

# Homework 3 Solutions

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## Problem 1

Since there is an absolute value in the objective function and  $x_3$  is unrestricted, we may divide the feasible domain into two parts:  $x_3 \geq 0$  and  $x_3 < 0$ .

For  $x_3 \geq 0$ ,  $|x_3| = x_3$ . Hence, we may construct a LP problem:

$$\begin{aligned} \text{Maximize} \quad & 3x_1 - 2x_2 + 4x_3 \\ \text{subject to} \quad & -x_1 + 2x_2 \leq -5 \\ & 3x_2 - x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Similarly, for  $x_3 < 0$ ,  $|x_3| = -x_3$ , and we can also construct a LP problem:

$$\begin{aligned} \text{Maximize} \quad & 3x_1 - 2x_2 - 4x_3 \\ \text{subject to} \quad & -x_1 + 2x_2 \leq -5 \\ & 3x_2 - x_3 \geq 6 \\ & x_1, x_2 \geq 0, x_3 \leq 0. \end{aligned}$$

Then convert those two LP problems above into standard forms:

$$\begin{aligned} \text{Minimize} \quad & -3x_1 + 2x_2 - 4x_3 \\ \text{subject to} \quad & -x_1 + 2x_2 + \xi_1 = -5 \\ & 3x_2 - x_3 - \xi_2 = 6 \\ & x_1, x_2, x_3, \xi_1, \xi_2 \geq 0 \end{aligned} \tag{1}$$

and

$$\begin{aligned} \text{Minimize} \quad & -3x_1 + 2x_2 - 4x_3 \\ \text{subject to} \quad & -x_1 + 2x_2 + \xi_1 = -5 \\ & 3x_2 + x_3 - \xi_2 = 6 \\ & x_1, x_2, x_3, \xi_1, \xi_2 \geq 0 \end{aligned} \tag{2}$$

Take the optimal solution as the one that solves (1) or (2) with a smaller optimal value.

## Problem 2.1

Solutions:

1. We need to show that if the feasible domain is bounded, then the LP problem has a bounded optimal value.

*Proof.* Consider the standard form of a LP problem.

$$\begin{aligned} & \text{Minimize} && c^T x \\ & \text{subject to} && Ax = b \\ & && x \geq 0 \end{aligned}$$

where  $A \in \mathbb{R}^{m \times n}$ ,  $x \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$ .

Denote its feasible domain as  $P := \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ . If  $P$  is bounded, then  $\exists M \geq 0$  such that  $\|x\| \leq M$  for all  $x \in P$ .

Recall Cauchy-Schwartz inequality,  $\forall x, y \in \mathbb{R}^n$ ,

$$|x^T y| \leq \|x\| \cdot \|y\|$$

Hence, we have

$$c^T x \geq -\|c\| \cdot \|x\| = \|c\|(-\|x\|) \geq \|c\| \cdot (-M).$$

Since  $c$  is a constant vector,  $M$  exists as a constant, we know  $c^T x$  has a lower bound, which proves that the LP problem is bounded.

□

2. Next we give a counterexample to show that the opposite direction is not true.

Consider the following LP problem:

$$\begin{aligned} & \text{Minimize} && 2x_1 + x_2 \\ & \text{subject to} && x_1 + x_2 \geq 1 \\ & && x_1, x_2 \geq 0 \end{aligned} \tag{3}$$

It is obvious that the feasible domain is not bounded but the problem is bounded.

## Problem 2.2

**ATTENTION:** The definitions of POLYHEDRON and POLYTOPE are on page 16, Chapter 2.

- (a) All of the claims are false. Counterexample is the unit plate ( $S = \{(x, y) | x^2 + y^2 \leq 1\}$ ) on  $\mathbb{R}^2$ .

- (b) (i) is true.

*Proof.*  $\forall x, y \in S$ , their affine combination is also in  $S$ . Hence,  $\forall \alpha \in (0, 1)$ ,  $\alpha x + (1 - \alpha)y \in S$ . Since it is the convex combination of  $x$  and  $y$ , it proves the  $S$  is convex. □

(ii) is false. For example, let  $S = \{(x, y) \in \mathbb{R}^2 | x + y = 1\}$ .  $S$  is a line on  $\mathbb{R}^2$  but it doesn't satisfy the definition of cone.

- (iii) is true.

