Exam 1 Solutions

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Problem 1

(a) False. Counterexample:

$S_1 = \{x = (x_1, x_2) \in \mathbb{R}^2 | x_2 = 0, x_1 \geqslant 0\}$ and $S_2 = \{x = (x_1, x_2) \in \mathbb{R}^2 | x_1 = 0, x_2 \geqslant 0\}$.

Both S_1 and S_2 are convex but $S_1 \cup S_2$ is not convex.

(b) True. $\text{int}(S)$ is open and the interior of an open set is itself.

(c) False. Counterexample:

Let $P = \mathbb{R}^2_+$, i.e. the first quadrant. It is a unbounded polyhedron, but it is not affine.

(d) False. We have shown in homework that the set of optimal solutions P_0 is a convex set. So if P_0 has two points, then any convex combination of them will be in P_0. Hence it won't have exactly two points in it.

(e) True. The objective function $c^T x$ is countinuous on P, which is a nonempty closed bounded set. So P is a nonempty compact set and $c^T x$ will reach its maximum and minimum. This is supported by Weierstrass theorem.

(f) True. Since $n = m + 1$, the solution space of $Ax = b$ is of Rank(1), which is a one-dimensional subspace. Hence, it has at most two vertices. In conclusion, there are at most two BFS of (LP).

(g) False. Consider an LP problem with a constant objective function. Then its optimal solutions are basic solutions but not basic.

For example,

$$\begin{aligned} \text{Minimize} \quad & 0 \\ \text{Subject to} \quad & x_1 + x_2 = 1 \\ & x_1, x_2 \geqslant 0. \end{aligned}$$

$m = 1$, but $x = (1/2, 1/2)^T$ is an optimal solution.

(h) False. A BS has $n - m$ nonbasic variables so it has at least $n - m$ zero components. Hence, a degenerate BS has more than $n - m$ zero components.

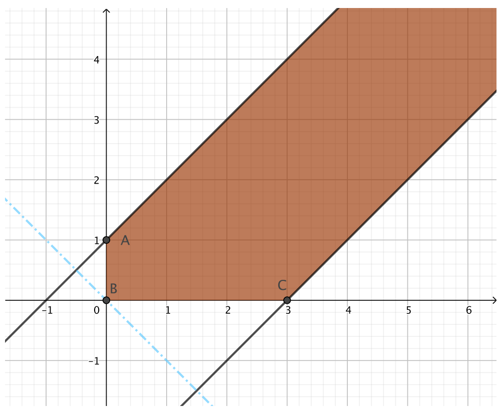
Problem 2

(a) The problem can be reformed as the following,

$$\begin{aligned}
\text{Minimize} \quad & -x_1 - x_2 \\
\text{Subject to} \quad & x_1 - x_2 \leqslant 3 \\
& x_1 - x_2 \geqslant -1 \\
& x_1, x_2 \geqslant 0.
\end{aligned}$$

We plot the figure.

Figure 1: Problem 2



From the figure we see that the problem is unbounded. This is to say that the objective value will goes to $-\infty$.

(b) There are 5 BS, 3 BFS(with '*').

$$\begin{gathered}
[-1, 0, 4, 0]^T \\
[3, 0, 0, 4]^T \; * \\
[0, 1, 4, 0]^T \; * \\
[0, -3, 0, 4]^T \\
[0, 0, 3, 1]^T \; *
\end{gathered}$$

(c) Start from BFS $x = [0, 1, 4, 0]^T$. The basic matrix is $B = \begin{bmatrix} A2 & A_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$ and so $B = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$. Hence the fundamental matrix

$$M = \begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 0 & -1 & 1 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Compute reduced cost

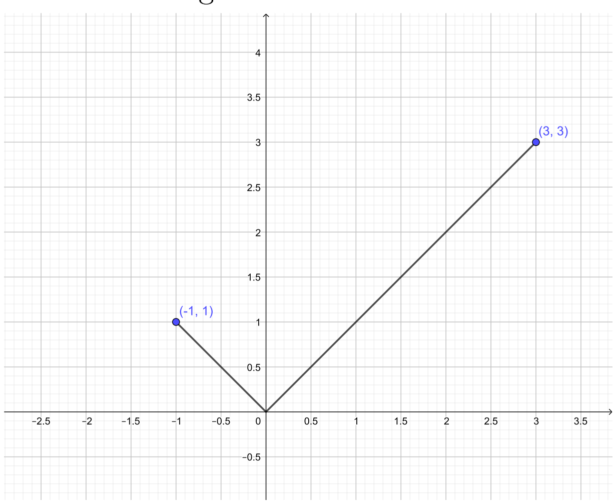
$$r_1 = c^T \mathbf{d}^1 = [-1, 0, -1, 0] \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = -2 < 0, \quad r_2 = c^T \mathbf{d}^4 = [-1, 0, -1, 0] \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = 1 > 0.$$

Hence we take $\mathbf{d}^1$. However, since $\mathbf{d}^1 > 0$ and $[B|N]d = 0$ (this is easy to check). This is to say that $\mathbf{d}^1$ is a extremal direction. Thus, $\forall \alpha > 0$ and $x + \alpha \mathbf{d}^1$ is feasible and $c^T(x + \alpha \mathbf{d}^1) \to -\infty$, as $\alpha \to +\infty$. Hence, the LP problem is unbounded.

Problem 3

(a) We plot the figure and it is clear that the optimal value $z^* = 3$ is attained at $x^* = 3$.

Figure 2: Problem 3



(b) Use the similar idea as in homework 2, let $x = x^+ - x^-$. The Standard LP problem is

$$\begin{aligned} \text{Minimize} \quad & -x^+ - x^- \\ \text{Subject to} \quad & -x^+ + x^- + a_1 \qquad = 1 \\ & x^+ - x^- + \qquad\quad a_2 = 3 \\ & x^+, x^-, a_1, a_2 \geqslant 0. \end{aligned}$$

(Actually, this problem is not equivalent to the original one since it is lacked of the cross term $x^+ \cdot x^- = 0$).

(c) Let's try to use Revised Simplex to solve it. Start from point $x = (x^+, x^-, a_1, a_2)^T = (0, 1, 0, 4)^T$. It is clear that our starting point x is a BFS and $B = [A_2, A_4] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.

Then $N = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ and The fundamental matrix $M = \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Compute

$$M^{-1} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{d}^1 = [1, 1, 0, 0]^T$ and $\mathbf{d}^3 = [0, -1, 1, -1]^T$. The reduced costs are $r^1 = c^T \mathbf{d}^1 = [-1, -1, 0, 0] \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = -2 < 0$, $r^3 = c^T \mathbf{d}^3 = [-1, -1, 0, 0] \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 1 > 0$.

Take $\mathbf{d}^1$. Since $\mathbf{d}^1 > 0$ and $A\mathbf{d}^1 = 0$, we know that $\mathbf{d}^1$ is an extremal direction and yields a contradiction that problem is unbounded.

In conclusion, the Revised Simplex method is not a good choice in solving this problem.