MA 515 Homework 6

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Problem 1

Problem 2

Proof. Suppose $x \neq y$. Let $V = span\{x, y\}$ functional $f : V \to \mathbb{R}$ such that $\forall s, t \in \mathbb{R}$,

$$f(sx + ty) = s\|x\| - t\|y\|.$$

Hence, $f(x) = \|x\|$, $f(y) = -\|y\|$ and $f(x) \neq f(y)$. By the theorem, there exists a functional $F : X \to \mathbb{R}$ such that $F = f$ on V and $\|f\|_\infty = \|F\|_\infty$, which is a contradiction. □

Problem 3

Problem 7

Proof.
1. $< x, x >= 1/2(\|x + x\|^2 - \|x\|^2 - \|x\|^2) = \|x\|^2 \geqslant 0$. And $< x, x >= 0$ if and only if $x = 0$.
2. It is also clear that $< x, y >=< y, x >, \forall x, y \in X$.
3. We will show $< x, y + z >=< x, y > + < x, z >$, $\forall x, y, z \in X$. By definition, we know

$$< x, y + z >= \frac{1}{2}(\|x + y + z\|^2 - \|x\|^2 - \|y + z\|^2).$$

and

$$\begin{aligned} \|x + y + z\|^2 =& 2\|x\|^2 + 2\|y + z\|^2 - \|x - y - z\|^2 \\ =& 2\|x + y\|^2 + 2\|z\|^2 - \|x + y - z\|^2 \end{aligned} \tag{1}$$

Also, with Parallelogram theorem, we have

$$\|x - y - z\|^2 + \|x + y - z\|^2 = 2\|x - z\| + 2\|y\|^2.$$

Hence, plug it in (1) and have

$$\|x+y+z\|^2 = \|x\|^2 + \|y+z\|^2 + \|x+y\|^2 + \|z\|^2 - \|x-z\|^2 - \|y\|^2.$$

which implies

$$\begin{aligned} < x, y+z > =& \frac{1}{2}(\|x+y\|^2 - \|x-z\|^2 + \|z\|^2 - \|y\|^2) \\ =& \frac{1}{2}(\|x+y\|^2 - \|y\|^2 - \|x\|^2 - \|x-z\|^2 + \|z\|^2 + \|x\|^2) \\ =& < x, y > + < x, z > \end{aligned}$$

4. We need to show that $< \lambda x, y >= \lambda < x, y >, \forall x, y \in X, \lambda \in \mathbb{R}$.

   To show this, we need a few steps. Firstly, it holds for $\lambda \in \mathbb{N}$ and it can be proved by induction. Also,

$$\begin{aligned} < x, -y > =& \frac{1}{2}(\|x-y\|^2 - \|x\|^2 - \|y\|^2) \\ =& \frac{1}{2}(-\|x+y\|^2 + \|y\|^2 + \|x\|^2) \\ =& - < x, y > \end{aligned}$$

   Hence, it holds for $\lambda = -1$ and so holds for $\lambda \in \mathbb{Z}$.

   Next we will show that it holds for $\lambda \in \mathbb{Q}$. Let $\lambda = p/q, (q \neq 0), p, q \in \mathbb{Z}$. Hence,

$$q < x, \lambda y >= q < x, \frac{p}{q} y >=< x, py >= p < x, y > .$$

   Both sides divided by q and we have $< \lambda x, y >= \lambda < x, y >, \forall \lambda \in \mathbb{Q}$.

   Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\forall \lambda \in \mathbb{R}$, there exists a sequence of rational numbers $\{\lambda_n\}_{n \in \mathbb{N}}$ such that $\lambda_n \to \lambda$. Hence, $< \lambda x, y >= \lambda < x, y >, \forall \lambda \in \mathbb{R}$.

   In conclusion, $< \cdot, \cdot >$ is an inner product.

□

Problem 8

(i) *Proof.* Suppose $\| \cdot \|_1$ is induced by inner product, i.e., $\| \cdot \|_1 = \sqrt{< \cdot, \cdot >}$. However, if so, then $\| \cdot \|$ must satisfy parallelogram identity. For $a = (1,1)^T, b = (-1,2)^T$,

$$2\|a\|_1^2 + 2\|b\|_1^2 = 26 \neq 18 = \|a-b\|_1^2 + \|a+b\|_1^2.$$

This is a contradiction.

□

(ii) Still, it breaks the parallelogram identity.

Let $f(x) = x, g(x) = 2x$. Hence, $\|f\|_\infty = 1, \|g\|_\infty = 2$. But $\|f+g\|_\infty = 3, \|f-g\|_\infty = 1$. So

$$\|f+g\|_\infty + \|f-g\|_\infty \neq 2\|f\|_\infty + 2\|g\|_\infty.$$

Problem 9

Proof. "$\Rightarrow$", proved in class. $\langle x, x_n \rangle$ converges $\langle x, x \rangle = \|x\|^2$. Hence,

$$\lim_{n\to+\infty} \|x_n - x\|^2 = \lim_{n\to+\infty} \langle x_n - x, x_n - x \rangle = \lim_{n\to+\infty} \|x_n\|^2 - 2\langle x, x_n \rangle + \|x\|^2 = 0$$

"$\Leftarrow$". If $x_n \to x$, then by Cauchy-Schwartz inequality,

$$0 \leqslant \lim_{n\to+\infty} |\langle x_n - x, x \rangle| \leqslant \lim_{n\to+\infty} \|x_n - x\|\|x\| = 0.$$

By squeeze theorem, $\lim_{n\to+\infty} \langle x_n - x, x \rangle = 0$. Hence, $x_n \rightharpoonup x$.

Also,

$$\begin{aligned} 0 = \overline{\lim_{n\to+\infty}} \|x_n - x\|^2 = \overline{\lim_{n\to+\infty}} &\langle x_n - x, x_n - x \rangle \\ &= \overline{\lim_{n\to+\infty}} \|x_n\|^2 - 2 \overline{\lim_{n\to+\infty}} \langle x, x_n \rangle + \|x\|^2 \\ &= \overline{\lim_{n\to+\infty}} \|x_n\|^2 - 2 \lim_{n\to+\infty} \langle x, x_n \rangle + \|x\|^2 \\ &= \overline{\lim_{n\to+\infty}} \|x_n\|^2 - \|x\|^2 \end{aligned}$$

Similarly, we have $\underline{\lim}_{n\to+\infty} \|x_n\|^2 - \|x\|^2 = 0$. Hence, $\lim_{n\to+\infty} \|x_n\| = \|x\|$. $\square$

Problem 10

Proof. (i) (Shown in class) Since $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$, for any $x \in \mathcal{H}$, it can be expressed as

$$x = \sum_{i=1}^{\infty} \alpha_i e_i \qquad \alpha_i \in \mathbb{R}.$$

Hence,

$$\langle x, e_i \rangle = \alpha_i \quad \text{and} \quad \langle x, x \rangle = \|x\|^2 = \sum_{i=1}^{\infty} \alpha_i^2 < +\infty.$$

Hence,

$$\lim_{n\to+\infty} \langle x, e_i \rangle = \lim_{n\to+\infty} \alpha_i = 0.$$

i.e., $e_n \rightharpoonup 0$.

(ii) ******

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$\square$