The Multiplicative Weights Update Method:

Applications to Linear Programming and Semidefinite Programming

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Based on paper by Arora, Hazan, and Kale (2012)

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Outline

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1. Algorithm Overview
2. MW Algorithm for Linear Programming
3. MW Algorithm for Semidefinite Programming

Multiplicative Weights (MW) Algorithm Overview

- A decision maker has a set of n possible decisions
- Begin with equal probabilities of choosing each decision
- In each iteration:
  1. Decision maker chooses a decision
  2. Obtains a (possibly negative) payoff
  3. Adjusts the probabilites with a multiplicative factor based on payoff

Notation

- t: iteration number (aka - "round")
- T: Number of iterations in total.
- $p^{(t)} \in \mathbb{R}^n$: probability vector for round t
  $p_i^{(t)}$ is the probability of selecting decision i in round t
- $m^{(t)} \in \mathbb{R}^n$: gain vector "revealed by nature" after the decision in round t is made
  (assume $m_i^{(t)} \in [-1, 1]$, $\forall i$, $\forall t$)

General MW Algorithm

Fix $\eta \leq \frac{1}{2}$. Initialize $w_i^{(0)} = 1, \ \forall i = 1, \ldots, n$.

For $t = 1, \ldots, T$:

1. Choose decision i with probability $p_i^{(t)} = \frac{w_i^{(t)}}{\sum_i w_i^{(t)}}$
2. Observe the payoff $m^{(t)}$
3. Update the weights: for each i,

$$w_i^{(t+1)} = w_i^{(t)} \exp(\eta m_i^{(t)})$$

Performance

- Expected gain is "not too much less" than the gain of the best decisions

$$\sum_{t=1}^{T} m^{(t)} \cdot p^{(t)} \geq \sum_{t=1}^{T} m_i^{(t)} - \eta \sum_{t=1}^{T} (m_i^{(t)})^2 \cdot p^{(t)} - \frac{\ln n}{\eta}$$

$(\forall\, i = 1, \ldots, n)$.

- MW has many applications in machine learning, game theory, online decision making, etc.

Outline

Linear Classifier Problem

MW algorithm can be used to solve a linear classifier problem

- m labeled examples (ℓ_j, a_j), with $\ell_j \in \{-1, 1\}$ and $a_j \in \mathbb{R}^n$
- Want to find a normal vector x to a hyperplane so that

$$\operatorname{sgn}(a_j^T x) = \ell_j \quad \forall j = 1, \ldots, m$$

- Equivalently, we want to find x such that

$$\ell_j a_j^T x \geq 0 \quad \forall j = 1, \ldots, m$$

(Redefine $a_j \to \ell_j a_j$)

MW Algorithm to Solve an LP

MW Algorithm can be used to solve the following LP:

$$\begin{aligned} a_j^T x &\geq 0 \quad \forall j = 1, \ldots, m \\ \mathbb{1}^T x &= 1 \\ x_i &\geq 0 \quad \forall i = 1, \ldots, n \end{aligned}$$

(x is analogous to the distribution p)

MW Algorithm to Solve the Linear Classifier Problem

Define $\rho = \max_j ||a_j||_\infty$, $\eta = \frac{\epsilon}{2\rho}$, $\epsilon > 0$.

Initialize $x_i^{(0)} = \frac{1}{n}, \forall i$

While $\exists\, j$ s.t. $a_j^T x^{(t)} < 0$

1. Find the index j such that $a_j^T x^{(t)} < 0$
2. Obtain payoff $m^{(t)} = \frac{a_j}{\rho}$
3. Update $x^{(t)}$: for each i,

