MA 515 Homework 4

Zheming Gao

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Problem 1

Proof. $\{T_n\}_{n\in\mathbb{N}}$ is a sequence of uniformly bounded linear operators and it satisfies

$$\lim_{n\to+\infty} T_n(x) := T(x)$$

for any $x \in X$. Now we want to show T is a bounded linear operator. First, T is linear because the limit operation on T_n preserves the linearity of T_n. Also, $\mathcal{D}(T)$ is X, which yields that T is a linear operator from X to Y.

Next we need to show T is bounded, or more precisely, $||T||_\infty \leqslant M$. Since $||T_n||_\infty < M$, we know for any $||x||_X = 1$, $||T_n(x)||_Y < M$. Hence,

$$\lim_{n\to+\infty} \|T_n(x)\| = ||T(x)|| \leqslant M.$$

which implies $\sup_{||x||_X=1} ||T(x)|| \leqslant M$, i.e., $||T||_\infty \leqslant M$.

$\square$

Problem 2

Proof. First we need to show that Λ is bounded. Take arbitrarily $x = \{x_n\}_{n\in\mathbb{N}} \in \ell^\infty$, and there exists $M > 0$ such that $|x_i| \leqslant M$ for any $i \geqslant 0$. Therefore,

$$\begin{aligned} ||\Lambda(x)||_{\ell^\infty} = ||y||_{\ell^\infty} &= \sup_{i\geqslant 1} |y_i| \\ &= \sup_{i\geqslant 1} \left| \frac{x_1 + \cdots + x_i}{i} \right| \\ &\leqslant \sup_{i\geqslant 1} \left| \frac{\sum_{j=1}^{i} |x_j|}{i} \right| \leqslant M. \end{aligned}$$

Hence, $||\Lambda||_\infty = \sup_{||x||_{\ell^\infty}=1} ||\Lambda(x)||_{\ell^\infty} \leqslant M$.

Next we need to find the value of $||\Lambda||_\infty$. From the definition,

$$||\Lambda||_\infty = \sup_{||x||_{\ell^\infty}=1} ||\Lambda(x)||_{\ell^\infty} = \sup_{||x||_{\ell^\infty}=1} \sup_{i\geqslant 1} \left| \frac{x_1 + \cdots + x_i}{i} \right|.$$

Also, $||x||_{\ell^\infty} = 1$ implies $|x_j| \leqslant 1, \forall j \geqslant 1$. Hence,

$$\sup_{||x||_{\ell^\infty}=1} \sup_{i \geqslant 1} \left| \frac{x_1 + \cdots + x_i}{i} \right| = \sup_{i \geqslant 1} \frac{i \cdot 1}{i} = 1.$$

Hence, $||\Lambda||_\infty = 1$.

□

Problem 3

Proof. It is enough to show that $\mathcal{N}(T) = \{0\}$. If so, then T is invertible and is a linear operator. Indeed, supposed there exists $x \neq 0, x \in \mathcal{N}(T)$, then $0 = \|T(x)\| \geqslant \|x\| > 0$, which yields a contradiction.

Next we need to show T^{-1} is bounded. Since T is surjective and invertible, we know T must be a bijection. If so, for any $y \in Y$, we have $\|y\| \geqslant b\|T^{-1}(y)\|$. Hence,

$$\sup_{\|y\| \neq 0} \frac{\|T^{-1}(y)\|}{\|y\|} \leqslant \frac{1}{b} < +\infty.$$

i.e., T^{-1} is a bounded linear operator.

□

Problem 4

Proof. We want to show $\|T_n(x_n) - T(x)\|_Y \to 0$ as $n \to +\infty$. By triangle inequality,

$$\begin{aligned} \|T_n(x_n) - T(x)\|_Y &\leqslant \|T_n(x_n) - T(x_n)\|_Y + \|T(x_n) - T(x)\|_Y \\ &\leqslant \|T_n - T\|_\infty \|(x_n)\|_X + \|T\|_\infty \|x_n - x\|_X \end{aligned}$$

and we know $\|x_n - x\| \to 0$ as $n \to +\infty$. Hence, $\|x_n\|_X$ is bounded. What's more, $\lim_{n \to +\infty} \|T_n - T\|_\infty = 0$.

In conclusion, $\lim_{n \to +\infty} ||T_n(x_n) - T(x)|| = 0$.

□

Problem 5

Proof. $S \circ T$ is a linear operator. Indeed, the domain of $S \circ T$ is X, which is a subspace of itself. Also, function composition preserves linearity.

