

MA 515 Homework 5

Zheming Gao

October 23, 2017

Problem 1

Proof. Let $V = \text{span}\{v_1, \dots, v_n\}$, where v_1, \dots, v_n are linearly independent elements in V . Then there exists n linearly independent elements $x_1, \dots, x_n \in X$ such that $T(x_i) = v_i, i = 1, \dots, n$. The existence promised by the fact that T is a linear operator. Let $Y_0 = \text{span}\{x_1, \dots, x_n\}$. Hence, $\dim(Y_0) = \dim(V) = n$.

Also, $\ker(T) \cap Y_0 = \{0\}$. Indeed, if $\exists y \neq 0, y \in Y_0$ such that $T(y) = 0$. Let $y = \sum_{i=1}^n \beta_i x_i$. Then there exists $\beta_i \neq 0$. By linearity of T , $T(y) = \sum_{i=1}^n \beta_i T(x_i) = \sum_{i=1}^n \beta_i v_i \neq 0$ and it yields a contradiction.

Next, we will show $\ker(T) + Y_0 = X$. Suppose not, for any $x \in X$, there exists $z \notin \ker(T) + Y_0, w \in \ker(T), r \in Y_0$ such that $x = z + w + r$. Let $r = \sum_{i=1}^n t_i x_i$, and $T(x) = \sum_{i=1}^n \alpha_i v_i$. Hence,

$$\sum_{i=1}^n \alpha_i v_i = T(x) = T(z) + T(w) + T(r) = T(z) + \sum_{i=1}^n t_i v_i.$$

, which implies that $T(z) = \sum_{i=1}^n (\alpha_i - t_i) v_i$.

However, $z \notin \ker(T) + Y_0$ and so $T(z) \notin \text{span}\{v_1, \dots, v_n\} \subset (\ker(T) + Y_0)$. Hence, it is a contradiction.

In conclusion, $\ker(T) + Y_0 = X$ and $\ker(T) \cap Y_0 = \{0\}$, and it implies that $X = \ker(T) \oplus Y_0$. \square

Problem 2

Proof. First we need to show that Λ is bounded. Take arbitrarily $x = \{x_n\}_{n \in \mathbb{N}} \in \ell^\infty$, and there exists $M > 0$ such that $|x_i| \leq M$ for any $i \geq 0$. Therefore,

$$\begin{aligned} \|\Lambda(x)\|_{\ell^\infty} &= \|y\|_{\ell^\infty} = \sup_{i \geq 1} |y_i| \\ &= \sup_{i \geq 1} \left| \frac{x_1 + \dots + x_i}{i} \right| \\ &\leq \sup_{i \geq 1} \left| \frac{\sum_{j=1}^i |x_j|}{i} \right| \leq M. \end{aligned}$$

Hence, $\|\Lambda\|_\infty = \sup_{\|x\|_{\ell^\infty}=1} \|\Lambda(x)\|_{\ell^\infty} \leq M$.

Next we need to find the value of $\|\Lambda\|_\infty$. From the definition,

$$\|\Lambda\|_\infty = \sup_{\|x\|_{\ell^\infty}=1} \|\Lambda(x)\|_{\ell^\infty} = \sup_{\|x\|_{\ell^\infty}=1} \sup_{i \geq 1} \left| \frac{x_1 + \cdots + x_i}{i} \right|.$$

Also, $\|x\|_{\ell^\infty} = 1$ implies $|x_j| \leq 1, \forall j \geq 1$. Hence,

$$\sup_{\|x\|_{\ell^\infty}=1} \sup_{i \geq 1} \left| \frac{x_1 + \cdots + x_i}{i} \right| = \sup_{i \geq 1} \frac{i \cdot 1}{i} = 1.$$

Hence, $\|\Lambda\|_\infty = 1$. □

Problem 3

Proof. It is enough to show that $\mathcal{N}(T) = \{0\}$. If so, then T is invertible and is a linear operator. Indeed, supposed there exists $x \neq 0, x \in \mathcal{N}(T)$, then $0 = \|T(x)\| \geq \|x\| > 0$, which yields a contradiction.

Next we need to show T^{-1} is bounded. Since T is surjective and invertible, we know T must be a bijection. If so, for any $y \in Y$, we have $\|y\| \geq b\|T^{-1}(y)\|$. Hence,

$$\sup_{\|y\| \neq 0} \frac{\|T^{-1}(y)\|}{\|y\|} \leq \frac{1}{b} < +\infty.$$

i.e., T^{-1} is a bounded linear operator. □

Problem 4

Proof. We want to show $\|T_n(x_n) - T(x)\|_Y \rightarrow 0$ as $n \rightarrow +\infty$. By triangle inequality,

$$\|T_n(x_n) - T(x)\|_Y \leq \|T_n(x_n) - T(x_n)\|_Y + \|T(x_n) - T(x)\|_Y \leq \|T_n(x_n) - T(x_n)\|_Y + \|T\|_\infty \|x_n - x\|_X.$$

and we know $\|x_n - x\| \rightarrow 0$ as $n \rightarrow +\infty$. Hence, it is enough to show that $\|T_n(x_n) - T(x_n)\|_Y \rightarrow 0$ as $n \rightarrow +\infty$.