$$x_i^{(t+1)} = x_i^{(t)} \exp(\eta m_i^{(t)})$$

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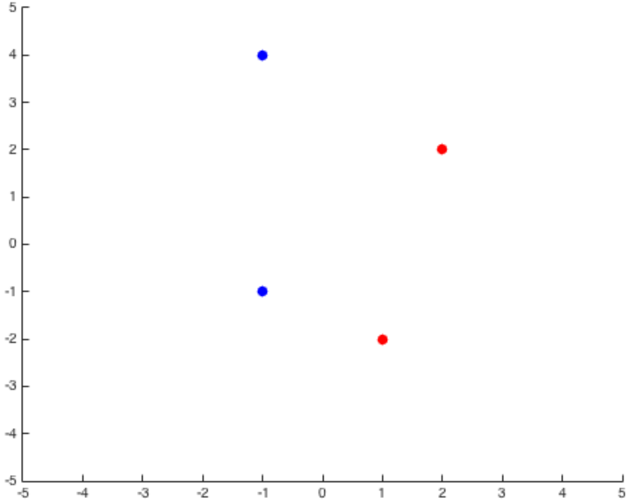
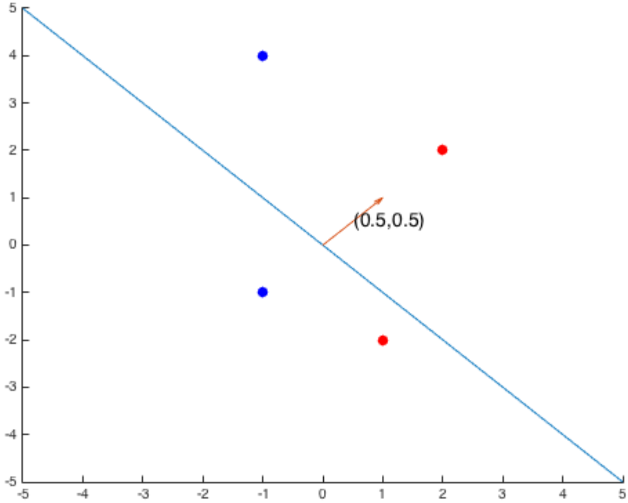
Linear Classifier: Toy Example

Find a separating hyperplane for the following points



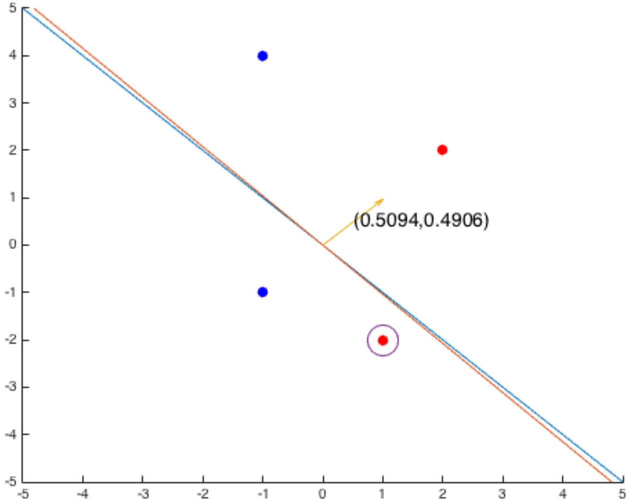
Linear Classifier: Toy Example

Initialize: $x^{(0)} = [\frac{1}{2}, \frac{1}{2}]$.



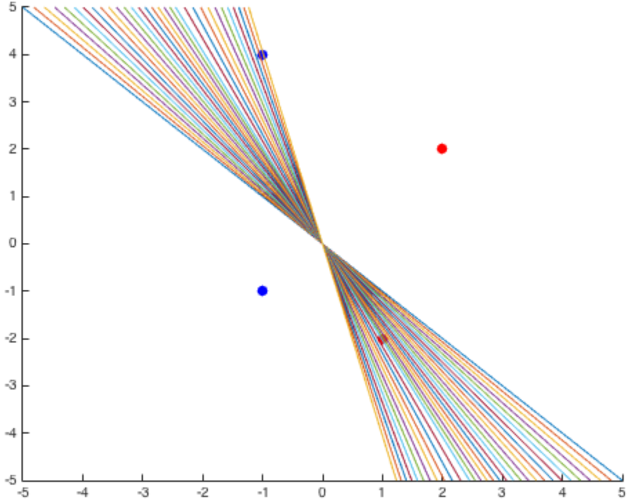
Linear Classifier: Toy Example

Misclassified example yields gain of $m^{(0)} = [\frac{1}{2}, -1]$.



