BlockTech Quant Assignment

This assignment is written taking into account that you have read and understood the explanation in the following link: https://www.investopedia.com/articles/investing/021215/examples-understand-binomial-option-pricing-model.asp.

1 Describing the Stochastic Process. Assume that we have a stock that is valued at \$100 per share. Its movement over time can be described by the following stochastic process. At each time step, it has either moved up by \$1 or down by \$1, both with a risk-neutral probability of 0.5. This price process can be modeled by a recombining binomial tree. In a recombining tree the order of up and down moves is irrelevant: given a certain number of up moves and a certain number of down moves, you will end up in the same node regardless of the order in which the moves are taken. We assume that the interest rates are **zero**.

(a) What is the maximum price that the stock can reach after 20 time steps?
(b) What is the minimum price that the stock can reach after 20 time steps?
(c) We assume that the volatility $\sigma(X_t)$ at time step t is defined by

$$\sigma(X_t) = \frac{\sqrt{\mathbb{E}[X_{t+1}^2|X_t] - \mathbb{E}[X_{t+1}|X_t]^2}}{X_t}, \tag{1}$$

where $\mathbb{E}[X_{t+1}|X_t]$ is the conditional expectation (under the risk-neutral probabilities) of the stock price X_{t+1} at time step $t+1$, given X_t. As an example, at the initial time step, the volatility is calculated as follows:

$$\sigma(X_0) = \frac{\sqrt{0.5 \cdot 101^2 + 0.5 \cdot 99^2 - (0.5 \cdot 101 + 0.5 \cdot 99)^2}}{100} = \frac{1}{100}.$$

Derive a closed-form formula for the volatility at time step t, depending on the stock price X_t. You can do this using pen and paper and taking a picture of it.
(d) Does the volatility increase or decrease when the price goes up? Does that make sense to you given the price process?

2 Pricing Options. In this exercise, we will price a number of options. Assume the setup of Question 1 and assume that the options expire at time step $t = 20$.

(a) Can you intuitively argue what the prices are of a call option with strike \$120 and a put option with strike \$80?
(b) Using the methodology described in the link, write a program to compute the prices of and 80-put, 85-put, 90-put, 95-put, 100-put, 100-call, 105-call, 110-call, 115-call and 120-call.

3 Alternative Binomial Tree. Similar to Question 1, we consider a binomial tree on time steps $t = 0, 1, \ldots, 20$, with the stock price X_t being valued at \$100 at $t = 0$ and with zero interest rates. At each time step t, the stock price either moves up by some absolute amount u or down by d, based on the risk-neutral probability q. The values of u, d and q may be different at each node of the tree. Each node can be uniquely identified by the pair (t, j), where j is defined as the difference between the number of up moves and the number of down moves, i.e. $j = moves_{up} - moves_{down}$.

Note that $-20 \leq j \leq 20$, because the options are assumed to expire at $t = 20$.

In the questions below, you will be asked to derive formulas for u, d and q at each node (t, j), based on certain conditions that the binomial tree should satisfy. These conditions are as follows:

i. The formula for the volatility $\sigma(X_t)$ in Equation (1) reduces to the expression

$$\sigma(X_t) = \frac{1.02^j}{X_t}.$$

ii. The price in a certain node is equal to the price in all other nodes with the same value of j.

iii. Given the current price X_t, the expected price in the next time step is equal to the current price.

iv. The tree is recombining, that is, given a certain number of $moves_{up}$ and $moves_{down}$, you will end up in the same node regardless of the order in which the moves are taken.

You can work on pen and paper. Please show your steps that you used to arrive at your answer.

(a) One of the conditions is implied by another one. Which one? Explain your reasoning.

(b) Assume that the up and down moves are always equal to \$1 and that $q \neq 0.5$. Under this assumption, there is one condition that cannot be satisfied for any value of t and j. Which one can **never** be satisfied?

(c) Assume now that we may freely choose the up moves u and down moves d. They may be different at each node (t, j). Based on the condition from the previous question, express the risk-neutral probability q in terms of u and d.

(d) Using conditions i and iii, derive a relation between u and d at each node (t, j).

(e) Using conditions ii and iv and your answer to the previous question, derive a closed-form formula for u at each node, depending **at most** on u_0, t and j. Here, u_0 is the up move at the initial node of the tree.

(f) Using your answers to the previous questions, derive a closed-form formula for the risk-neutral probabilities q at each node, depending **at most** on u_0, t and j. How would you easily check if your formulas for u, d and q make sense?

4 **Implementing the Alternative Model.** In this exercise you will implement the alternative binomial tree to price the options from Question 2.

(a) Use your formulas for u, d and q from Question 3 to price the same options from Question 2(b).

(b) The put-call parity is a fundamental equality of option pricing and is based on a simple no-arbitrage argument. This means that it should hold independently of the underlying stock price process. If you inspect your prices from the previous question with at least 10 decimal places, you will find an example of two options for which the put-call parity is not holding (but should). Which two options? Why do you think the put-call parity is not holding?