Homework 3 Solutions

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Problem 1

Since there is an absolute value in the objective function and x_3 is unrestricted, we may divide the feasible domain into two parts: $x_3 \geqslant 0$ and $x_3 < 0$.

For $x_3 \geqslant 0$, $|x_3| = x_3$. Hence, we may construct a LP problem:

$$\begin{aligned} \text{Manximize} \quad & 3x_1 - 2x_2 + 4x_3 \\ \text{subject to} \quad & -x_1 + 2x_2 \leqslant -5 \\ & 3x_2 - x_3 \geqslant 6 \\ & x_1, x_2, x_3 \geqslant 0 \end{aligned}$$

Similarly, for $x_3 < 0$,$|x_3| = -x_3$, and we can also construct a LP problem:

$$\begin{aligned} \text{Manximize} \quad & 3x_1 - 2x_2 - 4x_3 \\ \text{subject to} \quad & -x_1 + 2x_2 \leqslant -5 \\ & 3x_2 - x_3 \geqslant 6 \\ & x_1, x_2 \geqslant 0, x_3 \leqslant 0. \end{aligned}$$

Then convert those two LP problems above into standard forms:

$$\begin{aligned} \text{Minimize} \quad & -3x_1 + 2x_2 - 4x_3 \\ \text{subject to} \quad & -x_1 + 2x_2 + \xi_1 = -5 \\ & 3x_2 - x_3 - \xi_2 = 6 \\ & x_1, x_2, x_3, \xi_1, \xi_2 \geqslant 0 \end{aligned} \tag{1}$$

and

$$\begin{aligned} \text{Minimize} \quad & -3x_1 + 2x_2 - 4x_3 \\ \text{subject to} \quad & -x_1 + 2x_2 + \xi_1 = -5 \\ & 3x_2 + x_3 - \xi_2 = 6 \\ & x_1, x_2, x_3, \xi_1, \xi_2 \geqslant 0 \end{aligned} \tag{2}$$

Take the optimal solution as the one that solves (1) or (2) with a smaller optimal value.

Problem 2.1

Solutions:

1. We need to show that if the feasible domain is bounded, then the LP problem has a bounded optimal value.

   Proof. Consider the standard form of a LP problem.

   $$\begin{aligned} \text{Minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geqslant 0 \end{aligned}$$

   where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$.

   Denote its feasible domain as $P := \{x \in \mathbb{R}^n | Ax = b, x \geqslant 0\}$. If P is bounded, then $\exists M \geqslant 0$ such that $||x|| \leqslant M$ for all $x \in P$.

   Recall Cauchy-Schwartz inequality, $\forall x, y \in \mathbb{R}^n$,

   $$|x^T y| \leqslant ||x|| \cdot ||y||$$

   Hence, we have

   $$c^T x \geqslant -||c|| \cdot ||x|| = ||c||(-||x||) \geqslant ||c|| \cdot (-M).$$

   Since c is a constant vector, M exists as a constant, we know $c^T x$ has a lower bound, which proves that the LP problem is bounded.

   □

2. Next we give a counterexample to show that the opposite direction is not ture.

   Consider the following LP problem:

   $$\begin{aligned} \text{Minimize} \quad & 2x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \geqslant 1 \\ & x_1, x_2 \geqslant 0 \end{aligned} \tag{3}$$

   It is obvious that the feasible domain is not bounded but the problem is bounded.

Problem 2.2

ATTENTION: The definitions of POLYHEDRON and POLYTOPE are on page 16, Chapter 2.

(a) All of the claims are false. Counterexample is the unit plate ($S = \{(x, y)|x^2 + y^2 \leqslant 1\}$) on $\mathbb{R}^2$.

(b) (i) is true.

Proof. $\forall x, y \in S$, their affine combination is also in S. Hence, $\forall \alpha \in (0, 1)$, $\alpha x + (1 - \alpha)y \in S$. Since it is the convex combination of x and y, it proves the S is convex.

$\square$

(ii) is false. For example, let $S = \{(x, y) \in \mathbb{R}^2|x + y = 1\}$. S is a line on $\mathbb{R}^2$ but it doesn't satisfy the definition of cone.

(iii) is true.