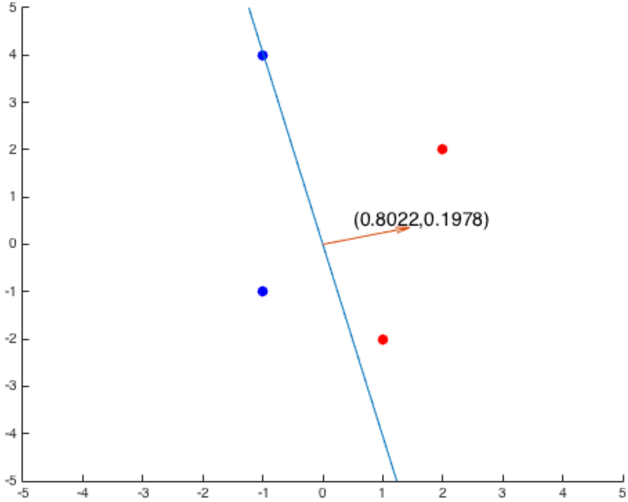
Linear Classifier: Toy Example

Continued iterations:



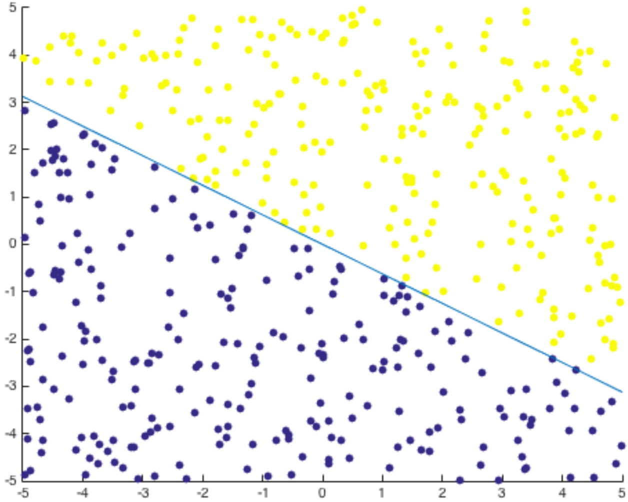
Linear Classifier: Toy Example

Final separating hyperplane, and solution to the LP:



Larger linear classifier problem

500 labeled examples



Outline

Matrix MW Algorithm

Solving an SDP using Multiplicative Weights algorithm:

Recall the LP:

$$\begin{aligned} a_j^T x &\geq 0 \quad \forall j \\ \mathbb{1}^T x &= 1 \\ x_i &\geq 0 \quad \forall i \end{aligned}$$

Now the SDP:

$$\begin{aligned} A_j \bullet X &\geq 0 \quad \forall j = 1, \ldots, m \\ \mathrm{Tr}(X) &= 1 \\ X &\in \mathbb{S}_+^n \end{aligned}$$

Matrix MW Algorithm

Define $\rho = \max_j ||A_j||$, $\eta = -\ln(1-\epsilon), \epsilon > 0$.

Initialize $W^{(0)} = I_n$ and $X^{(0)} = W^{(0)} / Tr(W^{(0)})$.

