MA 515 Homework 2

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Problem 1

(a) *Proof.* If $\exists \mathcal{O} \in Y$ such that $\mathcal{O}$ is open in Y and $E = \mathcal{O} \cap X$, then $\forall x \in E$, x must be in $\mathcal{O}$. Since $\mathcal{O}$ is open, $\exists r_0$ such that $B_Y(x, r_0) \subset \mathcal{O}$. i.e, $\{y \in Y | d(x,y) < r_0\} \subset \mathcal{O}$. With the fact that $X \subset Y$, $B_X(x, r_0) := \{y \in X | d(x,y) < r_0\} \subset \mathcal{O}$. Hence, E is open in X.

Conversely, if E is open in X, then for all $x \in E$, there exists $r_x > 0$ such that $\{y \in X | d(x,y) < r_x\} \subset E$. Since $X \subset Y$, let $\mathcal{O} := \{y \in Y | d(x,y) < r_x\}$ and it is clear that $\mathcal{O} \cap X \subset E$. What's more, $\forall x \in E$, x is also in $\mathcal{O} \cap X$. Hence, $E = \mathcal{O} \cap X$.

□

(b) *Proof.* If E is closed in X, then $X \setminus E$ is open in X. Use the result from (a), it is equivalent to say that there exists an open set $\mathcal{O} \subset Y$ such that $X \setminus E = \mathcal{O} \cap X$. And this implies,

$$E = X \setminus (O \cap X) = X \setminus \mathcal{O} \cup \phi = X \setminus \mathcal{O} = (X \setminus \mathcal{O}) \cap Y = (Y \setminus \mathcal{O}) \cap X.$$

Let $F = Y \setminus \mathcal{O}$ and F is closed in Y.

□

Problem 2

(a) We want to show $\overline{U \cup V} \subset \overline{U} \cup \overline{V}$. Take $x \in \overline{U}$, by definition of closure, $\forall \epsilon > 0$, $\exists y \in U \subset (U \cup V)$ such that $d(x,y) < \epsilon$. This implies that $x \in \overline{U \cup V}$. Hence, $\overline{U} \subset \overline{U \cup V}$. Similarly, $\overline{V} \subset \overline{U \cup V}$. In all, $\overline{U} \cup \overline{V} \subset \overline{U \cup V}$.

For the other direction, we prove it by contradiction. Suppose $\exists y \in \overline{U \cup V}$ such that $y \notin \overline{U} \cup \overline{V}$. Then there exists $\epsilon > 0$, such that, $d(x,y) \geqslant \epsilon$ and $d(x,z) \geqslant \epsilon$ for all $x \in U, z \in V$. i.e, $\forall w in U \cup V$, $d(w,y) > \epsilon$, which is $y \notin \overline{U \cup V}$ (contradiction).

In conclusion, $\overline{U \cup V} = \overline{U} \cup \overline{V}$.

(b) We want to show $\overline{U \cap V} \subset \overline{U} \cap \overline{V}$. If $x \in \overline{U \cap V}$, $\forall \epsilon > 0$, $\exists y \in U \cap V$ such that $d(x,y) < \epsilon$. Since $yy \in U \cap V \Rightarrow y \in U, y \in V$, we know that $x \in \overline{U}, x \in \overline{V}$. Hence, $x \in \overline{U} \cap \overline{V}$, i.e, $\overline{U \cap V} \subset \overline{U} \cap \overline{V}$.

Conversely, we prove it by contradiction. Suppose there exists $x \in \overline{U} \cap \overline{V}$, such that $x \notin \overline{U \cap V}$. Then, there exists $\epsilon > 0$ such that $\forall z \in U \cap V$, $d(x, z) \geqslant \epsilon$. Since $z \in U \cap V \Rightarrow z \in \overline{U}$, it implies that $x \in \overline{U}$, which leads to a contradiction.

In conclusion, $\overline{U \cap V} = \overline{U} \cap \overline{V}$.

Problem 5

Proof. For one direction, if f is not continuous, then by definition, there exists $\epsilon_0 > 0$, for each n, $\exists x_n \in X$, such that $|x_n - x| < 1/n$, but $\sigma(f(x_n), f(x)) \geqslant \epsilon_0$. And this implies a contradiction, for then $x_n \to x$ but $f(x_n)$ doesn't converge to $f(x)$.

For the other direction, we know f is continuous, i.e., $\forall \epsilon > 0$, $\exists \delta > 0$, such that $\sigma(f(y) - f(x)) < \epsilon$ holds for all $d(x, y) < \delta$. With $x_n \to x$, there exists integer $N_\delta > 0$, such that $d(x_n, x) < \delta$ holds for all $n > N_\delta$.

Hence, $\forall \epsilon > 0$, $\exists N_\delta > 0$ such that $\sigma(f(x_n) - f(x)) < \epsilon$ holds for all $n > N_\delta$. And this is equivalent to $f(x_n) \to f(x)$.

□

Problem 11

Proof.

1. Step 1

   We want to show that if E is totally bounded, then $\bar{E}$ is totally bounded. By definition, $\forall \epsilon > 0$, there exists finitely many points $a_1, a_2, \ldots, a_{N_\epsilon} \in E$ such that $E \subset \cup_{i=1}^{N_\epsilon} B(a_i, \epsilon/2)$. Since E is dense in $\bar{E}$, $\forall x \in \bar{E}$, $\exists y \in E$ such that $d(x, y) < \epsilon/2$. This is equivalent that $\exists a \in \{a_1, a_2, \ldots, a_{N_\epsilon}\}$ such that $\exists y \in B(a, \epsilon/2)$, and $d(x, y) < \epsilon/2$. With triangle inequality,

   $$d(a, x) \leqslant d(a, y) + d(y, x) < \epsilon/2 + \epsilon/2 = \epsilon.$$

   Hence, x is in $B(a, \epsilon)$, which leads to $x \in \cup_{i=1}^{N_\epsilon} B(a_i, \epsilon/2)$. Since x is arbitrarily chosen from $\bar{E}$, we conclude that $\bar{E} \subset \cup_{i=1}^{N_\epsilon} B(a_i, \epsilon/2)$. i.e, $\bar{E}$ is also totally bounded.

2. Step 2

   Conversely, let $\bar{E}$ is totally bounded, then $\forall \epsilon > 0$, $\exists a_1, \ldots, a_{N_\epsilon}$ such that $\bar{E} \subset \cup_{i=1}^{N_\epsilon} B(a_i, \epsilon/2)$. With the fact that E is dense in $\bar{E}$, for any a_i, there exists $b_i \in E$ such that $d(b_i, a_i) < \epsilon$. Hence, by triangle inequality, $\forall y \in B(a_i, \epsilon)$,

   $$d(y, b_i) \leqslant d(y, a_i) + d(a_i, b_i) < \epsilon/2 + \epsilon/2 = \epsilon.$$

   Hence, $y \in B(b_i, \epsilon)$, which implies $\bar{E} \in \cup_{i=1}^{N_\epsilon} B(b_i, \epsilon)$. Since $E \subset \bar{E}$ and $b_i \in E, \forall i$, we conclude that $E \subset \cup_{i=1}^{N_\epsilon} B(b_i, \epsilon)$, which is equivalent to that E is totally bounded.

□