Proof. If S is affine, then pick $x_0 \in S$, construct set $S_0 := \{x - x_0|x \in S\}$. It is clear to see that S_0 is a linear subspace (Use the definition and it is obvious). Then S_0 must be a solution set of a linear system, say $Ay = 0$ for all $y \in S_0$. This is equivalent to say that $\exists A$ such that $\forall x \in S$, $Ax = Ax_0$. Let $b = Ax_0$, and S can be expressed as the intersection of two sets: $P_1 := \{x|Ax \geqslant b\}$ and $P_2 := \{x|Ax \leqslant b\}$. Since P_1 and P_2 are both finite intersections of half spaces, we conclude that S is a finite intersection of half spaces, i.e., S is a polyhedron.

$\square$

(iv) is false because S may not be bounded. Use the same counterexample as for (ii).

(c) All of them are false. A counterexample is $S = \{(x,y)|x = 0, y \geqslant 0\} \cup \{(x,y)|y = 0, x \geqslant 0\}$. We see that S is the union of non-negative parts of x and y axis on $\mathbb{R}^2$. By definition, S is a cone. But it is neither convex nor affine. Also, S is not a polyhedron, so it is not a polytope, either.

(d) If S is a polyhedron, it is the intersection of finite half spaces. So we may express S as a set $\{x \in \mathbb{R}^n | Ax \leqslant b, A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m\}$, which means that S is an intersection of m half spaces.

For (i), it is ture and we would like to show that S is convex.

Proof. Take $x, y \in S$, and $\forall \alpha \in (0,1)$, the convex combination of x and y is in S. This is true because

$$A(\alpha x + (1-\alpha)y) = \alpha Ax + (1-\alpha)Ay \leqslant \alpha b + (1-\alpha)b = b.$$

Hence, S is a convex set.

$\square$

For (ii) and (iii), they are false and we will give a counterexample in $\mathbb{R}^2$.

Let $S := \{(x,y)|0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$. It is clear that S is a polyhedron, but S is neither affine nor a cone.

For (iv), it is false because we may give a unbounded polyhedron $S = \{(x,y)|x \geqslant 0, y \geqslant 0, x + y \geqslant 1\}$. And it is not a polytope.

(e) If S is a polytope, it must be a polyhedron. So (i) and (iv) are true.

(ii) and (iii) are false. We may raise the same counterexample as the first one(the unit square) in part (d).

Problem 2.3

Proof. To prove H is affine, take x, y from H arbitrarily and we need to show the affine combination of x and y is also in H. This is true, since for any $\alpha_1, \alpha_2 \in \mathbb{R}$ that satisfy $\alpha_1 + \alpha_2 = 1$,

$$a^T(\alpha_1 x + \alpha_2 y) = \alpha_1 a^T x + \alpha_2 a^T y = \alpha_1 \beta + \alpha_2 \beta = \beta.$$

which shows $\alpha_1 x + \alpha_2 y$ is in H.

For convexity, it follows from the fact that H is affine because the convex combination of two points is a special case of their affine combination.

□

Problem 2.4

1. *Proof.* We want to show $\forall x, y \in \cap_{i=1}^p C_i$, the convex combination of x, y is also in it.

   Since $x, y \in \cap_{i=1}^p C_i$, we know $x, y \in C_i$, $\forall i = 1, \ldots, p$. with the fact that C_i is convex, for any $\alpha \in (0,1)$, $\alpha x + (1-\alpha)y \in C_i$ holds for each index i. Hence, $\alpha x + (1-\alpha)y \in \cap_{i=1}^p C_i$.

   The claim is then proved.

   □

2. $\cup_{i=1}^p$ may not be convex. A counterexample is to let $C_1 = \{(x,y)|x = 0, y \in \mathbb{R}\}$, and $C_1 = \{(x,y)|y = 0, x \in \mathbb{R}\}$. It is clear that C_1, C_2 are convex and they are x and y axis in $\mathbb{R}^2$. However, $C_1 \cup C_2$ is not convex.

   Let $a = (1,0) \in C_2$, $b = (0,1) \in C_1$. $a, b \in C_1 \cup C_2$, but $\frac{1}{2}a + \frac{1}{2}b = (1/2, 1/2) \notin C_1 \cup C_2$.

Problem 2.5

(It will be easy to use the results from the problems we just solved. But it is fine if you use other methods to prove this claim.)

