MA 515 Homework 2

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Problem 1

(a) *Proof.* If $\exists \mathcal{O} \in Y$ such that $\mathcal{O}$ is open in Y and $E = \mathcal{O} \cap X$, then $\forall x \in E$, x must be in $\mathcal{O}$. Since $\mathcal{O}$ is open, $\exists r_0$ such that $B_Y(x, r_0) \subset \mathcal{O}$. i.e, $\{y \in Y | d(x, y) < r_0\} \subset \mathcal{O}$. With the fact that $X \subset Y$, $B_X(x, r_0) := \{y \in X | d(x, y) < r_0\} \subset \mathcal{O}$. Hence, E is open in X.

Conversely, if E is open in X, then for all $x \in E$, there exists $r_x > 0$ such that $\{y \in X | d(x, y) < r_x\} \subset E$. Since $X \subset Y$, let $\mathcal{O} := \{y \in Y | d(x, y) < r_x\}$ and it is clear that $\mathcal{O} \cap X \subset E$. What's more, $\forall x \in E$, x is also in $\mathcal{O} \cap X$. Hence, $E = \mathcal{O} \cap X$.

□

(b) *Proof.* If E is closed in X, then $X \setminus E$ is open in X. Use the result from (a), it is equivalent to say that there exists an open set $\mathcal{O} \subset Y$ such that $X \setminus E = \mathcal{O} \cap X$. And this implies,

$$E = X \setminus (O \cap X) = X \setminus \mathcal{O} \cup \phi = X \setminus \mathcal{O} = (X \setminus \mathcal{O}) \cap Y = (Y \setminus \mathcal{O}) \cap X.$$

Let $F = Y \setminus \mathcal{O}$ and F is closed in Y.

□

Problem 2

(a) Part I *Proof.* We want to show $\overline{U \cup V} \subset \overline{U} \cup \overline{V}$. Take $x \in \overline{U}$, by definition of closure, $\forall \epsilon > 0$, $\exists y \in U \subset (U \cup V)$ such that $d(x, y) < \epsilon$. This implies that $x \in \overline{U \cup V}$. Hence, $\overline{U} \subset \overline{U \cup V}$. Similarly, $\overline{V} \subset \overline{U \cup V}$. In all, $\overline{U} \cup \overline{V} \subset \overline{U \cup V}$.

For the other direction, we prove it by contradiction. Suppose $\exists y \in \overline{U \cup V}$ such that $y \notin \overline{U} \cup \overline{V}$. Then there exists $\epsilon > 0$, such that, $d(x, y) \geqslant \epsilon$ and $d(x, z) \geqslant \epsilon$ for all $x \in U, z \in V$. i.e, $\forall w in U \cup V$, $d(w, y) > \epsilon$, which is $y \notin \overline{U \cup V}$ (contradiction).

In conclusion, $\overline{U \cup V} = \overline{U} \cup \overline{V}$.

□

(a) Part II *Proof.* We want to show $\overline{U \cap V} \subset \overline{U} \cap \overline{V}$. If $x \in \overline{U \cap V}$, $\forall \epsilon > 0$, $\exists y \in U \cap V$ such that $d(x, y) < \epsilon$. Since $yy \in U \cap V \Rightarrow y \in U, y \in V$, we know that $x \in \overline{U}, x \in \overline{V}$. Hence, $x \in \overline{U} \cap \overline{V}$, i.e, $\overline{U \cap V} \subset \overline{U} \cap \overline{V}$.

Conversely, we prove it by contradiction. Suppose there exists $x \in \overline{U} \cap \overline{V}$, such that $x \notin \overline{U \cap V}$. Then, there exists $\epsilon > 0$ such that $\forall z \in U \cap V$, $d(x, z) \geqslant \epsilon$. Since $z \in U \cap V \Rightarrow z \in \overline{U}$, it implies that $x \in \overline{U}$, which leads to a contradiction.

In conclusion, $\overline{U \cap V} = \overline{U} \cap \overline{V}$.

□

(b) *Proof.* $\forall x \in \overline{U}$, there exists a sequence $\{x_n\}$ in U such that $x_n \to x$. Since $U \subset V$, then x_n also converges in $\overline{V}$. Hence, $x \in \overline{V}$. This proves that $\overline{U} \subset \overline{V}$.

□

Problem 3

Proof.