$\forall x_m \in X, m = 1, 2, \dots$, $\lim_{n \rightarrow +\infty} \|T_n(x_m) - T(x_m)\| = 0$ since $\{T_n\}$ converges to T . Hence, we have

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \|T_n(x_m) - T(x_m)\| = 0.$$

This implies $\|T_n(x_n) - T(x_n)\|_Y \rightarrow 0$ because $\{\|T_n(x_n) - T(x_n)\|\}_{n \in \mathbb{N}}$ is a subsequence of $\{\|T_n(x_m) - T(x_m)\|\}_{m, n \in \mathbb{N}}$.

In conclusion, $\lim_{n \rightarrow +\infty} \|T_n(x_n) - T(x_n)\| = 0$. □

Problem 5

Proof. $S \circ T$ is a linear operator. Indeed, the domain of $S \circ T$ is X , which is a subspace of itself. Also, function composition preserves linearity.

Next we need to show $S \circ T$ is bounded. Consider norm $\|S \circ T\|_\infty$,

$$\begin{aligned}\|S \circ T\|_\infty &= \sup_{\|x\|_X=1} \|(S \circ T)(x)\|_Z = \sup_{\|x\|_X=1} \|S(T(x))\|_Z \\ &\leq \sup_{\|x\|_X=1} \|S\|_\infty \|T(x)\|_Y \\ &= \|S\|_\infty \sup_{\|x\|_X=1} \|T(x)\|_Y \\ &= \|S\|_\infty \|T\|_\infty.\end{aligned}$$

Since both S and T are bounded linear operators, $\|S\|_\infty$ and $\|T\|_\infty$ are less than positive infinity and this leads to the conclusion that $\|S \circ T\|_\infty < +\infty$.

In conclusion, $S \circ T$ is a bounded linear operator. □

Problem 6

Proof. (a) T is a contraction mapping. Indeed, let $c = \|T\|_\infty < 1$. Hence, for any $x_1, x_2 \in X$ ($x_1 \neq x_2$),

$$\|T(x_1) - T(x_2)\| = \|T(x_1 - x_2)\| \leq \|T\|_\infty \|x_1 - x_2\| = c \|x_1 - x_2\|.$$

where $0 < c < 1$. Hence, there exists a unique $x_0 \in X$ such that $T(x_0) = x_0$. And this is equivalent to say that linear operator (which is easy to check) $\mathcal{N}(I - T) = \{x_0\}$. However, $\{0\} \in \mathcal{N}(I - T)$ always holds. Thus, $x_0 = 0$.

Next we need to show that $I - T$ is a one-to-one mapping. Suppose not, then there exists $y \in X$ and distinct $y_1, y_2 \in X$ such that $(I - T)(y_1) = (I - T)(y_2) = y$. By linearity, $(I - T)(y_1 - y_2) = 0$ and it yields that $y_1 = y_2$, which is a contradiction.

What's more, $I - T$ is surjective. Indeed, $I - T$ maps from X to X . For any element $x \in X$, \exists unique $y \in X$ such that $y = (I - T)(x)$. Hence, volume of the range of $I - T$ equals to the volume of domain. i.e., $|\mathcal{R}(I - T)| = |\mathcal{D}(I - T)| = |X|$. Also, $\mathcal{R}(I - T) \subset X$, which yields that $\mathcal{R}(I - T) = X$. In conclusion, $I - T$ is bijective.

(b) Let $S = \sum_{n=0}^{\infty} T^n$ and consider $\|S\|_\infty$.

$$\begin{aligned}
\|S\|_\infty &\leq \left\| \lim_{m \rightarrow +\infty} \sum_{n=0}^m T^n \right\|_\infty \\
&= \lim_{m \rightarrow +\infty} \left\| \sum_{n=0}^m T^n \right\|_\infty \\
&\leq \lim_{m \rightarrow +\infty} \sum_{n=0}^m \|T^n\|_\infty \\
&\leq \lim_{m \rightarrow +\infty} \sum_{n=0}^m \|T\|_\infty^n \\
&= \lim_{m \rightarrow +\infty} \sum_{n=0}^m c^n \\
&= \frac{1}{1-c} < +\infty.
\end{aligned}$$

We use triangle inequality above. Also, the limit and norm can exchange due to the continuity of norm.

Hence, S is bounded in $\|\cdot\|_\infty$. And it is obvious that S is a linear operator, so $S \in (B(X, X), \|\cdot\|_\infty)$.

(c) It is enough to check that $S \circ (I - T) = (I - T) \circ S = I$.

$$\begin{aligned}
S \circ (I - T) &= S - S \circ T = \sum_{n=0}^{\infty} T^n - \sum_{n=1}^{\infty} T^n \\
&= T^0 = I
\end{aligned}$$

$$\begin{aligned}
(I - T) \circ S &= S - T \circ S = \sum_{n=0}^{\infty} T^n - \sum_{n=1}^{\infty} T^n \\
&= T^0 = I
\end{aligned}$$

Hence, $S = (I - T)^{-1}$.

□

Problem 7

Proof. For each $n \in \mathbb{N}$, $\|T^n\|_\infty \leq \|T\|_\infty^n$. Since T is a bounded linear operator, $\|T\|_\infty < +\infty$. By triangle inequality and Taylor theorem,

$$\|S\|_\infty \leq \sum_{n=0}^{+\infty} \frac{\|T^n\|_\infty}{n!} \leq \sum_{n=0}^{+\infty} \frac{\|T\|_\infty^n}{n!} = e^{\|T\|_\infty}.$$

□