{ and } b = \begin{pmatrix} 12 \\ 5 \\ 11 \end{pmatrix}$$

(a)  $B = [A_3, A_4, A_5] = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 4 \end{pmatrix}$ , and  $N = [A_1, A_2] = \begin{pmatrix} -2 & 1 \\ -1 & 2 \\ 1 & -3 \end{pmatrix}$ .

The fundamental matrix  $M = \begin{bmatrix} B & N \\ \mathbf{0} & I \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & -1 & 2 \\ 1 & 0 & 4 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$  and  $M^{-1} = \begin{bmatrix} B^{-1} & -B^{-1}N \\ \mathbf{0} & I \end{bmatrix} =$

$$\begin{bmatrix} 2 & -2 & -1 & 3 & -1 \\ -1/2 & 3/2 & 1/2 & 0 & -1 \\ -1/2 & 1/2 & 1/2 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) Apply Gaussian elimination on matrix  $[B, N]$  and get reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -3 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

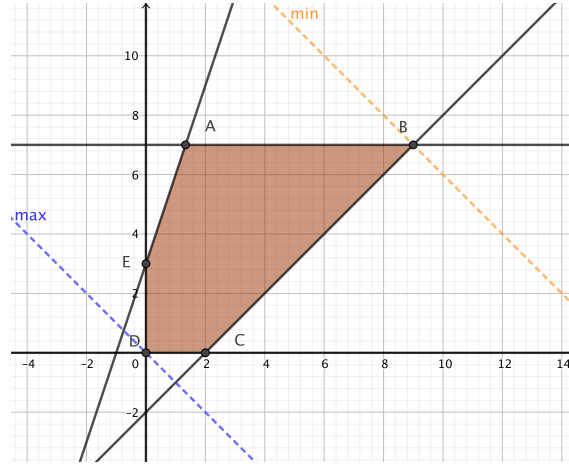
Hence,  $x_3 = 3x_1 - x_2 + 3$ ,  $x_4 = -x_2 + 7$ ,  $x_5 = -x_1 + x_2 + 2$ . Reform the LP problem using only two variables as the following:

$$\begin{array}{ll} \text{Minimize} & -x_1 - x_2 (+14) \\ \text{subject to} & 3x_1 - x_2 + 3 \geq 0 \\ & -x_2 + 7 \geq 0 \\ & -x_1 + x_2 + 2 \geq 0 \\ & x_1, x_2 \geq 0 \end{array}$$

(c) The feasible domain is part of the intersection of three hyperplanes on  $\mathbb{R}^5$ , hence, its dimension is reduced by 3 and can be represented in  $\mathbb{R}^2$ .

We plot the region of  $P$ .

Figure 1: Region  $P$ .



- (d) Basic feasible solution  $\mathbf{x} = (0, 0, 3, 7, 2)^T$ . And it is corresponding to point  $D = (0, 0)^T$  on fig.1.

(e)

$$B^{-1}A = \begin{bmatrix} -3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}, \quad B^{-1}b = \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix}$$

One explanation of this:

We see that  $B^{-1}A$  is the same with the reduced row echelon form of  $A$ . And  $B^{-1}b$  is exactly the basic feasible solution (positive entries). This is always true since

$$Ax = b \Leftrightarrow [B|N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b.$$

Since  $x_N = 0$ , we get  $Bx_B = b$  so that  $x_B = B^{-1}b$ . This implies that  $B^{-1}A$  is the reduced row echelon form of  $A$  for basic variable, since  $B^{-1}Ax = B^{-1}b = x_B$ .

(Other proper answers will be also acceptable).

- (f) From  $M^{-1}$  we know that  $\mathbf{d}^1 = (3, 0, -1, 1, 0)^T$  and  $\mathbf{d}^2 = (-1, -1, 1, 0, 1)^T$ . Reduced costs:

$$r^1 = [c_B^T | c_N^T] \mathbf{d}^1 = (1, 1, 2, -2, -1) \mathbf{d}^1 = -1, \quad r^2 = [c_B^T | c_N^T] \mathbf{d}^2 = (1, 1, 2, -2, -1) \mathbf{d}^2 = -1$$

- (g) From above, either direction leads to a potential reduction in the objective value, since  $r^1$  and  $r^2$  are both negative. Consider the nonnegativity constraint ( $x + \alpha \mathbf{d} \geq 0$ ), we get the step length for  $\mathbf{d}^1$  is  $\alpha_1 = 2$  and for  $\mathbf{d}^2$  is  $\alpha_2 = 3$ .

