MA 515 Homework 5

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Problem 1

Proof. Let $V = \text{span}\{v_1, \dots, v_n\}$, where $v_1, \dots, v_n$ are linearly independent elements in V. Then there exists n linearly independent elements $x_1, \dots, x_n \in X$ such that $T(x_i) = v_i, i = 1, \dots, n$. The existence promised by the fact that T is a linear operator. Let $Y_0 = \text{span}\{x_1, \dots, x_n\}$. Hence, $\dim(Y_0) = \dim(V) = n$.

Also, $\ker(T) \cap Y_0 = \{0\}$. Indeed, if $\exists y \neq 0, y \in Y_0$ such that $T(y) = 0$. Let $y = \sum_{i=1}^n \beta_i x_i$. Then there exists $\beta_i \neq 0$. By linearity of T, $T(y) = \sum_{i=1}^n \beta_i T(x_i) = \sum_{i=1}^n \beta_i v_i \neq 0$ and it yields a contradiction.

Next, we will show $\ker(T) + Y_0 = X$. Suppose not, for any $x \in X$, there exists $z \notin \ker(T) + Y_0, w \in \ker(T), r \in Y_0$ such that $x = z + w + r$. Let $r = \sum_{i=1}^n t_i x_i$, and $T(x) = \sum_{i=1}^n \alpha v_i$. Hence,

$$\sum_{i=1}^n \alpha_i v_i = T(x) = T(z) + T(w) + T(r) = T(z) + \sum_{i=1}^n t_i v_i.$$

, which implies that $T(z) = \sum_{i=1}^n (\alpha_i - t_i) v_i$.

However, $z \notin \ker(T) + Y_0$ and so $T(z) \notin \text{span}\{v_1, \dots, v_n\} \subset (\ker(T) + Y_0)$. Hence, it is a contradiction.

In conclusion, $\ker(T) + Y_0 = X$ and $\ker(T) \cap Y_0 = \{0\}$, and it implies that $X = \ker(T) \oplus Y_0$. $\square$

Problem 2

Proof. If T is continuous, then the preimage(i.e., $\ker(T)$) of $\{0\}$ is closed since $\{0\}$ is closed. Also, $\ker(T)$ is a subspace due to the linearity of T.

If $\ker(T)$ is a closed subspace in X, we need to show that T is continuous, or equivalently, bounded. Since Y is a finite-dimensional space, from the result of problem 1, we know there exists a finite-dimensional subspace $Y_0 \subset X$ such that $X = \ker(T) \oplus Y_0$. Hence, for any $x \in X$, there exists $y \in \ker(T), z \in Y_0$ such that $x = y + z$.

Consider the norm of T,

$$\|T\|_\infty = \sup_{\|x\| \leqslant 1} \|T(x)\| \leqslant \sup_{\|x\| \leqslant 1 y \in \ker} .$$

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$\square$

Problem 3

Proof. Denote the graph of f as $G(f) := \{(x, f(x)) | x \in X\} \subset X \times Y$. Let $\{z_n\}_{n\in\mathbb{N}} = \{(x_n, f(x_n))\}_{n\in\mathbb{N}} \subset G(f)$ that converges to $z = (x, y)$. It is enough to show that $y = f(x)$.

Indeed, $x = \lim_{n\to+\infty} x_n$ and so $\lim_{n\to+\infty} f(x_n) = f(x)$ due to the continuity of f. Also, $y = \lim_{n\to+\infty} f(x_n)$. Hence, $y = f(x)$.

$\square$

Question: Here we only need X, Y to be metric spaces. We didn't really need completeness. Is it correct?

Problem 4

(a) *Proof.* $\square$

(b) Let f be the following function,

$$f(x) = \begin{cases} \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Problem 5

Proof. $S \circ T$ is a linear operator. Indeed, the domain of $S \circ T$ is X, which is a subspace of itself. Also, function composition preserves linearity.