*Proof.* If  $S$  is affine, then pick  $x_0 \in S$ , construct set  $S_0 := \{x - x_0 | x \in S\}$ . It is clear to see that  $S_0$  is a linear subspace (Use the definition and it is obvious). Then  $S_0$  must be a solution set of a linear system, say  $Ay = 0$  for all  $y \in S_0$ . This is equivalent to say that  $\exists A$  such that  $\forall x \in S$ ,  $Ax = Ax_0$ . Let  $b = Ax_0$ , and  $S$  can be expressed as the intersection of two sets:  $P_1 := \{x | Ax \geq b\}$  and  $P_2 := \{x | Ax \leq b\}$ . Since  $P_1$  and  $P_2$  are both finite intersections of half spaces, we conclude that  $S$  is a finite intersection of half spaces, i.e.,  $S$  is a polyhedron. □

(iv) is false because  $S$  may not be bounded. Use the same counterexample as for (ii).

- (c) All of them are false. A counterexample is  $S = \{(x, y) | x = 0, y \geq 0\} \cup \{(x, y) | y = 0, x \geq 0\}$ . We see that  $S$  is the union of non-negative parts of  $x$  and  $y$  axis on  $\mathbb{R}^2$ . By definition,  $S$  is a cone. But it is neither convex nor affine. Also,  $S$  is not a polyhedron, so it is not a polytope, either.

- (d) If  $S$  is a polyhedron, it is the intersection of finite half spaces. So we may express  $S$  as a set  $\{x \in \mathbb{R}^n | Ax \leq b, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m\}$ , which means that  $S$  is an intersection of  $m$  half spaces.

For (i), it is true and we would like to show that  $S$  is convex.

*Proof.* Take  $x, y \in S$ , and  $\forall \alpha \in (0, 1)$ , the convex combination of  $x$  and  $y$  is in  $S$ . This is true because

$$A(\alpha x + (1 - \alpha)y) = \alpha Ax + (1 - \alpha)Ay \leq \alpha b + (1 - \alpha)b = b.$$

Hence,  $S$  is a convex set.

□

For (ii) and (iii), they are false and we will give a counterexample in  $\mathbb{R}^2$ .

Let  $S := \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$ . It is clear that  $S$  is a polyhedron, but  $S$  is neither affine nor a cone.

For (iv), it is false because we may give a unbounded polyhedron  $S = \{(x, y) | x \geq 0, y \geq 0, x + y \geq 1\}$ . And it is not a polytope.

- (e) If  $S$  is a polytope, it must be a polyhedron. So (i) and (iv) are true.  
(ii) and (iii) are false. We may raise the same counterexample as the first one (the unit square) in part (d).

## Problem 2.3

*Proof.* To prove  $H$  is affine, take  $x, y$  from  $H$  arbitrarily and we need to show the affine combination of  $x$  and  $y$  is also in  $H$ . This is true, since for any  $\alpha_1, \alpha_2 \in \mathbb{R}$  that satisfy  $\alpha_1 + \alpha_2 = 1$ ,

$$a^T(\alpha_1 x + \alpha_2 y) = \alpha_1 a^T x + \alpha_2 a^T y = \alpha_1 \beta + \alpha_2 \beta = \beta.$$

which shows  $\alpha_1 x + \alpha_2 y$  is in  $H$ .

For convexity, it follows from the fact that  $H$  is affine because the convex combination of two points is a special case of their affine combination. □

## Problem 2.4

1. *Proof.* We want to show  $\forall x, y \in \cap_{i=1}^p C_i$ , the convex combination of  $x, y$  is also in it.

Since  $x, y \in \cap_{i=1}^p C_i$ , we know  $x, y \in C_i, \forall i = 1, \dots, p$ . with the fact that  $C_i$  is convex, for any  $\alpha \in (0, 1)$ ,  $\alpha x + (1 - \alpha)y \in C_i$  holds for each index  $i$ . Hence,  $\alpha x + (1 - \alpha)y \in \cap_{i=1}^p C_i$ .

The claim is then proved. □

2.  $\cup_{i=1}^p$  may not be convex. A counterexample is to let  $C_1 = \{(x, y) | x = 0, y \in \mathbb{R}\}$ , and  $C_2 = \{(x, y) | y = 0, x \in \mathbb{R}\}$ . It is clear that  $C_1, C_2$  are convex and they are  $x$  and  $y$  axis in  $\mathbb{R}^2$ . However,  $C_1 \cup C_2$  is not convex.

Let  $a = (1, 0) \in C_2, b = (0, 1) \in C_1$ .  $a, b \in C_1 \cup C_2$ , but  $\frac{1}{2}a + \frac{1}{2}b = (1/2, 1/2) \notin C_1 \cup C_2$ .