Step 1 We would like to show that if $x \in K$, then $d_K(x) = 0$. By definition, $\forall x \in K$, $d_K(x) = \inf_{w \in K} d(x, w)$. Since $d(x, w) \geqslant 0$ and $d(x, w) = 0$ when $w = x$. Hence, $d_K(x) = \inf_{w \in K} d(x, w) = d(x, x) = 0$.

Step 2 Conversely, let's prove it by contradiction. When $d_K(x) = 0$, suppose $x \notin K$, which is to say that $x \in K^c$. Since K is closed, K^c is open. Then $\exists \delta > 0$ such that $B(x, \delta) \in K^c$, which implies $B(x, \delta) \cap = \phi$. Thus, $\forall w \in K$, $d(x, w) > \delta > 0$, and it implies $\inf_{w \in K} d(x, w) \geqslant \delta > 0$. This leads to a contradiction to the assumption.

□

Problem 4

Proof. Consider f as a continuous map from X to Y. $\forall F \subset Y$ closed, we have $Y \setminus F \subset Y$ is open in Y. Then $f^{-1}(Y \setminus F) = f^{-1}(Y) \setminus f^{-1}(F) = X \setminus f^{-1}(F)$ is open in X. This implies that $f^{-1}(F)$ is closed in X.

Conversely, $\forall E \subset Y$ open subset in Y, $Y \setminus E$ is closed in Y.

$$f^{-1}(E) = f^{-1}(Y \setminus (Y \setminus E)) = f^{-1}(Y) \setminus f^{-1}(Y \setminus E) = X \setminus f^{-1}(Y \setminus E).$$

Since $f^{-1}(Y \setminus E)$ is closed in X, $f^{-1}(E)$ is open in X. Hence, the preimage of any open set is also open and this satisfies the definition of continuous function. Thus, f is continuous.

□

Problem 5

Proof. For one direction, if f is not continuous, then by definition, there exists $\epsilon_0 > 0$, for each n, $\exists x_n \in X$, such that $|x_n - x| < 1/n$, but $\sigma(f(x_n), f(x)) \geqslant \epsilon_0$. And this implies a contradiction, for then $x_n \to x$ but $f(x_n)$ doesn't converge to $f(x)$.

For the other direction, we know f is continuous, i.e., $\forall \epsilon > 0$, $\exists \delta > 0$, such that $\sigma(f(y) - f(x)) < \epsilon$ holds for all $d(x, y) < \delta$. With $x_n \to x$, there exists integer $N_\delta > 0$, such that $d(x_n, x) < \delta$ holds for all $n > N_\delta$.

Hence, $\forall \epsilon > 0$, $\exists N_\delta > 0$ such that $\sigma(f(x_n) - f(x)) < \epsilon$ holds for all $n > N_\delta$. And this is equivalent to $f(x_n) \to f(x)$.

□

Problem 6

Proof. Prove by contradiction. Suppose a Cauchy sequence $\{x_n\}$ does not converges to x, though it has subsequence $\{x_{n_k}\}$ that converges to $x \in X$. Then $\exists \epsilon > 0$, for any $N > 0$, $d(x_n, x) \geqslant \epsilon$ for some $n > N$. Since $\{x_{n_k}\}$ converges to x, there exists $N_1 > 0$, such that $d(x_{n_k}, x) < \epsilon/2$, $\forall n_k > N_1$. Also, $\{x_n\}$ is Cauchy and it leads to $\exists N_2 > 0$, such that $d(x_m, x_n) < \epsilon/2$, $\forall m, n > N_2$.

Taking $N = \max\{N_1, N_2\}$ and for some $n, n_k > N$, we use triangle inequality and get,

$$\epsilon \leqslant d(x_n, x) \leqslant d(x_n, x_{n_k}) + d(x_{n_k}, x) < \epsilon/2 + \epsilon/2 = \epsilon.$$

which is obviously a contradiction.

□

Problem 7

Proof. To prove l^2 is complete, it is enough to show that all Cauchy sequences converge in l^2. We will show this step by step. Pick a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \subset l^2$, we construct a point x, and we need to show that $x \in l^2$ and $x_n \to x$.

1. Take a Cauchy sequence $\{x^n\}_{n\in\mathbb{N}} \subset l^2$, where the i^{th} element is $x^i := (x_1^i, x_2^i, \dots)$. Since $\{x^n\}$ is Cauchy, $\forall \epsilon > 0$, $\exists N > 0$, such that

$$|x_i^n - x_i^m| <= \left(\sum_{j=1}^{\infty} |x_j^n - x_j^m|^2 \right)^{1/2} < \epsilon. \quad \forall n, m > N, \forall i$$

Hence, $\{x_i^n\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$, which is a complete metric space. Hence, $x_i^n \to x_i$, $\forall i \geqslant 1$.