- (h) 1) If we take  $\mathbf{d}^1$ , then the new solution will be  $\bar{x} = x + \alpha_1 \mathbf{d}^1 = (9, 7, 0, 2, 0)^T \geq 0$ . The basis now is  $\bar{B} = [A_3, A_4, A_1]$  and  $\bar{N} = [A_5, A_2]$ . It is easy to check that  $\bar{B}\bar{x} = b$ . Hence  $\bar{x}$  is a basic feasible solution(BFS).  $\bar{x}$  is also an adjacent extreme point of  $x$ . (On fig.1,  $\bar{x}$  is the point  $C$ )
- 2) If we take  $\mathbf{d}^2$ , then the new solution will be  $\bar{x} = x + \alpha_2 \mathbf{d}^2 = (0, 4, 5, 0, 3)^T \geq 0$ . The basis now is  $\bar{B} = [A_4, A_5, A_2]$  and  $\bar{N} = [A_3, A_1]$ . It is easy to check that  $\bar{B}\bar{x} = b$ . Hence  $\bar{x}$  is a basic feasible solution(BFS).  $\bar{x}$  is also an adjacent extreme point of  $x$ . (On fig.1,  $\bar{x}$  is the point  $E$ )
- (i) 1) If we take  $\mathbf{d}^1$ , update  $\tilde{M} = \begin{bmatrix} \tilde{B} & \tilde{N} \\ \mathbf{0} & I \end{bmatrix}$ .

$$\tilde{M} = \begin{bmatrix} 1 & 1 & -2 & 1 & 1 \\ 0 & 1 & -1 & -1 & 2 \\ 1 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{M}^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 & -3 & 2 \\ -1/2 & 3/2 & 1/2 & 0 & -1 \\ -1/2 & 1/2 & 1/2 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

From the last two columns of  $\tilde{M}^{-1}$  we get  $\tilde{\mathbf{d}}^5 = \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ ,  $\tilde{\mathbf{d}}^2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ .

Let  $\tilde{c}^T = [\tilde{c}_B^T | \tilde{c}_N^T]$ . We get  $\tilde{c}^T \tilde{\mathbf{d}}^5 = 1 > 0$ , but  $\tilde{c}^T \tilde{\mathbf{d}}^2 = -2 < 0$ . So  $\tilde{x}$  is not an optimal solution since  $\tilde{\mathbf{d}}^2$  is a good direction of translation.

- 2) If we take  $\mathbf{d}^2$ , update  $\bar{M} = \begin{bmatrix} \bar{B} & \bar{N} \\ \mathbf{0} & I \end{bmatrix}$ .

$$\bar{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & -2 \\ 1 & -1 & 2 & 0 & -1 \\ 0 & 4 & -3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \bar{M}^{-1} = \begin{bmatrix} -5/2 & 7/2 & 3/2 & 1 & -3 \\ 3/2 & -3/2 & -1/2 & -1 & 2 \\ 2 & -2 & -1 & -1 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

From the last two columns of  $\bar{M}^{-1}$  we get  $\bar{\mathbf{d}}^5 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ ,  $\bar{\mathbf{d}}^2 = \begin{bmatrix} -3 \\ 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$ .

Let  $\bar{c}^T = [\bar{c}_B^T | \bar{c}_N^T]$ . We get  $\bar{c}^T \bar{\mathbf{d}}^3 = 1 > 0$ , but  $\bar{c}^T \bar{\mathbf{d}}^1 = -4 < 0$ . So  $\bar{x}$  is not an optimal solution since  $\bar{\mathbf{d}}^1$  is a good direction of translation.

- (j) 1) If basic variables are  $x_3, x_4, x_1$ , then the reduced row echelon form (RREF) of  $[\tilde{B}, \tilde{N}, -b]$  is