## Problem 2.5

(It will be easy to use the results from the problems we just solved. But it is fine if you use other methods to prove this claim.)

*Proof.* Let  $A \in \mathbb{R}^{m \times n}$  be  $\begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}$ , where  $a_i^T$  is the  $i$ th row of  $A$ . and  $b = [b_1, \dots, b_m]^T$ . Then the feasible domain  $P$  is equivalent to

$$\cap_{i=1}^m \{x \in \mathbb{R}^n | a_i^T x = b_i\} \cap \{x \in \mathbb{R}^n | x \geq 0\}.$$

Let  $P_i := \{x \in \mathbb{R}^n | a_i^T x = b_i\}$  and each  $P_i$  is a hyperplane. Also, it is obvious that  $\{x \in \mathbb{R}^n | x \geq 0\}$  is convex (use the definition and easy to prove). Use the results from 2.3 and 2.4, and we know that the  $P_i$  is convex and intersection of convex sets is also convex. Hence,  $P$  is convex. □

## Problem 2.6

*Proof.* Consider a LP problem in standard form.

$$\begin{aligned} & \text{Minimize} && c^T x \\ & \text{subject to} && Ax = b \\ & && x \geq 0 \end{aligned}$$

where  $A \in \mathbb{R}^{m \times n}$ ,  $x \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$ .

Denote its feasible domain as  $P := \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ . Let the supporting hyperplane  $H$  of feasible domain  $P$  be the following,

$$H := \{x \in \mathbb{R}^n | -c^T x = \beta\}.$$

and  $\forall x \in P$ ,  $-c^T x \leq \beta$ . since  $H \cap P \neq \emptyset$ , take any  $x^* \in H \cap P$ , we have  $c^T x^* = -\beta$ . Hence,  $\forall x \in P$ ,  $c^T x \geq c^T x^* = -\beta$ , which proves that  $x^*$  is an optimal solution to the LP problem. □

## Problem 2.8

From problem 2.7, the feasible region is

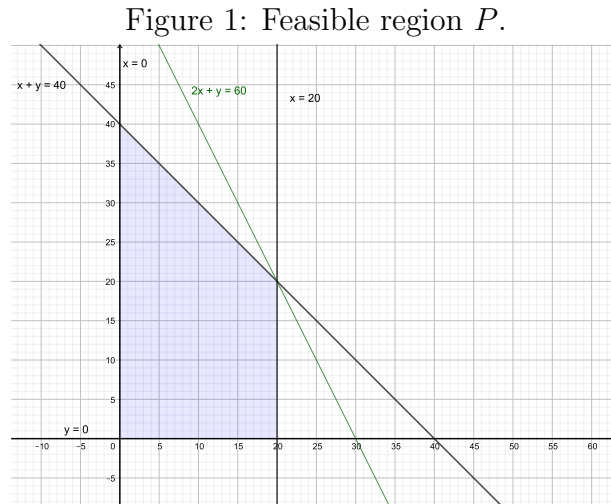


Figure 2: solution to (a)

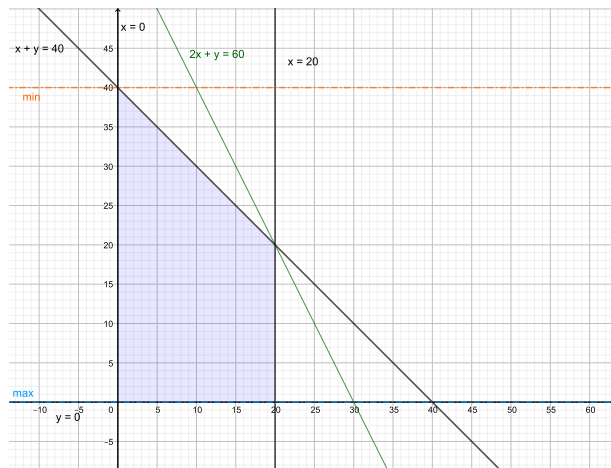


Figure 3: solution to (b)

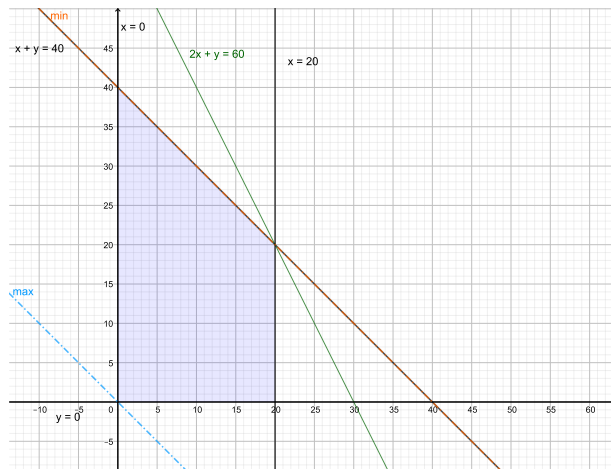


Figure 4: solution to (c)