Consider first N entries of the point in l^2. Denote $y_N^i := (x_1^i, x_2^i, \dots, x_N^i)$ and $y_N = (x_1, x_2, \dots, x_N)$. And it is clear that $\{y_N^n\}_{n\in\mathbb{N}}$ converges to y_N. This is obvious since this is in the finite dimensional case. $\forall 1 \leqslant i \leqslant N$, there exists N_i such that $|x_i^n - x_i| < \epsilon/\sqrt{N}$, $\forall n > N_i$. Take $\overline{N} = \max_{1\leqslant i\leqslant N}\{N_i\}$, we have

$$\left(\sum_{j=1}^{N} |x_j^n - x_j^m|^2 \right)^{1/2} < \epsilon.$$

Notice that N is arbitrarily chosen, so we may take limit of N on both sides of the inequality above and get

$$d(x^n, x) = \left(\sum_{j=1}^{\infty} |x_j^n - x_j^m|^2 \right)^{1/2} \leqslant \epsilon.$$

And this proves that $x^n \to x$ in l^2.

2. Next we need to show that $x \in l^2$. This is proved by the following

$$\sum_{i=1}^{\infty} |x_i|^2 = \sum_{i=1}^{\infty} \lim_{n\to\infty} |x_i^n|^2 = \lim_{n\to\infty} \sum_{i=1}^{\infty} |x_i^n|^2 < +\infty.$$

Infinite summation and limit can exchange due to dominate convergence theorem(DCT). Hence, $x \in l^2$.

In conclusion, $\{x^n\}$ converges to x in l^2, which proves that l^2 is complete.

□

Problem 8

Proof. Take a Cauchy sequence $\{x_n\}$ on $(X, \tilde{d})$. Then $\forall \epsilon > 0$, $\exists N > 0$, such that $\tilde{d}(x_m, x_n) < \epsilon$, for all $m, n > N$. Since $\tilde{d}(x, y) = d(x, y)/(1 + d(x, y))$, we know $d(x, y) < \epsilon/(1 - \epsilon)$. It is clear that $\epsilon/(1-\epsilon) \to 0$ as $\epsilon \to 0$. Hence, $\{x_n\}$ is also a Cauchy sequence on (X, d). With the fact that (X, d) is complete, $\{x_n\}$ is convergent and it leads to the completeness of $(x, \tilde{d})$.

□

Problem 9

Proof. Take a Cauchy sequence $\{z_n\}$ in (Y, d_Y). Since (Y, d_Y) is complete, we know that $\forall \epsilon > 0$, $\exists N$ such that $d_Y(z_m, z_n) < \epsilon$, $\forall m, n > N$.

With the fact that T is isometric, T is automatically injective, otherwise two distinct points will be mapped to the same point and it leads to a contradiction since the distance between two same points is 0. Also, $T(X) = Y$, i.e. T is surjective. Hence, T is a bijection from X to Y. Therefore there exists a inverse mapping of T such that $\forall y \in Y$, $\exists x \in X$ such that $T^{-1}(y) = x$. Apply this on the Cauchy sequence, $\exists \{x_n\}$ such that $T(x_n) = z_n$, for all $n \in \mathbb{N}$. Hence,

$$d_Y(z_m, z_n) < \epsilon \Leftrightarrow d_Y(T(x_m), T(x_n)) < \epsilon \Leftrightarrow d_X(x_m, x_n) < \epsilon.$$

And this implies that $\{x_n\}$ is Cauchy on (X, d_X) and so it converges. Let $x_n \to x \in (x, d_X)$, i.e, $\forall \epsilon > 0$, $\exists N$ such that $d_X(x_n, x) = d_Y(T(x_n), T(x)) < \epsilon$ for all $n > N$. Hence, $\{T(x_n)\} = \{z_n\}$ converges to $T(x)$ in (Y, d_Y). Thus, any Cauchy sequence on (Y, d_Y) converges in itself, which leads to the completeness of (Y, d_Y).

□

Problem 10

I didn't figure the answer by myself. Actually, I discussed this problem with other students in class and got the ideas from them.