Next we need to show $S \circ T$ is bounded. Consider norm $\|S \circ T\|_\infty$,

$$
\begin{aligned}
\|S \circ T\|_\infty = \sup_{\|x\|_X=1} \|(S \circ T)(x)\|_Z &= \sup_{\|x\|_X=1} \|S(T(x))\|_Z \\
&\leqslant \sup_{\|x\|_X=1} \|S\|_\infty \|T(x)\|_Y \\
&= \|S\|_\infty \sup_{\|x\|_X=1} \|T(x)\|_Y \\
&= \|S\|_\infty \|T\|_\infty.
\end{aligned}
$$

Since both S and T are bounded linear operators, $\|S\|_\infty$ and $\|T\|_\infty$ are less than positive infinity and this leads to the conclusion that $\|S \circ T|_\infty < +\infty$.

In conclusion, $S \circ T$ is a bounded linear operator.

□

Problem 6

Proof. (a) T is a contraction mapping. Indeed, let $c = \|T\|_\infty < 1$. Hence, for any $x_1, x_2 \in X(x_1 \neq x_2)$,

$$\|T(x_1) - T(x_2)\| = \|T(x_1 - x_2)\| \leqslant \|T\|_\infty \|x_1 - x_2\| = c\|x_1 - x_2\|.$$

where $0 < c < 1$. Hence, there exists a unique $x_0 \in X$ such that $T(x_0) = x_0$. And this is equivalent to say that linear operator (which is easy to check) $\mathcal{N}(I - T) = \{x_0\}$. However, $\{0\} \in \mathcal{N}(I - T)$ always holds. Thus, $x_0 = 0$.

Next we need to show that $I - T$ is a one-to-one mapping. Suppose not, then there exists $y \in X$ and distinct $y_1, y_2 \in X$ such that $(I - T)(y_1) = (I - T)(y_2) = y$. By linearity, $(I - T)(y_1 - y_2) = 0$ and it yields that $y_1 = y_2$, which is a contradiction.

What's more, $I - T$ is surjective. Indeed, $I - T$ maps from X to X. For any element $x \in X$, $\exists$ unique $y \in X$ such that $y = (I - T)(x)$. Hence, volume of the range of $I - T$ equals to the volume of domain. i.e., $|\mathcal{R}(I - T)| = |\mathcal{D}(I - T)| = |X|$. Also, $\mathcal{R}(I - T) \subset X$, which yields that $\mathcal{R}(I - T) = X$. In conclusion, $I - T$ is bijective.

(b) Let $S = \sum_{n=0}^{\infty} T^n$ and consider $\sum_{n=0}^{\infty} \|T^n\|_\infty$. Since X is a Banach space, by the theorem, $B(X, X)$ is also a Banach space. Hence, absolutely convergence implies convergence. It is enough to show that $\sum_{n=0}^{\infty} \|T^n\|_\infty$ exists.

$$
\begin{aligned}
\lim_{m \to +\infty} \sum_{n=0}^{m} \|T^n\|_\infty &\leqslant \lim_{m \to +\infty} \sum_{n=0}^{m} \|T\|_\infty^n \\
&= \lim_{m \to +\infty} \sum_{n=0}^{m} c^n \\
&= \frac{1}{1 - c} < +\infty.
\end{aligned}
$$

We use triangle inequality above. Also, the limit and norm can exchange due to the continuity of norm. Hence, $\sum_{n=0}^{\infty} \|T^n\|_\infty$ exists.

Hence, $S = \lim_{m\to+\infty} \sum_{n=0}^{m} T^n$ exists and so it is bounded because $\|S\|_\infty = \|\sum_{n=0}^{\infty} T^n\|_\infty \leqslant 1/(1-c)$. S is also linear.

(c) It is enough to check that $S \circ (I-T) = (I-T) \circ S = I$. Since S exists, we have $\lim_{N\to+\infty} T^N = 0$.

$$\begin{aligned} S \circ (I-T) = S - S \circ T &= \lim_{N\to+\infty} \sum_{n=0}^{N} T^n - \sum_{n=1}^{N+1} T^n \\ &= \lim_{N\to+\infty} I - T^{N+1} = I. \end{aligned}$$

$(I-T) \circ S$ is the same. Hence, $S = (I-T)^{-1}$.

□

Problem 7

Proof. For each $n \in \mathbb{N}$, $\|T^n\|_\infty \leqslant \|T\|_\infty^n$. Since T is a bounded linear operator, $\|T\|_\infty < +\infty$. By triangle inequality and Taylor theorem,

$$\|S\|_\infty \leqslant \sum_{n=0}^{+\infty} \frac{\|T^n\|_\infty}{n!} \leqslant \sum_{n=0}^{+\infty} \frac{\|T\|_\infty^n}{n!} = e^{\|T\|_\infty}.$$

□