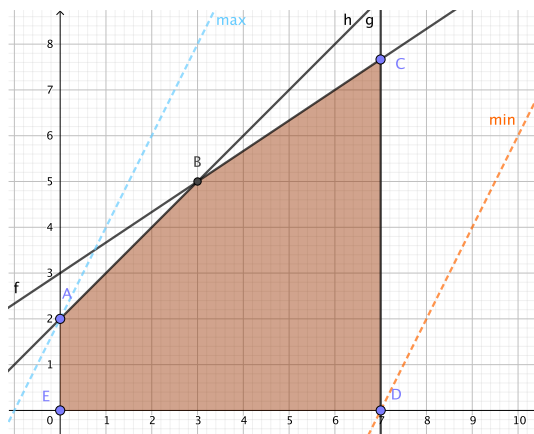
$$\begin{bmatrix} 1 & 0 & 0 & 3 & -2 & -9 \\ 0 & 1 & 0 & 0 & 1 & -7 \\ 0 & 0 & 1 & 1 & -1 & -2 \end{bmatrix}$$

and  $x_3 = 2x_2 - 3x_5 + 9$ ,  $x_4 = -x_2 + 7$ ,  $x_1 = x_2 - x_5 + 2$ . Reform the LP problem using only two variables as the following:

$$\begin{array}{llll} \text{Minimize} & -2x_2 + x_5(+12) \\ \text{subject to} & 2x_2 - 3x_5 + 9 & \geq 0 \\ & -x_2 + 7 & \geq 0 \\ & x_2 - x_5 + 2 & \geq 0 \\ & x_2, x_5 & \geq 0 \end{array}$$

We plot the region.

Figure 2: Region on  $x_2, x_5$ .



- 2) If basic variables are  $x_4, x_5, x_2$ , then the reduced row echelon form (RREF) of  $[\overline{B}, \overline{N}, -b]$  is

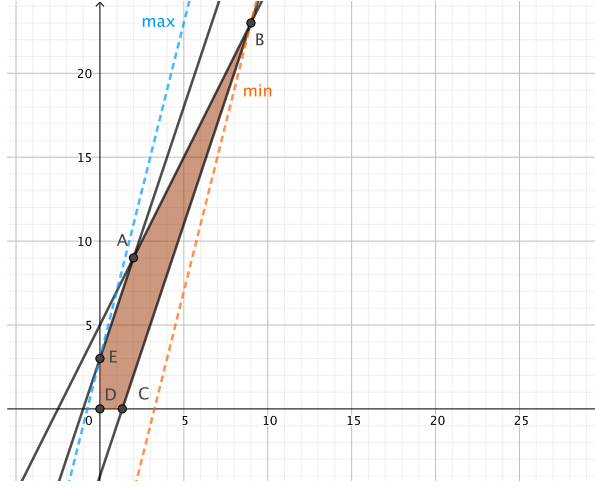
$$\begin{bmatrix} 1 & 0 & 0 & -1 & 3 & -4 \\ 0 & 1 & 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & 1 & -3 & -3 \end{bmatrix}$$

and  $x_4 = x_3 - 3x_1 + 4$ ,  $x_5 = -x_3 + 2x_1 + 5$ ,  $x_2 = -x_3 + 3x_1 + 3$ . Reform the LP problem using only two variables as the following:

$$\begin{array}{llll} \text{Minimize} & -4x_1 + x_3(+11) \\ \text{subject to} & x_3 - 3x_1 + 4 & \geq 0 \\ & -x_3 + 2x_1 + 5 & \geq 0 \\ & -x_3 + 3x_1 + 3 & \geq 0 \\ & x_1, x_3 & \geq 0 \end{array}$$

We plot the region.

Figure 3: Region on  $x_1, x_3$ .



(k) (Any proper answer will be acceptable).

(l) We can always express  $n - 2$  variables by using the rest 2 variables. Then, the LP problem can be reformed as an LP problem on  $\mathbb{R}^2$ .

## Problem 4 (3.4)

From point  $x = [1/2, 0, 0, 0, 1, 1]^T$ , we know that  $B = [A_5, A_6, A_1]$  and  $N = [A_2, A_3, A_4]$ .

Construct matrix  $M = \begin{bmatrix} B & N \\ \mathbf{0} & I \end{bmatrix}$  and reduced cost  $r = c_N^T - c_B^T B^{-1} N = [-1, -1, 1/2]$ .

Note that  $r_2, r_3$  are negative, and figure out the step length  $\alpha_2 = \alpha_3 = 1/2$ . Hence, we can pick either one from  $\mathbf{d}^2$  or  $\mathbf{d}^3$ . Let's pick  $\mathbf{d}^2$ .  $x_{\text{new}} = x + \alpha_2 \mathbf{d}^2 = [1/2, 1/2, 0, 0, 1]^T$ .

