SEMIDEFINITE OPTIMIZATION (MA 591) — HW 1

Convexity and positive semidefinite matrices

1. Let $K \subseteq \mathbb{R}^n$ be a full-dimensional convex cone. Find the simplest possible description of the set $K - K \stackrel{\text{def}}{=} \{\mathbf{x} - \mathbf{y} \,|\, \mathbf{x} \in K, \mathbf{y} \in K\}$. (You'll know it when you have it.) How does the answer change if we drop the assumption that K is full-dimensional?

2. Show that for every $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$, $x_0 \geq \|(x_1, \ldots, x_n)\|_2$ if and only if the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ x_1 & x_0 & & \\ \vdots & & \ddots & \\ x_n & & & x_0 \end{pmatrix}$$

is positive semidefinite. (All entires outside of the first row and column and the diagonal are zero.)

3. Let k be any positive integer, $\mathbf{A} \in \mathbb{S}^k_+$, and $\mathbf{B} \in \mathbb{S}^k$. Show that

$$\operatorname{tr}(\mathbf{A})\lambda_{\min}(\mathbf{B}) \leq \operatorname{tr}(\mathbf{AB}) \leq \operatorname{tr}(\mathbf{A})\lambda_{\max}(\mathbf{B}).$$

(Extra credit.) Also show that for every (not necessarily PSD) symmetric matrices $\mathbf{A}, \mathbf{B} \in \mathbb{S}^k$,

$$\sum_{i=1}^{k} \lambda_i(\mathbf{A})\lambda_{k+1-i}(\mathbf{B}) \leq \operatorname{tr}(\mathbf{AB}) \leq \sum_{i=1}^{k} \lambda_i(\mathbf{A})\lambda_i(\mathbf{B}),$$

where $\lambda_1, \ldots, \lambda_k$ denote the eigenvalues of the appropriate matrices in increasing order.

4. *Choose Your Own Adventure*™:

- If you are a statistician, show that the elementwise product of two positive semidefinite matrices of the same size is positive semidefinite. (*Hint:* a matrix is PSD if and only if it is a covariance matrix of some multivariate random variable.)
- If you are a linear programming fan, prove rigorously that a function $f\colon \mathbb{R}^n \mapsto \mathbb{R}$ is simultaneously convex and concave if and only if it is affine. (That is, iff it can be written as $f(\mathbf{x}) = \mathbf{a}^{\mathrm{T}}\mathbf{x} + b$ for some $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$.) *Note:* do not just assume that f is differentiable!
- If you are a geometer at heart, then prove rigorously that $\mathbb{S}^2_+$ is not a convex polyhedron.
- If linear algebra or matrix theory is your jam, then show that if the symmetric matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy $\mathbf{A} \succcurlyeq \mathbf{B} \succcurlyeq \mathbf{0}$, then they also satisfy $\mathbf{A}^{1/2} \succcurlyeq \mathbf{B}^{1/2}$, but $\mathbf{A}^2 \succcurlyeq \mathbf{B}^2$ does not necessarily hold. *Hint:* for the first half, show first that if $\mathbf{x}$ is an eigenvector of $\mathbf{A}^{1/2} - \mathbf{B}^{1/2}$, then

$$\mathbf{x}^{\mathrm{T}}(\mathbf{A}^{1/2} + \mathbf{B}^{1/2})(\mathbf{A}^{1/2} - \mathbf{B}^{1/2})\mathbf{x} = \mathbf{x}^{\mathrm{T}}(\mathbf{A} - \mathbf{B})\mathbf{x}$$

  holds. (Even though $(\mathbf{A}^{1/2} + \mathbf{B}^{1/2})(\mathbf{A}^{1/2} - \mathbf{B}^{1/2}) \neq \mathbf{A} - \mathbf{B}$ in general, so do not use that "identity".)

5. *(Extra credit.) Do both parts.*

(a) Let f and g be two convex, continuously differentiable functions on the real line, both of which attain a minimum. Prove using basic calculus that $f + g$ also attains a minimum.

(b) Show that the previous assertion *fails* for multivariate functions. That is, find two convex functions f and g with continuous gradients over $\mathbb{R}^n$, for which $f + g$ fails to attain a minimum even though both f and g have minima.

Due on August 30 (Wednesday), by the start of the class. (You may turn in typeset solutions by email.)