Proof. Let $A \in \mathbb{R}^{m \times n}$ be $\begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}$, where a_i^T is the ith row of A. and $b = [b_1, \cdots, b_m]^T$. Then the feasible domain P is equivalent to

$$\cap_{i=1}^m \{x \in \mathbb{R}^n | a_i^T x = b_i\} \cap \{x \in \mathbb{R}^n | x \geqslant 0\}.$$

Let $P_i := \{x \in \mathbb{R}^n | a_i^T x = b_i\}$ and each P_i is a hyperplane. Also, it is obvious that $\{x \in \mathbb{R}^n | x \geqslant 0\}$ is convex(use the definition and easy to prove). Use the results from 2.3 and 2.4, and we know that the P_i is convex and intersection of convex sets is also convex. Hence, P is convex.

□

Problem 2.6

Proof. Consider a LP problem in standard form.

$$\begin{aligned} \text{Minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geqslant 0 \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$.

Denote its feasible domain as $P := \{x \in \mathbb{R}^n | Ax = b, x \geqslant 0\}$. Let the supporting hyperplane H of feasible domain P be the following,

$$H := \left\{x \in \mathbb{R}^n | - c^T x = \beta\right\}.$$

and $\forall x \in P$, $-c^T x \leqslant \beta$. since $H \cap P \neq \phi$, take any $x^* \in H \cap P$, we have $c^T x^* = -\beta$. Hence, $\forall x \in P$, $c^T x \geqslant c^T x^* = -\beta$, which proves that x^* is an optimal solution to the LP problem.

□

Problem 2.8

From problem 2.7, the feasible region is

Figure 1: Feasible region P.

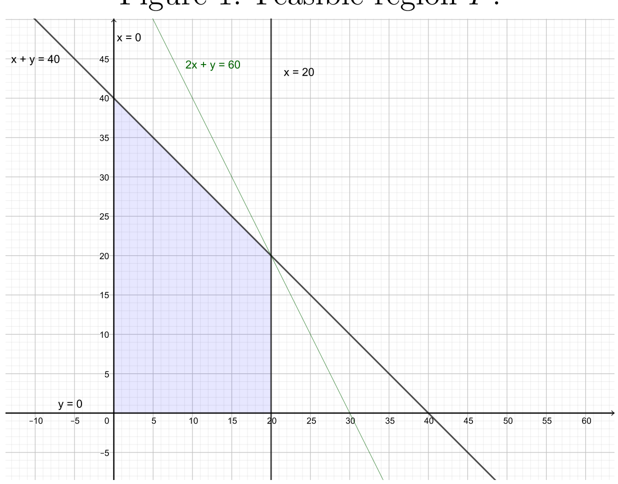

Figure 2: solution to (a)

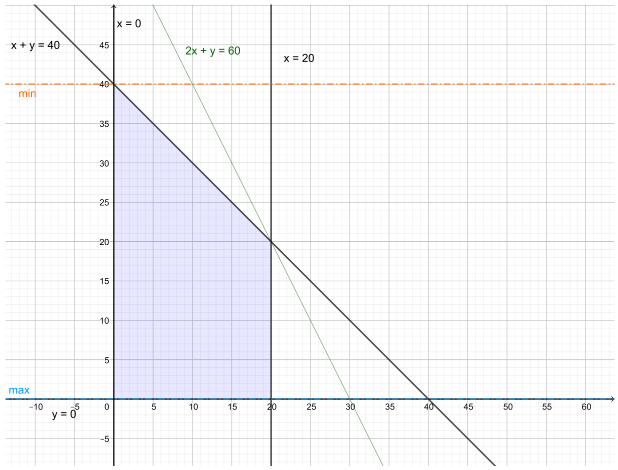

Figure 3: solution to (b)

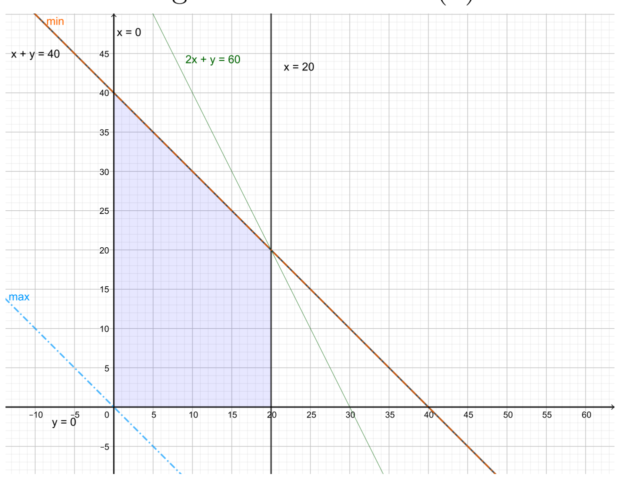

Figure 4: solution to (c)

