Homework 1 Solutions

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1 Problem 1

Solutions:

1.
$$det \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix} = a^2b^2 - abab = 0.$$

2.
$$det \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \cos^2\alpha + \sin^2\alpha = 1.$$

3.
$$det \begin{pmatrix} 1 & x & x \\ x & 2 & x \\ x & x & 3 \end{pmatrix} = 1 \times 2 \times 3 + x^3 + x^3 - 2x^2 - x^2 - 3x^2 = 2x^3 - 6x^2 + 6.$$

4. Similar to the problem above,
$$\begin{aligned} det \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} &= bc^2 + a^2c + ab^2 - a^2b - ac^2 - b^2c \\ &= ab(b-a) + ac(a-c) + bc(c-b). \end{aligned}$$

5. Change the order of columns. And apply the operations on matrix blocks:
$$\begin{aligned} det \begin{pmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{pmatrix} &= -det \begin{pmatrix} 0 & 2 & a & 1 \\ 0 & b & 0 & 2 \\ c & 4 & 5 & 3 \\ 0 & 0 & 0 & d \end{pmatrix} \\ &= -det \begin{pmatrix} 0 & 2 & a \\ 0 & b & 0 \\ c & 4 & 5 \end{pmatrix} \cdot d \\ &= abcd. \end{aligned}$$

6. Same idea as the problem above. Note that it won't change the determinant of a matrix if we add one row multiplied by a constant number on another row.

   suppose $a \neq 0$. Then, add $\frac{1}{a}$row1 on row2. The determinant doesn't change.

$$det \begin{pmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{pmatrix} = det \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & b+\frac{1}{a} & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{pmatrix}$$

   Then change orders of rows, and change orders of columns.

$$\begin{aligned} det \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & b+\frac{1}{a} & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{pmatrix} &= -det \begin{pmatrix} 0 & b+\frac{1}{a} & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \\ a & 1 & 0 & 0 \end{pmatrix} \\ &= det \begin{pmatrix} b+\frac{1}{a} & 1 & 0 & 0 \\ -1 & c & 1 & 0 \\ 0 & -1 & d & 0 \\ 1 & 0 & 0 & a \end{pmatrix} \\ &= det \begin{pmatrix} b+\frac{1}{a} & 1 & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{pmatrix} \cdot a \\ &= 1 + ab + ad + cd + abcd. \end{aligned}$$

2 Problem 2

Solution: All linear systems can be formed as $Ax = b$.

1. In this problem, $x \in \mathbb{R}^4$, A is a 4 by 4 matrix and b is a 4 by 1 vector.

$$[A|b] = \begin{bmatrix} 2 & -1/2 & -1/2 & 0 & 0 \\ -1/2 & 2 & 0 & -1/2 & 3 \\ -1/2 & 0 & 2 & -1/2 & 3 \\ 0 & -1/2 & -1/2 & 2 & 0 \end{bmatrix} \xrightarrow{Gauss \quad elimination} \begin{bmatrix} 1 & -1/4 & -1/4 & 0 & 0 \\ 0 & 1 & -1/15 & -4/15 & 8/5 \\ 0 & 0 & 1 & -2/7 & 12/7 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

   Thus, the solution to the linear system is $x = (1, 2, 2, 1)^T$.

2.

$$[A|b] = \begin{bmatrix} 2 & 3 & 5 & 1 & 3 \\ 3 & 4 & 2 & 3 & -2 \\ 1 & 2 & 28 & -1 & 8 \\ 7 & 9 & 1 & 8 & 0 \end{bmatrix} \xrightarrow{Gauss \quad elimination} \begin{bmatrix} 1 & 3/2 & 5/2 & 1/2 & 3/2 \\ 0 & 1 & 11 & -3 & 13 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

   We see $[A|b]$ is not in full rank so there is no solution to the linear system.

3 Problem 3

$A = P\Lambda Q$, $A^2 = P\Lambda QP\Lambda Q$. Since $QP = I$, $A^2 = P\lambda^2 Q$. Hence, $A^k = P\Lambda^k Q$.
It is enough to calculate what Λ^k. Since Λ is a diagonal matrix,

- k is a odd number,

  $\Lambda^k = \Lambda$. Then
  $$A^k = P\Lambda Q = \begin{pmatrix} 7 & -12 \\ 6 & -7 \end{pmatrix}.$$

- k is an even number,

  $\Lambda^k = I$. Then
  $$A^k = PIQ = PQ = I.$$

4 Problem 4

The linear system can be formed as $Ax = b$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

It is clear that A is invertible.(You may check that A is of full rank) To find the inverse of A, i.e. A^{-1}, there is one method by Gauss elimination

$$[A|I] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{Gauss \quad elimination} \begin{bmatrix} 1 & 0 & 0 & 1 & -1/2 & 0 \\ 0 & 1 & 0 & 1 & -1/2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} = [I|A^{-1}]$$

The solution is $x = A^{-1}b = (1/2, -3/2, 2)^T$.

5 Problem 5

1.

Proof. Proof goes here. Repeat as needed □