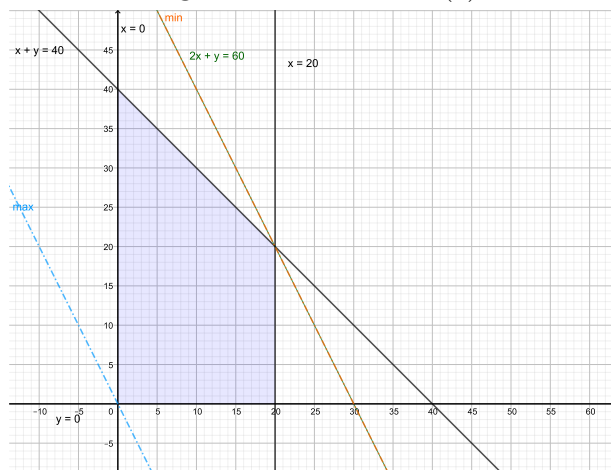


Figure 5: solution to (d)

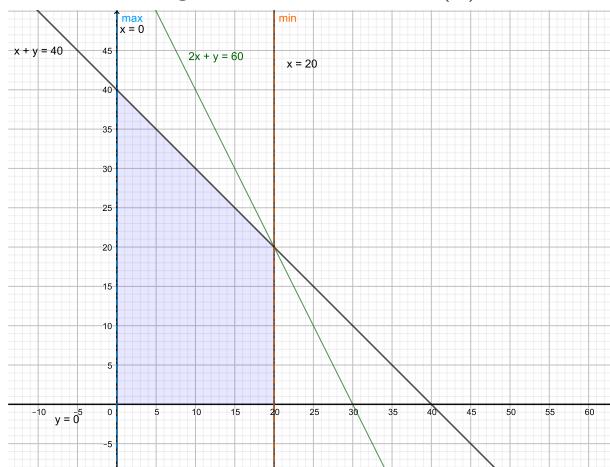


Figure 6: solution to (e)

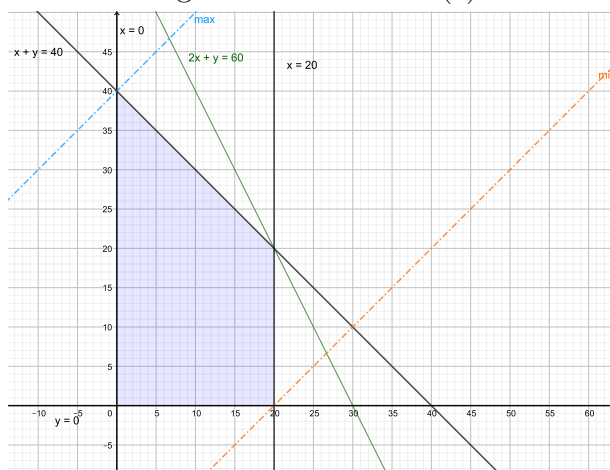




Table 1: optimal solutions

	max	min
a	0	-40
b	0	-40
c	0	-60
d	0	-20
e	40	-20

In our figures,  $x$  is  $x_1$  and  $y$  is  $x_2$ . Using the graphic method and we get the results in following figures.

The optimal values can be tracked in the table.

## Problem 2.9

*Proof.* Consider set  $B$  as the set of optimal solutions to a Lp problem, which as the standard form,

$$\begin{aligned} &\text{Minimize} && c^T x \\ &\text{subject to} && Ax = b \\ &&& x \geq 0 \end{aligned}$$

where  $A \in \mathbb{R}^{m \times n}$ ,  $x \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$ .

Denote its feasible domain as  $P := \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ .

Suppose the optimal value is  $M$ , i.e.,  $\forall x^*, y^* \in B$ ,  $c^T x^* = x^T y^* = M \leq c^T x$ ,  $\forall x \in P$ .

Hence, take arbitrary  $\alpha \in (0, 1)$ , and it is true that  $c^T(\alpha x^* + (1 - \alpha)y^*) = M$ . Hence,  $B$  is convex. □

This result is useful to us. We know if there are two different optimal solutions to the LP problem, then any convex combination of those two solutions are also optimal.