While $\exists\, j$ s.t. $A_j \bullet X^{(t)} < 0$:

1. Find the index j such that $A_j \bullet X^{(t)} < 0$
2. Obtain the payoff $M^{(t)} = \frac{A_j}{\rho}$
3. Update the weight matrix $W^{(t)}$ and $X^{(t)}$:

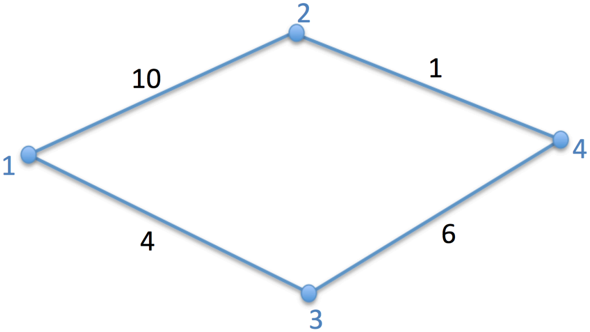
$$W^{(t+1)} = \exp\left(\eta \sum_{\tau=1}^{t} M^{(\tau)}\right)$$

$$X^{(t+1)} = \frac{W^{(t+1)}}{Tr(W^{(t+1)})}$$

Recall the MAXCUT problem

Given a weighted graph $G = (E, V)$, find a partition of the vertices into two vertex sets V_1 and V_2, i.e., $V = V_1 \dot{\cup} V_2$, that maximizes the "cut" edge weights.

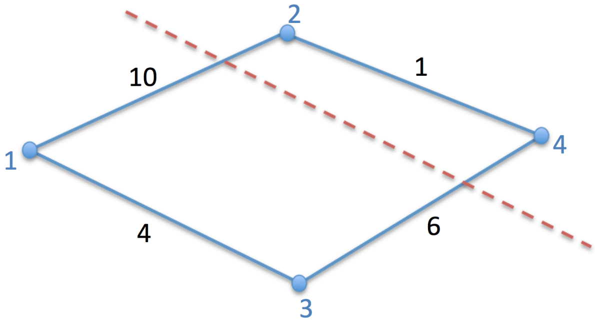
Figure 1: Toy example



Recall the MAXCUT problem

Given a weighted graph $G = (E, V)$, find a partition of the vertices into two vertex sets V_1 and V_2, i.e., $V = V_1 \dot{\cup} V_2$, that maximizes the "cut" edge weights.

Figure 2: Toy example



Recall the MAXCUT problem

For a graph $G = (E, V)$ with $|V| = n$ and $V = V_1 \dot\cup V_2$, the vector $x \in \mathbb{R}^n$ gives the partition:

$$x_i = \begin{cases} 1, & \text{if } v_i \in V_1 \\ -1, & \text{if } v_i \in V_2 \end{cases} \quad i = 1, \dots, n$$

Let $w_{ij} :=$ the weight on edge (i, j)

$$\max_{x \in \{-1,1\}^n} \sum_{(i,j) \in E} w_{ij} \frac{1 - x_i x_j}{2}$$

The MAXCUT relaxation

Introduce the auxiliary matrix variable $X = xx^T$:

$$\max_{x,X} \sum_{(i,j)\in E} w_{ij}\frac{1 - x_i x_j}{2}$$
$$\text{s.t. } X = xx^T$$
$$x_i^2 = 1$$

Then drop the rank 1 constraint for the relaxation:

$$\max_{X} \frac{1}{4} L \bullet X$$
$$\text{s.t. } X_{ii} = 1 \quad \forall\ i = 1 \ldots, n$$
$$X \succcurlyeq 0$$

$$L_{ij} = \begin{cases} \sum_{k\in V} w_{ik}, & i = j \\ -w_{ij}, & i \neq j \end{cases}$$

Reformulating the MAXCUT relaxation

Decision form: Does there exist a b^* such that

$$\begin{aligned} \frac{1}{4} L \bullet X &\geq b^* \\ X_{ii} &= 1 \quad \forall\ i = 1 \ldots, n \\ X &\succcurlyeq 0 \end{aligned}$$