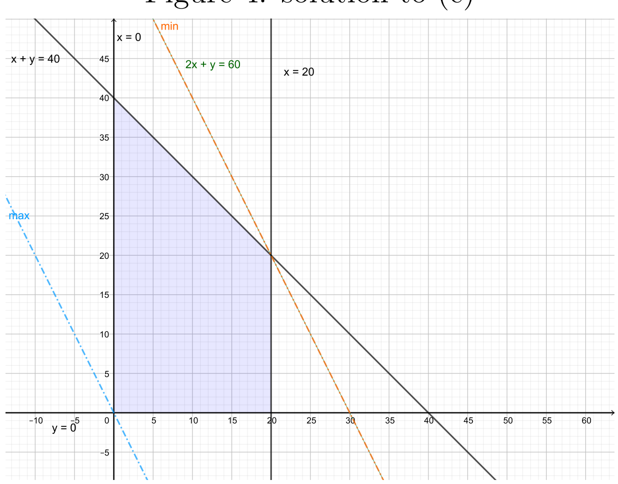

Figure 5: solution to (d)

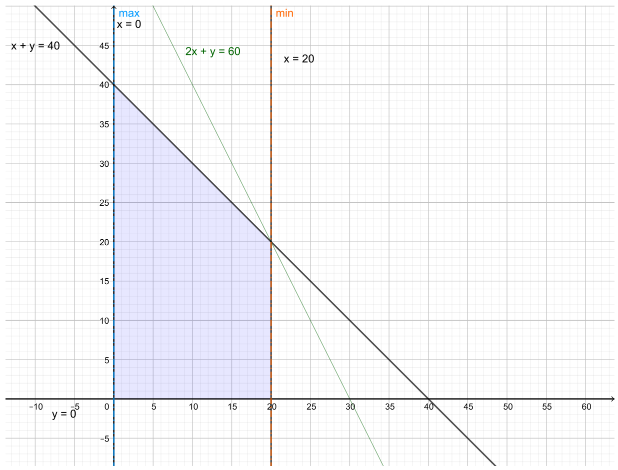

Figure 6: solution to (e)

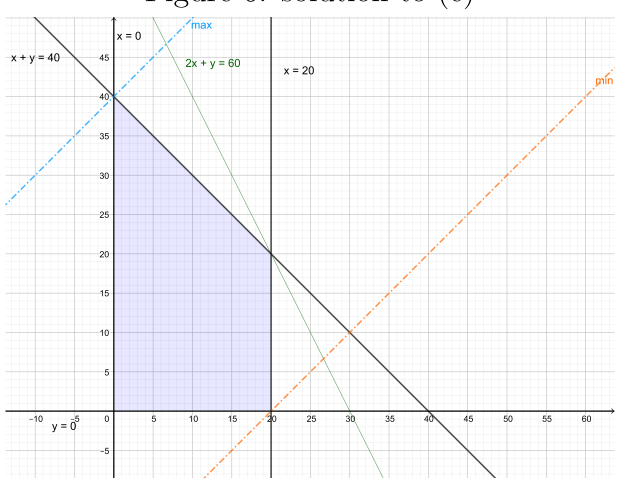

Table 1: optimal solutions

	max	min
a	0	-40
b	0	-40
c	0	-60
d	0	-20
e	40	-20

In our figures, x is x_1 and y is x_2. Using the graphic method and we get the results in following figures.

The optimal values can be tracked in the table.

Problem 2.9

Proof. Consider set B as the set of optimal solutions to a Lp problem, which as the standard form,

$$\begin{aligned} \text{Minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geqslant 0 \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$.

Denote its feasible domain as $P := \{x \in \mathbb{R}^n | Ax = b, x \geqslant 0\}$.

Suppose the optimal value is M, i.e., $\forall x^*, y^* \in B$, $c^T x^* = x^T y^* = M \leqslant c^T x$, $\forall x \in P$.

Hence, take arbitrary $\alpha \in (0, 1)$, and it is true that $c^T(\alpha x^* + (1 - \alpha) y^*) = M$. Hence, B is convex.

$\square$

This result is useful to us. We know if there are two different optimal solutions to the LP problem, then any convex combination of those two solutions are also optimal.