MA 515-001, Fall 2017, Homework 3

Due: Mon Oct 9, 2017, in-class.

Problem 1. Check if the following are normed spaces. In the negative case, identify which of the properties (i)-(iii) fails.

(a) Let $X = \mathbb{R}$ with

$$\|x\| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -2x & \text{if} \quad x < 0\,. \end{cases}$$

(b) Fix $\lambda \in \mathbb{R}$, let X be the space of all continuous function $f : [0, +\infty[\to \mathbb{R}$ such that

$$\|f\| = \sup_{t \geq 0} e^{\lambda t} \cdot |f(t)| < +\infty\,.$$

(c) Let $X = \mathbb{R}^2$. Given $p \geq 1$, define

$$\|x\| = (|x_1|^p + |x_2|^p)^{\frac{1}{p}} \qquad \forall x = (x_1, x_2)\,.$$

(d) Let $X = \mathbb{R}^2$. Given $p \in (0, 1)$, define

$$\|x\| = (|x_1|^p + |x_2|^p)^{\frac{1}{p}} \qquad \forall x = (x_1, x_2)\,.$$

Problem 2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Prove that the Cartersian product

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

is also a Banach space, with norm

$$\|(x, y)\| = \max\{\|x\|_X, \|y\|_Y\} \qquad \forall (x, y) \in X \times Y\,.$$

Problem 3. Let $(X, \|\cdot\|_X)$ be Banach spaces. Let $f : [0, +\infty) \to [0, +\infty)$ be an increasing continuous function such that

(i) $f(0) = 0, \qquad f(s) > 0 \qquad \forall s > 0\,;$

(ii) $f(s + t) \leq f(s) + f(t) \qquad \forall s, t \geq 0\,.$

Denote by

$$d_f(x, y) = f(\|x - y\|_X) \qquad x, y \in X\,.$$

(a) Show that (X, d_f) is a *complete* metric space.

(b) In addition, assume that

$$f(\lambda \cdot t) \ \leq \ \lambda \cdot f(t) \qquad \forall \lambda, t > 0\,.$$

Show that the unit ball

$$B_f(0,1) \ = \ \{x \in X \mid d_f(0,x) < 1\}$$

is convex.

Problem 4. Prove that a norm space $(X, \|\cdot\|)$ is complete if and only if every absolutely convergent series has a sum

$$\sum_{n=1}^{\infty} \|x_n\| \ < \ \infty \qquad \text{implies that} \qquad \sum_{n=1}^{\infty} x_n \ = \ \lim_{k\to\infty} \sum_{n=1}^{k} x_n \ \text{ exists}\,.$$

Problem 5. Fixed $p \geq 1$, recalling that

$$l^p \ = \ \left\{ x = \{x_i\}_{i\geq 1} \ \Big| \ \sum_{i=1}^{\infty} |x_i|^p < +\infty \right\}$$

and

$$\|x\|_p \ = \ \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{\frac{1}{p}}\,.$$

Let $e_k = \{x_i^k\}_{i\geq 1}$ be such that

$$x_k^k \ = \ 1 \qquad \text{and} \qquad x_i^k \ = \ 0 \qquad \forall i \neq k\,.$$

Show that the set

$$V \ \doteq \ span\{e_1, e_2, ...\} \ = \ \left\{ \sum_{i=1}^{n} \alpha_i \cdot e_{n_i} \ \Big| \ n, n_i \in \mathbb{Z}^+, \alpha_i \in \mathbb{R} \right\}$$

is dense in l^p.

Problem 6. Let $X = \mathbb{R}$ be a metric space and

$$C(X) \ = \ \left\{ f : X \to \mathbb{R} \ \Big| \ f \text{ is continuous and } \sup_{x\in X} |f(x)| < +\infty \right\}\,.$$

Let $\{f_n\}_{n\geq 1}$ be a sequence in $C(X)$ that converges to $f \in C(X)$ uniformly, i.e.,

$$\lim_{n\to\infty} \sup_{x\in X} |f_n(x) - f(x)| \ = \ 0\,.$$

Show that the sequence $\{f_n\}_{n\geq 1}$ is equicontinuous on X.

Problem 7. Given $a, b \in \mathbb{R}$ and $a < b$, consider the set of Hölder continuous of order $\alpha \in (0,1)$

$$C^{\alpha}([a,b]) \;=\; \left\{ f : [a,b] \to \mathbb{R} \;\middle|\; \frac{|f(x)| - f(y)|}{|x-y|^{\alpha}} \;\leq\; C \qquad \forall x \neq y \in [a,b] \;\text{ for some constant } C \right\}.$$

For every $f \in C^{\alpha}([a,b])$, denote by

$$\|f\|_{\alpha} \;=\; \max\left\{ |f(x)| + \frac{|f(x) - f(y)|}{|x-y|^{\alpha}} \qquad \forall x \neq y \in [a,b] \right\}.$$

(a) Show that $(C^{\alpha}([a,b]), \|\cdot\|_{\alpha})$ is a normed vector space.

(b) Consider the unit ball in $(C^{\alpha}([a,b]), \|\cdot\|_{\alpha})$

$$\overline{B}_{\alpha}(0,1) \;=\; \left\{ f \in C^{\alpha}([a,b]) \;\middle|\; \|f\|_{\alpha} \leq 1 \right\}.$$

Using Arzelà Ascoli theorem to prove that the closure of $\overline{B}_{\alpha}(0,1)$ has compact closure as a subset of $(C([a,b]), \|\cdot\|_{\infty})$.

Problem 8. Let (X,d) be a *compact* metric space and a map $T : X \to X$ such that

$$d(T(x), T(y)) \;<\; d(x,y) \qquad \forall x, y \in X.$$

Show that T has a fixed point.

Problem 9. Let $f : \mathbb{R} \to [0,1]$ be a contractive map. Using Banach contraction principle to show that the equation

$$e^{f(x)} \;=\; 4x$$

has a unique solution.

Problem 10. Let $(X, \|\cdot\|)$ be a normed vector space and let $\{e_1, e_2, ..., e_n\} \subset X$ be linear independent unique vector. Show that

(i) There exists $\beta_2 > 0$ such that

$$\|\lambda_1 \cdot e_1 + \lambda_2 \cdot e_2\| \;\geq\; \beta_2 \cdot (|\lambda_1| + |\lambda_2|) \qquad \forall \lambda_1, \lambda_2 \in \mathbb{R}.$$

(ii) There exists $\beta_n > 0$ such that

$$\left\| \sum_{i=1}^{n} \lambda_i \cdot e_i \right\| \;\geq\; \beta_n \cdot \sum_{i=1}^{n} |\lambda_i| \qquad \forall \lambda_i \in \mathbb{R}.$$