Next step:

From point  $x = [1/2, 1/2, 0, 0, 1]^T$ , we know  $B = [A_2, A_6, A_1]$  and  $N = [A_5, A_3, A_4]$ . Construct matrix  $M$  and reduced cost  $r = c_N^T - c_B^T B^{-1} N = [-1/2, -1, -1/2]$ . Find the most negative direction, pick  $\mathbf{d}^3$  with  $\alpha_3 = 1/2$ .  $x_{\text{new}} = x + \alpha_3 \mathbf{d}^3 = [1/2, 1/2, 1/2, 0, 0]^T$ .

Next step:

From point  $x = [1/2, 1/2, 1/2, 0, 0]^T$ , we know  $B = [A_1, A_2, A_3]$  and  $N = [A_4, A_5, A_6]$ . Compute the reduced cost  $r = c_N^T - c_B^T B^{-1} N = [1/2, 1/2, 1/2] \geq 0$ . Hence,  $x = [1/2, 1/2, 1/2, 0, 0]^T$  is the optimal solution.

## Problem 5 (3.8)

*Proof.* For a degenerate BFS  $x$  with  $p (< m)$  positive components, we have  $n - p$  zero components in it. And also,  $n - m$  of  $n - p$  will be nonbasic variables and there will be at most  $C(n - p, n - m)$  situations.

□

### Problem 6 (3.9)

*Proof.* We know  $\overline{M} = \begin{bmatrix} B & N \\ \mathbf{0} & I \end{bmatrix}$  and let  $W = \begin{bmatrix} B^{-1} & -B^{-1}N \\ \mathbf{0} & I \end{bmatrix} \in \mathbb{R}^{n \times n}$ . Thus, it is enough to check  $W\overline{M} = I$  and  $\overline{M}W = I$ .

Those are true since

$$W\overline{M} = \begin{bmatrix} B^{-1}B - B^{-1}N\mathbf{0} & B^{-1}N - B^{-1}NI \\ \mathbf{0}B + I\mathbf{0} & \mathbf{0}N + I * I \end{bmatrix} = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$$

Similarly,  $\overline{M}W = I$ .

In conclusion,  $\overline{M}^{-1} = W$ .

□

### Problem 7 (3.13)

From point  $x = [x_1, x_2, x_3, x_4]^T = [30, 0, 10, 0]^T$ , we know that  $B = [A_3, A_1]$  and  $N = [A_2, A_4]$ . Compute reduced cost  $r = c_N^T - c_B^T B^{-1}N = [-1/2, 3/2]$ .

Note that  $r_2$  is negative, so  $x_2$  enter the basis. Construct  $M^{-1} = \begin{bmatrix} 1 & -1/2 & -1/2 & 1/2 \\ 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

to figure out  $\mathbf{d}^2 = [-1/2, 1, -1/2, 0]^T$  and the step length  $\alpha_2 = 20$ . Hence,  $x_{\text{new}} = x + \alpha_2 \mathbf{d}^2 = [20, 20, 0, 0]^T$ .

Next step:

From point  $x = [x_1, x_2, x_3, x_4]^T = [20, 20, 0, 0]^T$ , we know that  $B = [A_1, A_2]$  and  $N = [A_3, A_4]$ . Compute reduced cost  $r = c_N^T - c_B^T B^{-1}N \geq 0$ . Hence, this is the optimal solution. The optimal value  $z^* = c^T x^* = -100$ .

### Problem 8 (3.14)

From point  $x = [x_1, x_2, x_3, x_4, x_5]^T = [30, 0, 10, 0, 0]^T$ , we know that  $B = [A_3, A_2, A_1]$  and  $N = [A_4, A_5]$ . Compute reduced cost  $r = c_N^T - c_B^T B^{-1}N = [2, -3]$ .

Note that  $r_5$  is negative, so  $x_5$  enter the basis. Construct  $M^{-1} = \begin{bmatrix} 1 & -1 & 1 & 1 & -1 \\ 0 & 1 & -2 & -1 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

to figure out  $\mathbf{d}^5 = [-1, 2, -1, 0, 1]^T$  and the step length  $\alpha_5 = 10$ . Hence,  $x_{\text{new}} = x + \alpha_5 \mathbf{d}^5 = [20, 20, 0, 0, 10]^T$ .

Next step:

From point  $x = [x_1, x_2, x_3, x_4, x_5]^T = [20, 20, 0, 0, 10]^T$ , we know that  $B = [A_1, A_2, A_5]$  and  $N = [A_3, A_4]$ . Compute reduced cost  $r = c_N^T - c_B^T B^{-1}N \geq 0$ . Hence, this is the optimal solution. The optimal value  $z^* = c^T x^* = -100$ .