Proof. For l^∞, the unit ball $B(0,1)$ is not totally bounded. Let $\epsilon = 1/4$ and $x_1, x_2, \ldots, x_n \in B(0,1)$ be arbitrarily finitely many points. Then define $v = \{v_n\}_{n\in\mathbb{N}}$ as:

$$v_i \in (-1,1) \setminus (x_i^{(i)} - 1/4, x_i^{(i)} + 1/4).$$

where $x_i^{(i)}$ is ith entry of x_i, $i \geqslant 1$.
Hence,

$$\sup_{n\geqslant 1} |v_n - x_j^{(n)}| \geqslant |v_j - x_j^{(j)}| \geqslant 1/4 \qquad \forall j \in \mathbb{N}.$$

Hence, $v \notin \cup_{i=1}^n B(x_i, 1/4)$.
In conclusion, $B(0,1)$ is not totally bounded.

□

Problem 11

Proof.

1. Step 1

   We want to show that if E is totally bounded, then $\bar{E}$ is totally bounded. By definition, $\forall \epsilon > 0$, there exists finitely many points $a_1, a_2, \ldots, a_{N_\epsilon} \in E$ such that $E \subset \cup_{i=1}^{N_\epsilon} B(a_i, \epsilon/2)$. Since E is dense in $\bar{E}$, $\forall x \in \bar{E}$, $\exists y \in E$ such that $d(x,y) < \epsilon/2$. This is equivalent that $\exists a \in \{a_1, a_2, \ldots, a_{N_\epsilon}\}$ such that $\exists y \in B(a, \epsilon/2)$, and $d(x,y) < \epsilon/2$. With triangle inequality,

   $$d(a,x) \leqslant d(a,y) + d(y,x) < \epsilon/2 + \epsilon/2 = \epsilon.$$

   Hence, x is in $B(a, \epsilon)$, which leads to $x \in \cup_{i=1}^{N_\epsilon} B(a_i, \epsilon/2)$. Since x is arbitrarily chosen from $\bar{E}$, we conclude that $\bar{E} \subset \cup_{i=1}^{N_\epsilon} B(a_i, \epsilon/2)$. i.e, $\bar{E}$ is also totally bounded.

2. Step 2

   Conversely, let $\bar{E}$ is totally bounded, then $\forall \epsilon > 0$, $\exists a_1, \ldots, a_{N_\epsilon}$ such that $\bar{E} \subset \cup_{i=1}^{N_\epsilon} B(a_i, \epsilon/2)$. With the fact that E is dense in $\bar{E}$, for any a_i, there exists $b_i \in E$ such that $d(b_i, a_i) < \epsilon$. Hence, by triangle inequality, $\forall y \in B(a_i, \epsilon)$,

$$d(y, b_i) \leqslant d(y, a_i) + d(a_i, b_i) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence, $y \in B(b_i, \epsilon)$, which implies $\bar{E} \in \cup_{i=1}^{N_\epsilon} B(b_i, \epsilon)$. Since $E \subset \bar{E}$ and $b_i \in E, \forall i$, we conclude that $E \subset \cup_{i=1}^{N_\epsilon} B(b_i, \epsilon)$, which is equivalent to that E is totally bounded.

□

Problem 12

Proof. Since K is compact and f is continuous, then $f(K)$ is compact on $\mathbb{R}$. This implies that $f(K)$ is closed and bounded and the existence of $\max f(K)$ in $f(K)$. Let $f_{\max} = \sup f(K) = \max f(K)$. Since $f_{\max} \in f(K)$, then there exist $x_{\max} \in K$ such that $f_{\max} = f(x_{\max})$.

□

Problem 13

Proof. Define $V = \overline{O} \setminus O$ and let $\epsilon := 1/2 \inf_{x \in K} d_Y(x)$. Since K is compact, it is totally bounded. This is to say that $\exists a_1, a_2, \ldots, a_{N_\epsilon} \in K$ such that $K \subset \cup_{i=1}^{N_\epsilon} B(a_i, \epsilon)$.

Denote $U = \cup_{i=1}^{N_\epsilon} B(a_i, \epsilon)$ and U is open. We need to show $\overline{U} \subset O$. This is true since for any $a_i \in \{a_1, \ldots, a_{N_\epsilon}\}$, $\overline{B}(a_i, \epsilon) \subset O$. With the results from problem 2, $\overline{U} \subset O$ is proved.

In conclusion, $K \subset U \subset \overline{U} \subset O$.

□