Observe that $X_{ii} = 1 \implies (e_i e_i^T) \bullet X = 1$. Rescale X by $\frac{1}{n}$. Arrive at the formulation:

$$\begin{aligned} \left(\frac{n}{4b^*} L - I\right) \bullet X &\geq 0 \\ \left(n(e_i e_i^T) - I\right) \bullet X &\geq 0 \quad \forall\ i = 1, \ldots, n \\ \mathrm{Tr}(X) &= 1 \\ X &\succcurlyeq 0 \end{aligned}$$

Reformulating the MAXCUT relaxation

Let $A_j = n(e_j e_j^T) - I$, $j = 1, \ldots, n$ and $A_{n+1} = \frac{n}{4b^*} L - I$.

$$\begin{aligned} A_j \bullet X &\geq 0 \quad \forall\, j = 1, \ldots, n \\ A_{n+1} \bullet X &\geq 0 \\ \mathrm{Tr}(X) &= 1 \\ X &\succcurlyeq 0 \end{aligned}$$

Matrix MW algorithm for MAXCUT (Large-Margin)

Define $\rho = \max_j ||A_j||$, $\eta = -\ln(1-\epsilon), \epsilon > 0$, $\delta > 0$.

Initialize $W^{(0)} = I_n$ and $X^{(0)} = W^{(0)}/Tr(W^{(0)})$.

While $\exists\, j$ s.t. $A_j \bullet X^{(t)} < -\delta$:

1. Find the index j such that $A_j \bullet X^{(t)} < -\delta$
2. Obtain the payoff $M^{(t)} = \frac{A_j}{\rho}$
3. Update the weight matrix $W^{(t)}$ and $X^{(t)}$:

$$W^{(t+1)} = \exp\left(\eta \sum_{\tau=1}^{t} M^{(\tau)}\right), \quad X^{(t+1)} = \frac{W^{(t+1)}}{Tr(W^{(t+1)})}$$

Toy MAXCUT problem

Number of nodes: 4. $b^* = 16$.

ϵ, δ	0.1	0.01	0.001	0.0001
T	20	328	3465	34827
$\lvert b^* - b^{(T)}\rvert$	0.4595	0.0482	0.0039	0.0002
$\frac{\lvert b^* - b^{(T)}\rvert}{\lvert b^*\rvert}$	2.88×10^{-2}	3.01×10^{-3}	2.44×10^{-4}	1.25×10^{-5}

Table 1: toy example

MAXCUT problem with 10 vertices

Number of nodes: 10. $b^* = 80$.

ϵ, δ	0.1	0.01	0.001	0.0001
T	19	633	7230	73228
$\lvert b^* - b^{(T)} \rvert$	1.9107	0.1908	0.0204	0.0011
$\frac{\lvert b^* - b^{(T)} \rvert}{\lvert b^* \rvert}$	2.39×10^{-2}	2.39×10^{-3}	2.55×10^{-4}	1.38×10^{-5}

Table 2: 10 nodes example

MAXCUT problem with 100 vertices

Number of nodes: 100. $b^* = 8190$.

ϵ, δ	0.1	0.01	0.001	0.0001
T	2777	49971		
$\lvert b^* - b^{(T)} \rvert$	3.2430	0.7318		
$\frac{\lvert b^* - b^{(T)} \rvert}{\lvert b^* \rvert}$	3.96×10^{-4}	8.94×10^{-5}		

Table 3: 100 nodes example

Conclusions

- The MW algorithm can be used to solve LP and SDP feasibility problems of certain form.
- Use the MW algorithm to solve Linear Classification problems.
- Matrix MW algorithm performance depends on ϵ and δ. (Parameter sensitive).

References

S. Arora, E. Hazan, and S. Kale. The multiplicative weights update method: A meta-algorithm and applications, *Theory of Computing*, 8 (2012), pp 121-164.

S. Arora and S. Kale. A combinatorial, primal-dual approach to semidefinite programs. In *Journal of the ACM*, 63 (2016).