Next we need to show $S \circ T$ is bounded. Consider norm $\|S \circ T\|_\infty$,

$$\begin{aligned} \|S \circ T\|_\infty = \sup_{\|x\|_X=1} \|(S \circ T)(x)\|_Z &= \sup_{\|x\|_X=1} \|S(T(x))\|_Z \\ &\leqslant \sup_{\|x\|_X=1} \|S\|_\infty \|T(x)\|_Y \\ &= \|S\|_\infty \sup_{\|x\|_X=1} \|T(x)\|_Y \\ &= \|S\|_\infty \|T\|_\infty. \end{aligned}$$

Since both S and T are bounded linear operators, $\|S\|_\infty$ and $\|T\|_\infty$ are less than positive infinity and this leads to the conclusion that $\|S \circ T|_\infty < +\infty$.

In conclusion, $S \circ T$ is a bounded linear operator.

$\square$

Problem 6

Proof. (a) T is a contraction mapping. Indeed, let $c = \|T\|_\infty < 1$. Hence, for any $x_1, x_2 \in X(x_1 \neq x_2)$,

$$\|T(x_1) - T(x_2)\| = \|T(x_1 - x_2)\| \leqslant \|T\|_\infty \|x_1 - x_2\| = c\|x_1 - x_2\|.$$

where $0 < c < 1$. Hence, there exists a unique $x_0 \in X$ such that $T(x_0) = x_0$. And this is equivalent to say that linear operator (which is easy to check) $\mathcal{N}(I - T) = \{x_0\}$. However, $\{0\} \in \mathcal{N}(I - T)$ always holds. Thus, $x_0 = 0$.

Next we need to show that $I - T$ is a one-to-one mapping. Suppose not, then there exists $y \in X$ and distinct $y_1, y_2 \in X$ such that $(I - T)(y_1) = (I - T)(y_2) = y$. By linearity, $(I - T)(y_1 - y_2) = 0$ and it yields that $y_1 = y_2$, which is a contradiction.

What's more, $I - T$ is surjective. Indeed, $I - T$ maps from X to X. For any element $x \in X$, $\exists$ unique $y \in X$ such that $y = (I - T)(x)$. Hence, volume of the range of $I - T$ equals to the volume of domain. i.e., $|\mathcal{R}(I - T)| = |\mathcal{D}(I - T)| = |X|$. Also, $\mathcal{R}(I - T) \subset X$, which yields that $\mathcal{R}(I - T) = X$. In conclusion, $I - T$ is bijective.

(b) Let $S = \sum_{n=0}^{\infty} T^n$ and consider $\|S\|_\infty$.

$$\begin{aligned}
\|S\|_\infty &\leqslant \|\lim_{m \to +\infty} \sum_{n=0}^{m} T^n\|_\infty \\
&= \lim_{m \to +\infty} \|\sum_{n=0}^{m} T^n\|_\infty \\
&\leqslant \lim_{m \to +\infty} \sum_{n=0}^{m} \|T^n\|_\infty \\
&\leqslant \lim_{m \to +\infty} \sum_{n=0}^{m} \|T\|_\infty^n \\
&= \lim_{m \to +\infty} \sum_{n=0}^{m} c^n \\
&= \frac{1}{1 - c} < +\infty.
\end{aligned}$$

We use triangle inequality above. Also, the limit and norm can exchange due to the continuity of norm.

Hence, S is bounded in $\|\cdot\|_\infty$. And it is obvious that S is a linear operator, so $S \in (B(X, X), \|\cdot\|_\infty)$.

(c) It is enough to check that $S \circ (I - T) = (I - T) \circ S = I$.

$$\begin{aligned}
S \circ (I - T) = S - S \circ T &= \sum_{n=0}^{\infty} T^n - \sum_{n=1}^{\infty} T^n \\
&= T^0 = I
\end{aligned}$$

$$\begin{aligned}(I-T)\circ S = S - T\circ S &= \sum_{n=0}^{\infty} T^n - \sum_{n=1}^{\infty} T^n \\ &= T^0 = I\end{aligned}$$

Hence, $S = (I-T)^{-1}$.

$\square$

Problem 7

Proof. For each $n \in \mathbb{N}$, $\|T^n\|_\infty \leqslant \|T\|_\infty^n$. Since T is a bounded linear operator, $\|T\|_\infty < +\infty$. By triangle inequality and Taylor theorem,

$$\|S\|_\infty \leqslant \sum_{n=0}^{+\infty} \frac{\|T^n\|_\infty}{n!} \leqslant \sum_{n=0}^{+\infty} \frac{\|T\|_\infty^n}{n!} = e^{\|T\|_\infty}.$$

$\square$